why did I lose pts on #3?

18.01 EXAM II

Tuesday, Oct. 20, 2009

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Recitation Instructor (Circle One): C. Breiner / I. Losev / X. Ma / S. Ramakrishnan / B. Rhoades

Recitation Hour: 1 PM

Instructions: You may not use calculators, notes, textbooks, or personal electronic devices. As a courtesy to other students, please turn off all cell phones. Read each question carefully. Whenever possible, include justification for your reasoning and show your work. Answers without any work or explanation will receive little or no credit. There are 6 questions and a 55 minute time limit on this exam. Good luck.

Question	Score	Maximum	
1	1	5	← disaster
2	5	5	← id values @ min / max
3	5	6	← copy error + more?
4	2	5	
5	4.5	5	
6	1	5	- did not remember / explain more
Total	19	31	

Basic area and volume formulas:

Volume of a Cone: $\frac{1}{3}\pi r^2 h$
Volume of a Sphere: $\frac{4}{3}\pi r^3$
Surface Area of a Sphere: $4\pi r^2$

thanks

πr^2

$2\pi r$

 Name: ______________

1. (a) Give a general expression for the quadratic approximation to a twice differentiable function $f(x)$ at $x = a$.

1

$a-x$
$x-a$

$$f(a) + f'(a)(a-x) + \frac{f''(a)}{2}\cancel{(a-x)}^2$$

$x-a$

$x-a$

I did flip it :(

grrr

(b) Use your answer from part (a) to give an approximate value for $\ln(1.2)$, where $\ln(x)$ is the natural log function.

0

$x = 0$

$\ln = \frac{1}{1+x}$

did I do "?

$$\cancel{\frac{1}{1+1.2} + \frac{1}{1.2}(1.2-0) + \frac{\frac{1}{1.2}}{2}(1.2-0)^2}$$

:(should have mem better focus more disaster

$$\ln(1.2) = \ln(1) + \ln'(1)(1.2-1) + \frac{\ln''(1)}{2}(1.2-1)^2$$

$(\ln(x))' = \frac{1}{x}$

$(\ln(x))'' - \frac{1}{x^2}$

$$\ln(1.2) = 0 + .2 - \frac{1}{2}(.2)^2$$

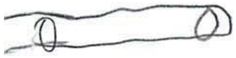
$$.2 - \frac{.04}{2}$$

$.18$

 Name: ______________________

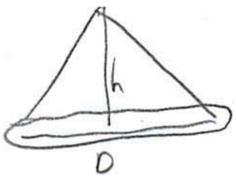
related rate

2. Salt is poured from a conveyer belt at a rate of 30 ft^3/min, forming a conical pile with a circular base whose height and diameter of base are always equal. How fast is the height of the pile increasing when the pile is 10 ft. high?



$30\,ft^3/min = \frac{dV}{dt}$

$\frac{dh}{dt} = ?$



$h = D = 2r$

$r = \frac{1}{2}h$

$h = 10$

$V = \frac{1}{3}\pi r^2 h$

$V = \frac{1}{3}\pi r^2 \cdot 2r$

$V = \frac{2}{3}\pi r^3$

$\frac{dV}{dt} = \frac{2}{3}\pi \cdot 3r^2 r'$

$\frac{dV}{dt} = 2\pi r^2 r'$

$\frac{dV}{dt} = \frac{1}{2}h^2\pi \frac{dh}{dt}$

$30 = \frac{1}{2}10^2 \pi \frac{dh}{dt}$ 100π

$\frac{dh}{dt} = \frac{30}{50\pi} = \frac{3}{50\pi}$ ft/min

figure out - why different??

$2r^2$

$2(\frac{1}{2}h)^2$

$2\frac{h^2}{4}$

$\frac{1}{2}h^2$

r or h

$V = \frac{1}{3}\pi(\frac{1}{2}h)^2 h$

$V = \frac{1}{12}\pi h^3$

$(\frac{1}{2}h)^2$

$(\frac{h}{2})^2$

$\frac{h^2}{4}$ $\frac{1}{4}h^2$

$\frac{1}{4}\cdot\frac{1}{3} = \frac{1}{12}$ $\frac{3}{12} = \frac{1}{4}$

$\frac{dV}{dt} = \frac{1}{12}\cdot 3\pi h^2 \frac{dh}{dt}$

$\frac{dV}{dt} = \frac{1}{4}\pi h^2 \frac{dh}{dt}$

$30 = \frac{1}{4}\pi \cdot 10^2 \frac{dh}{dt}$

$\frac{30}{25\pi} = \frac{6}{5\pi}$ ft/min

picked right one

$\frac{5}{5}$

bingo

 Name: ______________

3. Draw a careful picture of the graph of the function

$$f(x) = x - 3x^{1/3}.$$

$f(0) = 0$
$f(1) = -2$
$f(-1) = 2$

Be sure to indicate the coordinates of any local maxima and minima, the intervals on which the function is increasing and decreasing, and asymptotes (if any of these features occur). Computing inflection points may help you draw an accurate picture, but is not necessary.

$f'(x) = 1 - 3 \cdot \frac{1}{3} x^{-2/3} = 1 - x^{-2/3} = 1 - \frac{1}{\sqrt[3]{x^2}}$

$f''(x) = - -\frac{2}{3} x^{-5/3} = \frac{2}{3} x^{-5/3} = \frac{2}{3\sqrt[3]{x^5}}$

$1 - x^{-2/3} = 0$ +1

$-x^{-2/3} = 1$

$\frac{1}{\sqrt[3]{x^2}} = 1$

$1 = \sqrt[3]{x^2}$

$1 = x^2$

$x = \pm 1$ +1

$\frac{2}{3\sqrt[3]{x^5}} = 0$

$0 = \sqrt[3]{x^5}$

$0 = x^5$

$0 = x$

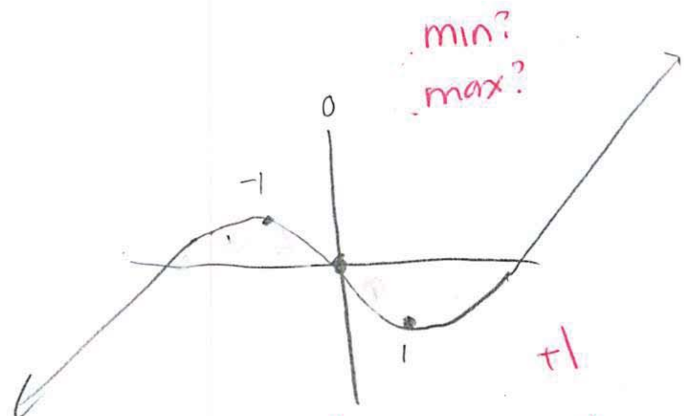
$+ \downarrow$ | -1 max | $- \downarrow$ | 0 inflection | $- \uparrow$ | 1 min | $+ \uparrow$ +1

$+ -$ = direction slope
$\uparrow \downarrow$ = concavity

increasing $(\infty, -1)(1, \infty)$
decreasing $(-1, 1)$ +1

$\frac{2}{3\sqrt[3]{1^5}} = \frac{2}{3} \uparrow \cup$ min

$\frac{2}{3}\sqrt[3]{-1^5} = \frac{2}{-3} = \downarrow \cap$ max



5/6

vertical tangent line at $x = 0$

 Name: ______________________

this time

4. A metal storage tank is to be made in the shape of a cylinder with a circular base and a hemispherical top. Find the dimensions of the tank which require the least amount of metal used to hold a fixed constant volume V.

$SA = 2\pi r h + \pi r^2 + \frac{1}{2} \cdot 4\pi r^2 = 2\pi r h + \pi r^2 + 2\pi r^2$

side bottom top half

$2\pi r h + 3\pi r^2$ ✓

$V = \pi r^2 h + 0 + \frac{1}{2}\frac{4}{3}\pi r^3$

main

✓ $\pi r^2 h + \frac{4}{6}\pi r^3$ copy error :(

$V = \pi r^2 h + \frac{2}{3}\pi r^2$

$\pi r^2 h = V - \frac{2}{3}\pi r^2$

$h = \frac{V}{\pi r^2} - \frac{2\cancel{\pi r^2}}{3\cancel{\pi r^2}}$

watch signs

$h = \frac{V}{\pi r^2} - \frac{2}{3}$ = fixed

$SA = 2\pi r\left(\frac{V}{\pi r^2} - \frac{2}{3}\right) + 3\pi r^2$

$\frac{2\cancel{\pi} \cancel{r} V}{\cancel{\pi} r^{\cancel{2}}} - \frac{4\pi r}{3} + 3\pi r^2$

$\frac{2V}{r}$ ← fixed $- \frac{4}{3}\pi r + 3\pi r^2$

$2Vr^{-1} - \frac{4}{3}\pi r + 3\pi r^2$

$\frac{dSA}{dr} = 2V \cdot -1 r^{-2} - \frac{4}{3}\pi + 3\pi 2r$

$\frac{-2V}{r^2} - \frac{4}{3}\pi + 6\pi r$

$0 = \frac{2V}{r^2} - \frac{4}{3}\pi + 6\pi r$

solve for V

$\frac{2V}{r^2} = \frac{4}{3}\pi - 6\pi r$

$2V = \frac{4}{3}\pi r^2 - 6\pi r^3$

$\boxed{V = \frac{2}{3}\pi r^2 - 3\pi r^3}$

constant

seems really wrong - where did I go off track

want dimensions

what is r?

$r^3\left(\frac{2}{3}\pi - 3\pi\right) = V$

$r = \sqrt[3]{\frac{V + \frac{2}{3}\pi}{3\pi}}$

ran out of time rush

2

$V = \pi r^2 h + \frac{2}{3}\pi r^2$

$SA = \pi r^2 + 2\pi r^2 + 2\pi r h$

sub h into SA

$SA = 3\pi r^2 + 2\pi r \left(\frac{V - \frac{2}{3}\pi r^3}{\pi r^2}\right) \rightarrow \frac{2V}{r} - \frac{4}{3}\pi r^2$

$\frac{d}{dr} = 6\pi r - \frac{2V}{r^2} - \frac{8}{3}\pi r$

$\frac{10}{3}\pi r = \frac{2V}{r^2}$ at a critical pt

^ sub V back in to get in terms r + h

r = h at critical pt

^ why? what causes this

no endpoints

2nd deriv - check sign <u>have to justify</u>

really focused

new

- happy w/ grade

- stay like this

 Name: ______________________

5. Explain why Newton's method eventually fails when finding zeroes of $f(x) = x^3 - 3x + 7$ with a starting value $x_1 = 2$. from pratice - kinda

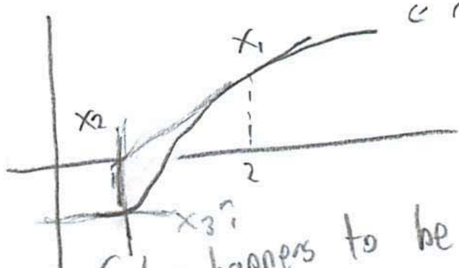
~~the estimations do not get closer to~~ 0

$f'(x) = 3x^2 - 3$

$x_1 = 2$

$x_2 = 2 - \frac{x^3 - 3x + 7}{3x^2 - 3} = \frac{2^3 - 3(2) + 7}{3(2)^2 - 3} = \frac{8 - 6 + 7}{12 - 3} = \frac{9}{9} = 1$ ✓

$x_3 = 1 - \frac{(1)^3 - 3(1) + 7}{3(1)^2 - 3} = \frac{1 - 3 + 7}{3 - 3} = \frac{5}{0}$ ← divide by 0 error



← random function

↑ slope happens to be 0 at x_3
which means newton's method will fail
where will you go next
nowhere because you flattened at ✓
never intercepts x-axis (-.5) ← needed the words

$\boxed{\text{looking for tangent line intercept x axis}}$

got this I hope

 Name: ____________

MVT??

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

6. Prove that

$$\overset{\oplus\ Y_1}{\sqrt{1+x}} < \overset{\oplus}{1}+\frac{1}{2}x^{\,Y_2}, \quad \text{if } x > 0. \quad x \oplus$$

for it to be less than

X	Y_1	Y_2
1	$\sqrt{2}$	$\frac{3}{2}$
2	$\sqrt{3}$	2
3	$\sqrt{4}$	$\frac{5}{2}$
4	$\sqrt{5}$	3

$(1+x)^{1/2}$

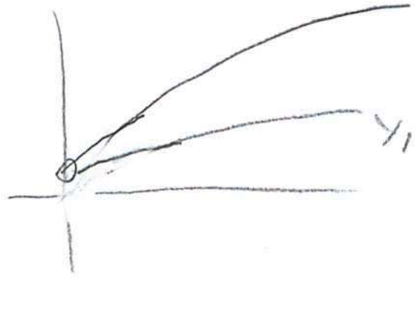
$Y_1' = \frac{1}{2}(1+x)^{-1/2} \cdot 1 \qquad Y_2' = \frac{1}{2}$

$\frac{1}{2(1+x)^2}$

denominator gets larger as $x \to \infty$ $(x > 0)$
means slope gets smaller $(m \to 0)$
but silll positive as $x \to \infty$
$x > 0$

So $Y_1 < Y_2$ at the start
and Y_1 slope will approch 0 as $x \to \infty$ $(x > 0)$
where as Y_2 will grow at a constant rate further increasing it's lead over Y_1

did not really use proper termonoloy



Approx picture

similar to pset problem

opps

t in class forgot

1.

Look at $f(x) = 1 + \frac{1}{2}x - \sqrt{1+x}$ $[0, w]$

$$f(w) = f(0) + f'(c)(w-0) \qquad c \in (0,w)$$

$$= 0 + \underbrace{f'(c) \cdot w}_{+}$$

$$f'(x) = \frac{1}{2} - \frac{1}{2\sqrt{1+x}}$$

$$= \frac{1}{2}\left(1 - \frac{1}{\sqrt{1+x}}\right) \leftarrow \text{why is this } +$$

$\left[\text{for } x > 0 \ \sqrt{1+x} > 1 \ \text{ so } 1 > \frac{1}{\sqrt{1+x}}\right]$

So other part + as well

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Question	Score	Maximum
1		5
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3		6
4		5
5		5
6		5
Total		31

Basic area and volume formulas:

Volume of a Cone: $\frac{1}{3}\pi r^2 h$

Volume of a Sphere: $\frac{4}{3}\pi r^3$

Surface Area of a Sphere: $4\pi r^2$

 Name: ________________

1. (a) Give a general expression for the quadratic approximation to a twice differentiable function $f(x)$ at $x = a$.

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$$

(b) Use your answer from part (a) to give an approximate value for $\ln(1.2)$, where $\ln(x)$ is the natural log function.

$$\ln(1.2) = \ln(1) + \ln'(1)(1.2-1) + \frac{\ln''(1)}{2}(1.2-1)^2$$

$$\left((\ln(x))' = \frac{1}{x} \quad (\ln(x))'' = \frac{-1}{x^2} \right)$$

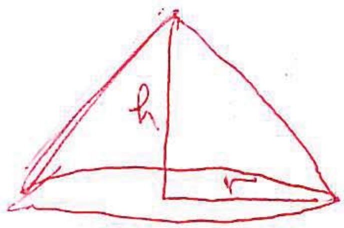
$$\Rightarrow \ln(1.2) = 0 + (0.2) - \frac{1}{2}(0.2)^2$$

$$= 0.2 - 0.04/2$$

$$= 0.18$$

Name: ______________________

2. Salt is poured from a conveyer belt at a rate of 30 ft^3/min, forming a conical pile with a circular base whose height and diameter of base are always equal. How fast is the height of the pile increasing when the pile is 10 ft. high?



$$h = 2r$$

$$V = \frac{1}{3}\pi r^2 h$$

$$= \frac{1}{3}\pi \left(\frac{h}{2}\right)^2 h$$

$$= \frac{1}{12}\pi h^3$$

$$\frac{dV}{dt} = \frac{1}{4}\pi h^2 \frac{dh}{dt}$$

$$\Rightarrow \quad \frac{dh}{dt} = \frac{dV}{dt}\left(\frac{4}{\pi h^2}\right)$$

$$= (30)\left(\frac{4}{10^2\pi}\right) = \frac{120}{100\pi}$$

$$= \frac{12}{10\pi}\ \frac{ft}{min}$$

 Name: ____________________

3. Draw a careful picture of the graph of the function

$$f(x) = x - 3x^{1/3}.$$

Be sure to indicate the coordinates of any local maxima and minima, the intervals on which the function is increasing and decreasing, and asymptotes (if any of these features occur). Computing inflection points may help you draw an accurate picture, but is not necessary.

$f(x) = 0$ $\qquad x - 3x^{1/3} = 0$ $\qquad x^{1/3}(x^{2/3} - 3) = 0$

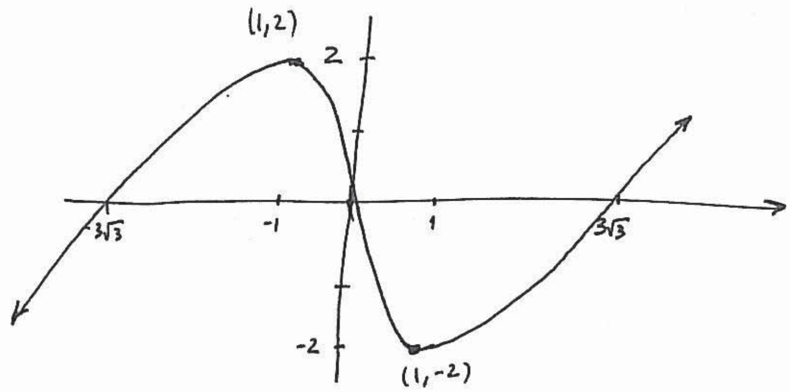
$x = 0, +3\sqrt{3}, -3\sqrt{3}.$

$f'(x) = 1 - 3 \cdot \frac{1}{3} x^{-2/3} = 1 - x^{-2/3}$ $\;\;$: critical pts at ± 1. $\Rightarrow$ at $\underset{\text{min}}{(1,-2)}$ $\;\underset{\text{max}}{(-1,+2)}$

Vertical tangent line at $x = 0$

inc. $\quad$ dec. $\quad$ inc.

$-1 \qquad 1$

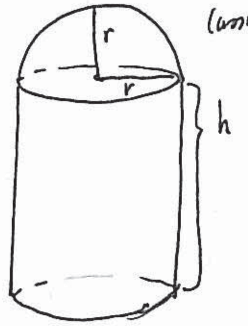


 Name: ____________________

4. A metal storage tank is to be made in the shape of a cylinder with a circular base and a hemispherical top. Find the dimensions of the tank which require the least amount of metal used to hold a fixed constant volume V.

Const $= V = \frac{1}{2}\left(\frac{4}{3}\pi r^3\right) + (\pi r^2)h. = \frac{2}{3}\pi r^3 + \pi r^2 h$

(assuming) amount of metal used is prop. to surface area → minimize SA.



$SA = \frac{1}{\cancel{2}}\left(\cancel{4}^{\,2}\pi r^2\right) + (2\pi r)h + \pi r^2 =$

$= 3\pi r^2 + 2\pi r h.$

$h = \frac{V - \frac{2}{3}\pi r^3}{\pi r^2}$

$SA = 3\pi r^2 + 2\pi r\left(\frac{V - \frac{2}{3}\pi r^3}{\pi r^2}\right) = 3\pi r^2 + \frac{2V}{r} - \frac{4}{3}\pi r^2$

$= \frac{5}{3}\pi r^2 + \frac{2V}{r}.$

$\frac{d}{dr}SA = \frac{10}{3}\pi r + (-1)\frac{2V}{r^2}$ set $=0$. , assume $r \neq 0$.

$\frac{\cancel{10}^{\,5}}{3}\pi r = \frac{\cancel{2}V}{r^2}$ $V = \frac{5}{3}\pi r^3 \Rightarrow r = \sqrt[3]{\frac{3V}{5\pi}}$

$\Downarrow$

$h = \frac{5/3\,\pi r^3 - 2/3\,\pi r^3}{\pi r^2} = \frac{\pi r^3}{\pi r^2} = r \Rightarrow \boxed{h = r = \sqrt[3]{\frac{3V}{5\pi}}}$

This is a critical pt. should show it is indeed a min.

Check boundaries: $r=0$, $h=0$.

$r = 0 \Rightarrow SA = 0$ ~~✗~~

$h = 0 \Rightarrow SA = 3\pi r^2$, $V = \frac{2}{3}\pi r^3 \Rightarrow r = \sqrt[3]{\frac{3V}{2\pi}}$

$SA = 3\pi\left(\frac{3V}{2\pi}\right)^{2/3}$ when $h=0$.

when $h=r$, $h = r = \sqrt[3]{\frac{3V}{5\pi}}$, so

$SA = 5\pi\left(\frac{3V}{5\pi}\right)^{2/3}$

Comparing $5^{1/3}$ to $\frac{3}{2^{2/3}}$. cube both sides.

$5 < \frac{27}{4}$, so SA at critical pt $<$ SA at bdry pt

b/c $20 < 27$

$\Rightarrow$ min attained at $\boxed{h = r = \sqrt[3]{\frac{3V}{5\pi}}}$

5. Explain why Newton's method eventually fails when finding zeroes of $f(x) = x^3 - 3x + 7$ with a starting value $x_1 = 2$.

$f'(x) = 3x^2 - 3$

For $x_1 = 2$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2 - \frac{f(2)}{f'(2)} = 2 - \frac{9}{9} = 1$$

$f(x_2) = f(1) = 1 - 3 + 7 = 5$

$f'(x_2) = f'(1) = 3 - 3 = 0$

The problem is that the slope of the tangent line at $x = 1$ is horizontal. Thus, it does not intersect the x-axis & we cannot continue Newton's method.

 Name: ____________________

6. Prove that

$$\sqrt{1+x} < 1 + \frac{1}{2}x, \quad \text{if } x > 0.$$

Set $g(x) = 1 + \frac{1}{2} - \sqrt{1+x}$. WTS $g(x) > 0$ for $x > 0$.

$g(0) = 1 + 0 - 1 = 0$. Sufficient to show $g'(x) > 0$ for all $x > 0$.

$$g'(x) = 0 + \frac{1}{2} - \frac{1}{2}\frac{1}{\sqrt{1+x}} = \frac{1}{2} - \frac{1}{2\sqrt{1+x}}$$

For $x > 0$, $\sqrt{1+x} > \sqrt{1} = 1$, so

$$\frac{1}{\sqrt{1+x}} < \frac{1}{1} = 1, \text{ and}$$

$$-\frac{1}{2\sqrt{1+x}} > -\frac{1}{2}\cdot 1 = -\frac{1}{2}$$

$\Rightarrow$

$$\underbrace{\frac{1}{2} - \frac{1}{2\sqrt{1+x}}}_{g(x)} > \frac{1}{2} - \frac{1}{2} = 0 //$$

//.

showed $g(x) > 0$ for $x > 0$.