18.01 EXAM III

Tuesday, Nov. 10, 2009

Name: Michael Plasmeier

E-mail: theplaz@mit.edu

Recitation Instructor (Circle One): C. Breiner / I. Losev / X. Ma / S. Ramakrishnan / B. Rhoades

Recitation Hour: ____________

Instructions: You may not use calculators, notes, textbooks, or personal electronic devices. As a courtesy to other students, please turn off all cell phones. Read each question carefully. Whenever possible, include justification for your reasoning and show your work. Answers without any work or explanation will receive little or no credit. There are 6 questions and a 50 minute time limit on this exam. Good luck.

Question	Score	Maximum
1	4	4
2	1	5
3	0	6
4	0.5	5
5	2	5
6	1	6
Total	9	31

← thought I got

≥ 11 = passing

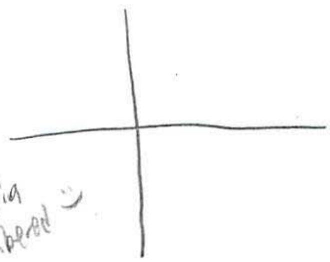
no usub on whole test!

 Name: ______________________

Like $y = x^2 - 4x$ $\quad y = 2x - x^2$ $\quad x^2 - 4x = 2x - x^2$?what is x where = ?

1. Compute the area between the curves $x = y^2 - 4y$ and $x = 2y - y^2$.

Find where intersect



intersection - remembered I think :)

$$y^2 - 4y = 2y - y^2$$

$+y^2 \quad +4y \quad +4y \quad +y^2$

what is y where =

$$2y^2 = 6y$$

$$y^2 = 3y \qquad x = 3^2 - 4(3)$$

$9 - 12$

$$x = -3$$

$$y = 3, 0 \qquad (-3, 3)$$

$$\int_0^3 (y^2 - 4y) - (2y - y^2)\, dy$$

$$\int_0^3 y^2 - 4y - 2y + y^2$$

$$\int_0^3 2y^2 - 6y\, dy$$

$$\frac{2y^3}{3} - \frac{6y^2}{2} \Big|_0^3$$

$$\frac{2}{3}y^3 - 3y^2 \Big|_0^3$$

$$\frac{2}{3}(3)^3 - 3(3)^2 - \left[\frac{2}{3}(0)^3 - 3(0)^2\right]$$

$$18 - 27$$

$-9 \rightarrow$ flip # $\rightarrow$ 9

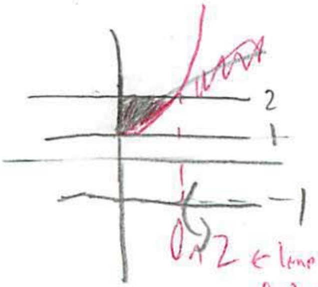
4. ✓

 Name: ______________________

?dash

2. Find the volume of the solid obtained by revolving the region bounded by the curves $y = e^x$, $y = 2$, and $x = 0$ about the line $y = -1$.

Setup integral only



$\sum (\text{area}) \cdot 2\pi$

$\lim_{n\to\infty} \sum^{n} x(2-e^x)2\pi$ missing n - but don't need

$2\pi \int_{0}^{e^2} x(2-e^x)\,dx$

Did not remember steps since studied inefficiently

go slow + re... it

washers

but must take into account distance to the axis

Forget

$2\pi \int_{0}^{e^2 \;\ln 2} x\left((2+1) - (e^x + 1)\right) d\cancel{x}\,y$

if do w/ shells

Wrong bounds

$\int_{0}^{\ln 2} \pi \left(3^2 - (1+e^x)^2\right) dx$

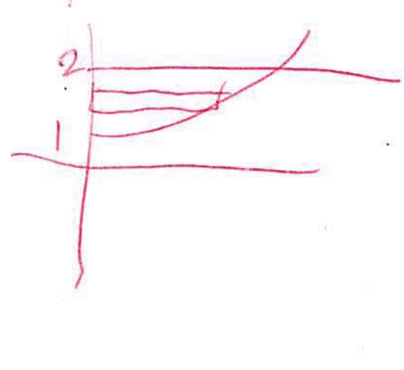
why 2 - since x_i

?2

why 3 - oh yeah

subtraction error

$\int_{1}^{2} 2\pi(y+1)\ln y\, dy$



3. Evaluate each of the following expressions

I did not get this one at all

(a)

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(1 + i \cdot \frac{3}{n}\right)^2 \frac{3}{n}$$

$\to$ see back

n

$\left(1+\frac{3n}{n}\right)^2 \frac{3}{n}$

$4^2 \cdot \frac{3}{n}$

$16 \cdot 0$

0

$\left(1+1\cdot\infty\right)^2 \cdot \frac{3}{\infty}$

$\infty \cdot \frac{3}{\infty}$

$\infty \cdot 0$

So still will $= 0$

but we are summing the values - will eventually not add more

0/3

(b) The value $f(4)$ for the continuous function f satisfying

is that the value or have to find ?!?

$$x \sin \pi x = \int_0^{x^2} f(t)\, dt$$

$\sin(0) = 0$

$4 \sin(4\pi) = \int_0^{4^2} f(t)dt$

$4 \cdot 0 = f(16) - f(0)$

$0 = f(16)$

to find $f(x)$

↓ differentiate both sides

0/3

?! what is it asking wtf

don't ever get know

$\frac{d}{dx} x \sin(\pi x) = \frac{d}{dx} \int_0^{x^2} f(t)\,dt$

constant

$\sin \pi x + \pi x(\cos \pi x) = f(t)\,dt \leftarrow f(x^2)\, 2x$

did not know or could integrate def. integral ? integrate → differentiate

FTC

$f(x^2) = \frac{1}{2x} \sin \pi x + \frac{\pi}{2} \cos \pi x$

so $f(4) = f(2^2) = \frac{1}{4} \sin 2\pi + \frac{\pi}{2} \cos 2\pi = \frac{\pi}{2}$

? given

*key was to recognize that it was Right Riemann sum

? since $\frac{3}{n}$

Consider $[0, 3]$, cut into n parts

Consider the function $f(x) = (1+x)^2$ ← ? how do I know that ????

Right Rieman sum

$$\sum_{i=1}^{n} \left(1 + i\,\frac{3}{n}\right)^2 \frac{3}{n}$$

← generic Riemann sum formula

$$\frac{b-a}{n} \sum f(x_i)$$

$x_i = a + i\frac{b-a}{n}$

← get rid of n

convert integral

$$\int_0^3 (1+x)^2\,dx$$

solve integral

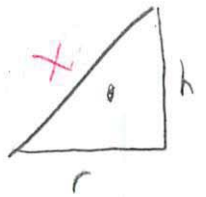
$$\int_0^3 x^2 + 2x + 1\,dx$$

$$\frac{x^3}{3} + \frac{2x^2}{2} + x \Big|_0^3$$

$$9 + 9 + 3 = 21$$

same as $\int_1^4 x^2\,dx$

4. (a) Find the centroid (i.e. center of mass) of a right triangle with height h and base r (assuming the triangle has uniform density).



~~Symetrical shape $\bar{x} = \bar{y}$~~ ← can be - does not have to be

$\int_0^r x \cdot h \, dx$ decide ||| or ≡ $\int_0^h r y \, dy$

$\int_0^r \cancel{f}\, x \, dx$ wrong integral $r \frac{y^2}{2} \Big|_0^h$

$f = \frac{h}{r}$ $\cancel{f} \frac{x^2}{2} \Big|_0^h$ $r\frac{(h^2)}{2}$

~~$\frac{h}{2}(r^2) - (\frac{h}{2} \cdot 0)^2$~~ $\frac{h}{r} \cdot \frac{h^2}{2}$

$\bar{x} = \frac{r^2 h}{2}$

$\bar{y} = \frac{r h^2}{2}$ $\left(\frac{r^2 h}{2}, \frac{r h^2}{2}\right)$

back

See ans sheet

0

Thanks I guess! from p-set which got from Brendon

(b) Use Pappus' Theorem and your answer in the previous part to find the volume of a cone with height h and base radius r.

Papus → Volume = ~~com~~ area · 2π dist travell by com ← which is $\frac{r}{3}$

~~r area around~~

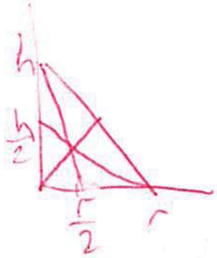
$\frac{1}{2}hr$ $\frac{r^2 h}{2} \cdot \cancel{2\pi}$ $2\pi \frac{r}{3}$ where is area

$V = \boxed{\frac{\pi r^2 h}{3}}$ 0.5

? (✓) have heard before

Can't ever get this right - my fav -> put w/ x+1

$area = \frac{1}{2} h r$



$$\frac{h/2}{r} = \frac{h/3}{2r/3} \quad (h/3, r/3)$$

i don't get

S S from something to something

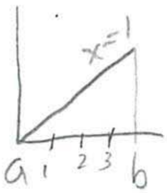
briner not helping me

Name: ______________________

5. Given a definite integral

Defining a scenario

$$\int_a^b f(x)\, dx,$$



let T_n be the *trapezoid* approximation with n intervals, M_n the *midpoint* approximation using n intervals, and S_{2n} the *Simpson's rule* approximation using $2n$ intervals. Prove that

$$\frac{1}{3}T_n + \frac{2}{3}M_n = S_{2n}.$$

n = 4 ? genericly ☺

complex way to ask

weird way of writing that they all =

$$T_n = \frac{\Delta x}{2}\left(y_0 + 2y_1 + 2y_2 + 2y_3 + y_4\right) \checkmark$$

$$M_n = \Delta x \left(\cancel{\frac{y_1 - y_0}{2} + \frac{y_2 - y_1}{2} + \frac{y_3 - y_2}{2} + \frac{y_4 - y_3}{2}} \right) ✗$$

$$\hookrightarrow \sum_{i=1}^{n} f(x_{2x_i - 1})$$ ← know the rules!

$$S_{2n} = \frac{\Delta x}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + y_4\right) \checkmark$$

diff. Δx than above

→ back

This was closer I though

T2n should have fixed

$$T_n = \frac{\frac{1}{4}}{2}\left(0 + 2 \cdot .25 + 2 \cdot .5 + 2 \cdot .75 + 1\right)$$

$$\frac{1}{8} \cdot 4 = \frac{1}{2}$$

$$M_n = \frac{1}{4}\left(\frac{.25 - 0}{2} + \frac{.5 - .25}{2} + \frac{.75 - .5}{2} + \frac{1 - .75}{2}\right)$$

$$\frac{1}{4}\left(.125 + .375 + .625 + .875\right) = \frac{1}{2}$$

$$S_n = \frac{1/4}{3}\left(0 + 4 \cdot .25 + 2 \cdot .5 + 4 \cdot .75 + 1\right)$$

$$\frac{1}{12} \cdot 6 = \frac{1}{2}$$

$$S_{2n} = \frac{1/8}{3}\left(0 + 4 \cdot .125 + 2 \cdot .25 + 4 \cdot .375 + 2 \cdot .5 + 4 \cdot .625 + 2 \cdot .75 + 4 \cdot .875 + 1\right) = \frac{1}{2}$$

Can't prove w/ #

$$\frac{1}{3} \cdot \frac{1}{2} + \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{2}$$

$$\frac{1}{6} + \frac{3}{6} = \frac{1}{2} \quad (\checkmark)$$

← proved w/ # but basically you are saying $\frac{1}{3}x + \frac{2}{3}x = x$ – all should be the same

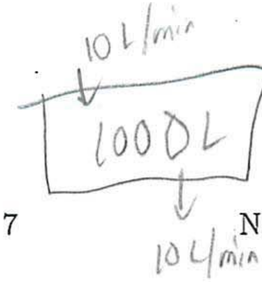
$$\frac{1}{3}T_n + \frac{2}{3}M_n = \frac{1}{3}T_n + \frac{4}{6}M_n$$

$$= \frac{b-a}{6n}\left(f(x_0) + f(x_{2n}) + 2\sum_{i=1}^{n-1} f(x_{2i}) + 4\sum_{i=1}^{n} f(x_{2i-1})\right)$$

what is this

$$= S_n$$

?, how does this prove anything



* did not understand how to set up ?

 Name: ____________________

6. A tank contains 1000 L of brine (that is, salt water) with 15 kg of dissolved salt. Pure water enters the top of the tank at a constant rate of 10 L / min. The solution is thoroughly mixed and drains from the bottom of the tank at the same rate so that the volume of liquid in the tank is constant.

$\left(1 - \frac{1}{n}\right)$

$S(0) = 15 = \frac{15}{1000}$

S = concentration of salt

$S(t) = ?$ ← part b

$S(t) = \frac{15}{1000} - \frac{ds}{dt}$

(a) Find a differential equation expressing the rate at which salt leaves the tank.

hard to visualize + write equation

~~$\frac{dS}{dt} = \frac{15}{(1000-10t)}dt$~~ ← changes

t in min

but it continously mixes

concentration if water filtered when leaving tank but leaving pure + amt salt changes

~~$\frac{ds}{dt} = 15 - \frac{1000-10t}{15}dt$~~

hint that uses ln

prob wrong since no log

variable

$\frac{ds}{dt} = -10 L/m \cdot \frac{s(t)}{1000}$

$= -\frac{s(t)}{100} \frac{kg}{min}$

1

not right d.e. but some good obs.

(b) Solve this differential equation to find an expression for the amount of salt (in kg) in the mixture at time t.

can't visualize the f b problem need to be for better at this

~~$S(t) = \int 15 - \frac{1000-10t}{15} dt$~~

~~$15t - \frac{1000t - 10t^2}{30}$~~

$\frac{ds}{s(t)} = -\frac{1}{100}dt$

Integrate

$\ln(s(t)) = -\frac{1}{100}t + c$

exponentiate

$s(t) = k \cdot e^{-1/100 t}$

$+ e^c = k$

$s(0) = 15$ so $k = 15$

$s(t) = 15e^{-1/100 t}$

(c) How long does it take for the total amount of salt in the brine to be reduced by half its original amount? (Recall $\ln 2 \approx .693$.) ← hint

prob wrong since no ln 2

$S(t) =$ ~~[illegible]~~

$.5 \cdot 15 = 15 - \frac{1000-10t^2}{30}$

$7.5 = 15 - \frac{1000-10t^2}{30}$ ✓

$-7.5 = -\left(\frac{1000-10t^2}{30}\right)$

$7.5 = \frac{1000-10t^2}{30}$ → back

$225 = 1000 - 10t^2$

$-775 = -10t^2$

$775 = 10t^2$

$7.75 = t^2$

$t = \sqrt{7.75}$

30 × 7.5 = 150 + 210.0 = 225

focus a lot on working backwards - just learn to solve the problem

$$e^{-1/100\,t} = \frac{1}{2}$$

$$\ln \qquad \ln$$

$$1/100\,t = \ln 2$$

$$t = 100 \ln 2$$

$$69.2 \text{ min}$$

18.01 EXAM III

Tuesday, Nov. 10, 2009

Name: Solutions.

E-mail: ____________________

Recitation Instructor (Circle One): C. Breiner / I. Losev / X. Ma / S. Ramakrishnan / B. Rhoades

Recitation Hour: ____________________

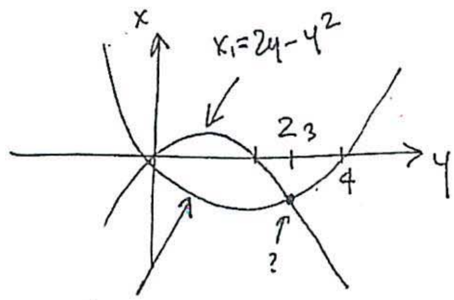
Instructions: You may not use calculators, notes, textbooks, or personal electronic devices. As a courtesy to other students, please turn off all cell phones. Read each question carefully. Whenever possible, include justification for your reasoning and show your work. Answers without any work or explanation will receive little or no credit. There are 6 questions and a 50 minute time limit on this exam. Good luck.

Question	Score	Maximum
1		4
2		5
3		6
4		5
5		5
6		6
Total		31

1. Compute the area between the curves $x = y^2 - 4y$ and $x = 2y - y^2$.

$y(y-4)$ $\qquad$ $y(2-y)$

$0, 4$ $\qquad$ $0, 2$



$x_2 = y^2 - 4y$ $\qquad$ height $= x_1 - x_2$

$y^2 - 4y = 2y - y^2$

$2y^2 - 6y = 0$

$y^2 - 3y = 0$

$y = 0$ or $y = 3$

$$\int_0^3 2y - y^2 - (y^2 - 4y)\,dy$$

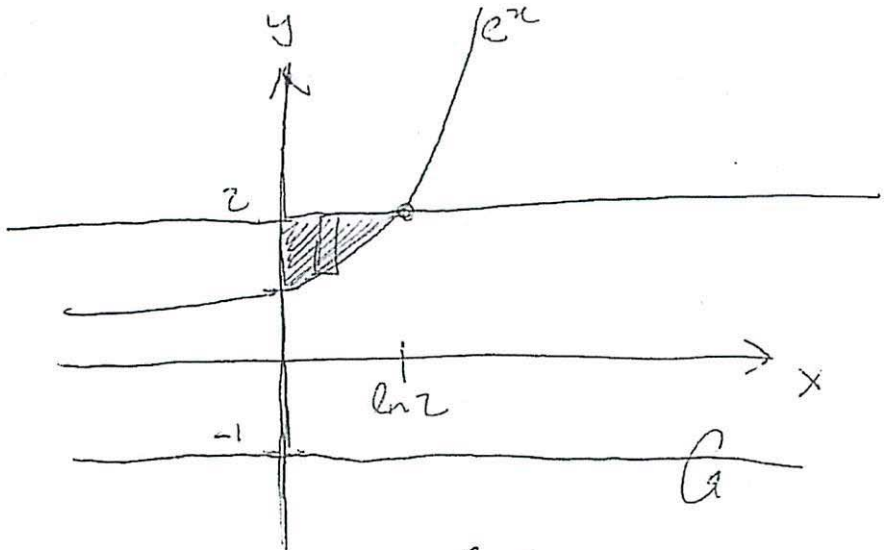
$$\int_0^3 6y - 2y^2\,dy = 3y^2 - \frac{2}{3}y^3 \Big|_0^3 = \cancel{27 - 2\cdot}$$

$$3\cdot 3^2 - 2\cdot 3^2 = $$

$$= 1\cdot 3^2 = \boxed{9}$$

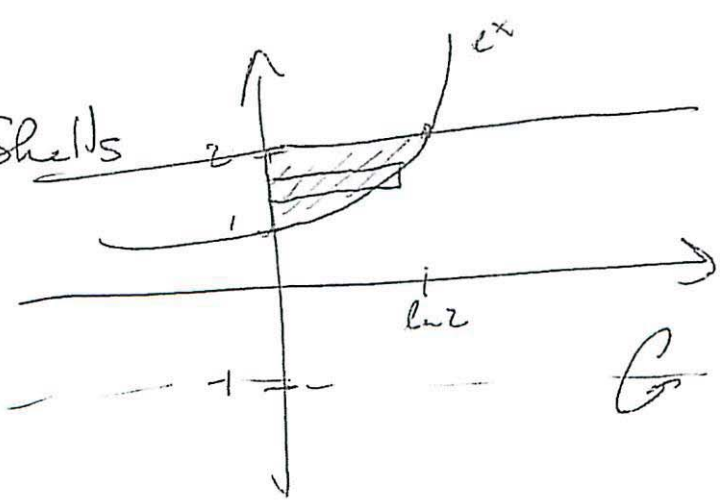
 Name: ______________________

2. Find the volume of the solid obtained by revolving the region bounded by the curves $y = e^x$, $y = 2$, and $x = 0$ about the line $y = -1$. You only need to give a definite integral expressing the volume. Do not solve the integral.



Washers: $\displaystyle\int_0^{\ln 2} \pi\left(3^2 - (1+e^x)^2\right)\,dx$

Shells



$\displaystyle\int_1^2 2\pi(y+1)\ln y\,dy$

3. Evaluate each of the following expressions

(a)

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(1 + i \cdot \frac{3}{n}\right)^2 \frac{3}{n}$$

Consider the interval $[0, 3]$, cut into n parts.

Consider the function $f(x) = (1+x)^2$.

Then the right RS is.

$$\sum_{i=1}^{n} \left(1 + i\frac{3}{n}\right)^2 \cdot \frac{3}{n}$$

So $\lim\limits_{n\to 0} \sum\limits_{i=1}^{n} (1+i\frac{3}{n})^2 \frac{3}{n} = \int_0^3 (1+x)^2 dx$

$= \int_0^3 1 \cdot dx + \int_0^3 2x\,dx + \int_0^3 x^2 dx = 3 + 9 + \frac{1}{3}27 = 21$

(b) The value $f(4)$ for the continuous function f satisfying

$$x \sin \pi x = \int_0^{x^2} f(t)\, dt$$

$\frac{d}{dx}(x \sin \pi x) = \frac{d}{dx} \int_0^{x^2} f(t) dt$

$\Rightarrow \sin \pi x + \pi x \cos \pi x = f(x^2) \cdot 2x$

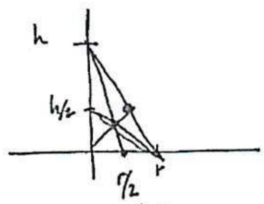
$\Rightarrow f(x^2) = \frac{1}{2x} \sin \pi x + \frac{\pi}{2} \cos \pi x$

So $f(4) = f(2^2) = \frac{1}{4} \sin 2\pi + \frac{\pi}{2} \cos(2\pi)$

$= \frac{\pi}{2}$.

 Name: ______________________

4. (a) Find the centroid (i.e. center of mass) of a right triangle with height h and base r (assuming the triangle has uniform density).



something like $(h/3, r/3)$

$$\frac{h/2}{r} = \frac{h/3}{2r/3} \checkmark$$

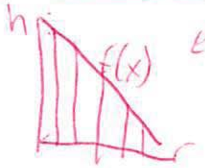
<See solutions to HW> COM

1a2

$$\frac{\sum_{i=1}^{n} m_i \cdot x_i}{\sum_{i=1}^{n} m_i} \to \lim_{n\to\infty} \frac{\sum x_i \overbrace{f(x_i)\Delta x}^{mass}}{f(x_i)\Delta x}$$

$$\frac{\int x f(x)\,dx}{Mass = Area}$$

integral way. Briner office hrs



← to find $\frac{\Delta y}{\Delta x} = \frac{h-0}{0-r} = \frac{-h}{r}$

$y = mx + b$

$h = -\frac{h}{r} \cdot 0 + b \quad b = h$

$0 = -\frac{h}{r} \cdot r + b \quad b = h$

$y = -\frac{h}{r}x + h$

x) $$\frac{\int_0^r x\left(-\frac{h}{r}x + h\right) dx}{\frac{1}{2}hr}$$

y) $$\frac{\sum m_i y_i}{m_i} = f(x_i)\,\Delta x \cdot \frac{f(x_i)}{2} \to$$

$$\frac{\int_0^r \frac{f(x)}{2} \cdot f(x)\,dx}{\frac{1}{2}hr}$$

(b) Use Pappus' Theorem and your answer in the previous part to find the volume of a cone with height h and base radius r.

now integrate

Area · dist travelled.

$$\frac{1}{2}hr \cdot \frac{2\pi r}{3} = \frac{\pi r^2 h}{3}.$$

$$\frac{\int_0^r -\frac{h}{r}x^2 + hx\,dx}{\frac{1}{2}hr}$$

$$\frac{-\frac{h}{r}\frac{x^3}{3} + h\frac{x^2}{2}\Big|_0^r}{\frac{1}{2}hr}$$

$$\frac{-\frac{h}{r} \cdot \frac{r^3}{3} + \frac{hr^2}{2}}{\frac{1}{2}hr}$$

$$\frac{-\frac{hr^2}{3} + \frac{hr^2}{2}}{\frac{1}{2}hr}$$

$$\frac{\frac{-2hr^2 + 3hr^2}{6}}{\frac{hr}{2}}$$

$$\frac{hr^2}{3 \cdot 2} \cdot \frac{2}{hr} \quad \frac{r}{3}$$

$$\frac{\int_0^r \frac{f(x)^2}{2}\,dx}{\frac{1}{2}hr} \rightarrow \frac{\int_0^r \frac{\left(-\frac{h}{r}x+h\right)^2}{2}\,dx}{\frac{1}{2}hr}$$

$$\frac{\int_0^r \frac{\left(\left(\frac{h^2}{r^2}\right)x^2 - 2\frac{h^2}{r}x + h^2\right)}{2}\,dx}{\frac{1}{2}hr}$$

Watch signs

$$\frac{\frac{\frac{h^2}{r^2}x^3}{3} - \frac{2h^2}{r}\frac{x^2}{2} + h^2x \Big|_0^r}{\frac{2}{\frac{1}{2}hr}}$$

$$\frac{\frac{\frac{h^2}{\cancel{r^2}}\frac{r^{\cancel{3}}}{3} - \frac{2h^2}{\cancel{r}}\cdot\frac{r^{\cancel{2}}}{\cancel{2}} + h^2r}{2}}{\frac{hr}{2}}$$

Wow a lot
of algebra
-made a lot of mistakes
-I am knowing the rules
now - but still a bit
Unsure

$$\frac{\frac{h^{\cancel{2}}\cancel{r}}{3} - h^{\cancel{2}}\cancel{r} + h^{\cancel{2}}\cancel{r}}{\cancel{2}} \cdot \frac{\cancel{2}}{\cancel{hr}}$$

$$\frac{h}{3} - h + h$$

$$\frac{h}{3}$$

should $= \frac{h}{3}$

 Name: ______________________

5. Given a definite integral
$$\int_a^b f(x)\,dx,$$
let T_n be the *trapezoid* approximation with n intervals, M_n the *midpoint* approximation using n intervals, and S_{2n} the *Simpson's rule* approximation using $2n$ intervals. Prove that
$$\frac{1}{3}T_n + \frac{2}{3}M_n = S_{2n}.$$

Easiest to divide $[a,b]$ into $2n$ intervals:

Let $x_0 = a$, $x_{2n} = b$, $x_i = a + \frac{(b-a)i}{2n}$.

Then $T_n = \frac{b-a}{2n}\left(f(x_0) + f(x_{2n}) + 2\sum_{i=1}^{n-1} f(x_{2i})\right)$

$$M_n = \frac{b-a}{n}\left(\sum_{i=1}^{n} f(x_{2i-1})\right)$$

$$S_n = \frac{b-a}{6n}\left(f(x_0) + 4f(x_1) + 2f(x_2) + \dots + f(x_{2n})\right)$$
$$= \frac{b-a}{6n}\left(f(x_0) + f(x_{2n}) + 4\sum_{i=1}^{n} f(x_{2i-1}) + 2\sum_{i=1}^{n-1} f(x_{2i})\right)$$

$$\frac{1}{3}T_n + \frac{2}{3}M_N = \frac{1}{3}T_n + \frac{4}{6}M_n$$
$$= \frac{b-a}{6n}\left(f(x_0) + f(x_{2n}) + 2\sum_{i=1}^{n-1} f(x_{2i}) + 4\sum_{i=1}^{n} f(x_{2i-1})\right)$$

$$= S_n$$

 Name: ____________________

6. A tank contains 1000 L of brine (that is, salt water) with 15 kg of dissolved salt. Pure water enters the top of the tank at a constant rate of 10 L / min. The solution is thoroughly mixed and drains from the bottom of the tank at the same rate so that the volume of liquid in the tank is constant.

(a) Find a differential equation expressing the rate at which salt leaves the tank.

Let $s(t)$: amount of salt in kg. at time t.

$$\frac{ds}{dt} = -10 \text{ L/min} \cdot \frac{s(t) \text{ kg}}{1000 \text{ L}} = -\frac{s(t)}{100} \text{ kg/min}$$

(b) Solve this differential equation to find an expression for the amount of salt (in kg) in the mixture at time t.

Use separation of variables.

$$\frac{ds}{s(t)} = \frac{-1}{100} dt \to \text{integrate: } \ln(s(t)) = -\frac{1}{100}t + C$$

Exponentiate both sides:

$$s(t) = k \cdot e^{-1/100 t}$$ where we've written $e^C = k$ for some constant k.

$s(0) = 15$, so $k = 15$

$$\boxed{s(t) = 15 e^{-1/100 t}}$$

(c) How long does it take for the total amount of salt in the brine to be reduced by half its original amount? (Recall $\ln 2 \approx .693$.)

We need $e^{-1/100 t} = 1/2$

$$\ln(e^{-1/100 t}) = \ln(1/2)$$

$$1/100\, t = \ln 2$$

$$t = 100 \cdot \ln 2 \approx \boxed{69.3 \text{ minutes}}$$