① - will pick 15 from these 25 qu and change #

18.01 Final Review

12/10

Calc exam 1 $\boxed{\text{Redoing}}$

1. f' for $\frac{x}{1-x^2}$ — easier for this

- can quotient or product

$x(1-x^2)^{-1}$

$1(1-x^2)^{-1} + -1(1-x^2)^{-2} \cdot -2x \cdot 1$ — make sur — forgot $-1-1 = -2$

Chain rule

$\frac{1}{1-x^2} + \frac{2x}{(1-x^2)^2}$

~~note same denom~~

$\cancel{\frac{1+2x}{1-x^2}}$

$\frac{1-x^2}{(1-x^2)^2} + \frac{2x}{(1-x^2)^2}$

factor and then cross out

$\frac{1-x^2+2x}{(1-x^2)^2} = \frac{2x}{(1-x^2}$

so easy

②

1. again $\frac{x}{1-x^2}$

$x(1-x^2)^{-1}$

$1 \cdot (1-x^2)^{-1} + -1(1-x^2)^{-2} \cdot -2\cancel{x} \cdot 1$ Small copy error

$\frac{1}{1-x^2} + \frac{2x^2}{(1-x^2)^2}$

get same denom

$\frac{(1-x^2)}{(1-x^2)^2} + \frac{2x^2}{(1-x^2)^2}$

$\frac{1-x^2+2x}{(1-x^2)^2} = \frac{2x}{1-x^2}$ ✓ ← factor then cross out

~~Did the top wrong~~ ~~misread ans~~ ~~[illegible]~~

can't just cross out duh

1. again quotient rule

$\frac{x \cdot -2x - [(1-x^2) \cdot 1]}{(1-x^2)^2}$ wrong way

$\frac{-2x^2 - 1 + x^2}{(1-x^2)^2}$

$\frac{-1 - x^2}{(1-x^2)^2}$

low de high − high de low / low low

$\frac{(1-x^2) \cdot 1 - [x \cdot -2x]}{(1-x^2)^2}$

$\frac{1-x^2 + 2x^2}{(1-x^2)^2}$

$\frac{1+x^2}{(1-x^2)}$ ← factor then cross out

and I can't even do this righ!

③ try #~~1~~ 1b

$\ln(\cos x) - \frac{1}{2}\sin^2 x$

take deriv

- hard
- did not do right

actually kinda easy

- chain rule

$\frac{1}{\cos x} \cdot -\sin x - \frac{1}{2} \cdot 2\sin x \cdot \cos x$

$-\frac{\sin x}{\cos x} - \sin x \cdot \cos x$

$-\tan x - \sin x \cos x$ ✓

before did not know $\frac{d}{dx} \ln x = \frac{1}{x}$

c) $f^{(5)}(x)$ for xe^x

- do this one carefully

$f'x = 1 \cdot e^x + e^x x$

$f'' = e^x + e^x + xe^x$

$f''' = e^x + e^x + e^x + xe^x$

$f'''' = ex + e^x + e^x + e^x + xe^x$

$f^5 = ex + ex + ex + ex + ex + xex$

$5ex + xex$ ✓ got it last ~~it~~

(4) #2

Find tangent line

$x^{2/3} + y^{2/3} = 4$ at $(-\sqrt{27}, 1)$

- take deriv
- eval at that pt
- but both x and y
- implicit differentiation?
  - how did that go again?

Well first check that pt is on line

$(-\sqrt{27})^{2/3} + 1^{2/3} \overset{?}{=} 4$

$\sqrt[3]{27} + 1 = 4$

$3 + 1 = 4$ ✓

$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3} \times y' = 0$ ↓ differentiate too

Solve for y'

$\frac{2}{3}y^{-1/3}y' = -\frac{2}{3}x^{-1/3}$

$y' = \frac{-\frac{2}{3}x^{-1/3}}{\frac{2}{3}y^{-1/3}} = -\frac{2}{3} \cdot \frac{3}{2} = -\frac{6}{6} = -1$

$y' = \frac{-x^{-1/3}}{y^{-1/3}}$

$y = \frac{-\sqrt{y}}{\sqrt{x}}$

③ try #1 ~~a~~ 1b

$\ln|\cos x| - \frac{1}{2}\sin^2 x$

take deriv

- hard

actually kinda easy

- did not do right

- chain rule

$\frac{1}{\cos x} \cdot -\sin x - \frac{1}{2} \cdot 2\sin x \cdot \cos x$

$-\frac{\sin x}{\cos x} - \sin x \cdot \cos x$

$-\tan x - \sin x \cos x$ ✓

before did not know $\frac{d}{dx} \ln x = \frac{1}{x}$

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$f'x = 1 \cdot e^x + e^x x$

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$f''' = e^x + e^x + e^x + xe^x$

$f'''' = e^x + e^x + e^x + e^x + xe^x$

$f^5 = e^x + e^x + e^x + e^x + e^x + xe^x$

$5e^x + xe^x$ ✓ got it last time did for product rule parts

(4) TR

Find tangent line

$x^{2/3} + y^{2/3} = 4$ at $(-\sqrt{27}, 1)$

- take deriv
- eval at that pt
- but both x and y
- implicit differentiation?
  - how did that go again?

Well first check that pt is on line

$(-\sqrt{27})^{2/3} + 1^{2/3} \overset{?}{=} 4$

$\sqrt[3]{27} + 1 = 4$

$3 + 1 = 4$ ✓

y' differentiate ten

$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\, y' = 0$

Solve for y'

$\frac{2}{3}y^{-1/3}\, y' = -\frac{2}{3}x^{-1/3}$

$y' = \dfrac{-\frac{2}{3}x^{-1/3}}{\frac{2}{3}y^{-1/3}}$ $= -\frac{2}{3}\cdot\frac{3}{2} = -\frac{6}{6} = -1$

$y' = \dfrac{-x^{-1/3}}{y^{-1/3}}$

$y = \dfrac{-\sqrt{y}}{\sqrt{x}}$

(5)

~~Now plug pt on line~~

plug that in w/ real #

$$\frac{-\sqrt[3]{3^{3/2}}}{1} = \frac{3^{-\frac{1}{2}}}{1} = \frac{1}{\sqrt{3}}$$

Now on line

$$x - 1 = \frac{1}{\sqrt{3}}\left(x + \sqrt{2}y\right)$$

$$y = \frac{1}{\sqrt{3}}x + \sqrt{4} + 1$$

$$\boxed{y = \frac{1}{\sqrt{3}}x + 4}$$

making same mistakes as before
I remember implicit as being easy

Lecture 5

$$x^2 + y^2 = 1$$

$$2x + 2yy' = 0$$

$$2yy' = -2x$$

$$y' = \frac{-2x}{2y} = \frac{-x}{y}$$ ✓

So y'
and solve for that
and plug in #

(6) Particle $y = t^3 - 3t + 3$

distance traveled in $t = 3$

- ~~POSI~~ distance $= \Delta x$

Can't just

$3^3 - 3(3) + 3$

since may go back and forth

← Integral

$f' = vel$
$f'' = acc$

physics $d = \frac{1}{2}at^2$

$x_0 + X \cdot t$

no must look for max/min

- which you do by taking deriv and setting = to 0

- I got this wrong on test
- Brain is working in same way as before

break into parts

$3t^2 - 3$

$3t^2 - 3 = 0$

$3t^2 = 3$

$t^2 = 1$

$t = 1$

~~2nd deriv for 1 or~~

find up or down by plugging

7

$\cancel{3(.5)^2}$

$(.5)^3 - 3(.5) + 3$

$\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} - 1.5 + 3$

$\frac{1}{8} - 1.5 + 3$

$\frac{1}{2} + \frac{1}{8}$

$\frac{4}{8} + \frac{1}{8}$

$\frac{5}{8}$

+

2nd deriv

$3 \cdot 2 +$

$6 +$

$2^3 - 3(2) + 3$ ← think supposed to plug in to 1st deriv to find slope up/down

$8 - 6 + 3$

$2 + 3$

5

+

$3(.5)^2 - 3$

$\frac{3}{4} - 3$

−

$3(2)^2 - 3$

$12 - 3$

9

+

Xean

So now take from 0, 1 and 1, 3

$0^3 - 3(0) + 3$

3

2

$1^3 - 3(1) + 3$

$1 - 3 + 3$

1

$3^3 - 3(3) + 3$

$27 - 9 + 3$

$\cancel{33}$ 21

~~32~~ 20

~~34~~ 22

crappy math very stupid error

(8) #4 Product rule for deriv
+ prove

$f g' + g f'$

Use def to prove
- just copy proof I'm guessing
- actually did ok on this

- just copying now

$$\lim_{h \to 0} \frac{f(x+h)\, g(x+h) - f(x)\, g(x)}{h}$$

← yeah $\lim \frac{f(x+h) - f(x)}{h}$

expand - ? how

oh +1 -1

$$\lim_{h \to 0} \frac{f(x+h)\, g(x+h) - f(x)\, g(x+h) + f(x)\, g(x+h) - f(x)\, g(x)}{h}$$

So can split + factor

$$\lim_{h \to 0} \frac{f(x+h) \quad f(x)}{h} \; g(x+h) + f(x) \; \frac{g(x+h) - g(x)}{h}$$

Contineous + differentiable

$f' \quad g \quad + \quad f \quad g'$

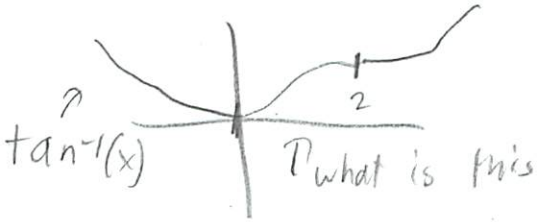
9 #5 Set of real # a, b, c where function differentiable

$\tan^{-1}(x) \quad x \le 0$

$ax^2 + bx + c \quad 0 < x < 2$

$x^3 - \frac{1}{4}x^2 + 5 \quad x \ge 2$

So want it to be continuous + no sharp breaks



= at $\tan^{-1}(0)$ ← what is??? calc says 0 ← oh can get from triangle

What angle to get 0

$= 0°$

$= (2)^3 - \frac{1}{4}(2)^2 + 5$

$8 - 1 + 5$

12

So $(0,0)$ $[2,12)$

need no right angles too

1st deriv need to be =

$\frac{d}{dx}\tan^{-1}(x) = ???$ $\frac{1}{x^2+1}$ need to memorize for integral too

$\frac{d}{dx} x^3 - \frac{1}{4}x^2 + 5 = 3x^2 - \frac{1}{4} \cdot 2x$

$3x^2 - \frac{1}{2}x$

$3(2)^2 - \frac{1}{2}(2)$

$12 - 1$

11

$\frac{1}{0^2+1} = 1$

(10)

so $(0,0)$ $(2,12)$

$d=1$ $d=11$

ax^2+bx+c

$2ax+b$

$0=a0^2+b\cdot 0+c$

$\boxed{0=c}$

$12=a(2)^2+2b+c$

$12=4a+2b$

$1=2a\,0+b$

$\boxed{1=b}$

$11=2a(2)+b$

$11=4a+1$

$10=4a$

$\boxed{a=\frac{10}{4}=\frac{5}{2}}$

✓ awsome

did this on my own (except for looking something up) and solved systems

(11) #6 Last qu

spent 2hrs on this

f satisfies $f(x+y) = f(x) + f(y) + x^2y + xy^2$

for all real # x y

$$\lim_{x \to 0} \frac{f(x)}{x} = 1$$

a) Find $f(0)$ ↑ what does this mean

have no clue on this one

$x = 0$

$y = 0$

Since $x \to 0$ $\lim_{x \to 0} \frac{f(x)}{x} = 1$

$f(x) \to x \to 0$ as $x \to 0$

$f(0) = 0$

b) Find f' ↑ so $f(x) = x$??

$x' = 1$ as $x \to 0$ so $f'(0) = 1$

if $f(x) = 0$

$\lim \frac{f(x)}{x} = ?$

$\lim \frac{f(x) - f(0)}{x - 0} = f'(0)$ use definition

(12)

c) $F'(x)$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\lim_{h \to 0} \cancel{f(x)} + f(h) \quad \text{circle 41} \quad + x^2 h + xh^2 - \cancel{f(x)}$$

$$\lim_{h \to 0} \frac{f(h)}{h} + x^2 + xh$$

$$1 + x^2 \quad 0$$

$$x^2 + 1$$

I don't get this at all
Still don't

(13) Test #2

1. ~~Test~~ quadratic approx

~~$f(x) + f'(x)$~~

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$$

- power series
- need to memorize

$2!$

$\underline{x-a}$

2. approx $\ln(1.2)$ ← ~~well is $\frac{1}{1.2}$ since that's~~ wrong

- what is a again?
- pick approx

$a = 0$

$\ln 1.2 = \ln(a)$

$f' = \frac{1}{1.2} x^{-1}$ ↓ normal

$f'' = \cancel{\ln(1.2)} \; -\frac{1}{x^2}$

not integrating

$\cancel{\ln(1.2)}$

$$\frac{1}{1.2} = \ln(1.2) + \frac{1}{1.2}($$

what is it w/ approx?

Yeah 1 is the easy so call it a

x is what you want ← why you do approxs

$$\ln(1.2) = \ln(1) + \frac{1}{1}(1.2-1) + \frac{1}{2}(1.2-1)^2$$

$$0 + .2 + \cancel{0} \; \frac{1}{2} \cdot .04$$

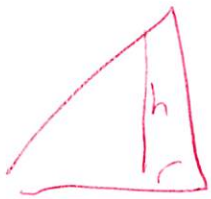
~~.24~~ .18

(14.) 2. Salt poured 30 ft³/min

Conical pile w/ circular base $h = d$

How fast height increasing when pile = 10 ft

oh related rates forgot this one



$h = 2r$

$V = \frac{1}{3}\pi r^2 h$

$\frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h$

$\frac{1}{3}\pi\frac{h^2}{4}h$

$V \quad \frac{1}{12}\pi h^3$, <u>call together</u>

differentiate

$\rightarrow \frac{dV}{dt} = \frac{1}{4}\pi h^2 \frac{dh}{dt}$

rearrange $\frac{dh}{dt} = \frac{dV}{dt}\left(\frac{4}{\pi h^2}\right)$

↑ change in height =

plug in

$$30\left(\frac{4}{\pi(10)^2}\right) = \frac{120}{100\pi} = \frac{12}{10\pi} \text{ ft/min}$$

- So many types of related rates
- need to review concept

(15) Related Rates Review
Lecture 12

- Something is changing $\frac{D}{dx}$
  - speed
  - or volume being added

find the speed of something else

Lots of situations
- like cone heigh's + widths

do different problems

---

3. Draw $f(x) = x - 3x^{1/3}$
- local min max
  ↑ ↓
  asymptotes
  Inflection pt

So this is a ~~com~~ long one

$f' = 1 - 3 \cdot \frac{1}{3} x^{-2/3}$

$1 - x^{-2/3}$

$f'' = -1 \cdot -\frac{2}{3} x^{-5/3}$

$\frac{2}{3} x^{-5/3}$

(16) Find where deriv = 0

$$1 - x^{-2/3} = 0$$

$$-x^{-2/3} = -1$$

$$x^{-2/3} = 1$$

$$x = 1$$

$x = -1$? Calc! yes

$1 - (-2)^{-2/3}$

$1 - \frac{2^{1/3}}{2}$

(+)

$1 - 0^{-2/3}$

(+) try like .25

(−) −.25

0 often special

$1 - (2)^{-2/3}$

$1 - .6$

(+)

? why so wrong

Oh simple math error

- duh

find where 2nd deriv = 0

$$\frac{2}{3} x^{-5/3} = 0$$

$\cdot \frac{3}{2}$ $\cdot \frac{3}{2}$ ✓

$$x = \pm 1$$

pratice $\sqrt[3]{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2}$

can't really do w/o calc

- ? how know +?

↓

$\frac{2}{3}(-2)^{-5/3}$

(−) (↓)

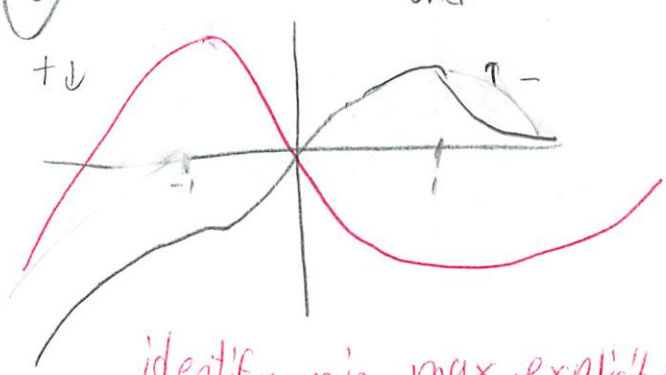
$\frac{2}{3}(0)^{-5/3}$

und

$\frac{2}{3}(2)^{-5/3}$

(↑) (+)

also do normal intercepts



these go wrong quickly

identify min max explicitly

(17) Metal storage tank

Cylier

Circular base

hemispherical top

dimesions least SA

max volume

SA = know dimensions $\frac{1}{2}(4\pi r^2) + 2\pi r h + \pi r^2 = 3\pi r^2 + 2\pi r h$

V = same know $= \frac{1}{2}\left(\frac{4}{3}\pi r^3\right) + (\pi r^2)h = \frac{2}{3}\pi r^3 + \pi r^2 h$

Note: $h = \dfrac{V - 2/3\pi r^3}{\pi r^2}$ ← relate ;

know write out SA

$$SA = 3\pi r^2 + 2\pi r\left(\frac{V - 2/3\pi r^3}{\pi r^2}\right) = 3\pi r^2 + \frac{2V}{r} - \frac{4}{3}\pi r^2$$

$$= \frac{5}{3}\pi r^2 + 2\frac{V}{r}$$

$$\frac{d\,SA}{dr} = \frac{10}{3}\pi r + (-1)\frac{2V}{r^2}$$

$$\frac{5}{3}\pi r = \frac{V}{r^2}$$

not concentrating anymore

Skip for now

- basically find max

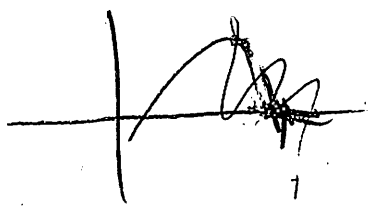
(18) Explain why Newton's method fails

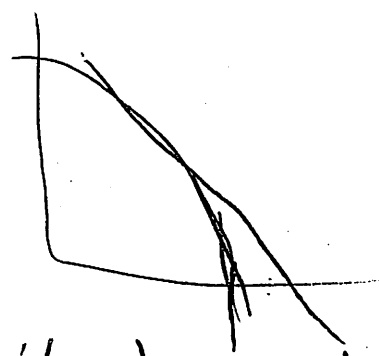
What is Newton's method again

- finding successively better roots ($y=0$) x axis

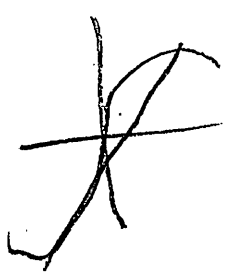
$$x_0 - \frac{f(x_0)}{f'(x_0)}$$

can lead to it or not





so it fails when it bounces back + forth



WP when fails

- bad starting pt
- iteration pt stationary
- yeah ossilates

5 $x^3 - 3x + 7$ $x_1 = 2$

$f' = 3x^2 - 3$

$$2 - \frac{(2)^3 - 3(2) + 7}{3(2)^2 - 3}$$

$$2 - \frac{8 - 6 + 7}{9}$$

(19)

$2 - 1 = 1$

$$1 - \frac{(1)^3 - 3(1) + 7}{3(1)^2 - 3}$$

$$1 - \frac{1 - 3 + 7}{3 - 3}$$ ← divide by 0

will get you to a pt where you must divide by 0

~~is deriv horiz~~

~~$\frac{\text{fun}}{\text{deriv}}$~~

close

slope ~~m~~ at $x = 1$ ← know that, visualize what you are doing

is horiz

does not intersect x axis

can't cont line

If I did not really learn it earlier will be confused on Fast passed review

-phy 5 topics so easier to re learn

(20) #6 "fun" problem

Prove that $\sqrt{1+x} < 1 + \frac{1}{2}x$ if $x > 0$

? how – +

taking deriv does not help

- but if in that chap is something related to that

- ~~does it~~ does <u>not</u> $= \sqrt{1} + \sqrt{x}$

I did not get before

tried table

tries MVT $\frac{f'(b) - f'(a)}{b-a} = f'(c)$

said similar to p-set problem

$g(x) = 1 + \frac{1}{2} - \sqrt{1-x}$ w/s $g(x) > 0$ $[0, w]$

$g'(x) = 0 + \frac{1}{2} - \frac{1}{2} \frac{1}{\sqrt{1-x}}$

$\frac{1}{2} - \frac{1}{2\sqrt{1+x}}$

~~find g(0) 1+0 1 = 0~~

$f(w) = f(0) + f'(c)(w-0)$

$c \in (0, w)$

$0 + f'(c) - w$

$\uparrow$ (+)

$\frac{1}{2}\left(1 - \frac{1}{\sqrt{1+x}}\right)$

for $x > 0$ $\sqrt{1+x} > 1$

So $1 > \frac{1}{\sqrt{1+x}}$

so (+) as well

(21) Exam #3

Compute area b/w curves

$x = y^2 - 4y$

$x = 2y - y^2$

What is the formula again?
I remember it was fairly easy

Subtract one from the other
and find the area of that

$y^2 - 4x - [2y - y^2]$

$2y^2 - 6y$

$\int 2y^2 - 6y \, dx$

$2\frac{y^3}{3} - 6\frac{y^2}{2}$

$\frac{2}{3}y^3 - 3y^2$

well it is definite. Need to find intersection pt
so set = to

23

$y^2 - 4y = 2y - y^2$

$+y^2 \quad +4y \quad +4y \quad +y^2$

$2y^2 = 6y$

what suppsed to do?

just do it

$y^2 = 3y$

$\sqrt{} \quad \sqrt{}$

$y = \pm\sqrt{3y}$

$\pm\sqrt{3}$

$3^2 = 3$

$y = 3$

$\cancel{y = -3}$ ⊗

$x = 0$

also

← * what is y where they are = *

Now find x for that

$x = 2(3) - (3)^2$

$6 - 9$

-3

don't need

$(3, 3)$

$\int_0^3 - - - -$

$\frac{2}{3}y^3 - 3y^2 \Big|_0^3$

$\frac{2}{3}(3)^3 - 3(3)^2 - \left[\frac{2}{3}(0)^3 - 3(0)^2\right]$

$18 - 27$

$-9 \rightarrow 9$

Seemed to have solved the algebra problem but not really mem trig and knowing context and not as many mental math errors

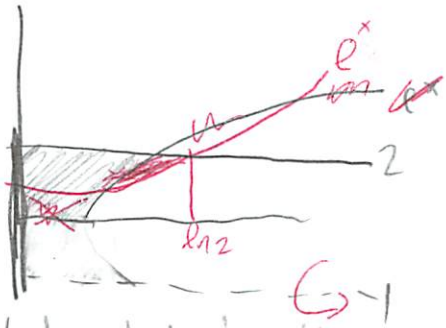
(23) Find the volume of solid from revolving

$y = e^x$

$y = 2$ about $y = -1$

$x = 0$

This should be fairly easy



but what is it again for empty space? (the -1)

So this is suptracting again

$$\int_0^2 (2 - e^x)\, 2\pi$$

why not 2π? - unsure of before

$$\int_0^{\ln 2} \pi\left(3^2 - (1 + e^x)^2\right) dx$$

$2 - -1$ $\quad$ $e^x - -1$ subtract each part

← wasters

(24) 3. Evaluate

$$\lim_{n \to \infty} \sum_{i=1}^{n} \left(1 + i\,\frac{3}{n}\right)^2 \frac{3}{n}$$

what is i?

I Did not get this one at all before and still have no clue what it is asking

Ok about integral definition

[0,3] into n parts

function $(1+x)^2$

Right Riemann sums

annoyed at all of these different methods

$$\sum_{i=1}^{n} \left(1 + i\,\frac{3}{n}\right)^2 \frac{3}{n}$$

just need to recognize it

$$\int_0^3 (1+x)^2\,dx$$

$u = 1+x$

$du = 1\,dx$

$$\int_0^3 u^2\,dx$$

$$\frac{u^3}{3}\Big|_0^3$$

$$\frac{(1+x)^3}{3} \to \frac{(1+(3))^3}{3} - \frac{(1+0)^3}{3}$$

$$21\tfrac{1}{3} - \tfrac{1}{3}$$

$$21$$

$$\begin{array}{r} 16 \\ \times\ 4 \\ \hline 64 \end{array} \qquad 3\overline{)64}\ \ 21\tfrac{1}{3}$$

(25) b. value $f(4)$ for f satisfies

$$x \sin \pi x = \int_0^{x^2} f(t)\,dt$$

- so what is it asking?
- got a 0 on this before

take deriv of both sides

$$1 \cdot \sin \pi x + x \cos \pi x \cdot \pi = f(x^2) \cdot 2x$$

$$f(x^2) = \frac{1}{2x} \sin \pi x + \frac{\pi}{2} \cos \pi x$$

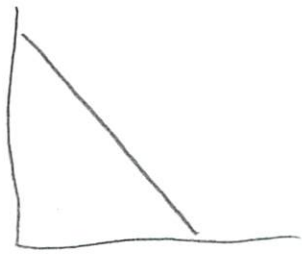
$$f(4) = f(2^2) = \frac{1}{2(2)} \sin \pi(2) + \frac{\pi}{2} \sin(2)\pi$$

$$\frac{\pi}{2}$$

stupid tricks

4. Find the centeroid - center of mass of right triangle

- got this wrong



How were we supposed to do this

a HW qu

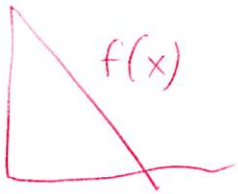
con

$$\frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i} \to \lim_{n \to \infty} \frac{x_i \overbrace{f(x_i)\,\Delta x}^{\text{mass}}}{f(x_i)\,\Delta x}$$

(20) $\int x f(x)\, dx$

mass = area

Integral way Briner Office Hrs



$$\frac{\Delta y}{\Delta x} = \frac{h-0}{0-r} = \frac{-h}{r}$$

$$y = mx + b$$

$$h = \frac{-h}{r} \cdot 0 + b$$

$$0 = \frac{-h}{r} \cdot r + b$$

$$y = \frac{-h}{r} x + h$$

I am always bad at these long ones

$$x)\ \frac{\int_0^r x\left(\frac{-h}{r}x + h\right) dx}{\frac{1}{2} hr}$$

$$y)\ \sum \frac{m_i y_i}{m_i} = f(x_i)\Delta x \cdot \frac{f(x_i)}{2}$$

$$\int_0^r \frac{f(x)}{2} \cdot f(x)\, dx$$

integrate

27

x) $\dfrac{\int_0^r \frac{-h}{r}x^2 + hx\,dx}{\frac{1}{2}hr}$

$\dfrac{-\frac{h}{r}\frac{x^3}{3} + \frac{hx^2}{2}\Big|_0^r}{\frac{1}{2}hr}$

$\dfrac{-\frac{h}{r}\cdot\frac{r^3}{3} + \frac{hr^2}{2}}{\frac{1}{2}hr}$

y) $\dfrac{-\frac{hr^2}{3} + \frac{hr^2}{2}}{\frac{1}{2}hr}$

$\dfrac{\frac{-2hr^2 + 3hr^2}{6}}{\frac{hr}{2}}$

$\dfrac{hr^2}{6}\cdot\dfrac{2}{hr}$

$\dfrac{r}{3}$

no clue on this one

(28) b) Use pappus theorm to find vol of cone

- what is Pappus origin?
- remember I liked it

~~Ex.~~ From wp

- oh just the rotate it around to get 2π

$$\frac{1}{2} hr \cdot \frac{2\pi r}{3} = \frac{\pi r^2 h}{3}$$

review

5. The trapizodal problem

- just remember the formulas
- really screwed this up on test
- but nailed it on makeup

Just know the formulas

$$\cancel{T_n = \frac{b-a}{2n}\left(f(x_0) + \right.}$$

put them in later

trapizodal $\frac{b-a}{n}\left(\frac{1}{2} y_0 + y_1 + y_2 + y_3 + \frac{1}{4} y_4\right)$

Simpson $\frac{b-a}{n}\left(\frac{1}{6} y_0 + \frac{4}{6} y_1 + \frac{2}{6} y_2 + \frac{4}{6} y_3 + \frac{1}{6} y_4\right)$

(29) #6 This was the hard related rates one

- ~~do later~~ So tank 1000 L brine

15 kg salt

↷ 10 L/min

drains at same rate, constant volume

$$\frac{ds}{dt} = -10\ L/min \cdot \frac{s(t)}{1000}\ \frac{kg}{L} = -\frac{s(t)}{100}\ kg/min$$

$s(t)$ = amt salt time t

- ∴ so losing 10 L of salt each min

think of salt in blocks - can't leave box

1 box drops out each min

Overall concentration ↓

- I thought concentration changes in time - not true - simpler than I think

#6b Solve the differential equation for amt salt at time t

Differential equation Review

Lesson 16

$$\frac{dy}{dx} = f(x)$$

$$y = \int f(x)$$ ← just definition of integral

Lecture 16

(30) Example #2 $\left(\frac{d}{dx} + x\right) y = 0$

~~Solve~~

Seperate variables

$$\frac{dy}{dx} + xy = 0$$

$$\frac{dy}{dx} = -xy$$

$\cdot dx \quad /y$

$$\frac{dy}{y} = -x\,dx$$

anti deriv both sides

$$\int \frac{dy}{y} = \int -x\,dx$$

$$\ln y = -\frac{x^2}{2} + C$$

exponent ← don't forget!

$$|y| = e^{-x^2/2} * e^{c}$$

↑ multiply - some reason

$$y = a e^{-x^2/2}$$

↑ replace w/ a

In general $\frac{dy}{dx} = f(x)\,g(x)$

$$\frac{dy}{g(x)} = f(x)\,dx$$

$$h(y)\,dy = f(x)\,dx \quad \text{where } h(x) = \frac{1}{g(y)}$$

(31) Back to problem already in progress

Solve $\frac{ds}{dt} = \frac{-s(t)}{100}$

Find amt salt at ~~m~~ t

ie ~~the~~ solve for $-s(t)$

Get all on lside

$$\frac{ds}{dt} = \frac{-s(t)}{100}$$

$\cdot dt$

$$ds = \frac{-s(t)}{100}\, dt$$

$\cdot 100 \qquad \cdot 100$

$$100\, ds = -s(t)\, dt$$

$$\frac{1\, ds}{-s(t)} = \frac{dt}{100} \quad ✓$$

$$\int \qquad \int$$

$\frac{1}{100}$ is a constant

$$-\ln(s) = \cancel{\ln(100)} + c$$

wrong

$\frac{1}{100} t + c$

$\cancel{\ln(s)}$

$$s(t) = \cancel{-100 + c} \quad k \cdot e^{-1/100 t} + c$$

↑ put constant out in front

Solve for 0

$\cancel{s(0) = k \cdot e^{0}}$ ← from problem

$\cancel{k} = \cancel{k}\ 15$

$$s(t) = 15 \cdot e^{-1/100 t}$$

grr - get it now - just have to execute right

(32) Exam 4 - most recent

1. Integrate

$$\int_1^4 \sqrt{t}\,\ln t\,dt$$

- ? by parts

$u = \ln t \qquad dv = \sqrt{t} = t^{1/2}\,dt$

$du = \frac{1}{t}\,dt \qquad v = \frac{2}{3}t^{3/2}$

$uv - \int v\,du$ ← remember this right

$\ln t \cdot \frac{2}{3}t^{3/2} - \int \frac{2}{3}t^{3/2} \cdot \frac{1}{t}\,dt$

← t^{-1} - add

ok got this wrong last time

$\ln t \cdot \frac{2}{3}t^{3/2} - \int \frac{2}{3}t^{1/2}$

$\ln t \cdot \frac{2}{3}t^{3/2} - \frac{2}{3}\cdot\frac{2}{3}t^{3/2}$

* $3 \cdot 3 = 9$ not 6!

$\ln t \cdot \frac{2}{3}t^{3/2} - \frac{4}{9}t^{3/2} \Big|_1^4$

$\ln 4 \cdot \frac{2}{3}(4)^{3/2} - \frac{4}{9}(4)^{3/2} - \left[\ln 1 \cdot \frac{2}{3}(1)^{3/2} - \frac{4}{9}(1)^{3/2}\right]$

↑ $4 \cdot 4 \cdot 4$

↑ 0? yeah ✓

$\ln 4 \cdot \frac{2}{3} \cdot 8 - \frac{4}{9} \cdot 8 - 0 - \frac{4}{9}$

just add across!

still in play

$\frac{16}{3}\ln 4 - \frac{32}{9} - \frac{4}{9}$

$\frac{16}{3}\ln 4 - \frac{28}{9}$ ← almost

$\frac{16}{4}$

$\sqrt{64}$

$8 \cdot \frac{2}{3}$

(33) #2

Compute

$$\int_0^{\pi/4} \tan^4 \sec^6\theta \, d\theta$$

$\frac{\cos^2}{\cos^2} + \frac{\sin^2}{\cos^2} = \frac{1}{\cos^2}$

$1 + \tan^2 = \sec^2$

$du \tan = \sec^2$

$du \sec = \tan \sec$

$u = \tan$

$du \sec^2$

$$\int_0^{\pi/4} u^4 \sec^4\theta \, 2u$$

$$\int_0^{\pi/4} u^4 (1+u^2)$$

I remember the tricks on this one :)
- good memory

missed stuff

modify too

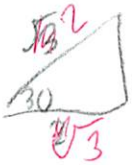
$$\int_0^{\pi/4} u^4 + u^6$$

$u = \tan 0 = 0$

$\tan \frac{\pi}{4} = 1$

$$\int_0^1 \frac{u^5}{5} + \frac{u^7}{7}$$

$$\frac{\tan^5\theta}{5} + \frac{\tan^7\theta}{7}$$



$$\frac{\tan^5(1)}{5} + \frac{\tan^7(\theta)}{7}$$

done on get this wrong - embarassing

expand inside

$$\int_0^1 u^4 (1 + 2u^2 + u^4)$$

$$\int_0^1 u^4 + 2u^6 + u^8$$

$$\frac{u^5}{5} + \frac{2u^7}{7} + \frac{u^9}{9} \Big|_0^1$$

(34) $\frac{\tan^5\theta}{5} + \frac{2\tan^7\theta}{7} + \frac{\tan^9\theta}{9} \Big|_0^1$

$\cancel{\frac{\tan^5 1}{5} + \frac{2\tan^7 1}{7} + \frac{\tan^9 1}{9}}$ $\left[\frac{\tan^5 0}{5} + \frac{2}{7}\tan^7 0 + \frac{\tan^9 0}{9}\right.$

how solve this

they don't plug tan in

why??

just $u = 1$

is that standard procss for when changing u?

doing this test much better since most recent

(35)

$$\int \frac{10}{(x-1)(x^2+9)}$$

$$\frac{10}{(x-1)(x^2+9)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+9}$$ since $-x^2$

$$\frac{10}{(x^2+9)} = A + \frac{(Bx+C)(x-1)}{x^2+9}$$

what x so this = 0 At $x=1$

$$\frac{10}{x-1} = \frac{A(x^2+9)}{(x-1)} + Bx+C$$

~~x=3~~

no x can make this = 0

$$\frac{10}{(1^2+9)} = A + \cancel{(B\cdot 1+C)}\ 0$$

$$\boxed{A=1}$$

$$10 = A(x^2+9) + (Bx+C)(x-1)$$

$$10 = Ax^2+9 + Bx^2 - Bx + Cx - C$$

$x^2)$ $0 = A + B$

$x)$ $0 = -B + C$ = $0 = -B + -1$

$1)$ $10 = 9 - C$ $\quad$ $+1 = -B$

$1 = -C$ $\quad$ $\boxed{B=-1}$

$\boxed{C=-1}$

Something is wrong ~~remember~~ Fixed

(36)

$$\int \frac{1}{x-1} + \int \frac{-1x-1}{x^2+9}$$

$$\int \frac{1}{x-1}\,dx + \int \frac{-x}{x^2+9}\,dx + \int \frac{-1}{x^2+9}\,dx$$ now integrate right

↑ screwed that up so what was right

$(x-1)^{-1}$

$v = x^2+9$

$dv = 2x\,dx$

$-\frac{1}{2}dv = -x\,dx$

$\ln(x-1) + \int \frac{1}{u} - \frac{1}{2}du$ $\quad - \cancel{\ln(x^2+9)}$

duh the tan thing

$-\frac{1}{2}\int \frac{1}{u}$ $\quad -\frac{1}{3}\int \frac{1}{\frac{x^2}{3}+3}$

$-\frac{1}{2}\ln(x^2+9)$ $\quad -\frac{1}{3}\tan^{-1}(x/3)$

$\ln(x-1) - \frac{1}{2}\ln(x^2+9) \cancel{-\ln(x^2+9)}$

$-\frac{1}{3}\tan^{-1}(x/3)$

✓ ✓ ✓ now

- just remember the trick
- had gotten this before

(37) 4. Did not get this at all

$$\int \frac{1}{(5-4x-x^2)^{5/2}}\,dx$$

See how well I remember the solution

need to complete the square

$-x^2 - 4x + 5$

$(-x-2)(x-2) + 9$

~~→ now want it to be~~

i want it to be 1

$-(x+2)^2 + 9$

↑ move − sign out front

(Factor out −)

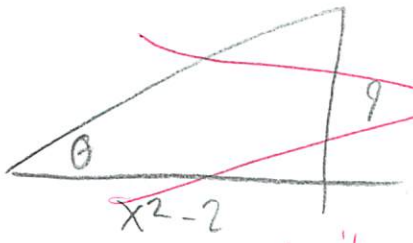
$$\int \frac{1}{(-(x+2)^2+9)^{5/2}}$$

now what?

↑ some sort of trig identity

− but what

+ I can do one word: triangle



$\tan\theta = \frac{9}{x^2-2}$

$(x^2-2)\tan\theta = 9$

write as

$$\int \frac{1}{((9-(x+2)^2)^{5/2}}$$

$\sin\theta = \frac{x+2}{3}$

$3\sin\theta = x+2$

if have opp x right

(38)

$x^2 + 2 = 3\sin\theta$

$x = 3\sin\theta - 2$

$dx = 3\cos\theta \, d\theta$

$$\int \frac{3\cos\theta \, d\theta}{(3^2 - x^2)^{5/2}}$$

not yet

factor out 9

$$\int \frac{3\cos\theta \, d\theta}{9^{5/2}(1 - \sin^2\theta)^{5/2}}$$

really complex problem

- no one could have solved this

$$\frac{1}{81}\int \frac{\cos\theta}{(\cos^2)^{5/2}}$$

$$\frac{1}{81}\int \frac{\cancel{\cos\theta}}{\cos^{5}\theta}$$

$$\frac{1}{81}\int \frac{1}{\cos^4\theta}$$

$$\frac{1}{81}\int \cos^{-4}\theta$$

$$\frac{1}{81}\int \sec^4\theta$$

split again

$$\frac{1}{81}\int (1 + \tan^2\theta)(\sec^2\theta)\, d\theta$$

$u = \cancel{\sec^2\theta}$ tan

$du = \sec^2$

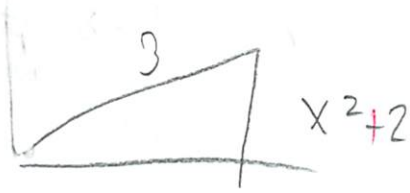
$$\frac{1}{81}\int (1 + u^2)\, du$$

(39)

$$\frac{1}{81}\, u + \frac{u^3}{3}\, du$$

$$\frac{1}{81}\left(\tan\theta + \frac{\tan^3\theta}{3}\right)$$

back to triangle



need to find this
oh just basic
$9-(x^2-2)^2$

$$\tan\theta = \frac{x^2-12}{9-(x^2-2)^2}$$

$$\frac{1}{81}\left(\frac{x^2+2}{9-(x^2-2)^2} + \frac{\left(\frac{x^2+2}{9-(x^2-2)^2}\right)^3}{3}\right)$$ and distribute

well can ~~factor~~
expand this

$$\frac{1}{81}\left(\frac{x^2-2}{\sqrt{5-4x-4x^2}} + \frac{(x^2-2)^3}{\sqrt{5-4x-4x^2}^{\,3}}\right) + C$$

boy made a bunch
of mistakes
— think I know this better now

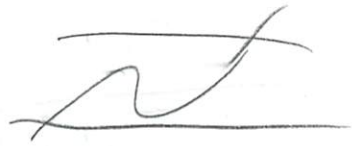
(40) 5a. Arc lenght

- always bad at geometry problems

$x = y + y^3$

from $y = 1$ to 4

$\int_1^4 (x + y^3)\, 2\pi$ ~~[illegible]~~ no radius



what is arc lenght formula again?

Oh this is parametric

$y = t$

$x = t + t^3$

? how did they do this?

$\int_1^4 \sqrt{x'^2 + y'^2}$

$\int_1^4 \sqrt{1 + 3t^2 + 1}$

5b. Surface area

- some but this is $2\pi \cdot$ radius

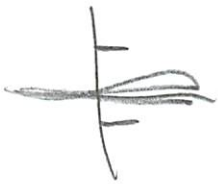
$x = a\cos^3 t$

$y = a\sin^3 t$

$0 \le t \le \frac{\pi}{2}$

- never really got parametric

About x axis



So radius $= y - a\sin^3 t$

(41)

$$2\pi \int_0^{\pi/2} a\sin^3 t \sqrt{(a\,3\cos^2 t \sin t)^2 + (a\,3\sin^2 t \cos t)^2}\,dt$$

can simplify more

$$2\pi \int_0^{\pi/2} 3a^2 \sin^3 t \sqrt{\sin^2 t \cos^2 t (\cos^2 t + \sin^2 t)}$$

factor

$$6a^2\pi \int_0^{\pi/4} \sin^4 t \cos t$$

And then problem #25 is the series example 12/15

- not in the mood at all for this

Given the function $f(x) = \square$ (

Find Taylor series for $f(x)$ in $x-a$ $a = \square$

- So what is a Taylor series
  - a function as an infinite sum
  - calc from values of derivative
  - use a finite # terms to apcoximate a function
  - the approximation I can never remember

how to learn the concepts too - using wf/online was helpful last time for physics

$$f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!}$$ ← almost ✓

x is actual
a is approx

← think of it this way

(42) Which is the same as

$$\boxed{\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n}$$

memorize

in this example

$f(x) = \sin x$

$a = 0$

*learn concepts

so find f'

f''

x is not defined just write in

- well approx at $a=0$

$$0 + \frac{1(x-0)^1}{1!} + \frac{0(x-0)^2}{2!} + \frac{-1(x-0)^3}{3!}$$

$$x - \frac{1}{3!}x^3 + \frac{x^5}{5!} - \frac{x^7}{7!}$$

extra credit that this also = (for sin x)

$$\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}(-1)^n$$

b) Now it asks for radius of convergency

- ie does it converge? for what x values?

(43) So use the ratio test

$$\lim_{n\to\infty} \left| \frac{b_{n+1}}{b_n} \right|$$

← fill in $(n+1)$

$$\left| \frac{x^{2(n+1)+1}}{(2(n+1)+1)!} \cdot \frac{(2n+1)!}{x^{2n+1}} \right|$$

$$\frac{x^{2n+2+1}}{x^{2n+1}}$$

$$|x^2| \frac{\frac{1}{(2n+3)!}}{1/(2n+1)!}$$

← then does that come ? the ÷ algebra?

$\frac{x^{2n+3}}{x^{2n+1}}$ oh I see

just write it out + do it

$$x^2 \frac{1}{(2n+3)(2n+2)}$$

Solve for x so that this is true for $x=0$

So set $=0$ - what x ~~DE~~ make this true?

<u>anything</u>

and note <1 so converges anywhere

So say $T(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = \sin x$

for all x

(44) Now prove

- how do you prove?
- oh w/ that formula

$$f(x) - T(x) = 0$$

$$f(x) - \lim_{k \to 0} T_k(x) = 0$$

$$\lim_{k \to 0} \underbrace{(f(x) - T_k(x))}_{R_k(x)} = 0$$

$$R_x(x) \leq \frac{M^{n+1}}{(k+1)!} |x-0|$$

for $|x-0| < d$

where $M \geq |f^{(k+1)}(x)|$ on $x \in (-d, d)$

So for $\sin x$, choose $M = 1$ (for any k, any x)

why in all world?

Max?

just pick 1 where this is true

approaches 0 as $n \to \infty$

so this does by squeeze theorm

I think that is it for this series

should do as pratice w/o looking

- and do it for all qu
- going to do pratice test

Last Lecture
Review

12/10

Talked to prof about past exams
- Has to talk to math office

OCW not to helpful

30 qu on past exams

He will pick 15 from them randomly

Series example will be on final last qu

$\boxed{}$ = hidden

- Given the function $f(x) = \boxed{}$

a) Find the Taylor series for $f(x)$ in $x-a$ $a = \boxed{}$

b) Find the radius of convergence for Taylor Series

c) Prove that $f(x) = T(x)$ ← taylor series in the interval of convergence

Example $f(x) = \sin x$
$a = 0$

a) $$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$ ← 1 pt, general

when $a = 0$

← compute this explicitly

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x-0)^n$$

$f(x) = \sin x \qquad f(0) = 0$

$f'(x) = \cos x \qquad f'(1) = 1$

$f''(x) = -\sin x \qquad f''(2) = 0$

$f'''(x) = -\cos x \qquad f'''(3) = -1$

then repeats every 4 terms

$$T(x) = 0 + \frac{1 \cdot (x-0)^1}{1!} + \frac{0\,(x-0)^2}{2!} + \frac{-1\,(x-0)^3}{3!} + \dots$$

can keep going

$$0 + x + 0 - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots$$

full credit (with ...)

extra

$$= \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}(-1)^n$$

b) Find the radius of convergence

– ratio test

$$\lim_{n\to\infty} \left| \frac{b_{n+1}}{b_n} \right|$$

$$\lim_{n\to\infty} \left| \frac{x^{2(n+1)+1}}{(2(n+1)+1)!} \cdot \frac{(2n+1)!}{x^{2n+1}} \right|$$

the $(-1)^n$ term does not matter

$$\lim_{n\to\infty} |x^2| \frac{1/(2n+3)!}{1/(2n+1)!}$$

$\leftarrow (2n+3)! = (2n+3)(2n+2)(2n+1)(2n) + \dots$

$(2n+1)!$

$$\lim_{n\to\infty} |x^2| \frac{1}{(2n+3)(2n+2)} < 1$$

$\leftarrow$ should be

Now solve for x so this is true

Notice $\lim_{n\to\infty} |x^2| \frac{1}{(2n+3)(2n+2)} = 0$ for any x

Do more than memorizing steps - know the math!

No matter the x value $\to$ converges
ie radius $= \infty$

$$T(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \quad \text{for all } x \quad \boxed{= \sin x}$$

hardest $\to$ part says Brubaker

c) Now prove $f(x) = T(x)$ for all x

- use Taylor's inequality
- want $f(x) - T(x) = 0$
- want $f(x) - \lim_{k\to 0} T_k(x) = 0$
- want $\lim_{k\to 0} \underbrace{\left(f(x) - T_k(x)\right)}_{R_k(x)} = 0$

- formula: $R_k(x) \le \frac{M}{(k+1)!} |x-0|$
  for $|x-0| < d$
  where $M \ge |f^{(k+1)}(x)|$ on $x \in (-d, d)$

just write

- for $\sin x$, choose $M=1$ (for any k, any x)

$$\lim_{k\to\infty} R_k(x) \le \lim_{k\to\infty} \frac{1}{(k+1)!} |x|^{k+1}$$

$= 0$ since factorials grow faster than a power

Will post e online so can see infinite solution

Could integrate $\sin x$ as integrating its polynomials
So people use as many terms as want of
taylor series to approx derivatives

Also works for complex #

$z = x + iy$ $\quad$ x, y real
$i = \sqrt{-1}$

e^{ix} = power series
$= \cos x + i \sin x$
$= 1 + (ix) + \frac{(ix)^2}{2!}$
$= 1 - \frac{x^2}{2!} + \ldots + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots\right)$

$e^{i\pi} = \cos \pi + i \sin \pi$
$-1 + 0$
$= -1$

→ take more math classes

Exam ≤ 15 questions

24 qu on past exams + 1 today
study all 25 qu
all those qu but w/ different #

All questions on

1. Definition of derivative ← limits of secants
2. Its applications - related rates, curve sketching
3. Definition of integrals ← Riemann sum
4. Its applications
5. Series

First 3-4 qu → take derivatives + integrals

Don't think as math as magic
Based in logic
Derive applications
Why things are true
Work as hard as possible to become a super genius
Is a week left!

18.01 EXAM IV

Thursday, December 3, 2009

Name: Michael Plasmeier

E-mail: mpleo2@mit.edu

Recitation Instructor (Circle One): C. Breiner / I. Losev / X. Ma / S. Ramakrishnan / B. Rhoades

Recitation Hour: _____

Instructions: You may not use calculators, notes, textbooks, or personal electronic devices. As a courtesy to other students, please turn off all cell phones. Read each question carefully. Whenever possible, include justification for your reasoning and show your work. Answers without any work or explanation will receive little or no credit. There are 5 questions and a 50 minute time limit on this exam. Good luck.

Question	Score	Maximum
1	3.5	5
2	1	5
3	3.5	5
4	0	7
5	2+1.5	6
Total	12	28

felt confident on this but did just like normal
- same plot on graph

 Name: _____

1. Compute the following integral:

$\frac{1}{t}$

$$\int_1^4 \sqrt{t}\,\ln t\,dt$$

by parts

$uv - \int v\,du$

$\int_1^4 t^{1/2} \ln t \, dt$

$u = \ln t$ $\qquad dv = t^{1/2}$

$du = \frac{1}{t}\,dt$ $\qquad v = \frac{2}{3}t^{3/2}$

$uv - \int v\,du$

$\ln t \cdot \frac{2}{3}t^{3/2} - \int \frac{2}{3}t^{3/2} \cdot t^{-1}\,dt$

$\frac{2}{3}t^{3/2}\ln t - \frac{2}{3}\int t^{+3/2}\,dt$ $\quad -1$

$\frac{2}{3}t^{3/2}\ln t - \frac{2}{3}\underbrace{t^{3/2}}_{\frac{3}{2}}$

← they add not multiply

← integrated wrong also

$\frac{2}{3}t^{3/2}\ln t - \frac{4}{9}\sqrt{t}^{3/2}$ $\qquad \frac{2}{3}\cdot\frac{2}{3} = \frac{4}{9}$ ← multiply across!

practice more by parts

$\frac{2}{3}(4)^{3/2}\cdot\ln(4) - \frac{4}{9}\sqrt{4}^{3/2} - \left[\frac{2}{3}(1)^{3/2}\cdot\ln(1) - \frac{4}{9}\sqrt{1}^{3/2}\right]$

$\frac{2}{3}\cdot 16 \cdot \ln 4 - \frac{8}{3} - \frac{2}{3}\ln(1) + \frac{4}{3}$

$4^{3/2} = 8$

-5

$\frac{32}{3}\ln 4 - \frac{28}{9} - \frac{2}{3}\ln 1$

prob know

$\boxed{\ln(1) = 0}$ ← how am I supposed to know

OPPS $4 \cdot 4 \cdot 4 \to \sqrt{\ }$

I took $\frac{1}{2}$ $\boxed{grd}$

almost had it
key was putting u^2
in and expanding

 Name: _____

2. Compute the following integral:

$$\int_0^{\pi/4} \tan^4\theta \sec^6\theta \, d\theta$$

$\frac{d}{dx}\tan = \sec^2$

$\frac{d}{dx}\sec = \sec\tan$

$\sin^2 + \cos^2 = 1$

$\tan^2 + 1 = \sec^2$

~~oh what did we say for evens - keep reducing till easy - cut in half somehow~~ no evens

$$\int_0^{\pi/4} \tan^4\theta \cdot \sec^4\theta \cdot \sec^2\theta \, d\theta$$

do this one

① $u = \sec\theta$ $\quad$ $v = \tan\theta$

$du = \sec\theta\tan\theta\, d\theta$ $\quad$ $dv = \sec^2\theta\, d\theta$

$v = \tan\theta$

$dv = \sec^2\theta$

④ $\int_0^{\pi/4} u^4 \sec^4 \, dv$

which method is it?

② $\int_0^{\pi/4} \tan^4\theta \sec^4\theta \, (\tan^2\theta + 1)$

$\int_0^{\pi/4} \tan^6\theta \sec^4\theta + \tan^4\theta \sec^4\theta$

- got more complex

$\int_0^{\pi/4} u^4 (\tan^2 + 1) \, dv$

key

almost

change to u^2

bounds

$\int_0^1 u^4 (1+u^2)^2 \, du$

have to change bounds

$\tan\frac{\pi}{4} = 1$

$\tan 0 = 0$

③ $\int_0^{\pi/4} \frac{\sin^4\theta}{\cos^4\theta} \cdot \frac{1}{\cos^6\theta}$

simple algebra

$\int_0^1 u^4(1 + 2u^2 + u^4) \, du$

$\int_0^1 u^4 + 2u^6 + u^8 \, du$

now integrate

$\frac{u^5}{5} + \frac{2u^7}{7} + \frac{u^9}{9} \Big|_0^1$

$\frac{1}{5} + \frac{2}{7} + \frac{1}{9} = \frac{188}{315}$

(1b) $\int_0^{\pi/4} u^2 \, du \, \sec^2\theta \tan^3\theta$

$u = \sec\theta$

$du = \sec\theta\tan\theta$

now what

$\int_0^{\pi/4} u^2 \, du \, \sec^2\theta \sec\theta \tan^3\theta$

$w = \tan\theta$

$dw = \sec^2\theta$

$\int_0^{\pi/4} u^2 \, du \, w^3 \, dw \sec\theta$

3. Compute the following integral:

$$\int \frac{10}{(x-1)(x^2+9)}\,dx$$

$$\frac{A}{x-1} + \frac{Bx+C}{x^2+9}$$

$$\frac{10}{(x^2+9)} = A + \frac{(Bx+C)(x-1)}{x^2+9}$$

$x=1$ $\quad$ $x=1$

$$\frac{10}{10} = 1 = A$$

$$\frac{10}{(x-1)} = \frac{A(x^2+9)}{(x-1)} + Bx+C$$

$x \neq$ anything

$$10 = 1(x^2+9) + (Bx+C)(x-1)$$

$$10 = x^2+9 + Bx^2 - Bx + Cx - C$$

2) $0 = 1 + B \leftarrow B = -1$

1) $0 = -B + C \leftarrow 0 = -1 + C \leftarrow C = 1$

0) $10 = 9 - C \leftarrow$ ✓ $10 = 9 - -1$ ✓

→ back

$$\int \frac{1}{x-1}\,dx + \int \frac{-1x}{x^2+9}\,dx + \int \frac{-1}{x^2+9}\,dx$$

$\int (x-1)^{-1}$

$\ln(x-1)$ +

$u = x^2+9$

$du = 2x\,dx$

$-\frac{1}{2}du = -x\,dx$

$-\frac{1}{2}\int \frac{du}{u}$

$-\frac{1}{2}\ln(x^2+1)$

$-\cancel{\int x^{-2}+9^{-1}}$

$\cancel{+\frac{x^{-1}}{-1} + \ln(9)}$

$\cancel{+\frac{1}{x} + \ln 9}$

may be illegal

$-\frac{1}{3}\tan^{-1}(x/3)$

$\ln(x-1) - \frac{1}{2}\ln(x^2+9)$ $\underbrace{\times \cancel{\frac{1}{x} + \ln 9}}_{\text{Wrong}}$ $+C$

I forgot $\frac{1}{x^2+1} = \tan^{-1}(x)$

but how to $\frac{1}{3}$?

divide 3

$$\frac{1}{\frac{x^2}{3}+3} = \frac{1}{x^2+1} = \tan^{-1}(x)$$

$$\frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)$$

How would you think about that

3.5

? should have been more

think got it

 Name: _____

Should have practiced more

Worth most pts

4. Compute the following integral:

Completely wrong

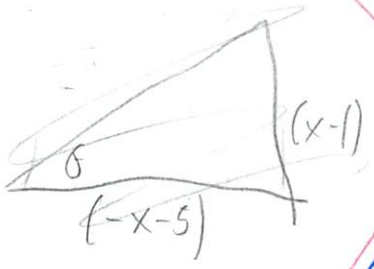
$$\int \frac{1}{(5-4x-x^2)^{5/2}}\,dx$$

$$\int \frac{1}{\sqrt{-x^2-4x+5}^{\,5}}$$

← ? triangle
no trig stuff, no a
but was no triangle qu
factor

$$\int \frac{1}{\sqrt{(-x-5)(x-1)}^{\,5}}$$

? no what



- what other methods
  - u-sub
  - by parts ← he said any function can do by
  - compound method

$$\frac{A}{x-5} + \frac{B}{x-1}$$

$u = -x^2-4x+5$

$du = -2x-4$

dV

$V =$ what does here?

$\frac{1}{u^{5/2}}$ ← where does du go

← Factor like this

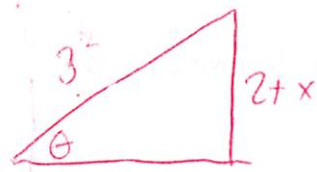
How we supposed to know

$$\int \frac{1}{\left(9-(2+x)^2\right)^{5/2}}\,dx$$

$x+2 = 3\sin\theta$

$x = 3\sin\theta - 2$

$dx = 3\cos\theta\, d\theta$



$\sin\theta = \frac{x+2}{3}$

$$\int \frac{3\cos\theta\, d\theta}{9^{5/2}\left(1-\sin^2\theta\right)^{5/2}}$$

← $\frac{3}{9^{5/2}} = \frac{3}{9\cdot 9\cdot 9\cdot 9\cdot 9\cdot\sqrt{2}} = \frac{3}{243} = \frac{1}{81}$

$\sqrt{59049}$

How did they do this, came out same but they have $\frac{1}{3^4}$

Factor out 9 (2 step) and triangle

$$\frac{1}{81}\int \frac{\cos\theta\, d\theta}{(\cos^2)^{5/2}}$$

← $(\cos^2\theta)^{5/2} = |\cos^5\theta|$

$$\frac{1}{81}\int \frac{\cos\theta\, d\theta}{|\cos^5\theta|}$$

$$\frac{1}{81}\int \cos^{-4}\theta$$

↓ realize

$$\frac{1}{81}\int \sec^4\theta$$

now must split again

$$\frac{1}{81}\int (1+\tan^2\theta)(\sec^2\theta)\, d\theta$$

$u = \tan\theta$

$du = \sec^2\theta$

$$\frac{1}{81}\int 1+u^2\, du$$

$$\frac{1}{81}\left(u+\frac{u^3}{3}\right)+C$$

← don't forget to plug back in

$$\frac{1}{81}\left(\tan\theta + \frac{\tan^3\theta}{3}\right)+C$$

→ and again use triangle

find 3th leg

$\sqrt{9-(x+2)^2}$

$\sqrt{5-4x-x^2}$

$\tan\theta = \frac{x+2}{\sqrt{5-4x-x^2}}$

$$\frac{1}{81}\left(\frac{x+2}{\sqrt{5-4x-x^2}} + \frac{(x+2)^3}{(5-4x-x^2)^{3/2}}\right)+C$$

 Name: _____

5. (a) Set up (but do not solve) the integral for the arc length along the curve $x = y + y^3$ from $y = 1$ to $y = 4$.

$$\int_1^4 \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

$dx = 1 + 3y^2$

$y = t$

$x = t + t^3$

how they did this

$$\int_1^4 \sqrt{1 + (1+3y^2)^2}\, dy$$

Same

$$\int_1^4 \sqrt{(x'(t))^2 + (y'(t))^2}$$

$$\int_1^4 \sqrt{(1+3t^2) + 1}$$

$(1+3y^2)(1+3y^2)$

$$\cancel{\int_1^4 \sqrt{1 + (1 + 6y^2 + 9y^4)}\, dy}$$

not

$$\cancel{\int_1^4 \sqrt{9y^4 + 6y^2 + 2}\, dy}$$

$$\cancel{\int_1^4 9y^2 + 6y + \sqrt{2}\, dy}$$

don't do

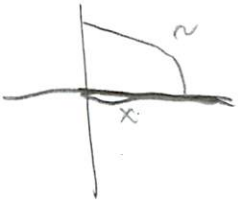
2/3

(b) Set up (but do not solve) the integral for the surface area of the surface obtained by rotating the curve given by

forgot diff $\cos^3$

$$x = a\cos^3 t, \quad y = a\sin^3 t, \quad 0 \leqslant t \leqslant \pi/2$$

about the x-axis. Here a is an arbitrary constant.



$dx = a\,3\cos^2 t \sin t$

$dy = a\,3\sin^2 t \cos t$

$$\int_0^{\pi/2} 2\pi \underset{\text{radius}}{\cancel{x}}\, y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$= \sin 3t$

duh

$$2\pi \int_0^{\pi/2} \cancel{2\pi x} \sqrt{(a\,3\cos^2 t \sin t)^2 + (a\,3\sin^2 t \cos t)^2}\, dt$$

$|a| \sin 3t$

$$2\pi \int_0^{\pi/2} 3a^2 \sin^3 t \sqrt{\underbrace{\sin^2 t \cos^2 t}_{\sin t \cos t} \underbrace{(\cos^2 t + \sin^2 t)}_{1}}$$

$1.5/3$

$$6a^2\pi \int_0^{\pi/2} \sin^4 t \cos t\, dt$$

simplify more

18.01 EXAM IV

Thursday, December 3, 2009

Name: SOLUTIONS.

E-mail: _____

Recitation Instructor (Circle One): C. Breiner / I. Losev / X. Ma / S. Ramakrishnan / B. Rhoades

Recitation Hour: _____

Instructions: You may not use calculators, notes, textbooks, or personal electronic devices. As a courtesy to other students, please turn off all cell phones. Read each question carefully. Whenever possible, include justification for your reasoning and show your work. Answers without any work or explanation will receive little or no credit. There are 5 questions and a 50 minute time limit on this exam. Good luck.

Question	Score	Maximum
1		5
2		5
3		5
4		7
5		6
Total		28

 Name: _____

1. Compute the following integral:

$$\int_1^4 \sqrt{t}\,\ln t\,dt$$

$u = \ln t$ $\qquad$ $v = \frac{2}{3}t^{3/2}$

$du = \frac{1}{t}\,dt$ $\qquad$ $dv = \sqrt{t}\,dt$

$$\int_1^4 \sqrt{t}\,\ln t\,dt = \frac{2}{3}t^{3/2}\ln t\Big|_1^4 - \int_1^4 \frac{2}{3}\underbrace{t^{3/2}\cdot t^{-1}}_{t^{1/2}}\,dt$$

$$= \left(\frac{2}{3}t^{3/2}\ln t - \frac{4}{9}t^{3/2}\right)\Big|_1^4 =$$

$$= \frac{16}{3}\ln 4 - \frac{28}{9}$$

$$\frac{28}{9} = \frac{4}{9}\left(4^{3/2}-1\right)$$

Name: _____

2. Compute the following integral:

$$\int_0^{\pi/4} \tan^4\theta \sec^6\theta \, d\theta$$

$\sec^2\theta = 1 + \tan^2\theta$

$$\int_0^{\pi/4} \tan^4\theta \sec^6\theta \, d\theta = \int_0^{\pi/4} \tan^4\theta \, (1+\tan^2\theta)^2 \sec^2\theta \, d\theta =$$

$$\left(\begin{array}{ll} u = \tan\theta & du = \sec^2\theta \, d\theta. \\ 0 \le \theta \le \pi/4 \Rightarrow & 0 \le u \le 1. \end{array} \right)$$

$$= \int_0^1 u^4 \, (1+u^2)^2 \, du = \int_0^1 u^4 \, (1 + 2u^2 + u^4) \, du =$$

$$= \int_0^1 u^4 + 2u^6 + u^8 \, du = \frac{1}{5}u^5 + \frac{2}{7}u^7 + \frac{1}{9}u^9 \Big|_0^1 =$$

$$= \frac{1}{5} + \frac{2}{7} + \frac{1}{9}$$

$$\left(= \frac{188}{315} \right).$$

 Name: _____

3. Compute the following integral:

$$\int \frac{10}{(x-1)(x^2+9)}\,dx$$

$$\int \frac{10}{(x-1)(x^2+9)}\,dx = \int \frac{1}{x-1} - \frac{x+1}{x^2+9}\,dx$$

$$= \int \frac{1}{x-1}\,dx - \int \frac{x}{x^2+9}\,dx - \int \frac{1}{x^2+9}\,dx$$

$$= \ln(x-1) - \frac{1}{2}\ln(x^2+9) - \frac{1}{3}\tan^{-1}(x/3) + C.$$

 Name: _____

4. Compute the following integral:

$$\int \frac{1}{(5-4x-x^2)^{5/2}}\,dx$$

$$\int \frac{1}{\left(9-(2+x)^2\right)^{5/2}}\,dx$$

$x+2 = 3\sin\theta$

$dx = 3\cos\theta\, d\theta$

$$= \int \frac{3\cos\theta\, d\theta}{(9)^{5/2}\left(1-\sin^2\theta\right)^{5/2}}$$

$$= \frac{1}{3^4}\int \frac{\cos\theta\, d\theta}{\cos^5\theta} = \frac{1}{3^4}\int \sec^4\theta\, d\theta$$

$$= \frac{1}{3^4}\int \left(1+\tan^2\theta\right)\left(\sec^2\theta\right) d\theta$$

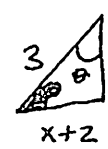
$u = \tan\theta$

$du = \sec^2\theta\, d\theta$

$$= \frac{1}{3^4}\int (1+u^2)\, du = \frac{1}{3^4}\left(u + \frac{u^3}{3}\right) + C.$$

$$= \frac{1}{3^4}\left(\tan\theta + \frac{\tan^3\theta}{3}\right) + C$$

$\sin\theta = \dfrac{x+2}{3}$



$\sqrt{9-(x+2)^2} = \sqrt{5-4x-x^2}$

$\therefore \Rightarrow \tan\theta = \dfrac{x+2}{\sqrt{5-4x-x^2}}$

$$\boxed{= \frac{1}{3^4}\left(\frac{x+2}{\sqrt{5-4x-x^2}} + \frac{(x+2)^3}{3\left(5-4x-x^2\right)^{3/2}} + C\right)}$$

 Name: _____

5. (a) Set up (but do not solve) the integral for the arc length along the curve $x = y + y^3$ from $y = 1$ to $y = 4$.

$y = t$

$x = t + t^3$

$$AL = \int ds = \int_{t=1}^{4} \sqrt{(x'(t))^2 + (y'(t))^2}\, dt =$$

$$= \int_1^4 \sqrt{(1+3t^2)^2 + 1}\, dt$$

(b) Set up (but do not solve) the integral for the surface area of the surface obtained by rotating the curve given by

$$x = a\cos^3 t, \quad y = a\sin^3 t, \quad 0 \leqslant t \leqslant \pi/2$$

about the x-axis. Here a is an arbitrary constant.

$$SA = 2\pi \int_{t=0}^{\pi/2} |a| \sin^3 t \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

$$= 2\pi \int_0^{\pi/2} |a| \sin^3 t \sqrt{(3a\cos^2 t \cdot \sin t)^2 + (3a\sin^2 t \cos t)^2}\, dt =$$

$$= 2\pi \int_0^{\pi/2} 3a^2 \sin^3 t \sqrt{\sin^2 t \cos^2 t\, (\cos^2 t + \sin^2 t)}\, dt$$

$$= 6a^2\pi \int_0^{\pi/2} \sin^4 t \cos t\, dt.$$

18.01 Practice Final Exam

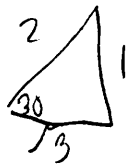
There are 19 problems, totaling 250 points. No books, notes, or calculators. This practice exam should take 3 hours.

Generally useful trigonometry:

$$\sin^2 x = \frac{1-\cos 2x}{2}; \qquad \cos^2 x = \frac{1+\cos 2x}{2}; \qquad \int \sec x = \ln(\sec x + \tan x)$$

$$\sec x = \frac{1}{\cos x}; \qquad \sin^2 x + \cos^2 x = 1; \qquad \tan^2 x + 1 = \sec^2 x$$

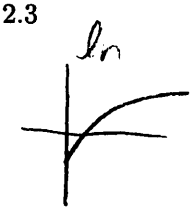
In a 30-60-90 right triangle, with hypotenuse 2, the legs are 1 and $\sqrt{3}$.



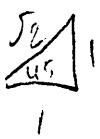
$\sqrt{2} = 1.41 \qquad \sqrt{3} = 1.73 \qquad \pi = 3.14 \qquad \ln 2 = .69 \qquad \ln 10 = 2.3$

$\ln(0) = -\infty$

$\ln(1) = 0$



Problem 1. (15) Evaluate each of the following:



a) $\dfrac{d}{dx}\dfrac{\ln x}{x^2}$; simplify your answer.

b) $\dfrac{d}{du}\sqrt{3\sin^2 u + 2}$

c) $\left.\dfrac{d^n}{dx^n} e^{kx}\right|_{x=0}$, k constant.

Problem 2. (10) Find the equation of the line tangent to the graph of $x^2y^2 + y^3 = 2$ at the point $(1,1)$ on the graph. (Give the equation in the form $y = mx + b$.)

Problem 3. (10) Using implicit differentiation, derive the formula for $D\cos^{-1} x$ by using the formula for $D\cos x$. (Let $y = \cos^{-1} x$.)

Problem 4. (10) Let $f(x) = \begin{cases} x^2 + x + a, & x \le 0 \\ bx + 2, & x > 0 \end{cases}$, a and b constants.
Find all values of a and b for which $f(x)$ is differentiable.

Problem 5. (15) On a night when the full moon is directly overhead, an outdoor Christmas tree 50 feet high is falling over. Its top is falling at the rate of 2 feet/sec, at the moment when it is 30 feet from the ground. At that moment, how rapidly is the shadow of the tree cast by the moon lengthening?

Problem 6. (15) Find the area of the largest rectangle whose base lies along the x-axis and whose top corners lie on the parabola $y = 1 - x^2$.

Problem 7. (15: 4,7,4) The graph of $y = y(x)$ has this property: at each point (x, y) on the graph, the normal line at that point passes through the fixed point $(1, 0)$. (The normal is the line perpendicular to the tangent line.)

a) Show that $y = y(x)$ satisfies the differential equation $y' = \dfrac{1-x}{y}$.

b) Using separation of variables, find all solutions to the differential equation. You can leave the solutions in implicit form, i.e., as equations connecting x and y.

c) Describe the curves which are their graphs. (You may have to use algebraic processes first (like completing the square) in order to change the equations into a form where you know what their graphs look like.)

Problem 8. (15) The cup of a wine-glass has the shape formed by rotating the parabola $y = x^2$ about the y-axis; its upper rim is a circle of radius 1. How much wine does it hold?

Problem 9. (10) Using the trapezoidal rule with three subdivisions ($n = 3$), estimate $\int_0^{\pi/2} \sin^2 x\, dx$. Do the work systematically, making a table of values first.

Problem 10. (15: 7,8) Let $F(x) = \int_0^x e^{-t^2} dt$.

a) Find $F'(1)$ and $F''(1)$.

b) Express $\int_1^2 e^{-u^2/4} du$ in terms of values of $F(x)$.

Problem 11. (15: 7,8) Between the two towers of a suspension bridge, each of the two main cables has the shape of the parabola $y = \frac{1}{10}x^2$ (units are kilometers). The two towers are 2 km. apart; the vertical cables from the main cable to the horizontal roadway are closely and equally spaced.

a) Set up a definite integral which gives the length of each main cable between the two towers.

b) What is the average length (to the nearest meter) of the vertical cables?

Problem 12. (20: 10,10) Evaluate

a) $\int_0^1 \frac{dx}{x^2 + 3x + 2}$; (begin by factoring the denominator).

b) $\int x^2 \ln x\, dx$

Problem 13. (10) Evaluate $\int_0^1 \frac{dx}{(x^2+1)^2}$ by making the substitution $x = \tan u$; remember the limits.

Problem 14. (15) Starting at the point where $r = 1$, the point P moves counterclockwise along the polar curve $r = e^{\theta/2\pi}$, in such a way that the line segment OP makes one complete revolution. (Here O denotes the origin.)
Sketch the curve, and find the total area swept out by OP as it makes the revolution.

Problem 15. (15) Evaluate (showing work):

a) $\lim_{x\to 0} \frac{\sin^2 x}{1 - \cos x}$

b) $\lim_{x\to 1} \frac{(\ln x)^2}{x - 1}$

c) $\lim_{x\to\infty} x^2 e^{-x}$

Problem 16. (10) Evaluate $\int_1^\infty \frac{dx}{x^{3/2}}$

Problem 17. (10) For what values of p does $\sum_1^\infty \frac{n}{\sqrt{4 + n^p}}$ converge? (Indicate reasoning.)

Problem 18. (15: 10,5)

a) Find by differentiating the function $f(x) = \sqrt{1+x}$ the first four non-zero terms of its Taylor series around $x = 0$. (Show work.)

b) Use the correct answer to (a) (or your own answer, if you don't know the correct answer) to calculate $\sqrt{1.2}$ to four decimal places.

Problem 19. (10) Find the Taylor series for $\tan^{-1} x$ around $x = 0$ by using term-by-term differentiation or integration on the appropriate geometric series. Give enough terms to make the pattern clear.

1

18.01 Practice Final Exam Solutions

1a) $\frac{d}{dx}\frac{\ln x}{x^2} = \frac{d}{dx}(x^{-2}\ln x) = \left(\frac{d}{dx}x^{-2}\right)\ln x + x^{-2}\left(\frac{d}{dx}\ln x\right)$

$= -2x^{-3}\ln x + x^{-2}x^{-1} = \boxed{\frac{1-2\ln x}{x^3}} \left(= \frac{\ln\left(\frac{e}{x^2}\right)}{x^3}\right)$

b) $\frac{d}{du}\sqrt{3\sin^2 u+2} = \frac{d}{du}(3\sin^2 u+2)^{\frac{1}{2}} = \frac{1}{2}(3\sin^2 u+2)^{-\frac{1}{2}}$

$\cdot \frac{d}{du}(3\sin^2 u+2) = \frac{1}{2}\frac{3\cdot 2\sin u\cos u}{\sqrt{3\sin^2 u+2}} = \boxed{\frac{3\sin u\cos u}{\sqrt{3\sin^2 u+2}}}$

c) $\frac{d}{dx}e^{kx} = ke^{kx}$ $\qquad \frac{d^2}{dx^2}e^{kx} = k^2e^{kx}$

$\frac{d^n}{dx^n}e^{kx} = (k)^n e^{kx}$ $\qquad \boxed{\frac{d^n}{dx^n}e^{kx}\Big|_{x=0} = (k^n)}$

2. $x^2y^2+y^3=2$ at $p=(1,1)$

Implicit differentiation $2xy^2\,dx + 2yx^2\,dy + 3y^2\,dy = 0$.

slope $\frac{dy}{dx} = -\frac{2xy^2}{2yx^2+3y^2}$. At $(1,1)$ slope $= -\frac{2}{5}$

Tangent line

$y = -\frac{2}{5}x + b$ To pass through $(1,1)$ $1 = -\frac{2}{5}\cdot 1 + b$

$b = \frac{7}{5}$

$\boxed{y = -\frac{2}{5}x + \frac{7}{5}}$

(2)

3. $y = \cos^{-1} x$ $\qquad \cos y = x$

Differentiate: $\qquad -\sin y \, dy = dx$

$$dy = \frac{dx}{-\sin y} = \frac{dy}{-\sqrt{1-\cos^2 y}} = \boxed{\frac{dx}{-\sqrt{1-x^2}}}$$

4. To be continuous at 0:

$$f(0) = \lim_{x \to 0^+} f(x) \qquad a = 2$$

To be differentiable at 0:

$$\lim_{x \to 0^-} f'(x) = \lim_{x \to 0^+} f'(x)$$

For $x < 0$

$f'(x) = (x^2 + x + a)' = 2x + 1$

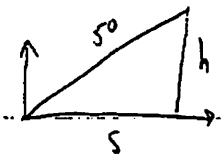
For $x > 0$ $\quad f'(x) = b$

$b = 2 \cdot 0 + 1 = 1$

$\boxed{a = 2 \qquad b = 1}$

(3)

5.



If "falling" means (prefered interpretation) loosing height, then

$\frac{dh}{dt} = -2$ f/s and as $h^2 + s^2 = 50^2$

$h = 30f$ $s = 40f$

$$2h\frac{dh}{dt} + 2s\frac{ds}{dt} = 0$$

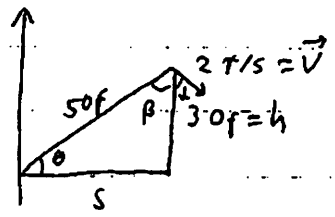
$$\frac{ds}{dt} = -\frac{h}{s}\frac{dh}{dt} = \frac{30f}{40f}\,2 \text{ f/s} = \boxed{\frac{3}{2}} \text{ f/s}$$

If falling means "has speed of", then

see next page

(4)

5' Soln 1:



$\frac{dh}{dt}$ = (v projected to vertical)

$= -v\cos\alpha$

$\alpha = 90 - \beta$ $\quad$ $\beta = 90 - \theta$, so

$\alpha = \theta$, $\quad \frac{dh}{dt} = -v\cos\theta$

$h^2 + s^2 = 50^2$ $\quad$ Differentiate: $2h\,dh + 2s\,ds = 0$

$$\frac{ds}{dt} = -\frac{h\frac{dh}{dt}}{s} = \frac{+h\,v\cos\theta}{s}$$

$h = 30f$

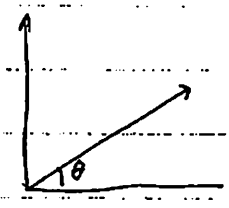
$s = 40f$

$v = 2$

$\cos\theta = \frac{40}{50}$

$$\frac{ds}{dt} = \frac{30 \cdot 2 \cdot \frac{40}{50}}{40} = \boxed{\frac{6}{5}}$$

Sol 2:



$s = r\cos\theta$ $\quad$ $ds = (dr)\cos\theta + r\sin\theta\,d\theta =$

$= r\sin\theta\,d\theta$

Also

$\vec{v} = r'\hat{r} + r\theta'\hat{\theta} =$ $\quad$ (as $r = \text{const} = 50$)

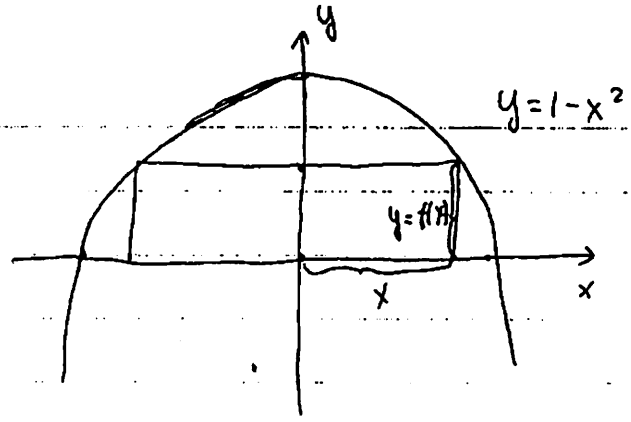
$= r\theta'\hat{\theta}$

$|\vec{v}| =$ [redacted] $r\frac{d\theta}{dt}$

So $\quad \frac{ds}{dt} = r\sin\theta\,\frac{d\theta}{dt} = \sin\theta\,|\vec{v}| = \frac{3}{5}\cdot 2 = \boxed{\frac{6}{5}}$

(5)

6



$A = (\overset{\text{base}}{2x})\,\overset{\text{height}}{f(x)} =$

$= 2x(1-x^2)$

Extrema: $(2x(1-x^2))' = 0$

$(x - x^3)' = 0$

$1 - 3x^2 = 0 \qquad x = \pm 1/\sqrt{3}$

This is a maximum (can check A'' or note that at bdry $x=0$ or $x=1$ $A=0$).

$A = \frac{2}{\sqrt{3}} \cdot \frac{2}{3} = \boxed{\frac{4}{3\sqrt{3}}}$

7. a) $p = (1,0)$. Line from (x_0, y_0) to $(1,0)$ has slope $k = \frac{y_0}{x_0 - 1}$. Line $\perp$ to it has slope

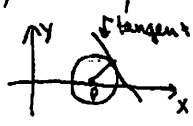
$-\frac{1}{k} = \frac{1-x_0}{y_0}$. This is tangent to $y = y(x)$, at (x_0, y_0) so

$y'(x) = \frac{1-x}{y}$

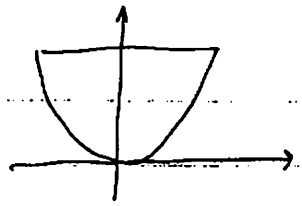
b) $\frac{dy}{dx} = \frac{1-x}{y}$ $\qquad y\,dy = (1-x)\,dx$. Integrate

$\frac{y^2}{2} = \left(x - \frac{x^2}{2}\right) + c$

c) $y^2 + x^2 - 2x = 2c$ $\qquad y^2 + (x-1)^2 = c_1$ -



circles centered at $p=(1,0)$ of varying radii

8. 

Upper rim $r=1$, so $x=1$, $y=1$

Need to compute volume

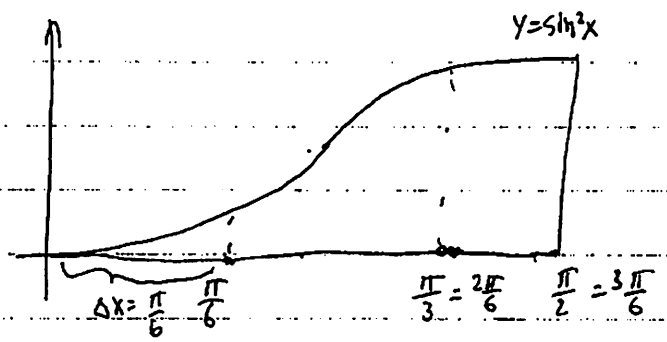
Method 1: Discs $\displaystyle\int_{y_{min}}^{y_{max}} \pi x^2\, dy$ $\quad x=\sqrt{y}$, $y_{min}=0$, $y_{max}=1$

$$\int_0^1 \pi y\, dy = \pi \frac{y^2}{2}\Big|_0^1 = \boxed{\frac{\pi}{2}}$$

Method 2: Shells $\displaystyle\int_{x_{min}}^{x_{max}} 2\pi x\, y\, dx$ $\quad y=1-x^2$, $x_{min}=0$, $x_{max}=1$

$$\int_0^1 2\pi (x-x^3)\, dx = 2\pi\left(\frac{x^2}{2}-\frac{x^4}{4}\right)\Big|_0^1 = 2\pi\,\frac{1}{4} = \boxed{\frac{\pi}{2}}$$

9.



x	y
0	0
$\frac{\pi}{6}$	$\left(\frac{1}{2}\right)^2=\frac{1}{4}$
$\frac{2\pi}{6}=\frac{\pi}{3}$	$\left(\frac{\sqrt{3}}{2}\right)^2=\frac{3}{4}$
$\frac{3\pi}{6}=\frac{\pi}{2}$	$1^2=1$

Trapezoidal $\displaystyle\sum_{n=1}^{3} \frac{f(\text{left})+f(\text{right})}{2}\cdot\frac{\pi}{6} = \frac{\pi}{6}\left(\frac{f(0)}{2}+f\left(\frac{\pi}{6}\right)+f\left(\frac{2\pi}{6}\right)+f\left(\frac{3\pi}{6}\right)\right)$

$$= \frac{\pi}{6}\left(0+\frac{1}{4}+\frac{3}{4}+\frac{1}{2}\right) = \frac{\pi}{6}\,\frac{3}{2} = \boxed{\frac{\pi}{4}}$$

Compare: $\displaystyle\int_0^{\frac{\pi}{2}} \sin^2 x\, dx = \frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi}{4}$

Answer is exactly correct because of symmetry

(7)

10 $F(x) = \int_0^x e^{-t^2}\, dt$

$F'(x) = e^{-x^2}$

a) By Fundamental Theorem $\boxed{F'(1) = e^{-1}}$ $F''(x) = \frac{d}{dx}(e^{-x^2})$

$= -2xe^{-x^2}$, $\boxed{F''(1) = -2e^{-1}}$

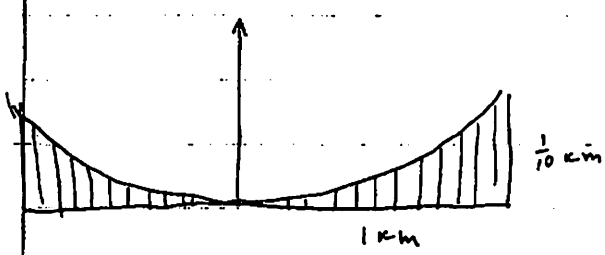
b) $A = \int_1^2 e^{-u^2/4}\, du$

Let $u = v/2$ $du = \frac{dv}{2}$ Substitute

$A = \int_{v_{start}}^{v_{finish}} e^{-v^2} \frac{dv}{2} = \frac{1}{2}\int_{\frac{1}{2}}^{1} e^{-v^2}\, dv =$

$= \boxed{\frac{1}{2}\left(F(1) - F(\tfrac{1}{2})\right)}$

11



a) length of main cable = arc length

$2\int_0^1 ds = 2\int_0^1 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx =$

$= \boxed{2\int_0^1 \sqrt{1+\left(\frac{1}{5}x\right)^2}\, dx}$

b) Average of $\frac{1}{10}x^2 = \frac{\int_0^1 \frac{1}{10}x^2\, dx}{1} = \frac{\frac{x^3}{30}\Big|_0^1}{1} = \boxed{\frac{1}{30}}$

12 a) $x^2+3x+2 = (x+1)(x+2)$

Now by cover-up

$$\frac{1}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2} \qquad A = \frac{1}{(-1)+2} = 1$$

$$B = \frac{1}{(-2)+1} = -1$$

$$\frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}$$

$$\int_0^1 \frac{1}{(x+1)(x+2)}\,dx = \int_0^1 \left(\frac{1}{(x+1)} - \frac{1}{(x+2)}\right) dx = \ln(x+1)\Big|_0^1 - \ln(x+2)\Big|_0^1$$

$$= -\ln 3 + \ln 2 + \ln 2 - \ln 1 = \boxed{\ln\frac{4}{3}}$$

b) $\int x^2 \ln x\,dx =$ | By parts $\quad v = \ln x \quad u' = x^2$

$v' = \frac{1}{x} \quad u = \frac{x^3}{3}$

$$= \frac{x^3}{3}\ln x - \int \frac{x^3}{3}\,\frac{1}{x}\,dx$$

$$= \frac{x^3}{3}\ln x - \frac{x^3}{9} + C = \boxed{\frac{x^3}{3}\left(\ln x - \frac{1}{3}\right) + C}$$

13 $x = \tan u \qquad dx = \sec^2 u\,du \qquad u_{start} = \tan^{-1} 0 = 0 \qquad u_{finish} = \tan^{-1} 1 = \frac{\pi}{4}$

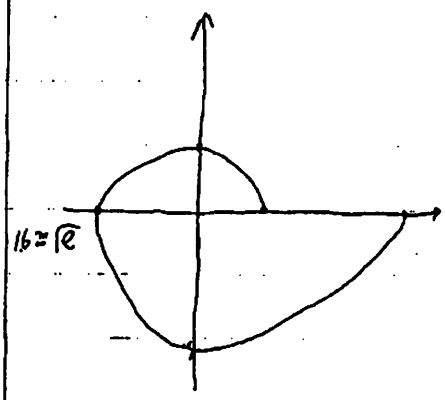
$$\int_0^1 \frac{dx}{(x^2+1)^2} = \int_{u_{start}}^{u_{finish}} \frac{\sec^2 u\,du}{(1+\tan^2 u)^2} = \int_0^{\frac{\pi}{4}} \frac{\sec^2 u}{\sec^4 u}\,du$$

$$= \int_0^{\frac{\pi}{4}} \cos^2 u\,du = \int_0^{\frac{\pi}{4}} \frac{\cos(2u)+1}{2}\,du = \left(\frac{u}{2} + \frac{\sin(2u)}{4}\right)\Big|_0^{\frac{\pi}{4}} = \boxed{\frac{\pi}{8} + \frac{1}{4}}$$

(9)

14 $\quad r = e^{\theta/2\pi}$ $\qquad r=1 \Rightarrow e^{\theta/2\pi} = 1$

$\theta/2\pi = 0, \ \theta = 0$



As θ goes from 0 to 2π

r increases from 1 to $e^{\frac{2\pi}{2\pi}} = e$

$$A = \int_0^{2\pi} \frac{1}{2} (r(\theta))^2 \, d\theta =$$

$$\int_0^{2\pi} \frac{1}{2} e^{\theta/\pi} \, d\theta = \frac{\pi}{2} e^{\theta/\pi} \Big|_0^{2\pi} = \boxed{\frac{\pi}{2}(e^2 - 1)}$$

15 a) $\lim_{x\to 0} \frac{\sin^2 x}{1-\cos x} = \lim_{x\to 0} \frac{1-\cos^2 x}{1-\cos x} = \lim_{x\to 0} \frac{(1-\cos x)(1+\cos x)}{1-\cos x} = \lim_{x\to 0} (1+\cos x) = \boxed{2}$

O.R

$$\lim_{x\to 0} \frac{\sin^2 x}{1-\cos x} \overset{\text{L'Hopital}}{=} \lim_{x\to 0} \frac{2\sin x \cos x}{\sin x} = \lim_{x\to 0} 2\cos x = \boxed{2}$$

O.R Taylor series $\quad$ top $= \sin^2 x = \left(x - \frac{x^3}{3} + \dots\right)^2 = x^2 - \frac{2x^4}{3} + \dots$

bot $= 1 - \cos x = 1 - \left(1 - \frac{x^2}{2} + \frac{x^4}{4!} \dots\right) = \frac{x^2}{2} + \frac{x^4}{4!} + \dots$

$$\frac{\text{top}}{\text{bot}} = \frac{x^2 - \frac{2x^4}{3} \dots}{\frac{x^2}{2} - \frac{x^4}{4!} \dots} = \frac{1 - \frac{2x^2}{3} + \dots}{\frac{1}{2} - \frac{x^2}{4!} + \dots} \qquad \lim_{x\to 0} \frac{\text{top}}{\text{bot}} = \frac{1}{\frac{1}{2}} = \boxed{2}$$

~~b) $\lim_{x\to 1} \frac{(\ln x)^2}{x-1} = \lim_{h\to 0} \frac{(\ln(h+1))^2}{h}$~~

$h = x - 1$

$x = h + 1$

~~by definition $= \left(\left(\ln(h+1)\right)^2\right)'\Big|_{h=0} = 2\ln(h+1)/h+1\Big|_{h=0} = 2\ln 1 = 0$~~

~~L'Hopital $= 2$~~

b) $\lim_{x \to 1} \frac{(\ln(x))^2}{x-1} \overset{\text{by definition}}{=} \left((\ln(x))^2\right)' \Big|_{x=1} =$

$= \frac{2\ln x}{x} \Big|_{x=1} = \boxed{0}$

OR $\lim_{x \to 1} \frac{(\ln(x))^2}{x-1} = \lim_{\substack{h \to 0 \\ h = x-1 \\ x = h+1}} \frac{(\ln(h+1))^2}{h} \overset{\text{L'Hopital}}{=} \lim_{h \to 0} 2\ln(h+1)\frac{1}{h+1} =$

$\frac{2\ln 1}{1} = \boxed{0}$

c) $\lim_{x \to \infty} x^2 e^{-x} = \lim_{x \to \infty} \frac{x^2}{e^x} \overset{\text{L'Hopital}}{=} \lim_{x \to \infty} \frac{2x}{e^x} \overset{\text{L'Hopital}}{=} \lim_{x \to \infty} \frac{2}{e^x} = 0$

16 $\int_1^\infty \frac{dx}{x^{3/2}} = \lim_{b \to \infty} \int_1^b \frac{dx}{x^{3/2}} = \lim_{b \to \infty} \left(\frac{x^{-1/2}}{(-1/2)} \right) \Big|_1^b =$

$= \lim_{b \to \infty} \left(+\frac{2}{1} - \frac{2}{1} b^{-1/2} \right) = \boxed{2}$

17. Rough estimate $\sqrt{4+n^p} \sim \sqrt{n^p} = n^{p/2}$

$\frac{n}{\sqrt{4+n^p}} \sim n^{1-p/2}$. So, let's do limit comparison

$b_n = n^{1-p/2}$ $\quad \lim_{n \to \infty} \frac{b_n}{a_n} = \frac{n}{\sqrt{n^p}} \Big/ \frac{n}{\sqrt{n^p+4}} = \frac{\sqrt{1+\frac{4}{n^p}}}{\sqrt{1}} =$

$= \sqrt{1+4n^{-p}} = 1$ for $p > 0$

Hence the series converges if and only if $1 - \frac{p}{2} < -1$, $\boxed{p > 4}$

Note: for $p < 0$

$\frac{n}{\sqrt{4+n^p}} > \frac{n}{\sqrt{5}}$, does not go to 0, so the series can not converge

(11)

18 a) $f(x) = (1+x)^{\frac{1}{2}}$ $\quad f'(x) = \frac{1}{2}(1+x)^{-\frac{1}{2}}$

$f''(x) = -\frac{1}{4}(1+x)^{-3/2}$ $\quad f'''(x) = \frac{3}{8}(1+x)^{-5/2}$

b) $(1+x)^{\frac{1}{2}} = f(x) \approx 1 + \frac{1}{2}x + \frac{1}{2} \cdot -\frac{1}{4}x^2 + \frac{1}{6}\,\frac{3}{8}x^3$

$(1.2)^{\frac{1}{2}} = 1 + \frac{1}{2}(0.2) - \frac{1}{2}\,\frac{1}{4}(0.2)^2 + \frac{1}{6}\,\frac{3}{8}(0.2)^3 =$

$= 1 + 0.1 - 0.005 + 0.0005 = \boxed{1.0955}$

19 $(\tan^{-1}(x))' = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n$

$\tan^{-1}(x) = \int (\tan^{-1}(x))'\,dx = \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$

(no constant, as $\tan^{-1}(0) = 0$)

18.01 Pratice ~~Test~~ Final

12/15

From web, 3 hrs to do

It gives some formulas, will assume have those

1. Evaluate

$$\frac{d}{dx}\frac{\ln x}{x^2}$$

$\ln x \cdot x^{-2}$

product rule

$$\frac{1}{x}x^{-2} + -2x^{-3}\ln x$$

neg

$$\frac{1}{x^3} - 2\frac{\ln x}{x^3}$$

Close
- stupid mistakes
bring down exponent

$$\frac{1 - 2\ln x}{x^3}$$

b. $\frac{d}{dv}\sqrt{3\sin^2 v + 2}$

chain rule twice

$(3\sin^2 v + 2)^{1/2}$

$\frac{1}{2}(3\sin^2 v + 2)^{-1/2} \cdot$ chain rule $\cdot\ 3 \cdot 2\sin v \cdot \cos v$

$$\frac{6\sin v\cos v}{2\sqrt{\sin^2 v + 2}} = \frac{3\sin v\cos v}{\sqrt{\sin^2 v + 2}}$$ ✓

② c) $\frac{d^n}{dx^n} e^{kx}\Big|_{x=0}$ k = constant

- general solution I guess
- like that 1 exam problem

$d^1 =$ $kx \cdot e^{kx}$

$d2 =$ chain rule

$k \cdot 1 \cdot e^{kx} + kxe^{kx}\, kx$

$e^{kx} + 2kx^2 e^{kx}$

d^3 $kxe^{kx} + 4kxe^{kx} + kxe^{kx} \cdot 2kx^2$

$k \cdot 1 \cdot e^{kx} + kxe^{kx}\, kx + 4k e^{kx} \cdot kxe^{kx} \cdot 4kx$

$+ k e^{kx} \cdot kxe^{kx}\, kx \cdot$

↑ this is chain rule spiraling out of control

try e^{2x}

$2e^{2x}$ ~~kxhp~~

just take coefficient

chain rule $u = 2x$

e^u

$u e^u \cdot 2$

$2e^u$

oh differentiate u I always forgot

So $\frac{d}{dx} 5x^2$ $u = x^2$

$du = 2x$) don't need chain rule

$5 \cdot 2x$

$10x$

③ e^{kx}

$u = kx$
$du = k$

d' ke^{kx}

$d'' = k \cdot k\, e^{kx}$

d''' $k^2 \cdot k\, e^{kx}$

$k^n e^{kx}$ ✓

So $\frac{d^n}{dx} e^{kx} \Big|_{x=0}$

so eval at $x=0$ ← don't forgtt to do

$k^n e^{k \cdot 0}$

k^n

don't make silly technical errors, should be able to do a chain rule

$(x^2+1)^3$ random pratice

$3(x^2+1)^2 \cdot 2x$

prduh dx

$6x(x^2+1)^2$

2. Find the equation of the tangent to graph $x^2y^2 + y^3 = 2$ at pt $(1,1)$ → in form $mx+b$

- so this is find deriv implicitly, plug pt in
- basic calculus, should have no problems
- just need to implement

$2xy^2 + 2yy'x^2 + 3y^2y' = 2$ ← differentiate that too

solve for y'

$2 - 2xy^2 = 2yy'x^2 + 3y^2y'$

$y'(2yx^2 + 3y^2) = 0 - 2xy^2$

(4) $\frac{dy}{dx} = y' = \frac{-2xy^2}{2yx^2 + 3y^2}$

Ok - now what

- plug pts in ??

- found equation of slope, right ✓ $\frac{dy}{dx}$

yes

$$\frac{dy}{dx} = \frac{-2(1)(1)^2}{2(1)(1)^2 + 3(1)^2}$$

$$\frac{dy}{dx} = \frac{-2}{5}$$

now points

$$y - 1 = -\frac{2}{5}(x - 1)$$

$$y - 1 = -\frac{2}{5}x + \frac{2}{5}$$

$$y = -\frac{2}{5}x + \frac{7}{5}$$ ✓ so made 1 mistake and was slightly unsure of next step

3. More implicit differentiation

- derive the formula for $D \cos^{-1} x$ by using the formula for $D \cos x$ let $y = \cos^{-1} x$

i don't really get formula for $D y$

is D a constant - or does it mean derive?

no clue on this one

(5) $y = \cos^{-1} x$

So $\cos y = x$ ← oh did not think of this

duh

$y^2 = x$

$y = \sqrt{x}$ ↓ inverse ?

if $f(x) = y$
$f^{-1}(y) = x$

So how does that help

? take deriv - what do they mean formula for $D \cos x$

differentiate implicitly

$-\sin y \, y' = x'$

$dy = \frac{dx}{-\sin y} = \frac{dx}{-\sqrt{1-\cos^2 y}} = \frac{dx}{-\sqrt{1-x^2}}$

↑ trig identity

- so basically wanted me to take derivative of inverse cosine by knowing deriv of reg cosign and converting inverse into regular and implicit differentiating it

← oh why not write $\frac{d}{dx}$ ← lower

- did not understand notation, did not link everything in my head, did not fully get inverses and need to recognize the trig identity
- check by knowing what the deriv of $\cos^{-1}$ is

⑥ #4 Let $f(x) = \begin{cases} x^2 + x + a & x \le 0 \\ bx + 2 & x > 0 \end{cases}$ a, b constants

Find <u>all</u> values a + b where differentiable

- that is where it is continuous + =
- so where =
- and were 1st deriv = at $x = 0$

$$x^2 + x + a = bx + 2$$
$$0^2 + 0 + a = b(0) + 2$$
$$a = 2$$

$$2x + 1 = b$$
$$2(0) + 1 = b$$
$$1 = b$$

✓ easy
- need to do them all like that
1. know concept
2. execute correctly

5. Night full moon overhead <u>related rates</u>
tree 50 ft tall falling over
top falling 2 ft/sec when 30 ft from ground
How rapidly is shadow lenghtening

- rusty on related rates
- need to clearly establish steps + process

(7) Look at notes try to get some steps

(1) Write out variables / pic

identify parts

$y = 30$ 50 trig

$dy = -2$

$x = ?$ $\sqrt{50^2 - 30^2}$ $x = \sqrt{50^2 - y^2}$

↑ constraint based on trig

$dx = ?$ ← what trying to find

↗ ? just solve

$\cancel{dx = \sqrt{50^2 - dy^2}}$ ← don't think that is legal

- but where is 30

~ this is more complex ~~this I thought~~

current $x = \sqrt{50^2 - 30^2}$

$x = 40$ ✓

that is my issue

but what is rate of x changing?

get equation for shadow → differentiate → plug in

oh $x^2 + -x^2 = 50$

implicit differentiation

$2y\frac{dy}{dt} + 2x\frac{dx}{dt} = 0$ ~~← what is this~~

Or for me ↑ so the dx and dy - not what from problem

- plug in now

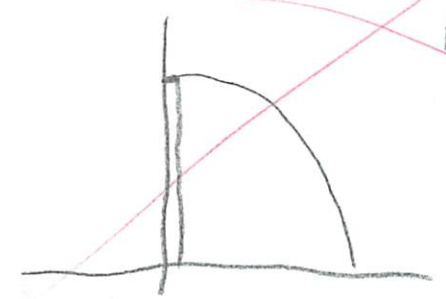
$2 \cdot 30 \cdot -2 + 2 \cdot 40 \cdot x' = 0$

Solve for x' $80x' = -120$ $x' = -\frac{3}{2}$ ft/sec

↗ Convert to (+) for word problem

8) Ok look back
- almost got it
- had one false statement
- remember it involved deriving something
  - implicitly?

6. Find area of largest rectangle whose base lies along x-axis ~~and~~ top corner on parabola $y = 1 - x^2$

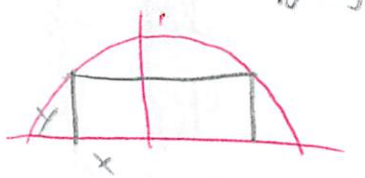


~~both corners?~~
~~that is impossible~~
~~but it is at $1 - b^2 \cdot b$~~

b = width

and want $b \to 0$

~~oh picture is~~



that makes it harder
and make a box w/ a significant large box

So box $l = 2x$
$h = 1 - x^2$

$2x \cdot 1 - x^2$

$A = 2x - 2x^3$

so how to do maximize - from hs
- so x $(0, 1)$ prob $\sim .25 - .5$

- is this something w/ integrating? $\int$
  - since area - don't think so

(9) So how going to do this maximize

extrema $2x(1-x^2)=0$

$2x - 2x^3$

? they differentitate

? to find intercepts

- oh right - this will make a curve

find the max

- find max - by setting deriv = 0 + yeah right

- they are doing things in order

- hint ↑ think

- will see if final is like that

need more introspecting time

- Single

$d = 2 - 6x^2$

~~Or integrate~~

$\cancel{\frac{2x^2}{2} - \frac{2x^4}{4}}$

$\cancel{x^2 - \frac{1}{2}x^4}$

~~Or just set = to 0~~

$\cancel{\frac{0}{2} = \frac{2x}{2} - \frac{2x^3}{2}}$

$\cancel{0 = x - x^3}$

$\cancel{x^3 = x}$ → now what

(10)

$0 = 2 - 6x^2$

$\frac{-2}{-6} = \frac{-6x^2}{-6}$

$\frac{1}{3} = x^2$

$x = \sqrt{\frac{1}{3}} \approx .577$

w/ calc

$A = 2x - 2x^3$

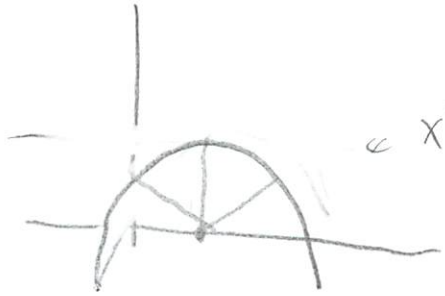
$2\sqrt{\frac{1}{3}} - 2\sqrt{\frac{1}{3}}^3$ ← how suppose to do calc less

$A = \frac{4\sqrt{3}}{9} = .7698$ ✓

Ok look back

- mistook picture / question
  - big mistake
- got formula ✓
- did not get at once should ~~set~~ find max
  - of that formula by settin deriv = 0
- then easy except for hard math w/o calculator

(11) Graph of $y = y(x)$ has properties
- at each point (x, y) normal line (perpendicular) passes through $(1,0)$



x^2+y^2 $(1-x)$

??? how to get $\to 1$

Change all x to $(x-1)$

a) Show that $y = y(x)$ satisfies

$$y' = \frac{1-x}{y}$$

So deriv of that

$2xx' + 2yy'$

$2yy' = -2xx'$

$y' = \frac{2xx'}{2y}$

$(x-1)^2 + y^2$

$x^2 - 2x + 1 + y^2 = 0$

$2xx' - 2x' + 2yy'$

$2yy' = -2xx' + 2x'$

$y' = \frac{2xx' + 2x}{2y}$

$y' = \frac{xx' + x}{y'}$

do not what they want

they do not try graph

at $p = (1,0)$

has slope $m = \frac{y_0}{x_0 - 1}$

$-\frac{1}{m} = \frac{-x_0+1}{y_0} = \frac{1-x_0}{y_0}$

$\angle$ line from (x_0, y_0) to $(1,0)$

$\perp$ slope $\to \left(-\frac{1}{m}\right)$

on did not think of that

the given problem

(12) Using seperation of variables find all solutions to differential equation ← what again

leave in implicit form

- this and FTC confusing
- the minor distinction

lot less qv pretty much getting it except 1 trick/ tech error

$$f'(x) = \frac{dy}{dx}$$

oh this

$$y' y = 1 - x$$

now what again?

$$\frac{dy}{dx} = y' = \frac{1-x}{y}$$

$\cdot dx$ $\quad \cdot y$

$$y\,dy = (1-x)\,dx$$

now what

$\int$ integrate $\int$

$$\frac{y^2}{2} = x - \frac{x^2}{2} + C$$

solve for y

$$y^2 = 2x - x^2$$

$$y = \sqrt{2x - x^2} + C$$

do you? they don't

(13)

c) Describe the curves which are the graphs

$y = \sqrt{2x - x^2}$ – may have to complete square

$\sqrt{-x^2 + 2x - 1 + 1}$

$\sqrt{(-x+1)(x-1) + 1}$ → or $y^2 = (-x+1)(x-1)+1$ etc

so how does that help
can find 0

$-x + 1 = 0$ $\quad$ $x - 1 = 0$

$-x = -1$ $\quad$ $x = 1$

$x = 1$

close to what I had

$y^2 + x^2 - 2x = 2c$ $\quad$ $y^2 + (x-1)^2 = c_1$

don't get

circles around $p = (1, 0)$
of varing radi,

what I had at start
So basially what I had in
Start – drawn as a circle

(14) Problem 8

Last for night

Cup of wine glass

← integrate

pick ~~disks~~ ~~shells~~ ①

$y = 1 - x^2$

$\int_0^1 2\pi \cdot 1 \cdot x^2$ ← radius

$2\pi \int_0^1 x^2$

$2\pi \frac{x^3}{3} \Big|_0^1$

$2\pi \frac{1^3}{3}$

$\frac{2\pi}{3}$ no units ?

~~what is disc~~

disc (across)

$x = \sqrt{y}$ ← why do this

$\int_0^1 \pi y \, dy$

$\pi \frac{y^2}{2} \Big|_0^1$

$\frac{\pi}{2}$

what I tried

② Shell

$\int_{x_{max}}^{x_{min}} 2\pi x y \, dx$ ← radius ← height

$y = 1 - x^2$

yeah think about this

this is the height of each x - think carefully

draw boxes should help think about

$2\pi \int x(1 - x^2)$

$2\pi \int x - x^3$

$2\pi \left(\frac{x^2}{2} - \frac{x^4}{4}\right) \Big|_0^1$

forgot →

$2\pi \left(\frac{1}{2} - \frac{1}{4}\right)$

$\frac{2\pi}{4} = \frac{\pi}{2}$

(15) So review

- got height wrong I did not
- but otherwise ended up doing it right

Ok done 2AM

getting really close - some stupid mistakes

little mistake on every problem

- is that all 15 pages

- went slow + ordered christmas gifts

+ took forever to get started

12/16 6:42AM

- only did 2 hr review session
- need better time management

- back to Problem 9.

Use trapizodal rule w/ $n=3$ to estimate $\int_0^{\pi/2} \sin^2 x\,dx$

- make table of values 1st
- ok forget formula
- don't need it - think

$0, \frac{\pi}{6}, \frac{2\pi}{6}, \frac{\pi}{3}, \frac{3\pi}{6}, \frac{\pi}{2}$

1 2 3

$$\frac{\frac{\pi}{2}-0}{3}\left(\sin^2\frac{\pi}{6} + \sin^2\frac{\pi}{3}, \sin^2\frac{\pi}{2}\right)$$

think this upper RS

30 60 90

$$\frac{\pi}{6}\left(\left(\frac{1}{2}\right)^2 \quad \left(\frac{\sqrt{3}}{2}\right)^2 \quad \frac{1^2}{1}\right)$$

(16) $\frac{\pi}{6}\left(\frac{1}{4} + \frac{3}{4} + 1\right)$

$\frac{2\pi}{6} = \frac{\pi}{3}$ $\frac{\pi}{4}$

Well I did not do rule right

$\frac{\pi}{6}\left(\frac{f(0)}{2} + f\left(\frac{\pi}{6}\right) + f\left(\frac{\pi}{3}\right) + \frac{f(\pi)}{2}\right)$

start and end $\frac{\text{both here}}{2}$

$\frac{\pi}{6}\left(\frac{0}{2} + \frac{1}{4} + \frac{3}{4} + \frac{1}{2}\right)$

$\frac{\pi}{6}\left(\frac{3}{2}\right)$

$\frac{3\pi}{12} \Rightarrow \frac{\pi}{4}$ ✓

So just forgot trapizodal rule form

can check by integrating

10. $F(x) = \int_0^x e^{-t^2}\,dt$

a. Find $F'(1)$ and $F''(1)$

So is this FTC? yes

$f(h) \cdot h'$

~~– but its not deriv of~~ – yes it is

$e^{-x^2} \cdot 1$ and eval at 1

$e^{-1^2} = e^{-1}$ ✓

(17) $f'' = -2x\, e^{-x^2-1}$

~~? write out exactly~~ take the deriv! duh - just do it!

Chain rule $\longrightarrow$ no deriv!

and eval at 1

$-2(1)\, e^{-(1)^2-1}$

$-2\, e^{-2}$ ✓

b) Express $\int_1^2 e^{-u^2/4}\, du$ in terms of $F(x)$

So $F(x) = \int_0^x -e^{-t^2}\, dt$

this is one of those do you really know it

$F(2) - F(1)$ $\quad \frac{1}{2t}$

but how $-\frac{u^2}{4} \to -t^2$

- can't just change t to be or say 2t

$2F(2) - 2F(1)$

No - not how they solved it

$A = \int_1^2 e^{-u^2/4}\, du$

$u = v/2 \quad du = \frac{dv}{2}$ sub in

$A = \int_{u_{start}}^{u_{finish}} e^{-v^2}\, \frac{dv}{2}$

(18)

$$\frac{1}{2}\int_{\frac{1}{2}}^{1} e^{-u^2}\,du$$

$$\frac{1}{2}\left(F(1)-F\left(\tfrac{1}{2}\right)\right)$$

is that close to my ans

$2(F(2)-F(1))$

will be diff - but maybe work out

What did they do?

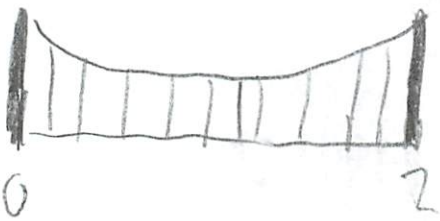
- u sub of $\frac{u}{2}$ ← think I kinda did, but had a different u or wrong way

I did 2f not $\frac{f}{2}$

- so that it was =

11. 2 towers suspension brige

in km



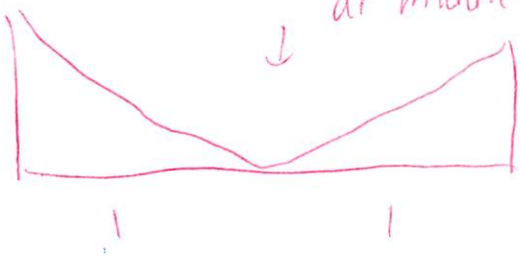
$y = \frac{1}{10}x^2$

So lenght of string

- arc lenght
- what is formula for that
- that was the end av I think

$\cancel{\int_0^2 \pi \frac{1}{10}x^2}$ ✓ formula

(19) Again some weirdness



- ? how supposed to know built like this - touching in middle
- but this makes it easier

$$2\int_0^1 ds$$

$$2\int_0^1 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$ ← oh that was formula

$$2\int_0^1 \sqrt{1+\left(\frac{1}{5}x\right)^2}$$ $\quad y = \frac{1}{10}x^2$

$$2\int_0^1 \left(1+\frac{1}{25}x^2\right)^{1/2}$$ $\quad \frac{dy}{dx} = \frac{1}{10}\cdot 2x = \frac{2}{10}x = \frac{1}{5}x$

$u = 1+\frac{1}{25}x^2$

$du = \frac{2}{25}x\,dx$ ← $\frac{du}{\frac{2}{25}} = \frac{25du}{2}$

just set up may be right

$$2\int_0^1 u^{1/2}\,\frac{25du}{2}$$

$$2\,\frac{2\left(1+\frac{1}{25}x^2\right)^{3/2}}{3}\cdot\frac{25}{2}\Big|_0^1$$

$$\frac{100}{5}\left(1+\frac{1}{25}x^2\right)^{3/2}$$ ← seems wrong

b) What is avg lenght of vertical cables

← just think about 1

- sum up all of the $\frac{\text{lenghts}}{n}$?
- right which is an integral

(20)

$$\int_0^1 \frac{1}{10} x^2 \, dx$$

$$\frac{1}{10} \int_0^1 x^2 \, dx$$

$$\frac{1}{10} \frac{x^3}{3} \Big|_0^1$$

$$\frac{1}{10} \frac{1^3}{3} - \frac{1}{10} \cdot \frac{0^3}{3}$$

$$\frac{1}{30}$$ ✓ got it right by self!

12. Evaluate

- ok here is techniques of integration section
- think learned this fairly well

$$\int \frac{dx}{x^2+3x+2}$$

? complete the square

$(x+1)(x+2)$ ← no just factor

? yes add

$$\frac{1}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{(x+2)}$$

? set $x = -2$ for this to be 0

? is this right

$$\frac{1}{-1} = \frac{A}{-1} + 0$$

$$-1 = -1A$$

$$A = 1$$

here set $x = -1$

$$\frac{1}{1} = 0 + \frac{B}{1}$$

$$B = -1$$

think did wrong

- will try next pg

(21)

$$\frac{1}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{(x+2)}$$

multiply both sides $x+1$

← much better - a really stupid forgetting of process
tiny mistake

$$\frac{1}{(x+2)} = A + \frac{B(x+1)}{(x+2)}$$ ← now set $x=-1$

$$\frac{1}{(-1+2)} = A + 0$$

$$\frac{1}{1} = A$$ ✓

now multiply both sides $x+2$

$$\frac{1}{x+1} = \frac{A(x+2)}{x+1} + B$$ ← set $x=-2$

$$\frac{1}{-2+1} = 0 + B$$

$$-1 = B$$ ✓ Much better

Now finish

$$\int \frac{1}{x+1}\,dy + \int \frac{-1}{x+2}\,dx$$

? not x^2 or else may be trig

~~$u = x+1$~~
~~$du = x\,dx$~~

oh $\ln(x+1) - \ln(x+2)$

✓ don't forget what answering

Don't forget definite integral → eval

(22)

$\ln(x+1) - \ln(x+2) \Big|_0^1$

$\ln(2) - \ln(3) - [\ln(1) - \ln(2)]$

see formulas test front pg

$2(\ln(2)) - \ln(3) - \ln(1)$

$\ln 1 = 0$

can add all of these

$\ln\left(\frac{4}{3}\right)$ ← how know?

12b $\int x^2 \ln x \, dx$

$uv = \int v u'$ ✓ got that - nope its −

$du = x^2$ $dv = \ln x$

$u = \frac{x^3}{3}$ $dv = \frac{1}{x}$ ✓ wrong way ✓ stupid errors really stupid

$\cancel{x^2 \cdot \frac{1}{x} + \int \frac{1}{x} \ln x}$

$\frac{x^3}{3} \cdot \ln x - \int \ln x \, x^2$ flip u and v

$u = \ln x$

$du = \frac{1}{x}$ ← not in there

do have to do by parts again

$dv = x^2$ back where started

$\frac{\ln(x) x^3}{3} -$

$\int \frac{x^3}{3} \cdot \frac{1}{x}$

$\frac{x^3}{3 \cdot 3} \to \frac{x^3}{9}$

- factor out $\frac{x^3}{3}\left(\ln x - \frac{1}{3}\right)$ +

(23) - Ok - got rule wrong → was −

- mixed up deriv

- ~~ok~~ and know either can be the ' one

- So don't have to remember which

13. $\int_0^1 \frac{dx}{(x^2+1)^2}$ by subbing in $x = \tan u$

$\frac{d}{dx} \tan = \sec^2$

~~but $\int \tan x = \frac{1}{(x-1)^2}$ ← ???~~

$\frac{d}{dx} \tan^{-1} x = \frac{1}{x^2+1}$

So $u = \tan^{-1} x$

$du = \frac{1}{x^2+1}$ ← but have this 2 ↑ ? how backwards inverse again

Write like this

$u = \tan^{-1} x$ ← or do they

$du = \frac{1}{x^2+1}$

$x = \tan u$

$dx = \sec^2 u \, du$

$\int_{u\,start}^{u\,finish} \frac{\sec^2 u \, du}{(1+\tan^2 u)^2}$

u start $= \tan^{-1} 0 = 0$

u finish $= \tan^{-1} 1 = \frac{\pi}{4}$

$\int_0^{\pi/4} \frac{\sec^2 u}{\sec^4 u} du$ ← note the trig identity

(24) $\int_0^{\pi/4} \cos^2 u \, du$

$\int_0^{\pi/4} \frac{\cos(2u)+1}{2} \, du$ ← double angle?

$\left(\frac{u}{2} + \frac{\sin(2u)}{4}\right)\Big|_0^{\frac{\pi}{4}}$

$\frac{\pi}{8} + \frac{1}{4}$

So got it was inverse
Forgot exact needing
Could not figure out inverse part
and would not have gotten 2x angle identity
reducing part
- perhaps - but would have thought had made
wrong turn

~~14. Starting at pt where r=1 the point~~
Polar qu which we skipped

15. Evaluate
- a) $\lim_{x \to 0} \frac{\sin^2 x}{1-\cos x}$ ← ok this is the normal evaluate

$\sin^2 x + \cos^2 x = 1$

(25) $\dfrac{1-\cos^2 x}{1-\cos x}$ So now what

Well $\lim_{x \to 0}$ ← So. that does not mean solve right at end

Split next ← what I thought

$$\frac{(1+\cos x)\cancel{(1-\cos x)}}{\cancel{1-\cos x}}$$

$1+\cos x$

↑ now $x=0$

$1+1=2$ (U) ← got it - just 1 hesitation

Should be able to do tese type of problems

b) $\lim_{x \to 1} \dfrac{(\ln x)^2}{x-1}$

↑ now how to do this

$\dfrac{\ln x \ \ln x}{x-1}$ ↑ what in all world

← can't just plug in 1

= by definition $\left((\ln(x))^2\right)' \Big|_{x=1}$

take derivative ↳ $\dfrac{2\ln x}{x}\Big|_{x=1} = \boxed{0}$ ← don't get

So they took deriv

↑ is this what we did at start of calc? last year

yeah L'Hopital's rule - we did not do this year

26

c) $\lim_{x \to \infty} x^2 e^{-x}$

? what to do ? take deriv ???

$2x e^{-x} + -1 e^{-x} \cdot x^2$

$\lim \frac{x^2}{e^x}$ ← only way L'Hopital which did not do

Review here: L'Hopital

if $\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = 0$ ← oh I remember when → $\frac{0}{0}$

then $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{f'(x)}{g'(x)}$

16. Evaluate $\int_1^\infty \frac{dx}{x^{3/2}}$

$x^{-3/2}$

$\left. \frac{x^{-1/2}}{-1/2} \right|_1^\infty$

$\frac{-2}{\sqrt{x}}$

$\frac{-2}{\sqrt{\infty}} - \frac{-2}{\sqrt{1}}$

$0 \quad +2$

2 ✓ got it :)

(27)

17. For what values of p does $\sum_{1}^{\infty} \frac{n}{\sqrt{4+n^p}}$

- is this one of those infinite sum problems?
- but I feel urge to solve it on my own
- no do ratio test p = x usually

$$\left| \frac{\frac{n+1}{\sqrt{4+(n+1)^p}}}{} \cdot \frac{\sqrt{4+n^p}}{n} \right|$$

I am always have loss how to reduce

~~$\frac{n+1}{\sqrt{4+}}$~~ well ~~value~~ for what values of p ~~$\frac{}{}$~~ is thing < 1

they reduced it first - or estimate it

$\sqrt{4+n^p} \cong \sqrt{n^p} = n^{p/2}$ ↑ I would never try something like that

$\frac{n}{n^{p/2}} = n^{1-p/2}$ ← would not have thought of that either

Now ratio test

$$\frac{(n+1)^{1-p/2}}{(n)^{1-p/2}}$$

they do $\frac{b_n}{a_n} = \frac{\frac{n}{\sqrt{n^p}}}{\frac{n}{\sqrt{n^p+4}}} = \frac{\sqrt{1+\frac{4}{n^p}}}{\sqrt{1}}$

↑ never seen that before

(28)

$= \sqrt{1+4n^{-p}} = 1$ for $p > 0$

So converges

$1 - \frac{p}{2} < -1$

$p > 4$

<u>note for $p < 0$</u>

$\frac{n}{\sqrt{4+n^p}} > \frac{n}{\sqrt{5}}$ does not go to 0

does not converge at here

don't think we never learned

18. Find by differentiating $f(x) = \sqrt{1+x}$ w/ first 4 terms taylor series

$f' = \frac{1}{2}(x+1)^{-1/2} \cdot 1$

$f(a) + \underline{f'(a)(x-a)} + \frac{f''(a)(x-a)^2}{2!} + \frac{f^3(a)(x-a)^3}{3!}$

~~a = 0~~

b) Find $\sqrt{1.2}$ so $x = .2$ $a = 0$

$f(0) + f'(0)(.2-0) + \frac{f''(0)(.2)^2}{2} + \frac{f'''(0)(.2)^3}{3 \cdot 2}$

(29)

$f = \sqrt{1+x}$

$f' = \frac{1}{2}(x+1)^{-1/2}$

$f'' = -\frac{1}{4}(x+1)^{-3/2}$

$f''' = -\frac{1}{4} \cdot -\frac{3}{2}(x+1)^{-5/2}$

$\frac{3}{8}(x+1)^{-5/2}$

$\sqrt{1} + \frac{1}{2}(1)^{-1/2}(.2) - \frac{\frac{1}{4}(1)^{-3/2}(.2)^2}{2} + \frac{\frac{3}{8}(1)^{-5/2}(.2)^3}{6}$

$1 + \frac{1}{2} \cdot \frac{1}{5} - \frac{\frac{1}{4} \cdot \frac{1}{25}}{2} + \frac{\frac{3}{8} \cdot \frac{1}{125}}{6}$

$1 + \frac{1}{10} - \frac{1}{200} + \ldots$ small

✓ coh actuall right

✓

11. Find the Taylor series for $\tan^{-1} x$ around $x=0$ by using term by term differentiation or integration on the appropriate geometric series. Give enough terms to make pattern clear.

- ok this is - show w/ the formula

So $\frac{1}{1-x} = \sum x^n$

$\frac{1}{1+x} = \sum (-x)^n$ ← I think $(-1)^n x^n$

seperate

(30) $\frac{1}{1+x^2} = (-x)^{2n}$

and that is deriv of $\tan^{-1}$

? So integrate

$\int (-x)^{2n}$ ← write as $\frac{(-1)^n (x)^{2n+1}}{2n+1}$

$$\sum \frac{(-x)^{2n+1}}{2n+1}$$

$$\frac{(-x)^1}{1} + \frac{(-x)^{2+1}}{3} + \frac{(-x)^5}{5} + \frac{(-x)^7}{7}$$

yeah got it almost
except split out
- but I relied on cheat sheet
- but had good idea how to solve from today
- and Brendan's office hrs

Pratice test done 9:19 PM

- took me 6 hrs
- need to do it in 3 tomorrow

do more related rates, series

I need to go into test knowing I've seen each type of problem, and just do them and hope not to make mistakes

Math Review Session Primer

12/16

Power Series for $\dfrac{x^2}{(1-2x)^2}$

Using $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ ← memorize

$$\frac{x^2}{(1-2x)^2} = x^2\left(\frac{1}{(1-2x)^2}\right)$$ ← close, not quite

$(1-x)^{-1}$ compared to $(1-x)^{-2}$

- One is the derivitive of the other

$$\frac{d}{dx}(1-x)^{-1} = +(1-x)^{-2}$$

So take $\frac{d}{dx}$ of the whole thing. $\frac{d}{dx}\sum_{n=0}^{\infty} x^n$

$$\therefore \frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} n x^{n-1} \qquad |x|<1$$

← could use ratio test further out

$$\frac{1}{(1-2x)^2} = x^2 \sum_{n=0}^{\infty} n(2x)^{n-1} \qquad |x| < \tfrac{1}{2}$$

sort-of a geometric series

$$\frac{x^2}{(1-2x)^2} = \sum_{n=0}^{\infty} n\, 2^{n-1}\, x^{n-1+2}$$

(2) first term $=0$, drop 1st term

$$\sum_{n=1}^{\infty} n\, 2^{n-1} x^{n+1}$$

- could use ratio test to check if unsure
  Or check the radius of convergence

Find the power series for

$\ln(5-x)$

(1) What of the ones you know

$\ln(1-x)$ ← could integrate $\frac{1}{1-x}$ also

But the 5 is trouble
- so pull it out

$\ln\left(5\left(1-\frac{x}{5}\right)\right)$

$\ln(5) + \ln\left(1-\frac{x}{5}\right)$

$$\ln(5) + -\sum_{n=0}^{\infty} \frac{\left(\frac{x}{5}\right)^{n+1}}{n+1}$$

$$\ln(5) - \sum_{n=1}^{\infty} \frac{\left(\frac{x}{5}\right)^{n}}{n}$$

what is radius of convergance?

$\left|\frac{x}{5}\right| < 1$

$|x| < 5$ ✓ did not know that was true

$$\ln(1-x) = -\int \frac{dx}{1-x} \quad |x|<1$$

$$-\int \frac{dx}{1-x} = -\int \sum_{n=0}^{\infty} x^n \, dx$$

$$\ln(1-x) = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} + C \quad \text{figure out} \quad |x|<1$$

find c, eval both at $x=0$

$$\ln(1) = 0 = -\sum_{n=0}^{\infty} \frac{0^{n+1}}{n+1} + C$$

*note $0^0 = 1$ $0^n = 0$

$0 = C$

$$\ln(1-x) = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$$

③

<u>final ans</u>

$$\ln(5) - \sum_{n=1}^{\infty} \frac{\left(\frac{x}{5}\right)^n}{n} \qquad |x| < 5$$

do ratio test to ✓

$$\left| \frac{x^{n+1}}{5^{n+1}(n+1)} \cdot \frac{5^n n}{x^n} \right|$$

-alternating sign does not matter drop $(-1)^n$

$$\lim_{n\to\infty} \left| \frac{n}{n+1} \cdot \frac{x}{5} \right|$$

$$\left| 1 \cdot \frac{x}{5} \right| \overset{\text{need}}{<} 1$$

$$\left|\frac{x}{5}\right| < 1$$

$$|x| < 5 \qquad -5 < x < 5$$

<u>could check end points</u>

radius $= 5$

interval $= (-5, 5)$ ← where series converges

eval endpoints to see if finite / converges to see if can inclu

<u>$x = -5$</u>

plug in

$$\ln(5) - \sum_{n=1}^{\infty} \frac{\left(\frac{-5}{5}\right)^n}{n}$$

does not matter

$$\frac{(-1)^n}{n}$$ do alternating series test

(4)

Alt Series test

$\sum (-1)^n a_n$ converges if $a_n \to 0$

$a_{n+1} < a_n$

$a_n > 0$

$\sum (-1)^{n+75} a_n$ ← still works same but $\sum (-1)^{2n}$ would not

So converges at -5 ✓

$\underline{x=5}$

$$\ln(5) - \sum \frac{\left(\frac{5}{5}\right)^n}{n}$$

$$- \sum_{n=1}^{\infty} \frac{1}{n}$$ ← diverges, harmonic series

$\rightsquigarrow$ interval $[-5, 5)$

Taylor Remainer

$f(x) = e^{5x}$

a) Find taylor Series at $a=0$

b) Find Radius of Convergence

c) Show for $T_k(x) = \sum_{n=0}^{k} \frac{f^{(n)}(0)}{n!}(x-0)^n$

$$\lim_{k\to\infty} T_k(x) = e^{5x}$$

(5) a) Find $f'(x)$ $f''(x)$ $f'''(x)$

Or could find a pattern

$f(x) = e^{5x}$

$f'(x) = 5e^{5x}$ } pay attention $\frac{d}{dx} 5x = 5$

- I always screw this up

$f''(x) = 5^2 e^{5x}$

$f'''(x) = 5^3 e^{5x}$

$f^{(n)}(x) = 5^{(n)} e^{5x}$

so $f(x) = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!} x^n$

$\sum_{n=0}^{\infty} \frac{5^n}{n!} \times 5^n e^{5(0)} x^n$

↑ should look like a polynomial otherwise forgot to eval

b) $\left| \frac{b^{n+1}}{b^n} \right|$

$\left| \frac{5^{n+1} x^{n+1}}{(n+1)!} \cdot \frac{(n)!}{5^n x^n} \right|$

$\lim_{n \to \infty} \left| \frac{5^1 x}{n+1} \right| < 1$? ✓

see pg 13b but

yes and it $= 0$ for any fixed x

$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}$

converges to $2.\ldots$

radius convergence ∞ ← so

$\boxed{* \text{ so } n \to \infty \to \frac{5}{\infty} \to \text{ go to } 0}$ it

(6) This means

$$\sum_{n=0}^{\infty} \frac{5^n (22)}{n!} = e^{5 \cdot 22}$$

for any x can eval there

$$\sum \frac{\pi^n}{n!} = e^{\pi}$$

← challenge qu to see if you get it - really quick qu

$$\boxed{e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}}$$

c) Now show $R_n \to 0$

$R_k(x) := f(x) - T_k(x)$ ← $T_k(x) = e^{5x}$ finite # of terms

↖ definition of

Want $\lim_{k \to \infty} R_k(x) = 0$

where does it = the function?

after checking it converges

as # terms → ∞ it gets closer + closer

$$|R_k(x)| \leq \frac{M}{(k+1)!} (x-a)^{k+1}$$

where $|M| \geq |f^{(n+1)}(c)|$ on some interval

↑ some certain interval

/ pick an arbitrary size

since to show → converges an infinite series

must show converges on every finite region

① Fix some d arbitrary

↑ #

② look at interval $|x| < d$

- this case $-d < x < d$

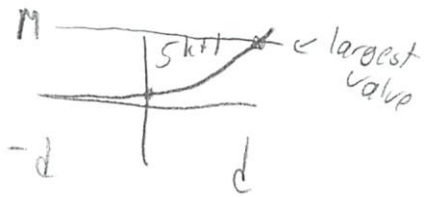
(7) $|R_k(x)| \le \frac{M}{(k+1)!} d^{k+1}$

what is M?

* Look at $|x| < d$ interval +
* Bound $f^{k+1}(x)$ on that interval

$f^{k+1} = 5^{k+1} e^{5x}$ ← in for n

↑ exponential



→ if know what is at d

bound on whole interval

So $|f^{(k+1)}(x)| \le 5^{k+1} e^{5d} = M$ ← bounding function on $|x| < d$

M is the maximum ← k does not nessily influence where the val function

- not always on an end point

- in recitation M was on other side

Picked an $x = d$ for Max

Plug in d for x get M

k is where stop # of terms

So write $\left| \frac{\overbrace{5^{k+1} e^{5d}}^{M} d^{k+1}}{(k+1)!} \right|$ ← goes to ∞ factorials grow way large

← remainder goes to 0

works for all d

So function is true

Practically

So basically, plug k+1 in for n and d in for x and see where it goes if goes to 0 → then works for all d

what if M different

as M

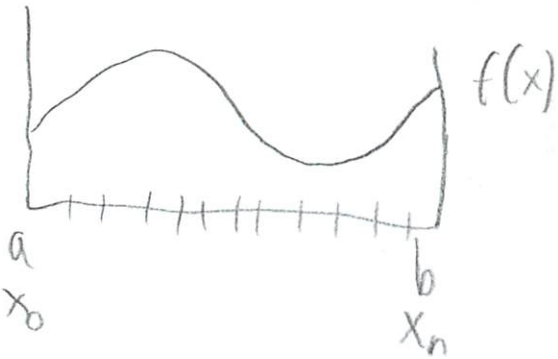
do not assign a value + show - keep letters

(8) More Pratice for this on P-Set

~~Def~~ Riemann Sums

- one definition for integral
- Some weird integrals can't take Riemann sum

- Trying to go from approx to actual value



$$\int_a^b f(x)\,dx$$

- partition into n pieces
- add rectangles
- # partitions $\to \infty$
- so width of each $\to 0$

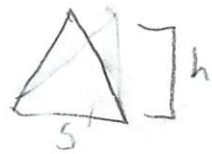
Upper + Lower Riemann sums converge to same thing

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\cdot \Delta x$$

$$\Delta x = \frac{b-a}{n}$$

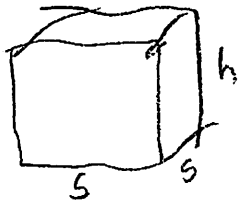
Find the volume of a pyrmid (note TA really screwed up redill at end)

- square base of sides s
- height = h



(9) 1st guess (very bad)

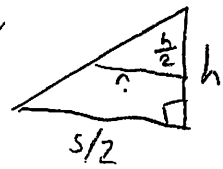
over approx



$V = s^2 h$

2nd guess

← but what is s here



$$V = s^2 \frac{h}{2} + \left(\frac{s}{4}\right)^2 \frac{h}{2}$$

$$\frac{\frac{s}{2}}{h} = \frac{?}{h/2}$$ cross multiply

$$h? = \frac{s}{2} \cdot \frac{h}{2}$$

$$\frac{h?}{h} = \frac{\frac{sh}{4}}{h}$$

$$\boxed{\frac{s}{4}}$$

3rd guess n pieces

height $= \frac{h}{n}$

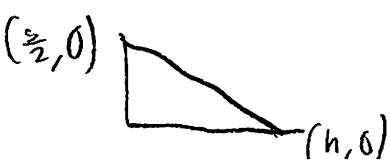
side cays linearly w/ h

want linear function so that $h = 0 \to \frac{s}{2}$

$h = h \to 0 = s$

$$s(h) = h - \frac{2h}{s}$$

turn on side



$$s(ht) = \frac{s}{2} - \frac{s}{2ht}$$

(10) square side lenght

at the ith interval

side lng (height) $= \frac{s}{2} - \frac{s}{2hi}$

Briner confused

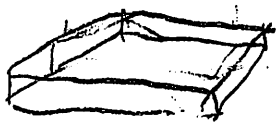
I am too

* s, h fixed
side len
height changing

$$= 2\left(\frac{s}{2} - \frac{sh}{2hi}\right) = s - \frac{sh}{hi}$$

need to add

height of each rectangle · side lenght 2
at a paticular height on pyrmid



$i \to 0 \to n-1$

n partitions let $i = 0, \ldots, n-1$

$$\sum_{i=0}^{n-1} \underbrace{\frac{h}{n}}_{\Delta = dh} \cdot \left(s - \frac{sh}{hi}\right)^2 = dh\left(s - \frac{s}{x}\right)$$

$n \to \infty$

$\frac{h}{n} \to 0$

$\int_0^h \left(s - \frac{s}{x}\right) dx$ ← small height

Δ = increments getting small

eval at different values of h

(11)

$$\int_0^h s^2 - \frac{2s^2}{x} + \frac{s^2}{x^2}\,dx$$

i get a log
Christee confused

↑ screwed up

Completly screwed it up
erase all of it

↓ better

restart

] want to do based off this $h - y$

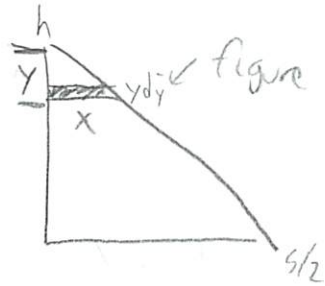


figure savores from what y is

$$\frac{x}{y} = \frac{s}{2h}$$ similar triangles

solve x in terms of y

$$x = \frac{ys}{2h}$$ side lenght = x

Area of savore at y from top

(i think)

$$\frac{y^2 s^2}{4h^2} \cdot 2^2 = \frac{y^2 s^2}{h^2}$$

Volume of rectangle

$$\int_0^h \frac{y^2 s^2}{h^2}\,dy$$ ← the little y - the change in y $= \frac{h}{n}$

(12) $\frac{x^3}{3} \cdot \frac{s^2}{h^2} \Big|_0^h = \frac{hs^2}{3}$ Seems fairly easy if know how to do

---

FTC

If you define $g(x) = \int_0^x f(t)\,dt$

$g'(x) = f(x)$ integral $g(x)$ $f(t)$ deriv

but why x and t
oh from $x \to 0$
$g(x) - g(0)$
$g(x)$?
but I think there is more funny biz in here

$g(x) = \int_0^{h(x)} f(t)\,dt$

then $g'(x) = f(h(x)) \cdot h'(x)$

right

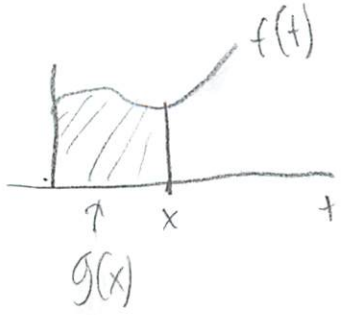
oh w/ v sub
have to change input to $v(x)$ not $v'(x)$

$g(x) = \int_{k(x)}^{h(x)} f(t)\,dt$ but w/ t

$g'(x) = f(h(x)) \cdot h'(x) - f(k(x)) \cdot k'(x)$ just

$\frac{d}{dx} \int_0^{4x} \sin(3t)\,dt$ using this formula

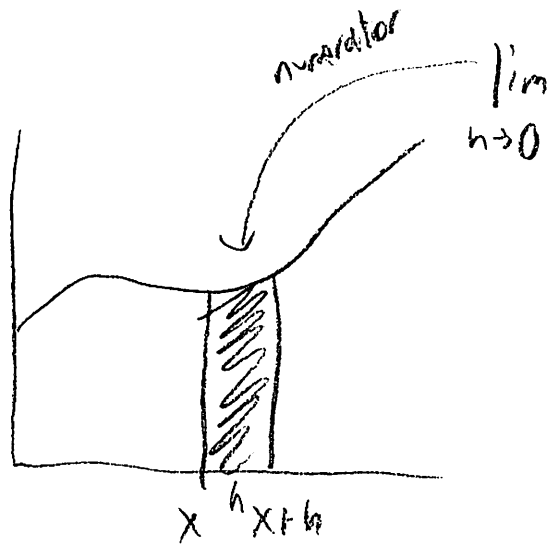
$4\sin(12x)$



$\cancel{\lim_{h \to 0} \frac{g(x+h) - g(x)}{h}}$

See next

(13)



$$\lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

divide by h
get height of the rectangle
that sits at $x = f(x)$
as h gets $\to 0$ ...
this is FTC

More FTC from Paul Lamar Notes

$$g(x+h) - g(x) = \int_a^{x+h} f(t)\,dt - \int_a^{x} f(t)\,dt$$

Yeah just $\int_x^{x+h} f(t)\,dt$

$$\cancel{\frac{g(x+h) - g(x)}{h}} = \frac{1}{h} \int_x^{x+h} f(t)\,dt$$

just add

never mind not going through proof

(13b) Added Work

Does $\frac{1}{1+n}$ converge?

$$\left| \frac{1}{1+(n+1)} \cdot \frac{1+n}{1} \right| < 1 ?$$

$$\left| \frac{1}{2+n} \cdot \frac{1+n}{1} \right| = \left| \frac{1+n}{2+n} \right| < 1$$

? how what do you do w/ this

put in lim as $n \to \infty$

$$\lim_{n \to \infty} \left| \frac{1+n}{2+n} \right| \quad \frac{\infty}{\infty} = 1 = \text{unsure}$$

---

try $\frac{d}{dx} \int_0^{4x} \sin(3t)\, dt$

$$\sin(3t\,4x) \cdot \overset{h'}{\cancel{\cos(3t) \cdot 3}}\ 4$$

and t disappears

$$\cancel{3 \sin 12xt \cdot \cos 3t}$$

$$\sin(3 \cdot 4x) \cdot 4 \quad \checkmark$$

The proof - don't have to know

Formulas for the Remainder Term in Taylor Series

In Section 11.10 we considered functions f with derivatives of all orders and their **Taylor series**

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

The nth partial sum of this Taylor series is the **nth-degree Taylor polynomial of f at a**:

$$T_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

We can write

$$f(x) = T_n(x) + R_n(x)$$

where $R_n(x)$ is the **remainder** of the Taylor series. We know that f is equal to the sum of its Taylor series on the interval $|x-a| < R$ if we can show that $\lim_{n\to\infty} R_n(x) = 0$ for $|x-a| < R$.

Here we derive formulas for the remainder term $R_n(x)$. The first such formula involves an integral.

$\boxed{1}$ **Theorem** If $f^{(n+1)}$ is continuous on an open interval I that contains a, and x is in I, then

$$R_n(x) = \frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)\,dt$$

Proof We use mathematical induction. For $n = 1$,

$$R_1(x) = f(x) - T_1(x) = f(x) - f(a) - f'(a)(x-a)$$

and the integral in the theorem is $\int_a^x (x-t)f''(t)\,dt$. To evaluate this integral we integrate by parts with $u = x - t$ and $dv = f''(t)\,dt$, so $du = -dt$ and $v = f'(t)$. Thus

$$\begin{aligned}\int_a^x (x-t)f''(t)\,dt &= (x-t)f'(t)\Big]_{t=a}^{t=x} + \int_a^x f'(t)\,dt \\ &= 0 - (x-a)f'(a) + f(x) - f(a) \qquad \text{(by FTC 2)} \\ &= f(x) - f(a) - f'(a)(x-a) = R_1(x)\end{aligned}$$

The theorem is therefore proved for $n = 1$.

Now we suppose that Theorem 1 is true for $n = k$, that is,

$$R_k(x) = \frac{1}{k!}\int_a^x (x-t)^k f^{(k+1)}(t)\,dt$$

We want to show that it's true for $n = k + 1$, that is

$$R_{k+1}(x) = \frac{1}{(k+1)!}\int_a^x (x-t)^{k+1} f^{(k+2)}(t)\,dt$$

Again we use integration by parts, this time with $u = (x - t)^{k+1}$ and $dv = f^{(k+2)}(t)$. Then $du = -(k + 1)(x - t)^k\, dt$ and $v = f^{(k+1)}(t)$, so

$$\begin{aligned}
&\frac{1}{(k+1)!}\int_a^x (x-t)^{k+1} f^{(k+2)}(t)\, dt \\
&\quad = \frac{1}{(k+1)!}(x-t)^{k+1} f^{(k+1)}(t)\Big]_{t=a}^{t=x} + \frac{k+1}{(k+1)!}\int_a^x (x-t)^k f^{(k+1)}(t)\, dt \\
&\quad = 0 - \frac{1}{(k+1)!}(x-a)^{k+1} f^{(k+1)}(a) + \frac{1}{k!}\int_a^x (x-t)^k f^{(k+1)}(t)\, dt \\
&\quad = -\frac{f^{(k+1)}(a)}{(k+1)!}(x-a)^{k+1} + R_k(x) \\
&\quad = f(x) - T_k(x) - \frac{f^{(k+1)}(a)}{(k+1)!}(x-a)^{k+1} \\
&\quad = f(x) - T_{k+1}(x) = R_{k+1}(x)
\end{aligned}$$

Therefore, (1) is true for $n = k + 1$ when it is true for $n = k$. Thus, by mathematical induction, it is true for all n.

To illustrate Theorem 1 we use it to solve Example 4 in Section 11.10.

EXAMPLE 1 Find the Maclaurin series for $\sin x$ and prove that it represents $\sin x$ for all x.

SOLUTION We arrange our computation in two columns as follows:

$$\begin{aligned}
f(x) &= \sin x & f(0) &= 0 \\
f'(x) &= \cos x & f'(0) &= 1 \\
f''(x) &= -\sin x & f''(0) &= 0 \\
f'''(x) &= -\cos x & f'''(0) &= -1 \\
f^{(4)}(x) &= \sin x & f^{(4)}(0) &= 0
\end{aligned}$$

Since the derivatives repeat in a cycle of four, we can write the Maclaurin series as follows:

$$\begin{aligned}
&f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots \\
&\quad = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}
\end{aligned}$$

With $a = 0$ in Theorem 1, we have

$$R_n(x) = \frac{1}{n!}\int_0^x (x-t)^n f^{(n+1)}(t)\, dt$$

Since $f^{(n+1)}(t)$ is $\pm\sin t$ or $\pm\cos t$, we know that $|f^{(n+1)}(t)| \leqslant 1$ for all t. We use the fact (see Exercise 65 in Section 5.2) that, for $a \leqslant b$,

$$\left| \int_a^b f(t)\,dt \right| \leqslant \int_a^b |f(t)|\,dt$$

Thus, for $x > 0$,

$$|R_n(x)| = \frac{1}{n!}\left| \int_0^x (x-t)^n f^{(n+1)}(t)\,dt \right| \leqslant \frac{1}{n!}\int_0^x (x-t)^n |f^{(n+1)}(t)|\,dt$$

$$\leqslant \frac{1}{n!}\int_0^x (x-t)^n\,dt = \frac{1}{n!}\frac{x^{n+1}}{n+1} = \frac{x^{n+1}}{(n+1)!}$$

For $x < 0$ we can write

$$R_n(x) = -\frac{1}{n!}\int_x^0 (x-t)^n f^{(n+1)}(t)\,dt$$

so

$$|R_n(x)| \leqslant \frac{1}{n!}\left| \int_x^0 |x-t|^n |f^{(n+1)}(t)|\,dt \right| \leqslant \frac{1}{n!}\int_x^0 (t-x)^n\,dt = \frac{(-x)^{n+1}}{(n+1)!}$$

Thus, in any case, we have

$$|R_n(x)| \leqslant \frac{|x|^{n+1}}{(n+1)!}$$

The right side of this inequality approaches 0 as $n \to \infty$ (see Equation 11.10.10), so $|R_n(x)| \to 0$ by the Squeeze Theorem. It follows that $R_n(x) \to 0$ as $n \to \infty$, so $\sin x$ is equal to the sum of its Maclaurin series. ∎

For some purposes the integral formula in Theorem 1 is awkward to work with, so we are going to establish another formula for the remainder term. To that end we need to prove the following generalization of the Mean Value Theorem for Integrals (see Section 6.5).

2 Weighted Mean Value Theorem for Integrals If f and g are continuous on $[a, b]$ and g does not change sign in $[a, b]$, then there exists a number c in $[a, b]$ such that

$$\int_a^b f(x)g(x)\,dx = f(c)\int_a^b g(x)\,dx$$

Proof Because g doesn't change sign, either $g(x) \geqslant 0$ or $g(x) \leqslant 0$ for $a \leqslant x \leqslant b$. For the sake of definiteness, let's assume that $g(x) \geqslant 0$.

By the Extreme Value Theorem (4.1.3), f has an absolute minimum value m and an absolute maximum value M, so $m \leqslant f(x) \leqslant M$ for $a \leqslant x \leqslant b$. Since $g(x) \geqslant 0$, we have

$$mg(x) \leqslant f(x)g(x) \leqslant Mg(x) \qquad a \leqslant x \leqslant b$$

and so

$$\boxed{3} \qquad m\int_a^b g(x)\,dx \leqslant \int_a^b f(x)g(x)\,dx \leqslant M\int_a^b g(x)\,dx$$

If $\int_a^b g(x)\,dx = 0$, these inequalities show that $\int_a^b f(x)g(x)\,dx = 0$ and so Theorem 2 is true because both sides of the equation are 0. If $\int_a^b g(x)\,dx \neq 0$, it must be positive and we can divide by $\int_a^b g(x)\,dx$ in (3):

$$m \leqslant \frac{\int_a^b f(x)g(x)\,dx}{\int_a^b g(x)\,dx} \leqslant M$$

Then, by the Intermediate Value Theorem (2.5.10), there exists a number c in $[a, b]$ such that

$$f(c) = \frac{\int_a^b f(x)g(x)\,dx}{\int_a^b g(x)\,dx} \qquad \text{and so} \qquad \int_a^b f(x)g(x)\,dx = f(c)\int_a^b g(x)\,dx$$

$\boxed{4}$ **Theorem** If $f^{(n+1)}$ is continuous on an open interval I that contains a, and x is in I, then there exists a number c between a and x such that

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

Proof The function $g(t) = (x-t)^n$ doesn't change sign in the interval from a to x, so the Weighted Mean Value Theorem for Integrals gives a number c between a and x such that

$$\int_a^x (x-t)^n f^{(n+1)}(t)\,dt = f^{(n+1)}(c)\int_a^x (x-t)^n\,dt$$

$$= -f^{(n+1)}(c)\left.\frac{(x-t)^{n+1}}{n+1}\right]_{t=a}^{t=x} = f^{(n+1)}(c)\frac{(x-a)^{n+1}}{n+1}$$

Then, by Theorem 1,

$$R_n(x) = \frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)\,dt$$

$$= \frac{1}{n!}f^{(n+1)}(c)\frac{(x-a)^{n+1}}{n+1} = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

The formula for the remainder term in Theorem 4 is called **Lagrange's form of the remainder term.** Notice that this expression

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

is very similar to the terms in the Taylor series except that $f^{(n+1)}$ is evaluated at c instead of at a. All we can say about the number c is that it lies somewhere between x and a.

In the following example we show how to use Lagrange's form of the remainder term as an alternative to the integral form in Example 1.

EXAMPLE 2 Prove that Maclaurin series for $\sin x$ represents $\sin x$ for all x.

SOLUTION Using the Lagrange form of the remainder term with $a = 0$, we have

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1}$$

where $f(x) = \sin x$ and c lies between 0 and x. But $f^{(n+1)}(c)$ is $\pm\sin c$ or $\pm\cos c$. In any case, $|f^{(n+1)}(c)| \leqslant 1$ and so

$$|R_n(x)| = \frac{|f^{(n+1)}(c)|}{(n+1)!} |x^{n+1}| \leqslant \frac{|x|^{n+1}}{(n+1)!}$$

By Equation 11.10.10 the right side of this inequality approaches 0 as $n \to \infty$, so $|R_n(x)| \to 0$ by the Squeeze Theorem. It follows that $R_n(x) \to 0$ as $n \to \infty$, so $\sin x$ is equal to the sum of its Maclaurin series.

EXAMPLE 3
(a) Approximate the function $f(x) = \sqrt[3]{x}$ by a Taylor polynomial of degree 2 at $a = 8$.
(b) How accurate is this approximation when $7 \leqslant x \leqslant 9$?

SOLUTION
(a)

$$f(x) = \sqrt[3]{x} = x^{1/3} \qquad f(8) = 2$$

$$f'(x) = \tfrac{1}{3}x^{-2/3} \qquad f'(8) = \tfrac{1}{12}$$

$$f''(x) = -\tfrac{2}{9}x^{-5/3} \qquad f''(8) = -\tfrac{1}{144}$$

$$f'''(x) = \tfrac{10}{27}x^{-8/3}$$

Thus the second-degree Taylor polynomial is

$$T_2(x) = f(8) + \frac{f'(8)}{1!}(x-8) + \frac{f''(8)}{2!}(x-8)^2$$

$$= 2 + \tfrac{1}{12}(x-8) - \tfrac{1}{288}(x-8)^2$$

The desired approximation is

$$\sqrt[3]{x} \approx T_2(x) = 2 + \tfrac{1}{12}(x-8) - \tfrac{1}{288}(x-8)^2$$

(b) Using the Lagrange form of the remainder term we can write

$$R_2(x) = \frac{f'''(c)}{3!}(x-8)^3 = \tfrac{10}{27}c^{-8/3}\frac{(x-8)^3}{3!} = \frac{5(x-8)^3}{81c^{8/3}}$$

where c lies between 8 and x. In order to estimate the error we note that if $7 \leqslant x \leqslant 9$, then $-1 \leqslant x - 8 \leqslant 1$, so $|x-8| \leqslant 1$ and therefore $|x-8|^3 \leqslant 1$. Also, since $x > 7$, we have

$$c^{8/3} > 7^{8/3} > 179$$

and so

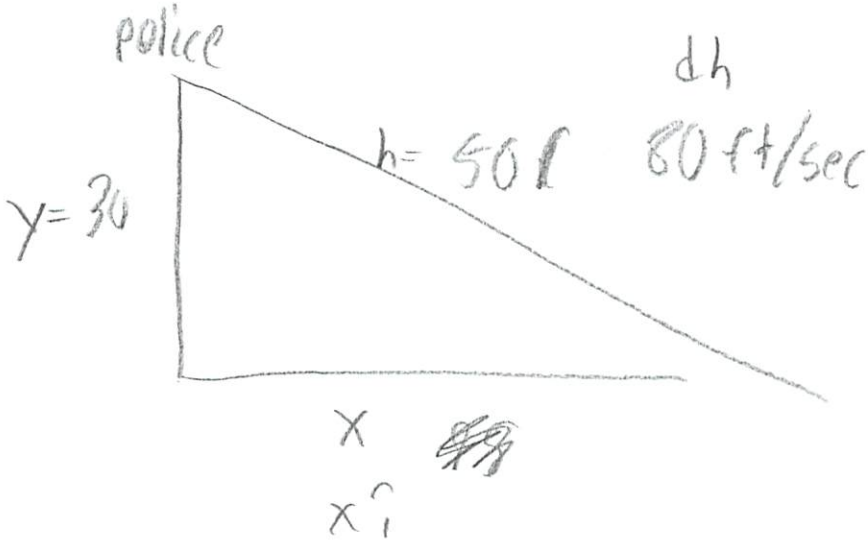
$$|R_2(x)| = \frac{5|x-8|^3}{81c^{8/3}} < \frac{5 \cdot 1}{81 \cdot 179} < 0.0004$$

Thus if $7 \leqslant x \leqslant 9$, the approximation in part (a) is accurate to within 0.0004.

(1) More Pratice 18.01 12/16
Last Hour 11PM

~~I think the~~

The tiny related rates is fairly easy



$x = \sqrt{50^2 - 30^2}$ ← same # different story

$x = 40$

$x^2 + x^2 = h^2$
50^2

$2xx' + 2xx' = 0$ ← note 0

$2 \cdot 30 \cdot y' +$

← fixed

$x^2 = h^2 - 30^2$

$2xx' = 2hh' - 0$

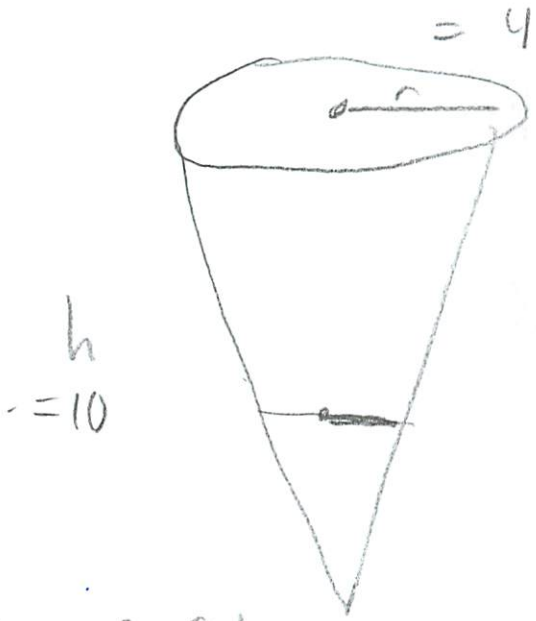
$2 \cdot 40 \cdot x' = 2 \cdot 50 \cdot 80$

$80x = 8000$ $= 100$ speeding 5 mph

✓ good

(2) Now a non trig one

Like a conical tank



- well the hard part is knowing the volume of a cone

$V = \frac{1}{3}\pi r^2 h$

$V = 12$ ft/min

at $h = 5$, dh?

$V_{total} = \frac{1}{3}\pi\, 4^2 \cdot 10 = \frac{160\pi}{3}$

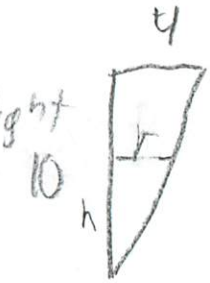
implicit

↓ static

$dV = \frac{1}{3}\pi \cdot 2r \cdot h'$

based on height

fill in r before differentiating



$dV = \frac{1}{3}\pi\, 2 \cdot \frac{4}{10} h\, h'$

$2 = \frac{1}{3}\pi \cdot 2 \cdot \frac{4}{10} \cdot 5 \cdot h'$

$2 = \frac{\pi \cdot 2 \cdot 4 \cdot 5}{3 \cdot 10} h'$

$2 = \frac{40\pi}{30} h'$

$2 = \frac{4}{3}\pi\, h' \rightarrow h' = \frac{2 \cdot 3}{4}\pi = \frac{6\pi}{4} = \boxed{\frac{3}{2}\pi}$

$\frac{10}{4} = \frac{h}{r}$

$\frac{4}{10} = \frac{r}{h}$

$r = \frac{4}{10} h$ ✓

Close but 1st time I got this problem type

③ Yeah the first time I really understood related rates whats going on

Ok now series - review today's review

$$\frac{x^2}{(1-2x)^2} \qquad x^2 \frac{1}{(1-2x)^2}$$

$$x^2 \frac{1}{1-x} = x^n$$

no must take deriv

duh - knew this was wrong

$$x^2 \frac{1}{(1-x)^2} = \cancel{x^n} \; n x^{n-1}$$

$$\frac{d}{dx} \frac{1}{1-x} = \frac{d}{dx} x^n$$

$$(1-x)^{-1}$$

$$-(1-x)^{-2} = n x^{n-1}$$

$$-\frac{1}{(1-x)^2}$$

$$x^2 \frac{1}{(1-2x)^2} \quad \cancel{(2x)^n}$$

$$n(2x)^{n-1}$$

now and back x^2

$$\frac{x^2}{(1-2x)^2} = \sum n \, 2^{n-1} x^{n-1+2}$$

$$n \, 2^{n-1} x^{n+1}$$

✓ done

(4) ~~b) Find the power series~~

Could use ratio test to check radius of convergence

Know every step - or will be confused

Often my problem is knowing next step

Now another one

$\ln(5-x)$

- know $\ln(1-x)$ ← by integrating $\frac{1}{1-x}$

now how to get in

$\ln\left(5\left(1-\frac{x}{5}\right)\right)$

$\ln(5) + \ln\left(1-\frac{x}{5}\right)$ ← know ln rules

now plugged in

$\ln(5) - \sum \frac{x^{n+1}}{n+1}$ ← skipped, make sure can solve for C

$\sum_{0}^{\infty} \frac{\left(\frac{x}{5}\right)^n + 1}{n+1}$

can now to 1

$\ln(5) - \sum_{1}^{\infty} \frac{\left(\frac{x}{5}\right)^n}{n}$

* make sure to change radius of convergence

$\left|\frac{x}{5}\right| < 1$

$|x| < 5$

(5) Ok do ratio test to (V)

$$\left(\frac{\left(\frac{x}{5}\right)^{n+1}}{n+1} \cdot \frac{n}{\left(\frac{x}{5}\right)^{n}} \right)$$

$\cancel{\frac{\frac{x}{5}}{n}}$ always bad

so this does not cancle

$$\left| \frac{n}{n+1} \quad \frac{x}{5} \right|$$

get this

~~the~~ need

$\frac{\infty}{\infty}$

$$\left| 1 \cdot \frac{x}{5} \right| < 1$$

$$\left| \frac{x}{5} \right| < 1 \quad \text{converges}$$

$$|x| < 5$$

need to check the end points by plugging into original

$\underline{5}$ $\quad \ln(5) - \sum \frac{\left(\frac{5}{5}\right)^n}{n} = \frac{1^n}{n} = \infty$ $\quad = 1$ ~~converges~~ diverges

$\underline{-5}$ $\quad \ln(5) - \sum \frac{\left(\frac{-5}{5}\right)^n}{n} = \frac{(-1)^n}{n}$ converges

) just memorize I guess

⑥ Taylor Reminder

$f(x) = e^{5x}$

a) find Taylor Series at $a=0$

- oh they are doing this again
- can write it all out

b) test ratio

let me try reducing

$$\left| \frac{5^{n+1} x^{n+1}}{(n+1)!} \quad \frac{n!}{5^n x^n} \right.$$

$$\left| \frac{5x}{n+1} \right|$$ ✓ where does < 1 as $n \to \infty$

$\frac{5x}{\infty} < = 0$ for any x ✓

c) Ok the remainder stuff

$R_n \to 0 := f(x) - T_k(x)$

after x terms

trying to see where it = function

after ya know where it converges

want $\lim R_n \to 0$

(7) know definition

$$|R_k(x)| \leq \frac{M}{(k+1)!}(x-a)^{k+1}$$

want to find M so that $M \geq f^{(n+1)}(c)$
on some interval
- oh just pick arbitrary d so look $|x| < d$

$$5^{k+1} \cdot e^{5d} = M$$

plug in
k+1 for n
and d for x

So $$\frac{\overbrace{5^{k+1} e^{5d}}^{M} \overbrace{d^{k+1}}^{(x-a)^{k+1}}}{(k+1)!}$$

Check that goes to 0 w/ factoral
So function true

ok going to study/memorize cheat sheet + go
to bed 12:03 AM

~~- only half time actually studying~~
~~need to get up~~

Cheat Sheet 18.01

Taylor Series $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$

$$f(a) + \frac{f'(a)(x-a)}{1} + \frac{f''(a)(x-a)^2}{2 \cdot 1}$$

Ratio Test $\lim_{n \to \infty} \left| \frac{b_{n+1}}{b_n} \right|$

<1 converges
$=1$ does not help
>1 diverges

$\lim (\underbrace{f(x) - T_k(x)}_{R_k(x)}) = 0$ $\quad R_k(x) \leq \frac{M r^{n+1}}{(k+1)!} |x-0|$

for $|x-0| < d$

where $M \geq |f^{(k+1)}(x)|$ on $x \in (-d, d)$

product rule ⊕	so $\lvert f^{(k+1)}(x) \rvert \leq 5^{k+1} e^{5d} = M$
chain rule ⊗	

n $\quad n$

M is the max of the d you picked

Newton's Method

Riemann Sum $\int_a^b f(x)\,dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \cdot \Delta x$

Trapizodal Rule

notation may be off

$$\frac{b-a}{n}\left(\frac{f(0)}{2} + f(1) + f(2) + \dots + \frac{f(n)}{2}\right)$$

Simpson's Rule $\frac{b-a}{6}\left(f(0) + 4f(1) + 2f(2) + 4f(3) + f(4)\right)$

Geometric Series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad |x|<1$$

$\ln(1-x)$ integrate $= -\int \frac{dx}{1-x}$

sub $-x$ for x — sub in x^2 for x

$\frac{1}{1-x} \to \frac{1}{1+x} \to \frac{1}{1+x^2}$

integrate

arc tan x

integrate

$\ln(1-x)$

are all Taylor series centered at $x=0$

$f(x) = \frac{f^{(n)}(a)}{n!}(x-a)^n$

- coefficient ratio always $= 1$ for $\sum 1\, x^n$

- look over more recitation
- office hrs

$0^0 = 1$

$0^x = 0 \quad x<0$

$\ln(1) = 0$

$\ln(0) = -\infty$

$x^1 = x$

Alternating Series Test

$$\sum (-1)^n a_n$$

converges if
$a_n \to 0$
$a_{n+1} < a_n$
$a_n > 0$

FTC

$g(x) = \int_0^x f(t)\,dt$

$g'(x) = f(x)$ given that

$g(x) = \int_{k(x)}^{h(x)} f(t)\,dt$

$g'(x) = f(h(x)) \cdot h'(x) - f(k(x)) \cdot k'(x)$

Arc lenght

$$\int_0^b \sqrt{1+\left(\frac{dy}{dx}\right)^2} \quad \text{or} \quad \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}$$

By Parts

$uv - \int u'v$

Quotient Rule

$$\frac{\text{low } d \text{ high} - \text{high } d \text{ low}}{\text{low}^2}$$ ← I always mix up

$$d \tan^{-1} x = \frac{1}{x^2+1}$$

$$\sin 2x = 2 \sin x \cos x$$

$$\cos 2x = \cos^2 x - \sin^2 x$$

$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$$

$\frac{1}{n}$ = harmonic series, diverges

$\frac{(-1)^n}{n}$ converges

) can do alternating series test