15.401 Finance Theory I

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Lecture 5: Forwards and Futures

Key concepts

Forward contracts

Futures contracts

Mark to market

Forward and futures prices

Commodity futures

Financial futures

Hedging with forwards and futures

forward - custom
future - standardized

Readings:

Brealey, Myers and Allen, Chapter 26

Bodie, Kane and Markus, Chapters 22, 23.1 - 23.2, 23.6

Use of Derivatives

Use of Derivatives Among Fortune 500 Companies (Source: International Swaps and Derivatives Association, 2009 Survey)

Sector	Interest rate	For Ex	Commodity	Credit	Equity
Basic materials	60	74	68	0	5
Consumer goods	46	53	26	1	6
Financial	116	117	75	93	97
Health care	17	14	1	1	5
Industrial goods	34	34	9	1	9
Services	66	69	31	1	8
Technology	55	59	10	4	11
Utilities	22	21	20	0	2
Total	416	441	240	101	143

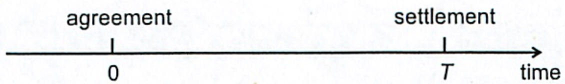
Market Size

Source: BIS, 2010 (in billions of US dollars)

	Notional amounts outstanding						Gross market values					
	Jun 2007	Dec 2007	Jun 2008	Dec 2008	Jun 2009	Dec 2009	Jun 2007	Dec 2007	Jun 2008	Dec 2008	Jun 2009	Dec 2009
Total	516,407	595,738	683,814	547,371	604,622	614,674	11,140	15,834	20,375	32,244	25,372	21,583
Foreign exchange contracts	48,645	56,238	62,983	44,200	48,775	49,196	1,345	1,807	2,262	3,591	2,470	2,069
Interest rates contracts	347,312	393,138	458,304	385,896	437,198	449,793	6,063	7,177	9,263	18,011	15,478	14,018
Equity-linked contracts	8,590	8,469	10,177	6,159	6,619	6,591	1,116	1,142	1,146	1,051	879	710
Commodity contracts	7,567	8,455	13,229	3,820	3,729	2,944	636	1,898	2,209	829	689	545
Credit default swaps	42,581	58,244	57,403	41,883	36,046	32,693	721	2,020	3,192	5,116	2,987	1,801
Unallocated	61,713	71,194	81,719	65,413	72,255	73,456	1,259	1,790	2,303	3,645	2,868	2,440

3/14 + 3/16

A forward contract is a commitment to buy (sell) at a future date a given amount of a commodity or an asset at a price agreed on today.



- The price fixed now for future exchange is the forward price
- The buyer obtains a "long position'" in the asset or commodity

Example. A tofu manufacturer needs 100,000 bushels of soybeans in 3 months.

- Current price of soybeans is $12.50/bu but may go up
- Wants to make sure that 100,000 bushels will be available
- Enter 3-month forward contract for 100,000 bushels of soybeans at $13.50/bu
- Long side buy 100,000 bushels from short side at $13.50/bu in 3 months

Features of forward contracts:

- Traded over the counter (not on exchanges)
- Custom tailored
- No money changes hands until maturity

Advantages of forward contracts:

- Full flexibility

Disadvantages of forward contracts:

- Counter party risk
- Illiquidity

A futures contract is an exchange-traded, standardized, forward-like contract that is marked to the market daily. Futures contracts can be used to establish a long (or short) position in the underlying commodity or asset.

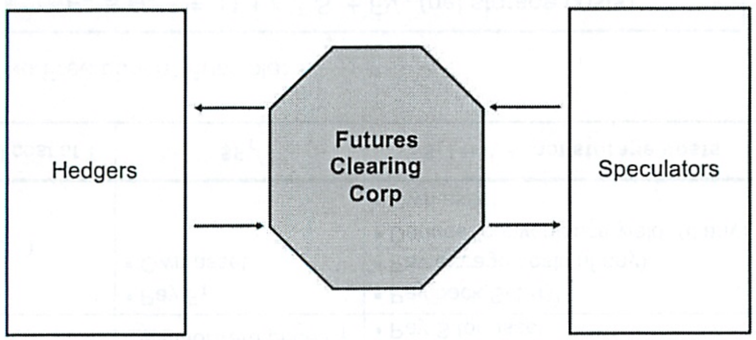
Features of futures contracts:

- Standardized contracts:
  - ① underlying commodity or asset (size, grade)
  - ② quantity
  - ③ maturity
- Traded on exchanges
- Guaranteed by the clearing house --- little counter-party risk
- Gains and losses settled daily --- marked to market
- Margin account required as collateral to cover losses

Example. NYMEX crude oil (light sweet) futures with delivery in Dec 2010 were traded at a price of $78.82/barrel on Sept 13, 2010.

- Each contract is for 1,000 barrels
- Tick size: $0.01 per barrel, $10 per contract
- Initial margin: $5,063
- Maintenance margin: $3,750
- No cash changes hands today (contract price is $0)
- Buyer has a "long" position (wins if prices go up)
- Seller has a "short" position (wins if prices go down)

Futures clearing house reduces counter party risk and improves liquidity



Example. Yesterday, you bought 10 December live-cattle contracts on the CME, at a price of \$0.7455/lb.

Contract size 40,000 lb

Agreed to buy 400,000 pounds of live cattle in December

Value of position yesterday:

$$(\$0.7455)(10)(40,000) = \$298,200$$

No money changed hands

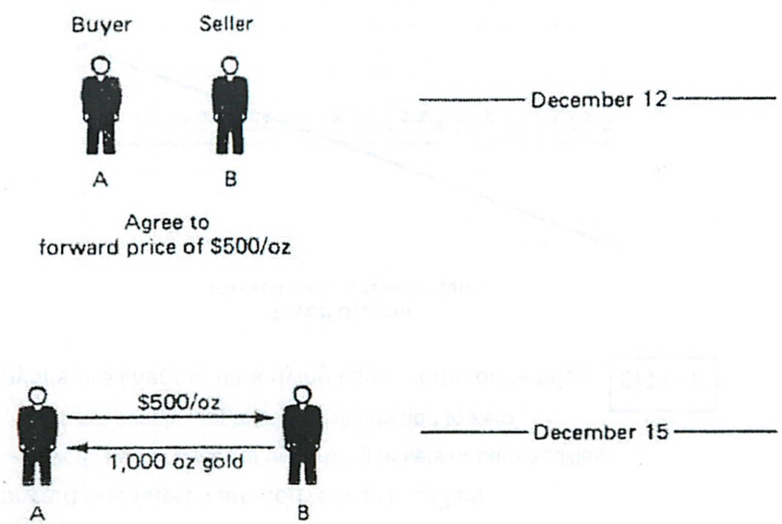
Initial margin required (5% - 20% of contract value)

Today, the futures price closes at \$0.7435/lb, 0.20 cents lower. The value of your position is now

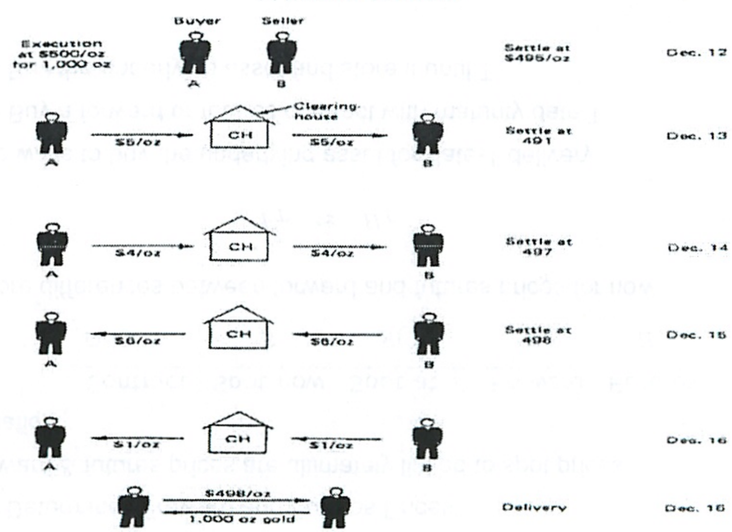
$$(\$0.7435)(10)(40,000) = \$297,400$$

which yields a loss to your position of \$800.

A forward contract



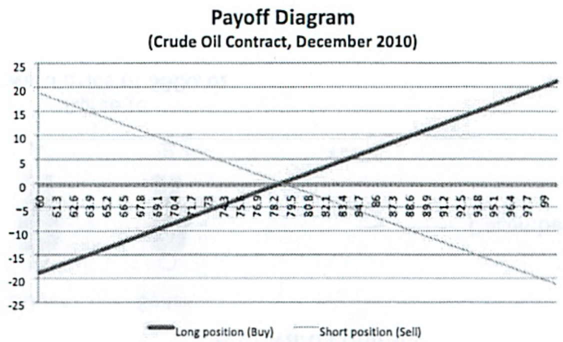
A futures contract



Forward and futures are derivative securities

- Payoffs tied to prices of underlying assets or commodities
- Zero net supply (aggregate positions add to zero)

Payoffs are linear in underlying asset/commodity price: $\boxed{S(T) - F}$



What Determines Forward and Futures Prices?

Forward & futures prices are ultimately linked to spot prices

Notation:

Contract	Spot now	Spot at T	Forward	Futures
Price	S	$S(T)$	F_T	H_T

Ignore differences between forward and futures prices for now

$$F_T \approx H_T$$

Two ways to buy the underlying asset for date-T delivery

1. Buy a forward or futures contract with maturity date T
2. Buy the underlying asset and store it until T

Date	Forward Contract	Outright Asset Purchase
0	▪ Pay \$0 for contract with forward price F_T	▪ Borrow S ▪ Pay S for asset
T	▪ Pay F_T ▪ Own asset	▪ Pay back $S(1+r)^T$ ▪ Pay storage costs (if any) ▪ Deduce "convenience yield" (if any) ▪ Own asset
Total cost at T	$\$F_T$	$\$S(1+r)^T$ + net storage costs

By "No Free Lunch" Principle:

$$F_T \approx H_T = (1+r)^T S + FV_T(\text{net storage costs})$$
$$= (1+r)^T S - FV_T(\text{net convenience yield})$$

Gold

Held for long-run investments

Easy to store → negligible cost of storage

No dividends or benefits

Two ways to buy gold at date T :

Buy now for S and hold until T

Buy forward, pay F and take delivery at T

• No-arbitrage condition requires that:

$$F \approx H = S(1+r)^T$$

Example. Gold quotes on 2009.09.02 were

Spot price \$976.60/oz

2010 February futures (CMX) \$979.80/oz

The implied interest rate for the 5-month period is r = 0.79%

Month	Last	Change	Prior Settle	Open	High	Low	Volume
Sept 2010	1245.0	+0.5	1244.5	1242.6	1247.0	1242.6	18
Oct 2010	1244.8	-0.2	1245.0	1246.6	1249.5	1241.1	4,327
Nov 2010	12460	-0.1	1246.1	1246.1	1249.9	1243.7	274
Dec 2010	1246.5	0.00	1246.5	1247.3	1251.0	1242.3	71,717
Feb 2010	1247.7	-0.4	1248.1	1246.8	1251.4	1245.2	875
Apr 2010	1248.8	-0.8	1249.6	1247.5	1251.9	1246.8	90
Jun 2010	1251.7	+0.5	1251.2	1251.0	1252.7	1251.0	370

Source: CME, Sept-13-2010,
http://www.cmegroup.com/trading/metals/precious/gold_quotes_globex.html

Oil

- Not held for long-term investment (unlike gold), but for future use
- Costly to store
- Additional benefits (convenience yield) for holding physical commodity (compared to holding futures)

Let the percentage holding cost be 'c' and convenience yield be 'y.'

$$F \approx H = S[1+r-(y-c)]^T$$
$$= S(1+r-\hat{y})^T$$

where $\hat{y} = y - c$ is the net convenience yield.

Example. Prices on 2009.09.02 are

- Spot oil price 68.05/barrel (light sweet)
- Dec 09 oil futures price 69.33/barrel (NYMEX)
- 3-month interest rate 0.33% (LIBOR)

Annualized net convenience yield: $\hat{y} = -7.41\ \%$

For commodity futures:

1. Contango: Futures prices increase with maturity
2. Backwardation: Futures prices decrease with maturity

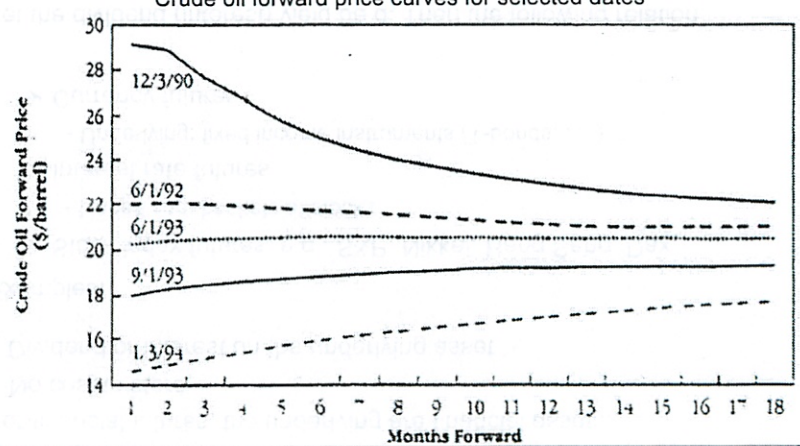
Another definition is one that adjusts for the time-value of money:

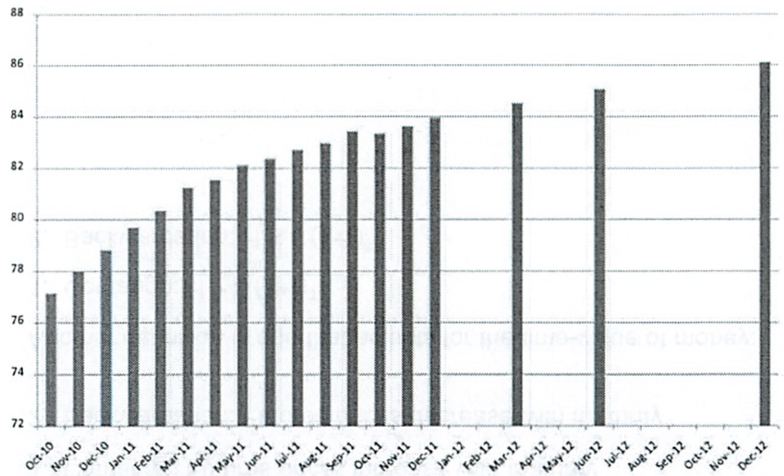
1. Contango: $H > S(1+r)^T$
2. Backwardation: $H < S(1+r)^T$

Backwardation occurs if net convenience yield exceeds interest rate:

$$\hat{y} - r = y - c - r > 0$$

Crude oil forward price curves for selected dates





Source: CME, Sept-13-2010,
http://www.cmegroup.com/trading/energy/crude-oil/light-sweet-crude.html

For financial futures, the underlying are financial assets.

No cost to store

Dividend or interest on the underlying asset

Examples:

- Stock index futures, e.g., S&P, Nikkei, Hang Seng, Dax, . . .
  - Underlying: baskets of stocks
- Interest rate futures
  - Underlying: fixed income instruments (T-bonds, . . .)
- Currency futures

Let the dividend (interest) yield be d. Then the following relation between the forward/futures price and spot price holds:

$$F = S(1 + r - d)^T \approx H$$

Stock index futures

Futures settled in cash (no delivery)

Underlying asset (basket of stocks) pay dividends

Example. Prices on 2009.09.02 are

S&P 500 closed at 994.75

S&P futures maturing in December closed at 989.70

3-month interest rate 0.33%

$$d = [1 + r - (F/S)^{12/3}]$$

$$= [1 + (0.0033) - (989.70/994.75)^4] = 2.35\%$$

The annual dividend yield is: d = 2.35%.

Since the underlying asset is a portfolio in the case of stock index futures, trading in the futures market is easier than trading in cash market when trading portfolios

Thus, futures prices may react more quickly to macro-economic news than the index itself

Index futures are very useful to market makers, investment bankers, stock portfolio managers:

- hedging market risk in block purchases & underwriting
- creating synthetic index funds
- Implementing portfolio insurance

Example. You have $1 million to invest in the stock market and you have decided to invest in a diversified portfolio. The S&P index is a good candidate. How would you do this?

a) One approach is to buy the S&P index in the cash market:
- Buy the 500 stocks
- Weights proportional to their market capitalization

b) Another way is to buy S&P futures:
- Deposit the money in your margin account
- Assuming S&P is at 1000 now and the each contract assigns $250 to each index point (see, e.g., WSJ), the number of contracts to buy is:

$$\frac{\$1{,}000{,}000}{(\$250)(1000)} = 4$$

Example. (Cont'd)

As the S&P index fluctuates, the future value of your portfolio (in $M) would be the following (ignoring interest payments and dividends):

S&P	Portfolio (a)	Portfolio (b)
900	0.90	0.90
1000	1.00	1.00
1100	1.10	1.10

Interest rate futures

Underlying assets are riskless or high grade bonds

Delivery is required (allowing substitutes)

Example. Consider a T-bond with annual coupon rate of 7% (with semi-annual coupon payments) that is selling at par. Suppose the current short rate is 5% (APR). What should be the 6-month forward price of the T-bond?

Coupon yield on the bond:

$$y = (7\%) / 2 = 3.5\%$$

Forward price:

$$F = S(1 + r - y)^T = (\$100)(1 + 2.5\% - 3.5\%) = \$99.00$$

Hedging with Forwards

Hedging with forward contracts is simple, because one can tailor the contract to match maturity and size of position to be hedged.

Example. Suppose that you, the manager of an oil exploration firm, have just struck oil. You expect that in 5 months time you will have 1 million barrels of oil. You are unsure of the future price of oil and would like to hedge the oil price risk.

Using a forward contract, you could hedge your position by selling forward 1 million barrels of oil. Let S(t) be the spot oil price at t (in months). Then,

Position	Value in 5 months (per barrel)
Long position in oil	$S(5)$
Short forward position	$F - S(5)$
Net payoff	F

One problem with using forwards to hedge is that they are illiquid.

If after 1 month you discover that there is no oil, then you no longer need the forward contract. In fact, by holding just the forward contract you are now exposed to the risk of oil-price changes.

In this case, you would want to unwind your position by buying back the contract. Given the illiquidity of forward contracts, this can be difficult and expensive.

To avoid problems with illiquidity of forwards, one may use futures contracts.

Example. (Continued) In the above example, you can sell 1 million barrels worth of oil futures. Suppose that the size of each futures contract is 1,000 barrels. The number of contracts you want to short is

$$\frac{1{,}000{,}000}{1{,}000} = 1{,}000$$

Example. We have \$10 million invested in government bonds and are concerned with movement of interest rates over the next six months. Use the 6-month T-bond futures to protect investment value.

Duration of the bond portfolio is 6.80 years

Current futures price is \$93 2/32 (for face value of \$100)

- The T-bond to be delivered has a duration of 9.20 years
  Each contract delivers \$100,000 face value of bonds
- Futures price for the total contract is \$93,062.50
- 6-month interest rate is 4%

Should we short or long the futures? <u>Short.</u> (Do you know why?)

How many contracts to short?

Match duration:

$$(\text{\# of contracts})(\$93{,}062.50)(9.20) = (\$10{,}000{,}000)(6.80)$$

Thus:

$$(\text{\# of contracts}) = \frac{\$10{,}000{,}000}{\$93{,}062.50} \; \frac{6.80}{9.20} = 79.42$$

Since futures contracts are standardized, they may not perfectly match your hedging need. The following mismatches may arise when hedging with futures:

Maturity

Contract size

Underlying asset

Thus, a perfect hedge is available only when

1. the maturity of futures matches that of the cash flow to be hedged
2. the contract has the same size as the position to be hedged
3. the cash flow being hedged is linearly related to the futures'

In the event of a mismatch between the position to be hedged and the futures contract, the hedge may not be perfect – basis risk.

Forward contracts

Futures contracts

Mark to market

Forward and futures prices

Commodity futures

Financial futures

Hedging with forwards/futures

15.401 – Spring 2011 – Financial Futures

1. Financial futures are:

   - Standardized contracts,
   - To deliver or receive,
   - A specified financial instrument,
   - At a specified price and date.

2. Commodity futures have been traded for a long time. Financial futures are increasingly popular, and are traded on many different underlying financial instruments, such as debt securities and equity indices.

3. The buyer of the contract agrees to accept delivery of the financial instrument (**long position**), and the seller of the contract agrees to deliver the financial instrument (**short position**).

4. Financial futures contracts are traded on organized exchanges, such as the CME Group which created by the merger of the Chicago Board of Trade (CBT) and the Chicago Mercantile Exchange (CME). Only members of a futures exchange can trade futures on the exchange floor. Floor brokers or commission brokers execute orders for customers. Professional traders or position traders or scalpers or locals, in contrast, trade futures for their own account.

5. To trade financial futures contracts, an investor must open an account with a brokerage house which executes futures trades. The investor must also establish a margin deposit before trades can be executed. These margin deposits often consist of Treasury Bills, which continue to earn interest for the investor.

6. The initial margin is a small percentage of the contract's full value, perhaps 3% to 20%. This allows an investor to take a substantial position in financial futures:

   - Without actually buying the asset, and
   - With huge financial leverage.

7. Remember that financial leverage "levers" both positive and negative rates of return. A successful futures trade can generate enormous positive rates of return. An unsuccessful futures trade, in contrast, can result in disaster. At the end of ***EVERY TRADING DAY***, the futures contract is "***marked to market***," with the value of the contract revised to match prices at the end of the day. If a trader's position has deteriorated below the maintenance margin, the investor will receive our old friend, the margin call. If the investor cannot meet the margin call, his or her position will be liquidated by the exchange.

8. The funny thing about financial futures, an investor's contract is actually with the exchange itself, not the counter party to a long or short position. For every futures contract there is one long and one short, but the exchange's clearinghouse stands between all trades, guaranteeing payment on all futures contracts. The clearinghouse also supervises delivery of contracts on settlement date. In this way, longs and shorts contract do not know, or need to know, the counter party. The Clearinghouse precludes a need to know.

9. Finally, very few futures contracts are actually delivered. When an investor chooses to close out a profit or loss position, he or she typically executes the opposite side of the same contract. For example, if an investor is long one December Treasury Bond futures contract, shorting the same December Treasury Bond futures contract closes the position. The investor receives the profit or must cover the loss when the position is closed.

15.401 Lecture 5 Futures

Futures Contracts

Metal & Petroleum Futures

	Open	High	hi lo	Low	Settle	Chg	Open interest
Copper-High (CMX)-25,000 lbs.; cents per lb.							
March	434.00	434.20		424.85	**432.50**	1.20	3,448
May	434.55	435.85		425.40	**433.85**	1.15	97,898
Gold (CMX)-100 troy oz.; $ per troy oz.							
March	1431.60	1432.60		1424.30	**1426.90**	-7.20	561
April	1432.30	1437.20		1423.60	**1427.20**	-7.30	304,213
June	1434.50	1438.60		1425.30	**1428.80**	-7.20	88,261
Aug	1430.00	1438.90		1428.00	**1430.20**	-7.20	26,214
Dec	1437.40	1442.50		1430.00	**1433.00**	-7.20	29,101
Dec'12	1450.00	1456.50		1450.00	**1450.30**	-6.70	10,919
Platinum (NYM)-50 troy oz.; $ per troy oz.							
March	...	...		...	**1802.60**	-17.80	15
April	1821.20	1821.20		1794.00	**1802.60**	-17.80	33,021
Silver (CMX)-5,000 troy oz.; cnts per troy oz.							
March	3581.5	3639.5		3565.0	**3565.3**	-20.2	1,596
May	3589.5	3655.0		3551.0	**3565.8**	-20.7	82,515
Crude Oil, Light Sweet (NYM)-1,000 bbls.; $ per bbl.							
April	104.85	105.79		103.33	**105.02**	-0.42	262,242
May	106.13	107.08		104.65	**105.95**	-0.78	235,586
June	106.72	107.63		105.25	**106.48**	-0.80	148,445
July	107.66	107.91		105.64	**106.86**	-0.81	83,255
Dec	106.77	107.77		105.56	**106.79**	-0.77	183,649
Dec'12	104.00	104.36		102.46	**103.56**	-0.46	134,818
Heating Oil No. 2 (NYM)-42,000 gal.; $ per gal.							
April	3.0469	3.0712		2.9829	**3.0111**	-.0546	84,927
May	3.0500	3.0802		2.9957	**3.0231**	-.0512	53,722
Gasoline-NY RBOB (NYM)-42,000 gal.; $ per gal.							
April	2.9950	3.0105		2.9323	**2.9467**	-.0572	78,176
May	2.9999	3.0177		2.9430	**2.9560**	-.0559	60,999
Natural Gas (NYM)-10,000 MMBtu.; $ per MMBtu.							
April	3.920	3.964		3.832	**3.864**	-.063	184,452
May	3.984	4.021		3.892	**3.924**	-.060	225,151
June	4.043	4.081		3.963	**3.986**	-.056	66,508
July	4.118	4.151		4.033	**4.062**	-.051	81,648
Oct	4.211	4.249		4.135	**4.168**	-.040	72,519
Jan'12	4.790	4.804		4.716	**4.740**	-.038	59,162

Agriculture Futures

	Open	High	hi lo	Low	Settle	Chg	Open interest
Corn (CBT)-5,000 bu.; cents per bu.							
March	710.75	712.00		694.50	**698.75**	-12.00	6,489
May	717.50	719.50		700.25	**705.50**	-12.00	708,426
Ethanol (CBT)-29,000 gal.; $ per gal.							
April	2.565	2.570		2.538	**2.561**	-.034	980
July	2.555	2.555		2.533	**2.546**	-.029	1,208
Oats (CBT)-5,000 bu.; cents per bu.							
March	371.50	373.00		371.50	**368.50**	-10.00	40
May	385.75	385.75		373.00	**375.00**	-11.00	8,824
Soybeans (CBT)-5,000 bu.; cents per bu.							
March	1389.25	1389.25		1368.00	**1375.50**	-12.25	3,872
May	1395.75	1396.00		1374.00	**1382.00**	-13.00	256,182
Soybean Meal (CBT)-100 tons; $ per ton.							
March	355.00	358.00		354.00	**355.40**	-2.70	1,945
May	364.10	364.80		358.50	**360.40**	-3.00	90,332
Soybean Oil (CBT)-60,000 lbs.; cents per lb.							
March	58.39	58.40		57.60	**57.97**	-.60	1,739
May	59.20	59.20		58.08	**58.48**	-.62	174,039
Rough Rice (CBT)-2,000 cwt.; cents per cwt.							
March	1350.00	1350.00		1342.00	**1352.50**	-16.00	34
May	1395.50	1396.00		1361.00	**1379.50**	-15.00	16,380
Wheat (CBT)-5,000 bu.; cents per bu.							
March	774.25	775.00		749.25	**751.25**	-17.75	267
May	801.00	803.25		778.25	**779.75**	-21.00	193,744
Wheat (KC)-5,000 bu.; cents per bu.							
March	883.25	886.50		865.00	**864.50**	-23.50	178
May	897.50	900.00		873.00	**874.00**	-24.50	88,708
Wheat (MPLS)-5,000 bu.; cents per bu.							
March	933.50	933.50		933.50	**912.00**	-28.50	36
May	942.75	943.00		918.25	**922.25**	-21.50	23,060
Cattle-Feeder (CME)-50,000 lbs.; cents per lb.							
March	129.900	131.400		129.850	**131.175**	1.375	7,119
Aug	134.800	136.300	▲	134.600	**136.125**	1.375	13,878
Cattle-Live (CME)-40,000 lbs.; cents per lb.							
April	113.550	115.725		113.500	**114.800**	1.200	118,321
June	114.825	116.650		114.725	**115.900**	1.150	118,640
Hogs-Lean (CME)-40,000 lbs.; cents per lb.							
April	86.700	89.400		86.200	**89.125**	2.725	64,603
June	99.875	102.200		99.325	**101.675**	2.200	74,598
Pork Bellies (CME)-40,000 lbs.; cents per lb.							
March	...	116.500		...	**116.500**	2.000	1
Lumber (CME)-110,000 bd. ft., $ per 1,000 bd. ft.							
March	297.90	300.50		295.40	**300.60**	...	774
May	318.00	321.40		316.30	**318.50**	-.30	5,946
Milk (CME)-200,000 lbs., cents per lb.							
March	19.55	19.63	▲	19.48	**19.54**	-.07	6,673
April	18.40	18.45		17.78	**17.96**	-.44	6,043
Cocoa (ICE-US)-10 metric tons; $ per ton.							
March	3,700	3,700		3,700	**3,703**	-18	85
May	3,670	3,734		3,614	**3,633**	-29	74,838
Coffee (ICE-US)-37,500 lbs.; cents per lb.							
March	282.55	287.00	▲	282.55	**287.15**	6.20	371
May	282.40	287.65	▲	281.30	**287.20**	6.05	79,044
Sugar-World (ICE-US)-112,000 lbs.; cents per lb.							
May	30.06	31.05		29.75	**30.70**	.71	238,594
July	27.60	28.39		27.36	**28.08**	.45	175,066
Sugar-Domestic (ICE-US)-112,000 lbs.; cents per lb.							
May	39.65	39.65		39.60	**39.60**	-.02	1,631
Jan'12	36.02	36.04		36.02	**36.20**	.22	1,586
Cotton (ICE-US)-50,000 lbs.; cents per lb.							
March	211.00	215.00		202.05	**207.04**	-7.46	517
May	213.76	215.75		207.14	**207.14**	-7.00	70,601
Orange Juice (ICE-US)-15,000 lbs.; cents per lb.							
March	178.00	180.05		178.00	**178.70**	1.00	676
May	175.00	178.50		174.50	**176.90**	2.35	25,188

Interest Rate Futures

	Open	High	hi lo	Low	Settle	Chg	Open interest
Treasury Bonds (CBT)-$100,000; pts 32nds of 100%							
March	120-160	120-240		120-000	**120-030**	-22.0	29,386
June	118-290	119-060		118-120	**118-160**	-22.0	601,937
Treasury Notes (CBT)-$100,000; pts 32nds of 100%							
March	119-285	120-040		119-210	**119-245**	-8.5	34,264
June	118-155	118-235		118-070	**118-110**	-8.5	1,492,320
5 Yr. Treasury Notes (CBT)-$100,000; pts 32nds of 100%							
March	117-190	117-240		117-132	**117-167**	-5.0	37,453
June	116-212	116-260		116-150	**116-190**	-4.7	1,182,864
2 Yr. Treasury Notes (CBT)-$200,000; pts 32nds of 100%							
March	109-145	109-152		109-125	**109-137**	-1.5	38,202
June	109-035	109-047		109-010	**109-022**	-2.0	823,832
30 Day Federal Funds (CBT)-$5,000,000; 100 - daily avg.							
March	99.853	99.858	▲	99.853	**99.855**	.005	67,302
May	99.865	99.865	▲	99.860	**99.860**	...	62,957
1 Month Libor (CME)-$3,000,000; pts of 100%							
March	99.7475	99.7475		99.7425	**99.7450**	...	6,880
April	99.7400	99.7400	▲	99.7400	**99.7375**	...	6,650
Eurodollar (CME)-$1,000,000; pts of 100%							
March	99.6900	99.6900		99.6875	**99.6875**	...	975,540
Sept	99.5400	99.5500		99.5300	**99.5350**	-.0100	1,350,091
Dec	99.3800	99.3900		99.3600	**99.3700**	-.0150	1,354,506
March'12	99.1250	99.1450		99.0950	**99.1150**	-.0200	1,280,797

Currency Futures

	Open	High	hi lo	Low	Settle	Chg	Open interest
Japanese Yen (CME)-¥12,500,000; $ per 100¥							
March	1.2152	1.2164		1.2069	**1.2096**	-.0055	112,112
June	1.2158	1.2172		1.2076	**1.2104**	-.0055	16,741
Canadian Dollar (CME)-CAD 100,000; $ per CAD							
March	1.0277	1.0300		1.0256	**1.0295**	.0017	119,633
June	1.0256	1.0278		1.0235	**1.0274**	.0017	40,389
British Pound (CME)-£62,500; $ per £							
March	1.6205	1.6209		1.6123	**1.6158**	-.0041	116,348
June	1.6191	1.6192		1.6107	**1.6141**	-.0040	19,429
Swiss Franc (CME)-CHF 125,000; $ per CHF							
March	1.0795	1.0797		1.0680	**1.0688**	-.0110	51,297
June	1.0800	1.0805		1.0688	**1.0695**	-.0110	11,683
Australian Dollar (CME)-AUD 100,000; $ per AUD							
March	1.0110	1.0127		1.0046	**1.0099**	-.0006	123,688
June	.9998	1.0012		.9933	**.9985**	-.0006	21,993
Mexican Peso (CME)-MXN 500,000; $ per 10MXN							
March	.82950	.83375		.82700	**.83275**	.00375	91,078
June	.82300	.82750		.82100	**.82675**	.00375	61,703
Euro (CME)-€125,000; $ per €							
March	1.3968	1.3988		1.3860	**1.3901**	-.0065	212,061
June	1.3949	1.3967		1.3840	**1.3880**	-.0065	39,871

Index Futures

	Open	High	hi lo	Low	Settle	Chg	Open interest
DJ Industrial Average (CBT)-$10 x index							
March	12085	12245		12065	**12199**	121	13,720
June	12025	12165		12025	**12136**	121	37
Mini DJ Industrial Average (CBT)-$5 x index							
March	12202	12207		12197	**12199**	121	83,574
June	12133	12139		12133	**12136**	121	1,370
S&P 500 Index (CME)-$250 x index							
March	1308.60	1325.50		1306.20	**1320.10**	11.00	319,095
June	1307.00	1320.50		1301.50	**1315.30**	11.10	38,354
Mini S&P 500 (CME)-$50 x index							
March	1308.75	1325.75		1306.00	**1320.00**	11.00	2,897,776
June	1303.75	1321.00		1301.25	**1315.25**	11.00	137,428
Nasdaq 100 (CME)-$100 x index							
March	2325.50	2348.00		2311.50	**2333.75**	9.25	19,405
June	2330.00	2344.50		2318.00	**2331.50**	9.50	1,369
Mini Nasdaq 100 (CME)-$20 x index							
March	2324.0	2348.0		2310.5	**2333.8**	9.3	360,571
June	2324.8	2345.3		2308.3	**2331.5**	9.5	4,447
Mini Russell 2000 (ICE-US)-$100 x index							
March	811.20	828.40		805.50	**824.20**	14.50	333,705
June	811.00	825.70		803.10	**821.60**	14.50	7,238
Mini Russell 1000 (ICE-US)-$100 x index							
March	727.80	735.40		724.60	**732.40**	6.30	17,475
U.S. Dollar Index (ICE-US)-$1,000 x index							
March	76.52	77.00		76.43	**76.80**	.29	44,547
June	76.79	77.27		76.69	**77.07**	.29	6,696

Source: Thomson Reuters

15.401 Derivitives I

3/14

Derivitives have been future for 15 years
- make a lot more $

forwards
futures
options

Very widley used
- interest rate
- Forex
- Commodity
- credit
- equity

To lock in an exchange rate for consumer ~~com~~ product cos
- sales overseas
- make purchases of raw materials

Utilities want to lock in interest rate on bonds
- and raw material

Also futures on S+P
250 x index
Settle = closing price $1315.30 \cdot 250 =$ value one contract
$= 328{,}825.00$
notional value

Open interest = how many contracts outstanding
= 38,354

(2)

Total value of contract

$$= 328{,}825 \cdot 38{,}354$$

$$\$ 12.6 \text{ billion}$$

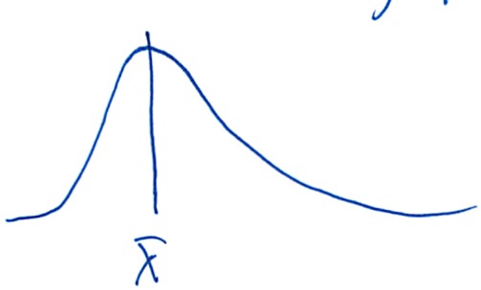
To show that this is a big market

Gross market value = net +/− of peoples positions

Forward

Committment to buy/sell at a ~~[illegible]~~ future date for a given amt of commodity at a price set today

buyer has long position



if prices go ↑ you look good
↓ bad

③

But you ~~[illegible]~~ lock in the price

- no variability for buyer
- price/availability risk

Speculators buy to try and predict price movements

Over the counter / not on exchanges

Custom tailored

No money changes hands until maturity

Good
Full flexibility

But counter party risk - massive

Illiquidity - between you + other party

If counter party fails you are out of luck - need to be careful who you are dealing with

---

Futures

Exchange traded, standardized, forward-like

marked to market daily

Can buy/sell hedge/speculate

page gives specifics

daily price limits

(4)

Position limit
etc

---

everyone knows the underlying commodity or asset (size, grade)
quantity
maturity

guarenteed by clearing house
make sure each person has enough money before buying in
gains + losses settled daily mark to market
- if lost \$, have to put more in

---

example

\$78.82 price
×1000
\$78,820 value of contract -

Go long / buy it

In Dec will have to pay 78,820 for oil

Inital margin (aka deposit) \$5,063
- have to deposit ~~\$~~ your Treasury bills in escrow

5

Someone else Short/sells \$78,820

\$5063 inital margin

Buyer

Day 1 Now price goes ↑ \$80

= \$80,000

making \$

up \$1,180

$$\begin{array}{r} 5063 \\ +\ 1180 \\ \hline 6243 \end{array}$$

Price ↑ \$80

= \$80,000

Lost \$1,180

Have to pay each day

New 5063 inital margin

− 1180 loss

3883 new margin

Futures are 0 sum gain

- Someone wins, someone loses

Day 2 ~~Still~~ \$81 → \$81,000

Still over maintance margin

\$81 → \$81,000

as soon as seller goes under maint. margin - get margin call - have to put in more \$ now

(e)

New margin 2,883

+ 807 add. cash

3,750

If you don't put $ up they will sell your position (all of it) and give you the extra $

To get out - go short or long ← the opposite on the same ~~exactly~~ contract
they you are out

Easy to get at

Since most have no desire to actually take delivery
So just get out

Don't have to put up much $ 5,000 for $ 80,000

Airlines - Some airlines agressivly hedge everything

①

<u>Basis risk</u> - hedging with a contract that does not match your exposure

* Need to be able to hold position (have $ for margin call) or will lose position

Every day they keep score

If a bunch of people lose position
- if prices falling
- and long people can't come up w/ $
- but then prices freefall
- that's why circuit breakers
- worse w/ feedback loop

Payoffs are linear $S(T) - F$

$S(T)$ ← spot price

F ← future price
- where made deal
- the settle price

Don't care where price is if did not enter into a contract

8

Need oil in Dec
- either buy now + store
- or futures contract

Not perfect substitutes!
- see table p15
- Opp. cost "cost of carry" if buying today
  - borrowing the $ to buy the asset
- but could have a convenience yield

- net storage costs = storage costs − convenience yield

"No free lunch principle"
Will calculate this tomorrow

15.401 Forwards + Futures

Not the same as owning the asset

Don't have to pay upfront

Some assets are difficult to store

But sometime missing out on underlying assets

- no dividends

Short futures = promise to deliver the asset

Less risk? Since arbitrage would be possible if price of futures gets far away from asset price

- immediatly

$$F_T \approx H_T = (1+r)^T S + FV_T \quad \text{net storage costs}$$

$$= (1+r)^T S - FV_T \quad \text{net convience yield}$$

Gold

No convience yield

$$F \approx H = S(1+r)^T$$

F: ~~future~~ forward price
- custom
- no exchanges

H: Futures price
- exchange
- liquid

S: Spot rate

(2)

Oil - Very costly to store
- Could be speculating
- Or held for hedging
- Convience Yield - can use it in your plant, if onsite

$$69.33 = 68.05 \cdot (1 + r - \hat{y})^{t}$$

↓ %!

$$\frac{69.33}{68.05} = (1.018810)^{12/3} - 1 \quad 3 \text{ months}$$

↑ 3 months occurs 4 times a year (I think that is wrong!)

$\hat{y} = y - c$

net convience yield

s/p

Contango - Future prices rise in time

Backwardization - Future prices drop in time

If no cost to store/convience yield the only different is interest rate

If holding asset is expensive, then price of future contract goes up over time →

If holding asset is better (bonds that pay coupons) future contract is worth less over time

③

Yield for interest rate future - just that currency's interest rate
- straight interest rate

Bonds, stocks - also have convience yield

Financial futures

- no cost to store
- dividend or interest on underlying assets
- Stock index futures
  - basket of stocks
- Interest rate futures
- Currency futures
- futures settled in cash (no delivery)
- Can find annualized dividend yield
- diff b/w spot + futures price
  - since dividend yield higher than interest rate
- dividends can repay what you borrow and more
- can go long stock + short futures if index higher than future price
  $d > r$ - price of stock ↑
  " " future contract ↓
- when $r > d$ then people don't want underlying stock
- What computers are good at
- prices much closer

(4)

Short selling is expensive (high transaction costs)
Futures caiser to trade
↓ trans costs
Open more hrs of the day
So can react more quickly to macro variablies

Hedge positions
Can create portfolio insurance
Can sell much caiser

\$250 is each index point — see contract terms
↑ Contract is \$250 · index price = value of contract
Exact same payoff as actually owning stack
– ignoring interest + dividends

T-Bond (CBT) \$100,000 32nds of 100%

	Open	Settle Close
June	118-29	118-16

118 + 11/32 = 118.~~34375~~ 500 % of Por

So value/ settle price = ~~118,343.75~~
118,500

(5) Coupon Yield

$$Y = \frac{7\%}{2} = 3.5\%$$

Forward Price

$$F = S(1 + r - y)T$$

$$= \$100(1 + 2.5\% - 3.5\%)$$

$$= \$99$$

Hedging w/ Forwards

- simple, since can tailor contract to match what you want

Can wait 5 months + then sell the oil at market price

Or buy a forward to hedge

- Can get exact terms of contract you want

But you can't sell easily if can find the oil

- Need to get someone to take the forward
- Or have to buy oil at market price

Counterparty risk

$$\# \text{ of contracts} = \frac{\text{amt}}{\text{amt per contract}}$$

↑ accurate

You own bonds. Afraid interest rates will ↑, prices ↓

Can manage w/ futures

Go short on futures

Can figure how many contracts to short

Have immunitized

$$\# \text{ Contracts} = \frac{\text{face}}{\text{contract price}} \cdot \text{durations} = \frac{10{,}000{,}000}{93\,062} \cdot \frac{6.80}{9.20} = 79.42$$

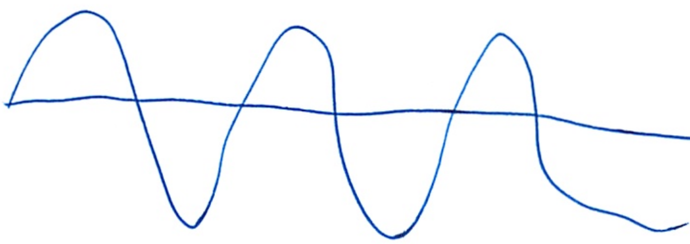
(6)

But can't buy .42 contracts

So may not to buy perfect hedge basis risk

If do perfectly hedge, you just get your coupons

- no capital gains



Use duration to get modified duration

If interest rates ↓ wish had not done this

Exam Median 64 so that will be line for A

Out of 80

15.401 Recitation

4: Forwards & Futures

Learning Objectives

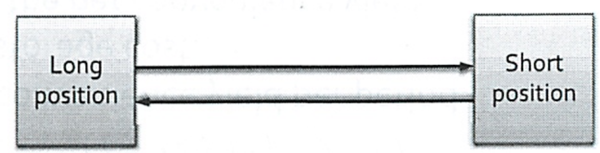
- ❑ Review of Concepts
  - ○ Forwards vs. futures
  - ○ Spot-future parity
  - ○ Hedging with F&F
- ❑ Examples
  - ○ Mark to market
  - ○ Oil!
  - ○ Stock futures

Review: forwards vs. futures

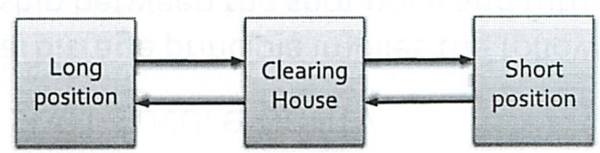
- ❑ Similarity:
  - ○ One party has the **obligation to buy** an asset at a fixed future date and at a price determined today; the counterparty has the **obligation to sell**.
- ❑ Difference:
  - ○ forwards
    - custom made
    - over the counter (OTC)
    - no payments until maturity
  - ○ futures
    - standardized
    - traded on exchange
    - marked-to-market daily

Review: forwards vs. futures

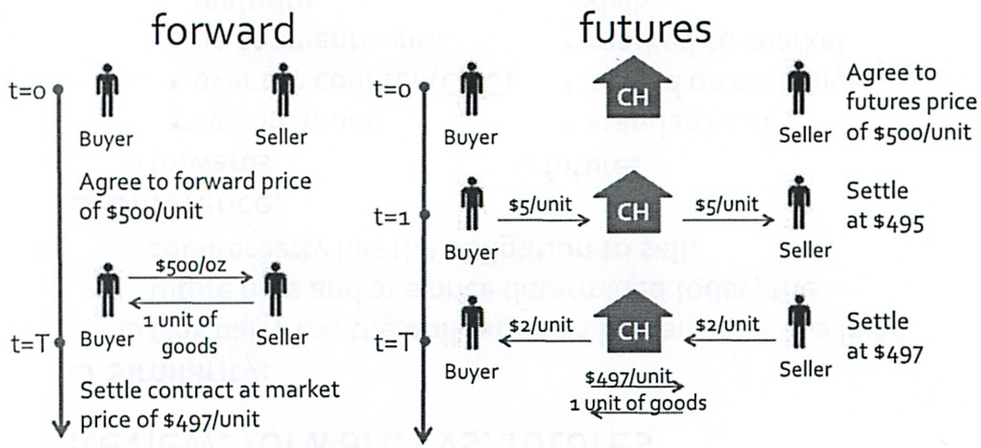
forwards



futures



Review: forwards vs. futures



In both cases, the seller earns $3/unit.

Review: spot-future parity

- The **no-arbitrage** principle implies the following relationship between the spot price and futures price:

$$F_0 = P_0\left(1 + r_f - y + c\right)^T$$

- y is the convenience yield per period
  c is the storage cost.
  $(y - c)$ is the net convenience yield
  T is the time to maturity of the futures contract.
- If the spot-future parity does not hold, there is an **arbitrage opportunity**.

Review: spot-future parity

- If net convenience yield is positive (storage costs are low and convenience yields are high), futures price will be lower than the spot price adjusted for the time value of money. This is known as **backwardation**.
- If net convenience yield is negative, adjusted futures price is higher than the spot price. This is known as **contango**.

Review: hedging with F&F

- Forward contract can be used to hedge future inflow and outflow of commodities against price fluctuations:
  - Pro: contracts are custom-made to fit exact needs
  - Con: illiquid
- Futures contract can be used in a similar way:
  - Pro: liquid
  - Con: standardized contract

Review: hedging with F&F

- Suppose that you just discovered gold at a remote location in Yukon, but it will take 6 months to build a mine and start production. You can secure your revenue today by taking a short position in a gold forward contract:

	CF at t=0	CF at t=6
Long position in gold	0	S_6
Short position in forward	0	$F_0 - S_6$
Net Cash Flow	0	F_0

- Your future net cash flow, F_0, is a fixed number at t=0.

Example 1: mark to market

- Assume the current futures price for silver for delivery 5 days from today is $10.10 per ounce. Suppose that over the next 5 days, the futures price evolves as follows:

Day	0	1	2	3	4	5
Futures Price	$10.10	10.2	10.25	10.18	10.18	10.21

- If you have a long position of 25,000 ounces at time 0, what are your cash flows for the next five days?

Example 1: mark to market

- Answer:

Day	Profit/loss per ounce		Daily proceeds
1	10.20 – 0.10 = 0.10		$2,500
2	10.25 – 10.20 = 0.05		1,250
3	10.18 – 10.25 = -0.07	x 25,000	-1,750
4	10.18 – 10.18 = 0.00		0
5	10.21 – 10.18 = 0.03		750
			Sum = $2,750

Example 2: oil!

- Oil is currently trading at $50 per barrel. The 1-year risk-free interest rate is 3.8%, and the 1-year forward price of oil is $50.40.
  a. What is the net convenience yield of oil?
  b. What would you do if the 1-year forward price is $51 instead?

Example 2: oil!

- Answer:
  a. $50.4 = 50 \times (1 + 0.038 - \hat{y})^1 \Rightarrow \hat{y} = 3\%$

Example 2: oil!

- Answer:
  b. there is an arbitrage opportunity:

	CF at t=0	CF at t=1
Sell 1 barrel forward	\$0.00	$\$51.00 - S_t$
Buy 1 barrel now & hold	-\$50.00	$\$1.50 + S_t$
Borrow \$50 today	\$50.00	-\$51.90
Net Cash Flow	\$0.00	\$0.60

Example 3: stock futures

- Stock price of Acme Inc. is trading at \$56, and is expected to pay \$10 of dividend in the next two years. The term structure of interest rates is flat at 3%. What is the 2-year forward price on company A's stock?

Example 3: stock futures

- Answer: the no-arbitrage principle gives
  $F_2 = 56 \cdot 1.03^2 - 10 = \$49.41$

15.401 Finance Theory I

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Lecture 6: Options

Introduction to options

Option payoffs

Corporate securities as options

Use of options

Basic properties of options

Binomial Option Pricing Model

Black-Scholes option pricing formula

Readings:

Brealey, Myers and Allen, Chapters 20 - 21

Bodie, Kane and Markus, Chapters 20 - 21

Option types:

Call: The right to buy an asset (the underlying asset) for a given price (exercise price) on or before a given date (expiration date)

Put: The right to sell an asset for a given price on or before the expiration date

Exercise styles:

European: Owner can exercise the option only on expiration date

American: Owner can exercise the option on or before expiration date

Key elements in defining an option:

- Underlying asset and its price S
- Exercise price (strike price) K
- Expiration date (maturity date) T (today is 0)
- European or American

Example. A European call option on IBM with exercise price $100. It gives the owner (buyer) of the option the right (not the obligation) to buy one share of IBM at $100 on the expiration date. Depending on the share price of IBM on the expiration date, the option's payoff is:

IBM Price at T	Action	Payoff
⋮	Not Exercise	0
80	Not Exercise	0
90	Not Exercise	0
100	Not Exercise	0
110	Exercise	10
120	Exercise	20
130	Exercise	30
⋮	Exercise	$S_T - 100$

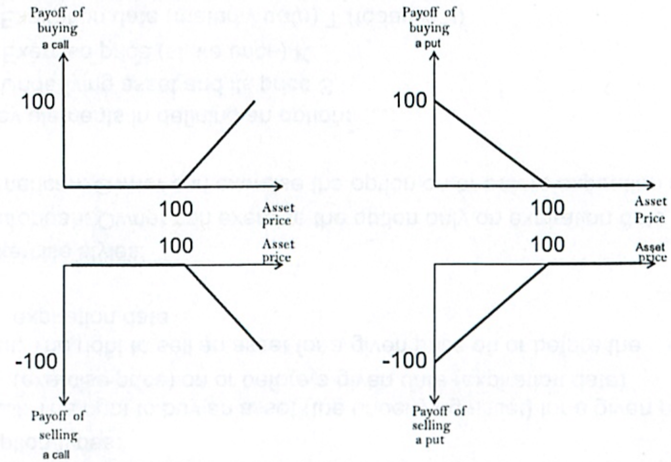
The payoff of a call option is never negative, the floor is zero

Sometimes, it is positive

Actual payoff depends on the price of the underlying asset:

$$CF_T\text{ (call)} = \max\,[S_T - K, 0]$$

Option payoffs plotted as a function of the price of the underlying asset at expiration:

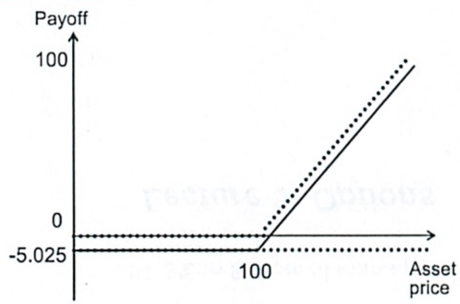


The net payoff from an option, however, must includes its cost.

Example. A European call on IBM shares with an exercise price of $100 and maturity of three months is trading at $5. The 3-month interest rate, not annualized, is 0.5%. What is the price of IBM that makes the call break-even?

At maturity, the call's net payoff is as follows:

IBM Price	Action	Payoff	Net payoff
...	Not Exercise	0	-5.025
80	Not Exercise	0	-5.025
90	Not Exercise	0	-5.025
100	Not Exercise	0	-5.025
110	Exercise	10	4.975
120	Exercise	20	14.975
130	Exercise	30	24.975
...	Exercise	S_T - 100	S_T - 100- 5.025



The break even point is given by (with "C" representing the call premium):

$$\begin{aligned} \text{Net payoff} &= \max[S_T - K, 0] - C(1+r)^T \\ &= S_T - \$100 - (\$5)(1+0.005) \\ &= \$0 \end{aligned}$$

or

$$S_T = \$105.025$$

Price of underlying asset = S

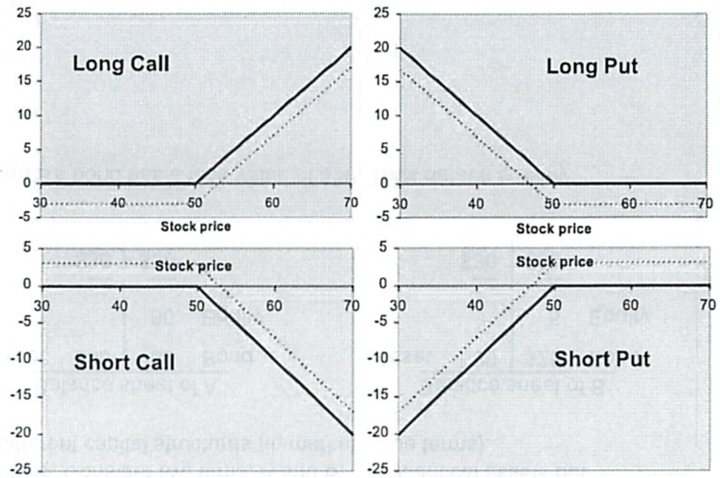
Exercise price = K, Maturity = T, Interest rate = r

Call option (price = C)

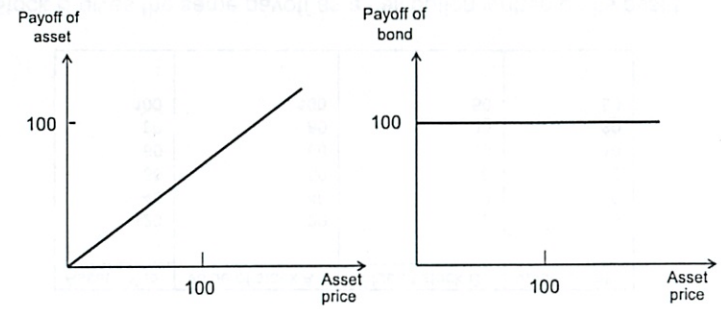
	if S < K	if S = K	if S > K
Payoff	0	0	S - K
Profit	- $C(1+r)^T$	– $C(1+r)^T$	S - K - $C(1+r)^T$

Put option (price = P)

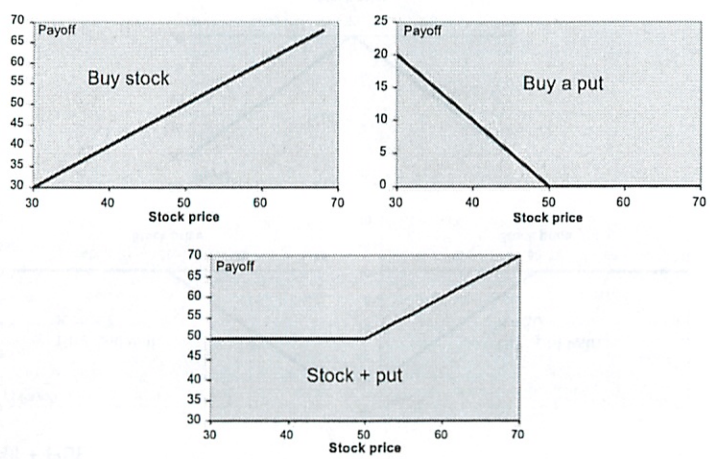
	if S < K	if S = K	if S > K
Payoff	K - S	0	0
Profit	K - S - $P(1+r)^T$	- $P(1+r)^T$	- $P(1+r)^T$



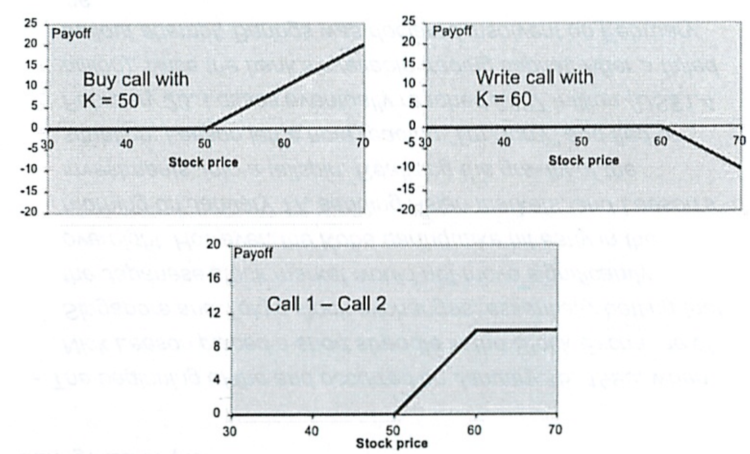
Example. The underlying asset and the bond (with face value $100) have the following payoff diagram:



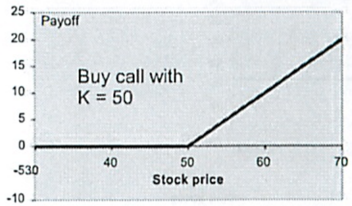
Stock + Put

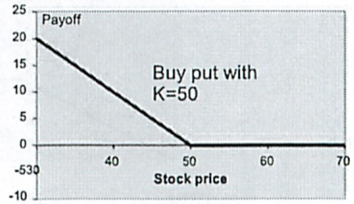


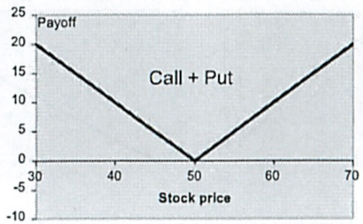
Call 1 (buy) - Call 2 (write)



Call + Put







Barings Bank plc:

- *The beginning of the end occurred on January 16, 1995, when Nick Leeson placed a* ***short straddle*** *in the Stock Exchange of Singapore and Tokyo stock exchanges, essentially betting that the Japanese stock market would not move significantly overnight. However, the Kobe earthquake hit early in the morning on January 17, sending Asian markets, and Leeson's investments, into a tailspin. Realizing the gravity of the situation, Leeson left a note reading "I'm Sorry" and fled on February 23. Losses eventually reached £827 million (US$1.4 billion), twice the bank's available trading capital. After a failed bailout attempt, Barings was declared insolvent on February 26*

Example. Consider two firms, A and B, with identical assets but different capital structures (in market value terms).

Balance sheet of A

Asset	$30	$0	Bond
		30	Equity
	$30	$30	

Balance sheet of B

Asset	$30	$25	Bond
		5	Equity
	$30	$30	

Firm B's bond has a face value of $50. Thus default is likely

Example. (Continued)

Consider the value of stock A, stock B, and a call option on the underlying asset of firm B with an exercise price of $50:

Asset value	Value of stock A	Value of stock B	Value of call
⋮	⋮	⋮	⋮
$20	20	0	0
40	40	0	0
50	50	0	0
60	60	10	10
80	80	30	30
100	100	50	50
⋮	⋮	⋮	⋮

Stock B gives the same payoff as a call option written on its asset

Thus B's common stock is really a call option on the firm's asset

Indeed, many corporate securities can be viewed as options:

Equity	A call option on the firm's assets with the exercise price equal to its bond's redemption (or face) value
Bond	A portfolio combining the firm's assets and a short position in the call with the exercise price equal its bond's redemption value

$$\text{Equity} \equiv \text{Max}\,[\,0,\ A - B\,]$$
$$\text{Debt} \equiv \text{Min}\,[\,A,\ B\,] \ = A - \text{Max}\,[\,0,\ A - B\,]$$
$$A = D + E$$

Warrant	Call options on the firm's stock
Convertible bond	A portfolio combining straight bonds and a call on the firm's stock with the exercise price related to the conversion ratio
Callable bond	A portfolio combining straight bonds and a call written on the bonds

For convenience, we refer to the call's underlying asset as stock. It could also be a bond, foreign currency or some other asset.

Notation:

- S: Price of stock now
- S_T: Price of stock at T
- B: Price of discount bond of par \$1 and maturity T ($B \leq 1$)
- C: Price of a European call with strike K and maturity T
- P: Price of a European put with strike K and maturity T

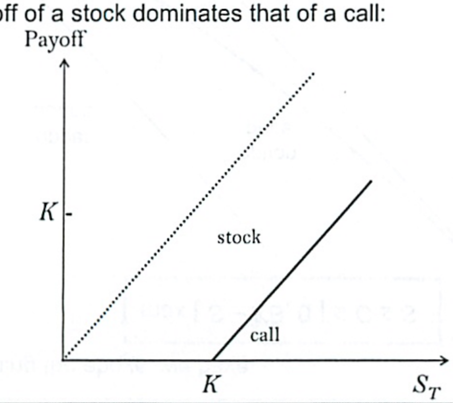
For our discussions:

- Consider only European options (no early exercise)
- Assume no dividends (option cash flow occurs only at maturity)

First consider European options on a non-dividend paying stock

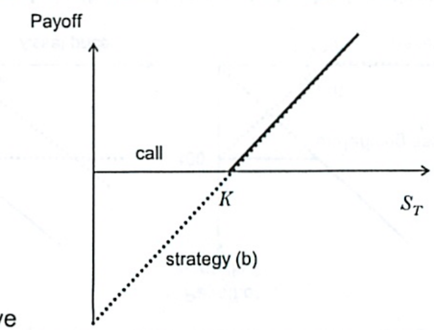
1. $C \geq 0$
2. $C \leq S$

The payoff of a stock dominates that of a call:



3. $C \geq S - KB$
   - Strategy (a): Buy a call
   - Strategy (b): Buy a share of stock and borrow KB

The payoff of strategy (a) dominates that of strategy (b):

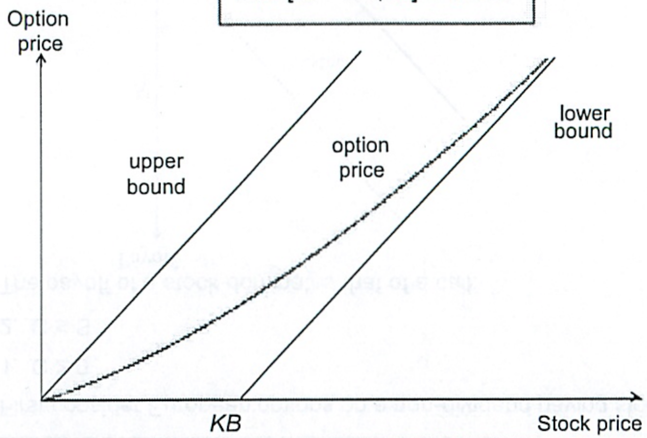


Since $C \geq 0$, we have

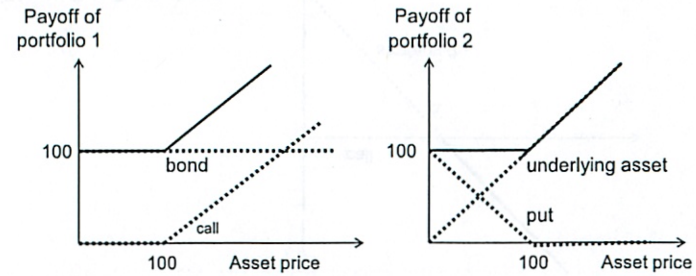
$$C \geq \max[S - KB,\ 0]$$

4. Combining the above, we have

$$\boxed{\max [\, S - KB, 0\,] \le C \le S}$$



Portfolio 1: A call with strike \$100 and a bond with par \$100
Portfolio 2: A put with strike \$100 and a share of the underlying asset



Their payoffs are identical, so their prices must also be identical:

$$\boxed{C + K/(1+r)^T = P + S}$$

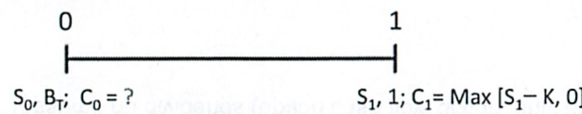
This is called the put-call parity relationship, or simply put-call parity.

Option value increases with the volatility of underlying asset

Example. Two firms, A and B, with the same current price of \$100. B has higher volatility of future prices. Consider call options written on A and B, respectively, with the same exercise price \$100.

	Good state	bad state
Probability	p	$1-p$
Stock A	120	80
Stock B	150	50
Call on A	20	0
Call on B	50	0

Clearly, the call option on stock B should be more valuable.



Determinants of option value:

Key factors in determining option value:

1. price of underlying asset - S
2. strike price - K
3. time to maturity - T
4. interest rate - r
5. volatility of underlying asset - σ

Additional factors that can sometimes influence option value:

6. expected return on the underlying asset
7. investors' attitude toward risk, …

In order to have a complete option pricing model, we need to make additional assumptions about

1. Price process of the underlying asset (stock)
2. Other factors

We will assume, in particular, that:

Prices do not allow arbitrage

Prices are ``reasonable"

A benchmark model --- Price follows a binomial process (↑ or ↓).

$$S_0 \begin{cases} S_{up} \\ S_{down} \end{cases}$$

t = 0 t = 1 time

Example. Valuation of a European call option on a stock.

Current stock price is \$50

There is one period to go

Stock price will either go up to \$75 or go down to \$25

There are no cash dividends

The strike price is \$50

One period borrowing and lending rate is 10%

The stock and bond present two investment opportunities:

$$50 \begin{cases} 75 \\ 25 \end{cases} \qquad 1 \begin{cases} 1.1 \\ 1.1 \end{cases}$$

The option's payoff at expiration is:

$$C_0 \begin{cases} 25 \\ 0 \end{cases}$$

What is C_0, the value of the option today?

Form a portfolio of stocks and bonds that replicates the call's payoff:

- a shares of the stock
- b dollars in the riskless bond

such that

$$75a + 1.1b = 25$$
$$25a + 1.1b = 0$$

Unique solution: $a = 0.5$ and $b = -11.36$

That is:

- buy half a share of stock and sell \$11.36 worth of bond
- the payoff of this portfolio is identical to that of the call option
- present value of the call must equal the current cost of this "replicating portfolio" which is

$$(\$50)(0.5) - \$11.36 = \$13.64$$

Number of shares needed to replicate one call option is called the option's hedge ratio or delta.

In the above problem, the option's delta is a = 1/2.

Determining the Replicating Portfolio

The option's delta or hedge ratio can be calculated as:

↓profit

$$\Delta = \frac{Call_{up} - Call_{down}}{Stock_{up} - Stock_{down}}$$

↑raw

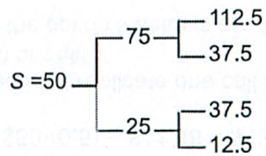
The position in the riskless bond can be calculated as:

↓profit

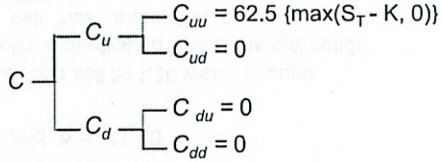
$$B = \frac{Stock_{up} \times Call_{down} - Stock_{down} \times Call_{up}}{(Stock_{up} - Stock_{down}) \times (1 + r)}$$

↑raw

More than one period:



Call price process:



Terminal value of the call is known, and

C_u and C_d denote the option value next period when the stock price goes up and goes down, respectively

Compute the current value by working backwards: first C_u and C_d and then C

Step 1. Start with Period 1:

1. Suppose the stock price goes up to $75 in period 1:
   - Construct the replicating portfolio at node (t =1, up):

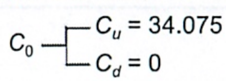
$$112.5a + 1.1b = 62.5$$
$$37.5a + 1.1b = 0$$

   - Unique solution: a = 0.833, b = - 28.4
   - The cost of this portfolio: (0.833)(75) - 28.4 = 34.075
   - The exercise value of the option: 75 - 50 = 25 ≤ 34.075
   - Thus, C_u = 34.075

2. Suppose the stock price goes down to $25 in period 1. Repeat the above for node (t =1, down):
   - The replicating portfolio: a = 0, b = 0
   - The call value at the lower node next period is C_d = 0

Step 2. Now go back one period, to Period 0:

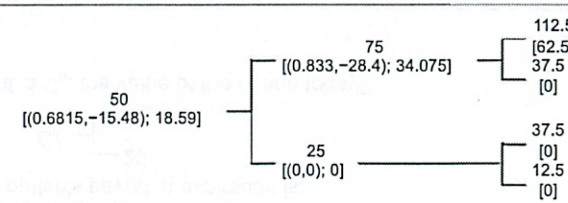
- The option's value next period is either 34.075 or 0:



- If we can construct a portfolio of the stock and bond to replicate the value of the option next period, then the cost of this replicating portfolio must equal the option's present value.
- Find a and b so that

$$75a + 1.1b = 34.075$$
$$25a + 1.1b = 0$$

- Unique solution: a = 0.6815, b = -15.48
- The cost of this portfolio: (0.6815)(50) - 15.48 = 18.59
- The present value of the option must be C_0 = 18.59
- It is greater than the exercise value 0 (thus no early exercise)



Play Forward:

Period 0: Spend $18.59 and borrow $15.48 at 10% interest rate to buy 0.6815 shares of the stock

Period 1:

- When the stock price goes up, the portfolio value becomes 34.075. Re-balance the portfolio to include 0.833 stock shares, financed by borrowing 28.4 at 10%
  - ✓ One period later, the payoff of this portfolio exactly matches that of the call
- When the stock price goes down, the portfolio becomes worthless. Close out the position.
  - ✓ The portfolio payoff one period later is zero

Thus

No early exercise.

Replicating strategies give payoffs identical to those of the call.

The initial cost of the replicating strategy must equal the call price.

What we have used to calculate the call option's value:

current stock price - S

magnitude of possible future changes of stock price – volatility - σ

interest rate - r

strike price - K

time to maturity - T

What we have not used:

probabilities of upward and downward stock price movements

investor's attitude towards risk

Questions on the Binomial Model

What is the length of a period?

Price can take more than two possible values.

Trading takes place continuously.

Response: The length of a period can be arbitrarily small.

If we let the period-length get smaller and smaller, we obtain the Black-Scholes option pricing formula:

$$\boxed{C(S, K, T) = SN(x) - KR^{-T}N\left(x - \sigma\sqrt{T}\right)}$$

where

x is defined by:

$$x = \frac{\ln\left(\frac{S}{KR^{-T}}\right)}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T}$$

T is in units of a year

R = 1+r, where r is the annual riskless interest rate

σ is the volatility of annual returns on the underlying asset

- N (•) is the cumulative normal density function

An interpretation of the Black-Scholes option pricing formula:

The call is equivalent to a levered long position in the stock

$SN(x)$ is the amount invested in the stock

$KR^{-T}N\left(x-\sigma\sqrt{T}\right)$ is the dollar amount borrowed

The option delta is $N(x) = C_S$

Black-Scholes formula

Example. Consider a European call option on a stock with the following data:

- $S = 50$, $K = 50$, $T = 30$ days
- The volatility σ is 30% per year
- The current annual interest rate is 5.895%

Then

$$x = \frac{\ln\left(50/50(1.05895)^{-\frac{30}{365}}\right)}{(0.3)\sqrt{\frac{30}{365}}} + \frac{1}{2}(0.3)\sqrt{\frac{30}{365}} = 0.0977$$

$$\begin{aligned} C &= 50N(0.0977) - 50(1.05895)^{-\frac{30}{365}} N\left(0.0977 - 0.3\sqrt{\tfrac{30}{365}}\right) \\ &= 50\,(0.53890) - 50(0.99530)(0.50468) \\ &= 1.83 \end{aligned}$$

Black Monday (10/19/1987)

The Black Monday decline was the largest one-day percentage decline in stock market history. The Dow Jones Industrial Average (DJIA) dropped 22.6%.

Before the crash, many program traders synthetically replicated put options on the index by buying (selling) when the price rises (falls), using the Black&Scholes formula (delta hedging).

U.S. Congressman Edward J. Markey, who had been warning about the possibility of a crash, stated that "Program trading was the principal cause." This is the most popular but not entirely undisputed view.

Implied Volatility

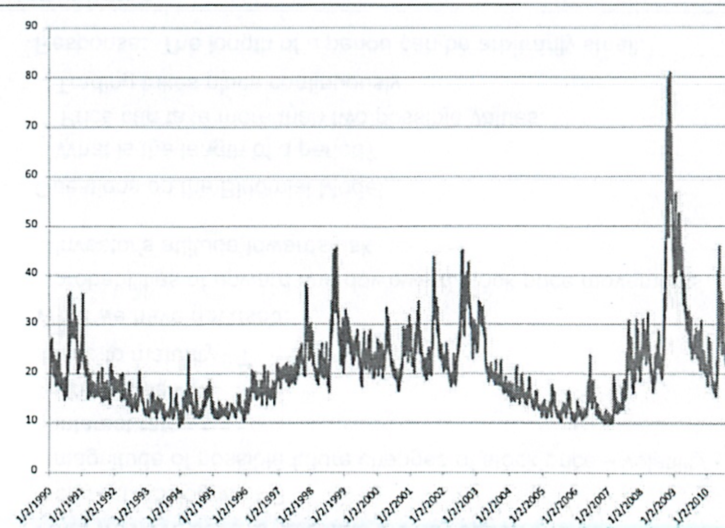
For every level of volatility, σ , there is a corresponding Black-Scholes option price, C_0

Similarly, for any option price, C_0, there is a corresponding volatility, σ , using Black-Scholes

This is called the "implied volatility"

According to the Black-Scholes model, what should be true about the implied volatility of options on the same stock (with different strikes and expirations)?

VIX Index (1/2/1990—9/10/2010)



- Introduction to options
- Option payoffs
- Corporate securities as options
- Use of options
- Basic properties of options
- Binomial Option Pricing Model
- Black-Scholes option pricing formula

3/28
Handout

15.401 – Options
Spring 2011

1. Options are very different than financial futures and forwards. Futures and forwards must be delivered. The holder of an option has a choice. If exercising the option is profitable, do so. If it is not, let it expire worthless.

2. Options are of two types, calls and puts:

a. A call option gives the holder the right to purchase a specific financial instrument at a set price (exercise or strike price) on or before a set date. The holder of the call buys the option for a cash premium and chooses to exercise or not. The writer of the call sells the option for a cash premium, but is exposed to possible exercise.

b. A put option gives the holder the right to sell a specific financial instrument at a set price (exercise or strike price) on or before a set date. The holder of the put buys the option for a cash premium and chooses to exercise or not. The writer of the put sells the option for a cash premium, but is exposed to possible exercise.

3. Buyers/holders of call options expect market prices to increase. The call is in the money if the underlying asset's market price exceeds the exercise price, at the money if market price equals exercise price, and out of the money if market price is less than exercise price.

4. Buyers/holders of put options, in contrast, expect market prices to decrease. The put is in the money if the underlying asset's market price is less than the exercise price, at the money if market price equals exercise price, and out of the money if market price exceeds exercise price.

5. Like futures, options are also traded on organized exchanges like the Chicago Board Options Exchange (CBOE) and the NYSE's London International Financial Futures Exchange (NYSE Liffe). The exchange again operates a clearinghouse to ensure writers of options satisfy their obligations. Buyers and writers of options also close their positions by taking the opposite side of the same option (if you own a call, you sell the same call). Profit or loss upon closing the position is the difference between sales price and purchase price.

6. European options can only be exercised at expiration, but American options can be exercised at any time up to the expiration date. The structure of European options makes them easier (!) to analyze. American options are much more difficult, acting like a sequence of options, precluding any closed form solutions. This statement may be nonsense right now, but after we look at option pricing models you'll understand it a bit more.

7. Let's look at how option trading is reported. Keys to understanding trading are:

   - The company whose stock the option is written on
   - Current stock price for the company
   - Strike price or exercise price on the option
   - Expiry date on the option
   - Call option – volume traded and last call premium
   - Put option – volume traded and last put premium

   Notice the large number of options, exercise prices, and expiry dates for some companies.

15.401 - Monday, March 28, 2011

Option Chains | **Call Options** | **Data Pulled 21-Mar-11 1:30 p.m. EDT**

AAPL - Apple Inc.

Last $338.32 ↑ $7.66

Expiration	Strike	Last	Volume	Open Interest	'T' from 3-21
16-Apr-11	$290.00	$49.40	234	3,892	26 days: 0.0712
16-Apr-11	$300.00	$40.11	907	15,821	26 days: 0.0712
16-Apr-11	$310.00	$31.38	401	6,376	26 days: 0.0712
16-Apr-11	$320.00	$23.15	1,290	13,088	26 days: 0.0712
16-Apr-11	$330.00	$15.95	3,025	17,361	26 days: 0.0712
16-Apr-11	$340.00	$9.90	7,164	26,588	26 days: 0.0712
16-Apr-11	$350.00	$5.42	11,242	35,333	26 days: 0.0712
16-Apr-11	$360.00	$2.53	6,098	24,411	26 days: 0.0712
16-Apr-11	$370.00	$0.96	1,852	18,541	26 days: 0.0712
21-May-11	$290.00	$53.35	5	343	61 days: 0.1671
21-May-11	$300.00	$45.60	94	2,002	61 days: 0.1671
21-May-11	$310.00	$37.68	474	1,240	61 days: 0.1671
21-May-11	$320.00	$29.95	242	6,422	61 days: 0.1671
21-May-11	$330.00	$23.55	406	5,429	61 days: 0.1671
21-May-11	$340.00	$17.74	1,129	10,396	61 days: 0.1671
21-May-11	$350.00	$12.90	1,572	10,239	61 days: 0.1671
21-May-11	$360.00	$8.90	1,413	8,959	61 days: 0.1671
21-May-11	$370.00	$5.73	500	6,863	61 days: 0.1671

time % of year

Option Chains | **Put Options** | **Data Pulled 21-Mar-11 1:30 p.m. EDT**

AAPL - Apple Inc.

Last $338.32 ↑ $7.66

Expiration	Strike	Last	Volume	Open Interest	'T' from 3-21
16-Apr-11	$290.00	$1.22	1,015	6,368	26 days: 0.0712
16-Apr-11	$300.00	$1.88	3,017	13,212	26 days: 0.0712
16-Apr-11	$310.00	$2.95	3,392	12,290	26 days: 0.0712
16-Apr-11	$320.00	$4.76	3,279	19,271	26 days: 0.0712
16-Apr-11	$330.00	$7.55	3,807	26,326	26 days: 0.0712
16-Apr-11	$340.00	$11.50	6,236	26,098	26 days: 0.0712
16-Apr-11	$350.00	$17.00	1,249	12,120	26 days: 0.0712
16-Apr-11	$360.00	$24.17	216	6,285	26 days: 0.0712
16-Apr-11	$370.00	$32.34	135	3,432	26 days: 0.0712
21-May-11	$290.00	$4.78	306	2,614	61 days: 0.1671
21-May-11	$300.00	$6.45	391	5,864	61 days: 0.1671
21-May-11	$310.00	$8.67	265	2,726	61 days: 0.1671
21-May-11	$320.00	$11.20	344	10,956	61 days: 0.1671
21-May-11	$330.00	$14.85	435	4,328	61 days: 0.1671
21-May-11	$340.00	$19.05	1,303	5,079	61 days: 0.1671
21-May-11	$350.00	$23.95	291	4,728	61 days: 0.1671
21-May-11	$360.00	$30.31	22	3,230	61 days: 0.1671
21-May-11	$370.00	$36.89	121	2,153	61 days: 0.1671

15.401 Options

3/28

- Most important topic in the class
- but he is not good at it

- have a choice if you want delivery
- but it costs $ upfront
- if don't want to use it, it expires worthlessly

- Call - option to buy something at a known price by a certain day
  - 0 sum game - someone sold the option
    - is the counterparty
  - expect market prices to ↑
- put - option to sell at a certain price by a certain day
  - again a counterparty
  - expect market prices to ↓

- Chicago exchange
- NYSE Liffe
- clearinghouse protects againts counterparty risk
- European option - can only redeam on expiration date
  - easier to analysie

(2)

American option - can exercise any time

See back of sheet for options chain
- a list of avaliable options

Last = price of option
- price last sold at

No arbitrage

$$k = \underbrace{290 + 49.40}_{339.40 \text{ of value}} = C$$

Stock price \$338.32 ← must be lower

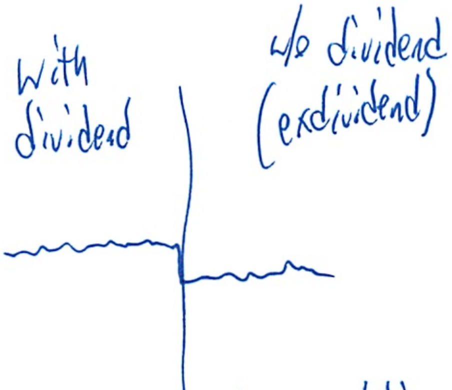
Call value $\geq \max(S-k, 0)$

Can get into options at much less \$
- options levered
- so ~~price~~ higher risk

Open Interest - # Contracts outstanding

It is never in your interest to exercise options early
- except if the stock pays dividends

(3)



since taking money out of the firm

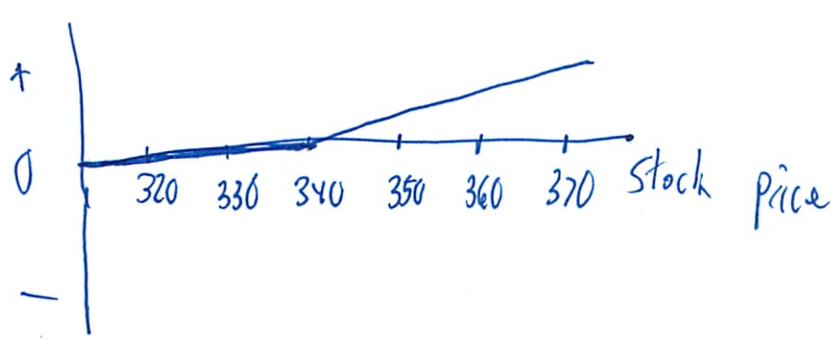
April Apple Call at \$340

exersize price

stock price on expiration date

S_T	Intrinsic Value Call	
320	0	- let expire, walk away
330	0	
340	0	- ~~won't~~ indifferent due to trans costs
350	10	
360	20	
370	30	

Payoff (<u>not</u> profit)



worst you can do once you buy it → 0

④

But they cost $!

Some extra preimum

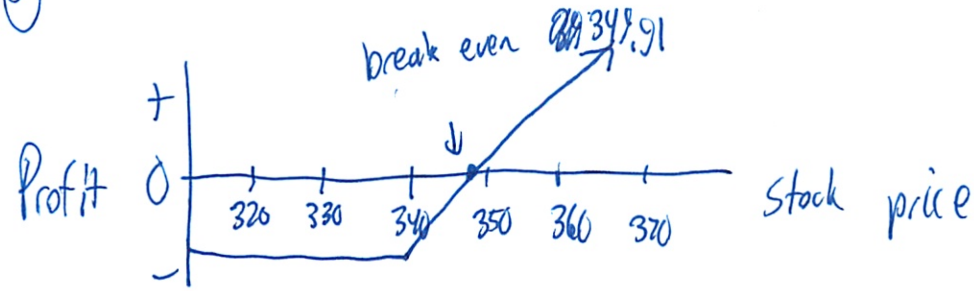
S_T	Intrinsic Call Value	Call Premia		$(\times (1+r)^T$
		\$ 9.90	← sunk cost Same for all amts	$9.90(1.01)^{.0712}$ ← need to back out preimum by discounting PV
				need an interest rate
				= \$9.91
				Same for all

So you put up $ today

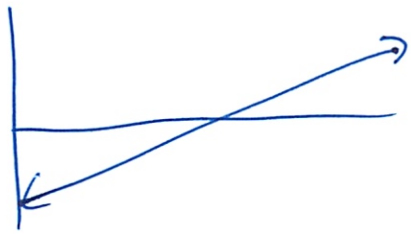
So changes profit

Profit
-9.91
-9.91
-9.91
.09
10.09
20.09

5



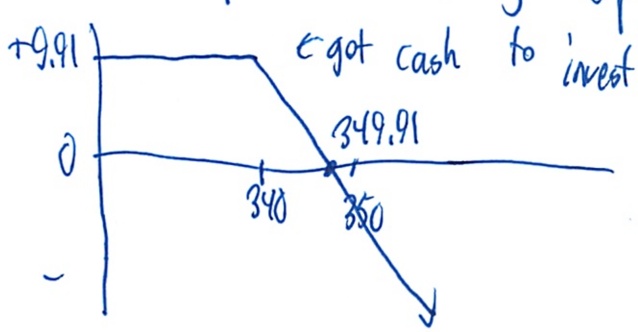
For stocks



? no downside limit

Expire on third Friday of trading month
- busy day

For the person selling option



just the mirror of other party

Call Premium from supply + demand of market
- Comes from stock price
(oh just the last)

6

Now a Put: May Apple Put at $350

	S_T	Intrinsic Value Put	Put Premia	$P_x(1+r)^T$	Net Profit
in the money	320	30	23.95	23.99	6.01
	330	20	↓	↓	-3.99
	340	10			-13.99
at the money	350	0			-23.99
out of the money	360	0			-23.99
	370	0			

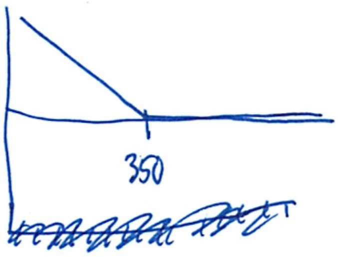
$23.95 \cdot (1.01)^{.1671}$

remember this is all of 3/21/11

ST is what-ifs - price at that $

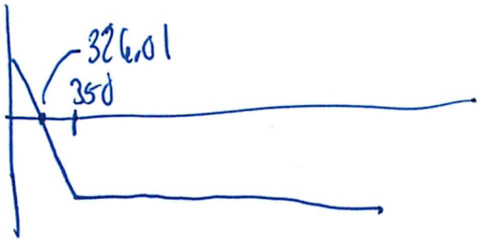
~~If also owned stock - ret~~

Proceeds



Since stock price 338.32 already have 11.68 payoff
- already well "into the money"
(but would lose $ due to the premium)

Profits



8

Powerpoint

- Lots of people traded
- But no one knew how to price

S = underlying stock price

k = strike/exercise price

T = maturity date

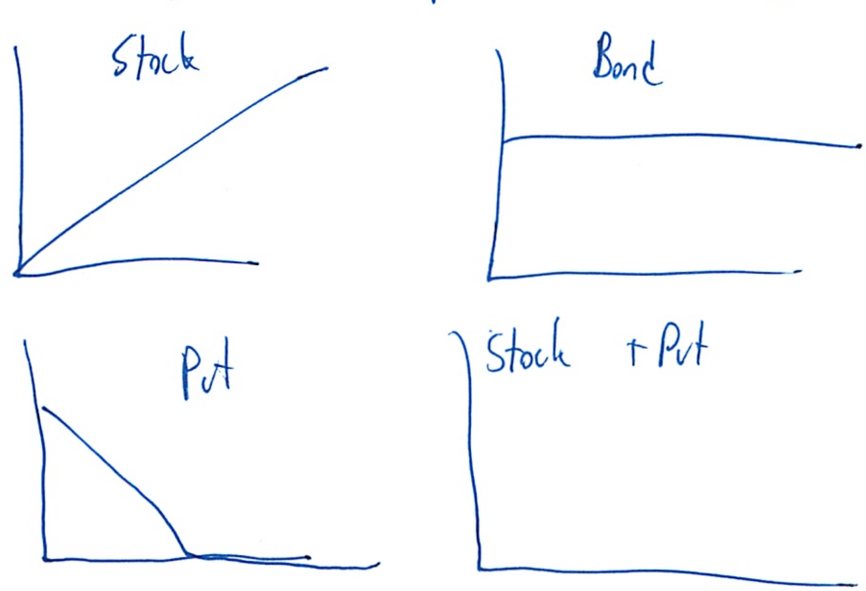
Payoff depends on price of underlying assets

Slides p 8 - chart w/ variables what happens

Would be very profitable if had perfect info

- tempting to manipulate market

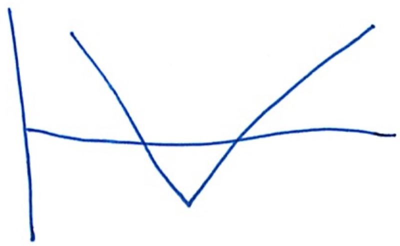
Pricing: Put together a portfolio that peforms the same way



(8)

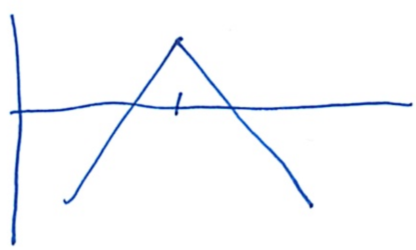
Nail together a call bought + written
(see slides)

Or nail together a call + put
- slide is wrong - no premium



← fixed

Short straddle (earthquake)



Options are everywhere inside companies

Look at it as option on value of company
- Sometimes best to default

Really call option on firm's assets
- if no extra after paying off bonds

(Don't really understand)

Bond - get min (assets, FV of bond)
- if company does ~~best~~ bad - get assets

(9)

Warent - call option firm sends

Convertible bond - is a call option

- - -

$$\max(S - KB, 0) \le C \le S$$

Call Preimum low at high prices

- since very volitiley
- time matters
- more time = more voltille

		S_T	Call value
$k = 50$	$S = 50$	75	25
		25	0

more voltatle	50	90	40
		10	0

- no downside
- ~~But~~ so higher volitility ↑ options pricing
  - unlike stocks

(1c)

When few traders bid-ask spreads get larger
prices become less reliable

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15.401 – Options day #2 – March 30, 2011

The Binomial Option Pricing Model

"Option Pricing: A Simplified Approach," Cox, Ross, and Rubenstein, 1979, Journal of Financial Economics, 7, 229-263.

15.401 - Binomial Option Pricing - March 30, 2011

$S_0 = \$30 \qquad K = \$30 \qquad r = 5\%$ per period

$S_1 = \$40$ or $\$20 \qquad Div_1 = \$0$

$\exists$ 2 investment opportunities:

Stock	Bond
$\$30 \nearrow \$40$, $\searrow \$20$	$\$1 \nearrow \$1.05$, $\searrow \$1.05$

And the call option's payoff at expiry is:

$$C_0 \begin{array}{l} \nearrow \$10 \quad (\$40 - \$30) \\ \searrow \$0 \end{array}$$

Determine the replicating portfolio:

$$\text{Option's } \Delta = \frac{\text{Call}_{up} - \text{Call}_{down}}{\text{Stock}_{up} - \text{Stock}_{down}} = \frac{\$10 - \$0}{\$40 - \$20} = 0.50$$

$$\text{Bond} = \frac{\text{Stock}_{up} \times \text{Call}_{down} - \text{Stock}_{down} \times \text{Call}_{up}}{(\text{Stock}_{up} - \text{Stock}_{down}) \times (1+r)}$$

$$= \frac{\$40 \times \$0 - \$20 \times \$10}{(\$40 - \$20) \times (1.05)} = -\frac{\$200}{\$21} = -9.5238$$

Check the replicating portfolio is correct:

T=1	$S_1 = \$40$	$S_1 = \$20$
Value of 1 call	\$10	\$0
0.50 shares	\$20	\$10
Repay \$9.5238 + interest	− \$10	− \$10
Σ Payoff	\$10	\$0

OK

Value of Replicating Portfolio at $T=0$ is Value of call:

$C_0 = S_0 \times \Delta - \text{Bond} = \$30 \times 0.50 - \$9.5238$

$C_0 = \$5.4762$

this must be the value of the call at $T=0$, otherwise Arbitrage opportunities exist

Value the Put Option:

$$C_0 + \frac{K}{(1+r)^T} = P_0 + S_0$$

$$P_0 = C_0 + \frac{K}{(1+r)^T} - S_0 = \$5.4762 + \frac{\$30}{1.05} - \$30$$

$P_0 = \$4.0476$

What about 2 periods, 2 stages:

$S_0 = \$430 \qquad K = \$430 \qquad r = .75\% = .0075$ per period

$S_1 = \$527$ or $\$350.85$

$S_2 = S_1$ +/– per chart:

$$\begin{tikzcd}[column sep=large] \underline{T=0} & \underline{T=1} & \underline{T=2} \\ & & \$645.89 \\ & \$527 \arrow[ur] \arrow[dr] & \\ \$430 \arrow[ur] \arrow[dr] & & \$430 \\ & \$350.85 \arrow[ur] \arrow[dr] & \\ & & \$286.27 \end{tikzcd}$$

To find Call Value at $T=0$ start with $T=1$ and work back

@ $\underline{S_1 = \$527}$ Option's $\Delta = \dfrac{C_u - C_d}{S_u - S_d} = \dfrac{\$215.89 - \$0}{\$645.89 - \$430} = 1.00$

Up state $\quad$ $\text{Bond} = \dfrac{S_u \times C_d - S_d \times C_u}{(S_u - S_d) \times (1 + r)} = \dfrac{\$645.89 \times \$0 - \$430 \times \$215.89}{(\$645.89 - \$430) \times 1.0075}$

$B = -\$426.80$ at $T=1 \longrightarrow$ Repay $\$430$ at $T=2$

Value of Replicating Portfolio and Call at $T=1$ is:

$C_1 = S_1 \times \Delta - \text{Bond}_1 = \$527 \times 1.00 - \$426.80$

$C_1 = \underline{\$100.20}$

At $S_1 = \$350.85$:

down state $C_u = \$\theta$, $C_d = \$\theta$, $C_1 = \$\theta$

Now we can value the Call at $T = \theta$, $C_0 =$

$$\text{Option's } \Delta = \frac{C_u - C_d}{S_u - S_d} = \frac{\$100.20 - \$0}{\$527 - \$350.85} = 0.5688$$

$$\text{Bond} = \frac{S_u \times C_d - S_d \times C_u}{(S_u - S_d) \times (1+r)} = \frac{\$527 \times \$0 - \$350.85 \times \$100.20}{(\$527 - \$350.85) \times (1.0075)}$$

$B = -\$198.09$ at $T = \theta \rightarrow$ Repay $\$199.58$ at $T = 1$

Value of Replicating Portfolio and Call at $T = \theta$:

$C_0 = S_0 \times \Delta - B = \$430 \times .5688 - \$198.09$

$C_0 = \$46.49$

Use Put - Call parity to Value Put at

time $T = \theta$

$$P_0 = C_0 + \frac{K}{(1+r)^T} - S_0$$

What about >2 periods:

you can break the option's "T" into smaller & smaller intervals → just use the binomial method to work back from the final interval to T = 0. Difficult by hand, easy with computing power.

the binomial method gets more & more accurate as the intervals get smaller and smaller.

<u>And</u> you can estimate the up (u) and down (d) movements in stock prices given the standard deviation of stock returns (σ):

$$1 + \text{up movements} = u = e^{\sigma\sqrt{h}}$$

$$1 + \text{down movements} = d = 1/u$$

where: σ = std dev of continuously compounded stock returns
h = interval as a fraction of a year

15.402 Binomial Options Pricing

P-Set up today, due Mon

$S_0 = 30$

$k = 30$ ← what is k? strike/exercise price of underlying security when contract formed

$r = 5\%$ /period

Div = 0

$S_1 = 40$ or 20

Stock

$30 → $40 → +10

$30 → $20 → −10

Bond

$1 → $1.05 → +.05

Call

C_0 → $40 → +10 − fees

C_0 → $30 → 0 − fees

(2)

To price: build replicating portfolio

just profit ↓0

$$\frac{\text{Call up} - \text{Call down}}{\text{Stock up} - \text{Stock down}}$$

raw prices

Want options Δ $\frac{C_u - C_d}{S_u - S_d}$ (Stock) $= \frac{\$10 - 0}{\$40 - 20} = .50$

think always fraction

So need half a share of stock at time 0

<u>Bond</u> $\frac{S_u \times C_d - S_d \times C_u}{(S_u - S_d) \times (1+r)} = \frac{40 \cdot 0 - 20 \cdot 10}{(40-20)(1.05)} = -9.5238$

always −

So <u>short</u> 9.5238 bonds (of \$1 FV)

In period 1 two things can happen

	$S_1 = 40$	$S_1 = 20$
Own 1 call	\$10	\$0
Own .5 stock	\$20	\$10
Repay 9.5238 at time 1	−10	−10
	\$10	\$0

③

Value of call at $T=0$, C_0

$S_0 \cdot \Delta - \text{Bond}$ ← well — bond always
— technically ⊕ a ⊖ bond value

$30 \cdot .5 - 9.5238$

$= 5.4762$ call pricing

Put-Call parity

- once have value of call
- then have value of put

$$P_0 = C_0 + \frac{K}{(1+r)^T} - S_0$$

$$= 5.4762 + \frac{30}{(1.05)^1} - 30$$

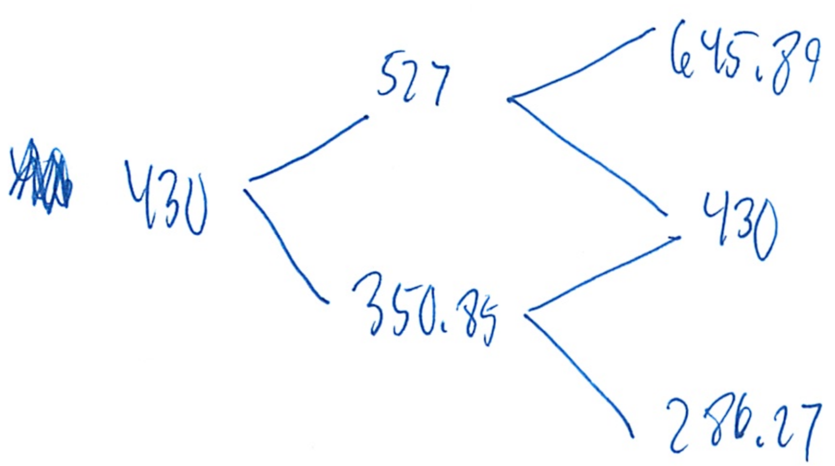
$$= 4.0476$$

(4)

Another example this time $t_x = 2$

~~Start at t_1 and work backwards to t_0~~

Start w/ payoffs $t=2$ then work backwards $t=1$
$t=1$ $t=0$



$k = 430$
↑ just something we picked does not need to match stock

What is value of call in good case
bad case

527

$$\text{Option } \Delta = \frac{C_u - C_d}{S_u - S_d} = \frac{215.89 - 0}{645.89 - 430} = 1.00$$

↙ 645.89 − 430 ↙ exercise price

← going both ways $527 <^{C_u}_{C_d}$

$$B = \frac{S_u \times C_d - S_d \times C_u}{(S_u - S_d)(1+r)} = -426.80$$

Pay of 426.80 · 1.0075 = 430 at $t=2$

↑ figuring out at $t=1$ using $t=2$ figures
then $t=0$ $t=1$

(5)

<u>down $t=1$</u>

- no pay off at either branch

$$350.85 \begin{cases} 430 \\ 286.27 \end{cases}$$

- So $C_u = C_d = 0$
- So ~~stock price pai~~ Options $\Delta = 0$

So ~~[illegible]~~ C_1 bottom $= 0$

<u>Call$_1$</u> top $= C_1$

$S_1 \cdot \Delta - B_1$ + technically (+) ↓ always short the bond

$527 \cdot 1.00 - 426.80$

$= 100.20$

Now march back to $t=0$

$$\text{Option } \Delta = \frac{C_u - C_d}{S_u - S_d} = \frac{100.20 - 0}{527 - 350.85} = .5688$$

$$B = \frac{S_u \cdot C_d - S_d \cdot C_u}{(S_u - S_d) \cdot (1+r)} = \frac{527 \cdot 0 - 350.85 \cdot 100.20}{(527 - 350.85)(1.0075)} = -198.09$$

at $t=0$

(6)

So now $Call_0 = C_0$

$S_0 \cdot \Delta_0 - B_0$

$= 430 \cdot .5688 - 198.09$

$= 46.49$

Then find Put price at $t=0 \to P_0$

$$P_0 = C_0 + \frac{K}{(1+r)^T} - S_0$$

$$= 46.49 + \frac{430}{(1.0075)^T} - 430$$

↑ period is always 1 in binomial options

$= \_\_\_\_\_\_$

As time periods ↑, becomes exponentially more complex
And can't go to ∞

Can estimate movements with σ (st dev of stock returns)

Say $\sigma = .25$

$h = 18$ days $= \frac{18}{365} = .0493$ days

$U = e^{.25\sqrt{.0493}}$

7

No one uses Binomial much

Use Black Sholes

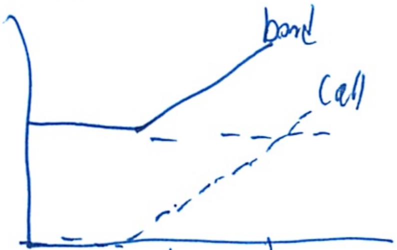
Or use calulator/computer

Or some people see in their heads

Slides (page 22)

Put - call parity

Payoff



Asset price

Call w/ 100 strike
Bond par 100

Option value increase when asset is more voltatle

Where people think stock will go flows to option

Should be no arbitrage possible

He does not use linear algebra/ solve simlanteous

Uses formulas he showed

Charts of more than 1 period....
As well as more write ups...

8

Must rebalance your portfolio every time period
- to keep riskless hedge

Thus
- no early calls
- replicating ~~a~~ ... (missed)

Lenght of period can get smaller + smaller
Risk neutral portfolio we built

MIT Sloan School of Management

Finance Theory I 15.401
Craig Stephenson Spring 2011

Problem Set 4: Futures and Forwards
(Due: Monday, April 4th, by 5:00 p.m.)

Unless specified as an "Excel Problem", all of these problems can (and should) be solved with a pencil, paper, and a simple calculator. Do not use Excel (except to check your work, if you want). Show your work cleanly and write out the formulas that you used to solve the problems. Circle your final answers.

Problem 1

The current price of silver is \$13.50 per ounce. The storage costs are \$0.10 per ounce per year payable quarterly at the beginning of each quarter and the interest rate is 5% APR compounded quarterly (1.25% APR per quarter). Calculate the future price of silver for delivery in nine months. Assume that silver is held for investment only and that the convenience yield of holding silver is zero.

Problem 2

You are employed by a U.S.-based company. Your company's balance sheet includes a receivable in the amount of € 60 million, and your obligor will pay you one year from today. Assume that the current 1-year U.S. dollar and Euro interest rates are 0.25% and 1.25%, respectively. These are both effective annual rates. The current spot \$/€ exchange rate is S = 1.40 U.S. dollars per Euro (\$1.40/€).

(a) If you save \$100 for one year, you will have \$100.25 at the end of the year. Suppose that instead you convert the \$100 into Euros and then save the corresponding amount at the euro savings rate of 1.25%. How many Euros will you have at the end of one year?

(b) Based on your answer to (a), what is the proper forward exchange rate for a 1-year forward contract to exchange U.S. dollars for Euros? (Express your answer as \$ per €.)

(c) Since you are a US company, you prefer not to be exposed to exchange rate risk. Assume that properly priced 1-year \$/€ forward contracts are available for trading. What transactions would you need to enter into in order to hedge your exposure to the risk of changes in the \$/€ exchange rate?

Problem 3

The spot price of gold (that is, the price for immediate delivery) is currently \$1420/ounce. The three-year interest rate is 1%. Assume gold is held for investment purposes and has zero convenience yield.

(a) Suppose that you enter into a forward contract to buy 100 ounces of gold in three years. Compute the forward price (in \$ per ounce) for delivery of gold in three years. Assume that there is no risk of default by either side of the forward contract (there is zero counterparty risk). Denote this forward price by $F_3(0)$: the forward price at t = 0 for delivery of gold at t = 3.

(b) Suppose that after one year (t = 1), the spot price of gold is \$1500/ounce and the two-year interest rate is 2%. Compute the forward price at t = 1 for delivery of gold at time t = 3: $F_3(1)$.

(c) At t = 0, you entered into a contract to buy 100 ounces of gold at t = 3 for a price of $F_3(0)$ per ounce. Now at t = 1, gold prices have gone up, so this is good news for you since you have locked in the price. What is the difference in cash flows between your forward contract and a newly initiated forward contract which is an agreement to buy gold at t = 3 at the price of $F_3(1)$? Be specific about the timing and magnitude of the cash flows. How much would you need to be paid in order to sell the forward contract that you bought at t = 0?

(d) Suppose that at t = 2, the price of gold falls to \$1460/ounce, and the one-year interest rate is 3%. Compute the t = 2 value of the contract you entered into at t = 0.

5

First year interest rate was .33

$$FV_{t=1} = PV_{t=0}(1 + .0033)$$

$$= 143420(1.0033)$$

$$= 143893.24$$

forward contract did not pay anything for!

Minus price of gold at year 3 at 1420/oz

$$PV_{t=1} = \frac{1420 \cdot 100}{(1+.02)^1} \leftarrow 2 \text{ year rate}$$

$$= 139215.69$$

So PV cashflow = $143893.29 - 139215.69$

$$= 4677.6$$

For other one

~~Cost = 153,000~~

PV price of gold $\dfrac{1500 \cdot 100}{(1.02)^1} = 147{,}058$

~~So PV contract = 153,000 - 147,058 = 5942~~

Rewrite

Michael Plasmeier

15.401 P-Set 4

4/2

9.5/10

1. Silver 13.50 / oz

Storage costs .10 / oz / year

5% APR compounded quarterly

- 1.25% / qt

Future price silver in 9 months

$$F = S(1 + r - \tilde{y})^T$$

futures price
- exchange
- liquid

spot rate

net convenience yield
y - c
as % of price

$$= 13.50\left(1 + .0125 - \left(\frac{0 - \frac{3}{4}\,.10}{13.50}\right)\right)^3$$ ← 3 quarters

$$= 14.2945$$

-.5

(2)

2. Will get € 60 million in 1 year

US interest rate .25%

EUR 1.25%

$\$/€ = \$1.40$ dollars per euro

a) If save \$100 for a year will have \$100.25

Suppose you save in Euros instead.

Convert to euros

$$\text{Euros}/\$ = \frac{1}{\$/\text{EUR}} = \frac{1}{1.4} = .7143$$

$$100 \text{ dollars} \cdot \frac{.7143 \text{ Euros}}{\text{dollars}} = 71.43 \text{ euros}$$

Save

$$71.43 \cdot (1 + .0125) = 72.32 \text{ euros}$$ ✓

b) So what would proper forward rate be?
- didn't really learn
- need no arbitrage - so must convert back to = return would have made in US

③

Indifferent b/w €72.32 euros or $100.25 USD

$$\frac{100.25 \text{ USD}}{72.32 \text{ EUR}} = 1.3860 \checkmark$$

forward exchange rate \$ → EUR
in t=1 year

c) To hedge – just own both the actual position and a future or a forward contract.

You need to buy (go long) a 1-year forward contract for dollars – since the US company knows it will need to buy dollars with euros in a year.

If it wanted to sell, it should short futures.

$$\# \text{ contracts} = \frac{\text{face value}}{\text{contract price}} \cdot \text{durations}$$

still don't totally get – recitation slides

(4)

3. Spot price Gold = \$1420/oz

3 year interest rate = 1%

a) Enter a forward for 100 oz gold in 3 years

0 Counterparty risk

0 convenience yield

Find $F_3(0)$

delivery, price at time

$$F_3(0) = S(1 + r - \bar{y})^T$$

$= 1420 \cdot 100(1 + .01 - 0)^1$ ← rate ~~is for~~ 3 years

rates are annualized - (convention)

$= 143{,}420$

b) At year 1 $t=1$ spot price = 1500/oz, $i = 2\%$

$F_3(1) = 1500 \cdot 100(1 + .02)^1$

$= 153{,}000$

c) At $t=0$ bought contract - good deal!

What are cash flows

First split interest rate?

When are we finding the cash flow for? at $t=1$?

(5)

Remember we paid nothing for contract

For first contract $F_3(0)$ get gold at 1420

$$\frac{1420 \cdot 100}{(1+.02)^1} = 139,215.69$$

↑ using current interest rate

Second contract $F_3(1)$

$$\frac{1500 \cdot 100}{(1.02)} = 147,058$$

So someone must pay you 147,058 − 139215.69 = 7842.31 for your $F_3(0)$ contract at $t=1$

✓

d) $t=2$ Gold 1460/oz ~~r~~ 3% for last year,
What is value of $F_3(0)$ at $t=2$?

What the gold will cost you

$$\frac{1420 \cdot 100}{(1.03)^1} = 137\,864.08$$

what it would cost you if you contracted today

$$\frac{1460 \cdot 100}{(1.03)} = 141\,747.57$$

⑥

So contract is worth $141,747.57 - 137,864.08$
$= 3883.49$ at $t=2$

- which makes sense - less than before

✓

MIT Sloan School of Management

Finance Theory I — 15.401
Craig Stephenson — Spring 2011

Solutions to Problem Set 4: Futures and Forwards
(Due: Monday, April 4th, by 5:00 p.m.)

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Problem 1

The current price of silver is \$13.50 per ounce. The storage costs are \$0.10 per ounce per year payable quarterly at the beginning of each quarter and the interest rate is 5% APR compounded quarterly (1.25% APR per quarter). Calculate the future price of silver for delivery in nine months. Assume that silver is held for investment only and that the convenience yield of holding silver is zero.

Solution: The formula on page 15 of the lecture notes tells us:

$$H_T = (1 + r)^T S \;+/-\; FV_T \;(+\, FV_T \text{ with net storage costs; } -\, FV_T \text{ with net convenience yield})$$

In this problem there are net storage costs of \$0.10 per ounce per year, which equals \$0.025 per ounce per quarter, and r = 1.25% per quarter. With these data:

$$H_T = \$13.50 \times (1.0125)^3 + \$0.025 \times (1.0125)^3 + \$0.025 \times (1.0125)^2 + \$0.025 \times (1.0125) = \$14.09$$

Problem 2

You are employed by a U.S.-based company. Your company's balance sheet includes a receivable in the amount of € 60 million, and your obligor will pay you one year from today. Assume that the current 1-year U.S. dollar and Euro interest rates are 0.25% and 1.25%, respectively. These are both effective annual rates. The current spot \$/€ exchange rate is S = 1.40 U.S. dollars per Euro (\$1.40/€).

(a) If you save \$100 for one year, you will have \$100.25 at the end of the year. Suppose that instead you convert the \$100 into Euros and then save the corresponding amount at the euro savings rate of 1.25%. How many Euros will you have at the end of one year?

Solution: Convert your \$ into Euro at the spot rate

$$\$100 \times 1€ / \$1.40 = €71.43$$

And invest your Euro proceeds at 1.25% for one year

$$€71.43 \times 1.0125 = €72.32$$

(b) Based on your answer to (a), what is the proper forward exchange rate for a 1-year forward contract to exchange U.S. dollars for Euros? (Express your answer as \$ per €.)

Solution: After converting $ to Euro and investing you have €72.32 at the end of one year. By investing $100 you have $100.25 at the end of one year. This means the forward rate should be

$$F = \$100.25 / €72.32 = 1.3862€ / \$1$$

(c) Since you are a US company, you prefer not to be exposed to exchange rate risk. Assume that properly priced 1-year $/€ forward contracts are available for trading. What transactions would you need to enter into in order to hedge your exposure to the risk of changes in the $/€ exchange rate?

Solution: Since the forward rate should be 1.3862€ / $1 and you are receiving Euro but want $, you can enter into a forward contract with a money center bank which makes a market in Euro. You want to sell €60,000,000 forward at 1.3862€ / $1, which results in $ proceeds of $83,172,000. In one year take your Euro and deliver them to satisfy the forward contract, receiving the desired $.

Alternatively you could borrow the present value of €60,000,000 at the Euro interest rate of 1.25% for one year, receiving Euro proceeds of €59,259,259.26. Take this amount and buy $ at the spot rate of 1€ / $1.40, receiving $82,962,962.96. Now invest these $ at the $ interest rate for one year, receiving $ proceeds of $83,170,370.37. This amount should be, must be, the same $ proceeds as you received from selling the Euro forward at 1.3862€ / $1. The reason these amounts do not precisely match is rounding in the calculations which produces the Euro forward rate. If you take these calculations to more decimals, the $ proceeds under either alternative are the same.

Problem 3

The spot price of gold (that is, the price for immediate delivery) is currently $1420/ounce. The three-year interest rate is 1%. Assume gold is held for investment purposes and has zero convenience yield.

(a) Suppose that you enter into a forward contract to buy 100 ounces of gold in three years. Compute the forward price (in $ per ounce) for delivery of gold in three years. Assume that there is no risk of default by either side of the forward contract (there is zero counterparty risk). Denote this forward price by $F_3(0)$: the forward price at t = 0 for delivery of gold at t = 3.

Solution: The formula on page 15 of the lecture notes tells us:

$$F_T = (1 + r)^T S +/- FV_T \text{ (+ } FV_T \text{ with net storage costs; - } FV_T \text{ with net convenience yield)}$$

In this case there are zero storage costs and convenience yield. With this information:

$$F_T = (1 + r)^T S = (1.01)^3 \times \$1,420 = \$1,463.03$$

The forward price at t = 0 for delivery of gold at t = 3 is $1,463.03.

(b) Suppose that after one year ($t = 1$), the spot price of gold is $1500/ounce and the two-year interest rate is 2%. Compute the forward price at $t = 1$ for delivery of gold at time $t = 3$: $F_3(1)$.

Solution: The only changes from part (a) above are we have moved forward to time $t = 1$, the spot price of gold is now $1,500, and interest rates have increased to 2%. These data produce the following forward price at $t = 1$ for delivery of gold at $t = 3$:

$$F_T = (1 + r)^T S = (1.02)^2 \times \$1,500 = \$1,560.60$$

(c) At $t = 0$, you entered into a contract to buy 100 ounces of gold at $t = 3$ for a price of $F_3(0)$ per ounce. Now at $t = 1$, gold prices have gone up, so this is good news for you since you have locked in the price. What is the difference in cash flows between your forward contract and a newly initiated forward contract which is an agreement to buy gold at $t = 3$ at the price of $F_3(1)$? Be specific about the timing and magnitude of the cash flows. How much would you need to be paid in order to sell the forward contract that you bought at $t = 0$?

Solution: By entering into the forward contract at $t = 0$ to buy gold at $t = 3$, you are committed to buy gold at $t = 3$ for $1,463.03. Now at $t = 1$, a forward contract to buy gold at $t = 3$ costs $1,560.60. You are a winner! You can buy gold at $t = 3$ for $1,560.60 - $1,463.03, or $97.57 less than the current price. Your forward contract at $1,463.03 is a valuable contract, which people would like to own. If some approached you about purchasing your valuable forward contract, you would have to be paid the present value at $t = 1$ of your valuable contract, which is worth $97.57 at $t = 3$. This present value is:

$$\$97.57 \times 1 / (1.02)^2 = \$93.78$$

So you would need to be paid $93.78 at $t = 1$ to sell the forward contact you entered into at $t = 0$.

(d) Suppose that at $t = 2$, the price of gold falls to $1460/ounce, and the one-year interest rate is 3%. Compute the $t = 2$ value of the contract you entered into at $t = 0$.

Solution: This is the same thing as (c) above, except we've now moved to $t = 2$. The spot price of gold is $1,460.00, so the forward price for gold at $t = 3$ is

$$F_T = (1 + r)^T S = (1.03) \times \$1,460 = \$1,503.80$$

You still own the forward contract you purchased at $t = 0$ for gold at $1,463.03, so you still own a valuable forward contract, with a value at $t = 3$ of $1,503.80 - $1,463.03, or $40.77. The present value of this $t = 3$ value at $t = 2$ is

$$\$40.77 \times 1 / (1.03) = \$39.58$$

So you would need to be paid $39.58 at $t = 2$ to sell the forward contact you entered into at $t = 0$.

15.401 – April 4, 2011 – Black-Scholes Option Pricing

We've seen that call option values are a function of 5 variables, current stock price, exercise price, time to expiry, variance of stock returns, and the risk free rate of return. Fisher Black and Myron Scholes isolated these 5 factors in their revolutionary 1973 article in the Journal of Political Economy, titled "The Pricing of Options and Corporate Liabilities." Black and Scholes used a replicating portfolio and an equilibrium argument to derive a partial differential equation, which was solved, generating the famous Black-Scholes Option Pricing Model for European call options:

$$C = SN(d_1) - Ke^{-rT}N(d_2)$$

where:

C = the value of the call option (the call premia),
S = the underlying stock's price,
N(·) = the cumulative distribution function for a standardized normal random variable,
K = the exercise price of the call option,
e = base e antilog, or 2.7183,
r = the risk free rate of return for one year assuming continuous compounding, and,
T = the time remaining to expiry of the call option, expressed as a fraction of a year.

To determine the equilibrium value of a European call option, you input values into the formula. The values of d_1 and d_2, however require the following formulae, which are then applied to the standard normal cumulative distribution table:

$$d_1 = \frac{\ln(S/K) + (r + \frac{1}{2}\sigma^2)(T)}{\sigma\sqrt{T}}$$

And

$$d_2 = d_1 - \sigma\sqrt{T}$$

where σ represents the standard deviation of the continuously compounded rate of return on the underlying stock.

These calculated values of d_1 and d_2 are applied to the standard normal cumulative distribution table to determine the cumulative probability from minus infinity to d_1 and d_2.

To price European puts, merely use the Put-Call parity relationship:

$$P = C + \frac{K}{e^{rT}} - S$$

where:

P = Price of a put option (the put premia),
C = Price of a call option,
K = Exercise price of the option,
e^{rT} = present value operator for a continuously compounded sum discounted at interest rate r for T years, and
S = Price of the underlying stock.

Notice the σ term in Black-Scholes. Remember this is the standard deviation of the continuously compounded rate of return on the underlying stock. Given σ, you can calculate the value of calls and puts. Conversely, given the value of calls and puts, you can calculate σ. Calculating σ in this manner produces implied standard deviations, telling you the market's expectation for future stock volatility. This is valuable information to investors. There is, unfortunately, no closed form algebraic solution to calculating σ, which must be done in an iterative process on a computer.

Table 1: Table of the Standard Normal Cumulative Distribution Function $\Phi(z)$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
-3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002
-3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0003
-3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005
-3.1	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007
-3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
-2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
-2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
-2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
-2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
-2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
-2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
-2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
-2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
-2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
-2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
-1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
-1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
-1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
-1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
-1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
-1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
-1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
-1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
-1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
-1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
-0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
-0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
-0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
-0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
-0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
-0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
-0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
-0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
-0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
-0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

15.401 Black Scholes
Option Pricing

- **binomial** - point in time where up + down node
  - gets more accurate at periods get shorter + shorter
  - wheres the value as you go contineous
  - have complex diff eq.
    - based off heat transfer formula

$$\boxed{C = SN(d_1) - ke^{-rT} N(d_2)}$$

example

$S_0 = \$90$ stock price $r = .007$ risk free interest rate

$k = \$80$ strike/exercise price $\sigma = .25$

↑ underlying asset

$T = 47$ days $(/365$ days$)$

↑ waiting period
you don't own asset
gives you \$

(2)

$$C = \underbrace{SN(d_1)}_{\text{roughly the Delta section}} - \underbrace{Ke^{-rT} N(d_2)}_{\text{roughly the bond section}}$$

$$d_1 = \frac{\ln\left(\frac{90}{80}\right) + \left(.007 + \frac{1}{2}(.25)^2\right)\left(\frac{47}{365}\right)}{.25\sqrt{\frac{47}{365}}}$$

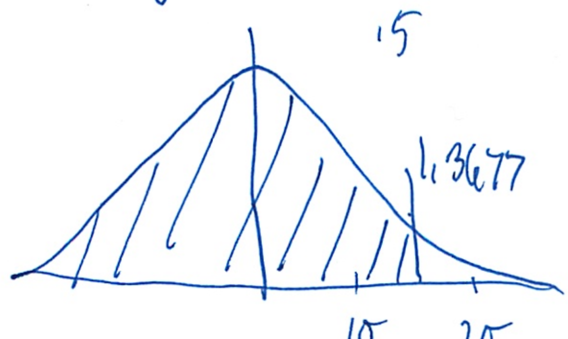
$= 1.3677$

$d_2 = 1.3677 - \sigma\sqrt{T}$

$= 1.3677 - .25\sqrt{.1288}$

$= 1.2780$

Now go to Normal



Want area before it CDF $\phi(1.3677)$

③

$1.36 = .9131$
$1.37 = .9147$ $\Big\rangle \Delta = .0016$

$\Delta = .0012$

$N(d_1) = .9131 + .0012$
$\quad .9143$

$1.27 = .8980$
$1.28 = .8997$ $\Big\rangle \Delta = .0017$

$N(d_2) = .8980 + .0014 = .8994$

Now just plug in for

$C = 90 \cdot .9143 - 80 \times e^{-.007 \cdot .1288} \cdot .8994$

$= 82.2870 - 71.8872$

$= 10.40$

(4)

Put-call parity always holds, so

$$P = C + \frac{K}{e^{rT}} - S$$

$$= 10.40 + \frac{80}{e^{.007 \cdot .1288}} - 90$$

$$= .33$$

~~[illegible]~~

does not seem very likley since price above exercise price, low σ

American options w/ significant dividends - add another block for dividend

---

σ of contineous returns

- what is it??
- can calc daily return each day for a year
- and then find the σ to estimate
- σ is supposed to be contineous
- implied volatility

(5)

Stock option often an employeee perk
- not an expense
- so don't show up on balance sheet before

- do now req on income / balance sheet so now told
- must use Black - Scholes pricing

Explained on slide again
implied volitility

Straddle - use apple data for the options
- buy a call and buy a put
- same stock
- same k - exercise price
- same T - same time to expiration date

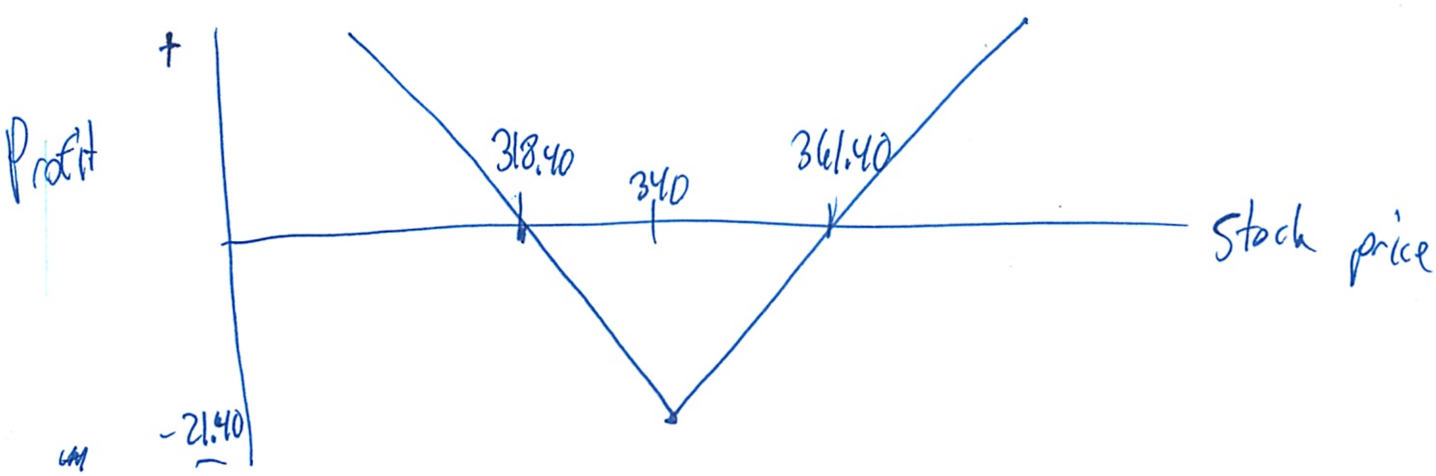
Say k = 340
T = April 16, 2011

Current call premia \$9.90
put " \$11.50

(6)

Payoff chart

Stock price	Payoff call	Payoff put	Premia	Profit
300	0	40	~~[illegible]~~	18.60
310	0	30	-21.40	8.60
320	0	20		-1.40
330	0	10		-11.40
340	0	0		-21.40
350	10	0		-11.40
360	20	0		-1.40
370	30	0		8.60
380	40	0		18.60

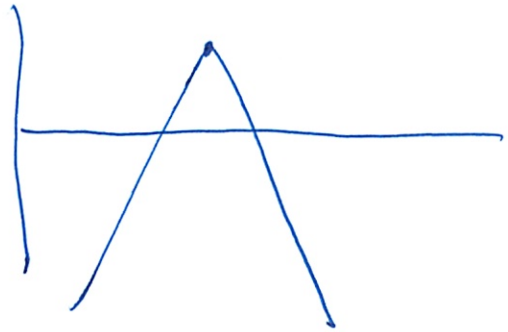


The risk is you lose your primeum if it does not move.

You just want movement

7

In reverse straddle its the exact opposet



Can keep the $ if it does not move

~ can capture the premium when you buy saddle

But other side tempting to create movement

If people don't want to trade the other side then put/call premia moves ~~[illegible]~~

Part C Risk

Chapter 7: Introduction to Risk and Return

Chapter 8: Portfolio Choice

Chapter 9: Capital Asset Pricing Model

Premise in Previous Discussions

1. A rich set of traded securities allow us to price a particular CF (asset) by arbitrage (specifically the lack of profitable arbitrage)
   - Time and risk
2. Pricing of risky CFs has the following properties:
   - CFs with "same risk" are discounted at the same rate
   - "Riskier" CFs are discounted at higher rates

Unanswered Questions

1. How do we measure risk?
2. How do financial markets determine the price of risk?

Goals for Part C

1. Quantifying risk (Chapter 7)

2. Portfolio choice (Chapter 8):
   - Diversifiable risk versus non-diversifiable risk
   - Optimal risk/return trade-off

3. Capital Asset Pricing Model (CAPM) (Chapter 9):
   - How to determine the price of risk (the risk adjusted discount rate)

15.401 Finance Theory I

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Lecture 7: Introduction to Risk and Return

Key concepts

Asset returns
Measuring risk
Investor preferences
Estimating risk and return
Historic asset returns and risks

Readings:

Brealey, Myers and Allen, Chapter 7, 8

Bodie, Kane and Markus, Chapters 5.2 - 5.4

Asset returns

P_0 is the asset price at the beginning of period

P_1 is the asset price at the end of period → uncertain (random variable)

D_1 is the dividend paid at the end of period → uncertain

$$\text{Return } \tilde{r}_1 = \frac{\tilde{D}_1 + \tilde{P}_1 - P_0}{P_0} = \frac{\tilde{D}_1 + \tilde{P}_1}{P_0} - 1$$

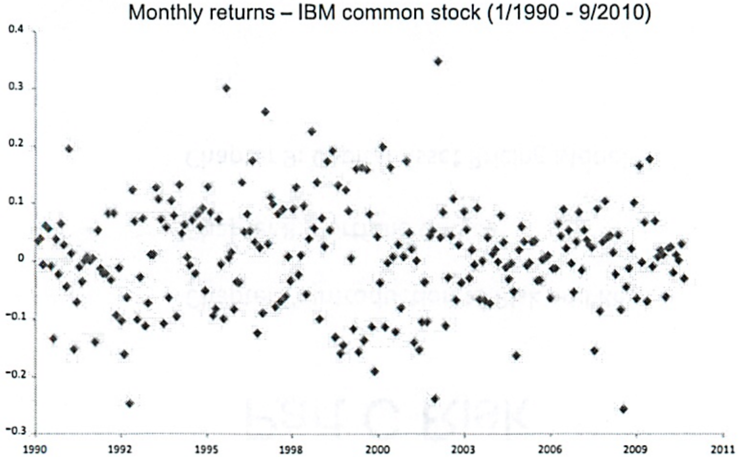
$$\text{Expected Return} = E[\tilde{r}_1]$$

$$\text{Excess Return} = \tilde{r}_1 - r_F$$

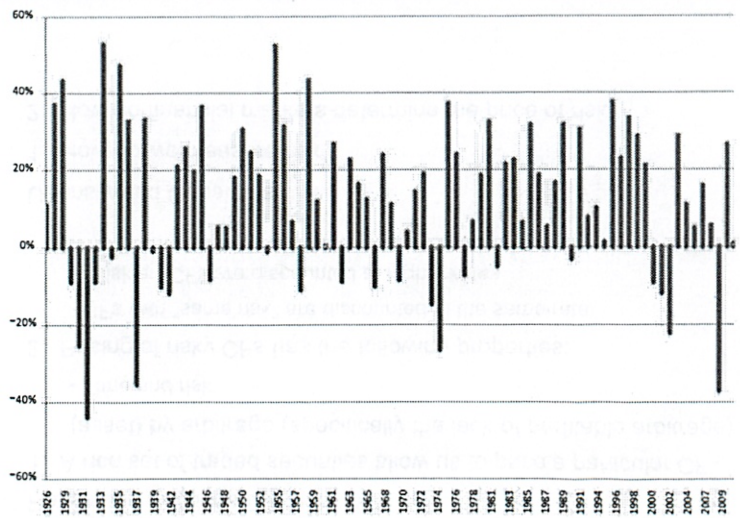
$$\text{Risk Premium} = E[\tilde{r}_1] - r_F = \pi$$

Asset returns

Monthly returns – IBM common stock (1/1990 - 9/2010)



S&P500 Annual returns, 1925-2010



Basic statistics

Mean, variance, standard deviation (SD):

$$\bar{r} = \mathrm{E}[\tilde{r}]$$

$$\sigma^2 = \mathrm{E}[(\tilde{r} - \bar{r})^2]$$

$$\sigma = \sqrt{\sigma^2}$$

Sample estimators:

$$\hat{r} = \frac{1}{T}\sum_{t=1}^{T} r_t$$

$$\hat{\sigma}^2 = \frac{1}{T-1}\sum_{t=1}^{T}(r_t - \hat{r})^2$$

$$\hat{\sigma} = \sqrt{\hat{\sigma}^2}$$

Other statistics

Median: 50th percentile (probability of 1/2 that r_t < median)

Skewness: Is the distribution symmetric?

- Negatively skewed: big losses are more likely than big gains
- Positively skewed: big gains are more likely than big losses

Kurtosis: Does the distribution have fat tails?

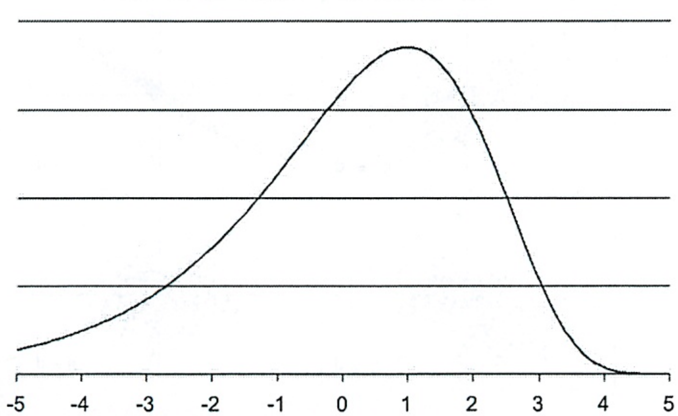
Correlation: How closely do two variables move together?

$$\mathrm{Cov}[\tilde{r}_i, \tilde{r}_j] = \mathrm{E}[(\tilde{r}_i - \bar{r}_i)(\tilde{r}_j - \bar{r}_j)] = \sigma_{ij} \quad \text{Covariance}$$

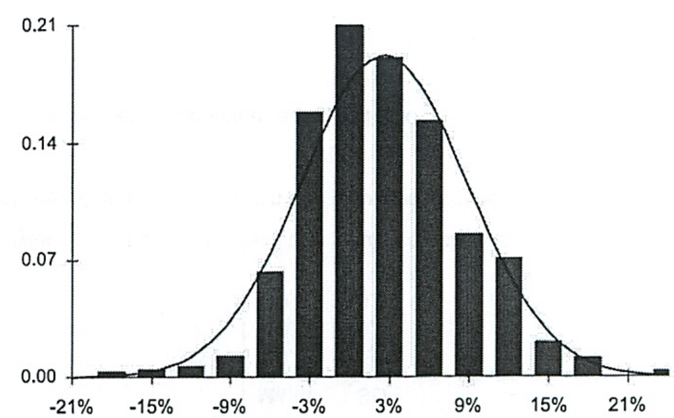
$$\mathrm{Corr}[\tilde{r}_i, \tilde{r}_j] = \frac{\mathrm{E}[(\tilde{r}_i - \bar{r}_i)(\tilde{r}_j - \bar{r}_j)]}{\sigma_i \sigma_j} = \rho_{ij} \quad \text{Correlation}$$

$$\beta_{ij} = \frac{\sigma_{ij}}{\sigma_j^2} \quad \text{Beta}$$

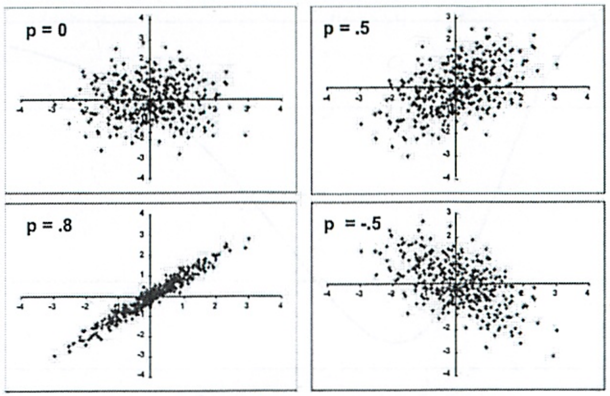
Negatively skewed distribution



GM Monthly Returns



Correlation between two random variables



(Slope of the scattered plot gives the beta.)

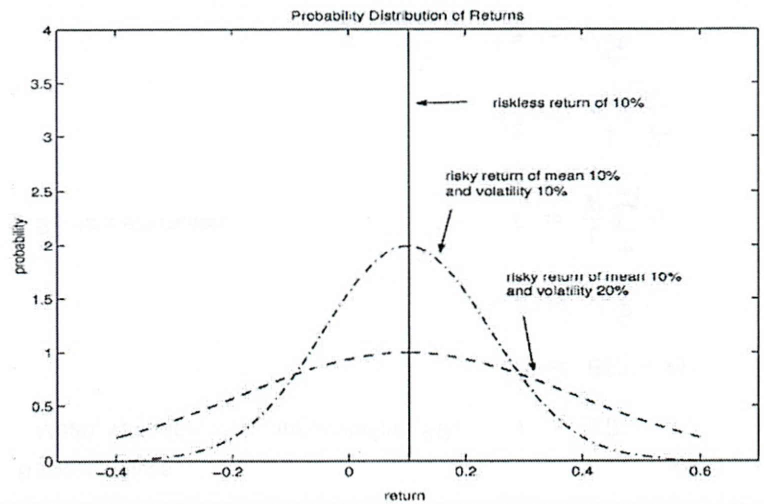
Example. Moments of return distribution. Consider three assets:

	Mean	SD
$\tilde{r}_0$ (%)	10.0	0.00
$\tilde{r}_1$ (%)	10.0	10.00
$\tilde{r}_2$ (%)	10.0	20.00

Between Asset 0 and 1, which one would you choose?

Between Asset 1 and 2, which one would you choose?

Investors care about expected return and risk.



Assumptions on investor preferences for 15.401

1. Higher mean in return is preferred:

   $$\bar{r} = E[\tilde{r}]$$

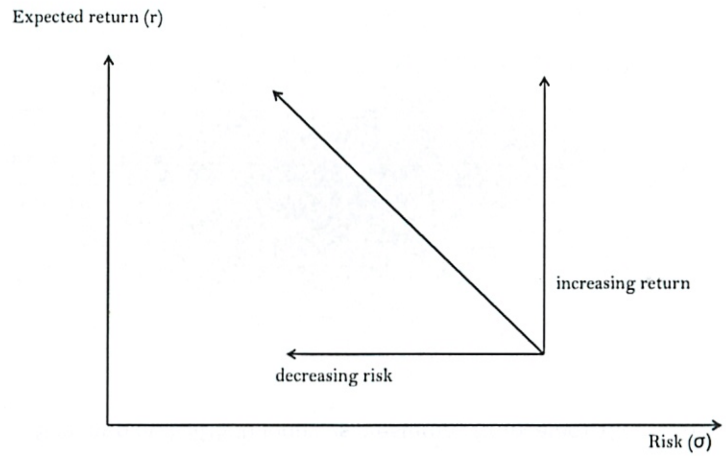
2. Higher standard deviation (SD) in return is disliked:

   $$\sigma = \sqrt{E[(\tilde{r} - \bar{r})^2]}$$

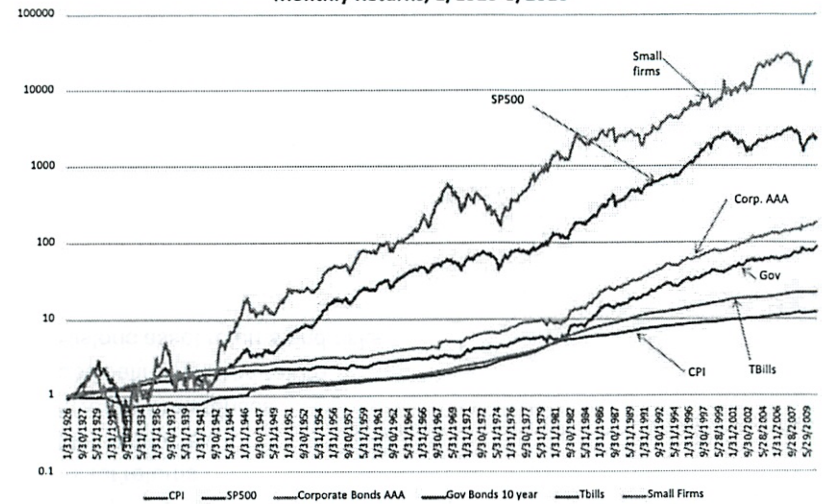
3. Investors care only about mean and SD (or variance)

Under 1-3, standard deviation (SD) gives a measure of risk.

Investor Preference for Return and Risk



Monthly Returns, 1/1925-8/2010



1. Riskier assets on average earn higher returns

	Mean (%)	Std(%)
Inflation	3.08	4.20
Treasury Bills	3.81	3.13
US Gov. Bonds (10 yr)	5.60	8.14
US Corp. Bonds (AAA)	6.49	7.04
S&P 500 stocks	11.74	20.52

Source: Global Financial Data and WRDS, Annual returns, 1925-2009.

	Start (1925)	End (2009)
Inflation	$ 1	$ 11.98
Treasury Bills	$ 1	$ 22.33
US Gov. Bonds (10 yr)	$ 1	$ 77.01
US Corp. Bonds (AAA)	$ 1	$ 165.23
S&P 500 stocks	$ 1	$ 2,382.65

Source: Global Financial Data and WRDS, Annual returns, 1925-2009.

2. Returns on risky assets can be highly correlated to each other

	Inflation	Treasury Bills	US Gov. Bonds	US Corp. Bonds	S&P500
Inflation	1.00	0.42	-0.10	-0.07	0.00
Treasury Bills		1.00	0.31	0.30	0.01
US Gov. Bonds			1.00	0.89	0.02
US Corp. Bonds				1.00	0.12
S&P 500					1.00

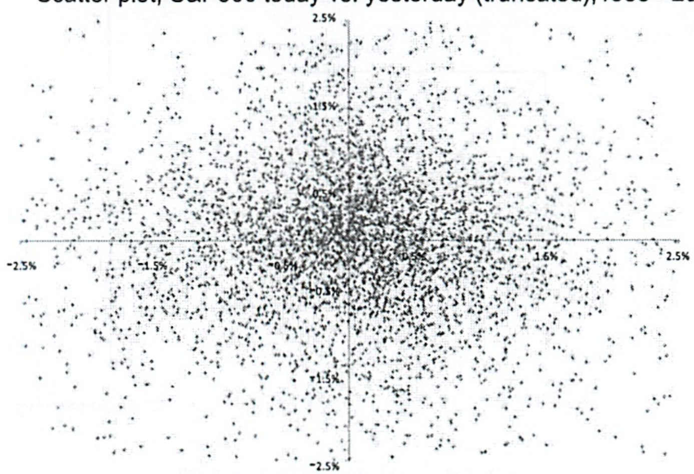
Source: Global Financial Data and WRDS, 1925-2009, annual series

3. Returns on risky assets are serially uncorrelated

	Autocorrelation
Inflation	0.63
Treasury Bills	0.92
US Gov. Bonds (10 yr)	-0.06
US Corp. Bonds (AAA)	0.14
S&P 500 stocks	0.01

Source: Global Financial Data and WRDS, 1925-2009.

Scatter plot, S&P500 today vs. yesterday (truncated),1988 - 2010



Asset returns
Measuring risk
Investor preferences
Estimating risk and return
Historic asset returns and risks

Risk + Return

- lack of arbitrage was a principle
- but securities are at diff risk
- require a diff risk adj. rate of return

*how do you measure + price? *

return $\tilde{r}_1 = \dfrac{\overbrace{\tilde{D}_1}^{\text{dividend}} + \tilde{P}_1 - P_0}{P_0} = \dfrac{\tilde{D}_1 + \tilde{P}_1}{P_0} - 1$

Want $E[\tilde{r}_1]$ = expected return

Excess return $\tilde{r}_1 - \underbrace{r_F}_{\text{risk free rate}}$

Risk preimum = $E[\tilde{r}_1] - r_F = \Pi$

So over long time risky assets have to pay more than risk-less assets over long period of time

Calc rate of return over time

- with dividend calculated in
- in % terms
- slide 8

(2)

Stats

$$\bar{r} = E[\tilde{r}]$$

$$\sigma^2 = E[(\tilde{r} - \bar{r})^2]$$

$$\sigma = \sqrt{\sigma^2}$$

predicting future

Sample estimator

$$\hat{r} = \frac{1}{T}\sum_{t=1}^{T} r_t$$

$$\hat{\sigma}^2 = \frac{1}{T-1}\sum_{t=1}^{T}(r_t - \hat{r})^2$$

$$\hat{\sigma} = \sqrt{\hat{\sigma}^2}$$

measuring past

Use data from past to try and project future

Skewness: Is dist symmetric?

- Neg skew - big losses more likely
- Pos skew - big gains " "

Dist is not very normal

Most you can lose is your initial investment

But unlimited upside

(3)

Kurtosis: Does the dist have fat tails

- Unprobable occurances occur more frequently then normal dist says

Correlation: How closely do two variables move together?

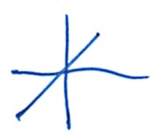
Covariance $E[(\tilde{r}_i - \bar{r}_i)(\tilde{r}_j - \bar{r}_j)] = \sigma_{ij}$

Correlation $\dfrac{E[(\tilde{r}_i - \bar{r}_i)(\tilde{r}_j - \bar{r}_j)]}{\sigma_i \sigma_j} = \rho_{ij}$

Beta $= \dfrac{\sigma_{ij}}{\sigma_j^2}$ = measure of risk of asset vs some market measure j

$\rho = 0$ random 

$\rho = .5$ kinda correlated 

$\rho = 1$ perfectly correlated 

$\rho = -.5$ one goes up, other goes down kinda 

$\rho = -1$ perfectly neg correlated 

How risky is asset?

How risky asset vs rest of portfolio?

④

Have 3 assets. Which would you choose

	Mean	σ
r_0	10	0
r_1	10	10
r_2	10	20

r_0 - guarenteed

r_1 - risky but no higher return

If was 10 we don't really know - each person
12 ~~we have~~ has diff risk comfortness
14

If stock is not selling, price will fall for r_1 r_2
And at some point ~~the~~ price will ~~the~~ be low enough to buy it

Slide 15 chart of returns

Higher return perefered $\bar{u}$

But lower st dev σ perfered

People want high return, low risk

⑤

Small firms - have to rebalance every year
- very voltatile
- you must rebalance!

Log scale on p 18

Correlation chart p21

Corp bonds longer than 10 - so don't fit chart p 19

p20 - need to rebalance each year!

p22 - common stocks serially uncorrelated
?day after day

if goes ↑ one day, the next day is unpredictable

since stock prices driven by info flow - which is random

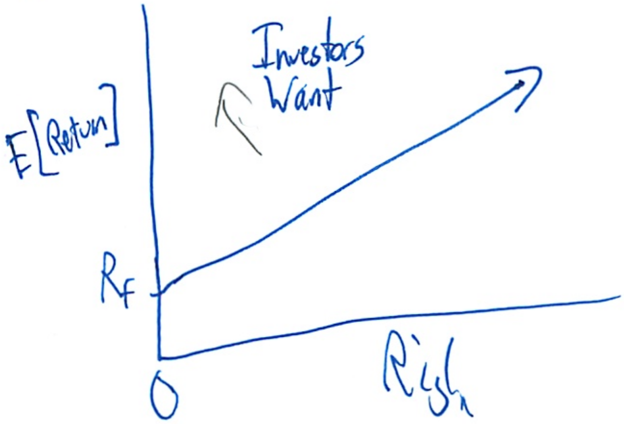
random walk w/ upward drift

but T-Bills driven by inflation
- inflation takes a while to change

S+P looks random

people are looking at data - want to get ahead

6



← Correlation get smaller

15.401 Recitation

5: Options

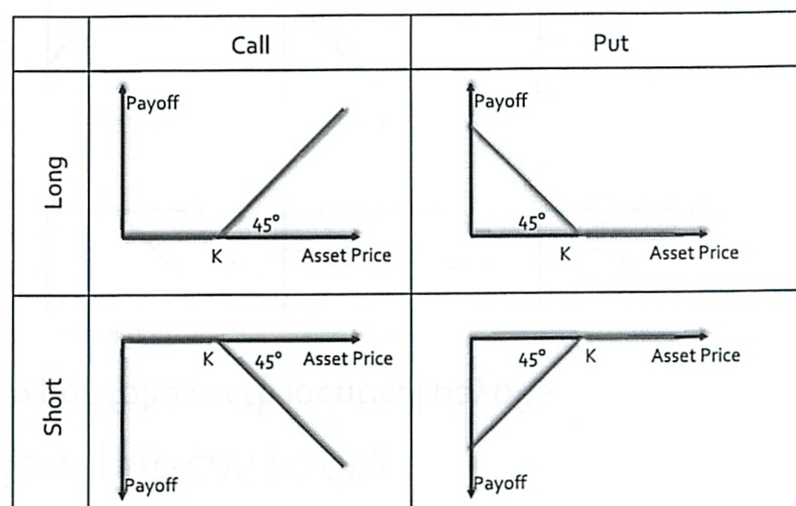
Learning Objectives

- Review of Concepts
  - Payoff profile
  - Put-call parity
  - Valuation of options
  - Binomial tree
- Examples
  - Payoff replication
  - Arboreal Corporation

Review: elements of a call/put option

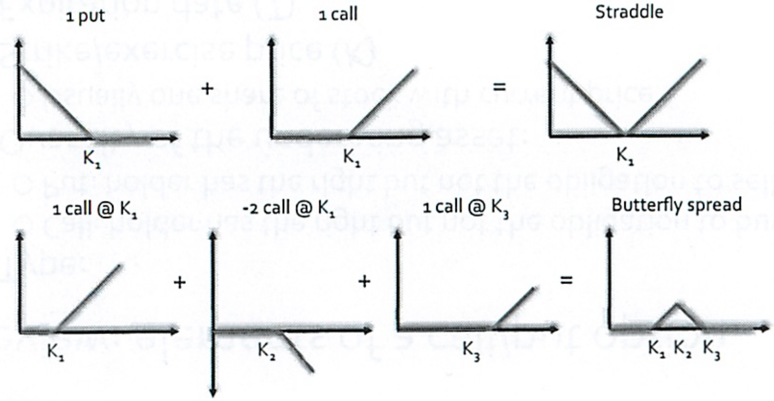
- Type:
  - Call: holder has the right but not the obligation to buy
  - Put: holder has the right but not the obligation to sell
- Quantity of the underlying asset:
  - Usually one share of stock with current price S
- Strike/exercise price (K)
- Expiration date (T)
- Style:
  - European: can only be exercised at T
  - American: can be exercised at any time between 0 and T.

Review: payoff profile



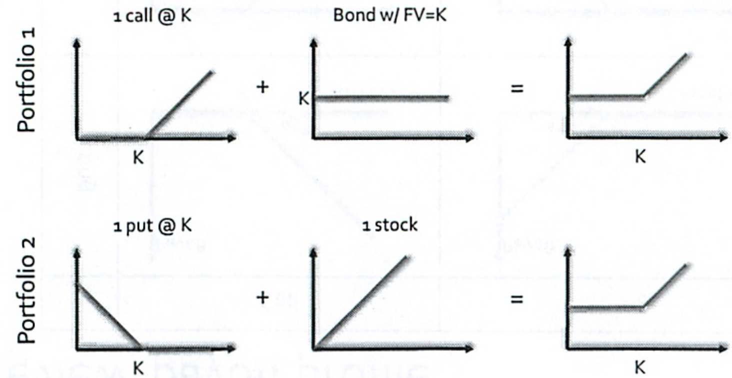
Review: payoff profile

- The payoff of a portfolio of options is the sum of payoffs of the individual components:



Review: put-call parity

- Two portfolios with identical payoffs



Review: put-call parity

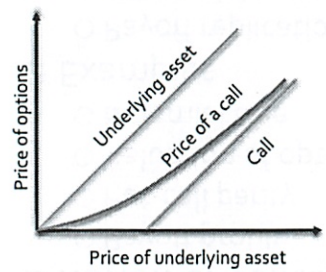
- No arbitrage implies that the two portfolios must have the same cost:

$$C + PV(K) = P + S$$

$$C + \frac{K}{(1+r)^T} = P + S$$

- This is the **put-call parity**.
- Note: the call and put must have the same exercise price (K).

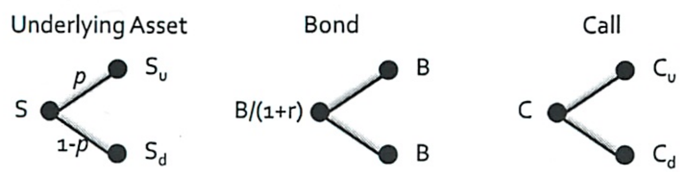
Review: value of an option



	Value of call	Value of put
Strike price (K)	Decrease	Increase
Price of underlying asset (S)	Increase	Decrease
Volatility of the underlying asset (σ)	Increase	Increase
Maturity (T)	Increase	Increase
Interest rate (r)	Increase	Decrease

Review: binomial tree

- Idea: if there are only two states of the world next period, we can price options given the underlying asset and a risk-free asset ("bond") by replication:



Review: binomial tree

- Replication:

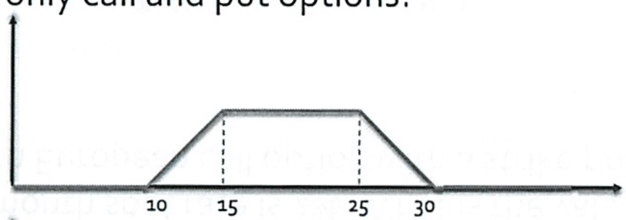
	CF at t = 0	CF at t=1 ("up" state)	CF at t=1 ("down" state)
A shares of underlying asset	$-A \times S$	$A \times S_u$	$A \times S_d$
Bond (FV=B)	$-B/(1+r)$	B	B
Total	$-A \times S - B/(1+r)$	$A \times S_u + B$	$A \times S_d + B$
Replication	$= -C$	$= C_u$	$= C_d$

- $A = (C_u - C_d) / (S_u - S_d)$
- $B = C_u - A \times S_u$
- $C = A \times S + B/(1+r)$

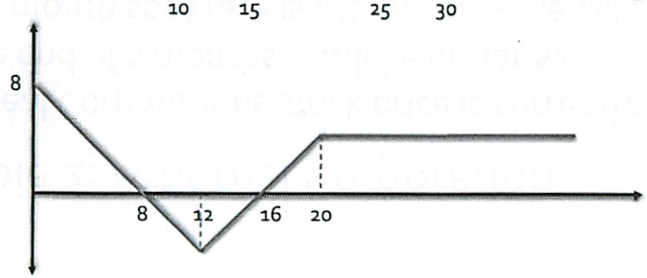
Example 1: payoff replication

- How would you replicate the following payoff profile using only call and put options?

a)



b)

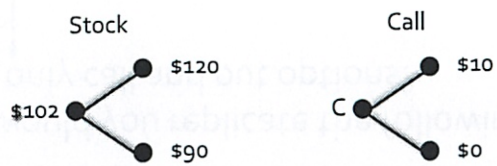


Example 1: payoff replication

- Answer:
  a) Long 1 call (K=10)
  Short 1 call (K=15)
  Short 1 call (K=25)
  Long 1 call (K=30)
  b) Long 1 put (K=8)
  Short 1 call (K=8)
  Long 2 calls (K=12)
  Short 1 call (K=20)

Example 2: Arboreal Corporation

- Arboreal Corporations stock price is currently \$102. At the end of 3 months it will be either \$120 or \$90. The 3-month spot rate is 2%. What is the value of a 3-month European call option with a strike price of \$110?



Example 2: Arboreal Corporation

- The call can be replicated with:
  - Long 1/3 stock: costs \$34
  - Short bond with FV=30: costs -\$30/(1+2%) = -\$29.41
- The price of the call must be

$$C = 34 - 29.41 = \$4.59$$

MIT Sloan School of Management

Finance Theory I — 15.401
Craig Stephenson — Spring 2011

Problem Set 5: Options
(Due: Wednesday, April 13th, by 5:00 p.m.)

Unless specified as an "Excel Problem", all of these problems can (and should) be solved with a pencil, paper, and a simple calculator. Do not use Excel (except to check your work, if you want). Show your work cleanly and write out the formulas that you used to solve the problems. Circle your final answers.

Problem 1

(a) In a commonly-used options trading strategy, code named "Alpha," the trader buys an at the money call option and sells an out of the money call option at a higher exercise price on the same underlying security expiring in the same month. Assume you decide to execute this strategy, buying a Google, Inc. June call at \$585 for \$26.60, and selling a Google, Inc. June call at \$610 for \$14.90. Draw the payoff and profit (net after premium) diagram from this strategy.

(b) Once you have completed the diagram from part (a), do an internet search and find the name for this options trading strategy. What are the benefits of this strategy?

Problem 2

The common stock of Hudson Motors will either fall to \$71.42 or rise to \$140 in the next year, a significant change from its current value of \$100. The current one-year interest rate is 5%.

(a) What is the delta of a one-year call option on Hudson with an exercise price of \$100?

(b) What is the value of the one-year call option on Hudson at \$100?

(c) What is the value of the one-year put option on Hudson at \$100?

? strike price

Problem 3

The common stock of Hathaway Browne currently trades for \$61.50, and Hathaway does not pay cash dividends. The standard deviation of continuously compounded annual returns on Hathaway stock is 31.0%, and the current annual risk-free rate of interest is 1.5%.

(a) Based on these facts and using the Black-Scholes formula, what is the price of a call option on Hathaway with an exercise price of \$65.00, which expires in 159 days?

(b) What is the price of a put option on Hathaway at \$65.00 which expires in 159 days?

(c) Excel Problem: Assume the call option at \$65.00 which expires in 159 days is actually trading for \$4.60. Given this call price, what is the implied standard deviation of continuously compounded annual returns for Hathaway's stock? To solve this write the Black-Scholes formula in Excel and either plug in numbers or use Solver to find the required σ.

Problem 4

After a successful career at the Walt Disney Company and regular investing in the employee stock ownership program, Marc Davis now owns 40,000 shares of Disney stock, which closed at \$42.43 per share on April 5, 2011. These shares represent all of Davis' financial assets, and since he plans to retire on October 5, 2011 (his 66th birthday), the value of these shares is critically important to Davis.

After reading analyst reports and thinking about the likely performance of Disney stock, Davis believes the stock has potential to increase in value by October 5, but of course, it could also decrease in value. He thinks the standard deviation of returns to Disney stock is 22.5%, and the 6-month risk-free rate of interest is 0.15%. Ignore income taxes in this problem.

(a) Davis is understandably worried that a decrease in the value of Disney stock will reduce the quality of his retirement lifestyle. Davis could sell his Disney shares now and invest the proceeds until October 5. If he selects this course of action, how much money will Davis have on October 5?

(b) Alternatively Davis could purchase options at an exercise price of \$42.50 expiring September 24, 2011, as insurance against a decrease in the price of Disney stock. What type of option should he purchase if he decides to purchase this insurance, and what is the price of this option?

(c) What are the advantages and disadvantages of using options on Disney stock to buy this insurance?

Michael Plasmeier   15.401 P-Set 5   4/9

8.25/10

1. Buys call @ 585 for 26.60

Sell call @ 610 for 14.90

Draw payoff or profit

Stock	Payoff 1	Payoff 2	Profit 1	Profit 2	Total
570	0	0	-26.60	14.90	-11.70
580	0	0	-26.60	14.90	-11.70
590	5	0	-21.60	14.90	-6.70
600	15	0	-11.60	14.90	3.3
610	25	0	-1.60	14.90	13.3
620	35	-10	8.4	4.90	13.3
630	45	-20	18.4	-5.10	13.3

Call - right to buy asset

Put - right to sell asset

Long - buying the selling - hope rises

Short - selling asset - hope price falls "writing"

- so liable for payouts

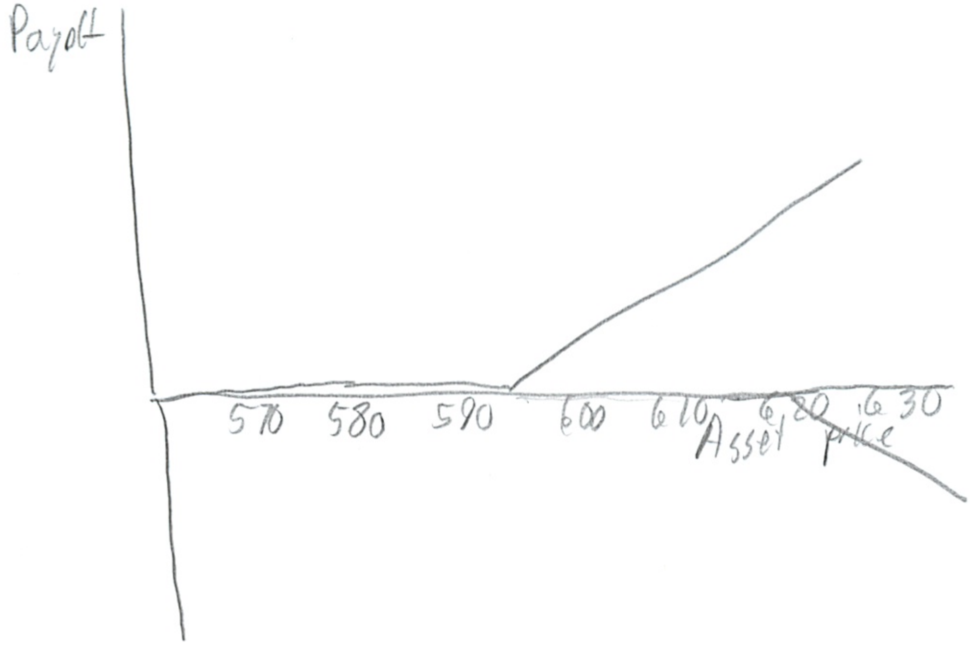
at the money - strike price = price underlying security

in the money - strike < market for call

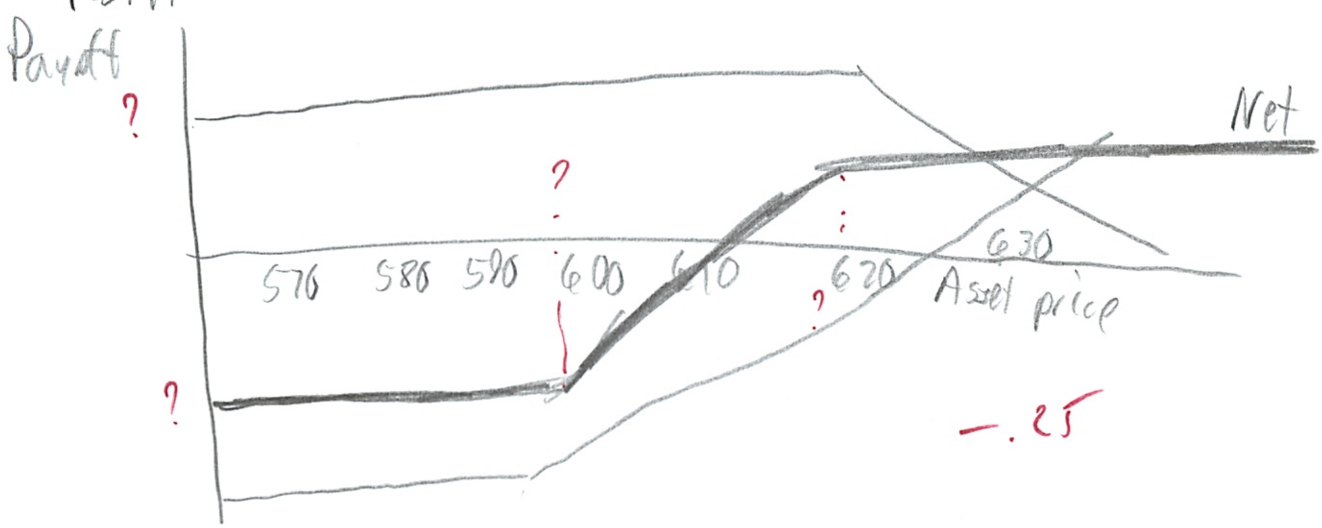
Out of the money - strike > market for call

②

Payoff



Profit



b) Bull call spread

- moderatly bullish to bullish
- risk reduction — but not very bullish
- doubly hedged strategy
  - losses/gains limited both ways
- always net cash outflow at purchase

(3)

Volitity does not effect much

Time decay goes both ways

2. Binomical

$$100 \begin{cases} 140 \\ 71.42 \end{cases} \qquad r = 5\%$$

a) $\Delta = \dfrac{\overset{\downarrow \text{profits}}{C_u - C_d}}{\underset{\uparrow \text{raw values}}{S_u - S_d}}$

$$= \frac{(140-100)-0}{140-71.42}$$

$$= \frac{40-0}{68.58}$$

$$= .583$$

↑ so, would need .583 shares of stock

b) Value of call option

$$B = \frac{\text{Stock up} \cdot \text{Call down} - \text{Stock down} \cdot \text{Call up}}{(\text{Stock up} - \text{Stock down}) \cdot (1+r)}$$

(4)

↓ raw stock prices ↓ profit

$$B = \frac{140 \cdot 0 - 71.42 \cdot 40}{(140 - 71.42)(1 + .05)} = -39.67$$

↑ raw stock prices

↑ so short 39.67 bonds (always short)

Value Call = $S_0 \cdot \Delta$ - Bond

$= 100 \cdot .5832 - 39.67$

$= 18.651$

c) Value Put

Use put-call parity

$$P_0 = C_0 + \frac{K}{(1+r)^t} - S_0$$

← strike price / exercise price of underlying security

So $K = S_0$?

– Price option will be exercised at

$$= 18.65 + \frac{100}{(1+.05)^1} - 100$$

$= 13.881$

(5)

3. Common stock Hataway Browne

$S_0 = 61.50$

No dividends

$\sigma = 31\%$

$r = 1.5\%$

a) Black-Sholes price $K = \$65$ exercise/strike price $T = \frac{159 \text{ days}}{365}$

$$C = S\,N(d_1) - K e^{-rT} N(d_2)$$

↑call price

$$T = \frac{159}{365} = .435$$

$d_1 = x$ ← in instructions

$$= \frac{\ln\left(\frac{S}{KR^{-T}}\right)}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T}$$

← diff in notes than in class - may work all same

$$= \frac{\ln\left(\frac{61.50}{65\cdot(1.015)^{-.435}}\right)}{.31\sqrt{.435}} + \frac{1}{2}\cdot .31\cdot\sqrt{.435}$$

$= -.0136$

⑥

In class we had this

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + \left(r + \frac{1}{2}(\sigma)^2\right)(T)}{\sigma\sqrt{T}}$$

$$= \frac{\ln\left(\frac{61.50}{65}\right) + \left(.015 + \frac{1}{2}(.31)^2\right)\left(\frac{159}{365}\right)}{.31\sqrt{\frac{159}{365}}}$$

$= .1202$

? So which is it?

Wikipedia uses class formula

$d_2 = d_1 - \sigma\sqrt{T}$

$= .1202 - .31\sqrt{\frac{159}{365}}$ $-.5$

$= -.08425$

$C = S N(d_1) - K e^{-rT} N(d_2)$

$= 61.50 \cdot \phi(.1202) - 65 \cdot e^{-.015 \cdot .435} \cdot \phi(-.08425)$

CDF normal normal

$= 61.50 \cdot .5478 - 65 \cdot e^{-.015 \cdot .435} \cdot .4664$

(7)

$= 3.57$

b) Put price

Put - call parity

$$P_0 = C_0 + \frac{K}{(1+r)^T} - S_0$$

$$= 3.57 + \frac{65}{(1.015)^{.435}} - 61.50$$

$$\approx 6.65$$

c) Excel problem. Solve for σ

Trades for 4.60

$$4.60 = 61.50 \cdot \Phi(d_1) - 65\, e^{-.015 \cdot .435} \cdot \Phi(d_2)$$

$$4.60 = 61.50 \cdot \Phi\left(\frac{\ln\left(\frac{61.50}{65}\right) + (.015) + \frac{1}{2}(\sigma)^2)(.435)}{\sigma \cdot \sqrt{.435}} \right) \cdots$$

$$- 65\, e^{-.015 \cdot .435} \cdot \Phi(\cdots - \sigma\sqrt{.435})$$

Solve for σ

8.

I did on Wolfram Alpha instead

Need to do Guess + check

P	Value
.3	3.57
.31	3.73 ← previous may be wrong slightly
.4	5.18
.36	4.53
.365	4.62
.364	4.60

↑ So a p.364

(9).

4. 40,000 shares Disney

$S_0 = 42.43$ at 4/5/2011

Retires 10/5/2011

thinks will ↑

$\sigma = .225$

$r = .15\%$ ∈ 6 months

a) How much $ if sold shares now and invests ∈ in what ∈ bonds?

$40{,}000 \cdot 42.43$

$= 1{,}697{,}200$

FV of that value

$1{,}697{,}200(1+.0015)^1$

$= 1{,}699{,}745.80$

(10).

b) Could purchase options at 42.50 exercisize price
What options to use?

Well he thinks value will likly increase, so he could

purchase calls (by first selling the stock)

× −1

calls are not insurance against decrease in price ...

If he exercizes the call then he can sell the shares at market price, If not he can just use the money

$$C = S\,\Phi\left(\frac{\ln\left(\frac{S}{K}\right) + \left(r + \frac{1}{2}(\sigma)^2\right)(t)}{\sigma\sqrt{T}}\right) \ldots$$

$$- K \cdot e^{-rT} \cdot \Phi\left(\ldots\ldots - \sigma\sqrt{T}\right)$$

(11) $T = 172 \text{ days}/365$

$$= 42.43\,\Phi\left(\frac{\ln\left(\frac{42.43}{42.50}\right) + \left(\overset{\text{annual rate}}{.003} + \frac{1}{2}\cdot .225^2\right)\left(\frac{172}{365}\right)}{.225\cdot\sqrt{\frac{172}{365}}}\right)$$

$$- 42.50 \cdot e^{-.003\cdot\frac{172}{365}}\,\Phi\left(\ldots - .225\cdot\sqrt{\tfrac{172}{365}}\right)$$

$$= 42.43\,\Phi(.0859) - 42.50\,e^{-.003\cdot\frac{172}{365}}\,\Phi(-.0684)$$

$$= 42.43 \cdot .534227 - 42.50 \cdot .99858 \cdot .472734$$

$$= 2.60459$$

So could purchase as many shares of this as he would like. He would need to save \$ to actually buy shares. No wait he could resell the option at exercise, so he could buy up to

$$\frac{1697200}{2.60} = 652{,}769.23 \text{ contracts}$$

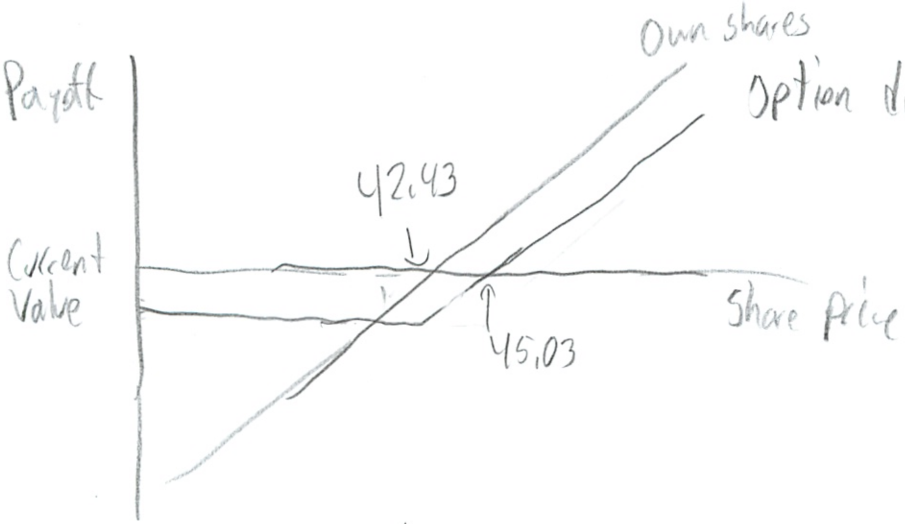
If stock go up, he could make money above $42.43 + 2.6 = 45.03$

If stock go down he only loses the fee on the contracts he buys.

(12)

c. Well it depends how he thinks the share price will go. If he thinks the price will go up, then he could buy options. This will also protect him if shares tank.

X



The option limits his downside, but reduce, by the fee his profit if stock prices go up.

He owns stock. To insure against downside he needs to purchase put options.

MIT Sloan School of Management

Finance Theory I 15.401
Craig Stephenson Spring 2011

Problem Set 5: Options
(Due: Wednesday, April 13th, by 5:00 p.m.)

Unless specified as an "Excel Problem", all of these problems can (and should) be solved with a pencil, paper, and a simple calculator. Do not use Excel (except to check your work, if you want). Show your work cleanly and write out the formulas that you used to solve the problems. Circle your final answers.

Problem 1

(a) In a commonly-used options trading strategy, code named "Alpha," the trader buys an at the money call option and sells an out of the money call option at a higher exercise price on the same underlying security expiring in the same month. Assume you decide to execute this strategy, buying a Google, Inc. June call at $585 for $26.60, and selling a Google, Inc. June call at $610 for $14.90. Draw the payoff and profit (net after premium) diagram from this strategy.

Solution

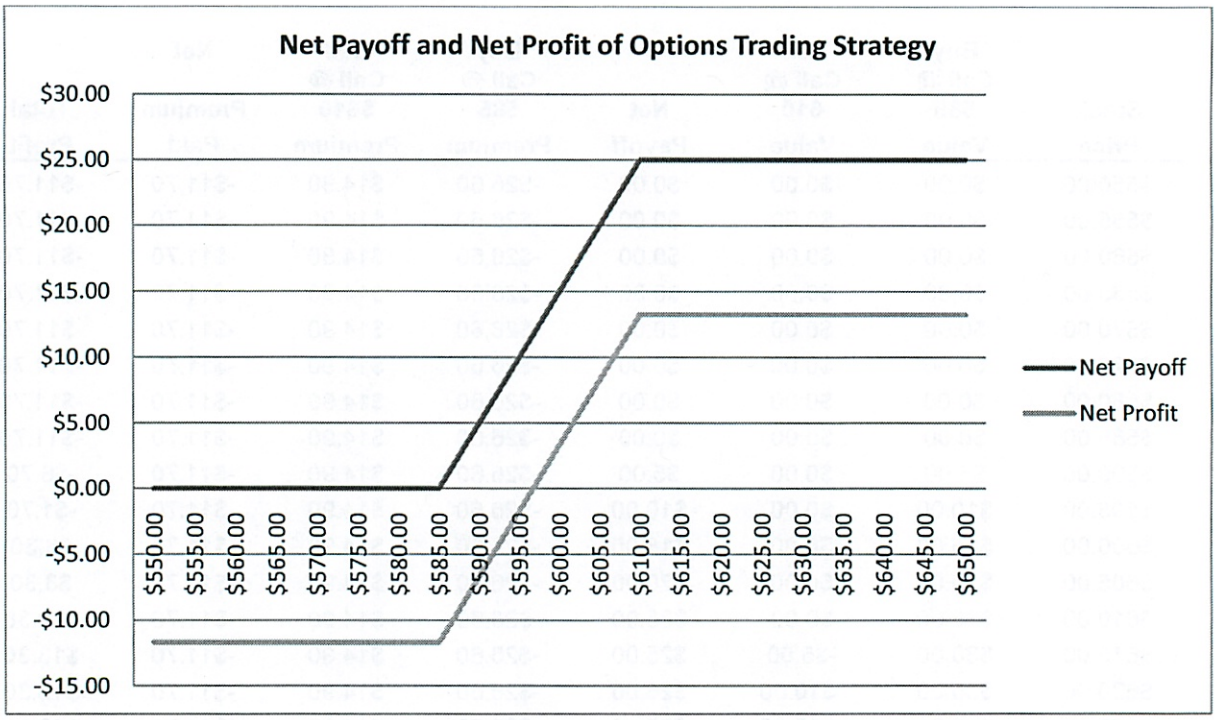
Stock Price	Buy Call @ 585 Value	Sell Call @ 610 Value	Net Payoff	Buy Call @ 585 Premium	Sell Call @ $610 Premium	Net Premium Paid	Total Profit
$550.00	$0.00	$0.00	$0.00	-$26.60	$14.90	-$11.70	-$11.70
$555.00	$0.00	$0.00	$0.00	-$26.60	$14.90	-$11.70	-$11.70
$560.00	$0.00	$0.00	$0.00	-$26.60	$14.90	-$11.70	-$11.70
$565.00	$0.00	$0.00	$0.00	-$26.60	$14.90	-$11.70	-$11.70
$570.00	$0.00	$0.00	$0.00	-$26.60	$14.90	-$11.70	-$11.70
$575.00	$0.00	$0.00	$0.00	-$26.60	$14.90	-$11.70	-$11.70
$580.00	$0.00	$0.00	$0.00	-$26.60	$14.90	-$11.70	-$11.70
$585.00	$0.00	$0.00	$0.00	-$26.60	$14.90	-$11.70	-$11.70
$590.00	$5.00	$0.00	$5.00	-$26.60	$14.90	-$11.70	-$6.70
$595.00	$10.00	$0.00	$10.00	-$26.60	$14.90	-$11.70	-$1.70
$600.00	$15.00	$0.00	$15.00	-$26.60	$14.90	-$11.70	$3.30
$605.00	$20.00	$0.00	$20.00	-$26.60	$14.90	-$11.70	$8.30
$610.00	$25.00	$0.00	$25.00	-$26.60	$14.90	-$11.70	$13.30
$615.00	$30.00	-$5.00	$25.00	-$26.60	$14.90	-$11.70	$13.30
$620.00	$35.00	-$10.00	$25.00	-$26.60	$14.90	-$11.70	$13.30
$625.00	$40.00	-$15.00	$25.00	-$26.60	$14.90	-$11.70	$13.30
$630.00	$45.00	-$20.00	$25.00	-$26.60	$14.90	-$11.70	$13.30
$635.00	$50.00	-$25.00	$25.00	-$26.60	$14.90	-$11.70	$13.30
$640.00	$55.00	-$30.00	$25.00	-$26.60	$14.90	-$11.70	$13.30
$645.00	$60.00	-$35.00	$25.00	-$26.60	$14.90	-$11.70	$13.30
$650.00	$65.00	-$40.00	$25.00	-$26.60	$14.90	-$11.70	$13.30

The options trading strategy produces a positive payoff on the bought call option when Google stock is greater than the $585 exercise price, but a negative payoff on the sold call option when Google stock is greater than the $610 exercise price. The net payoff is maximized at $25 at all values of Google stock above $610.

Entering into this options trading strategy requires a net initial investment of $11.70;a $26.60 cash outflow buying the call at $585 and an inflow of $14.90 selling the call. The total profit from this strategy has a floor at a loss of $11.70, and a ceiling at a profit of $13.30.

Since you were not given the precise number of days remaining on these call options, and not given the interest rate, it was not possible to calculate the future value of the net premium paid at the expiration date. If this information had been provided, the net premium paid amount of $11.70 would increase to the amount calculated by $C \times (1+r)^T$. For example, if the options had 64 days to expiry and the annual interest rate was 1%, the future value of the net premium paid would be $-\$11.70 \times (1.01)^{.1753} = -\$11.72$, and the profit table and chart would change to include this amount as the new premium paid.

Here is the net payoff and net profit chart for this strategy:



(b) Once you have completed the diagram from part (a), do an internet search and find the name for this options trading strategy. What are the benefits of this strategy?

Solution

This options trading strategy is a Bull Call Spread. This strategy is not a high-risk and high-return strategy, as the trader has a floor on losses at $11.70, and a ceiling on profits at $13.30. The trader will profit as long as Google's stock stays above $596.70, but this profit hits a maximum of $13.30 when Google's stock price reaches $610. The reason for this is at $610 the sold call options create value for the owner and loss exposure for the seller, so as the price of Google stock rises above $610, the positive payoff from the bought call at $585 is exactly offset by the negative payoff from the sold call at $610.

Problem 2

The common stock of Hudson Motors will either fall to $71.42 or rise to $140 in the next year, a significant change from its current value of $100. The current one-year interest rate is 5%.

(a) What is the delta of a one-year call option on Hudson with an exercise price of $100?

Solution

This is a simple one period Binomial Option Pricing Model exercise, where the stock price at t=0 is $100, the up stock price in t=1 is $140, the down stock price in t=1 is $71.42, and the exercise price of the call option is $100.

The call option's delta is calculated as:

________________ ____________

The replicating portfolio for this call option requires the purchase of 0.5833 shares of Hudson Motors' common stock.

(b) What is the value of the one-year call option on Hudson at $100?

Solution

To determine the value of the one-year call option on Hudson you also need to calculate the position in the riskless bond, B, calculated as:

____________________________ __________________

The replicating portfolio also requires the sale of \$39.6728 of bonds. Knowing the Δ and the B allows calculation of the value of the call option:

$$Call_0 = Stock_0 \times \Delta - B = \$100 \times 0.5833 - \$39.6728 = \$18.66$$

The present value of the call option must equal the current cost of the replicating portfolio, or \$18.66.

(c) What is the value of the one-year put option on Hudson at \$100?

Solution

Using the put-call parity relationship:

$$P = C + K/e^{rT} - S = \$18.66 + \$100/e^{.05 \times 1} - \$100 = \$13.78$$

So the value of the put option on Hudson Motors' common stock is \$13.78.

Problem 3

The common stock of Hathaway Browne currently trades for \$61.50, and Hathaway does not pay cash dividends. The standard deviation of continuously compounded annual returns on Hathaway stock is 31.0%, and the current annual risk-free rate of interest is 1.5%.

(a) Based on these facts and using the Black-Scholes formula, what is the price of a call option on Hathaway with an exercise price of \$65.00, which expires in 159 days?

Solution

The inputs to the Black-Scholes option pricing formula in this exercise are:

S = \$61.50 K = \$65.00 T = 159 / 365 = 0.435616

r = 1.5% = 0.0150 σ = 31% = 0.31

Using these inputs d_1 is calculated as:

$$d_1 = \frac{\ln(S/E) + (r + \frac{1}{2}\sigma^2)(T)}{\sigma\sqrt{T}} = -0.1363$$

And d_2 is calculated as:

$$d_2 = d_1 - \sigma\sqrt{T} = -0.3409$$

$N(d_1) = 0.4458$ and $N(d_2) = 0.3666$, which when input to the Black-Scholes formula produce the value of the call option on the common stock of Hathaway Browne:

$$V = SN(d_1) - Ke^{-rT}N(d_2) = \$61.50 \times 0.4458 - \$65 \times e^{-.015 \times 0.4356} \times 0.3666 = \$3.74$$

The value of the call option is $3.74.

(b) What is the price of a put option on Hathaway at $65.00 which expires in 159 days?

Solution

Using the put-call parity relationship:

$$P = C + K/e^{rT} - S = \$3.74 + \$65/e^{.015x0.435616} - \$61.50 = \$6.82$$

So the value of the put option on Hathaway Browne's common stock is $6.82.

(c) Excel Problem: Assume the call option at $65.00 which expires in 159 days is actually trading for $4.60. Given this call price, what is the implied standard deviation of continuously compounded annual returns for Hathaway's stock? To solve this write the Black-Scholes formula in Excel and either plug in numbers or use Solver to find the required σ.

Solution

Since the value of the call option in part c ($4.60) is greater than the calculated value of the call option in part a ($3.74), the σ in part c must be higher than in part a. Option values are a positive function of σ. Solving for σ given the call option value produces an implied standard deviation of 0.3632, that is, given the parameters of this exercise, to generate a call value of $4.60 requires a σ of 36.32%. Higher standard deviations of returns on the underlying asset result in higher option values.

Problem 4

After a successful career at the Walt Disney Company and regular investing in the employee stock ownership program, Marc Davis now owns 40,000 shares of Disney stock, which closed at $42.43 per share on April 5, 2011. These shares represent all of Davis' financial assets, and since he plans to retire on October 5, 2011 (his 66th birthday), the value of these shares is critically important to Davis.

After reading analyst reports and thinking about the likely performance of Disney stock, Davis believes the stock has potential to increase in value by October 5, but of course, it could also decrease in value. He thinks the standard deviation of returns to Disney stock is 22.5%, and the 6-month risk-free rate of interest is 0.15%. Ignore income taxes in this problem.

(a) Davis is understandably worried that a decrease in the value of Disney stock will reduce the quality of his retirement lifestyle. Davis could sell his Disney shares now and invest the proceeds until October 5. If he selects this course of action, how much money will Davis have on October 5?

Solution

Davis can sell the shares now for $42.43 per share and invest the proceeds for 6 months at 0.15% (0.0015). Since the investment period is only ½ of a year, the proceeds in 6 months are:

Since Davis owns 40,000 shares, his total proceeds on October 5, 2011 are $42.26 x 40,000 = $1,698,400.

(b) Alternatively Davis could purchase options at an exercise price of $42.50 expiring September 24, 2011, as insurance against a decrease in the price of Disney stock. What type of option should he purchase if he decides to purchase this insurance, and what is the price of this option?

Solution

Davis can buy insurance against a decrease in the price of Disney stock by purchasing put options, which give him the right to sell Disney stock at $42.50. If the stock price rises, Davis will sell his shares at this higher market price, but if the stock price declines, Davis will exercise his put options and sell his shares for $42.50.

The inputs to the Black-Scholes option pricing formula in this problem are:

S = \$42.43	K = \$2.50	$T = 0.4712$ (172/365 of a year)
$r = 0.15\% = 0.00150$	$\sigma = 22.5\% = 0.225$	

When these variables are input to Black-Scholes the results are $d_1 = 0.0711$, $d_2 = -0.0833$, $N(d_1) = 0.5284$, $N(d_2) = 0.4669$, the value of the call is $2.59, and the value of the put is $2.63. To purchase insurance against a decline in the price of Disney common stock, Davis can purchase put options at a exercise price of $42.50, which will cost $2.71 per put. With 40,000 shares to protect, Davis will pay $2.71 x 40,000 = $105,200 for this insurance.

If you used T=0.50 for ½ a year until October 5, the value of the put is $2.71 and the total cost is $108,400. One problem with options is they have contractual and common expiry dates, so although Davis wants October 5, he can only get September 24, 11 days earlier.

(c) What are the advantages and disadvantages of using options on Disney stock to buy this insurance?

Solution

Advantage: buying put options allows Davis to wait and see if the stock price increases. If this occurs, Davis will sell his shares at the higher market price. If the stock price remains the same or decreases, however, Davis will exercise his put options and sell his stock at $42.50 per share. Buying the put options gives him a payoff floor of $42.50 x 40,000 = $1,700,000.

Disadvantage: buying put options requires Davis to pay $2.63 per put, or $105,200 in total, which is a significant amount. This is the cost of his insurance, over $100,000.

15.401 Finance Theory I

Craig Stephenson

MIT Sloan School of Management

Lecture 8: Portfolio Theory

- Portfolios
- Portfolio returns
- Diversification
- Systematic vs. non-systematic risks
- Optimal portfolio choices
- Sharpe ratio

Readings:

- Brealey, Myers and Allen, Chapter 7, 8
- Bodie, Kane and Markus, Chapters 6.2, 7, 8

What is a portfolio?

A portfolio is simply a collections of assets:

- 'n' assets, each with share price P_i (i = 1, 2, … , n)
- A portfolio is a collection of N_i shares of each asset i
- Total value of portfolio:

$$V = N_1P_1 + N_2P_2 + \cdots + N_nP_n = \sum_{i=1}^{n} N_iP_i$$

- A typical portfolio has V > 0. Define portfolio weights:

$$w_i = \frac{N_iP_i}{N_1P_1 + N_2P_2 + \cdots + N_nP_n} = \frac{N_iP_i}{V}$$

- A portfolio can then also be defined by its asset weights

$$\{w_1, w_2, \dots, w_n\}, \quad w_1 + w_2 + \cdots + w_n = 1$$

- When V = 0, we have an arbitrage portfolio

Example. Your investment account of \$100,000 consists of three stocks: 200 shares of stock A, 1,000 shares of stock B, and 750 shares of stock C. Your portfolio is summarized by the following weights:

Asset	Shares	Price/Share	Dollar Investment	Portfolio Weight
A	200	\$50	\$10,000	10%
B	1,000	\$60	\$60,000	60%
C	750	\$40	\$30,000	30%
Total			\$100,000	100%

Example (cont). Your broker informs you that you only need to keep $50,000 in your investment account to support the same portfolio of 200 shares of stock A, 1,000 shares of stock B, and 750 shares of stock C; in other words, you can buy these stocks on margin. You withdraw $50,000 to use for other purposes, leaving $50,000 in the account. Your portfolio is summarized by the following weights:

Asset	Shares	Price/Share	Dollar Investment	Portfolio Weight
A	200	$50	$10,000	20%
B	1,000	$60	$60,000	120%
C	750	$40	$30,000	60%
Riskless Bond	-$50,000	$1	-$50,000	-100%
Total			$50,000	100%

Example. You decide to purchase a home that costs $500,000 by paying 20% of the purchase price and getting a mortgage for the remaining 80% What are your portfolio weights for this investment?

Asset	Shares	Price/Share	Dollar Investment	Portfolio Weight
Home	1	$500,000	$500,000	500%
Mortgage	1	-$400,000	-$400,000	-400%
Total			$100,000	100%

Leverage ratio = asset / net investment = $500,000 / $100,000 = 5

What happens to your total assets if your home price declines by 15%?

Why not pick the best asset instead of forming a portfolio?

- We don't know which stock is best
- Portfolios provide diversification, reducing unnecessary risks
- Portfolios can enhance performance by focusing bets
- Portfolios can customize and manage risk/reward trade-offs

How do we chose the "best" portfolio?

- What does "best" mean?
- What characteristics of a portfolio do we care about?
  - risk and reward (expected return)
  - higher expected returns are preferred
  - higher risks are not preferred

US and JP stock markets:

	mean	volatility
US	$E[R_1] = 13.6\%$	$\sigma_1 = 15.4\%$
JP	$E[R_2] = 15.0\%$	$\sigma_2 = 23.0\%$

and with correlation $\rho_{12} = 27\%$.

If an investor holds $w_1 = 60\%$ in the US and $w_2 = 40\%$ in JP what is the mean and volatility of the portfolio?

Portfolio mean:

$E(R_p) = 0.6x*0.136 + 0.4*0.150 = 14.2\%$

Portfolio variance:

$Var(R_p) = (0.6)^2*(0.154)^2 + (0.4)^2*(0.230)^2 + 2*0.6*0.4*0.27*0.154*0.230 = 0.02159$

$\sigma_p = 0.147 = 14.7\%$

This portfolio has higher expected return and lower risk than the US market alone!

This example illustrates the benefits from diversification

A portfolio's characteristics are determined by the returns of its assets and its weights in them.

Mean returns:

Asset	1	2	...	n
Mean Return	$\bar{r}_1$	$\bar{r}_2$	...	$\bar{r}_n$

Variances and co-variances:

	r_1	r_2	...	r_n
r_1	σ_1^2	σ_{12}	...	σ_{1n}
r_2	σ_{21}	σ_2^2	...	σ_{2n}
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
r_n	σ_{n1}	σ_{n2}	...	σ_n^2

Covariance of an asset with itself is its variance: $\sigma_{nn} = \sigma_{n^2}$

Example. Monthly stock returns on IBM (r_1) and Merck (r_2):

Mean returns

$\bar{r}_1$	$\bar{r}_2$
0.0149	0.0100

Covariance matrix

	r_1	r_2
r_1	0.007770	0.002095
r_2	0.002095	0.003587

Note: $\sigma_1 = 8.81\%$, $\sigma_2 = 5.99\%$ and $\rho_{12} = 0.40$.

The portfolio return is a weighted average of the individual returns:

$$r_p = w_1 \tilde{r}_1 + w_2 \tilde{r}_2$$

Example. Suppose you invest \$600 in IBM and \$400 in Merck for a month. If the realized return is 2.5% on IBM and 1.5% on Merck over the month, what is the return on your total portfolio?

The portfolio weights are

$w_{IBM} = \$600/\$1000 = 60\%$ and $w_{Merck} = \$400/\$1000 = 40\%$

$$r_p = \frac{(600)(0.025) + (400)(0.015)}{1000}$$

$$= (0.6)(0.025) + (0.4)(0.015)$$

$$= 0.021 = 2.1\%$$

Expected return on a portfolio with two assets

Expected portfolio return:

$$\bar{r}_p = w_1\bar{r}_1 + w_2\bar{r}_2$$

Unexpected portfolio return:

$$r_p - \bar{r}_p = w_1(\tilde{r}_1 - \bar{r}_1) + w_2(\tilde{r}_2 - \bar{r}_2)$$

Variance of return on a portfolio with two assets

The variance of the portfolio return:

$$\sigma_p^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_{12}$$

which is also the sum of all entries of the following table

	$w_1\tilde{r}_1$	$w_2\tilde{r}_2$
$w_1\tilde{r}_1$	$w_1^2\sigma_1^2$	$w_1w_2\sigma_{12}$
$w_2\tilde{r}_2$	$w_1w_2\sigma_{12}$	$w_2^2\sigma_2^2$

Example. Consider again investing in IBM and Merck stocks

Mean returns

$\bar{r}_1$	$\bar{r}_2$
0.0149	0.0100

Covariance matrix

	$\tilde{r}_1$	$\tilde{r}_2$
$\tilde{r}_1$	0.007770	0.002095
$\tilde{r}_2$	0.002095	0.003587

Consider the equally weighted portfolio: $w_1 = w_2 = 0.5$

Mean of portfolio return: $\bar{r}_p$ = (0.5)(0.0149) + (0.5)(0.0100) = 1.25%

Variance of portfolio return:

	$w_1\tilde{r}_1$	$w_2\tilde{r}_2$
$w_1\tilde{r}_1$	$(0.5)^2(0.007770)$	$(0.5)^2(0.002095)$
$w_2\tilde{r}_2$	$(0.5)^2(0.002095)$	$(0.5)^2(0.003587)$

σ_p^2 = $(0.5)^2(0.007770) + (0.5)^2(0.003587) + (2)(0.5)^2(0.002095)$

= 0.003888

σ_p = 0.0623 = 6.23%

Expected portfolio return when n = 3:

$$\bar{r}_p = w_1\bar{r}_1 + w_2\bar{r}_2 + w_3\bar{r}_3$$

The variance of the portfolio return:

$$\sigma_p^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + w_3^2\sigma_3^2 + 2w_1w_2\sigma_{12} + 2w_1w_3\sigma_{13} + 2w_2w_3\sigma_{23}$$

Example. IBM, Merck and Intel returns have covariance matrix:

	$\tilde{r}_{IBM}$	$\tilde{r}_{Merck}$	$\tilde{r}_{Intel}$
$\tilde{r}_{IBM}$	0.007770	0.002095	0.001189
$\tilde{r}_{Merck}$	0.002095	0.003587	0.000229
$\tilde{r}_{Intel}$	0.001189	0.000229	0.009790

What is the risk (Std Dev) of the equally weighted portfolio?

σ_p^2 = $\left(\frac{1}{3}\right)^2$ × (Sum of all entries of covariance matrix) = 0.003130

σ_p = 5.59%

For each asset individually:

σ_{IBM} = 8.81%, σ_{Merck} = 5.99%, σ_{Intel} = 9.89%

We now consider a portfolio of n assets: $\{w_1, w_2, \ldots, w_n\}$, $\sum_i w_i = 1$

1. The return on the portfolio is:

$$\tilde{r}_p = w_1\tilde{r}_1 + w_2\tilde{r}_2 + \cdots + w_n\tilde{r}_n = \sum_{i=1}^{n} w_i\tilde{r}_i$$

2. The expected return on the portfolio is:

$$\bar{r}_p = E[r_p] = w_1\bar{r}_1 + w_2\bar{r}_2 + \cdots + w_n\bar{r}_n = \sum_{i=1}^{n} w_i\bar{r}_i$$

3. The variance of portfolio return is:

$$\sigma_p^2 = Var[\tilde{r}_p] = \sum_{i=1}^{n}\sum_{j=1}^{n} w_iw_j\sigma_{ij}, \quad \sigma_{ii} = \sigma_i^2$$

4. The volatility (Std Dev) of portfolio return is:

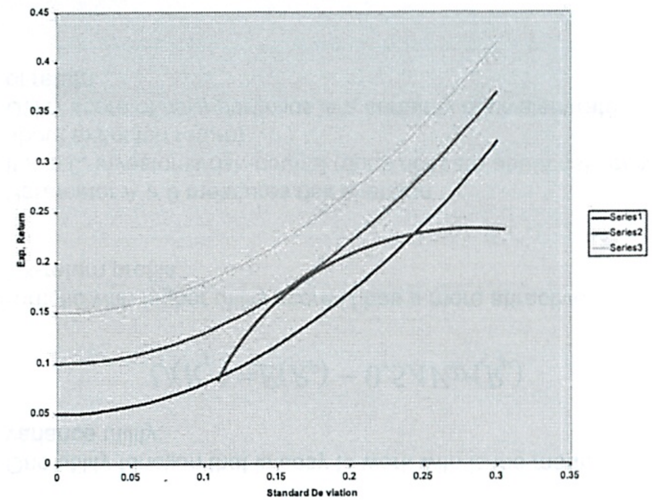
$$\sigma_p = \sqrt{V[\tilde{r}_p]} = \sqrt{\sigma_p^2}$$

Optimal portfolio reconciles what is

desirable - described by the *indifference* or utility curves

with what is

feasible - described by the upper portion of the investment opportunity set, the *efficient frontier*



1. Risk comes in two types:
   - Diversifiable (non-systematic, unique)
   - Non-diversifiable (systematic, common)
2. Diversification reduces (diversifiable) risk.
3. Investors hold frontier portfolios.
   - Large asset base improves the portfolio frontier.
4. When there is a risk-free asset, frontier portfolios are linear combinations of
   - the risk-free asset, and
   - the tangent portfolio.

Portfolios

Portfolio returns

Diversification

Systematic vs. non-systematic risks

Optimal portfolio choices

CML

Sharpe ratio

Risk-Return Tradeoff

Recall two of the Finance Axioms:

- Investors prefer more return to less return
- Investors are risk-averse

This means that investors prefer an investment :

- with a higher expected return $E(R_i)$
- with a lower variance and standard deviation, s_i

Investors optimally **trade off** risk and return

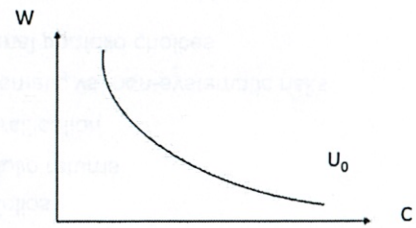
In order to maximize their expected utility.

Indifference Curves: Review

A person likes 2 goods: wine (W) and cheese (C).

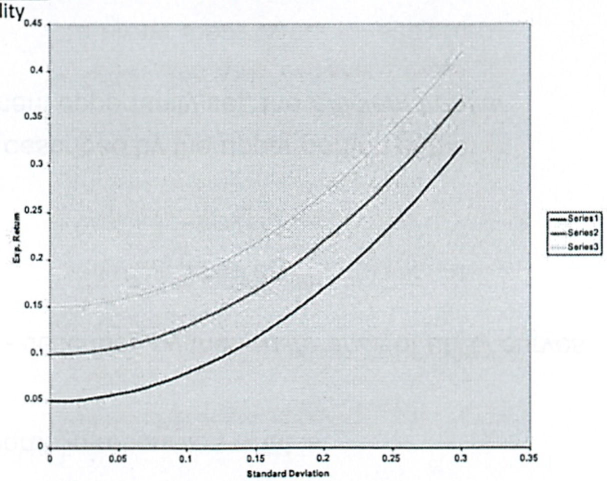
An indifference curve gives all the combinations of W and C that give the same utility level $U_0 = U(W,C)$.

People like to be on the highest possible indifference curve (people prefer more to less)



Indifference Curves in Finance

Indifference curve: A set of $(E(R_P),\sigma_P)$ combinations that give an investor the same expected utility



Example of utility function

One utility function that is easy to work with is the **mean-variance** utility:

$$U(R_p) = E(R_P) - 0.5A\,Var(R_p)$$

Portfolio with higher utility score U has a more attractive risk-return profile

Parameter A > 0 measures **risk aversion**
If A = 0, investor is risk-neutral (does not care about risk, only about expected return)
Utility score of risky portfolios is a **certainty equivalent rate of return**

When there exists a safe (risk-free) asset, each portfolio consists of the risk-free asset and risky assets.

Observation: A portfolio of risk-free and risky assets can be viewed as a portfolio of two portfolios:

1) the risk-free asset, and
2) a portfolio of only risky assets.

Example. Consider a portfolio with \$40 invested in the risk-free asset and \$30 each in two risky assets, IBM and Merck:

- w_0 = 40% in the risk-free asset
- w_1 = 30% in IBM and
- w_2 = 30% in Merck

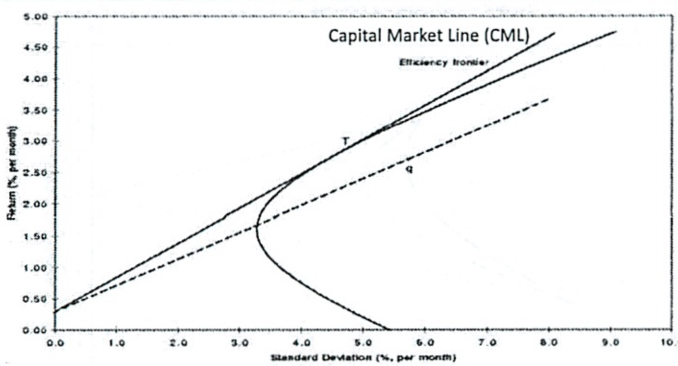
We can also view the portfolio as follows:

1) 1 - x = 40% in the risk-free asset
2) x = 60% in a portfolio of only risky assets which has
   a) 50% in IBM
   b) 50% in Merck

Consider a portfolio p with x invested in a risky portfolio q, and 1-x invested in the risk-free asset. Then,

$$\bar{r}_p = (1-x)r_F + x\bar{r}_q$$

$$\sigma_p^2 = x^2\sigma_q^2$$



With a risk-free asset, frontier portfolios are combinations of:

1) the risk-free asset
2) the tangent portfolio (consisting of only risky assets).

The frontier is also called the Capital Market Line (CML).

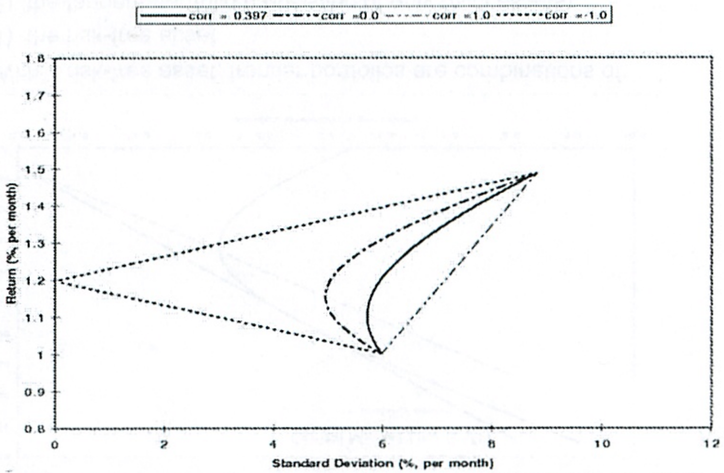
Sharpe ratio

A measure of a portfolio's risk-return trade-off, equal to the portfolio's risk premium divided by its volatility:

$$\boxed{\text{Sharpe Ratio} \equiv \frac{\bar{r}_p - \bar{r}_F}{\sigma_p} \quad \text{(higher is better!)}}$$

The tangent portfolio has the highest possible Sharpe ratio of all portfolios. So are all the portfolios on the CML.

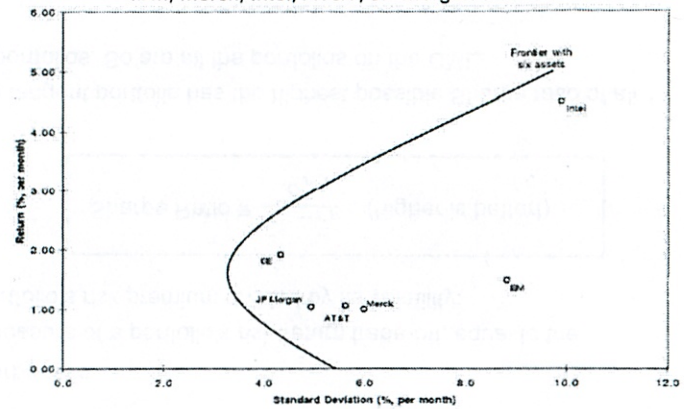
2. Perfect correlation between two assets ($\rho_{12} = \pm 1$):



Solving optimal portfolios "graphically":

Portfolio frontier from stocks of

IBM, Merck, Intel, AT&T, JP Morgan and GE



Given an expected return, the portfolio that minimizes risk (measured by Std Dev or variance) is a mean-variance frontier portfolio.

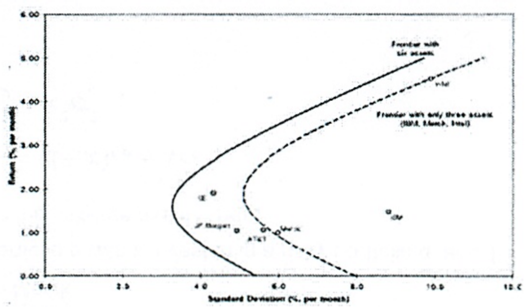
The locus of all frontier portfolios in the mean-Std Dev plane is called portfolio frontier.

The upper part of the portfolio frontier gives the efficient frontier portfolios.

To obtain the efficient portfolios, we need to solve the constrained optimization problem (P).

- Use Excel Solver to solve numerically

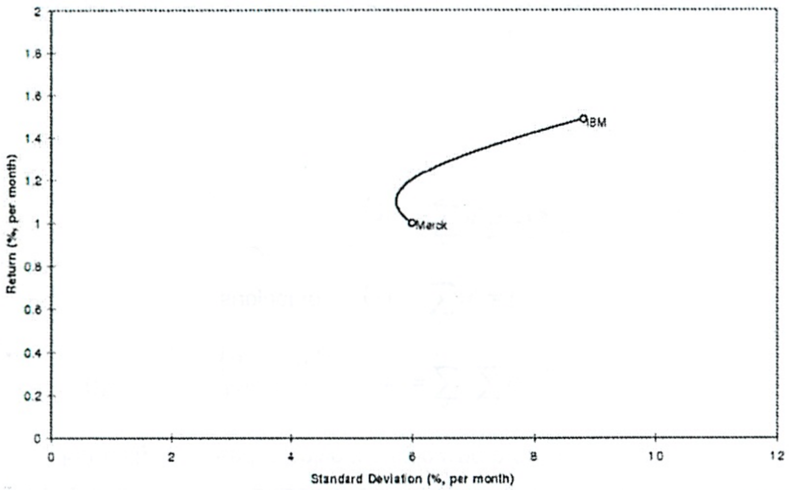
Portfolio frontier of IBM, Merck, Intel, AT&T, JP Morgan and GE



When more assets are included, the portfolio frontier improves, i.e., moves toward upper-left: higher mean returns and lower risk.

Intuition: Since one can always choose to ignore the new assets, including them cannot make one worse off.

Portfolio frontier when short sales are not allowed



With short sales

When short sales are allowed, portfolio weights are unrestricted.

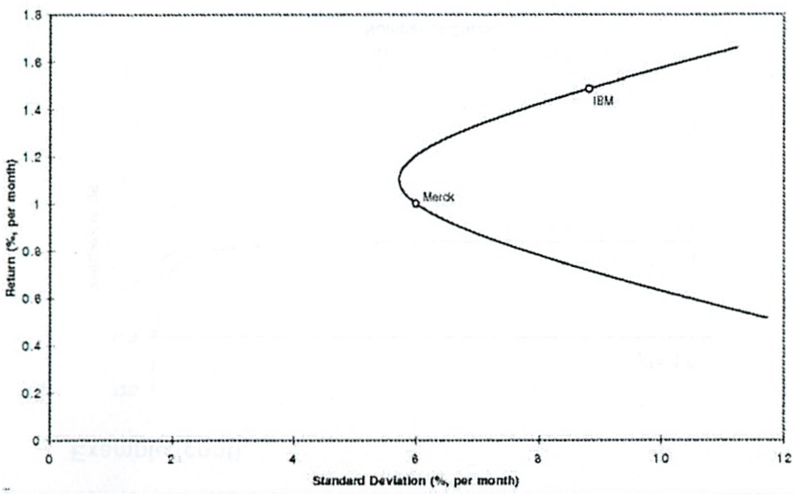
Example. (Cont)

Covariances	$\tilde{r}_{IBM}$	$\tilde{r}_{Merck}$
$\tilde{r}_{IBM}$	0.007770	0.002095
$\tilde{r}_{Merck}$	0.002095	0.003587
Mean	1.49%	1.00%
SD	8.81%	5.99%

Portfolios of IBM and Merck

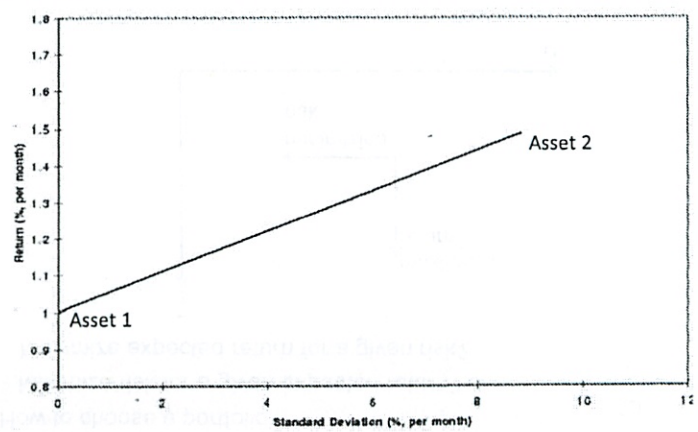
Weight in IBM (%)	-40	-20	0	20	40	60	80	100	120	140
Mean return (%)	0.80	0.90	1.00	1.10	1.20	1.29	1.39	1.49	1.59	1.69
SD (%)	7.70	6.69	5.99	5.72	5.95	6.62	7.61	8.81	10.16	11.60

Portfolio frontier when short sales are allowed

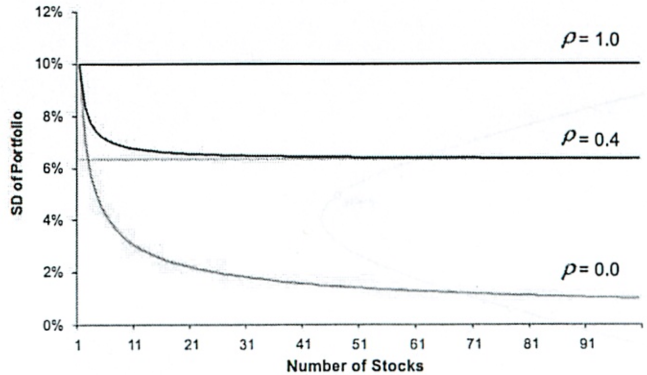


Special situations (without short sales)

Asset 1 is risk-free:



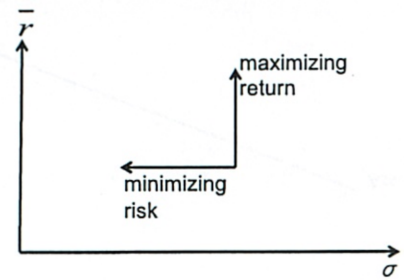
Example (cont).



How to choose a portfolio:

Minimize risk for a given expected return? or

Maximize expected return for a given risk?



Formally, we need to solve the following problem:

(P) : $$\underset{\{w_1,\ldots,w_n\}}{\text{Minimize}} \quad \sigma_p^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_{ij}$$

subject to (1) $$\sum_{i=1}^{n} w_i = 1$$

(2) $$\sum_{i=1}^{n} w_i \bar{r}_i = \bar{r}_p$$

$$\bar{r}_p = w\bar{r}_1 + (1-w)\bar{r}_2$$

$$\sigma_p^2 = w^2\sigma_1^2 + (1-w)^2\sigma_2^2 + 2w(1-w)\sigma_{12}$$

Without short sales

Example. IBM and Merck:

Covariances	$\tilde{r}$ IBM	$\tilde{r}$ Merck
$\tilde{r}$ IBM	0.007770	0.002095
$\tilde{r}$ Merck	0.002095	0.003587
Mean (%)	1.49	1.00
SD (%)	8.81	5.99

Portfolios of IBM and Merck

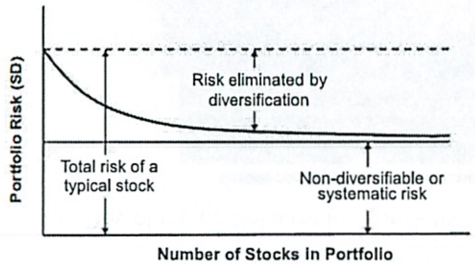
Weight in IBM (%)	0	10	20	30	40	50	60	70	80	90	100
Mean return (%)	1.00	1.05	1.10	1.15	1.20	1.25	1.30	1.34	1.39	1.44	1.49
SD (%)	5.99	5.80	5.72	5.78	5.95	6.23	6.62	7.08	7.61	8.19	8.81

Two kinds of risks

Certain risks cannot be diversified away.

Impact of diversification on portfolio risk

$-- \bar{\rho}=1.0$ and $— \bar{\rho}=0.5$



1. Diversification benefit has a limit
2. Remaining risk known as non-diversifiable (market, systematic, common)
3. Risk comes in two kinds:
   - Diversifiable risks
   - Non-diversifiable risks
4. Sources of non-div. risks
   - Business cycle
   - Inflation
   - Volatility
   - Credit
   - ForEx rates
   - ...

Diversification

Example. An equally-weighted portfolio of n assets:

	w_1r_1	$\cdots$	w_nr_n
w_1r_1	$w_1^2\sigma_1^2$	$\cdots$	$w_1w_n\sigma_{1n}$
$\vdots$	$\vdots$	$\ddots$	$\vdots$
w_nr_n	$w_nw_1\sigma_{n1}$	$\cdots$	$w_n^2\sigma_n^2$

A typical variance term: $\left(\frac{1}{n}\right)^2 \sigma_{ii}$.

- Total number of variance terms: n

A typical covariance term: $\left(\frac{1}{n}\right)^2 \sigma_{ij}, (i \neq j)$

- Total number of covariance terms: $n^2 - n$

Diversification

Add all the terms:

$$\sigma_p^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} w_iw_j\sigma_{ij} = \sum_{i=1}^{n}\left(\frac{1}{n}\right)^2\sigma_{ii} + \sum_{i=1}^{n}\sum_{j\neq i}^{n}\left(\frac{1}{n}\right)^2\sigma_{ij}$$

$$= \left(\frac{1}{n}\right)\left(\frac{1}{n}\sum_{i=1}^{n}\sigma_i^2\right) + \left(\frac{n^2-n}{n^2}\right)\left(\frac{1}{n^2-n}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\sigma_{ij}\right)$$

$$= \left(\frac{1}{n}\right)\text{(average variance)} + \left(\frac{n^2-n}{n^2}\right)\text{(average covariance)}$$

As n becomes very large:

Contribution of variance terms goes to zero

Contribution of covariance terms goes to "average covariance"

Diversification

Example (cont). The average stock has a monthly standard deviation of 10% and the average correlation between stocks is 0.40. If you invest the same amount in each stock, what is the variance of the portfolio?

$$\text{Cov}[R_i, R_j] = \rho_{ij}\sigma_i\sigma_j = 0.40 \times 0.10 \times 0.10 = 0.004$$

$$\text{Var}[R_p] = \frac{1}{n}0.10^2 + \frac{n-1}{n}0.004 \approx 0.004 \text{ if } n \text{ large}$$

$$\sigma_p \approx \sqrt{0.004} = 6.3\%$$

What if the correlation is 1.0? 0.0?

Portfolio variance is the weighted sum of **all** the variances and covariances of its assets:

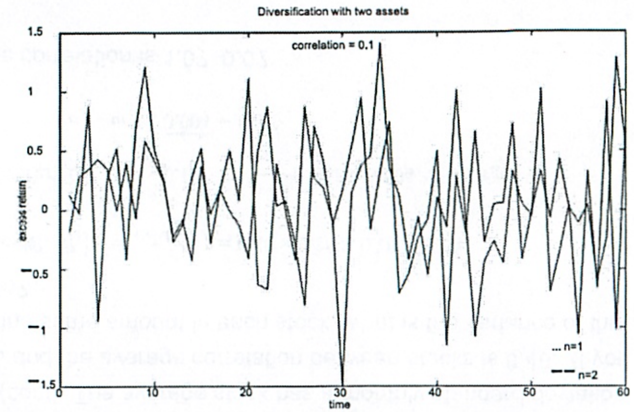
	$w_1 r_1$	$w_2 r_2$	$\cdots$	$w_n r_n$
$w_1 r_1$	$w_1^2\sigma_1^2$	$w_1 w_2 \sigma_{12}$	$\cdots$	$w_1 w_n \sigma_{1n}$
$w_2 r_2$	$w_2 w_1 \sigma_{21}$	$w_2^2\sigma_2^2$	$\cdots$	$w_2 w_n \sigma_{2n}$
...	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$w_n r_n$	$w_n w_1 \sigma_{n1}$	$w_n w_2 \sigma_{n1}$	$\cdots$	$w_n^2\sigma_n^2$

In order to calculate return variance of a portfolio, we need

a) portfolio weights
b) individual variances
c) all covariances

Diversification reduces risk.

1. Two assets:

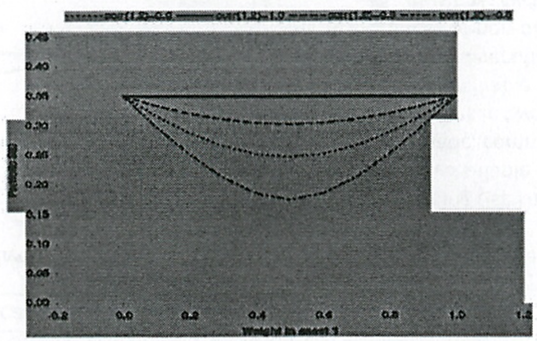


Example. Two assets with the same annual return Std Dev of 35%

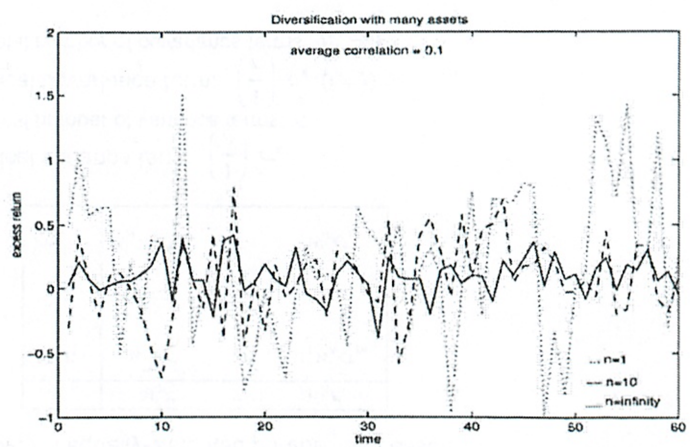
Consider a portfolio p with weight w in asset 1 and 1- w in asset 2.

$$\sigma_p = \sqrt{w^2\sigma_1^2 + (1-w)^2\sigma_2^2 + 2w(1-w)\sigma_{12}}.$$

Std Dev of the portfolio return is less than the SD of each individual asset.



2 Multiple assets:



15.401 Portfolio Theory

4/11

Finance in 1950s - all descriptive / qualatative

Price of risk - next Wed

- Capital Asset pricing model

Portfolio

$$\underset{\text{value}}{V} = \sum_{i=1}^{n} N_i P_i$$

Weight of each asset

$$W_i = \frac{N_i P_i}{V}$$

a portfolio can be detefined by asset weights

$$\sum_{i=1}^{n} W_i = 1$$

$V=0$ arbitrage portfolio

- playing w/ other people's money

Can buy stuff on margin

- borrowing
- aka leveing
- but only part of portofolio ~50%

E)

Weight is % of money you have invested

So if borrowed $, can own > 100%

Leverage ratio = $\frac{\text{asset}}{\text{net investment}} = \frac{500{,}000}{100{,}000} = 5$

- Small changes in asset prices go to you
- ~~man~~ loan is still there
- leverge ratio bounce around a lot

- we don't know which stock is best
- diversity - spread risk over assets
- Can enhance performance by focusing bets
- need to manage risk/return tradeoff
- amt of risk depends on correlation b/w stocks

$E[\underset{}{\overset{\text{ or } R_p}{\text{portfolio}}}] = \sum_{i=1}^{n} W_i E[R_i]$ = portfolio mean

$\sigma_p^2 = Var(R_p) = \cancel{\sum_{i=1}^{n} (W_i)^2 (\sigma_i)^2 + 2 \underbrace{\sum_{i=1}^{n} (W_i)^2 + \rho_{12} + \sum_{i=1}^{n} \sigma_i}_{Cov\,\sigma_{12}}}$ ← generized wrong

$= W_1^2 \sigma_1^2 + W_2^2 \sigma_2^2 + 2 W_1 W_2 \overbrace{\rho_{12} \cdot \sigma_1 \cdot \sigma_2}^{\sigma_{12}}$ ← for 2

(3)

This portfolio has higher expected return and lower risk than ~~the~~ either stock alone

~~Highest~~

	~ 1	very highly correlated	- not much risk ~~effect~~ low portofolio effect
ρ	0	not correlated	lower risk
	~ -1	neg correlated	lowest risk

Var, Cov matrix

$$
\begin{array}{c|cccc}
 & r_1 & r_2 & \cdots & r_n \\
\hline
r_1 & \sigma_1^2 & \sigma_{12} & - & \sigma_{1n} \\
r_2 & \sigma_{21} & \sigma_2^2 & - & \sigma_{2n} \\
\vdots & \vdots & \vdots & \ddots & \\
r_n & \sigma_{n1} & \sigma_{n2} & & \sigma_n^2
\end{array}
$$

← Cov matrix

← var

$\sigma_1^2 = \text{var}_1 = \sigma_{11}$

$\sigma_{12} = \sigma_{21}$

So slide 11 - returns for IBM

E[return] is just weighted sums of individual returns

$$\sigma_p^2 = w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + 2 w_1 w_2 \sigma_{12}$$

(4) where

$\sigma_{12} = \rho_{12} \cdot \sigma_1 \cdot \sigma_2$

As # stocks in portfolio grows, the cov dominate since there are so many of them
- since they grow ~~exponentially~~ $n^2 - n$

The lower the correlation, the lower the risk

So w/ 3 assets

↓ mean

$\bar{r}_p = w_1 \bar{r}_1 + w_2 \bar{r}_2 + w_3 \bar{r}_3$

$\sigma_p^2 = w_1^2 \sigma_1^2 + w_2 \sigma_2^2 + w_3^2 \sigma_3^2 + 2 w_1 w_2 \sigma_{12} + 2 w_1 w_3 \sigma_{13} + 2 w_2 w_3 \sigma_{23}$

↑ now 9 terms

$\sigma_p = \left(\frac{1}{3}\right)^2 \cdot$ Sum of all entries in table
- if all 3 equally weighted

The mean, st dev for individual stocks comes from lookat ~~several~~ ~~stocks~~ a big list of daily/monthly returns

$Var[r_p] = \sigma_p^2 = \sum_{i=1}^{n} \sum_{j=1} w_i w_j \sigma_{ij}$ ↓ so this works for Var as well like this $\sigma_{ii} = \sigma_i^2 = Var_i$

5

Volatility = st dev

$$\sigma_p = \sqrt{V[r_p]} = \sqrt{\sigma_p^2}$$

In last 10 years its much eaiser to by international Stocks

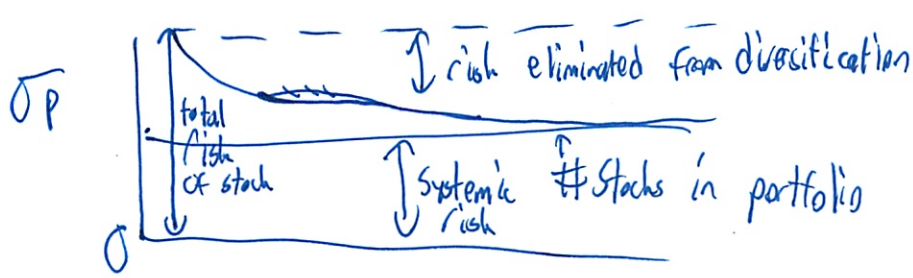
Interest chart p19

- What happens to σ_p when you decrease correlation b/w stocks
- lower better

Neg correlation is very desirable

- but not many assets are like this

Certain risks can't be diversified away



so some factors affect many/all stocks

So σ_{12} is often $\sim .5$ and $\sim .7$

6

- biz cycle
- inflation
- volatility
- credit
- forex

affects all

But somethings only 1 stock

- managers
- engineers
- weather in certain location

"ideosyncratic"

If everyone moves together, you are not reducing your risk

~~This is reducing~~

But get rid of / reduce ideosyncratic risks

Hard to get rid of the stuff that effects everyone

Could also do this table

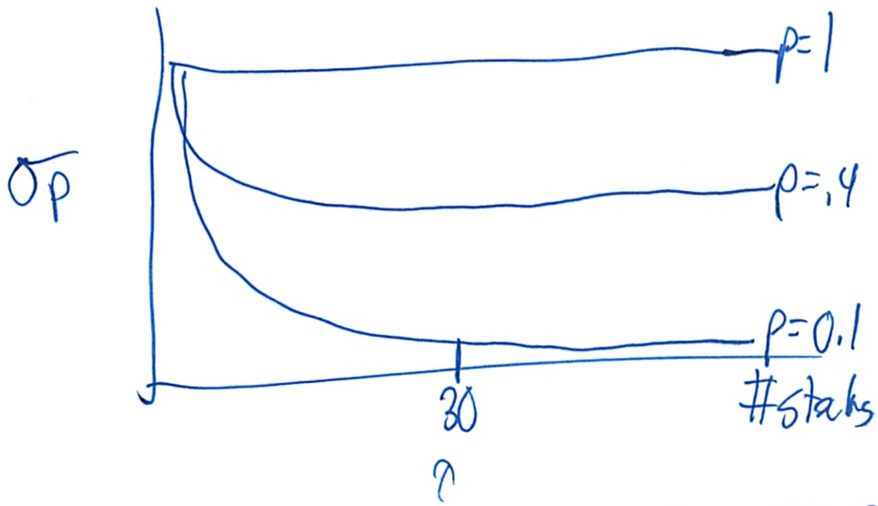
$$\begin{array}{c|cccc} & w_1 r_1 & \cdots & & w_n r_n \\ \hline w_1 r_1 & w_1 w_1 \sigma_{11}^2 & \cdots & & w_1 w_n \sigma_{n1} \\ \vdots & \vdots & \ddots & & \\ \vdots & \vdots & & \ddots & \\ w_n r_n & w_n w_1 \sigma_{1n} & & & (w_n)^2 (\sigma_{nn})^2 \end{array}$$

$\sigma_{n1} = \sigma_{1n}$

7

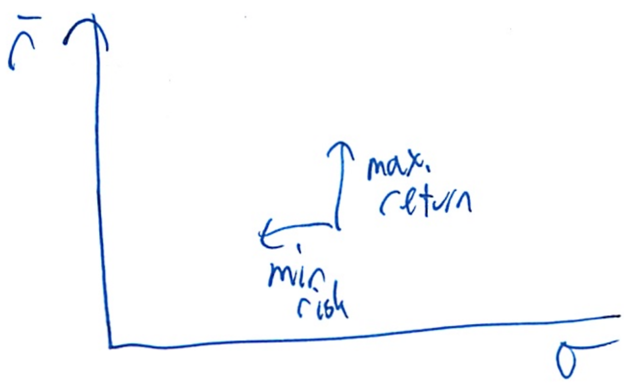
If cov is 0 - the portfolio is riskless
- everything washes out

If cov is ⊖ - math blows up



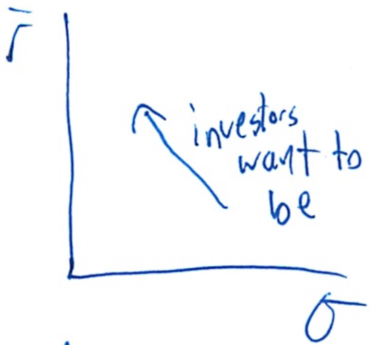
Once get to 30-50 assets you are rid of all risk

Investors want



15.401 More Portfolio

4/13



Var and covar comes out of matrix

Can look at diff. weights of portfolio

- mean return is simple weighted avg
- but σ is more complicated
  - less than the σ of the pieces

← per unit of return, less risk

↑ ~~pe~~ for a certain risk, more return

Portfolio fronteir - the best tradeoff of 2 stocks

- investors only go to top half of curve
  - want more return at same risk
- is an optimal point

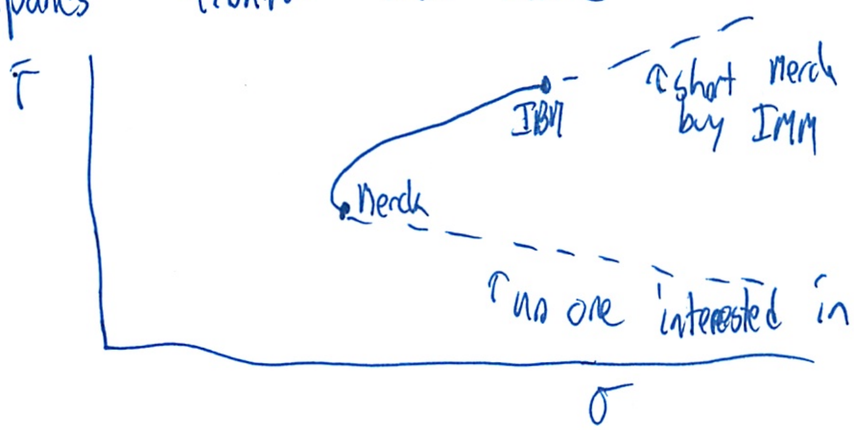
-1 correlation is the best

- but very hard/impossible to achieve

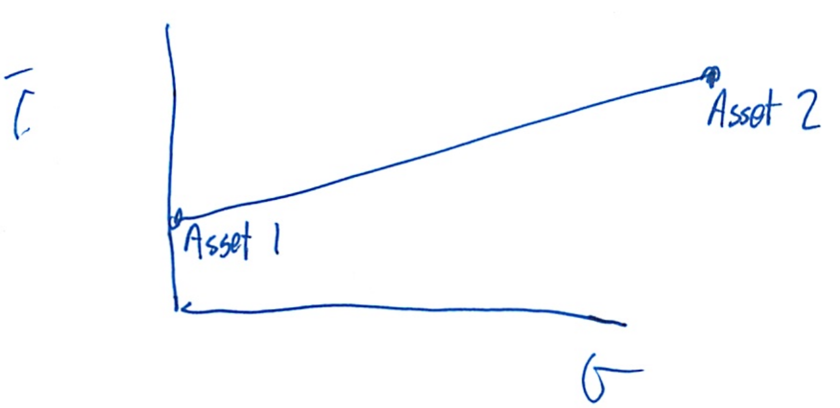
(2)

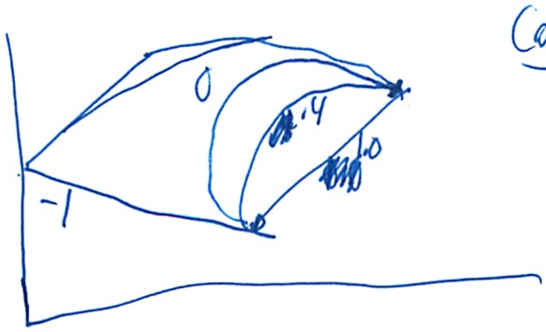
Can also short stocks
- so not limited $0 \to 100$
- can go below 0
- or above 100

Expands frontier even more



Special situation: Asset 1 is risk-free





Correlations

- kinda drawn wrong, see PPT

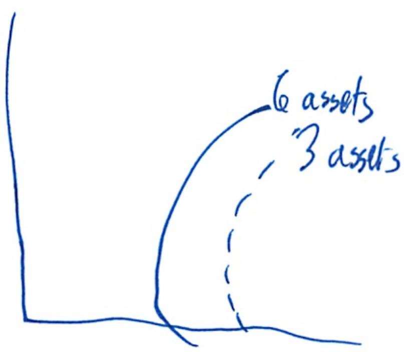
③

As you add more risky stocks
- lowish cov
- portfolio shifts to left

But how do you mix and match the weights to get to the fronteir?

- called ... (missed)

As add assets sift frontier up and to left

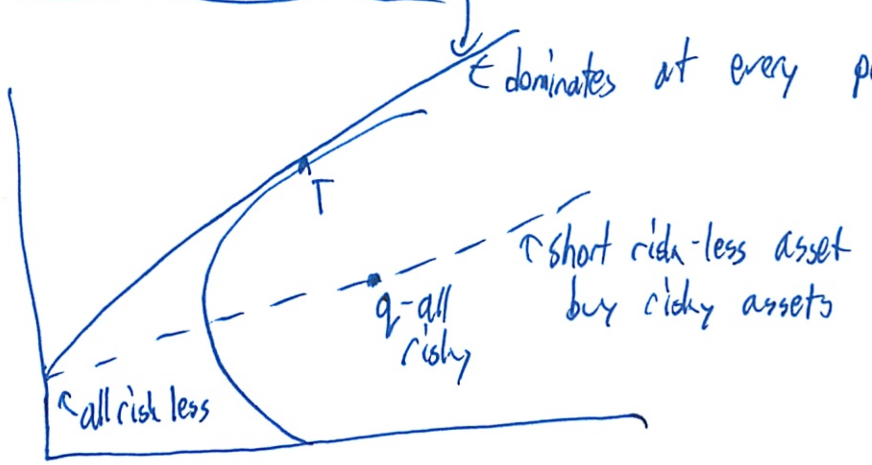


It's like free risk reduction

People can invest only in risk-free asset
Consider it a portfolio of portfolios
- the portfolio of risky assets
- the zero risk asset
0 correlation w/ other assets
risk only from risky portion
add in mean return

4

Capital Market Line - the frontier of the portfolios



If not on frontier - you are missing out

Slide up + down line depending on risk your willing to take

Assumes you can borrow at risk free rate of return

P-set due 4/25

With lower correlation - CML is less steep
- taking more risk for same return

Adding CML - gives you more possibilities

Sharpe ratio - A measure of risk-return tradeoff - equal to portfolio risk premium divided by volatility

$$= \frac{\bar{r}_p - r_F}{\sigma_p}$$

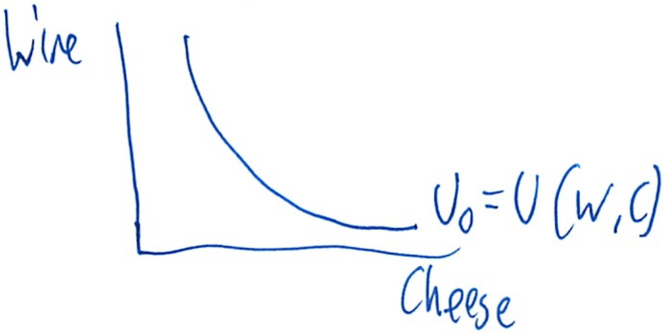
5

tangent portfolio has highest possible Sharpe ratio

- so too CML

investors have their own optimal trade off risk + return

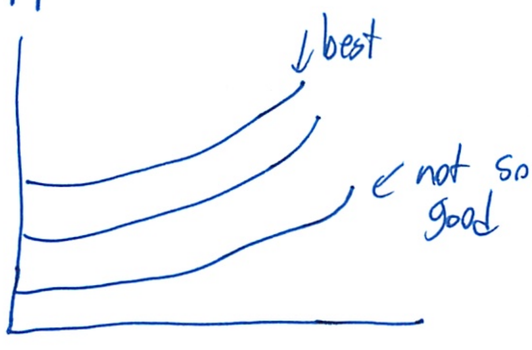
Like econ utility curve



People happy anywhere on indifference curve

<u>Investors</u> $(E(R_p), \sigma_p)$

- want upper left



- for each unit of risk - want more return
  - Since people are risk adverse - not risk neutral
  - and non-linear return - but more return than increase of risk (badly explained)

6

Mean - Varience utility

$$U(R_p) = E(R_p) - .5 A \cdot Var(R_p)$$

- higher U is more attractive risk - return profile
- A = risk aversion
  - >0 risk adverse
  - $=0$ risk neutral

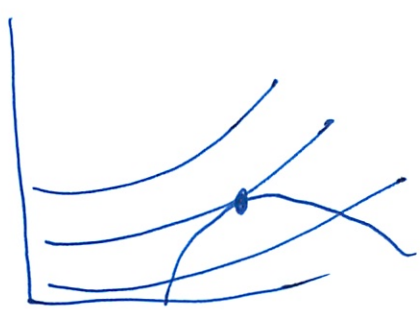
about equivilant rate of return

Optimal portfolio

- desirable - described by indifference curves 

- feasible - limited by efficent fronteir 

So find best tradeoff b/w them



Add risk free → Next Pg

But if add a risk free ~~rate~~ return
- you can go a little bit higher

7

Remember risk comes in 2 types

- diversifable
  - unique
  - non-systemic
- non-diversifable
  - systemic
  - macro economic factors

wash out w/ portfolio

still there