18.01 EXAM I

Friday, September 25, 2009

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Recitation Instructor: Briner

Recitation Hour: R05 ?

Instructions: You may not use calculators, notes, textbooks, or personal electronic devices. As a courtesy to other students, please turn off all cell phones. Read each question carefully. Whenever possible, include justification for your reasoning and show your work. Answers without any work or explanation will receive little or no credit. There are 6 questions and a 55 minute time limit on this exam. Good luck.

Issues

plug in point if have it
do product rule
$\frac{d}{dx} \ln(x) = \frac{1}{x} x'$

Question	Score	Maximum
1	3	8
2	2	5
3	0	5
4	2	5
5	0	5
6	2	6
Total	9	34

 Name: Michael Plasmeier

1. Compute the following derivatives. (Simplify your answers when possible.)

(a) $f'(x)$ where $f(x) = \dfrac{x}{1-x^2}$

$$\frac{1(1-x^2) - (-2x \cdot x)}{(1-x^2)^2} \to \frac{1-x^2+2x^2}{(1-x^2)^2} \to$$

think I got it, reduction more

$$\frac{1-x^2}{(1-x^2)^2} \quad \boxed{\cancel{+\frac{2x}{(1-x^2)^2} \to \frac{1}{(1-x^2)} + \frac{2x}{(1-x^2)^2}}}$$

+2/2

? done $\to$ $\dfrac{1-x^2}{(1-x^2)^2}$

(b) $f'(x)$ where $f(x) = \ln(\cos x) - \dfrac{1}{2}\sin^2(x)$

$$\frac{d}{dx} = \ln(\cos x) \cdot \sin x - \frac{1}{2} \cdot 2\sin(x) \cdot \cos x$$

$$-\cancel{\sin x} \ln(\cos x) - \sin x \cos x$$

?, more w/ the ln

did not know $\to \dfrac{1}{x} \; x'$

$$\frac{1}{\cos x} - \sin x - \frac{1}{2} \cdot 2\sin x \cos x \quad 1/3$$

$$\boxed{-\tan x - \sin x \cos x}$$

start

(c) $f^{(5)}(x)$, the fifth derivative of f, where $f(x) = xe^x$ $\boxed{\text{product rule}}$ $1 \cdot e^x + e^x \cdot x$

$f' = x + xe^x = x^2e^x$

$f'' = x^2 + xe^x = x^3e^x$

$f''' = 3x^3 + xe^x = x^4e^x$

$f'''' = 4x^4 + xe^x = x^5e^x$

$f^{(5)} = 5x^5 + xe^x = x^6e^x$

what was I thinking?

think got it

? duh

no

(1) $e^x + xe^x = e^x + xe^x$

(2) $e^x + e^x + xe^x = 2e^x + xe^x$

expands to

this always expands each th

0/3

(3) $2e^x + e^x + xe^x = 3e^x + xe^x$

(4) $4e^x + xe^x$

(5) $\boxed{5e^x + xe^x}$

go slow + do it right

+ otherwise product rule

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2. Find the equation of the tangent line to the "astroid" curve defined implicitly by the equation

$$x^{2/3} + y^{2/3} = 4$$

don't distribute $\cancel{(x^{2/3}+y^{2/3})^3 = 4^3}$ ← wrong

don't solve for y'

at the point $(-\sqrt{27}, 1)$.

Could check to make sure on curve

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}y' = 0$$

$$\frac{2}{3}y^{-1/3}y' = -\frac{2}{3}x^{-1/3}$$

$$y' = \frac{-\frac{2}{3}x^{-1/3}}{\frac{2}{3}y^{-1/3}} =$$

$\frac{2}{3} \cdot \frac{3}{2} = \frac{6}{6} = 1$

think got it
not sure
missing
Su

$$y' = -\frac{\sqrt[3]{y}}{\sqrt[3]{x}} \quad \cancel{\frac{\sqrt{y}}{\sqrt{x}}}$$

illegal

$\frac{1}{2} \cdot \frac{5}{5} = \frac{5}{10}$ ✓

2 plug in have point

$$y - 1 = -\frac{\sqrt{y}}{\sqrt{x}}\left(x - -\sqrt{27}\right)$$

$$y = -\frac{\sqrt{y}}{\sqrt{x}}\left(x + \sqrt{27}\right) + 1$$

which is?

$$-\frac{1}{-3^{3/2(-1/2)}}$$

$$\frac{-3^{3/2(-1/2)}}{1}$$

$$= \frac{3^{-1/2}}{1} = \frac{1}{\sqrt{3}}$$

$$\frac{1}{\sqrt{3}}\left(x + \sqrt{27}\right) + 1$$

$$\frac{1}{\sqrt{3}} \cdot \sqrt{27} + 1$$

$$\frac{1}{\sqrt{3}} \cdot \sqrt{3 \cdot 9} + 1$$

$$\frac{1}{\sqrt{3}} \cdot 3\sqrt{3} + 1$$

$$3 + 1$$

$$4$$

$$\boxed{\frac{1}{\sqrt{3}}x + 4}$$

A

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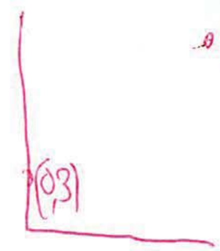
3. A particle is moving along a vertical axis so that its position y (in meters) at time t (in seconds) is given by the equation

$$y(t) = t^3 - 3t + 3, \quad t \geq 0.$$

Determine the total distance traveled by the particle in the first three seconds.

not / linear – can tell if velocity ever = 0 – does at 1

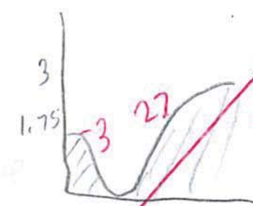


$y(3) = 3^3 - 3(3) + 3$

← can't even cube

$27 - 9 + 3$

21



? integration ??

$y(2) = 2^2 - 3(2) + 3$

$4 - 6 + 3$

1

oh was close – I made a mistake cubing

$y(1) = 1^2 - 3(1) + 3$

$1 - 3 + 3$

0

$y(0) = 3$

$+|3-0| + |0-21| = 6m$

sec 0→1 sec 1→3

$y(.5) = .25 - 1.5 + 3$

$3.25 - 1.5$

1.75

22m

$\frac{d}{dx} = 3t^2 - 3 = v$

$a = 6t$

$\int = \frac{t^4}{4} - \frac{3t^2}{2} + 3t + c$

0.

$\frac{3^4}{4} - \frac{3(3)^2}{2} + 3(3)$

$20.25 + 40.5 + 9$

69.75 m

$27 \cdot 3 = 81$

$\frac{81}{4}$

$4\overline{)81}$

$2\overline{)81}$

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4. State the product rule for the derivative of a pair of differentiable functions f and g using your favorite notation. Then use the DEFINITION of the derivative to prove the product rule. Briefly justify your reasoning at each step.

$$\frac{d}{dx}\left(f(x)\cdot g(x)\right) = \left(\frac{d}{dx}f(x)\right)g(x) + \left(\frac{d}{dx}g(x)\right)\cdot f(x)$$

No, want $(fg)'$ opps

$$\cancel{f'g'}\ (fg)' = f'\cdot g + g'\cdot f$$

$$\lim_{h\to 0}\frac{f(x+h)\,g(x+h) - f(x)\,g(x)}{h}$$ use limit formula ✓

!

$$\cancel{\frac{f(x+h)^3 + f(x+h)^2 g(x+h) + f(x+h)g(x+h)^2 + g(x+h)^3 \big/ f(x)^2 + 2f(x)g(x) + g(x)^2}{h}}$$ expand

$$\boxed{f(x+\cancel{h})\,g(x) + g(x+\cancel{h})\,f(x)}$$ cancle out

I think remember not to start

$$\lim_{h\to 0} f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)$$

$$\lim_{h\to 0}\left(g(x+h)\frac{f(x+h)-f(x)}{h} + f(x)\frac{g(x+h)-g(x)}{h}\right)$$

$g(x)$ differentiable thus contineous $g(x+h)=g(x)$

$$g(x)\,f'(x) + f(x)\,g'(x)$$

$\frac{2}{5}$

$$\tan = \frac{\sin}{\cos} \quad \frac{\frac{1}{\sqrt{x-1}}}{-\frac{1}{\sqrt{x-1}}} = -1$$

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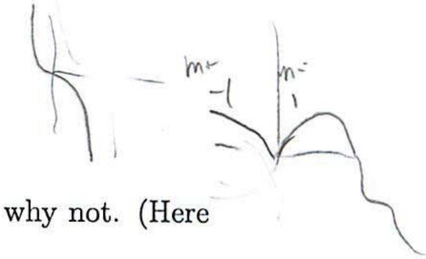
$\frac{0}{5}$

5. Does there exist a set of real numbers a, b and c for which the function

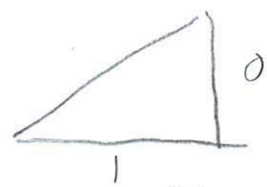
$$f(x) = \begin{cases} \tan^{-1}(x) & x \leq 0 \\ ax^2 + bx + c, & 0 < x < 2 \\ x^3 - \frac{1}{4}x^2 + 5, & x \geq 2 \end{cases}$$



is differentiable (i.e. everywhere differentiable)? Explain why or why not. (Here $\tan^{-1}(x)$ denotes the inverse of the tangent function.)

- must be contineous first of all to be differentiable

$\tan^{-1}(0) =$



what angle is tangent 0 $= 0^\circ$

$a(0)^2 + b(0) + c = 0$

must be 0

$\boxed{c = 0}$

like guessed at end

$\tan^{-1}$ we not remember

$2^3 - \frac{1}{4} 2^2 + 5 =$

$8 - 1 + 5 = 12$ 2are = here

$a(2)^2 + b(2) + 0 = 12$ not the x value the f(x) value

$4a + 2b = 12$

$2a + b = 6$

~~$a + \frac{1}{2} b = 3$~~

derivities must =

$\cancel{\frac{d}{dx} \tan^{-1}(x) = \frac{1}{x-1} = -1}$

$\cancel{\frac{1}{3} \cdot 2(0) + \frac{1}{3} = 0}$

$\cancel{\frac{2}{3} + \frac{1}{3} \neq -1}$

Do not match up - not differentiable at $x = 0$

$\frac{d}{dx} \tan^{-1} x \Big|_0 = \frac{d}{dx} ax^2 + bx$

$\boxed{1 = b}$

Must check both to make sure b + a are =

$\frac{d}{dx} ax^2 + bx + c \Big|_2 = \frac{d}{dx} x^3 - \frac{1}{4}x^2 + 5 \Big|_2$

$4a + b = 11$

$\boxed{a = \frac{5}{2}}$

??

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non linear

6. Suppose that f satisfies the equation $f(x+y) = f(x) + f(y) + x^2y + xy^2$ for all real numbers x and y. Suppose further that

$$\lim_{x\to 0} \frac{f(x)}{x} = 1.$$

(a) Find $f(0)$.

$f(x) = f(x) + f(0) + x^2 \cdot 0 + x \cdot 0^2$

$f(0+0) = f(0) + f(0) + 0^2 \cdot 0 + 0 \cdot 0^2$

0 +2 score

Since $x \to 0$ and $\lim_{x\to 0} \frac{f(x)}{x} = 1$

$f(x) \to 0$ as $x \to 0$

(b) Find $f'(0)$.

if $f(x) = 0$

$\lim_{x\to 0} \frac{f(x)}{x} =$

$\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = f'(0)$

def of derivitve

~~$f'(0) + f'(0) + 2xy' + x2yy'$~~

~~$0 + 0 + 2\cdot 0 \cdot 0 + 0 \cdot 2 \cdot 0 \cdot 0$~~

0 0 score

$f'(x) \to x' = 1$ as $x \to 0$

so $f'(0) = 1$

opproches 0 as $y = x$

y=x

zoomed in

(c) Find $f'(x)$.

~~$f'(x) + f'(y) + 2xy + x2yy'$~~

~~$y' = \frac{-f'(y) - f'(y) + 2xy}{2xy}$~~

~~$\boxed{y' = \frac{-f'(x)}{2xy} - \frac{f'(y)}{2xy} + 1}$~~

0 score

$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

rule #1

$\lim_{h\to 0} \frac{f(x) + f(h) + x^2h + xh^2 - f(x)}{h}$

$\lim_{h\to 0} \left(\frac{f(h)}{h} + x^2 + xh\right)$

$1 + x^2 + 0$

$\boxed{x^2 + 1}$

all the other ones I knew how to start must be the "fun" problem

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Total		34

1. Compute the following derivatives. (Simplify your answers when possible.)

(a) $f'(x)$ where $f(x) = \dfrac{x}{1-x^2}$

$$f'(x) = \frac{1\,(1-x^2) - (x)(-2x)}{(1-x^2)^2} = \frac{1-x^2+2x^2}{(1-x^2)^2} = \frac{1+x^2}{(1-x^2)^2}$$

(b) $f'(x)$ where $f(x) = \ln(\cos x) - \dfrac{1}{2}\sin^2(x)$

$\frac{1}{u}\cdot u'$

standard

$$f'(x) = \frac{1}{\cos x}(-\sin x) - \frac{1}{2}\cdot 2\sin x\cos x =$$

$$= \boxed{-\tan x - \sin x \cos x}$$

or $-\sin x\left(\dfrac{1+\cos^2 x}{\cos x}\right)$

easy in retrospect

(c) $f^{(5)}(x)$, the fifth derivative of f, where $f(x) = xe^x$

$f'(x) = e^x + xe^x = 1\cdot e^x + xe^x$

$f^{(2)}(x) = e^x + e^x + xe^x = 2e^x + xe^x$

$f^{(3)}(x) = 2e^x + e^x + xe^x = 3e^x + xe^x$

$f^{(4)}(x) = 4e^x + xe^x$

$\boxed{f^{(5)}(x) = 5e^x + xe^x}$

think got it

induction argument for general case:

$(f^{(k)}(x) = ke^x + xe^x$

$\Rightarrow f^{(k+1)}(x) = ke^x + e^x + xe^x$

$= (k+1)e^x + xe^x)$

$\frac{d}{dx}$ $\cancel{x}\ln x = \frac{1}{x}\, x'$

$1 \cdot \cancel{e^x + e^x}$.

$1\cdot \cancel{e^x + e^x e^x}$

1 $1 \cdot e^x + e^x x$

2 $e^x + e^x + xe^x$

3 $e^x + e^x + e^x + xe^x$

$3e^x + xe^x$

$5e^x + xe^x$ slow + do it right

$\frac{-\sqrt[3]{1}}{+\sqrt[3]{27}}$ $\frac{-1}{\sqrt{3}}$ ← still

$-\frac{1}{\sqrt{3}}(x + \sqrt{27})$

$-\frac{1}{\sqrt{3}}x + 3 + 1$

$\frac{1}{\sqrt{3}\cdot\sqrt{27}}$ ← would not have known that

$\sqrt{3}\cdot 3\cdot\sqrt{3}$

$3\cdot 3 + 1$

4

$\sqrt{3}\cdot\sqrt{3} = 3$

 Name: ______________________

2. Find the equation of the tangent line to the "astroid" curve defined implicitly by the equation
$$x^{2/3} + y^{2/3} = 4$$
at the point $(-\sqrt{27}, 1)$.

Check point is on curve:

$$-\sqrt{27} = -3^{3/2}$$

why?

$$\left(-3^{3/2}\right)^{2/3} + (1)^{2/3} = +3 + 1 = 4 \,//.$$

✓

Use implicit differentiation to get $dy/dx \Big|_{(-\sqrt{27},\,1)}$

$$\cancel{\tfrac{2}{3}} x^{-1/3} + \cancel{\tfrac{2}{3}} y^{-1/3} \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x^{-1/3}}{y^{-1/3}} = -\frac{-3^{3/2(-1/3)}}{1} = \frac{+3^{-1/2}}{1} = \frac{1}{\sqrt{3}}.$$

$$y - 1 = \frac{1}{\sqrt{3}}\left(x + \sqrt{27}\right)$$

$$y = \frac{1}{\sqrt{3}} x + \sqrt{9} + 1$$

$$\boxed{y = \frac{1}{\sqrt{3}} x + 4}$$

Seems straight forward

 Name: ____________________

3. A particle is moving along a vertical axis so that its position y (in meters) at time t (in seconds) is given by the equation

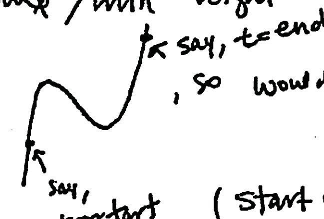
$$y(t) = t^3 - 3t + 3, \quad t \geq 0.$$

Determine the total distance traveled by the particle in the first three seconds.

Compute total dist up and down y-axis
(i.e. total dist, projected onto y-axis).

For this need to find max/min hight

(cubics look like this:



, so would want $\mathrm{I} + \mathrm{I} + \mathrm{I}$)

(start may be elsewhere — we don't know yet)

So look for min/max.
→ set $f'(x)=0$.
$y'(t)$

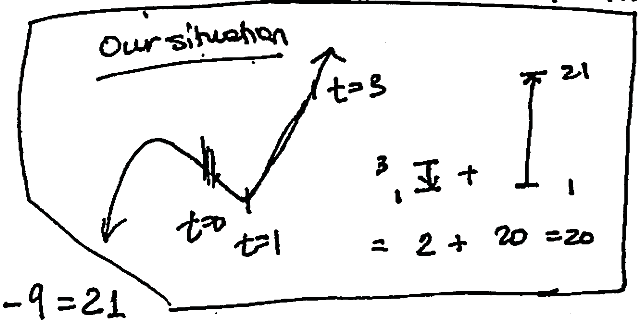
$y'(t) = 3t^2 - 3 \ (=0) \Rightarrow \quad 3t^2 - 3 = 0 \quad \Rightarrow \quad t^2 - 1 = 0 \quad t^2 = 1$

$\boxed{t = \pm 1}$

$y''(t) = 6t$
(2nd der test)

$t = +1$ – max
$t = -1$ – min

so for us, start t is after max
but before min.



$y(0) = 3$

$y(1) = 1 - 3 + 3 = 1$ (min

$y(3) = 3^3 - 9 + 3 = 27 + 3 - 9 = 30 - 9 = 21$

Total dist $= (3 - 1) + (21 - 1) = 2 + 20 = \boxed{22}$

 Name: ______________________________

4. State the product rule for the derivative of a pair of differentiable functions f and g using your favorite notation. Then use the DEFINITION of the derivative to prove the product rule. Briefly justify your reasoning at each step.

If f, g are both diff. functions of x,

$$(f \cdot g)'(x) = f'(x)\,g(x) + f(x)\,g'(x) \,..$$

$$(f \cdot g)'(x) = \lim_{h \to 0} \frac{f(x+h)\,g(x+h) - f(x)\,g(x)}{h} =$$

$$= \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)\,g(x+h) - f(x)\,g(x)}{h} =$$

$$= \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h}\, g(x+h) + f(x)\, \frac{g(x+h) - g(x)}{h} \right)$$

(f, g are differentiable $\Rightarrow$ continuous $\Rightarrow \lim_{h \to 0} g(x+h) = g(x) \ldots$)

$$= f'(x)\, g(x) + f(x)\, g'(x) \quad //.$$

5. Does there exist a set of real numbers a, b and c for which the function

$$f(x) = \begin{cases} \tan^{-1}(x) & x \leq 0 \\ ax^2 + bx + c, & 0 < x < 2 \\ x^3 - \frac{1}{4}x^2 + 5, & x \geq 2 \end{cases}$$

is differentiable (i.e. everywhere differentiable)? Explain why or why not. (Here $\tan^{-1}(x)$ denotes the inverse of the tangent function.)

First do continuity

$\tan^{-1}(0) = 0 \quad (\tan 0 = \frac{0}{1} = 0). \quad \Rightarrow c = 0.$

@2, $\quad 2^3 - \frac{1}{4}2^2 + 5 = 8 - 1 + 5 = 7 + 5 = 12.$

$\Rightarrow a2^2 + b \cdot 2 = 12 \Rightarrow \underline{2a + b = 6}$

$b = 6 - 2a$ //.

Finally, differentiability (want slopes to match @ $x = 0, 2$).

$f'(x)\Big|_{x=2} = 3x^2 - \frac{1}{2}x\Big|_{x=2} = 3 \cdot 4 - \frac{1}{2} \cdot 2 = 12 - 1 = 11$ //.

So want $\quad 2ax + b\Big|_{x=2} = 2a \cdot 2 + b = 4a + b = 11.$

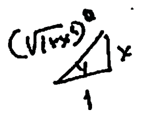
$b = 6 - 2a$

$4a + b = 11$
$2a + b = 6$
$\Rightarrow 2a = 11 - 6 = 5$
$a = 5/2$
$b = 6 - 5 = 1.$

$a = 5/2,\ b = 1,\ c = 0$
are our only option so far.
See if it works for diff. @ $x = 0$.

What is derivative of $\quad \frac{5}{2}x^2 + x$

$\tan^{-1}x$? $\quad \frac{1}{x^2+1}$ (also could do trick $\tan y = x$ and take derivative to remember)



$\tan y = x$

$\frac{\cos x(\cos x) + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$

$\sec^2 y \frac{dy}{dx} = 1 \quad \Rightarrow \frac{dy}{dx} = \frac{1}{\sec^2 y} = \cos^2 y = \frac{1}{1+x^2}$

$\frac{dy}{dx}\Big|_{x=0}$ should be $\frac{1}{1+0} = 1.$

$2ax + b\,\big|_{x=0} = b = 1$ ✓ great!

$$\boxed{a = 5/2,\ b = 1,\ c = 0}$$

 Name: ____________________

6. Suppose that f satisfies the equation $f(x+y) = f(x) + f(y) + x^2y + xy^2$ for all real numbers x and y. Suppose further that

$$\lim_{x\to 0} \frac{f(x)}{x} = 1.$$

(a) Find $f(0)$.

since $x \to 0$ and $\lim_{x\to 0} \frac{f(x)}{x} = 1$, $f(x) \to x \Rightarrow 0$. as $x \to 0$. . . .

$\Rightarrow \boxed{f(0)=0}$.

(b) Find $f'(0)$.

$f'(x) \to x' = 1$ as $x \to 0$, so $\boxed{f'(0) = 1.}$

(c) Find $f'(x)$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\cancel{f(x)} + f(h) + x^2h + xh^2 - \cancel{f(x)}}{h}$$

$$= \lim_{h\to 0} \left(\frac{f(h)}{h} + x^2 + xh \right)$$

$$= 1 + x^2 + 0 = \boxed{x^2+1} \quad //.$$