Syllabus

Wednesday, September 09, 2009
1:03 PM

18.01 Syllabus – Fall 2009

			DIFFERENTIATION
0.	W	Sept. 9	Recitation: Review of Graphing
1.	R	Sept. 10	Derivatives, slope, velocity, rate of change
2.	F	Sept. 11	Limits, continuity, trigonometric limits
3.	T	Sept. 15	Derivatives of products, quotients, sine, cosine
4.	R	Sept. 17	Chain rule, higher derivatives
5.	F	Sept. 18	Implicit differentiation, inverses PS 1 DUE
6.	T	Sept. 22	Exponential and log, logarithmic differentiation
7.	R	Sept. 24	Hyperbolic functions, Review for Exam 1
8.	F	Sept. 25	EXAM 1 (covering lectures 1–7)
			APPLICATIONS OF DIFFERENTIATION
9.	T	Sept. 29	Linear and quadratic approximation
10.	R	Oct. 1	Curve sketching
11.	F	Oct. 2	Max-min problems PS 2 DUE
12.	T	Oct. 6	Related rates
13.	R	Oct. 8	Newton's method and other applications
14.	F	Oct. 9	Mean value theorem, inequalities PS 3 DUE
	T	Oct. 13	COLUMBUS DAY SCHEDULE - Monday Recitation
15.	R	Oct. 15	Differentials, antiderivatives
16.	F	Oct. 16	Differential equations, separation of variables PS 4 DUE
17.	T	Oct. 20	EXAM 2 (covering lectures 9–16)
			INTEGRATION WITH APPLICATIONS
18.	R	Oct. 22	Definite integrals
19.	F	Oct. 23	First fundamental theorem of calculus
20.	T	Oct. 27	Second fundamental theorem of calculus, defn. of log
21.	R	Oct. 29	Areas between curves, volume by slicing
22.	F	Oct. 30	Volume by disks and shells PS 5 DUE
23.	T	Nov. 3	Work, average value, probability
24.	R	Nov. 5	Numerical integration
25.	F	Nov. 6	Improper integrals, Review for Exam 3 PS 6 DUE
26.	T	Nov. 10	EXAM 3 (covering lectures 18–25)
	W	Nov. 11	VETERAN'S DAY - No recitation
			TECHNIQUES OF INTEGRATION
27.	R	Nov. 12	Trigonometric integrals
28.	F	Nov. 13	Integration by inverse substitution, completing the square
29.	T	Nov. 17	Partial fractions
30.	R	Nov. 19	Integration by parts, reduction formulas
31.	F	Nov. 20	Parametric equations, arc length, surface area PS 7 DUE
32.	T	Nov. 24	Polar coordinates, area in polar coordinates
	R	Nov. 26	THANKSGIVING BREAK BEGINS – No lecture Thurs./Fri.
33.	T	Dec. 1	Review for Exam 4 PS 8 DUE
34.	R	Dec. 3	EXAM 4 (covering lectures 27–32)
			INFINITE SERIES
35.	F	Dec. 4	Infinite series, convergence tests
36.	T	Dec. 8	Taylor series
37.	R	Dec. 10	More on series, Review for Final Exam
38.	TBA	Dec. 14–18	FINAL EXAM

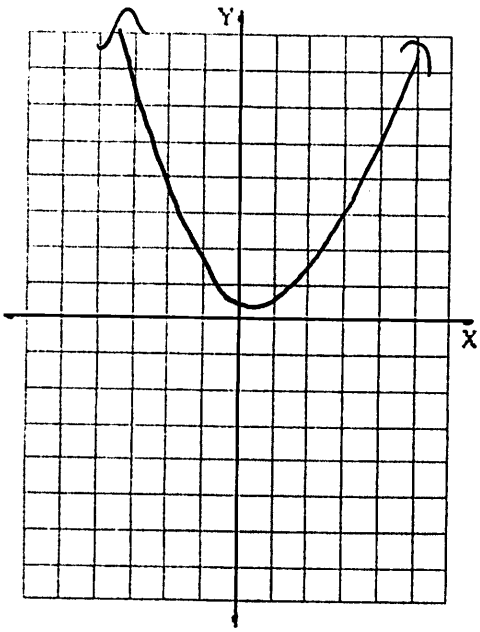
Worksheet 1: Review of Graphing

18.01 Fall 2009

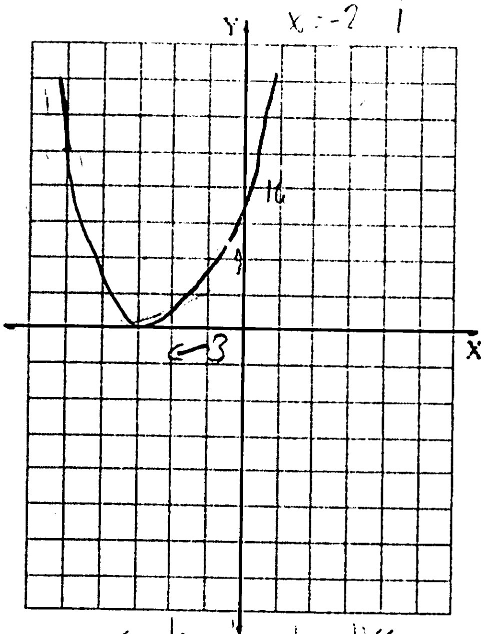
$x=0$ $y=9$
$x=1$ 16
$x=2$ 25
$x=-1$ 4
$x=-2$ 1

Problem 1. Graph the following functions.

a) $y = x^2$

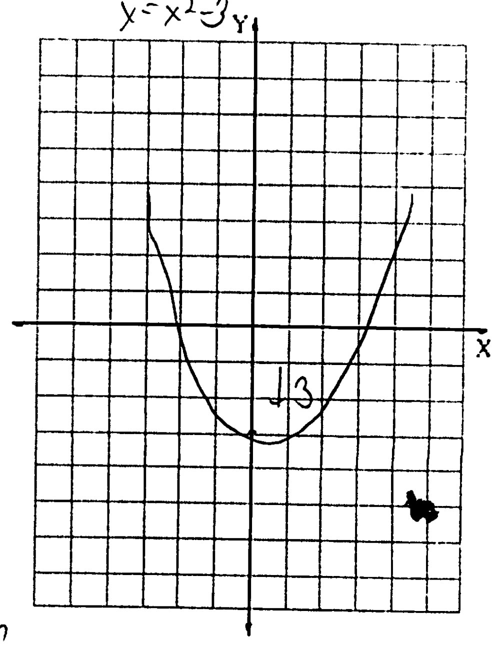


b) $y = (x+3)^2$



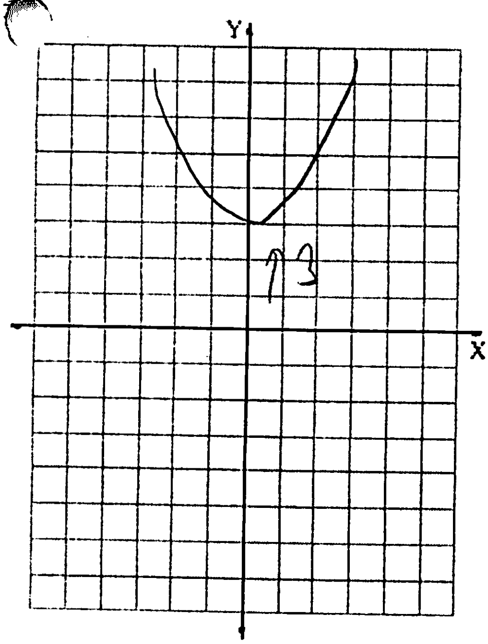
c) $y + 3 = x^2$

$y = x^2 - 3$



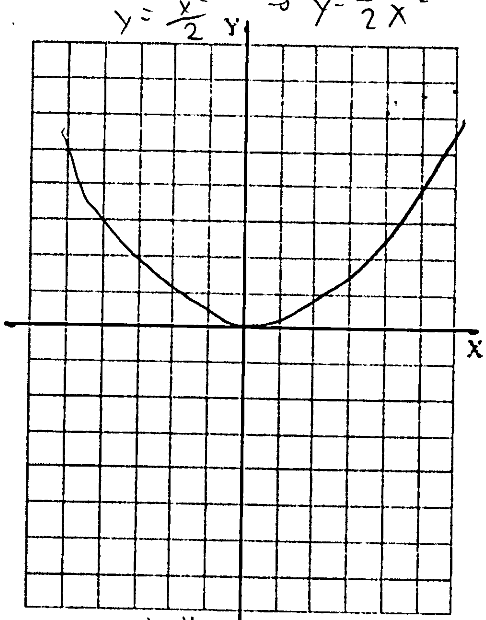
scale is also different ??

d) $y = x^2 + 3$

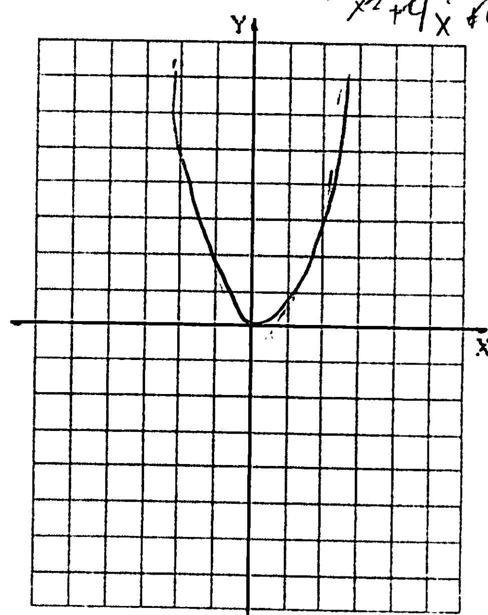


e) $2y = x^2$

$y = \frac{x^2}{2} \rightarrow y = \frac{1}{2}x^2$



f) $y = (2x)^2$ ~~$4 + 2x + 2x + x^2$~~ $x^2 + 4x + 4$



Shallower

Taking (a) as the base function, identify the transformations performed.

narrower by $\frac{1}{2}$

$4x^2$ - also, can be pulling up by 4

- does not work with

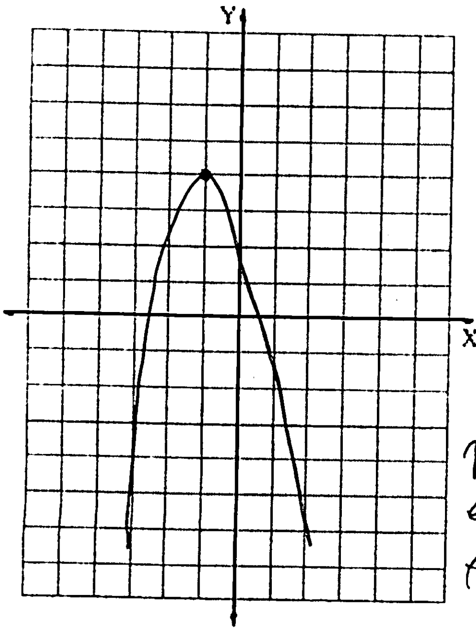
$y = \log(2x)$

$y = \log(2) + \log(x)$

Problem 2. a) Graph the following parabolas. Label min/max and zeroes.

What method can you use to make the second one easier to graph?

$y = -2(x+1)^2 + 4$



$\uparrow 4$

$\leftarrow 1$

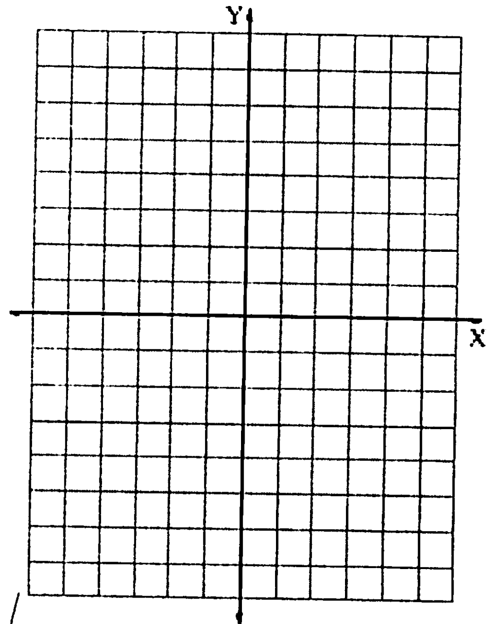
flipped

narrower

(over 1
−down 2) ← good way to think about

$y = 5 + 12x + 2x^2$

$2x^2 + 12x + 5$



completing the square

- make leading coefficien[t]

$2(x^2 + 6x + \_) + 5$

goal $(x + \_)^2$

$2(x^2 + 6x + 9) + 5 - 18$

$2(x+3)^2 - 13$

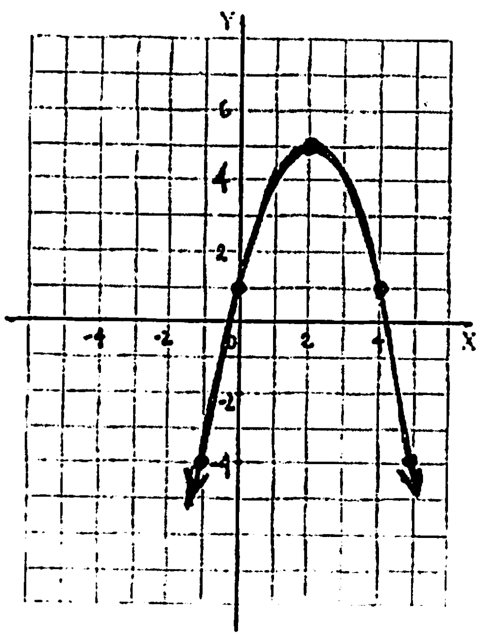
$\leftarrow 3$

$\downarrow 13$

narrower

ie vertical stretch by 2

b) Find the equation of the following parabola



$\uparrow 5$

$\rightarrow 2$

flipped

normal stretch

$y = -(x-2)^2 + 5$

c) Identify the zeroes and (vertical) asymptotes of

quadratic ↓ how it behaves near 0 is this

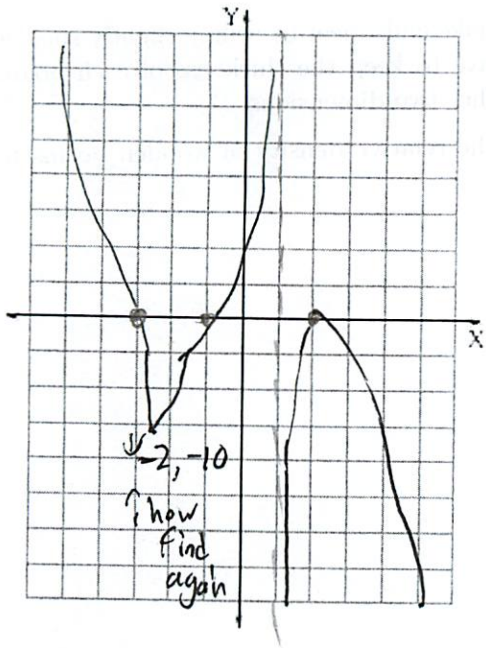
linear linear

$$y = \frac{-2(x-2)^2(x+1)(x+3)}{(x-1)}$$

co-efficient "multiplicity"

and do your best to sketch it on the grid below.

$x = 1$



zeros - numerator only where $x = 0$

$x = 2$

$x = -1$

$x = -3$

test pts

$x = 100$ $-$ $+$ $+$ $-$

$x = 1.5$ $+$ $++$ $+$

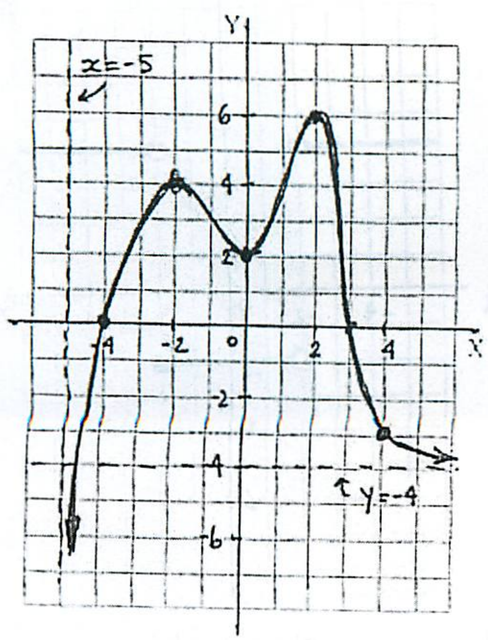
$x = .5$ $+$ $++$ $+$

$x = -.5$ $++$ $+$ $+$

$x = -2$ $+$ $-$

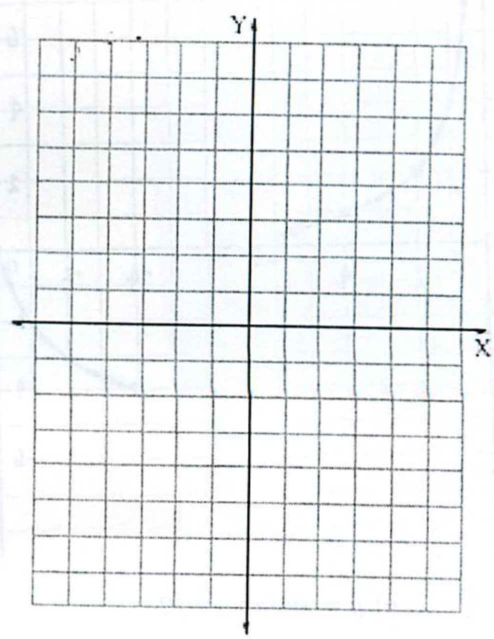
$x =$

Problem 3. Given the graph of $y = g(x)$, plot $y = -0.5g(x+2) + 2$ next to it.



↑2

←2



Office time 4-5

Problem 4. Milled Ideal Trees converts wooden beams into large rectangular pressed sawdust boards of a uniform thickness for a furniture manufacturer.

Until now, they have always bought cylindrical beams of a certain radius at a fixed cost of \$4.25 per meter.

Because of changes in the beam industry, their raw materials supplier will now only offer narrower beams of 3/4-ths the radius at the slightly lower price of \$4.00 per meter.

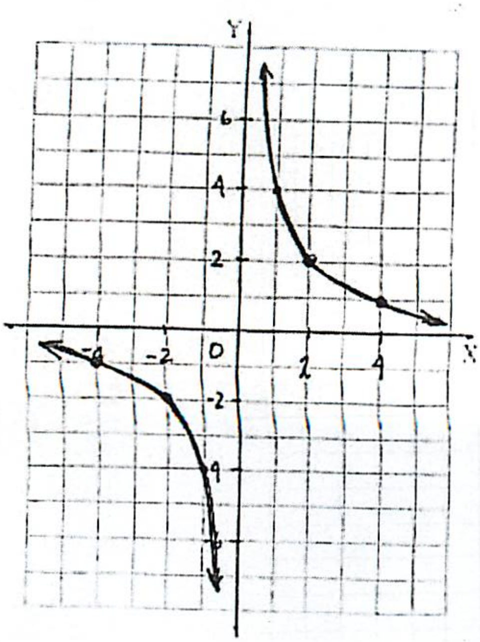
To make up for the change in costs of supplies, Milled Ideal Trees will try to make ends meet by selling slightly smaller boards at the same price. Because of furniture industry standards, they have to keep the thickness of each board constant, so they decide to go with a ten percent reduction on each of the other two dimensions.

By what factor does the enterprise's cost of materials per board change if the relative density of wooden beams to pressed boards stays the same?

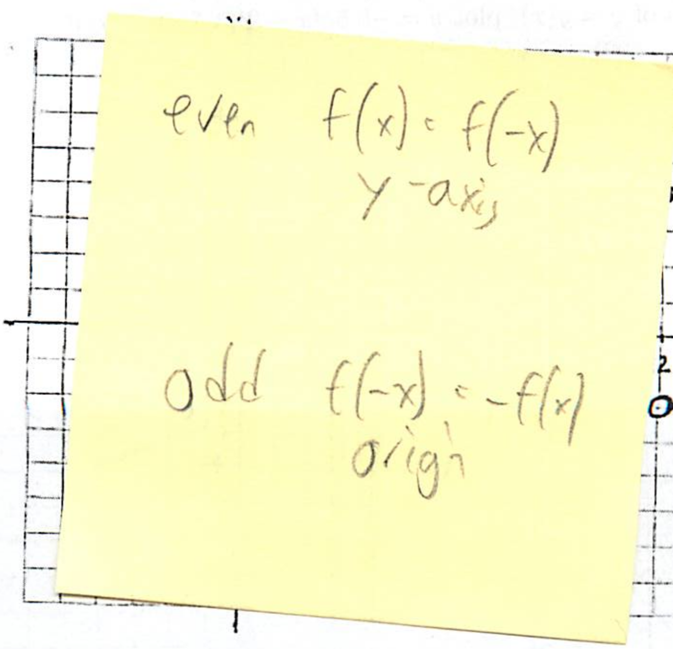
Problem 5. You will be given the graphs and equations of some functions.

Identify their symmetry by labeling each as **even**, **odd**, or **neither**

a)



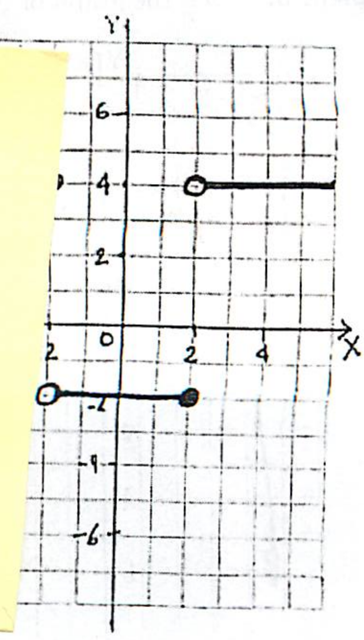
b)



even $f(x) = f(-x)$
y-axis

odd $f(-x) = -f(x)$
origin

c)



d) $y = x$

e) $y = \sin(x) - \pi/2$

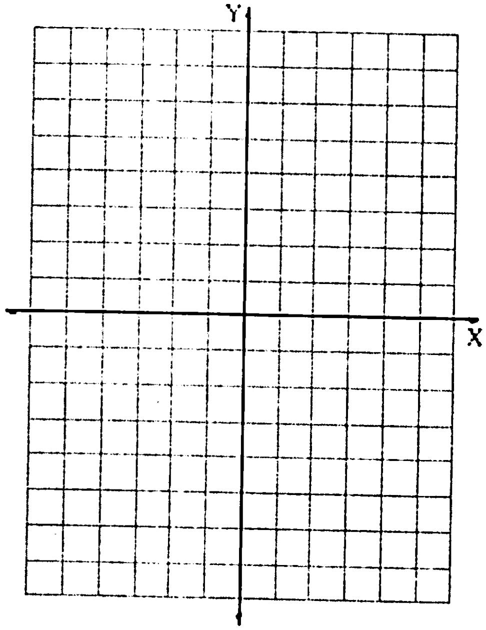
f) $y = \sin(x - \pi/2)$

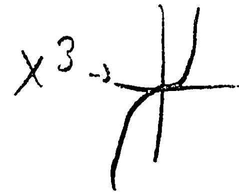
g) $y + 1 = |x|$

even → $f(x) = f(-x)$ y-axis
Odd $f(-x) = -f(x)$ origin

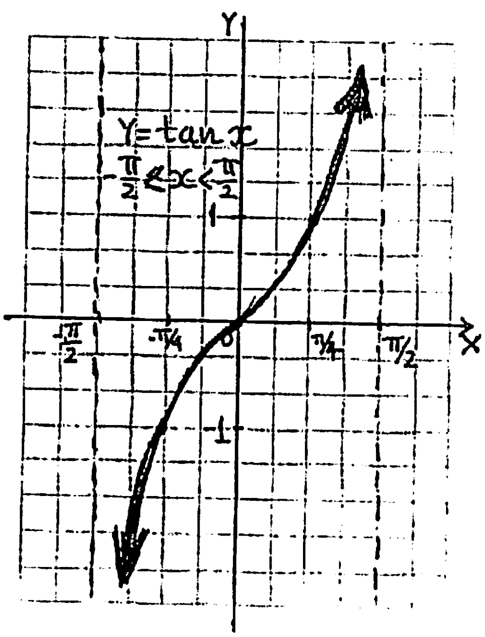
Problem 6. a) Draw the graph of $y = x^3 + 1$ reflected through the line $y = x$. What is the equation of the resulting function?

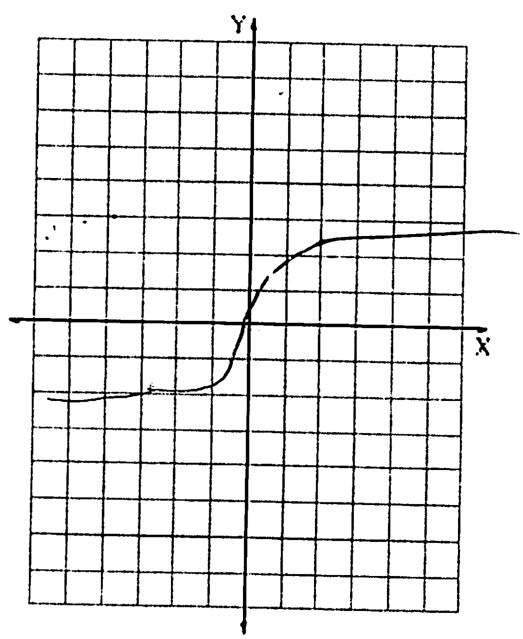
??





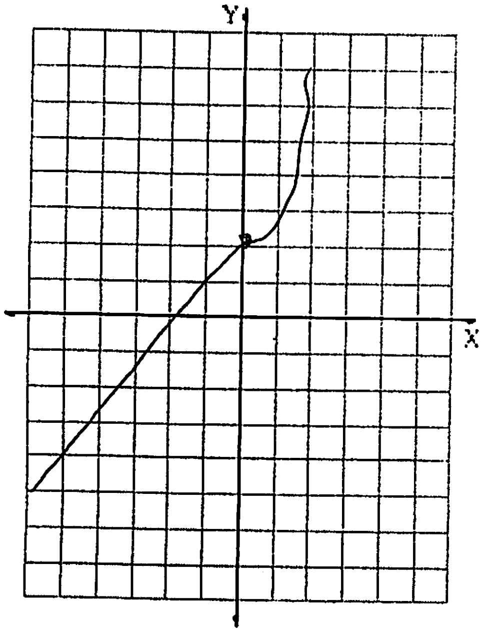
b) Given the graph of $y = \tan x$ on the domain $-\pi/2 < x < \pi/2$, plot $y = \arctan(x + 3)$. Label asymptotes.



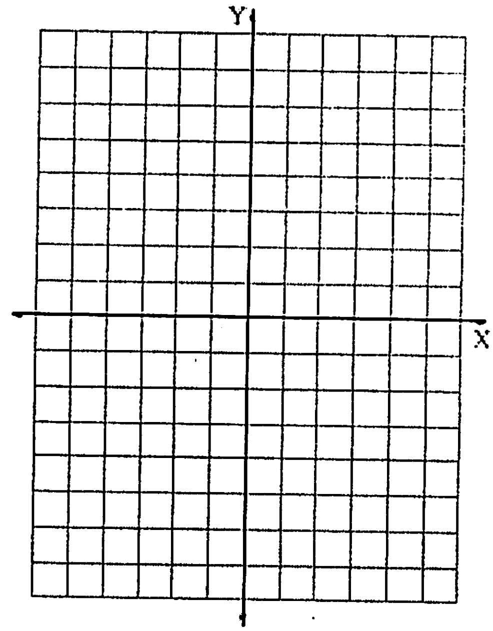


Problem 7. Graph

a) $y = \begin{cases} x+2 & x<0 \\ 3x^2+2 & x \geq 0 \end{cases}$



b) $y = \lfloor x + 0.5 \rfloor$?



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Problem 1. Graph the following functions.

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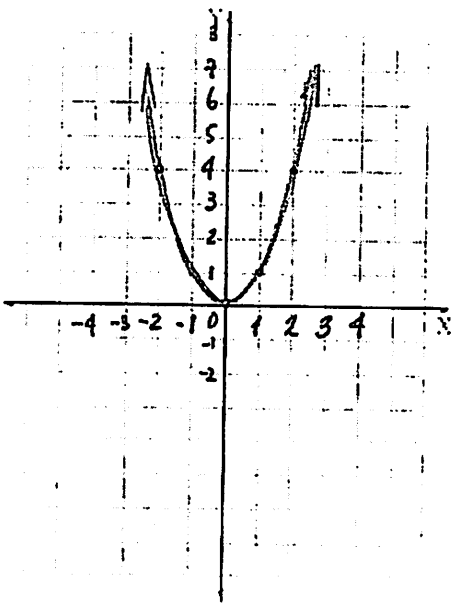
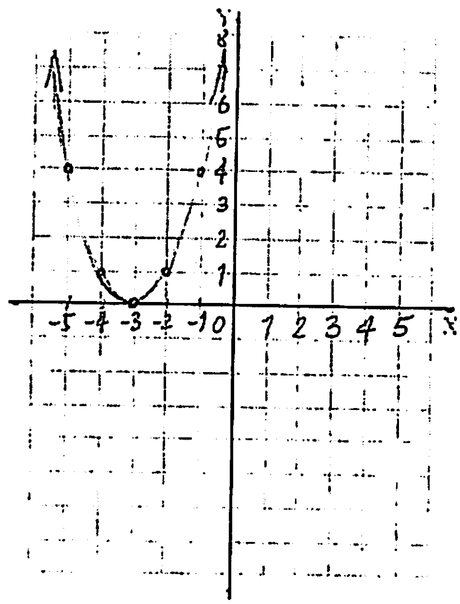
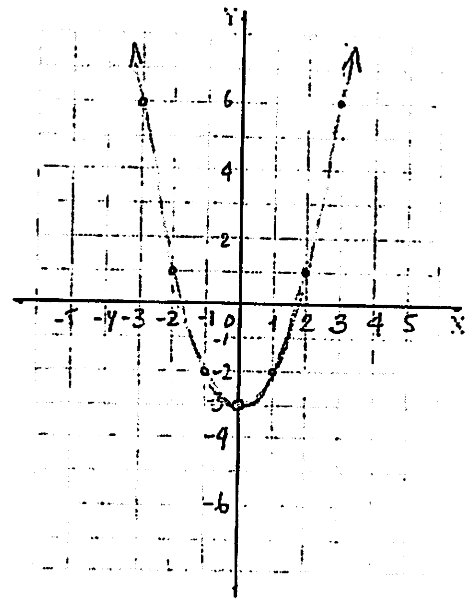


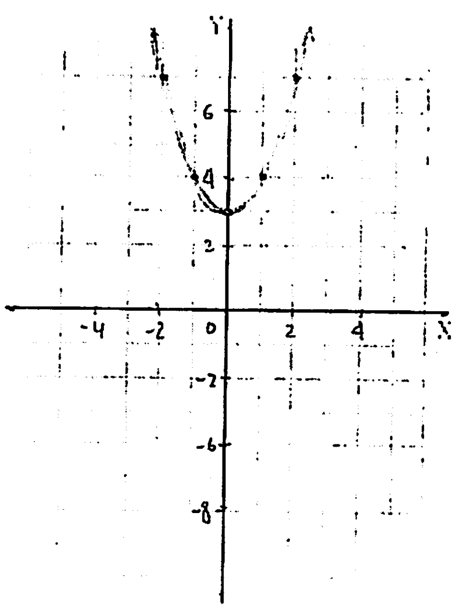
b) $y = (x+3)^2$



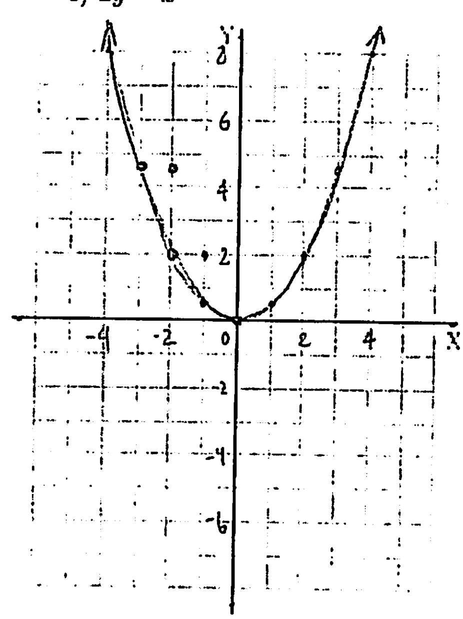
c) $y + 3 = x^2$



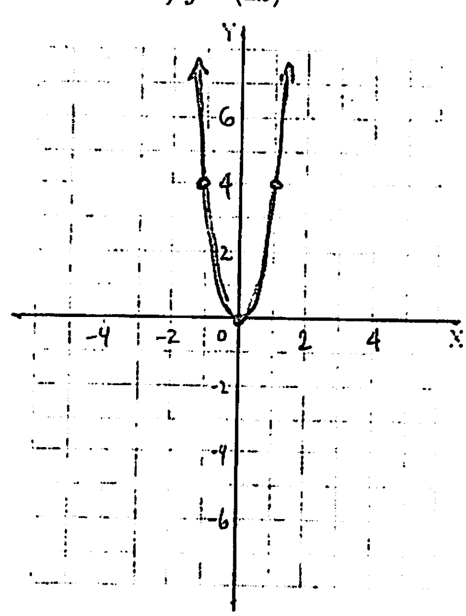
d) $y = x^2 + 3$



e) $2y = x^2$



f) $y = (2x)^2$



Taking (a) as the base function, identify the transformations performed.

a) base b) shift left by 3 c) shift down by 3 d) shift up by 3 e) shrink y-axis by 2
f) shrink x-axis by 2

Problem 2. a) Graph the following parabolas. Label min/max and zeroes.
What method can you use to make the second one easier to graph?

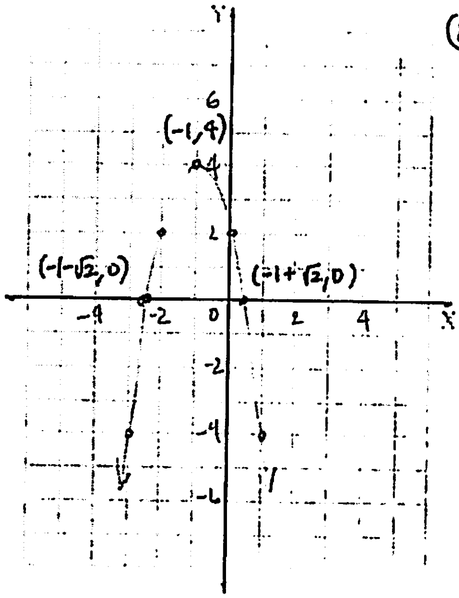
$y = -2(x+1)^2 + 4$

$-2(x+1)^2 = -4$

$(x+1)^2 = 2$

$x+1 = \pm\sqrt{2}$

$x = -1 \pm \sqrt{2}$



$y = 5 + 12x + 2x^2$

complete the square!

$2(x^2 + \underset{3\cdot 2}{6}x + 3^2 - 3^2) + 5 =$

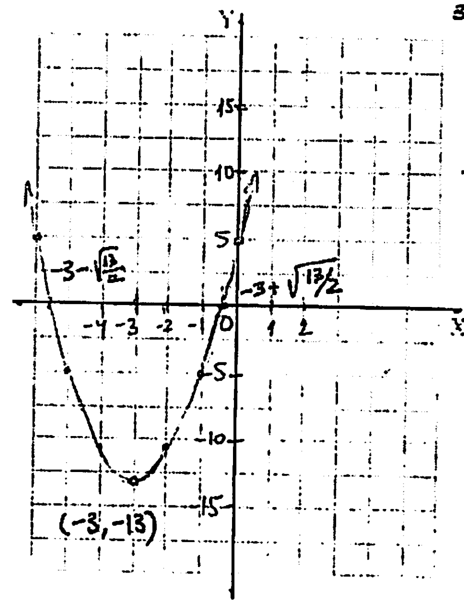
$= 2(x+3)^2 + 5 - 18$

$= 2(x+3)^2 - 13.$

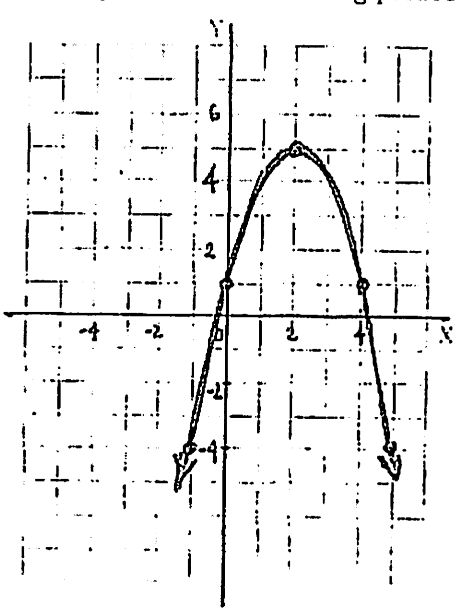
$2(x+3)^2 = 13$

$x+3 = \pm\sqrt{13/2}$

$x = -3 \pm \sqrt{\frac{13}{2}}$



b) Find the equation of the following parabola



parabola opens down $\Rightarrow$ leading coefficient is negative.

max @ $(2,5)$ $\Rightarrow$ know general form must be

$y = a(x-2)^2 + 5$

use one pt on graph to determine a:

say, $(x,y) = (0,1)$

$1 = a(0-2)^2 + 5 = a(-2)^2 + 5$

$a = \frac{1-5}{4} = \frac{-4}{4} = -1$ (good - negative)

$\boxed{y = -1(x-2)^2 + 5}$

(Check: $-1(4-2)^2 + 5 = -1(2)^2 + 5 =$

$= -4 + 5 = 1$ //)

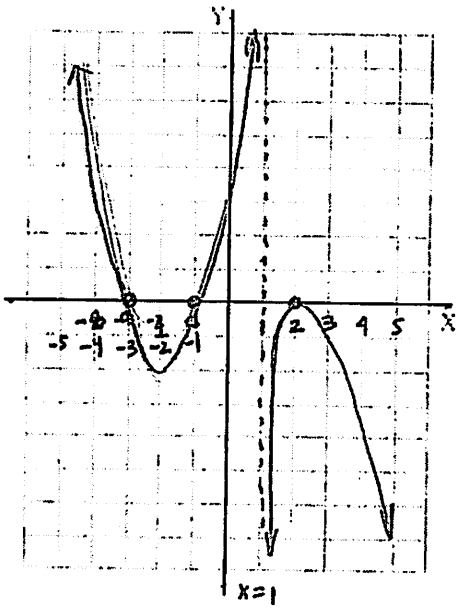
$\boxed{y = -(x-2)^2 + 5}$

c) Identify the zeroes and (vertical) asymptotes of

$$y = \frac{-2(x-2)^2(x+1)(x+3)}{(x-1)}$$

and do your best to sketch it on the grid below.

should just touch axis.

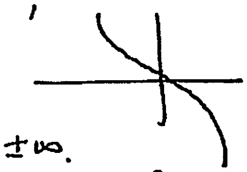


zeroes : $x = +2$ (double), -1, -3

V. asympt. (zeroes of denom) : $x = 1$

overall, graph behaves like

$$\frac{-2x^4}{x} = -2x^3,$$ which looks like



This tells us behavior at $\pm\infty$.

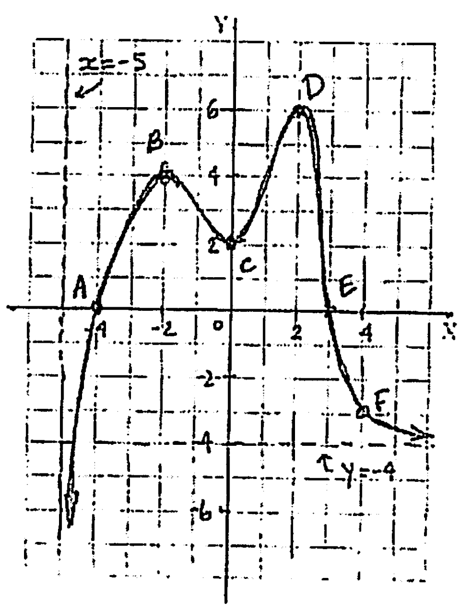
Examine sign (+/–) of factors ($(x-2)^2 \geq 0$ always)

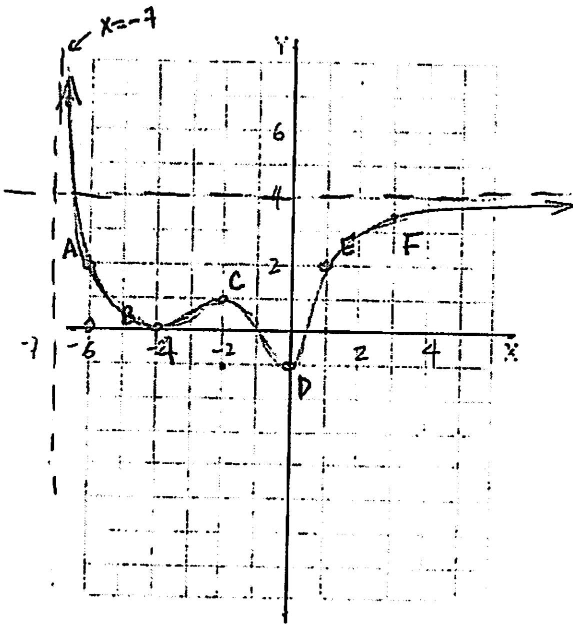
x+3, x+1, x–1

–3, –2, –1, 0, 1

$\cdot(-2)$ + – – + + sign of overall function

Problem 3. Given the graph of $y = g(x)$, plot $y = -0.5g(x+2) + 2$ next to it.





Y=4, approached from below.

^{3} Ex. $(-4)(-0.5)+2 = 2+2=4$

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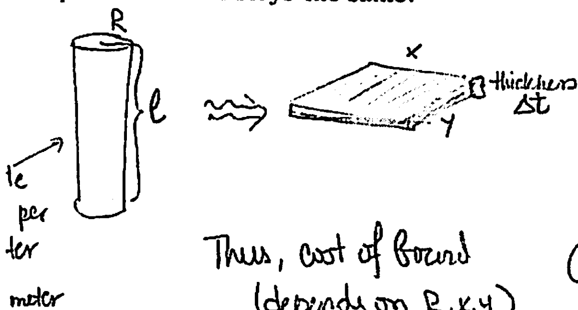


Cost of beam : $C_{meter}\,\ell$.

$Vol = \pi R^2 \ell \implies$ price / unit volume of wood $= C_{meter}\ell / \pi R^2 \ell = C_{meter}/\pi R^2 = P_{unit}$

cost of board $=$ Price of wood / unit vol $\cdot$ vol $= P_{unit\ Vol} \cdot x \cdot y \cdot \Delta t$

Thus, cost of board (depends on R, x, y) C_{meter}, Δt

$C(C_{meter}, R, x, y, \Delta t) = \dfrac{C_{meter} \cdot x \cdot y \cdot \Delta t}{\pi R^2}$

$x_{new} = 0.9\, x_{old}$

$R_{new} = \frac{3}{4} R_{old}$

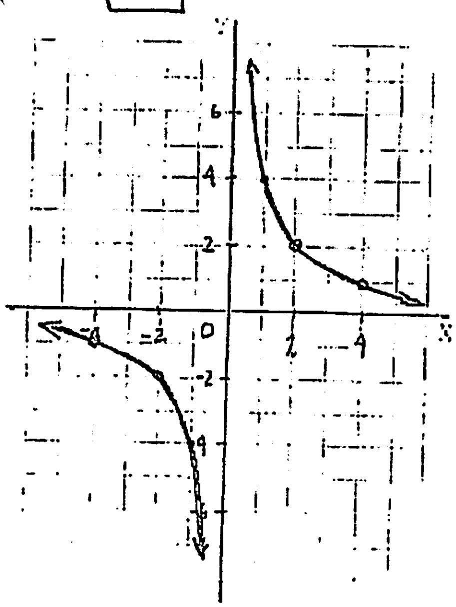
$C_{new}/C_{old} = \dfrac{4/4.25 \cdot 0.9 \cdot 0.9 \cdot 1}{(3/4)^2} = \boxed{1.355}$

(vs. 1.67 if did not change x, y)

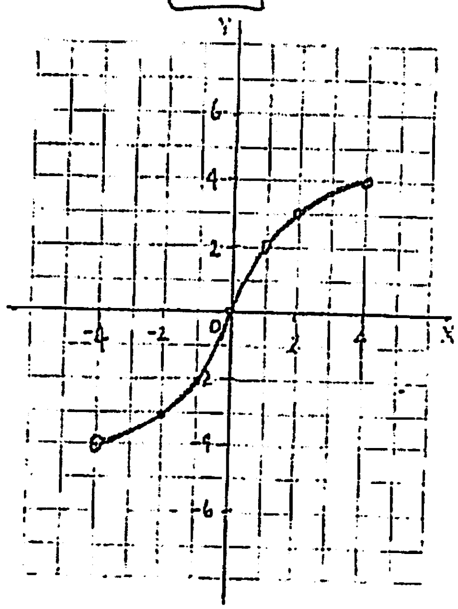
Problem 5. You will be given the graphs and equations of some functions.

Identify their symmetry by labeling each as even, odd, or neither

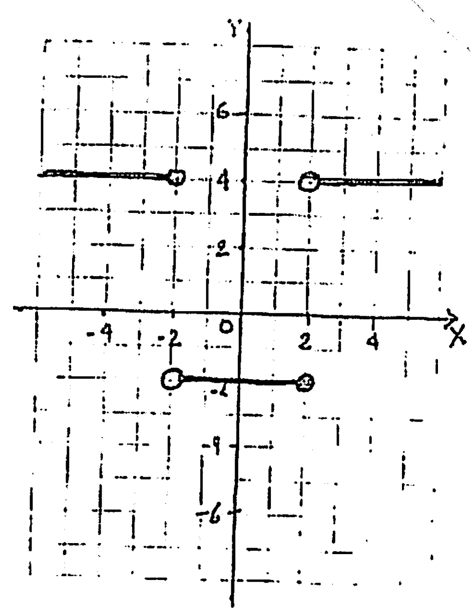
a) $\boxed{\text{odd}}$



b) $\boxed{\text{odd}}$



c) $\boxed{\text{neither}}$!



d) $y = x$

$\boxed{\text{odd}}$

e) $y = \sin(x) - \pi/2$

$\boxed{\text{neither}}$

f) $y = \sin(x - \pi/2)$

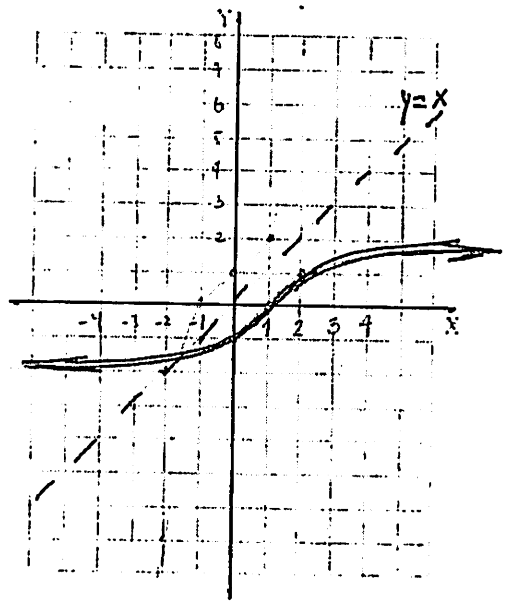
$-\cos x$

$\boxed{\text{even}}$

g) $y + 1 = |x|$

$\boxed{\text{neither}}$

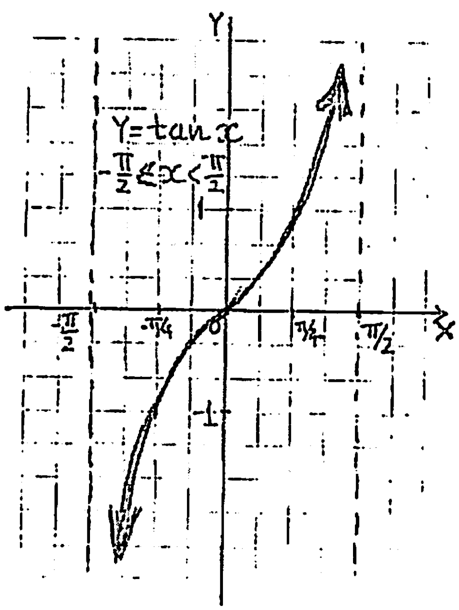
Problem 6. a) Draw the graph of $y = x^3 + 1$ reflected through the line $y = x$. What is the equation of the resulting function?

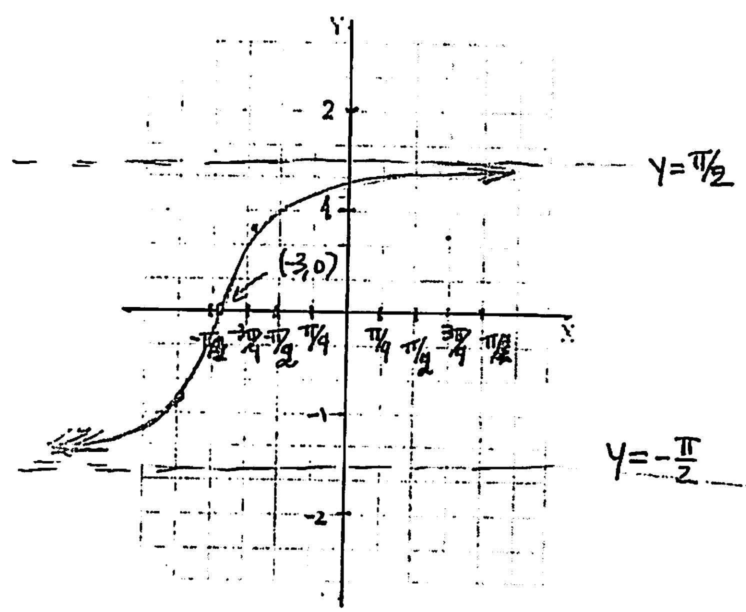


$x = y^3 + 1$

$\boxed{y = \sqrt[3]{x-1}}$

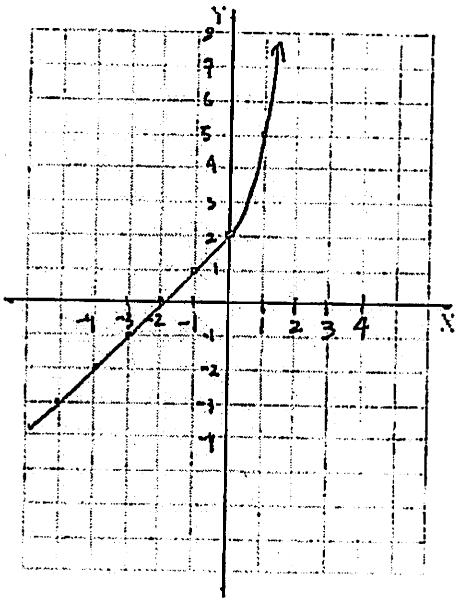
b) Given the graph of $y = \tan x$ on the domain $-\pi/2 < x < \pi/2$, plot $y = \arctan(x + 3)$. Label asymptotes.



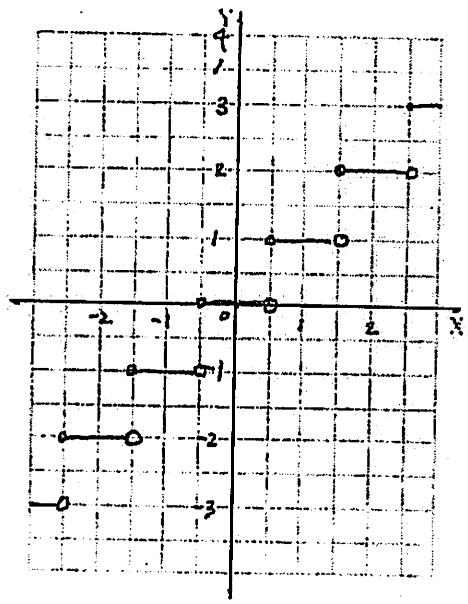


Problem 7. Graph

a) $y = \begin{cases} x+2 & x<0 \\ 3x^2+2 & x \geq 0 \end{cases}$



b) $y = \lfloor x + 0.5 \rfloor$



Textbook Chap 1 Review

Thursday, September 10, 2009
12:39 PM

distance formula

distance b/w 2 points

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

---

even $f(x) = f(-x)$ y axis

odd $f(-x) = -f(x)$ origin

Midpoint

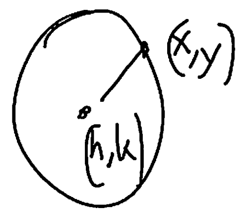
$$x = \frac{1}{2}(x_1 + x_2)$$
$$y = \frac{1}{2}(y_1 + y_2)$$

Slope

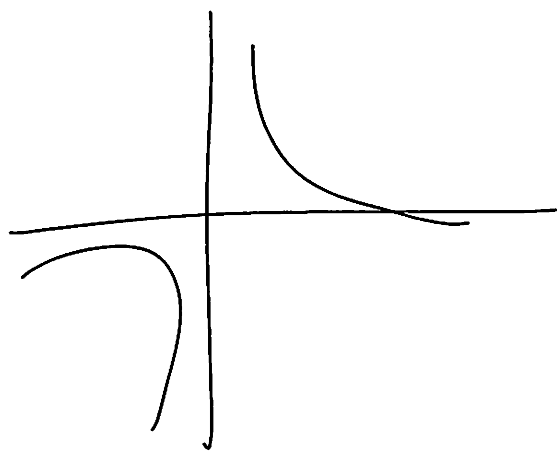
$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$y - y_0 = m(x - x_0)$$

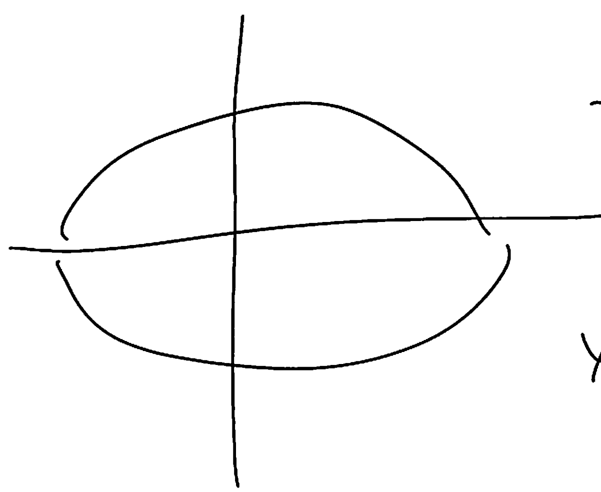
$r = \sqrt{(x-h)^2 + (y-k)^2}$



$y = \sqrt{x}$ $\quad$ $y = x^{1/2}$



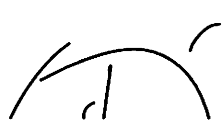
$y = \frac{1}{x}$



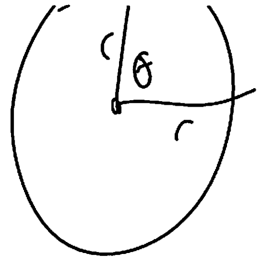
$y = \sqrt{25 - x^2}$

$y = -\sqrt{25 - x^2}$

Radian



natural



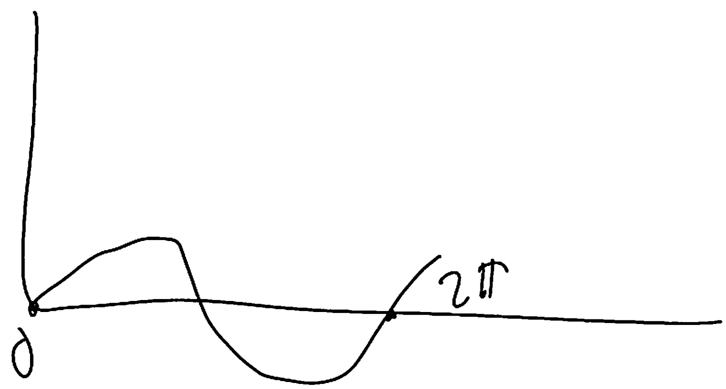
convinent

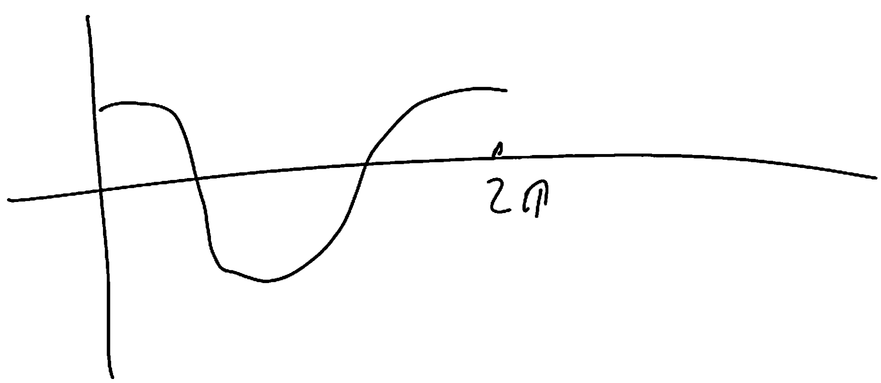
$\leftarrow$ all = ah-ha

$$2\pi r = 360^\circ$$

---

sin





PSET 1

Thursday, September 10, 2009
12:58 PM

long
1-2 weeks
get started
early

http://math.mit.edu/18.01/psets/Pset1.pdf

18.01 FALL 2009 – Problem Set 1

Due Friday 9/18/08, 1:45 pm in 2-106

18.01 Supplementary Notes, Exercises and Solutions are for sale at Copy Tech in the basement of Building 11. This is where to find the exercises labeled 1A, 1B, etc. You will need these for the first day's homework.

Web site: *http://math.mit.edu/18.01* Links to syllabus, course information, and problem sets. As the semester progresses, we'll also post announcements, exam info, etc.

Part I consists of exercises given in the Supplementary Notes and solved in section S of the Notes. Of course, you should attempt to solve problems without referring to solutions in advance. These problems will be graded without many comments.

Part II consists of problems for which solutions are not given; it is worth more points. Some of these problems are longer multi-part exercises posed here because they do not fit conveniently into an exam or short-answer format. See the guidelines below (and also on the website) for which types of collaboration are acceptable, and follow them.

To encourage you to keep up with the homework as it pertains to lectures, both Part I and Part II problems are listed with the accompanying lecture in which the material will be covered.

Part I (30 points)

Notation for Listing HW: 2.1 = Section 2.1 of the Simmons book;
2.4/13 = Section 2.4 Problem 13 in Simmons
Notes G = section G of the Notes;
1A-3 = Exercise 1A-3 in Section E (Exercises) of the Notes (solved in section S)

Recitation 0. Wed. Sept. 9: Graphing functions.
Read: Notes G, sections 1-4 HW: 1A-1b, 2b, 3abe, 6b, 7b

Lecture 1. Thurs., Sept. 10: Derivative; slope, velocity, rate of change.
Read: 2.1-2.4 HW: 1B-1, 1C-1a (using definition of derivative), 3abc, 4ab (using work in 3), 5, 6 (trace axes on answer sheet)

Lecture 2. Fri. Sept. 11: Limits and continuity; some trigonometric limits
Read: 2.5 (bottom p.70-73; concentrate on examples, skip the $\epsilon - \delta$ def'n)
Read: 2.6 to p. 75; learn def'n (1) and proof "differentiable $\Longrightarrow$ continuous" at the end.
Read: Notes C HW: 1C-2, 1D-1acdfg, 3acde, 6a, 8a (remembering "diff $\Longrightarrow$ cont.")

Lecture 3. Tues. Sept. 15: Differentiation formulas: products and quotients;
Derivatives of trigonometric functions.
In the following exercises, an *antiderivative* of $f(x)$ is any $F(x)$ for which $F'(x) = f(x)$.
Read: 3.1, 3.2, 3.4 HW: 1E-1ac, 2b, 3, 4b, 5ac; 1J-1e, 2

Lecture 4. Thurs. Sept. 17: Chain rule; higher derivatives.
Read: 3.3, 3.6 HW: 1F-1ab, 2, 6, 7bc; 1J- 1akm 1G-1bc, 5ab

Lecture 5. Fri. Sept. 18 Implicit differentiation; inverse functions.
Read: 3.5, Notes G section 5 HW: To be given on Problem Set 2.

Part II (43 points)

Directions and Rules: Collaboration on problem sets is encouraged, **but**

a) Attempt each part of each problem yourself. Read each portion of the problem before asking for help. If you don't understand what is being asked, ask for help interpreting the problem and then make an honest attempt to solve it.

b) Write up each problem independently. On both Part I and II exercises you are expected to write the answer in your own words.

Part II (43 points)

Directions and Rules: Collaboration on problem sets is encouraged, but

a) Attempt each part of each problem yourself. Read each portion of the problem before asking for help. If you don't understand what is being asked, ask for help interpreting the problem and then make an honest attempt to solve it.

b) Write up each problem independently. On both Part I and II exercises you are expected to write the answer in your own words.

c) Write on your problem set whom you consulted and the sources you used. If you fail to do so, you may be charged with plagiarism and subject to serious penalties.

d) Do not consult materials from previous semesters.

0. (3 points) Write the names of all the people you consulted or with whom you collaborated and the resources you used, or say "none" or "no consultation". This includes visits outside recitation to your recitation instructor. If you don't know a name, you must nevertheless identify the person, as in, "tutor in Room 2-106," or "the student next to me in recitation." Optional: note which of these people or resources, if any, were particularly helpful to you.

This "Problem 0" will be assigned with every problem set. Its purpose is to make sure that you acknowledge (to yourself as well as others) what kind of help you require and to encourage you to pay attention to how you learn best (with a tutor, in a group, alone). It will also greatly help us to know what resources you find useful.

1. (Wed, 3 pts) Express $(x-1)/(x+1)$ as the sum of an even and an odd function. (Simplify as much as possible.)

2. (Thurs, 4 pts) a) Use the following table of approximate square roots to give an approximate value of $\sqrt{102}$. You should begin by finding an approximate answer for the tangent line to $\sqrt{x}$ at $x = 100$, and using your answer to compute an approximate value of $\sqrt{102}$.

x	$\sqrt{x}$
100	10
101	10.049875
100.1	10.004998
100.01	10.000499

b) Use your tangent line from part (a) to give an estimate for $\sqrt{400}$. Is your approximation larger or smaller than the correct answer? Draw a picture to illustrate your answer. (Later, we'll use calculus to give a precise method for such approximations and associated error estimates.)

3. (Thurs, 4 pts) A 15 ft. tall street lamp is placed at the very top of a hill. Suppose the 200 ft. tall hill has an outline shaped like a parabola, and so approximated by the equation $y = 200 - x^2$. What is the lowest height on the hill y at which you can read a book on a cloudy night?

4. (Thurs, 6 pts) 3.1/21 (in Simmons, on parabolic mirrors)

5. (Thurs, 4 pts: 2 + 2)

a) A water cooler is leaking so that its volume at time t in minutes is $(10-t)^2/5$ liters. Find the average rate at which water drains during the first 5 minutes.

b) At what rate is the water flowing out 5 minutes after the tank begins to drain?

6. (Thurs, 5 pts) A prospective student sits in on the last 5 minutes of Thursday's lecture, and sees on the board that the slope of the tangent line can be defined by

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h},$$

but doesn't understand what this formula, or its parts, signify. Write a paragraph (or two) explaining the parts of this formula to this student.

7. (Friday, 8 pts: 1 + 1 + 1 + 1 + 1 + 1 + 2) Simmons 2.5/19d (put $u = 1/x$), 19f, 19g, 20c, 20g (show work); 22a (needs a calculator), 22b (see the proof on page 73).

8. (Tuesday, 6 pts: 2 + 2 + 2)

a) If u, v and w are differentiable functions, find the formula for the derivative of their product, $D(uvw)$.

b) Generalize your work in part (a) by guessing the formula for $D(u_1u_2\cdots u_n)$—the derivative

5. (Thurs, 4 pts: 2 + 2)

a) A water cooler is leaking so that its volume at time t in minutes is $(10 - t)^2/5$ liters. Find the average rate at which water drains during the first 5 minutes.

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8. (Tuesday, 6 pts: 2 + 2 + 2)

a) If u, v and w are differentiable functions, find the formula for the derivative of their product, $D(uvw)$.

b) Generalize your work in part (a) by guessing the formula for $D(u_1 u_2 \cdots u_n)$—the derivative of the product of n differentiable functions.

c) Then prove your formula by mathematical induction (i.e., prove its truth for the product of $n + 1$ functions, assuming its truth for the product of n functions).

Thursday, September 10, 2009
1:05 PM

Ben Brubaker

brubaker@math.mit.edu

2-267
Tue 11-12
Thur 10-12
or appt

Read section before PSET
Can't cover everything in lecture

Use paper notebooks
- transcribe to that
- psets on paper
- do this class on paper
- get a binder + paper

Learn critical thinking

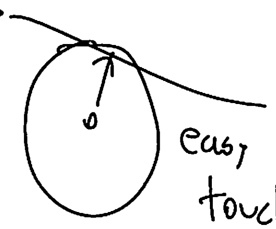
Tangent Problem

Thursday, September 10, 2009
1:15 PM

1st half of course

best local approx to a curve using a line

Circle



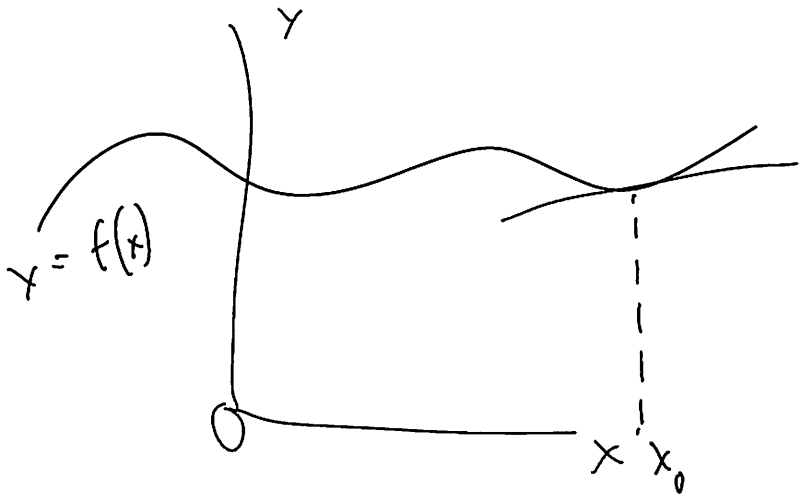
easy way to find line
touches circle in 1 pt

Picture under microscope
Circle looks more like a line as zoom in

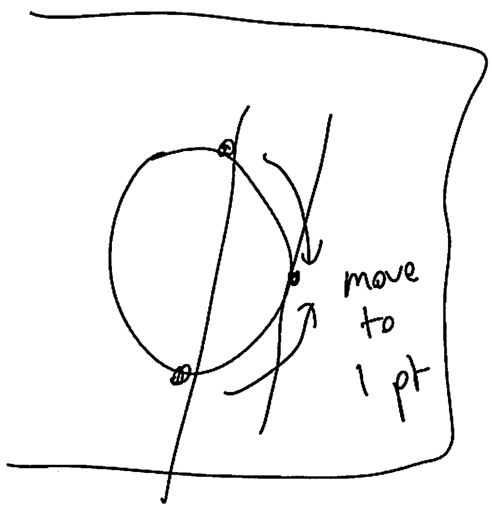
Does not $x^2+y^2=1$

General curves

- for $y = f(x)$

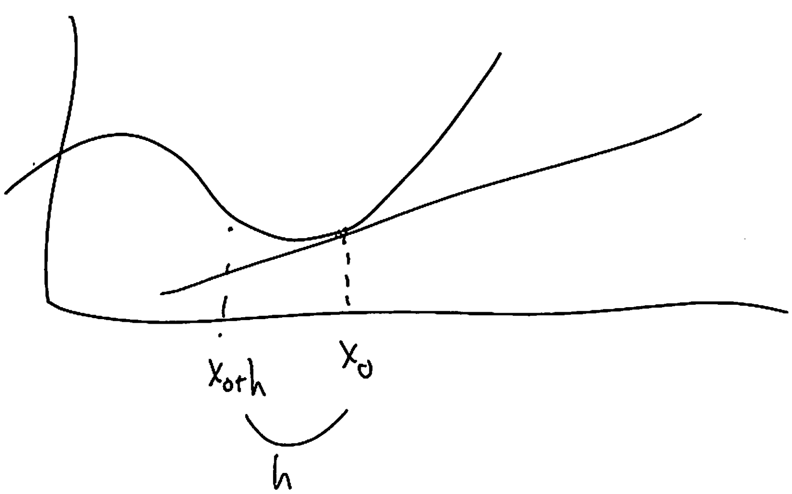


tangent line must include the pt
$(x_0, f(x_0))$



find slope of
tangent line

Define tangent line to $f(x)$ at x_0
to be limit of secent lines
through $(x_0, f(x_0))$
and a pt on the graph which
is closer and closer to x_0



as get closer + closer get a
better approximation

better approximation

Formula
- limit of slopes of secant lines

$$\text{slope} = \frac{\Delta y}{\Delta x} = \frac{f(x_0+h) - f(x_0)}{(x_0+h) - x_0}$$

$$\Downarrow$$

$$\frac{f(x_0+h) - f(x_0)}{h}$$

---

take limit as h gets closer

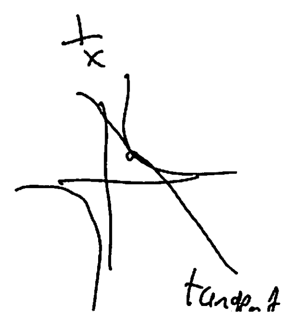
$$\boxed{\lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}}$$

Slope of tangent line

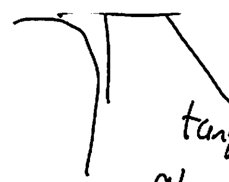
deriv

Example

- slope of tangent line $f(x) = \frac{1}{x}$ at $x = x_0$



$f(x) = \frac{1}{x}$ at $x = x_0$



tangent at $x = x_0$ to $f(x) = \frac{1}{x}$

$$\lim_{h \to 0} \frac{\frac{1}{x_0+h} - \frac{1}{x_0}}{h}$$

↑ can't just plug in 0 - get $\frac{0}{0}$

$$\lim_{h \to 0} \frac{\frac{x_0 - (x_0+h)}{x_0(x_0+h)}}{h}$$

$$\frac{\frac{-h}{x_0(x_0+h)}}{h} \begin{array}{l} \cdot \frac{1}{h} \\ \cdot \frac{1}{h} \end{array}$$

↑ to get rid of h

$$\lim_{h \to 0} \frac{-1}{x_0(x_0+h)}$$

$$\boxed{\frac{-1}{x_0^2}} = f'(x_0)$$

"derivitive of $f(x)$ at pt $x = x_0$"

aka slope of tangent line

$$\boxed{\frac{-1}{x_0^2}} = f'(x_0)$$

"derivitive of $f(x)$
at pt $x = x_0$"
aka slope of tangent line

---

1. Works for any point $x = x_0$
arbitary choice of pt
- could build a function
- input $x_0 \rightarrow$ output $f'(x_0)$

$$x_0 \rightarrow \frac{-1}{x_0^2}$$

1a. Called:
The derivitive of $f(x)$
$= f'(x)$

2. Solved tangent problem for $f(x) = \frac{1}{x}$
at $x = x_0$

$$y - f(x_0) = f'(x_0)(x - x_0)$$

- have computed these quantities

$$y - \frac{1}{x_0} = \left(\frac{-1}{x_0^2}\right)(x - x_0)$$

- for any choice $x = x_0$
- point slope form $y - y_0 = m(x - x_0)$

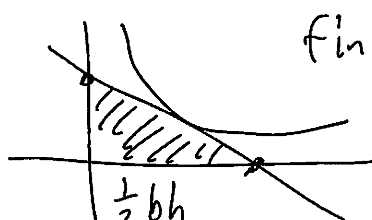
if $x_0 = 1$

$$y - 1 = -1(x - 1)$$

pt-slope form

$$\boxed{y = -x + 2}$$ ← a line

for any x_0
- but can define x_0 and get a line



$\frac{1}{2}bh$

$4/2 = 2$ area

for any triangle of $\frac{1}{x}$

find y-int
- (where $x = 0$)

solve for y

$$y = \frac{2}{x_0}$$

x-int
- where $y = 0$

$$x = 2x_0$$

Goals 1. library of derivitives
polynomials
trig functions
exponentials + logs
combinations
2. explain why useful physics, modeling
graphic function

More reading

Thursday, September 10, 2009
2:24 PM

from Supplement p 4

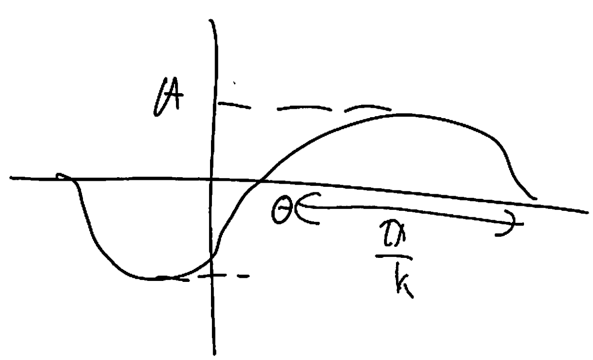
periodicity
- just that it goes around 100%

sinusoidal wave
- pure wave
- just sin curve
  - period 2π

  $A \sin 2x$
    ↑ stretches to $\pm A$ ↕

  $A \sin kx$
    ↑ $k > 0$ period $= \frac{2\pi}{k}$

  $A \sin k(x - \theta)$
    ↑ moves graph ← →



$a \sin(kx) + b \sin(kx)$

$y = x$

for reflecting

inverse

Switch

y and x

right

$y = x^2$

$y^2 = x$

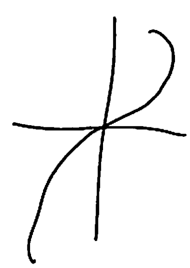
$y = \sqrt{x}$

* must restrict domain

$y = \sin x$

$x = \sin y$

$y = \sin^{-1}(x)$ ← arc sin



← not a function

Continuity + Discontinuity

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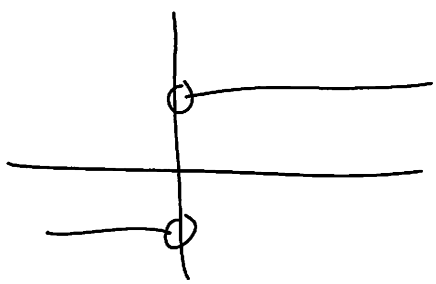
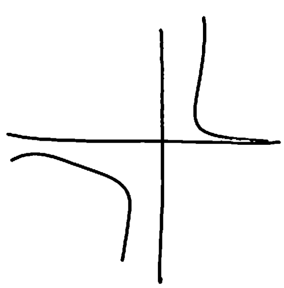
← right hand limits (+)

→

left hand limits (−)



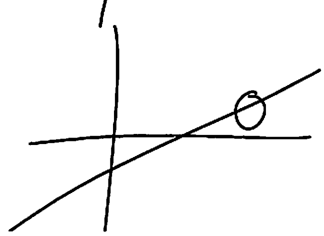
$\lim_{x \to 0^-} = -2$ $\lim_{x \to 0^+} = 2$



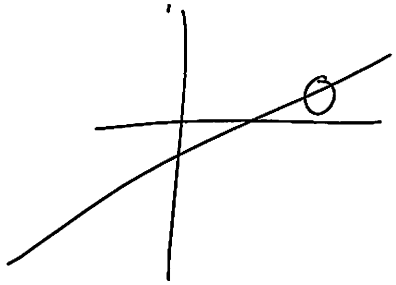
$\lim_{x \to 0^-} = -\infty$

$\lim_{x \to 0^+} = \infty$

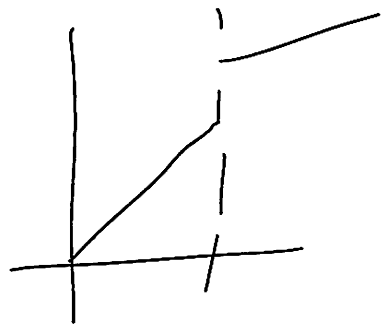
Continuity



removable



removable



jump

infinite

Exponential + Logarithms
Thursday, September 10, 2009
2:51 PM

exponentials
- simple growth + decay processes

logarithm
log = log 10

3 orders of magnitude = 10^3

Logs

supplemental reading

$y = a^x$
$x = a^y$ ↓ flip
$\parallel$
$y = \log_a x$

Only 1 or 2 common bases

base 10 - sci + engineering
good w/ decimal system

$e \approx 2.718$
simple for differentiation + integration
"natural"

base 2
good for computers

$\ln(a\,b) = \ln(a) + \ln(b)$
$\ln(a^b) = b\ln(a)$

$\ln(e^x) = x$

other bases can work
$a^x = (e^{\ln a})^x = e^{x \ln a}$

$\log_a x = \dfrac{\ln x}{\ln a}$

* to get rid of logs exponitate

Approximations

Supplemental Reading

$\approx$ approx = to

? is this just tangent line?

$$f(x) \approx f(a) + \underbrace{f'}_{\text{tangent}}(a)\,(x-a)$$

or sum of geometric sequence

$$\frac{1}{1-x} = 1 + x + x^2 + x^n$$

Properties of Integrals

<u>definite</u>

$$\boxed{\int_b^a f(x)\,dx}$$

$$\int_b^a f(x)\,dx = -\int_b^a f(x)\,dx$$

$$\int_a^a f(x)\,dx = 0$$

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

$$\int_a^b (f+g)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

$$\int_a^b C\,f(x)\,dx = C\int_a^b f(x)\,dx$$

Fundemental Theorm of Calculus

differentiation + integration

(1) indefinite integration can be reversed by differentiation - not by blocks

indefinate / antiderivitive integral

$F' = f$

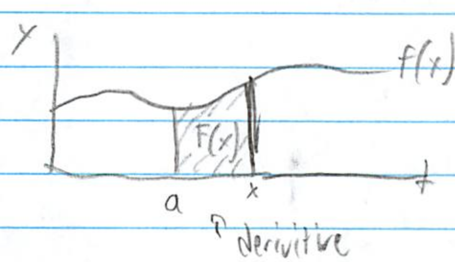
definate integral over an interval is = to diff between endpoints of interval

(2) compute definite integral by using many antiderivitives
- makes definite integrals simple

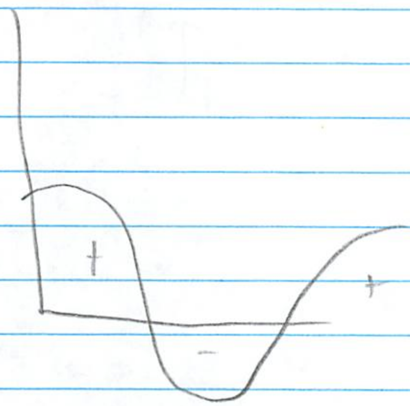
(1) $\int_a^b f(x)\,dx = F(b) - F(a) = F(x)\Big]_a^b$

$= \int f(x)\,dx\Big]_a^b$

(2)



Integration



net signed area
* area under a curve

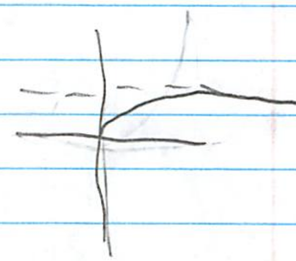
- anti derivitve - very similar to indefinite integral

*Reverse of differentiation

Limit

- get closer + closer

$$\lim_{n \to \infty} \frac{n}{n+1} = 1$$



- will never get to 1
  only approches

Differentiation How To
Textbook Chap 3

$\frac{d}{dx} c = 0$
constant

$\frac{d}{dx} x^2 \to 2x^{2-1} \to 2x$ $\qquad$ $\frac{d}{dx} x^3 \to 3x^2$

binomial theorm

$(a+b)^n = (a+b)(a+b) \cdots (a+b)$

$\frac{d}{dx} 3x^7 \to 3 \cdot 7 x^6$

$21 x^6$

---

product + quotient rules

$\frac{d}{dx} uv = u \frac{dv}{dx} + v \frac{du}{dx}$

Lecture 2

(Textbook 2.3 2.4)

9/11

yesterday

- did tangent problem to curve $y = f(x)$

  * dynamic process of computing sucessive secant line approx *

slope of tangent line $f'(x_0)$

$$\lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

slope of secant line through 2 pts
$(x_0, f(x_0))$
$(x_0+h, f(x_0+h))$

* won't work for every function f must be nice

Warmup $f(x) = x^n$ n → positive integer

$$f'(x) = \lim_{h \to 0} \frac{(x_0+h)^n - (x_0)^n}{h}$$

* need to use algebra to get to situation where can compute the limit

expand

$$\frac{\underbrace{(x_0+h)(x_0+h)\cdots(x_0+h)}}{h}$$

$x_0^n + n x_0^{n-1} \cdot h + (*) x_0^{n-2} h^2 + \dots + h^n$

after cancling ← pascal's triangle work

$$\lim_{x \to 0} \frac{n x_0^{n-1} h^1 + (\cancel{*}) x_0^{n-2} h^2 + \dots + h^n}{h}$$

every term has an h

$n x_0^{n-1}$ - independent of h

↑ proof of derivitive

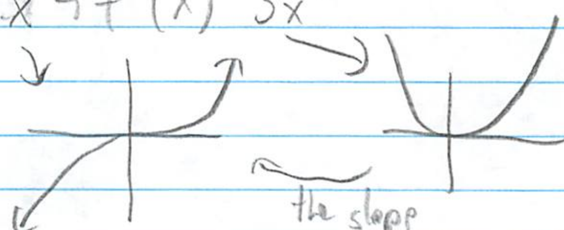
hard since pratical + theoritcal → so different

$$\boxed{\text{any point in our function} \quad f'(x) = n x^{n-1}}$$

prof have big picture understanding vs HS math teacher

$f(x) = x^3 \to f'(x)\ 3x^2$

squeezed



the slope at each point

different notation for derivitive

$f'(x)$ - function

$f'(x_0)$ - point x_0

$\frac{d}{dx} f(x)$ - function

↑ action of taking deriv

$\left.\frac{d}{dx} f(x)\right|_{x=x_0}$ ← at point

Some properties of deriv are easy to show

1. $\frac{d}{dx} c = 0$ ← deriv constant = 0

Taking 1 hr to study/review makes it so much clearer

2. $\frac{d}{dx}\left(c \cdot f(x)\right) = c \frac{d}{dx}\left(f(x)\right)$

3. $\frac{d}{dx}\left(f(x) + g(x)\right) = \frac{d}{dx}\left(f(x)\right) + \frac{d}{dx}\left(g(x)\right)$

Using these 3 rules - can differentiate any polynomial

$$g(x) = 13x^7 + 5x^4 - 2x + 1$$
$$13 \cdot 7x^6 + 5 \cdot 4x^3 - 2$$
$$91x^6 + 20x^3 - 2$$

Do all these events - have fun on your own don't need a grp - like architecture better

<u>How to prove #2</u>

$$\frac{d}{dx}\left(c \cdot f(x)\right) = \lim_{h \to 0} \frac{c \cdot f(x+h) - c\,f(x)}{h}$$

$$\lim_{h \to 0} c\left(\frac{f(x+h) - f(x)}{h}\right)$$

reasonable property of limits

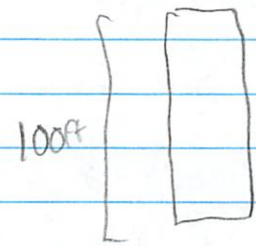
$$c \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$c \cdot \frac{d}{dx}\left(f(x)\right)$$

Perfer this to pre calc + calc teacher

Why take derivitives?

- wishing well 100 ft tall
- drop penny
- model flight



force of gravity
constant acc $= -32\ ft/s^2$

position $\to s(t)$

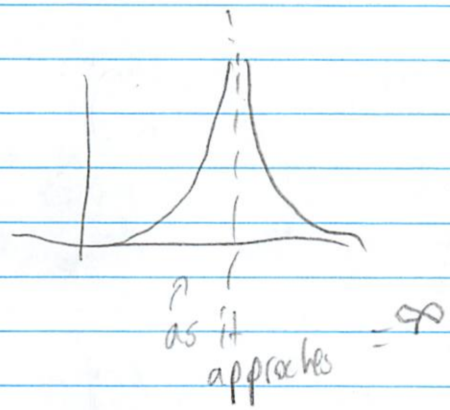
$s(t) = 100 - 16t^2$

velocity (instantaneous) $= s'(t_0)$
acceleration $= s''(t_0) = v'(t_0)$

$s'(t) = -32t$
$s''(t) = -32 \to -32\ ft/s^2$

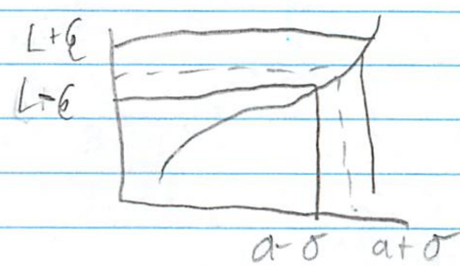
2.5 Concept of a Limit

Reading



ε does not really quality

$|f(x) - L| < \varepsilon$
$|x - a| < \delta$



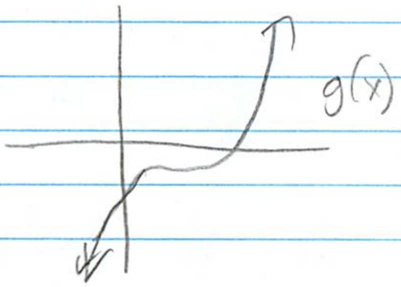
2.6 Contineous Function

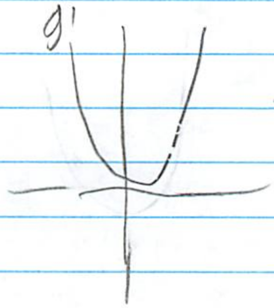
- w/o gaps + sudden changes
- I know this stuff fairly well

Recitation 2

9/14/09

1. Graph $g'(x)$ + write approx equation for der'v

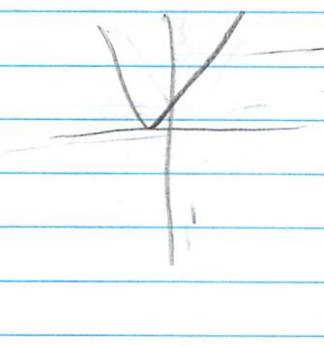




all (+) slope

$$\cancel{y - y_1 = m(x - x_1)}$$
$$\cancel{y - y_1 = 8(x - x_1)}$$
$$\cancel{y - 8 = 8(x-1)}$$
$$\cancel{-8 = 8(x-1)}$$
$$\cancel{-1 = (x-1)}$$
$$\cancel{x = 1}$$

Confused
finding where tangent line hits axis



continuous
not differentiable

2a. Find the value of $f'(2)$

$m = 6$ ← just read

← plug into point slope form

$y - 3 = 6(x - 2)$

$y - 3 = 6x - 12$ ← can leave it at pt slope

$y = 6x - 9$

look the same in a small neighborhood

2b Derivitive → ∞ ~~looks the same~~

Slope → ∞

2bb - streight diagonal w slope of -4

- asymptote $y = -4$

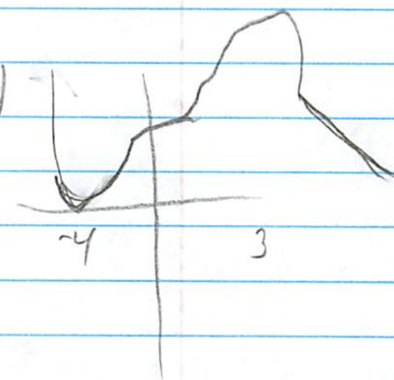
2c ?

Local max at $x = 3$

Local min at $x = -4$

↑ fastest at $x = 2$

Decreasing fastest as approaches $x \to -5$



answers go on site

Worksheet 2: Intro to Derivatives

18.01 Fall 2009

Problem 1. Let

$$y = \frac{1}{1-x}$$

Use the definition of the derivative to find the formula for dy/dx

$$\lim_{h\to 0} \frac{\frac{1}{1-(x+h)} - \frac{1}{1-x}}{h}$$

$$\frac{1-x-(1-x-h)}{(1-x-h)(1-x)} \Big/ h$$

$$1-x-1+x+h \,/\, h$$

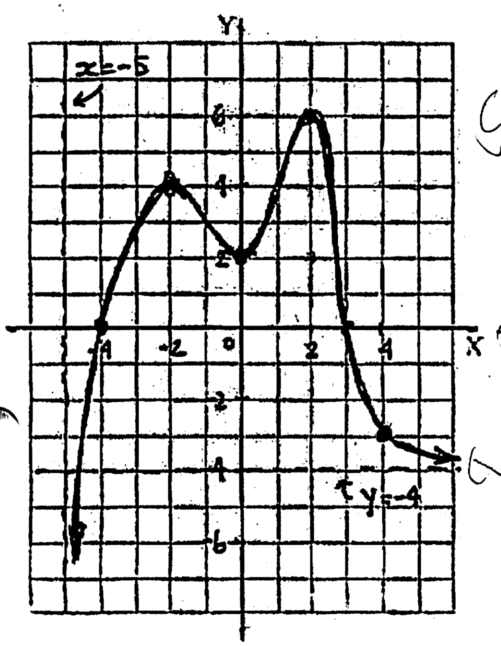
$$\frac{h}{(1-x-h)(1-x)} \cdot \frac{1}{h}$$

$$\frac{1}{(1-x-h)(1-x)}$$

$$\frac{1}{(1-x)^2}$$

only continues

Problem 2. Suppose the following graph from the first worksheet is the graph of the derivative $g'(x)$ of a certain function $y = g(x)$ defined on the interval $x > -5$.



$g'(x)$

← $g(x)$ changes direction?

a) What is the formula for the tangent line to $y = g(x)$ at the point (2,3)? How do the graphs of $y = g(x)$ and this tangent line look like relative to each other in a small neighborhood of $x = 2$?

b) What is the behavior of $y = g(x)$ as $x \to -5^+$ and $x \to \infty^-$?

c) At which values of x does $g(x)$ attain maximum and minimum values? Where, if anywhere, is it increasing fastest? decreasing fastest?

d) Suppose you could find out the value of $g(x)$ at any *one* value of x. Which point could you ask for to determine whether $g(x)$ is only negatively-valued for positive values of x?

d*) Looking ahead/discussion: what if you could only find out the value *at* $x = 0$. What are some cases in which you could still answer the above question, using only that value and this relatively sketchy graph?

Worksheet 2: Intro to Derivatives

18.01 Fall 2009

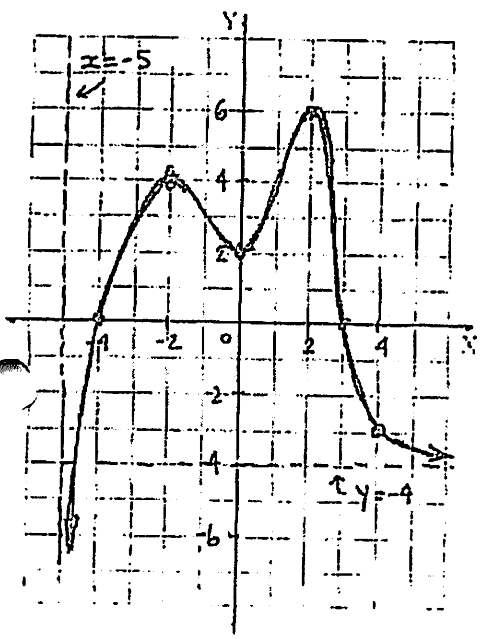
Problem 1. Let

$$y = \frac{1}{1-x}$$

Use the definition of the derivative to find the formula for dy/dx

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{\frac{1}{1-(x+h)} - \frac{1}{1-x}}{h} = \lim_{h\to 0} \frac{1-x-(1-(x+h))}{h(1-(x+h))(1-x)} = \lim_{h\to 0} \frac{h}{h(1-(x+h))(1-x)} = \frac{1}{(1-x)^2}$$

Problem 2. Suppose the following graph from the first worksheet is the graph of the derivative $g'(x)$ of a certain function $y = g(x)$ defined on the interval $x > -5$.



deriv = 6

a) What is the formula for the tangent line to $y = g(x)$ at the point (2,3)? How do the graphs of $y = g(x)$ and this tangent line look like relative to each other in a small neighborhood of $x = 2$?

From graph above, $g'(2) = 6$ ⇒ tangent line formula/eq.: $Y = 6x + b$. Solve for b: $3 = 6\cdot 2 + b \Rightarrow b = 3 - 12 = -9$.

$\boxed{Y = 6x - 9}$

b) What is the behavior of $y = g(x)$ as $x \to -5^+$ and $x \to \infty$?

as $x \to -5^+$, $g(x) \to$ (finite or ∞) approx. in a spike) $\boxed{\text{(local)}}$ (as $x \to \infty$, $g(x) \to -4x$) (so tends to $-\infty$)

c) At which values of x does $g(x)$ attain maximum and minimum values? Where, if anywhere, is it increasing fastest? decreasing fastest?

No global max/mins. local max : $x = 3$ increasing fastest : $x = 2$

local min : $x = -4$ decreasing fastest : nowhere (not attained)

d) Suppose you could find out the value of (*) $g(x)$ at any *one* value of x. Which point could you ask for to determine whether $g(x)$ is only negatively-valued for positive values of x?

(*) some ~~~~ (other) antiderivative – so that do not have pt from (a)...

ask for value at local max – @ $x = 3$.

d*) Looking ahead/discussion: what if you could only find out the value at $x = 0$. What are some cases in which you could still answer the above question, using only that value and this relatively sketchy graph?

Would know answer if val. at $x=0$ were either $\geq a$ for some not too negative (or positive) a, or (for answer – "no") were sufficiently negative (for answer "yes") that you would know the answer.

Basically, have a bound on increase in $g(x)$ from $x=0$ to $x=3$. By a very nominal/first approximation

Know this (looking ahead) by counting boxes below and above graph e.g. $\{ 6 \leq \Delta g \leq 18$ ... $x > -6 \to$ "no"

Office Hrs
Briner

8/14

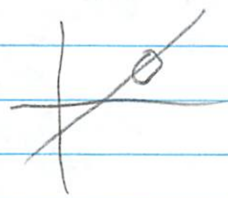
Limit - can fill in if removable
- defined elsewhere
- check that

most simplify

$\lim_{x \to 3} \frac{x^2-9}{x-3}$ $\frac{\cancel{(x-3)}(x+3)}{\cancel{(x-3)}}$ $x+3$ $\rightarrow 6$

discontinuous

continuous



* removable

& more stuff elsewhere

Lecture 3

9/15/09

Office hrs Thur 2:30 - 4 2-267

Last weed defined derivitive
computed it
proved some elementry properties of derivitive

1st qu on exam → to define tangent line to a curve
to model pennies in a well falling
- instentanous rate of change

Compute derivitive of a trig function

sin x is complicated function

$$\frac{d}{dx}(\sin x) = \lim_{h \to 0} \frac{\sin(x+h) - \sin x)}{h}$$

make easier → trig identities

$$\sin(x+h) = \sin x \cos h + \cos x \sin h$$

← factor

$$\lim_{h \to 0} \frac{\sin x(\cos h - 1) + \cos x \sin h}{h}$$

← split into 2 provided both limits exist

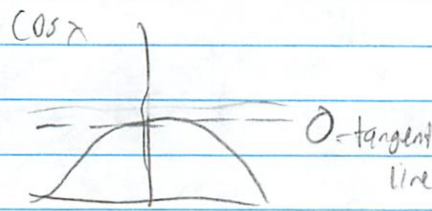
$$\lim_{h \to 0} \sin x \left(\frac{\cos h - 1}{h}\right) + \lim_{x \to 0} \cos x \left(\frac{\sin h}{h}\right)$$

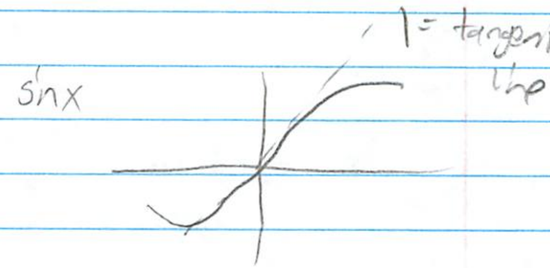
pull out stuff not dependended on h

$$\sin x \lim_{h \to 0} \left(\frac{\cos(h) - 1}{h}\right) + \cos x \lim_{h \to 0} \left(\frac{\sin h}{h}\right)$$

$$\qquad \qquad \underset{\displaystyle \frac{d}{dx}(\cos x)\Big|_{x=0}}{\cap} \qquad \qquad \underset{\displaystyle \frac{d}{dx}(\sin x)\Big|_{x=0}}{\cap}$$

Sketchy answer





Strategy

show $\cos x < \frac{\sin x}{x} < 1$ for small values of x

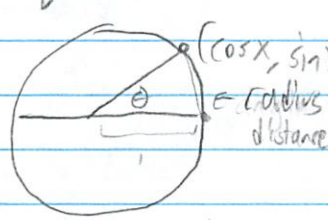
Take limit of everything
↑ squeeze theorm

$$\lim_{x \to 0} \cos x \leq \lim_{x \to 0} \frac{\sin x}{x} \leq \lim_{x \to 0} 1$$

$$1 \leq \underline{\quad} \leq 1$$

↑ can fill in 1

For Inequality



curvey line longer $\frac{\sin x}{x} < 1$

Other inequality follows similarly w/ squeeze theorm

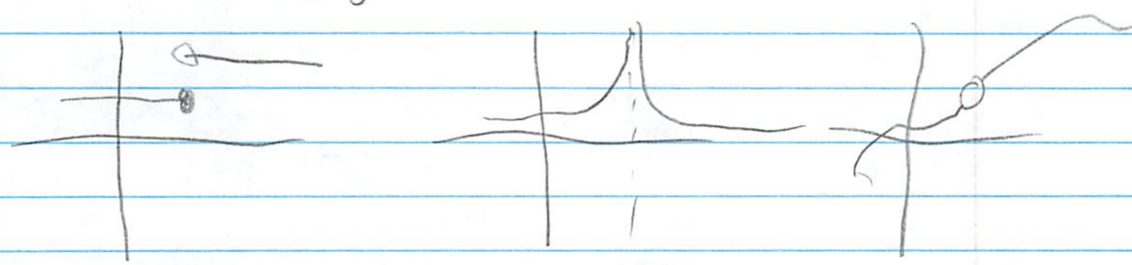
Combo of squeese theorm

$\lim_{h \to 0} \frac{\sin h}{h} = 1$ - how have to do w/ cos p73 in book

Algebric trik to show $\lim_{h \to 0} \frac{\cos h - 1}{h} = 0$

Now can prove $\frac{d}{dx} \sin x = \cos x$

When does computing the derivitive fail?



$f(x) = \begin{cases} 3 \text{ if } x < 4 \\ 2 \text{ in } x \leq 4 \end{cases}$

$\frac{(x-2)\, x^5 + 17^3 + 2x^2 + 1}{(x-2)}$

contineous →
3 statements in 1
- assume $f(x_0)$ is defined
- $\lim_{x \to x_0} f(x)$ is defined
- $\lim_{x \to x_0} f(x) = f(x_0)$

- all pictures : functions not contineous at some point

Contineous means that : use limit to define at x_0

$f(x)$ contineous at x_0 if $\lim_{x \to x_0} f(x) = f(x_0)$

If $f(x)$ is contineous for all x_0 in its domain + has derivitive at $x = x_0$

$\rightarrow f(x)$ is contineous

~~Now trying to take deriv of non contineous function~~

If $A \to B$
then
$not(A) \to not(B)$

Assume

pf: $f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$ = some # L

want to show that $f(x)$ is contineous at $x = x_0$

$\to \lim_{x \to x_0} f(x) = f(x_0)$

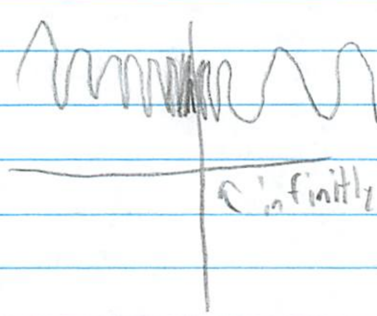
$\to \lim_{x \to x_0} \left(f(x) - f(x_0) \right) = 0$

$\lim_{x \to x_0} f(x) - f(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \cdot (x - x_0)$

$= \underbrace{\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}}_{L} \cdot \underbrace{\lim_{x \to x_0} (x - x_0)}_{0}$

Wierd

$\sin\left(\frac{1}{x}\right)^{+5}$ at $x = 0$



infinitly often   limit does not exist

Addition

Property of derivitive   Friday $\frac{d}{dx}\left(f(x) + g(x) \right) = \frac{d}{dx} f(x) + \frac{d}{dx} g(x)$

Multiplication

product rule $\frac{d}{dx}\left(f(x)\,g(x)\right)$

- How can we guess it?

$x^2 \to x \cdot x$

$\frac{d}{dx}(x^2) = 2x = \frac{d}{dx}(x \cdot x)$

$\frac{d}{dx}(5x) = 5$

- answer is semetric w/ respect to advancing role

$$\frac{d}{dx}\left(f(x)\,g(x)\right) = \frac{d}{dx}\left(f(x)\right) \cdot g(x) + \frac{d}{dx}\left(g(x)\right) \cdot f(x)$$

Recetation

9/16

Product rule
squeeze theorm
quotient rule

product rules $\quad f, g$ differentiable

$$(fg)' = f'g + g'f$$

$$\frac{d}{dx}(x^2 \sin x)$$

$$2x \sin x + (\cos x)x^2$$

quoitent rule

$$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$$

$f'(\cos x) = -\sin x$

chain rule

$$\frac{\cos x \cos x + (\sin x \sin x)}{\cos^2 x}$$

* $\cos^2 x + \sin^2 x = 1$

$$\frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$\frac{\cos^2 x}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x}$$

$$1 + \tan^2 x$$

so $f'(\tan x) = \sec^2 x$

$$\sec^2 x$$

* can make trig derivs w/ product/quotient rule

Mathmatical Induction

1. show base is true ($n=2$)
2. Assume kth case is true ($n=k$)
3. Show case $k+1$ is true ← treat as a case of 2 functions

domonoes falling in a row

Worksheet 3: Limits and Derivatives

study finding deriv from notes → l'hopitals

18.01 Fall 2009

Problem 1. Find the derivative of $\cos x$ from the definition of the derivative. You will probably need to use the trigonometric limits mentioned in lecture.

$\boxed{\lim_{h\to 0} \frac{1-\cosh}{h} = 0}$ $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ $\frac{\cos(x+h)-\cos(x)}{h}$

$\frac{\cosh(\cos x-1)}{h} - \frac{\sin x \sinh}{h}$ $\frac{\cosh(\cos x-1)-(\sin x \sinh)}{h}$ $\frac{\cos x \cosh - (\sin x \sinh) - \cosh}{h}$

$(\cos x)\lim \frac{\cosh - 1}{h} - \lim \frac{\sin x \sinh}{h} \to \cos x \cdot 0 - \lim \frac{\sin x \sinh}{h} \to -\sin x \cdot 1) \to -\sin x$

Problem 2. Compute the following limits 2 functions w/ finite rules

a) $\lim_{x\to 0} \frac{x^2}{\sin x}$ · product of 2 separate limits

$\boxed{\frac{\sin h}{h} = 1}$ can't do $x^2 / \frac{1}{\sin x}$ goes to ∞ $x \cdot \frac{x}{\sin x}$ reciprocal = 1 can't pull out of function $x \cdot 1 = 0$

b) $\lim_{x\to 5}(5y + 3x + 2)$

$\lim 3y + \lim 3x + \lim 2$

$y5 + 3(5) + 2$

$5y + 17$

c) $\lim_{s\to 0} \frac{\cos(\pi/2+s)-\cos(\pi/2)}{s}$ $\frac{\cos\frac{\pi}{2}\cos s - (\sin\frac{\pi}{2}\sin s) - \cos\frac{\pi}{2}}{s} \to \frac{\cos\frac{\pi}{2}(\cos s - 1) - \sin\frac{\pi}{2}\sin s}{s}$

deriv of cos at $\frac{\pi}{2}$

-1

d) $\lim_{x\to\pi/3} \frac{\cos(x)-0.5}{x-\pi/3}$ ← $\cos\frac{\pi}{3}$

set $x = \frac{\pi}{3} + h$ ← derivative lim as $h \to 0$

Problem 3. Suppose $f'(3) = 5$, $g'(3) = 2$, and $f(3) = g(3) = 2$. Compute

a) $(fg)'(3)$ $f' g' = 3 \cdot 2 = 6$

b) $(f/g)'(3)$ $\frac{f'}{g'} = \frac{3}{2} = 1.5$

c) $(3f - g'(3) * g)'(3)$

$(3\cdot 2 - 2 \cdot 2)'(3)$

$2'(3)$

office 4-5

Lecture 3
Derivatives of Products, Quotients, Sine, and Cosine

Derivative Formulas

There are two kinds of derivative formulas:

1. Specific Examples: $\frac{d}{dx}x^n$ or $\frac{d}{dx}\left(\frac{1}{x}\right)$
2. General Examples: $(u+v)' = u' + v'$ and $(cu) = cu'$ (where c is a constant)

A notational convention we will use today is:

$$(u+v)(x) = u(x) + v(x); \quad uv(x) = u(x)v(x)$$

Proof of $(u+v) = u' + v'$. (General)

Start by using the definition of the derivative.

$$\begin{aligned}
(u+v)'(x) &= \lim_{\Delta x \to 0} \frac{(u+v)(x+\Delta x) - (u+v)(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{u(x+\Delta x) + v(x+\Delta x) - u(x) - v(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \left\{ \frac{u(x+\Delta x) - u(x)}{\Delta x} + \frac{v(x+\Delta x) - v(x)}{\Delta x} \right\} \\
(u+v)'(x) &= u'(x) + v'(x)
\end{aligned}$$

Follow the same procedure to prove that $(cu)' = cu'$.

Derivatives of $\sin x$ and $\cos x$. (Specific)

Last time, we computed

$$\begin{aligned}
\lim_{x \to 0} \frac{\sin x}{x} &= 1 \\
\frac{d}{dx}(\sin x)|_{x=0} &= \lim_{\Delta x \to 0} \frac{\sin(0+\Delta x) - \sin(0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\sin(\Delta x)}{\Delta x} = 1 \\
\frac{d}{dx}(\cos x)|_{x=0} &= \lim_{\Delta x \to 0} \frac{\cos(0+\Delta x) - \cos(0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\cos(\Delta x) - 1}{\Delta x} = 0
\end{aligned}$$

So, we know the value of $\frac{d}{dx}\sin x$ and of $\frac{d}{dx}\cos x$ at $x = 0$. Let us find these for arbitrary x.

$$\frac{d}{dx}\sin x = \lim_{\Delta x \to 0} \frac{\sin(x+\Delta x) - \sin(x)}{\Delta x}$$

Recall:

$$\sin(a+b) = \sin(a)\cos(b) + \sin(b)\cos(a)$$

So,

$$\begin{aligned}\frac{d}{dx}\sin x &= \lim_{\Delta x\to 0}\frac{\sin x\cos\Delta x + \cos x\sin\Delta x - \sin(x)}{\Delta x}\\ &= \lim_{\Delta x\to 0}\left[\frac{\sin x(\cos\Delta x - 1)}{\Delta x} + \frac{\cos x\sin\Delta x}{\Delta x}\right]\\ &= \lim_{\Delta x\to 0}\sin x\left(\frac{\cos\Delta x - 1}{\Delta x}\right) + \lim_{\Delta x\to 0}\cos x\left(\frac{\sin\Delta x}{\Delta x}\right)\end{aligned}$$

Since $\dfrac{\cos\Delta x - 1}{\Delta x} \to 0$ and that $\dfrac{\sin\Delta x}{\Delta x} \to 1$, the equation above simplifies to

$$\boxed{\frac{d}{dx}\sin x = \cos x}$$

A similar calculation gives

$$\boxed{\frac{d}{dx}\cos x = -\sin x}$$

Product formula (General)

$$(uv)' = u'v + uv'$$

Proof:

$$(uv)' = \lim_{\Delta x\to 0}\frac{(uv)(x+\Delta x) - (uv)(x)}{\Delta x} = \lim_{\Delta x\to 0}\frac{u(x+\Delta x)v(x+\Delta x) - u(x)v(x)}{\Delta x}$$

Now obviously,

$$u(x+\Delta x)v(x) - u(x+\Delta x)v(x) = 0$$

so adding that to the numerator won't change anything.

$$(uv)' = \lim_{\Delta x\to 0}\frac{u(x+\Delta x)v(x) - u(x)v(x) + u(x+\Delta x)v(x+\Delta x) - u(x+\Delta x)v(x)}{\Delta x}$$

We can re-arrange that expression to get

$$(uv)' = \lim_{\Delta x\to 0}\left(\frac{u(x+\Delta x) - u(x)}{\Delta x}\right)v(x) + u(x+\Delta x)\left(\frac{v(x+\Delta x) - v(x)}{\Delta x}\right)$$

Remember, the limit of a sum is the sum of the limits.

$$\left[\lim_{\Delta x\to 0}\frac{u(x+\Delta x) - u(x)}{\Delta x}\right]v(x) + \lim_{\Delta x\to 0}\left(u(x+\Delta x)\left[\frac{v(x+\Delta x) - v(x)}{\Delta x}\right]\right)$$

$$(uv)' = u'(x)v(x) + u(x)v'(x)$$

Note: we also used the fact that

$$\lim_{\Delta x\to 0} u(x+\Delta x) = u(x) \qquad \text{(true because } u \text{ is continuous)}$$

This proof of the product rule assumes that u and v have derivatives, which implies both functions are continuous.

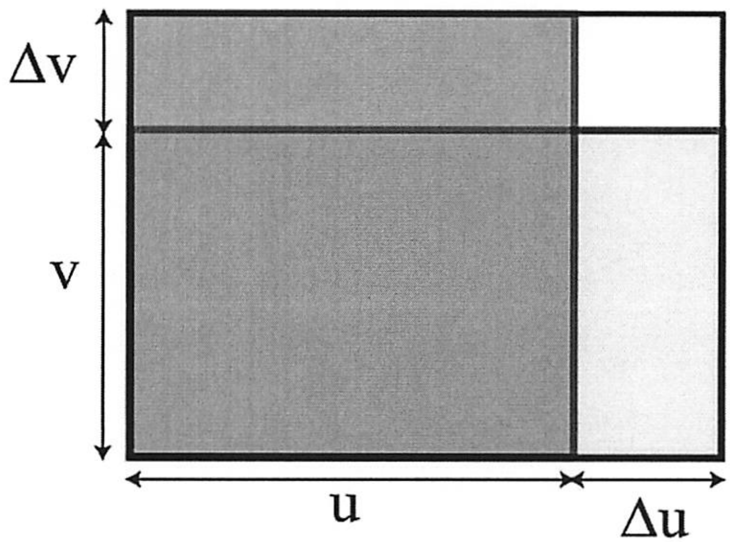


Figure 1: A graphical "proof" of the product rule

An intuitive justification:

We want to find the difference in area between the large rectangle and the smaller, inner rectangle. The inner (orange) rectangle has area uv. Define Δu, the change in u, by

$$\Delta u = u(x + \Delta x) - u(x)$$

We also abbreviate $u = u(x)$, so that $u(x + \Delta x) = u + \Delta u$, and, similarly, $v(x + \Delta x) = v + \Delta v$. Therefore the area of the largest rectangle is $(u + \Delta u)(v + \Delta v)$.

If you let v increase and keep u constant, you add the area shaded in red. If you let u increase and keep v constant, you add the area shaded in yellow. The sum of areas of the red and yellow rectangles is:

$$[u(v + \Delta v) - uv] + [v(u + \Delta u) - uv] = u\Delta v + v\Delta u$$

If Δu and Δv are small, then $(\Delta u)(\Delta v) \approx 0$, that is, the area of the white rectangle is very small. Therefore the difference in area between the largest rectangle and the orange rectangle is approximately the same as the sum of areas of the red and yellow rectangles. Thus we have:

$$[(u + \Delta u)(v + \Delta v) - uv] \approx u\Delta v + v\Delta u$$

(Divide by Δx and let $\Delta x \to 0$ to finish the argument.)

Quotient formula (General)

To calculate the derivative of u/v, we use the notations Δu and Δv above. Thus,

$$\begin{aligned}
\frac{u(x+\Delta x)}{v(x+\Delta x)} - \frac{u(x)}{v(x)} &= \frac{u+\Delta u}{v+\Delta v} - \frac{u}{v} \\
&= \frac{(u+\Delta u)v - u(v+\Delta v)}{(v+\Delta v)v} \quad \text{(common denominator)} \\
&= \frac{(\Delta u)v - u(\Delta v)}{(v+\Delta v)v} \quad \text{(cancel } uv - uv\text{)}
\end{aligned}$$

Hence,

$$\frac{1}{\Delta x}\left(\frac{u+\Delta u}{v+\Delta v} - \frac{u}{v}\right) = \frac{(\frac{\Delta u}{\Delta x})v - u(\frac{\Delta v}{\Delta x})}{(v+\Delta v)v} \longrightarrow \frac{v(\frac{du}{dx}) - u(\frac{dv}{dx})}{v^2} \qquad \text{as } \Delta x \to 0$$

Therefore,

$$(\frac{u}{v})' = \frac{u'v - uv'}{v^2}$$

.

Limits Involving Trigonometric Functions

Limits Involving Trigonometric Functions

The trigonometric functions sine and cosine have four important limit properties:

$$\lim_{x \to c} \sin x = \sin c$$

$$\lim_{x \to c} \cos x = \cos c$$

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

$$\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$$

You can use these properties to evaluate many limit problems involving the six basic trigonometric functions.

Example 1: Evaluate

$$\lim_{x \to 0} \frac{\cos x}{\sin x - 3}.$$

.

Substituting 0 for *x*, you find that cos *x* approaches 1 and sin *x* − 3 approaches −3; hence,

$$\lim_{x \to 0} \frac{\cos x}{\sin x - 3} = -\frac{1}{3}$$

Example 2: Evaluate

$$\lim_{x \to 0^+} \cot x.$$

Because cot *x* = cos *x*/sin *x*, you find

$$\lim_{x \to 0^+} \cos x / \sin x.$$

The numerator approaches 1 and the denominator approaches 0 through positive values because we are approaching 0 in the first quadrant; hence, the function increases without bound and

$$\lim_{x \to 0^+} \cot x = +\infty,$$

and the function has a vertical asymptote at $x = 0$.

Example 3: Evaluate

$$\lim_{x \to 0} \frac{\sin 4x}{x}.$$

Multiplying the numerator and the denominator by 4 produces

$$\begin{aligned} \lim_{x \to 0} \frac{\sin 4x}{x} &= \lim_{x \to 0} \frac{4 \sin 4x}{4x} \\ &= \left(\lim_{x \to 0} 4 \right) \cdot \lim_{x \to 0} \frac{\sin 4x}{4x} \\ &= 4 \cdot 1 \end{aligned}$$

$$\lim_{x \to 0} \frac{\sin 4x}{x} = 4$$

Example 4: Evaluate

$$\lim_{x \to 0} \frac{\sec x - 1}{x}.$$

.

Because sec x = 1/cos x, you find that

$$\begin{aligned} \lim_{x \to 0} \frac{\sec x - 1}{x} &= \lim_{x \to 0} \frac{\frac{1}{\cos x} - 1}{x} \\ &= \lim_{x \to 0} \frac{1 - \cos x}{x \cos x} \\ &= \lim_{x \to 0} \left(\frac{1}{\cos x} \right) \cdot \left(\frac{1 - \cos x}{x} \right) \\ &= \left[\lim_{x \to 0} \frac{1}{\cos x} \right] \cdot \left[\lim_{x \to 0} \frac{1 - \cos x}{x} \right] \\ &= 1 \cdot 0 \end{aligned}$$

$$\lim_{x \to 0} \frac{\sec x - 1}{x} = 0$$

| | |

Lecture

Product + Chain Rule

9/17

P-set due 1:45 in 2-106

Office hrs today 2-267

Product rule

know

$$\boxed{\frac{d}{dx}\tan x = \sec^2 x}$$

if f + g are differentiable

$$\left[\frac{d}{dx}\left(f(x)\cdot g(x)\right)\right] + \left[f(x)\frac{d}{dx}\left(g(x)\right)\right]$$

or $(fg)' = f'g + g'f$

$$\frac{d}{dx}\, x\tan x = 1\cdot\tan x + x(\sec^2 x)$$

know deriv of

poly nomial
trig functions

How do we prove it

$$\frac{d}{dx}\left(f(x)g(x)\right) = \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

want to know
take deriv
any function
r e^x

clever algebra $\lim_{h\to 0} f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)$

factor $\lim_{h\to 0} g(x+h)\left(\frac{f(x+h)-f(x)}{h}\right) + f(x)\left(\frac{g(x+h)-g(x)}{h}\right)$

split into 2 limit $\lim_{h\to 0} g(x+h)\left(\frac{f(x+h)-f(x)}{h}\right) + \underbrace{\lim_{h\to 0} f(x)\left(\frac{g(x+h)-g(x)}{h}\right)}_{f(x)\lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = f(x)g'(x)}$

$$\underbrace{\lim_{h\to 0} g(x+h)}_{g(x)} \cdot \underbrace{\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}}_{\frac{d}{dx} f(x)} + f(x) \frac{d}{dx}\left(g(x)\right) \Rightarrow$$

True if $g(x)$ is a continuous function

if $g(x)$ is differentiable it is continuous

Quotient rule

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{gf' - fg'}{g^2}$$

$\boxed{\left(f \cdot \frac{1}{g}\right)' \text{ w/ product rule}}$

Provided we know $\left(\frac{1}{g}\right)' = \frac{-g'}{g^2}$ — can we use product rule proof or quotient rule

You would have to prove rules on the test

annoying if we assume f + g differentiable

$g(x_0) \neq 0$

then can't use product rule

need to assume $\left(\frac{1}{g}\right)'$ differentiable

— Read proof on simmons p.90

— linear combo

— can test it for $\frac{x}{1}$ to see if correct ($=1$)

— deriv behaves addition

$$\frac{1 \cdot x' \cdot 1' x}{1^2} = \frac{1^2 - 0}{1^2} = 1$$

Chain Rule

applies to composition of functions
no analog in arimithic - only functions

$t \to \boxed{g} \to g(t)$

$g(t) = x$

$x \to \boxed{f} \to f(x) = y$

$t \longrightarrow g(t) \longrightarrow f(g(t))$

example $g(t) = t^2$
$f(x) = \sin(x)$

$f(g(t))$ [inside out]

$f(t^2)$

$\sin(t^2)$ ← old calculatures use to have to do that

$\frac{d}{dt}\underbrace{(f(g(t)))}_{y} = \frac{dy}{dt}$ ← can cancle notation like fraction $= \frac{dy}{\cancel{dx}} \cdot \frac{\cancel{dx}}{dt}$

notation of $\lim_{h \to 0} \frac{f(x+h) - fx}{h}$

not a fraction
but behaves like one

derivitive of y with respect to t at $t = t_0$

$x_0 = g(t_0)$
$y_0 = f(x_0)$

$$\lim_{t\to t_0} \frac{f(g(t)) - f(g(t_0))}{t - t_0}$$

$$\lim_{t\to t_0} \frac{y-y_0}{t-t_0} = \lim_{t\to t_0} \left(\frac{y-y_0}{x-x_0} \cdot \frac{x-x_0}{t-t_0}\right)$$

$x = g(t)$
$y = f(x)$

$$\overbrace{\lim_{t\to t_0} \left(\frac{y-y_0}{x-x_0}\right)}^{*} \left(\lim_{t\to t_0} \frac{x-x_0}{t-t_0}\right)$$

-rewrite in terms of t

$$\underbrace{\lim_{t\to t_0} \frac{g(t)-g(t_0)}{t-t_0}}_{g' \text{ at } t_0}$$

as $t \to t_0$, $x \to x_0$
because assume g is differentiable $\to$ contineous

trick of continuity

$$* = \lim_{x\to x_0} \frac{y-y_0}{x-x_0} = \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}$$

$$= f'(x_0)$$

$$\frac{d}{dt}\left(f(g(t))\right) = f'(x_0) \cdot g'(t_0)$$

in terms of t

$$f'(g(t_0))\, g'(t_0)$$

ex $g(x) = t^2$ $\quad g'(x) = 2t$
$f(x) = \sin x$ $\quad f'(x) = \cos x$

evaluate $\frac{d}{dt}\left(f(g(t))\right)\Big|_{t=\frac{\pi}{2}}$

$$f'(x_0)\, g'(t_0) \Big|_{t_0 = \frac{\pi}{2}}$$

$$g'\left(\frac{\pi}{2}\right) = \pi$$

$$f'(x_0) = f'(g(t_0)) = f'\left(\frac{\pi^2}{4}\right)$$

$$= \cos\left(\frac{\pi^2}{4}\right)$$

final $\pi \cos\left(\frac{\pi^2}{4}\right)$

ex2 $f(t) = \cos(t^2)$

$$t \to t^2$$

$$\shortparallel$$

$$x \to \cos x$$

$$f(x) = \frac{d}{dx} \cos x \Big|_{x = x_0} \cdot \frac{d}{dt}(t^2)\Big|_{t = t_0}$$

$$-\sin(t^2) \cdot 2t \Big|_{t = t_0}$$

peeling off the layers of the onions

$$d(\text{outer skin})\Big|_{\text{inner onion}}$$ ← insert whats left

repeat → ∞

$$\frac{d}{dx}\left(\cos(\tan(x^3))\right) = -\sin(\tan(x^3)) \cdot \sec^2(x^3) \cdot 3x^2$$

Lecture 4
Chain Rule, and Higher Derivatives

Chain Rule

We've got general procedures for differentiating expressions with addition, subtraction, and multiplication. What about composition?

Example 1. $y = f(x) = \sin x, x = g(t) = t^2$.
So, $y = f(g(t)) = \sin(t^2)$. To find $\dfrac{dy}{dt}$, write

$t_0 = t_0$	$t = t_0 + \Delta t$
$x_0 = g(t_0)$	$x = x_0 + \Delta x$
$y_0 = f(x_0)$	$y = y_0 + \Delta y$

$$\frac{\Delta y}{\Delta t} = \frac{\Delta y}{\Delta x} \cdot \frac{\Delta x}{\Delta t}$$

As $\Delta t \to 0$, $\Delta x \to 0$ too, because of continuity. So we get:

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} \leftarrow \textbf{The Chain Rule!}$$

In the example, $\dfrac{dx}{dt} = 2t$ and $\dfrac{dy}{dx} = \cos x$.

$$\text{So,} \quad \frac{d}{dt}\left(\sin(t^2)\right) = \left(\frac{dy}{dx}\right)\left(\frac{dx}{dt}\right) = (\cos x)(2t) = (2t)\left(\cos(t^2)\right)$$

Another notation for the chain rule

$$\frac{d}{dt}f(g(t)) = f'(g(t))g'(t) \qquad \left(\text{or} \quad \frac{d}{dx}f(g(x)) = f'(g(x))g'(x)\right)$$

Example 1. (continued) Composition of functions $f(x) = \sin x$ and $g(x) = x^2$

$$\begin{array}{rcccl} (f \circ g)(x) & = & f(g(x)) & = & \sin(x^2) \\ (g \circ f)(x) & = & g(f(x)) & = & \sin^2(x) \\ \text{Note:} \quad f \circ g & \neq & g \circ f. & & \textit{Not Commutative!} \end{array}$$

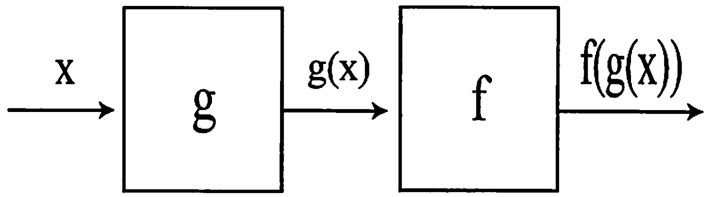


Figure 1: Composition of functions: $f \circ g(x) = f(g(x))$

Example 2. $\frac{d}{dx}\cos\left(\frac{1}{x}\right) = ?$

Let $u = \frac{1}{x}$

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} \\ \frac{dy}{du} &= -\sin(u); \qquad \frac{du}{dx} = -\frac{1}{x^2} \\ \frac{dy}{dx} &= \frac{\sin(u)}{x^2} = (-\sin u)\left(\frac{-1}{x^2}\right) = \frac{\sin\left(\frac{1}{x}\right)}{x^2} \end{aligned}$$

Example 3. $\frac{d}{dx}\left(x^{-n}\right) = ?$

There are two ways to proceed. $x^{-n} = \left(\frac{1}{x}\right)^n$, or $x^{-n} = \frac{1}{x^n}$

1. $\frac{d}{dx}\left(x^{-n}\right) = \frac{d}{dx}\left(\frac{1}{x}\right)^n = n\left(\frac{1}{x}\right)^{n-1}\left(\frac{-1}{x^2}\right) = -nx^{-(n-1)}x^{-2} = -nx^{-n-1}$

2. $\frac{d}{dx}\left(x^{-n}\right) = \frac{d}{dx}\left(\frac{1}{x^n}\right) = nx^{n-1}\left(\frac{-1}{x^{2n}}\right) = -nx^{-n-1}$ (Think of x^n as u)

Higher Derivatives

Higher derivatives are derivatives of derivatives. For instance, if $g = f'$, then $h = g'$ is the second derivative of f. We write $h = (f')' = f''$.

Notations

$f'(x)$	Df	$\frac{df}{dx}$
$f''(x)$	$D^2 f$	$\frac{d^2 f}{dx^2}$
$f'''(x)$	$D^3 f$	$\frac{d^3 f}{dx^3}$
$f^{(n)}(x)$	$D^n f$	$\frac{d^n f}{dx^n}$

Higher derivatives are pretty straightforward — just keep taking the derivative!

Example. $D^n x^n = ?$
Start small and look for a pattern.

$$\begin{aligned}
Dx &= 1 \\
D^2x^2 &= D(2x) = 2 \quad (= 1\cdot 2) \\
D^3x^3 &= D^2(3x^2) = D(6x) = 6 \quad (= 1\cdot 2\cdot 3) \\
D^4x^4 &= D^3(4x^3) = D^2(12x^2) = D(24x) = 24 \quad (= 1\cdot 2\cdot 3\cdot 4) \\
D^nx^n &= n! \leftarrow \text{we guess, based on the pattern we're seeing here.}
\end{aligned}$$

The notation $n!$ is called "n factorial" and defined by $n! = n(n-1)\cdots 2\cdot 1$

Proof by Induction: We've already checked the base case ($n = 1$).

Induction step: Suppose we know $D^n x^n = n!$ (n^{th} case). Show it holds for the $(n+1)^{st}$ case.

$$\begin{aligned}
D^{n+1}x^{n+1} &= D^n\left(Dx^{n+1}\right) = D^n\left((n+1)x^n\right) = (n+1)D^n x^n = (n+1)(n!) \\
D^{n+1}x^{n+1} &= (n+1)!
\end{aligned}$$

Proved!

Michael Plasmeier

67/73

18.01 FALL 2009 – Problem Set 1

Due Friday 9/18/08, 1:45 pm in 2-106

18.01 Supplementary Notes, Exercises and Solutions are for sale at Copy Tech in the basement of Building 11. This is where to find the exercises labeled 1A, 1B, etc. You will need these for the first day's homework.

Web site: *http://math.mit.edu/18.01* Links to syllabus, course information, and problem sets. As the semester progresses, we'll also post announcements, exam info, etc.

Part I consists of exercises given in the Supplementary Notes and solved in section S of the Notes. Of course, you should attempt to solve problems without referring to solutions in advance. These problems will be graded without many comments.

Part II consists of problems for which solutions are not given; it is worth more points. Some of these problems are longer multi-part exercises posed here because they do not fit conveniently into an exam or short-answer format. See the guidelines below (and also on the website) for which types of collaboration are acceptable, and follow them.

To encourage you to keep up with the homework as it pertains to lectures, both Part I and Part II problems are listed with the accompanying lecture in which the material will be covered.

Part I (30 points)

Notation for Listing HW: 2.1 = Section 2.1 of the Simmons book;
2.4/13 = Section 2.4 Problem 13 in Simmons
Notes G = section G of the Notes;
1A-3 = Exercise 1A-3 in Section E (Exercises) of the Notes (solved in section S)

Recitation 0. Wed. Sept. 9: Graphing functions.
Read: Notes G, sections 1-4 HW: ~~1A-1b, 2b, 3abe,~~ 6b, 7b

Lecture 1. Thurs., Sept. 10: Derivative; slope, velocity, rate of change.
Read: 2.1-2.4 HW: 1B-1, 1C-1a (using definition of derivative), 3abe, 4ab (using work in 3), 5, 6 (trace axes on answer sheet)

Lecture 2. Fri. Sept. 11: Limits and continuity; some trigonometric limits
Read: 2.5 (bottom p.70-73; concentrate on examples, skip the $\epsilon - \delta$ def'n)
Read: 2.6 to p. 75; learn def'n (1) and proof "differentiable $\Longrightarrow$ continuous" at the end.
Read: Notes C HW: ~~1C-2, 1D-1acdfg, 3acde, 6a, 8a~~ (remembering "diff $\Longrightarrow$ cont.")

Lecture 3. Tues. Sept. 15: Differentiation formulas: products and quotients; Derivatives of trigonometric functions.
In the following exercises, an *antiderivative* of $f(x)$ is any $F(x)$ for which $F'(x) = f(x)$.
Read: 3.1, 3.2, 3.4 HW: 1E-1ac, 2b, 3, 4b, 5ac; 1J-1e, 2

Lecture 4. Thurs. Sept. 17: Chain rule; higher derivatives.
Read: 3.3, 3.6 HW: 1F-~~1ab,~~ 2, 6, 7bc; 1J- 1akm 1G-1bc, 5ab

Lecture 5. Fri. Sept. 18 Implicit differentiation; inverse functions.
Read: 3.5, Notes G section 5 HW: To be given on Problem Set 2.

18.01 P-Set 1

9/18/08

This P-set is ~365 days late ☺ opps

Please use other side of paper!! Too much wasting

for scanning

Part 1

Rescitation 0

1A-1b Complete the square + sketch

$$y = 3x^2 + 6x + 2$$
$$y = 3(x^2 + 2x) + 2$$
$$3(x^2 + 2x + 1) + 2 - 3 \leftarrow (3 \cdot 1)$$
$$3(x+1)^2 - 1$$

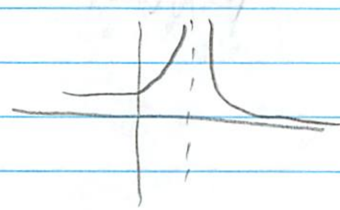
↓ 1

skinny

← 1

2b $\dfrac{2}{(x-1)^2}$

? $x=1$



3a $\dfrac{x^3 + 3x}{1 - 4x^4}$ $f(x) = -f(x)$ x

$f(-x) = -f(x)$ ✓ odd ← ~~denom does not matter to +/-~~

$$\boxed{\frac{\text{odd}}{\text{even}} = \text{odd}}$$

3b $\sin^2 x$ sin is odd odd × odd = even

3c $\frac{\tan x}{1+x^2}$ $\frac{\leftarrow \frac{\text{odd}}{\text{even}}}{\text{even}}$ $\frac{\text{odd}}{\text{even}}$ odd

3e $\ln(x^2)$ even

6b Express in form $A\sin(x+c)$

$\sin x - \cos x$

$a \sin kx + b\cos kx$

$a = A\cos k\phi$ $A = \sqrt{a^2+b^2}$

$b = -A\sin k\phi$ $\tan k\phi = \frac{-b}{a}$

$1 = A\cos k\phi$ $-1 = -A\sin k\phi$

$1 \sin k\phi$

$A = \sqrt{1^2 + -1^2}$ $\tan = \frac{--1}{1} = 1$

$A = \sqrt{2}$ $\tan^{-1}(1) = \frac{\pi}{4}$

i don't really get

$\sqrt{2}\sin(x - \frac{\pi}{4})$

7b Find period, amp, phase angle, sketch

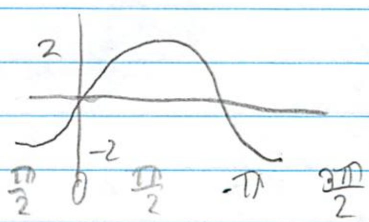
$b = -4\cos(x + \frac{\pi}{2})$

↑ flipped

amp $= 4$

period $= \frac{2\pi}{1} = 2\pi$

$\leftarrow \frac{\pi}{2}$



Lecture 1

1B-1 Test tube knocked 400 ft
drops $16t^2$ ft/sec $\qquad p = -16t^2 + 400$

a. avg speed sec 0-2

$$\frac{-16(2)^2+400-(-16(0)^2+400)}{2-0} = -32 \text{ ft/sec}$$

b. avg speed last 2 sec

$0 = -16t^2 + 400$
$t = 5$

$$\frac{-16(5)^2+400-(-16(3)^2+400)}{5-3} = -128 \text{ ft/sec}$$

c. $f'(t) = -16 \cdot 2t$
$-32t$
$-32(5) = -160$ ft/sec

1c-1a. Calc rate of change of area of disk w/ respect to radius

$$\frac{\pi(r+h)^2 - \pi r^2}{h} \rightarrow \frac{\pi(r^2 + 2rh + h^2) - \pi r^2}{h}$$

$$\frac{\pi r^2 + 2\pi r h + \pi h^2 - \pi r^2}{} \quad \frac{2\pi r h + h^2\pi}{h} \rightarrow 2\pi r + h\pi$$

$\pi(2r+h) \rightarrow 2\pi r$ as $h \to 0$
??

3a. Calc derivative using definition

$$f(x) = \frac{1}{(2x+1)}$$

$$\frac{1}{h}\left[\frac{1}{2(x+h)+1} - \frac{1}{2x+1}\right]$$

← some algebra reducing thing or to get same denom ← yeah

$$\frac{1}{h}\left[\frac{2x+1-(2(x+h)+1)}{(2(x+h)+1)(2x+1)}\right]$$

$$\frac{1}{h}\left[\frac{-2h}{2(x+h)+1)(2x+1)}\right]$$

$$\frac{-2}{(2(x+h)+1)(2x+1)} \to \frac{-2}{(2x+1)^2} \quad \text{as } h \to 0$$

3b $f(x) = 2x^2 + 5x + 4$

$$\frac{2(x+h)^2 + 5(x+h) + 4 - (2x^2 + 5x + 4)}{h}$$

$$\frac{2(x^2 + 2xh + h^2) + 5x + 5h + 4 - 2x^2 - 5x - 4}{h}$$

$$\frac{\cancel{2x^2} + 4xh + 2h^2 + \cancel{5x} + 5h + \cancel{4} - \cancel{2x^2} - \cancel{5x} - \cancel{4}}{h}$$

$$\frac{4xh + 2h^2 + 5h}{h} \to 4x + 2h + 5$$

← why - what are we doing here?

$4x + 5$ as $h \to 0$

3e Find where slope $= +1, -1, 0$

a. $\frac{-2}{(2x+1)^2}$ ← where does $+1, -1, 0$ = that

$\cancel{1 = \frac{-2}{(2x+1)^2}}$

* < 0 so no $+1$ or 0 ←?

$-1 = \frac{-2}{(2x+1)}$

← what are we doing here?

$-1(2x+1) = -2$ ✓ that - know your algebra :)

$2x+1 = 2$

$2x = 1$

$x = \frac{1}{2}$

b. $4x+5 = 1$
$4x = -4$
$x = -1$ where slope $= 1$

$4x+5 = 0$
$4x = -5$
$x = \frac{-5}{4}$
where slope $= 0$

$4x+5 = -1$
$4x = -6$
$x = \frac{-6}{4} = -\frac{3}{2}$
where slope $= -1$

4a Find tangent line

$\frac{1}{(2x+1)}$ at $x=1$ well deriv $= \frac{-2}{(2x+1)^2}$ $\frac{-2}{(2\cdot1+1)^2}$ $\frac{-2}{9}$

$y = -\frac{2}{9}x + b$

* find $f(1) \rightarrow \frac{1}{3}$ and do $-\frac{2}{9}(x-1) + \frac{1}{3}$

$\rightarrow \frac{-2x+5}{9}$

why?

to find x intercept I guess - knever did it like that

4b $f = 2x^2 + 5x + 4$ at $x = a$
$f' = 4x + 5$

$$(4(a)+5)(x-a) + 2a^2 + 5a + 4$$
$\uparrow f'(a)$ $\uparrow$ $\uparrow f(a)$

in For some reason

$$(4ax + 5x - 4a^2 - \cancel{5a} + 2a^2 + \cancel{5a} + 4$$
$$-2a^2 + 4ax + 5x + 4$$
$$(4a+5)x - 2a^2 + 4$$

5. Find tangent at origin

$$\cancel{y = 1 + (x-1)^2}$$
$$\cancel{y' = 2(x-1)}$$

$$\cancel{2(0-1)(x-0) + 1 + (x-1)^2}$$
$$\cancel{-2(x-0) + 1 + (x-1)^2}$$
$$\cancel{-2x + 0 + 1 + x^2 + 2x + 2}$$
$$\cancel{x^2 + 3}$$

$\uparrow$ This is tangent at $x = 0$
and does NOT go through origin

Want tangent line through $(a, 1 + (a-1)^2)$

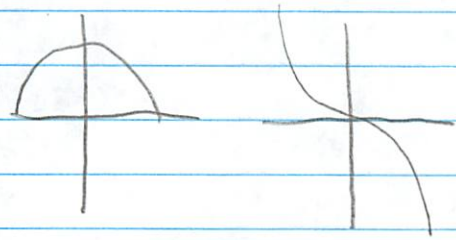
$$y = 2(a-1)(x-a) + 1 + (a-1)^2$$
– now want through origin

$\uparrow$ Kinda get

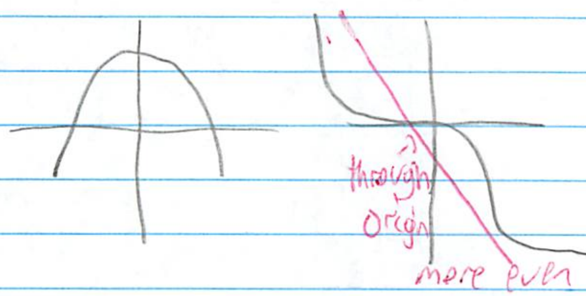
$$0 = 2(a-1)(0-a) + 1 + (a-1)^2$$
$$-2a^2 + 2a + 1 + a^2 - 2a + 1$$
$$-a^2 + 2$$
$$a = \pm\sqrt{2}$$

$$y = 2(\sqrt{2}-1)^2$$
$$y = 2(-\sqrt{2}-1)^2$$

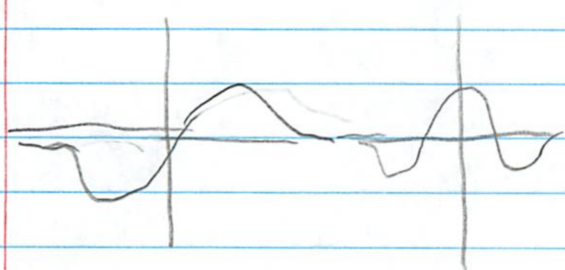
6. Graph Derivitive



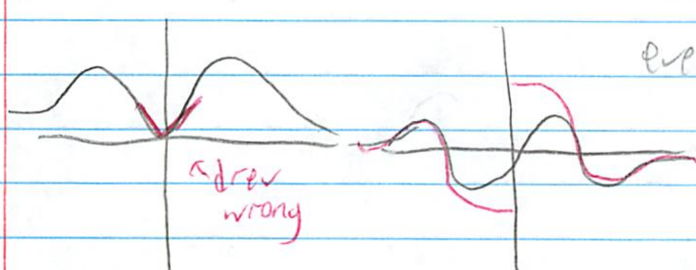
semicircle



parabola



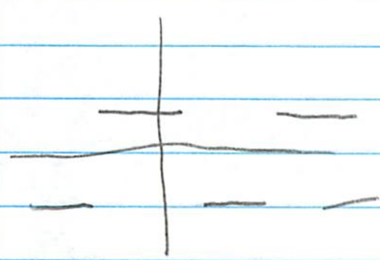
odd function



even function

??

periodic -6



Lecture 2

1c-2 $f(x) = (x-a)\,g(x)$ Use definition to find that $f'(x) = g(x)$ if $g(x)$ is contineous

?? don't get what asking

$$\frac{f(x) - f(a)}{x-a}$$

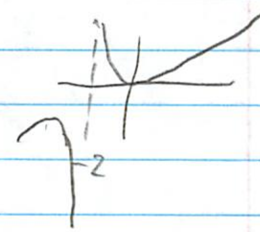
$$\frac{(x-a)\,g(x) - 0}{x-a}$$

$$g(x) \to g(a) \quad \text{as} \quad x \to a$$

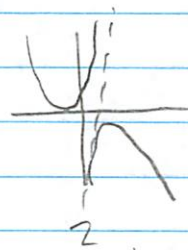
1D-1a Calc limit

a) $\lim_{x \to 0} \frac{4}{x-1}$ $\quad \frac{4}{-1} = -4$

c) $\lim_{x \to -2} \frac{4x^2}{x+2}$ undefined both $\pm\infty$ possible

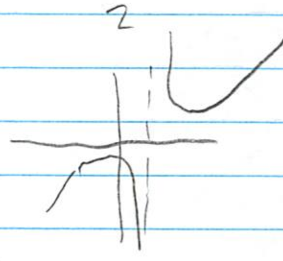


d) $\lim_{x \to 2^+} \frac{4x^2}{2-x}$



$-\infty$

f) $\lim_{x \to \infty} \frac{4x^2}{x-2}$



∞

g $\lim_{x \to \infty} \frac{4x^2 - 4x}{x-2}$ $\quad -\infty$

learn more
removable/non removable

3a Find points of discontinuity

a $\frac{x-2}{x^2-4}$ at $x = \pm 2$

removable ← if limit exists

since $\frac{\cancel{x-2}}{(x+2)\cancel{(x-2)}}$

c $\frac{x^4}{x^3}$ $x = 0$

removable

d $f(x) = \begin{cases} x+a & x>0 \\ a-x, & x<0 \end{cases}$ at $x=0$ ~~removable~~ removable

since at same pt = ??

e $f'(x)$ for the $f(x)$ in d

$\begin{cases} x, & x>0 \\ -x, & x<0 \end{cases}$ at $x=0$ ~~removable~~ jump ← $\lim_{x\to a^+} \neq \lim_{x \to a^-}$

1D-6a Find values where differentiable, constants a+b

↑ can take derivitive of

$\begin{cases} x^2+4x+1 & x\geq 0 \\ ax+b & x<0 \end{cases}$

Office Hours

$f'(x) \begin{cases} 2x+4, & \cancel{x\geq 0} \\ a, & \cancel{x<0} \end{cases}$

$4x+b = x^2+4x+1$

$x^2+1 =$

2nd

$f(0) = 0^2+4\cdot 0+1 = 1$ Match function values

$f(0^-) = \lim_{x\to 0} ax+b = b$ so $b=1$ by continuity

Next match slopes

Chinteen says do this 1st

$f'(0) = 2(0)+4$

$a = 4$ ← deriv

lots of line w/ that deriv

find which line you need

differentiable → continuous
→ must match graph and from left + right ↑

derivitives → need to match up at $x=0$

$f'(0^+) = \lim_{x \to 0} 2x+4 = 4$ and $f'(0) = a$

so $a = 4$ since $f'(0)$ exists

↑ i did not get what doing

8a. Find values of constants for $a + b$ where function is contineous but not differentable

$$\begin{cases} ax+b & x>0 \\ \sin 2x & x \leq 0 \end{cases}$$

$f(0) = \sin(2 \cdot 0) = 0$

$f(0^+) = \lim_{x \to 0} ax+b = b$

↑ continuity implies $b=0$
ie where line connects

slope $f'(0^-) = \lim_{x \to 0} 2\cos(2x) = 2$

$f'(0^+) = \lim_{x \to 0} a = a$

Need $a \neq 2$ so f not differentable
↑ angle (see picture

↑ contineous but not differentiable

~~i don't get~~
looked at

Lecture 3

1E-1a Find derivitive

$x^{10} + 3x^5 + 2x^3 + 4$

$10x^9 + 3 \cdot 5x^4 + 2 \cdot 3x^2$

$10x^9 + 15x^4 + 6x^2$

c $\frac{x}{2} + \pi^3$

$\frac{1}{2}x + \pi^3$ Opps never finished

$\frac{1}{2}$

X

2b Find antiderivitive

$x^6 + 5x^5 + 4x^3$

? like integratiding

$\frac{x^7}{7} + \frac{x^4}{2} + x + C$

3 Find pts where slope tangent line is horizontal

$y = x^3 + x^2 - x + 2$ ↑ deriv = 0

$3x^2 + 2x - 1$

$0 = 3x^2 + 2x - 1$ ← factor

$x = -1 \quad x = \frac{1}{3}$

4b Find where · a + b f(x) is differentable

$$b = f(x) = \begin{cases} ax^2 + bx + 4 & x \le 1 \\ 5x^5 + 3x^4 + 7x^2 + 8x + 4, & x > 1 \end{cases}$$

- must be contineous at x = 1 ← ? know it

$y = 5(1)^5 + 3(1)^4 + 7(1)^2 + 8(1) + 4$

$y = 27$

$27 = ax^2 + bx + 4$

$23 = ax^2 + bx$

? $a + b = 23$

$f'(1^-) = \lim_{x \to 1} 2ax + b = 2a + b$

$f'(1^+) = \lim 25x^4 + 12x^3 + 14x + 8 = 59$

← forgot had to be differentiable as well – match up

$2a + b = 59$

$a = 59 - 23 = 36$

$36 + b = 23$

$b = -13$

5a $\frac{x}{1+x} \rightarrow$ quotient rule $\left(\frac{u}{v}\right)$ $\frac{u'v - v'u}{v^2}$

$\frac{1 \cdot (1+x) - 1 \cdot x}{(1+x)^2} \rightarrow \frac{1 + x - x}{(1+x)^2} \rightarrow \frac{1}{(1+x)^2}$

c $\frac{x+2}{(x^2-1)}$ $\frac{x \cdot (x^2-1) - 2x(x+2)}{(x^2-1)^2}$

$\frac{x^3 - x - 2x^2 - 4x}{(x^2-1)^2}$

$\frac{x^3 - 2x^2 - 5x}{(x^2-1)^2}$

? why not right ??

Lecture 4

1F-lab Find derivitive $(\overbrace{x^2+2}^{u})^2$

method 1
U substitution

~~$u^2 \to 2u$~~

Well offical notation

$$\frac{d}{dx}u^2 = \overbrace{\frac{du}{dx}}^{\text{part 1}}\overbrace{\frac{d}{du}u^2}^{\text{part 2}} = 2x \cdot 2u =$$

u restore + distribute

$$4x(x^2+2)$$
$$4x^3+8x$$

method 2 - distribute

$$\frac{d}{dx}(x^2+2)^2 = \frac{d}{dx}(x^4+4x^2+4)$$
$$4x^3+8x$$

b. $(x^2+2)^{100}$ u sub is easier w/ large #
distributing is simpler

$$\frac{d}{dx}u^{100} = \frac{du}{dx}\frac{d}{du}u^{100} = 2x \cdot 100u^{99}$$

$2x \cdot 100(x^2+2)^{99}$ ← offical ans
$200x^{198}$ + really large #

1J-1e derivitive $\frac{\sin x}{x}$ $\lim_{h\to 0} \frac{\frac{\sin(x+h)}{(x+h)} - \frac{\sin x}{x}}{h}$

$\lim_{h\to 0} \frac{\frac{\sin(x+h)}{(x+h)}}{h} - \lim_{h\to 0} \frac{\frac{\sin x}{x}}{h} \to \frac{1}{h}$

C_1

$\frac{\cdot x}{\cdot x} \frac{\sin(x) + \sin(h)}{x + h^2}$

$\frac{x\cos x - \sin x}{x^2}$ ← ? how

2. Calculate $\lim_{x\to\frac{\pi}{2}} \frac{\cos x -}{x - \frac{\pi}{2}}$ by relating it to value of $(\cos x)'$ $= -\sin x$

relate to (cos x

$= -$

$\cos\frac{\pi}{2} = 0$ $\lim_{x\to\frac{\pi}{2}} \frac{\cos x - \cos\frac{\pi}{2}}{x - \frac{\pi}{2}} = \frac{d}{dx} \cos x \Big|_{x=\pi/2}$

↑ chain rule

$-\sin x \Big|_{x=\frac{\pi}{2}} = -1$

↑

$(\cos x)'$

2. Find deriv $x^{10}(x^2+1)^{10}$

$$x^{10} \cdot 10(x^2+1)^9 \cdot 2x$$

$$\cancel{x^{10} \cdot 10x^{18} \cdot 10 \cdot 2x}$$

$$10(3x^2+1)\, x^9 (x^2+1)^9$$ ← what's going on here?

6. Show derivitive even function is odd
derivite odd function is even

or use z

$$f(x) = f(-x)$$
differentiate

$$f'(x) = f(-z)\frac{dz}{dx} = f'(z)$$
because $\frac{dz}{dx} = -1$

$$g(x) = -g(x)$$

$$g'(x) = -g'(z)\frac{dz}{dx} = g'(z)$$
because $\frac{dz}{dx} = -1$

? I don't get how this proves

7b. Evaluate derivitive $\frac{dm}{dv}$ $\frac{m_0}{\sqrt{1-v^2/c^2}}$

$$\frac{m_0}{(1-v^2/c^2)^{1/2}} \rightarrow m_0(1-v^2/c^2)^{-1/2}$$

chain rule

$$m_0 \cdot -\frac{1}{2}(1-v^2/c^2)^{-3/2} \cdot \frac{-2v}{c^2}$$

$$\frac{m_0 v}{c^2(1-v^2/c^2)^{3/2}}$$

7c $\quad \frac{dF}{dr} = \frac{mg}{(1+r^2)^{3/2}} \quad \frac{dF}{dr}$

$mg(1+r^2)^{-3/2}$

$mg \cdot -\frac{3}{2}(1+r^2)^{-5/2} \cdot 2r$

$\frac{-3mgr}{(1+r^2)^{5/2}}$ ✓ Got it

1J-1a Derivitive $\sin(5x^2)$

$\cos(5x^2) \cdot 10x$

$10x\cos(5x^2)$ ✓ Got it

k $\tan^2(3x)$ ← can't do

$\cancel{\sec^2(3x) \cdot 3}$

$\cancel{3\sec^2(3x)}$ 2 ? what here

? $\cancel{6\tan(3x)\sec^2(3x)}$

$\cancel{\frac{6\sin x}{\cos^3 x}}$

do tan seperitly

or $2\tan(3x) \cdot (\tan 3x)'$

$2\tan(3x)(\sec 3x)^2 3x'$

$2\tan(3x)(\sec 3x)^2 \cdot 3$

$6\tan(3x)\sec^2(3x)$

$\frac{6\sin x}{\cos^3 x}$

m $\cos(2x)$ verify that have same deriv

$\cos^2 x - \sin^2 x$

$2\cos^2 x$

-use chain rule + trig doble angle rules

$(\cos^2 x - \sin^2 x)' = 2\cos x \sin x - 2\sin x \cos x = -4\cos x \sin x$

$(2\cos^2 x)' = 2 \cdot 2 \cdot \cos x(\cos x)' = -4\cos x \sin x$

$(\cos(2x))' = -2\sin(2x) = -2(2\sin x\cos x) = -4\cos x \sin x$

Find y''

16-1b $\frac{x}{x+5}$ → quotient rule

$$\frac{1 \cdot (x+5) - 1(x)}{(x+5)^2}$$

$x + 5 - x$

$$\frac{5}{(x+5)^2}$$

again

$$\frac{0 \cdot (x+5)^2 - 2\cancel{(x+5)} \cdot 1 \cdot 5}{(x+5)^{\cancel{4}\,3}}$$

$$\frac{-10}{(x+5)^3}$$

c $\frac{x-5}{x+5}$

$$\frac{1 \cdot (x+5) - 1 \cdot -5}{(x+5)^2}$$

$$\frac{-5}{(x+5)^2}$$

$$\frac{0 \cdot (x+5)^2 - 2(x+5) \cdot 1 \cdot -5}{(x+5)^4}$$

$$\frac{-2\cancel{(x+5)} \cdot -5}{\cancel{(x+5)}^{\cancel{4}}}$$

$$\frac{-10}{(x+5)^3}$$

16-5a $y = u(x)\,v(x)$

Find y'' and y'''

Use Liveets formula

Product rule

4-182

$y'\ u'v + v'u$

$y'' = u''v + u'v' + u'v' + u'v''$

$= u''v + 2u'v' + uv''$

$y''' = u'''v + u''v' + 2u''v' + 2u'v'' + u'v'' + uv'''$

group terms

$u'''v + 3u''v + 3u'v'' + uv'''$

↙ Leibniz' formula

Use it to check your answer

$y^{(p+q)},\ f\ y = x^p(1+n)^{ox}$

$y^{(p+q)} = u^{(p+q)}v + \binom{p+q}{1}u^{(p+q)}v^{(1)}$

$+ \binom{p+q}{2}u^{(p+q-2)}v^{(2)} + \ldots + u_1{(p+q)}$

$\binom{n}{k} = \frac{n!}{k!(n-k)!}$

where $m! = m(m-1)$

induction

prove

$p=1$

so $p \neq 1$

is true

whats $u = x^p$

$v = (1+x)^q$

each time you go down a degree

only 1 term survives

terms close to the end will $= 0$

8/11

Part II (43 points)

Directions and Rules: Collaboration on problem sets is encouraged, **but**

a) Attempt each part of each problem yourself. Read each portion of the problem before asking for help. If you don't understand what is being asked, ask for help interpreting the problem and then make an honest attempt to solve it.

b) Write up each problem independently. On both Part I and II exercises you are expected to write the answer in your own words.

c) **Write on your problem set whom you consulted and the sources you used.** If you fail to do so, you may be charged with plagiarism and subject to serious penalties.

d) Do not consult materials from previous semesters.

0. (3 points) Write the names of all the people you consulted or with whom you collaborated and the resources you used, or say "none" or "no consultation". This includes visits outside recitation to your recitation instructor. If you don't know a name, you must nevertheless identify the person, as in, "tutor in Room 2-106," or "the student next to me in recitation." Optional: note which of these people or resources, if any, were particularly helpful to you.

This "Problem 0" will be assigned with every problem set. Its purpose is to make sure that you acknowledge (to yourself as well as others) what kind of help you require and to encourage you to pay attention to how you learn best (with a tutor, in a group, alone). It will also greatly help us to know what resources you find useful.

1. (Wed, 3 pts) Express $(x-1)/(x+1)$ as the sum of an even and an odd function. (Simplify as much as possible.)

2. (Thurs, 4 pts) a) Use the following table of approximate square roots to give an approximate value of $\sqrt{102}$. You should begin by finding an approximate answer for the tangent line to $\sqrt{x}$ at $x = 100$, and using your answer to compute an approximate value of $\sqrt{102}$.

x	$\sqrt{x}$
100	10
101	10.049875
100.1	10.004998
100.01	10.000499

b) Use your tangent line from part (a) to give an estimate for $\sqrt{400}$. Is your approximation larger or smaller than the correct answer? Draw a picture to illustrate your answer. (Later, we'll use calculus to give a precise method for such approximations and associated error estimates.)

3. (Thurs, 4 pts) A 15 ft. tall street lamp is placed at the very top of a hill. Suppose the 200 ft. tall hill has an outline shaped like a parabola, and so approximated by the equation $y = 200 - x^2$. What is the lowest height on the hill y at which you can read a book on a cloudy night?

4 (Thurs, 6 pts) 3.1/21 (in Simmons, on parabolic mirrors)

5. (Thurs, 4 pts: 2 + 2)

a) A water cooler is leaking so that its volume at time t in minutes is $(10-t)^2/5$ liters. Find the average rate at which water drains during the first 5 minutes.

b) At what rate is the water flowing out 5 minutes after the tank begins to drain?

6. (Thurs, 5 pts) A prospective student sits in on the last 5 minutes of Thursday's lecture, and sees on the board that the slope of the tangent line can be defined by

$$\lim_{h \to 0} \frac{f(x+h)-f(x)}{h},$$

but doesn't understand what this formula, or its parts, signify. Write a paragraph (or two) explaining the parts of this formula to this student.

7. (Friday, 8 pts: 1 + 1 + 1 + 1 + 1 + 1 + 2) Simmons 2.5/19d (put $u = 1/x$), 19f, 19g, 20c, 20g (show work); 22a (needs a calculator), 22b (see the proof on page 73).

8. (Tuesday, 6 pts: 2 + 2 + 2)

a) If u, v and w are differentiable functions, find the formula for the derivative of their product, $D(uvw)$.

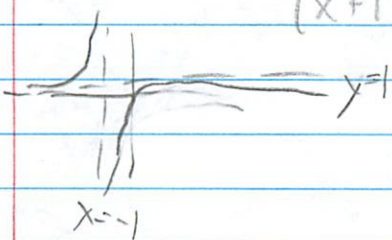
b) Generalize your work in part (a) by guessing the formula for $D(u_1 u_2 \cdots u_n)$—the derivative of the product of n differentiable functions.

c) Then prove your formula by mathematical induction (i.e., prove its truth for the product of $n+1$ functions, assuming its truth for the product of n functions).

Part 2

0. ~~No one as of yet~~ See side bar
~~Perhaps ask at end~~

1. Express $\frac{(x-1)}{(x+1)}$ as sum of even and odd function



- any function can be writen as sum of even + odd

Google → wikipedia

$$f_{even} = \frac{\frac{(x-1)}{(x+1)} + \frac{(-x-1)}{(-x+1)}}{2} \rightarrow$$

a problem in 1a goes through this

$$f_{odd} = \frac{\frac{(x-1)}{(x+1)} - \frac{(-x-1)}{(-x+1)}}{2}$$

$$\frac{1}{2}\left(\frac{x-1}{x+1} + \frac{-(x+1)}{(-x+1)}\right) \qquad \frac{1}{2}\left(\frac{x-1}{x+1} + \frac{x+1}{x-1}\right) \;\checkmark$$

common denom. doing what

; simply further

$$\frac{(x-1)(-x+1) + (-x-1)(x+1)}{2(x+1)(1-x)} + \frac{(x-1)(-x+1) + (x+1)+(x+1)}{2(x+1)(1-x)}$$

1.5

$$\frac{(x-1)^2+(1+x)^2}{2(x+1)(x-1)} + \frac{(x-1)^2-(x+1)^2}{2(x+1)(x-1)} = \frac{2x^2-2x+2x+2}{2(x^2-1)}$$

can break up

$$\rightarrow \frac{x^2+1}{x^2-1} + \frac{-2x}{x^2-1} = \boxed{\frac{x^2+1}{x^2-1} + \frac{2x}{1-x^2}}$$

2. Approx value of $\sqrt{102}$

Find tangent line at $\sqrt{x}$ at 100

$\lim_{x \to 100} \sqrt{x} = 10$

~~$\sqrt{101} = 10.049875$~~
~~$\sqrt{102} \approx 10.099750$~~

~~? not sure what it is asking~~

find tangent line

$\frac{10.049 - 10}{101 - 100} = \frac{.049}{1}$

tangent = .05

~~y = .05x + 5~~

pt slope form $y - 10 = .05(x - 100)$

$y = .05(102) + 5$

10.1 ✓

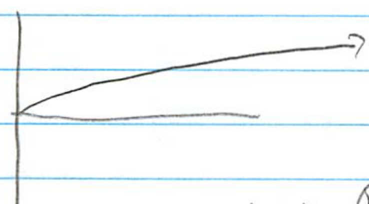
~~? how get algebraically~~

$y - b = m(x - a)$

find y-intercept

Christine Bremer office Hrs

b) $\sqrt{400}$ gets closer + closer to what?



height of tangent at a is close to height of graph at a

$y = .05(400) + 5$

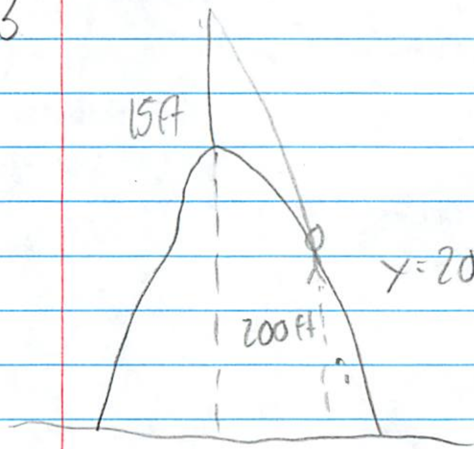
25 ✓

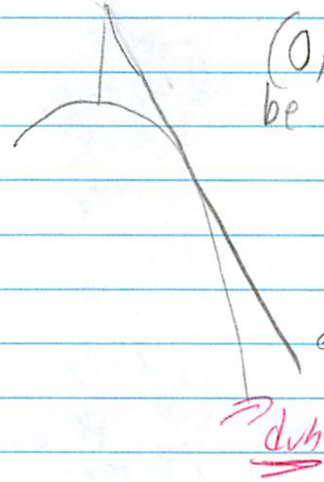
actual $\sqrt{400} = 20$ larger 4

/4

3

Office Hours Bliner





~~?? no clue algebraically~~

slope $= -2x$

pt slope form

$$215 - y = -2x(0 - x)$$

or $y - 215 = -2x_0(x)$ ← a line need x_0

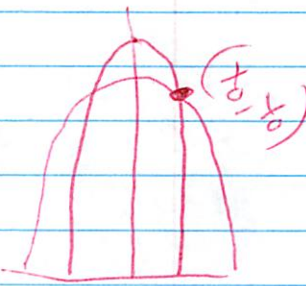
$$x = -2x^2 + 215$$

↑ for a certain x_0 value

get 2 lines

* interception point *

for every x value different line



↓ set = to

$$-2x^2 + 215 = 200 - x^2$$
$$+2x^2 \quad -200 \quad -200 \quad +2x^2$$

$$15 = x^2$$
$$x = \sqrt{15}$$

$$y = 200 - (\sqrt{15})^2$$
$$y = 185$$

$$(\sqrt{15}, 185)$$

$$\boxed{185 \text{ ft}}$$ ✓

now can find right tangent

$$y - 215 = -2\sqrt{15}(x)$$

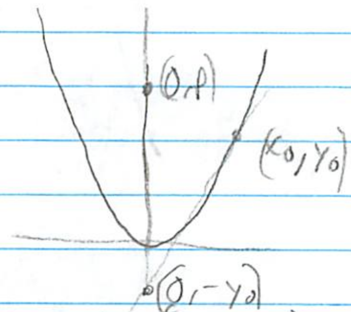
= 4

4. Simmons 3.1 #21

p = constant

$x^2 = 4py$ ← parabola

4-182 office hrs



a. tangent at (x_0, y_0) has y-intercept $-y_0$

Find equation into line $x=0$

$$\frac{f(x_0+h) - f(x_0)}{h} \quad \text{or} \quad 2x = 4py'$$

$$\frac{2x}{4p} = \frac{x}{2p} = y'$$

← $\boxed{\text{know how}}$

point slope form

$$y - y_0 = \frac{x_0}{2p}(x - x_0)$$

$$y - y_0 = \frac{x_0}{2p}(-x_0)$$

$$y = -\frac{x_0^2}{2p} + y_0$$

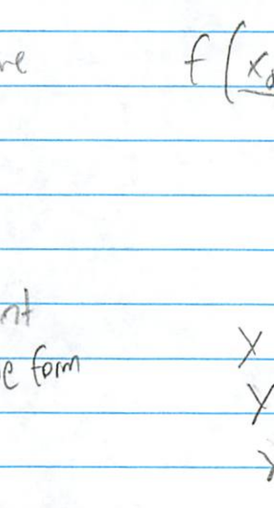
plug into original

$$y_0 = \frac{x_0^2}{2p} = -\frac{x_0^2}{2p} + y_0 \quad \leftarrow \text{is } -2y_0$$

$$= -2y_0 + y_0 = -y_0 \checkmark$$

b. Show triangle (x_0, y_0) $(0, -y_0)$ $(0, p)$ isoceles   use distance formula

y-haight



$$\text{dist}\left((0-y_0), (0-p)\right) \cong (0,p)\,(x_0, y_0)$$

$$d = \sqrt{(x_0 - 0)^2 + (y_0 - p)^2} = \sqrt{(0-0)^2 + (p - -y_0)^2}$$

$$\sqrt{x_0^2 + (y_0 - p)^2} = \sqrt{(p + y_0)^2}$$

$$y_0 + p = \sqrt{x_0^2 + p^2 - 2py_0 + y_0^2}$$

$$(y_0 + p)^2 = x_0^2 + p^2 - 2py_0 + y_0^2$$

$$y_0^2 + 2py_0 + p^2 = 4py_0 + p^2 - 2py_0 + y_0^2 \checkmark$$

↑ sub from def

$$y_0^2 + 2py_0 + p^2 = 2py_0 + y_0^2 + p^2$$

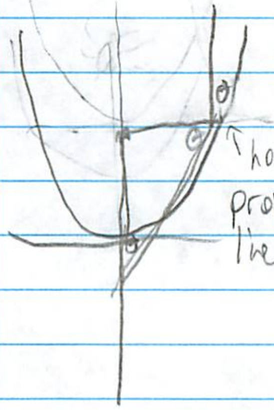
= 4

Show is = to

need to know setup

4-182
office hrs

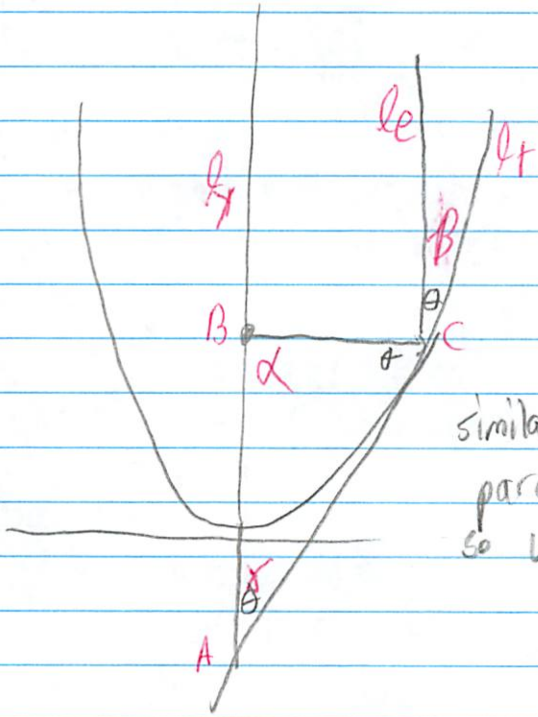
C



angle of incidence = angle of reflection

how to prove line is vertical

parallel lines
so vertical



similar angles
parallel to axis
so vertical

explain more = 1

$\angle A = \gamma \quad \angle BCA = \alpha$

angle between reflected ray + tangent line $= \beta$

① $\alpha = \beta$ angle of incidence = angle reflection

② $\alpha = \gamma$ since ABC is isocoeles and angles at base

therefore $\gamma = \beta$ - so reflected ray parallel to tangent line ie is vertical

1

5. a. Water Cooler leaking

$$V = (10-t)^2/5$$

Find avg rate drain in 5 min

$$\frac{(10-5)^2/5 - (10-0)^2/5}{5-0}$$

$$\frac{5-20}{5} = \frac{-15}{5} = -3 \text{ liters/minute}$$

b. Rate at $t=5$

$$\frac{100-20t+t^2}{5}$$

$$\frac{1}{5}t^2 - 4t + 20$$

$$\frac{1}{5}\cdot 2t - 4$$

$$\frac{2}{5}t - 4$$

$$\frac{2}{5}\cdot 5 - 4$$

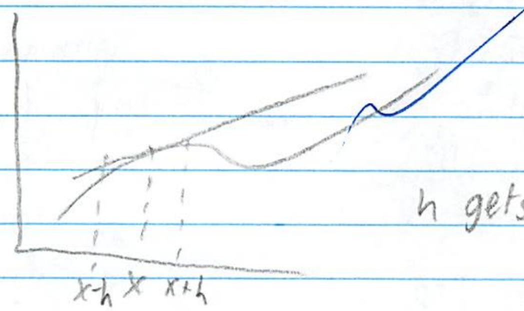
$$-2 \text{ liters/minute}$$

4

Class notes
last year's notes
wikipedia

6. This formula is the definition of differentiation.
It is a dynamic process to find the best local approximation of a tangent line to a curve.
So, you compute sucessive closer secent line approximations until you find the slope of the tangent line. You want h to get smaller and smaller, so that the space between the points on the secont line is smaller and smaller and closer to a tangent line (touches only at 1 point)

Understand much better
Great question



h gets smaller & smaller.

You find the slope between x+h and h. You do this by subtracting f(x) for the y value at point x from f(x+h). In order to account for the difference h provides, you divide by it.

5

Simmons 2.5

7. 19d. Evaluate limit

Rules

$\lim_{x \to 0} \frac{\sin x}{x} = 1$

$\lim_{x \to 0} \frac{1-\cos x}{x} = 0$

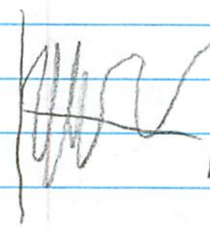
can just assume these

$\lim_{x \to \infty} x \sin \frac{1}{x}$ - if don't know how to do fast write it out

$\lim_{h \to 0} \frac{(x+h) \sin \frac{1}{x+h} - (x) \sin \frac{1}{x}}{h}$

$\frac{x \sin \frac{1}{x+h} + h \sin \frac{1}{x+h} - x \sin \frac{1}{x}}{h}$

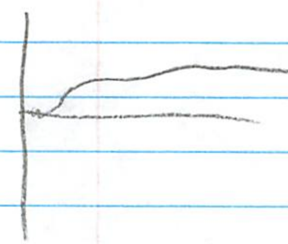
Or graph / see visually



or $\sin\left(\frac{1}{x}\right)\left(\frac{1}{x}\right)$

why does it say put $v = \frac{1}{x}$

calc says undefined



Use u-subsitution

let $u = \frac{1}{x}$

make it look like a rule
put x in terms of u

$\lim_{u \to 0^+} \frac{\sin u}{u} = 1$

$1 = \frac{1}{x}$

$x = 1$

office hrs
4-182

7. IIf $\lim_{x\to 0} \frac{\sin^2 x}{3x^2} \rightarrow \cancel{\frac{\cos^2 x - 1}{3x^2}} \; \cancel{\frac{\cos^2 x}{3x^2} - \frac{-1}{3x^2}}$

0

$\cancel{\frac{\sin^2 x}{x^2} \cdot \frac{1}{3} = \left(\frac{\sin x}{x^2}\right)^2 \cdot \frac{1}{3} = 1 \cdot \frac{1}{3}}$

~~inteterminate form~~

~~l'hopitals rule~~

$\cancel{\frac{\sin x \, \sin x}{x^2} \cdot \frac{1}{3}}$

$\cancel{\frac{2 \sin x \cos x}{2x} \cdot \frac{1}{3}} \quad \cancel{1 \cdot \cos x \cdot \frac{1}{3}}$

$\cancel{1 \cdot 0 \cdot \frac{1}{3}}$

$\cancel{0}$

$\lim_{x\to 0} \frac{\sin x \; \sin x}{3 \, x \; x} \rightarrow \lim_{x\to 0} \frac{1}{3} \cdot \lim_{x\to 0} \frac{\sin x}{x} \cdot \lim_{x\to 0} \frac{\sin x}{x}$

office hrs 4-182

$\frac{1}{3} \cdot 1 \cdot 1$

$\frac{1}{3}$ ✓

1

l'Hopital

7. 20g $\lim_{x \to 0} \frac{3x^2+4x}{\sin 2x} \to \frac{6x+4}{2\cos(2x)}$ $\frac{6x+4}{2 \cdot 1}$ $\frac{4}{2} = 2$ ✓

1 ⓥ

$\frac{360}{\pi}$

2π

20e $\lim_{x \to 0} \frac{3x + \sin x}{x} \to \lim \frac{3x}{x} + \lim \frac{\sin x}{x}$

not a problem

$3 + 1$

X 4 ⓥ Finally got it down

20c $\lim_{x \to 0} \frac{x^2}{1-\cos^2 x}$ $\frac{x^2}{\sin^2 x} = 1$

1

18g $\lim_{x \to 0} \frac{\sin 2x}{\sin 3x} \cdot \frac{2x}{2x}$ $\frac{2x \sin 2x}{2x \sin 3x}$ $\frac{2x \cdot 3x}{\sin 3x \, 3x}$ $\frac{2x}{3x} = \frac{2}{3}$ ✓

1

22a $\lim_{\theta \to 0} \frac{1-\cos\theta}{\theta^2}$

θ	y
.1	.4597
.1	.49958
.01	.5

$\lim = .5$ ✓

1

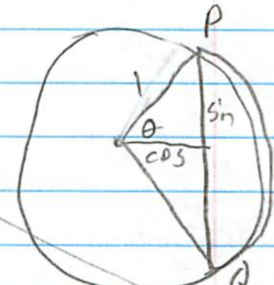
b. calculate → no $-\infty$ limit does not exist

23b $\frac{\sin\theta}{\theta} = 1$ $\sin\theta \cong \theta$ for small θ

wrong problem

1. $\sin(.1) = .099833$
$\sin(.01) = .0099998$
$\sin(.001) = .001$

$\frac{\text{chord } PQ}{\text{arc } PQ} \to 1$ as $\widehat{PQ} \to 0$



$\frac{1}{3}$

22b $\lim_{\theta \to 0} \frac{1-\cos\theta}{\theta^2} = \cancel{\lim_{\theta \to 0} \frac{1-\cos\theta}{\theta} \cdot \frac{1}{\theta}}$ ← ~~bit unhelpful~~

0

4-182 $\frac{\times(1+\cos\theta)}{\times(1+\cos\theta)}$ ✓ $\frac{1-\cos^2\theta}{\theta^2(1+\cos\theta)} = \lim_{\theta \to 0} \frac{\sin^2\theta}{\theta^2(1+\cos\theta)}$

$$\left(\lim_{\theta \to 0} \frac{1}{1+\cos\theta}\right)\left(\lim_{\theta \to 0} \frac{\sin\theta}{\theta}\right)^2$$

↑ direct substitution ↑

$$\frac{1}{2} \cdot 1^2$$

$$\frac{1}{2}$$ ✓

2✓

8. Find product rule for 3 functions u, v, w

take first 2 multiply by deriv of third

$$uv\frac{dw}{dx} + uw\frac{dv}{dx} + vw\frac{du}{dx}$$ ✓ 2

b. Generalize

Answer $u_1' u_2 \ldots u_n +$ $u_1 u_2' \ldots u_n + \ldots +$ $u_1 u_2 \ldots u_n'$

✓ did not put in values for n

$$u_{(n-1)} \cdot \frac{du_n}{dx} + u_{(n-1)}\frac{du_n}{dx} + \ldots n$$

0.5

wikipedia

$$\frac{d}{dx}\left[\prod_{i=1}^{k} f_i(x) = \sum_{i=1}^{k} \frac{d}{dx} f_i(x) \prod_{j \neq i} f_j(x)\right]$$

? don't get and would have never discovered

$$\sum_{i=1}^{n} u_i' \frac{u_1 u_2 \ldots u_n}{u_n}$$

c.

$$\frac{d}{dx} x^{n+1} = \frac{d}{dx}(x^n \cdot x)$$

$$= x\frac{d}{dx} x^n + x^n \frac{d}{dx} x$$ product rule

?? wikipedia

$$x(nx^{n-1}) + x^n \cdot 1$$

next page

$$(n+1)x^n$$

1

1 / 3.5

? I don't get

- first time I saw mathematical induction - need a model to follow

Concepts seem simpliar now
that I've done it + reviewed it

c) Prove mathmatical induction

base case: already did for $n = 2, 3$

inductive case: assume true for $n = k-1$
prove for $n = k$

$$D(U_1, U_2 \dots U_k) = D((U_1, U_2 \dots U_{k-1}) \cdot U_k) =$$
$$= D(U_1, U_2 \dots U_{k-1}) U_k + U_1 U_2 \dots U_{k-1} D(U_k)$$
$$= \left(\sum_{i=1}^{k-1} U_i' \frac{U_1 \dots U_{k-1}}{U_i} \right) U_k + (U_1, U_2, \dots U_{k-1})(U_k')$$

for inductive hypothesis

$$= \left(\sum_{i=1}^{k-1} U_i' \frac{U_1 - U_{k-1}, U_k}{U_i} \right) + U_k' \frac{U_1 U_2 \dots U_{k-1} U_k}{U_k}$$

$$= \sum_{i=1}^{k} U_i' \frac{U_1 \dots U_k}{U_i}$$

18.01. Fall 2009. Problem Set 1

Part II

SOLUTIONS - 5 pages total ①

"out of 3 for prob 0"

Problem 0: Collaborators $\boxed{+3\text{pts}}$ (/3 for #0)

Problem 1: Express $\frac{x-1}{x+1}$ as the sum of an even and an odd function. Simplify.

[redacted] $f(x) = \frac{x-1}{x+1}$

Even/odd decomposition: $f(x) = \frac{f(x)+f(-x)}{2} + \frac{f(x)-f(-x)}{2} = \frac{\frac{x-1}{x+1}+\frac{-x-1}{-x+1}}{2} + \frac{\frac{x-1}{x+1}-\frac{-x-1}{-x+1}}{2} =$

$= \frac{(x-1)(-x+1)+(-x-1)(x+1)}{2(x+1)(1-x)} + \frac{(x-1)(-x+1)-(x+1)(x+1)}{2(x+1)(1-x)} = \frac{(x-1)^2+(x+1)^2}{2(x+1)(x-1)} + \frac{(x-1)^2-(x+1)^2}{2(x+1)(x-1)} =$

$= \frac{2x^2-\cancel{2x}+\cancel{2x}+2}{2(x^2-1)} + \frac{\cancel{x^2}-2x+\cancel{1}-\cancel{x^2}-2x-\cancel{1}}{2(x^2-1)} = \frac{x^2+1}{x^2-1} + \frac{-2x}{x^2-1} = \boxed{\frac{x^2+1}{x^2-1} + \frac{2x}{1-x^2}}$

$\boxed{+3\text{pts}}$ (/3 ~~out~~ for #1)

Problem 2 Use table of approx. square roots to give an approx. value of $\sqrt{102}$. Begin by finding an approx answer for the tangent line to $\sqrt{x}$ @ $x=100$.

	x	$\sqrt{x}$	$(f(x+h)-f(x))/h$
	100	10	
$h=1$	101	10.049875	0.049875
$h=0.1$	100.1	10.004998	0.04998
$h=0.01$	100.01	10.000499	0.0499

tend to 0.05, use 0.0499 as slope

This is a thinking-it-out type of problem (goal: understand what is going on w/ approximating tangent line by secant lines)

Begin by computing difference quotient for each h

(This step not actually necessary - could have just used $h=0.01$ to get best approximation to slope of tangent line straightaway)

(This way, sort of see that some sort of limit might get reached - not ~~???~~ rigorous)

$y = \cancel{???}\ 0.0499x + b$

$b = 10 - 0.0499 \cdot 100 =$
$= 10 - 4.99 = 5.01$ $\Longrightarrow$ $\boxed{y = 0.0499x + 5.01}$

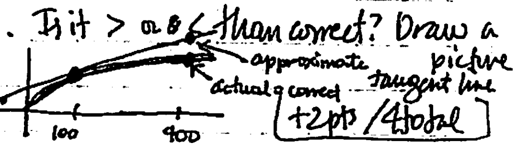
+2pts / 4 total

Using this $\sqrt{102} \approx 0.0499 \cdot 102 + 5.01 = 4.99 + 5.01 + 2 \cdot 0.0499 = \boxed{10.0998}$

(Note: could have also used secant lines w/ one of the other slopes to get better answer)

(b) Use the tangent line from part (a) to give estimate for $\sqrt{900}$. Is it > or < than correct? Draw a picture

$\sqrt{900} \approx 0.0499 \cdot 400 + 5.01 = 4.99 \cdot 4 + 5.01$
$= 10 + 4.99 \cdot 3 = 10 + 12 + 2.97 = \boxed{24.97 \text{ larger than act}}$

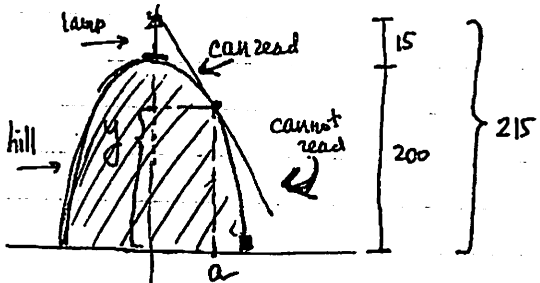


$\boxed{+2\text{pts} / 4\text{total}}$

(2)

Problem 3 15 ft tall lamp placed at top of parabolic hill, which is approx. by $y = 200 - x^2$.

What is lowest height y @ which can read book on a cloudy night?



$y = f(x) = 200 - x^2$.

Goal: find $x = a$ at which tangent line to $y = f(x)$ has y-intercept at $(0, 215)$.

Tangent line eq: $y = \underset{f'(a)}{m}x + \underset{\text{y-int.}}{b} \Rightarrow y = f'(a)x + 215.$

$f'(a) = -2x\big|_{x=a} = -2a.$ $\Rightarrow y = -2ax + 215$

Solve for a by plugging in point $(a, f(a)) = (a, 200 - a^2)$

$200 - a^2 = -2a(a) + 215$ $\quad a^2 = 215 - 200 = 15 \Rightarrow a = \sqrt{15} \Rightarrow y = f(a) = 200 - 15 = 185.$

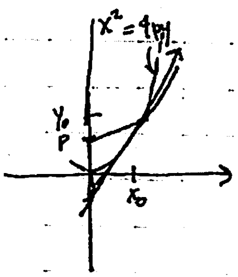
$\boxed{\text{Ans: } y = 185\text{ft}}$

$\boxed{+4\text{pts } /4}$

Problem 4 3.1/21 Simmons – parabolic mirrors.

Consider $x^2 = 4py$, (x_0, y_0) – point on parabola

a) Show that tangent at x_0, y_0 has y-int. $-y_0$.



Equation of tangent line at (x_0, y_0) is $y - y_0 = f'(x_0)(x - x_0)$, where $f(x) = \frac{x^2}{4p} = y$.

y-intercept is $y_0 - x_0 f'(x_0) = y_0 - 2\left(\frac{x_0}{4p}\right) \cdot x_0 =$

$= y_0 - 2\left(\frac{x_0^2}{4p}\right) = y_0 - 2y_0 = \boxed{-y_0}$ $\quad f'(x) = \frac{2x}{4p}$!

$\boxed{+2\text{pts } / 6 \text{ total}}$

$\Rightarrow$

Problem 4 (cont.)

$\boxed{b)}$ Show that $\Delta\big((x_0,y_0),\ (0,-y_0),\ (0,p)\big)$ is isosceles.

Use distance formula to ~~chute~~ compare edge lengths: (for square of distance/length)

$$\big(\text{Dist}\,((0,-y_0),\ (0,p))\big)^2 = (p+y_0)^2. = (*)$$

$$\big(\text{Dist}\,((x_0,y_0),\ (0,p))\big)^2 = (p-y_0)^2 + (-x_0)^2 = (p-y_0)^2 + x_0^2 =$$

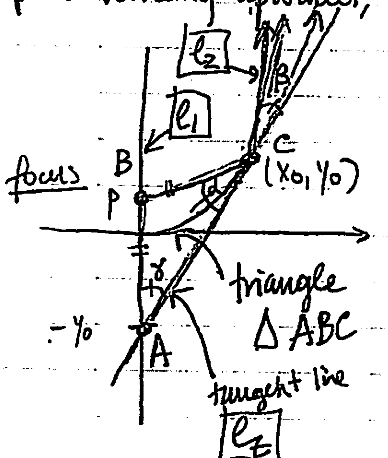
$$= p^2 - 2py_0 + y_0^2 + 4py_0 = p^2 + 2py_0 + y_0^2 = \underline{(p+y_0)^2} = (*)$$

Same

$\therefore$ Δ has two equal edges ~~or~~ (edges of same length) $\Rightarrow$ is ~~...~~ isosceles //

+2pts / 6 total

$\boxed{c)}$ Use part (b) to show that after reflection, ray from focus points vertically upwards, assuming angle of incidence = angle of reflection



Call the triangle from the previous part (redrawn at left) ΔABC. Label $\angle A - \gamma$, $\angle BCA - \alpha$ and the angle between the reflected ray and the tangent line $- \beta$ (see figure at right)

Label the line of the y-axis l_1 and the line of the reflected ray l_2.
Label the tangent line l_t

Know: ① $\underline{\alpha = \beta}$ (this is given – angle of incidence = angle of reflection)

② $\underline{\alpha = \gamma}$ (from part b), since ΔABC is isosceles and these are angles at base)

Therefore,
$\therefore$ $\underline{\gamma = \beta}$. But then, $\angle l_1, l_t = \angle l_2, l_t$, so l_1 is parallel to l_2

(the angle that reflected ray makes w/ tangent line = angle that the y-axis makes w/ tangent line, so reflected ray must ~~...~~ be parallel to tangent line – i.e. is vertical

+2pts / 6 total //

Problem 5

a) Leaking water cooler, volume at time t is $(10-t)^2/5$ liters.
Find average rate at which water drains during first 5 minutes

Let $V(t) = \dfrac{(10-t)^2}{5}$ Ave rate, $0 \le t \le 5$ $= \dfrac{V(5)-V(0)}{5} = \dfrac{\frac{(10-5)^2}{5} - \frac{10^2}{5}}{5}$

$$= \frac{5^2 - 10^2}{5^2} = \frac{\cancel{5^2} - 2^2\cancel{5^2}}{\cancel{5^2}} = 1 - 4 = -3 \quad (\text{lt}/\text{min})$$

Ans: -3 liters/min | +2pts / 4 total

b) At what rate is water flowing out 5 mins after tank begins to drain?

$$V'(t)\Big|_{t=5} = \frac{2(10-t)(-1)}{5}\Big|_{t=5} = \frac{-2(10-5)}{5} = -\frac{2(\cancel{5})}{\cancel{5}} = -2$$

Ans: -2 liters/min | +2pts / 4 total

Problem 6 Write a paragraph or two explaining parts of formula and entire formula

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

+ 5 pts / 5 total

< insert explanation of limit formula for derivative of a function here.
(Have explanation of infinitesimal h) >

Problem 7

19(d) $\lim_{x\to\infty} x \sin(1/x) = \lim_{x\to\infty} \dfrac{\sin 1/x}{1/x} = \lim_{(1/x)\to 0} \dfrac{\sin(1/x)}{(1/x)} = \boxed{1}$ +1 pt / 8 total

19(f) $\lim_{x\to 0} \dfrac{\sin^2 x}{3x^2} = \lim_{x\to 0} \dfrac{1}{3}\cdot\dfrac{\sin x}{x}\cdot\dfrac{\sin x}{x} = \dfrac{1}{3}\cdot 1\cdot 1 = \boxed{\tfrac{1}{3}}$ +1 pt / 8 total

19(g) $\lim_{x\to 0} \dfrac{\sin 2x}{\sin 3x} = \lim_{x\to 0} \dfrac{2}{3}\cdot\dfrac{\sin 2x}{2x}\cdot\dfrac{3x}{\sin 3x} = \dfrac{2}{3}\cdot 1\cdot 1 = \boxed{\tfrac{2}{3}}$ +1 pt / 8 total

20(c) $\lim_{x\to 0} \dfrac{x^2}{1-\cos^2 x} = \lim_{x\to 0} \dfrac{x^2}{\sin^2 x} = \lim_{x\to 0} \dfrac{x}{\sin x}\cdot\dfrac{x}{\sin x} = 1\cdot 1 = \boxed{1}$ +1 pt / 8 total

20(g) $\lim_{x\to 0} \dfrac{3x^2+4x}{\sin 2x} = \lim_{x\to 0} \dfrac{2x}{\sin 2x}(1.5x+2) = 1(0+2) = \boxed{2}$ +1 pt / 8 total

22(a) Let $f(\theta) = \dfrac{1-\cos\theta}{\theta^2}$ Eval @ some small pos. values of θ: $f(0.1) = 0.499583472$ $f(0.01) = 0.499995833$ +1 pt / 8 total

22(b) $\lim_{\theta\to 0} \dfrac{1-\cos\theta}{\theta^2} = \lim_{\theta\to 0} \dfrac{(1-\cos\theta)(1+\cos\theta)}{\theta^2(1+\cos\theta)} = \lim_{\theta\to 0} \dfrac{1-\cos^2\theta}{\theta^2(1+\cos\theta)} = \lim_{\theta\to 0} \dfrac{\sin^2\theta}{\theta^2(1+\cos\theta)} = 1\cdot 1\cdot\dfrac{1}{1+1} = \boxed{\tfrac{1}{2}}$ +2 pt / 8 total

Problem 8

a) If u, v, w are differentiable functions, find the formula for the derivative of their product, $D(uvw)$

Use the product rule twice. Recall $D'(uv) = u'v + uv'$

$$D'(uvw) = D'(uv)w + uv\,\underbrace{D(w)}_{w'} = (u'v + uv')w + uv(w') = \boxed{u'vw + uv'w + uvw'}$$

$\boxed{+2\text{pts} \,/\, 6 \text{ total}}$

b) Generalize part (a) by guessing formula for $D(u_1 \ldots u_n)$

$$D(u_1 u_2 \cdots u_n) = u_1' u_2 \cdots u_n + u_1 u_2' \cdots u_n + \ldots + u_1 u_2 \cdots u_n'$$

$$= \sum_{i=1}^{n} u_i' \frac{u_1 u_2 \cdots u_n}{u_i}$$

explicit closed form formula ... nicer than above.

both answers ok.

c) Prove by mathematical induction

Base case: already did for $n = 2, 3$.

Inductive case: assume true for $n = k-1$ functions. Prove for $n = k$.

$$D(u_1 u_2 \cdots u_k) = D\left((u_1 u_2 \cdots u_{k-1}) \cdot u_k\right) = D(u_1 u_2 \cdots u_{k-1})\, u_k + u_1 u_2 \cdots u_{k-1}\, D(u_k)$$

$$= \left(\sum_{i=1}^{k-1} u_i' \frac{u_1 \cdots u_{k-1}}{u_i}\right) u_k + (u_1 u_2 \cdots u_{k-1})(u_k') =$$

from inductive hypothesis

$$= \left(\sum_{i=1}^{k-1} u_i' \frac{u_1 \cdots u_{k-1} u_k}{u_i}\right) + u_k' \frac{u_1 u_2 \cdots u_{k-1} u_k}{u_k}$$

$$= \sum_{i=1}^{k} u_i' \frac{u_1 \cdots u_k}{u_i} \quad //$$

$\boxed{+2\text{pts} \,/\, 6 \text{ total}}$

Lecture
Examples Chain

9/18

Exam 1 next Fri 2-3 pm walker gym
↑walker memorial

Pset 2a posted today

today examples

study the proofs - very important - know them

Chain rule

$$\frac{d}{dx}(x^2+3x+7)^5 \qquad \leftarrow u^5 \to 5u^4$$

$$5(x^2+3x+7)^4 \times (2x+3)$$

↑ now $\frac{d}{dx}$ inside

Multiply

- could combine several rules

$$\frac{d}{dx}(x^2 \sin 5x)$$

apply product rule

- need to understand order opps

↙composition of functions - use chain rule
↑take deriv inside

$$2x \sin 5x + \sin(5x)' x^2$$

$$\boxed{2x\sin 5x + 5\cos(5x)\, x^2}$$

Chain rule proof p 95

No reason to stop at 1 deriv

$$f(x) \to f'(x) \to f''(x) \to f'''(x)$$

2nd derivitive

$f^{(n)}(x) \to$ take n derivitives

$$\frac{d^n}{dx^n}(f(n))$$

Implicit differentation

- no harder
- different letters

$$\frac{d}{dx}\left(x^{m/n}\right) \quad \leftarrow \text{rational \#}$$

know $\frac{d}{dx}(x^n) = n x^{n-1}$ ← pascal's triangle

or quotient rule $x^{-n} = -n x^{-n-1}$

write it as $y = x^{m/n}$

$y(x) = x^{m/n}$ ← do both sides of equality

$y(x)^n = x^m$ ← raise both sides to nth power

now take derivitive w/ respect to x of both sides

$$\frac{d}{dx}\left(y(x)^n\right) = m x^{m-1}$$

↑ chain rule

$$n\, y(x)^{n-1} \cdot y'(x) = m x^{m-1}$$

can now solve equation for $y'(x)$

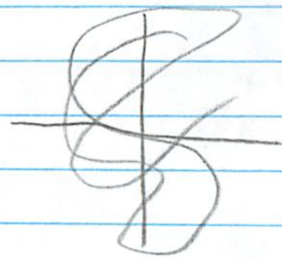
$$y'(x) = \frac{m x^{m-1}}{n\, y(x)^{n-1}}$$

Now need final ans in terms of x

- can sub in original y

$$\frac{m x^{m-1}}{n\underbrace{\left(x^{m/n}\right)^{n-1}}_{x^{m-\frac{m}{n}}}} = \frac{m}{n} x^{m/n} - 1$$

Example 2 $\quad f(x, y) = 0$



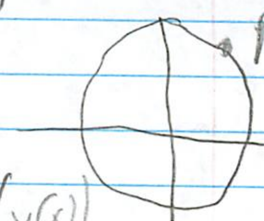
← calculus only cares about local properties
work with a section at a time

- Use implicit differentiation

lame example

$$x^2 + y^2 - 1 = 0$$

- unit circle
- not a function



think of y as a function of x $(y(x))$

$2x + 2y(x) \cdot y'(x) = 0$ take deriv of both sides
consider y as a function of x

$$y'(x) = \frac{-x}{y}$$

$p(x_0, y_0) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

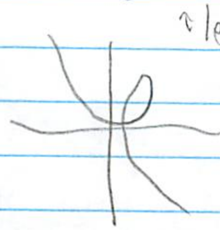
↑ don't need to sub back in for y

lame - could also solve for y

$$y = \begin{cases} \sqrt{1-x^2} & \text{upper hemi} \\ -\sqrt{1-x^2} & \text{lower hemi} \end{cases}$$

less lame example

Folium of descartes $\quad x^3 + y^3 = 6xy$
↑ leaf



implicit differentation easy
solving for y hard

Inverse Functions (derivitves of)

$x \to \boxed{f} \to f(x) = y$

$x = f^{-1}(y)$ $\boxed{\text{inverse}} \leftarrow y$

$f(f^{-1}(x)) = x = f^{-1}(f(x))$

Strategy
- start with $f(x)$
- apply inverse $f^{-1}(y) = x$

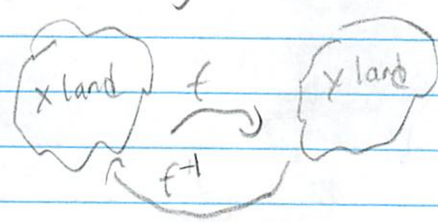
Plan: Given $y = f(x)$, solve for x

Example $y = \sqrt{x}$ Domain $x \geq 0$
Range $y \geq 0$

inverse ↓

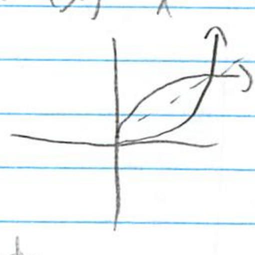
$y^2 = x$ ← $y \geq 0$ Domain
resulting x always ≥ 0

Obsessed writing x as independent



writing as a function like on calc $\boxed{\text{why?}}$

$f'(x) = x^2$ if $f(x) = \sqrt{x}$



reflections across $y = x$

Symmetry property exists
- reversed roles x and y

Use implicit diff

$$y^2(x) = x$$
$$2\,y(x)\,y'(x) = 1$$ take deriv both side
$$y'(x) = \frac{1}{2} \cdot \frac{1}{y(x)}$$ solve for y'

$$\boxed{\frac{d}{dx} x^{1/2} = \frac{1}{2} x^{-1/2}}$$

$$y'(x) = \frac{1}{2\sqrt{x}}$$

Trig functions + inverses

- appear in real life
- next week

Lecture 5
Implicit Differentiation and Inverses

Implicit Differentiation

Example 1. $\frac{d}{dx}(x^a) = ax^{a-1}$.
We proved this by an explicit computation for $a = 0, 1, 2, \ldots$. From this, we also got the formula for $a = -1, -2, \ldots$. Let us try to extend this formula to cover rational numbers, as well:

$$a = \frac{m}{n}; \quad y = x^{\frac{m}{n}} \quad \text{where } m \text{ and } n \text{ are integers.}$$

We want to compute $\frac{dy}{dx}$. We can say $y^n = x^m$ so $ny^{n-1}\frac{dy}{dx} = mx^{m-1}$. Solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{m}{n}\frac{x^{m-1}}{y^{n-1}}$$

We know that $y = x^{\left(\frac{m}{n}\right)}$ is a function of x.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{m}{n}\left(\frac{x^{m-1}}{y^{n-1}}\right) \\
&= \frac{m}{n}\left(\frac{x^{m-1}}{(x^{m/n})^{n-1}}\right) \\
&= \frac{m}{n}\frac{x^{m-1}}{x^{m(n-1)/n}} \\
&= \frac{m}{n}x^{(m-1)-\frac{m(n-1)}{n}} \\
&= \frac{m}{n}x^{\frac{n(m-1)-m(n-1)}{n}} \\
&= \frac{m}{n}x^{\frac{nm-n-nm+m}{n}} \\
&= \frac{m}{n}x^{\frac{m}{n}-\frac{n}{n}} \\
\text{So, } \frac{dy}{dx} &= \frac{m}{n}x^{\frac{m}{n}-1}
\end{aligned}$$

This is the same answer as we were hoping to get!

Example 2. Equation of a circle with a radius of 1: $x^2+y^2 = 1$ which we can write as $y^2 = 1-x^2$. So $y = \pm\sqrt{1-x^2}$. Let us look at the positive case:

$$\begin{aligned}
y &= +\sqrt{1-x^2} = (1-x^2)^{\frac{1}{2}} \\
\frac{dy}{dx} &= \left(\frac{1}{2}\right)(1-x^2)^{\frac{-1}{2}}(-2x) = \frac{-x}{\sqrt{1-x^2}} = \frac{-x}{y}
\end{aligned}$$

Now, let's do the same thing, using *implicit* differentiation.

$$\begin{aligned} x^2+y^2 &= 1 \\ \frac{d}{dx}\left(x^2+y^2\right) &= \frac{d}{dx}(1) = 0 \\ \frac{d}{dx}(x^2)+\frac{d}{dx}(y^2) &= 0 \end{aligned}$$

Applying chain rule in the second term,

$$\begin{aligned} 2x+2y\frac{dy}{dx} &= 0 \\ 2y\frac{dy}{dx} &= -2x \\ \frac{dy}{dx} &= \frac{-x}{y} \end{aligned}$$

Same answer!

Example 3. $y^3+xy^2+1=0$. In this case, it's not easy to solve for y as a function of x. Instead, we use implicit differentiation to find $\frac{dy}{dx}$.

$$3y^2\frac{dy}{dx}+y^2+2xy\frac{dy}{dx} = 0$$

We can now solve for $\frac{dy}{dx}$ in terms of y and x.

$$\begin{aligned} \frac{dy}{dx}(3y^2+2xy) &= -y^2 \\ \frac{dy}{dx} &= \frac{-y^2}{3y^2+2xy} \end{aligned}$$

Inverse Functions

If $y=f(x)$ and $g(y)=x$, we call g the *inverse function* of f, f^{-1}:

$$x = g(y) = f^{-1}(y)$$

Now, let us use implicit differentiation to find the derivative of the inverse function.

$$\begin{aligned} y &= f(x) \\ f^{-1}(y) &= x \\ \frac{d}{dx}(f^{-1}(y)) &= \frac{d}{dx}(x) = 1 \end{aligned}$$

By the chain rule:

$$\frac{d}{dy}(f^{-1}(y))\frac{dy}{dx} = 1$$

and

$$\frac{d}{dy}(f^{-1}(y)) = \frac{1}{\frac{dy}{dx}}$$

So, implicit differentiation makes it possible to find the derivative of the inverse function.

Example. $y = \arctan(x)$

$$\begin{aligned}
\tan y &= x \\
\frac{d}{dx}[\tan(y)] &= \frac{dx}{dx} = 1 \\
\frac{d}{dy}[\tan(y)]\frac{dy}{dx} &= 1 \\
\left(\frac{1}{\cos^2(y)}\right)\frac{dy}{dx} &= 1 \\
\frac{dy}{dx} &= \cos^2(y) = \cos^2(\arctan(x))
\end{aligned}$$

This form is messy. Let us use some geometry to simplify it.

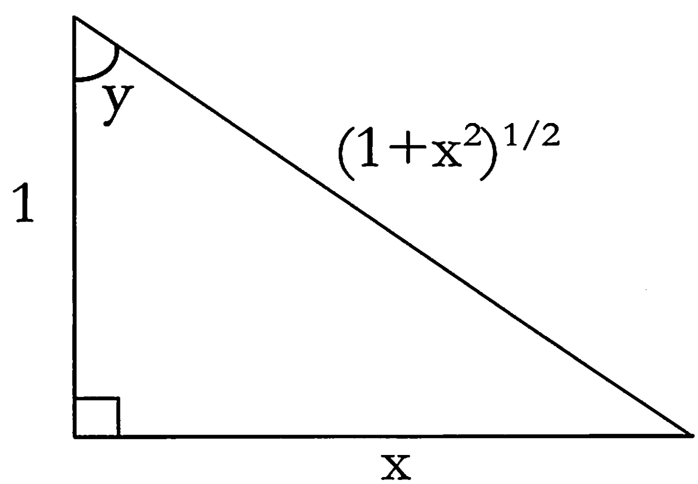


Figure 1: Triangle with angles and lengths corresponding to those in the example illustrating differentiation using the inverse function *arctan*

In this triangle, $\tan(y) = x$ so

$$\arctan(x) = y$$

The Pythagorian theorem tells us the length of the hypotenuse:

$$h = \sqrt{1+x^2}$$

From this, we can find

$$\cos(y) = \frac{1}{\sqrt{1+x^2}}$$

From this, we get

$$\cos^2(y) = \left(\frac{1}{\sqrt{1+x^2}}\right)^2 = \frac{1}{1+x^2}$$

So,

$$\frac{dy}{dx} = \frac{1}{1+x^2}$$

In other words,

$$\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}$$

Graphing an Inverse Function.

Suppose $y = f(x)$ and $g(y) = f^{-1}(y) = x$. To graph g and f together we need to write g as a function of the variable x. If $g(x) = y$, then $x = f(y)$, and what we have done is to trade the variables x and y. This is illustrated in Fig. 2

$f^{-1}(f(x)) = x$	$f^{-1} \circ f(x) = x$
$f(f^{-1}(x)) = x$	$f \circ f^{-1}(x) = x$

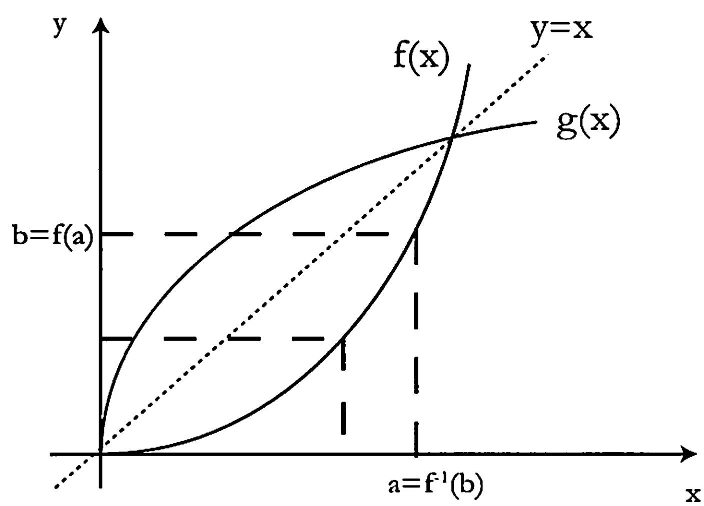


Figure 2: You can think about f^{-1} as the graph of f reflected about the line $y = x$

Recetation

9/21

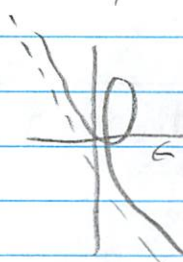
Pset 2a on website (do before exam)

Office hrs this week M 2-3
W - 3-4
R 3:30-5:30

2 review sessions W + T

Implicit Differentiation

$x^3 + y^3 = 6xy$



← 2 tangent lines at this point

x=1 asymptote

Ex y is not a function of x
- can't isolate y + write it out
- so can't take deriv easily

use ① - implicit differentiation

★ Could also use if 2 variables depend on 3rd variable hidden

$\frac{dx}{dt}$ and $\frac{dy}{dt}$ ← use ②

$\frac{d}{dx}(x^3) = 3x^2$

$\frac{d}{dx}(y^3)$ changes as x changes $= 3y^2 \frac{dy}{dx}$ (chain rule)

If did not depend on x = 0 → would = 0

y'

product rule

$$\frac{d}{dx}\left(\overbrace{6x}^{\text{group}}y\right) = 6y + 6x\frac{dy}{dx}$$

$$3x^2 + 3y^2\frac{dy}{dx} = 6y + 6x\frac{dy}{dx}$$

solve for $\frac{dy}{dx}$

$$\frac{dy}{dx} = ?$$

$\frac{1}{\cos x} \to \frac{\sin x}{\cos^2 x} \to \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x} = \sec x \tan x$

$f^2(x) = f(x)f(x)$

$f^{(2)}(x) = f''(x)$

Worksheet 4: More Derivatives

18.01 Fall 2009

Problem 1. Compute the following derivatives:

a) $\sin(1/(4x+2))$

$\sin\left(\frac{1}{4x+2}\right)$ $\quad \cos\left(\left(4x+2\right)^{-1}\right) \cdot -1(4x+2)^{-2} \cdot 4 \to \boxed{-4\cos(4x+2)^{-1} \cdot (4x+2)^{-2}}$

b) $\sec^2(x + \sec x)$

$\frac{1}{\cos^2(x+\sec x)}$ $\to$ ~~$-1(\cos x)^{-2} \cdot -\sin x$~~ ~~$\frac{\sin x}{\cos x} \tan x \cdot \frac{1}{\cos x} \sec x$~~

$\sec(x+\sec x)\sec(x+\sec x)$

$(\sec(x+\sec x))[\sec(x+\sec x)\tan(x+\sec x) -$

$(1+\sec x\tan x)]$

$2\sec(x+\sec x) \cdot (\sec(x+\sec x)\tan(x+\sec x))$

$\cdot (1+\sec x\tan x)$

$2\sec^2(x+\sec x)\tan(x+\sec x)(1+\sec x\tan x)$

c) $\tan \sin x$

$\sec^2(\sin x)\cos x \cdot 1$

Problem 2. Suppose $f(3) = 3$, $f'(3) = 5$, $g(3) = 4$, $g'(3) = 5$.

What is $(f^2(x)g(x))'|_{x=3}$?

$f^2(x) \cdot g'(x) + \underbrace{f^2(x)'}_{2f(x)\cdot f'(x)} \cdot g(x)$

$3^2 \cdot 5 + 2 \cdot 3 \cdot 5 \cdot 4$

$45 + 120 = 165$

Problem 3. a) Determine the tangent line to the unit circle at $\theta = \pi/6$, measured from the positive x-axis. First compute the slope by implicit differentiation, and then sanity-check your answer using trigonometry.

$x^2 + y^2 = 1$

$2x + 2y \cdot y' = 0$

$y' = \frac{-2x}{2y}$ $\quad \frac{\pi}{6} = 30°$ $\quad \pi/6$ sin x

$x = \sin(30)$

$y = \cos(30)$

$-2\left(\frac{\sqrt{3}}{2}\right) / \frac{1}{2} = -\sqrt{3}$

$\boxed{-\sqrt{3}}$

b) What is dy/dx on the curve $x^{3/2} + y^{5/2} = 1$ at the point $(2^{-2/3}, 2^{-2/5})$?

$\frac{3}{2}x^{\frac{1}{2}} + \frac{5}{2}y^{3/2} \cdot y' = 0$ $\quad y' = \frac{-\frac{3}{2}x^{1/2}}{\frac{5}{2}y^{3/2}}$

✓ plug in

$\frac{-\frac{3}{2}(2^{-2/3})^{1/2}}{\frac{5}{2}(2^{-2/5})^{3/2}}$

$\boxed{y - \frac{1}{2} = -\sqrt{3}\left(x - \frac{\sqrt{3}}{2}\right)}$ pt slope form

Problem 4. Find the derivative of $\arctan(x)$ (a.k.a. $\tan^{-1}(x)$)

Problem 5. $d^{97}(\sin x)/dx^{97} = ?$

I need to! [know the rules
recognize them
be able to do the algebra]

Lecture 6

9/27/01

Exam Fri
- mix of hw qu (1+2a)
- worksheets from recetation
- proofs in lecture
- old sample test
- one "fun" question

look at solutions
- answers are posted

On Fri - implicit differentation to compute more derivitives

$y = \sqrt{x}$

$y^2 = x$

$2y \cdot y' = 1$

$y' = \frac{1}{2y}$

functions go from 1 land to another

$y = \arctan(x)$ $\quad y = \tan^{-1}(x)$

hard to do
check lecture notes

inverse of $\tan(x)$

Not $\frac{1}{\tan x}$

"The angle whose tangent is x"

$\tan^{-1}(1)$ = angle whose tangent is $1 = \frac{\pi}{4}$

Can use implicit differentation to show

$$\frac{d}{dx}\left(\tan^{-1}(x)\right) = \frac{1}{1+x^2}$$

$y(x) = \sin^{-1}(x)$

$\sin(y(x)) = \sin(\sin^{-1}(x))$ ← take sin on both sides

$\sin(y(x)) = x$ ↓ simplify

$\cos(y(x))\, y'(x) = 1$ ↓ $\frac{d}{dx}$

$y'(x) = \dfrac{1}{\cos(y(x))}$ ← plug back in

$y'(x) = \dfrac{1}{\cos(\sin^{-1}(x))}$

angle whose sin is x

want $\theta = x$

so $\frac{opp}{hyp} = x$

so $\frac{x}{1} = x$

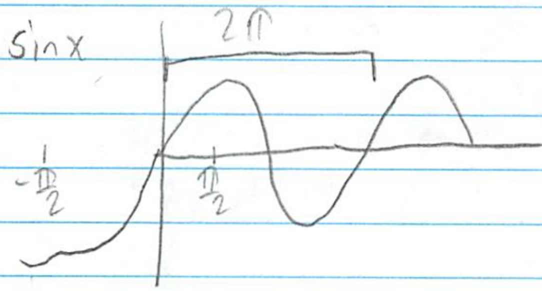
①

② now take cos of it → $\frac{adj}{hyp}$

know adj $= \sqrt{1-x^2}$

$\dfrac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$

$\boxed{y'(x) = \dfrac{1}{\sqrt{1-x^2}}}$ ← deriv of inverse trig functions



$\sin(x) = \sin(x+2\pi)$

↑ repeats every

which to go back?

* want to only take a chunk ("domain") to try horiz line test

for sin $[-\frac{\pi}{2}, \frac{\pi}{2}]$

Defining exponential + log functions

Pick # a >1
define $f(x) = a^x$

eg $2 \to 2^x$ (not x^2)

$2^{17/5} = \sqrt[5]{2^{17}}$

$2^{\pi} = \lim 2^{p/q}$
as $p/q \to \pi$

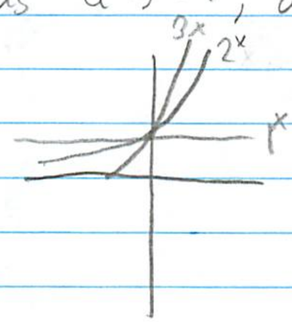
So estimate at a pi value 3.1415926

Some properties
1. $a^x a^y = a^{x+y}$
2. $(a^x)^y = a^{xy}$

Graph $a^0 = 1$ no matter what a

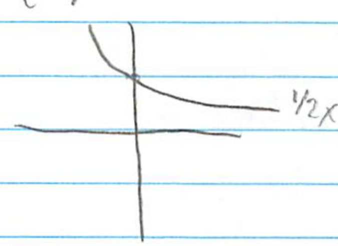
as $a \to -\infty$, $a^x \to 0$

as $a \to \infty$, $a^x \to \infty$ really fast
faster than x^n for any n



$a < 1$ eg $\frac{1}{2}$

$\left(\frac{1}{2}\right)^{-500} = 2^{500}$



$a = -1$
- depends on even or odd
- not contineous - <u>Bad idea</u>
- switches even/odd

Derivitive

$$\frac{d}{dx}(a^x) = \lim_{h\to 0} \frac{(a^{x+h}) - a^h}{h}$$

factor out a^x

$$a^x \lim_{h\to 0}\left(\frac{a^{h+0} - 1^0}{h}\right)$$

deriv of a^x at $x=0$

have some hard limit

think about deriv at 0

$a=1$, $\frac{d}{dx}(a^x)\big|_{x=0} \in [0,\infty)$

Thus there exists an $a \in [1,\infty)$ so that $\frac{d}{dx}(a^x)\big|_{x=0} = 1$

that # is called e

$e \approx 2.71828$

$$\frac{d}{dx}(e^x)\Big|_{x=0} = 1$$

$$\boxed{e^{i\pi}+1=0}$$

$$\frac{d}{dx}(e^x) = e^x\left(\frac{d}{dx}(e^x)\Big|_{x=0}\right) = e^x$$

function - derivitive is itself

Inverse functions to a^x

Define $\log_a(x)$ to be inverse function of a^x

e^x is special $\to$ Inverse $= \ln(x)$

$\frac{d}{dx}(\ln x) = ?$ $\qquad e^{\ln(x)} = x = \ln(e^x)$

Use implicit differentiation

$y(x) = \ln(x)$

So $e^{y(x)} = e^{\ln(x)}$

$e^{y(x)} = x$

$e^{y(x)} \cdot y'(x) = 1$

$y'(x) = \frac{1}{e^y} = \frac{1}{e^{\ln(x)}} = \boxed{\frac{1}{x}}$

Lecture 6: Exponential and Log, Logarithmic Differentiation, Hyperbolic Functions

Taking the derivatives of exponentials and logarithms

Background

We always assume the base, a, is greater than 1.

$$a^0 = 1; \quad a^1 = a; \quad a^2 = a \cdot a; \quad \dots$$

$$\begin{aligned} a^{x_1+x_2} &= a^{x_1}a^{x_2} \\ (a^{x_1})^{x_2} &= a^{x_1x_2} \\ a^{\frac{p}{q}} &= \sqrt[q]{a^p} \quad \text{(where } p \text{ and } q \text{ are integers)} \end{aligned}$$

To define a^r for real numbers r, fill in by continuity.

Today's main task: find $\dfrac{d}{dx}a^x$

We can write

$$\frac{d}{dx}a^x = \lim_{\Delta x \to 0} \frac{a^{x+\Delta x} - a^x}{\Delta x}$$

We can factor out the a^x:

$$\lim_{\Delta x \to 0} \frac{a^{x+\Delta x} - a^x}{\Delta x} = \lim_{\Delta x \to 0} a^x \frac{a^{\Delta x} - 1}{\Delta x} = a^x \lim_{\Delta x \to 0} \frac{a^{\Delta x} - 1}{\Delta x}$$

Let's call

$$M(a) \equiv \lim_{\Delta x \to 0} \frac{a^{\Delta x} - 1}{\Delta x}$$

We don't yet know what $M(a)$ is, but we can say

$$\frac{d}{dx}a^x = M(a)a^x$$

Here are two ways to describe $M(a)$:

1. Analytically $M(a) = \dfrac{d}{dx}a^x$ at $x = 0$.

   Indeed, $M(a) = \displaystyle\lim_{\Delta x \to 0} \frac{a^{0+\Delta x} - a^0}{\Delta x} = \frac{d}{dx}a^x\bigg|_{x=0}$

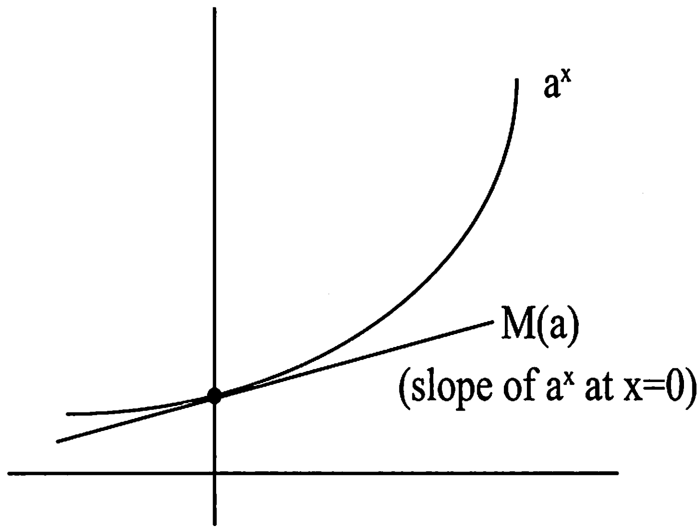


Figure 1: Geometric definition of $M(a)$

2. Geometrically, $M(a)$ is the slope of the graph $y = a^x$ at $x = 0$.

The trick to figuring out what $M(a)$ is is to beg the question and define e as the number such that $M(e) = 1$. Now can we be sure there is such a number e? First notice that as the base a increases, the graph a^x gets steeper. Next, we will estimate the slope $M(a)$ for $a = 2$ and $a = 4$ geometrically. Look at the graph of 2^x in Fig. 2. The secant line from $(0, 1)$ to $(1, 2)$ of the graph $y = 2^x$ has slope 1. Therefore, the slope of $y = 2^x$ at $x = 0$ is less: $M(2) < 1$ (see Fig. 2).

Next, look at the graph of 4^x in Fig. 3. The secant line from $(-\frac{1}{2}, \frac{1}{2})$ to $(1, 0)$ on the graph of $y = 4^x$ has slope 1. Therefore, the slope of $y = 4^x$ at $x = 0$ is greater than $M(4) > 1$ (see Fig. 3).

Somewhere in between 2 and 4 there is a base whose slope at $x = 0$ is 1.

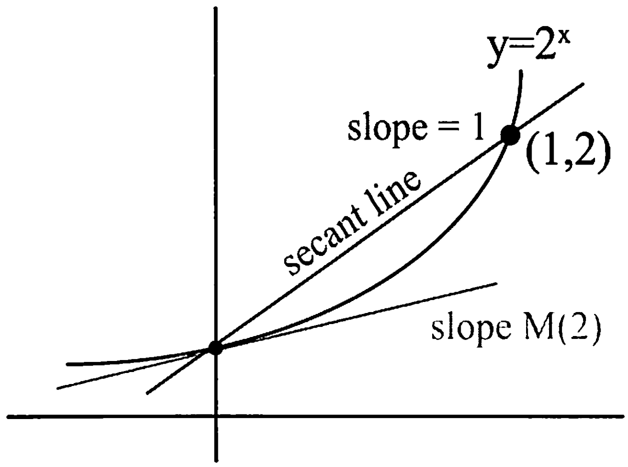


Figure 2: Slope $M(2) < 1$

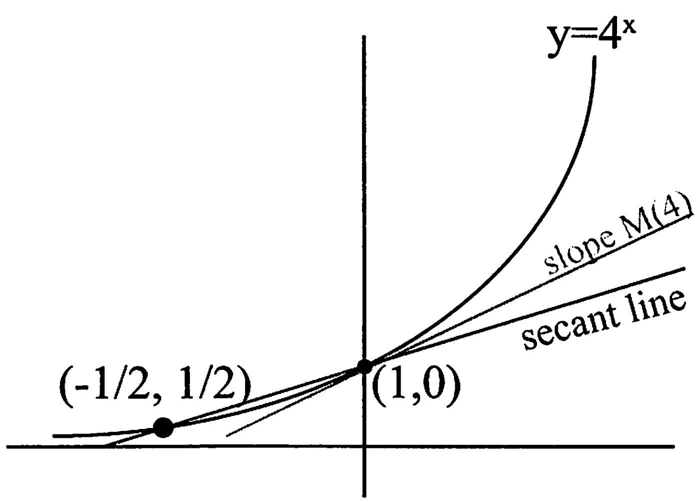


Figure 3: Slope $M(4) > 1$

Thus we can *define* e to be the unique number such that

$$M(e) = 1$$

or, to put it another way,

$$\lim_{h\to 0} \frac{e^h - 1}{h} = 1$$

or, to put it still another way,

$$\frac{d}{dx}(e^x) = 1 \quad \text{at } x = 0$$

What is $\frac{d}{dx}(e^x)$? We just defined $M(e) = 1$, and $\frac{d}{dx}(e^x) = M(e)e^x$. So

$$\boxed{\frac{d}{dx}(e^x) = e^x}$$

Natural log (inverse function of e^x)

To understand $M(a)$ better, we study the natural log function $\ln(x)$. This function is defined as follows:

$$\boxed{\text{If } y = e^x, \text{ then } \ln(y) = x}$$

(or)

$$\boxed{\text{If } w = \ln(x), \text{ then } e^x = w}$$

Note that e^x is always positive, even if x is negative.
Recall that $\ln(1) = 0$; $\ln(x) < 0$ for $0 < x < 1$; $\ln(x) > 0$ for $x > 1$. Recall also that

$$\ln(x_1 x_2) = \ln x_1 + \ln x_2$$

Let us use implicit differentiation to find $\frac{d}{dx}\ln(x)$. $w = \ln(x)$. We want to find $\frac{dw}{dx}$.

$$\begin{aligned}
e^w &= x \\
\frac{d}{dx}(e^w) &= \frac{d}{dx}(x) \\
\frac{d}{dw}(e^w)\frac{dw}{dx} &= 1 \\
e^w\frac{dw}{dx} &= 1 \\
\frac{dw}{dx} &= \frac{1}{e^w} = \frac{1}{x}
\end{aligned}$$

$$\boxed{\frac{d}{dx}(\ln(x)) = \frac{1}{x}}$$

Finally, what about $\frac{d}{dx}(a^x)$?

There are two methods we can use:

Method 1: Write base e and use chain rule.

Rewrite a as $e^{\ln(a)}$. Then,

$$a^x = \left(e^{\ln(a)}\right)^x = e^{x\ln(a)}$$

That looks like it might be tricky to differentiate. Let's work up to it:

$$\frac{d}{dx}e^x = e^x$$

and by the chain rule,

$$\frac{d}{dx}e^{3x} = 3e^{3x}$$

Remember, $\ln(a)$ is just a constant number– not a variable! Therefore,

$$\frac{d}{dx}e^{(\ln a)x} = (\ln a)e^{(\ln a)x}$$

or

$$\boxed{\frac{d}{dx}(a^x) = \ln(a)\cdot a^x}$$

Recall that

$$\frac{d}{dx}(a^x) = M(a)\cdot a^x$$

So now we know the value of $M(a)$: $M(a) = \ln(a)$.

Even if we insist on starting with another base, like 10, the natural logarithm appears:

$$\frac{d}{dx}10^x = (\ln 10)10^x$$

The base e may seem strange at first. But, it comes up everywhere. After a while, you'll learn to appreciate just how natural it is.

Method 2: Logarithmic Differentiation.

The idea is to find $\frac{d}{dx}f(x)$ by finding $\frac{d}{dx}\ln(f(x))$ instead. Sometimes this approach is easier. Let $u = f(x)$.

$$\frac{d}{dx}\ln(u) = \frac{d\ln(u)}{du}\frac{du}{dx} = \frac{1}{u}\left(\frac{du}{dx}\right)$$

Since $u = f$ and $\frac{du}{dx} = f'$, we can also write

$$\boxed{(\ln f)' = \frac{f'}{f} \quad \text{or} \quad f' = f(\ln f)'}$$

Apply this to $f(x) = a^x$.

$$\ln f(x) = x \ln a \implies \frac{d}{dx}\ln(f) = \frac{d}{dx}\ln(a^x) = \frac{d}{dx}(x\ln(a)) = \ln(a).$$

(Remember, $\ln(a)$ is a constant, *not* a variable.) Hence,

$$\frac{d}{dx}(\ln f) = \ln(a) \implies \frac{f'}{f} = \ln(a) \implies f' = \ln(a) f \implies \frac{d}{dx}a^x = (\ln a)a^x$$

Example 1. $\frac{d}{dx}(x^x) = ?$

With variable ("moving") exponents, you should use either base e or logarithmic differentiation. In this example, we will use the latter.

$$\begin{aligned} f &= x^x \\ \ln f &= x \ln x \\ (\ln f)' &= 1 \cdot (\ln x) + x\left(\frac{1}{x}\right) = \ln(x) + 1 \\ (\ln f)' &= \frac{f'}{f} \end{aligned}$$

Therefore,

$$f' = f(\ln f)' = x^x\left(\ln(x) + 1\right)$$

If you wanted to solve this using the base e approach, you would say $f = e^{x \ln x}$ and differentiate it using the chain rule. It gets you the same answer, but requires a little more writing.

Example 2. Use logs to evaluate $\lim_{k\to\infty}\left(1 + \frac{1}{k}\right)^k$.

Because the exponent k changes, it is better to find the limit of the logarithm.

$$\lim_{k\to\infty} \ln\left[\left(1+\frac{1}{k}\right)^k\right]$$

We know that

$$\ln\left[\left(1+\frac{1}{k}\right)^k\right] = k\ln\left(1+\frac{1}{k}\right)$$

This expression has two competing parts, which balance: $k \to \infty$ while $\ln\left(1+\frac{1}{k}\right) \to 0$.

$$\ln\left[\left(1+\frac{1}{k}\right)^k\right] = k\ln\left(1+\frac{1}{k}\right) = \frac{\ln\left(1+\frac{1}{k}\right)}{\frac{1}{k}} = \frac{\ln(1+h)}{h} \quad (\text{with } h = \frac{1}{k})$$

Next, because $\ln 1 = 0$

$$\ln\left[\left(1+\frac{1}{k}\right)^k\right] = \frac{\ln(1+h) - \ln(1)}{h}$$

Take the limit: $h = \frac{1}{k} \to 0$ as $k \to \infty$, so that

$$\lim_{h\to 0} \frac{\ln(1+h) - \ln(1)}{h} = \frac{d}{dx}\ln(x)\Big|_{x=1} = 1$$

In all,

$$\lim_{k\to\infty} \ln\left(1 + \frac{1}{k}\right)^k = 1.$$

We have just found that $a_k = \ln[\left(1+\frac{1}{k}\right)^k] \to 1$ as $k \to \infty$.

If $b_k = \left(1 + \frac{1}{k}\right)^k$, then $b_k = e^{a_k} \to e^1$ as $k \to \infty$. In other words, we have evaluated the limit we wanted:

$$\lim_{k\to\infty} \left(1 + \frac{1}{k}\right)^k = e$$

Remark 1. We never figured out what the exact numerical value of e was. Now we can use this limit formula; $k = 10$ gives a pretty good approximation to the actual value of e.

Remark 2. Logs are used in all sciences and even in finance. Think about the stock market. If I say the market fell 50 points today, you'd need to know whether the market average before the drop was 300 points or 10,000. In other words, you care about the percent change, or the ratio of the change to the starting value:

$$\frac{f'(t)}{f(t)} = \frac{d}{dt}\ln(f(t))$$

Recitation

9/23

Review tonight 7-9
tomorrow 7:30-9

yesterday

$f(x) = a^x$
$f'(x) = a^x \cdot f'(0)$ ← based on deriv at 0

$$\lim_{h\to 0} \frac{a^{x+h} - a^x}{h} = a^x \lim_{h\to 0} \frac{a^h - a^0}{h}$$

e is the # so that if $f(x) = e^x$, $f'(0) = 1$

$$\frac{d}{dx}(e^x) = e^x$$ only function deriv = itself

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

---

exponent rules

1. $a^m a^n = a^{m+n}$
2. $\frac{a^m}{a^n} = a^{m-n}$
3. $(a^m)^n = a^{mn}$
4. $a^x = b$ if and only if $\log_a b = x$

* logs are inverse of exponential functions *

$g(x) = a^x$
$g^{-1}(x) = \log_a x$

$$\boxed{g(\underbrace{g^{-1}(x)}_{=x}) = g^{-1}(g(x))}$$

Determine log rule

assume $\log_a m = k$ then $a^k = m$

$\log_a n = L$ $a^L = n$

First

$MN = a^k \cdot a^l = a^{k+l}$

So $k + L = \log_a MN$

$k + L = \log_a M + \log_a N$

1. $\log_a (MN) = \log_a M + \log_a N$
2. $\log_a \frac{M}{N} = \log_a M - \log_a N$
3. $\log_a (M^J) = J \log_a M$ ← if J is integer
4. $\log_a M = \frac{\ln M}{\ln a}$

prove 4

$\log_a M = k$

$a^k = M$

log bothsides

$\ln M = \ln a^k$

$\ln M = k \ln a$

$k = \frac{\ln M}{\ln a} = \log_a M$

Goal $\frac{d}{dx}(\log_a x)$

same $\frac{d}{dx}\left(\frac{\ln x}{\ln a}\right)$ don't need quotient rule $\ln a$ is a #

$$\frac{d}{dx}\frac{\ln x}{\ln a} = \frac{1}{\ln a} \cdot \frac{d}{dx}\ln x = \frac{1}{x \ln a}$$

$\log_x X = 1$

$\log_x X^8 = 8$

$\log_{x^2} X = \frac{1}{2}$

derivitive of x^x

- can't use a^x rule — HW problem

use somthing here

$$\frac{d}{dx}\left(\log_a(x^2+2x)\right)$$

$$\frac{1}{(x^2+2x)\ln a} \cdot (2x+2)$$

↑ last problem on hw

$$\frac{d}{dx}\ln(f(x)) = \frac{f'}{f}$$

$y =$ ugly $f(x)$

ln ln both sides

$\frac{y'}{y} = \sim$ implicit diff. Unwind

$$\frac{d}{dx}(a^x)$$

$y = a^x$

$\ln(y) = \ln(a^x)$

$\frac{d}{dx}\ln(y) = \frac{d}{dx}\ln(a^x)$ ← constant

chain rule

rule #3

$\frac{1}{y} \cdot y' = \ln a$

$\frac{y'}{a^x} = \ln a$

$y' = (\ln a) a^x$

$(a^x)'\big|_{x=0}$

$$\boxed{\frac{d}{dx}(a^x) = a^x \ln a}$$

Piecewise Function

$$f(x) = \begin{cases} g(x) & x < 1 \\ h(x) & x \geq 1 \end{cases}$$

for f to be contineous
- must = same amount where match $g(1) = h(1)$

for f to be differentiable

$$g'(1) = h'(1)$$

$$f(x) = \begin{cases} ax + b & x > 1 \\ x^2 - 3x + 2 & x \leq 1 \end{cases}$$

(plug in 1

$a + b = 0$ or $a = -b$ contineous
- infinite # pts $(a, -a)$

take both deriv
eval at 1 $a = 2 \cdot 1 - 3 = -1$ differentiable
$b = 1$

Part 2
2B

$$\frac{d}{dy}\left(\sin^2 y \cos^2 y\right) = g(y) = \ldots (2y)$$

not $\frac{d}{dx}$ so can't implicit differentiation
- no need

$\frac{dy}{dy} = 1$

no reason to use chain rule

Use trig identies to find $f(2y) = g(2y)$
some double thing

$$[f(2y)]' = 2f'(2y)$$

chain rule
order of opps thing

$f(y) = \sin y$ $f'(y) = \cos y$

$f(2y) = \sin(2y)$ $f'(2y) = \cos(2y)$

$[f(2y)]' = 2\cos(2y)$ $2\cos(2y)$

Reading

8.3 e / ln

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

e is the # for which $\lim_{h \to 0} \frac{e^h - 1}{h} = 1$

$$e = \lim_{h \to 0} (1 + h)^{1/h}$$

Derivs of
arc trig functions

Deriv

$\sin^{-1} = \frac{1}{\sqrt{1-x^2}}$ $\csc^{-1} = -\frac{1}{x\sqrt{x^2-1}}$

$\cos^{-1} = -\frac{1}{\sqrt{1-x^2}}$ $\sec^{-1} = \frac{-1}{x\sqrt{x^2-1}}$

$\tan^{-1} = \frac{1}{1+x^2}$ $\cot^{-1} - \frac{1}{1+x^2}$

Lecture 7

9/24/09

Office Hours 3:30 - 5:30 Breiner 2-172
2:30 - 4 Brubaker

Exponentials

$f(x) = a^x$

$a^{x_1 + x_2} = a^{x_1} a^{x_2}$

$(a^{x_1})^{x_2} = a^{x_1 x_2}$

① $\log_a(x_1 x_2) = \log_a(x_1) + \log_a(x_2)$

called e $\lim_{h \to 0} \frac{e^h - 1}{h} = 1$

$\frac{d}{dx}(e^x) = e^x$

Inverse is $\ln(x)$

a^x → inverse $\log_a = x$

$\log_a(a^x) = x$

$a^{\log_a(x)} = x$

$\frac{d}{dx} a^x = e^{x \ln a} = e^{x \ln(a)} \cdot \ln a = a^x \ln a$

should work for any a - including e

$a^{\log_a(x_1 + x_2)} = x_1 + x_2 = a^{\log_a(x_1) + \log_a(x_2)}$

② $\log_a(x_1^{x_2}) = x_2 \log_a(x_1)$

$a^x = e^{x \ln a} = e^{\ln(a^x)}$

$\frac{d}{}$

implicit diferentation

$y(x) = a^x$

$\log a \quad \log a$

$\log a(y(x)) = \log_a(a^x)$

identity of logs

$\frac{\ln y_x}{\ln a}$

try showing

$\frac{\ln(y(x))}{\ln(a)} = x$

including prooving w def. derivative

$\frac{1}{y(x)} \cdot y'(x) \cdot \frac{1}{\ln a} = 1$

Can do

$y'(x) = \ln a \cdot y(x)$

$\ln a (a^x)$

x^n

$\sin x \cos x$

e^x

$+ - \times \div$

composition

inverse

How to make new functions?

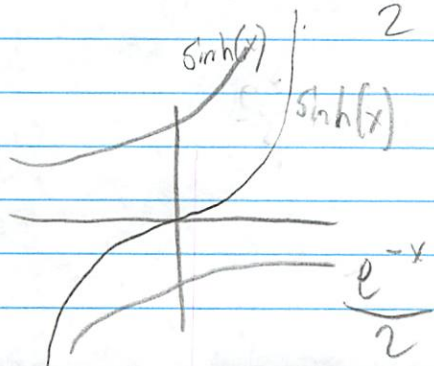
Hyperbolic trig Functions

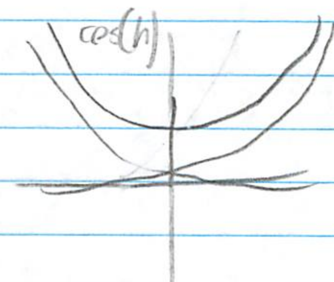
$\sinh(x) = \frac{e^x - e^{-x}}{2}$

kinda like taking even/odd of any function

$\cosh(x) = \frac{e^x + e^{-x}}{2}$

Graps not like trig functions





But properties are a lot like trig functions

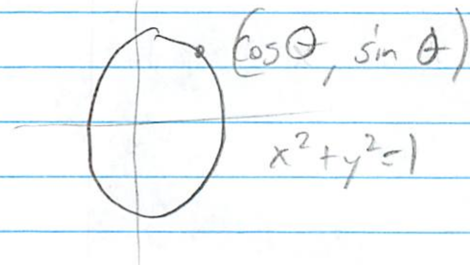
$$\frac{d}{dx} \sinh(x) = \cosh(x)$$

$$\frac{d}{dx} \cosh(x) = \sinh(x)$$

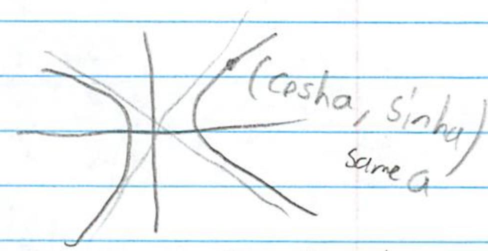
identies similar - but different

Identity $\boxed{\cosh^2(x) - \sinh^2(x) = 1}$

? why called hyperbolic



$\sin^2 x + \cos^2 x$



$x^2 - y^2 = 1$

$\sinh^2 - \cosh^2 = 1$

Exam Review

2 or 3 of the 6 proofs

① Should prove $\frac{d}{dx}(x^n) = n x^{n-1}$

and should also know it

Key: expand $(x+h)^n$ - pascal's triangle

$\frac{1}{3} - \frac{1}{2}$ of exam

② $\frac{d}{dx} \sin(x)$ $\cos(x)$

↳ $\lim_{h \to 0} \frac{\sin h}{h} = 1$

$\lim_{h \to 0} \frac{\cos h - 1}{h} = 0$

③ $\frac{d}{dx} e^x = e^x$ $\lim_{h\to 0} \frac{e^h - 1}{h} = 1$

also ④ sum rule - no key step
prove ⑤ constant multiple rule "
⑥ product rule add + subtract $\pm f(x+h)\,g(x)$

potential division by 0 - [quotient rule, chain rule] **Not** on exam to prove

$\frac{d}{dx} \ln(\sec x))$ ← by rules
← chain rules $\sec x = \frac{1}{\cos} = -\frac{1}{\sin x} = -\csc$

$\frac{d}{dx} \left(\frac{1}{(x^3+2x)}\right)^5$ power rule, chain rule

$\frac{d}{dx} \left(\frac{e^{-x^2}}{\sqrt{x}}\right)$ product rule w/ $x^{-1/2}$
simplify

compute tangent line
implicit differentiation
$f(x,y) = 0$
$x^2 + y^2 - 1 = 0$
~~inverse func~~ inverse function

no squeeze theorm

limits
- trig
- or secretly computed as derivtives

$\lim_{x\to 1} \frac{\ln(x)}{x-1} = \lim_{x\to 1} \frac{\ln x - \ln 1}{x-1} =$

$\frac{d}{dx} \ln x \Big|_{x=1} = \frac{1}{x}\Big|_{x=1} = 1$

Continuiety vs differentiablity
implies (proof -1 line)

piecewise problem
derivitives are instanteneous rate of change

re related rates

Lecture 7: Continuation and Exam Review

Hyperbolic Sine and Cosine

$$\frac{e^{x}-e^{-x}}{2}$$

Hyperbolic sine (pronounced "sinsh"):

$$\boxed{\sinh(x) = \frac{e^x - e^{-x}}{2}}$$

$$\frac{e+e^{-x}}{2}$$

Hyperbolic cosine (pronounced "cosh"):

$$\boxed{\cosh(x) = \frac{e^x + e^{-x}}{2}}$$

$$\frac{d}{dx}\sinh(x) = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{e^x - (-e^{-x})}{2} = \cosh(x)$$

Likewise,

$$\frac{d}{dx}\cosh(x) = \sinh(x)$$

(Note that this is different from $\frac{d}{dx}\cos(x)$.)

Important identity:

$$\cosh^2(x) - \sinh^2(x) = 1$$

Proof:

$$\cosh^2(x) - \sinh^2(x) = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2$$

$$\cosh^2(x) - \sinh^2(x) = \frac{1}{4}\left(e^{2x} + 2e^x e^{-x} + e^{-2x}\right) - \frac{1}{4}\left(e^{2x} - 2 + e^{-2x}\right) = \frac{1}{4}(2+2) = 1$$

Why are these functions called "hyperbolic"?
Let $u = \cosh(x)$ and $v = \sinh(x)$, then

$$u^2 - v^2 = 1$$

which is the equation of a hyperbola.

Regular trig functions are "circular" functions. If $u = \cos(x)$ and $v = \sin(x)$, then

$$u^2 + v^2 = 1$$

which is the equation of a circle.

Exam 1 Review

General Differentiation Formulas

$$
\begin{aligned}
(u+v)' &= u'+v' \\
(cu)' &= cu' \quad \text{constant} \\
(uv)' &= u'v+uv' \quad \text{(product rule)} \\
\left(\frac{u}{v}\right)' &= \frac{u'v-uv'}{v^2} \quad \text{(quotient rule)} \\
\frac{d}{dx}f(u(x)) &= f'(u(x))\cdot u'(x) \quad \text{(chain rule)}
\end{aligned}
$$

You can remember the quotient rule by rewriting

$$\left(\frac{u}{v}\right)' = (uv^{-1})'$$

and applying the product rule and chain rule.

Implicit differentiation

Let's say you want to find y' from an equation like

$$y^3 + 3xy^2 = 8$$

Pick 1 to go with

Instead of solving for y and then taking its derivative, just take $\frac{d}{dx}$ of the whole thing. In this example,

product rule — y' term + $x'y$

$$
\begin{aligned}
3y^2y' + 6xyy' + 3y^2 &= 0 \quad \text{subtract} \\
(3y^2+6xy)y' &= -3y^2 \quad \text{factor} \\
y' &= \frac{-3y^2}{3y^2+6xy}
\end{aligned}
$$

Note that this formula for y' involves both x and y. Implicit differentiation can be very useful for taking the derivatives of inverse functions.

For instance,

$$y = \sin^{-1}x \Rightarrow \sin y = x$$

Implicit differentiation yields

$$(\cos y)y' = 1$$

know proof intermediate

and

$$y' = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}$$

Specific differentiation formulas

You will be responsible for knowing formulas for the derivatives and how to deduce these formulas from previous information: x^n, $\sin^{-1} x$, $\tan^{-1} x$, $\sin x$, $\cos x$, $\tan x$, $\sec x$, e^x, $\ln x$.

For example, let's calculate $\frac{d}{dx}\sec x$: know it or quotient rule

yeah

$$\frac{d}{dx}\sec x = \frac{d}{dx}\frac{1}{\cos x} = \frac{-(-\sin x)}{\cos^2 x} = \tan x \sec x$$

$\frac{0 - -\sin x}{\cos^2 x}$ $\frac{\sin}{\cos}$ $\frac{1}{\cos}$

know to do it

tan sec

You may be asked to find $\frac{d}{dx}\sin x$ or $\frac{d}{dx}\cos x$, using the following information:

$$\lim_{h\to 0}\frac{\sin(h)}{h} = 1$$
$$\lim_{h\to 0}\frac{\cos(h)-1}{h} = 0$$

Remember the definition of the derivative:

$$\frac{d}{dx}f(x) = \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

Tying up a loose end

How to find $\frac{d}{dx}x^r$, where r is a real (but not necessarily rational) number? All we have done so far is the case of rational numbers, using implicit differentiation. We can do this two ways:

1st method: base e

$$\begin{aligned}
x &= e^{\ln x} \\
x^r &= \left(e^{\ln x}\right)^r = e^{r\ln x} \\
\frac{d}{dx}x^r &= \frac{d}{dx}e^{r\ln x} = e^{r\ln x}\frac{d}{dx}(r\ln x) = e^{r\ln x}\frac{r}{x} \\
\frac{d}{dx}x^r &= x^r\left(\frac{r}{x}\right) = rx^{r-1}
\end{aligned}$$

$\log_a b = x$

$a^x = b$

2nd method: logarithmic differentiation

$$\begin{aligned}
(\ln f)' &= \frac{f'}{f} \\
f &= x^r \\
\ln f &= r\ln x \\
(\ln f)' &= \frac{r}{x} \\
f' = f(\ln f)' &= x^r\left(\frac{r}{x}\right) = rx^{r-1}
\end{aligned}$$

Finally, in the first lecture I promised you that you'd learn to differentiate *anything*— even something as complicated as

$$\frac{d}{dx}e^{x\tan^{-1}x}$$

So let's do it!

$$\frac{d}{dx}e^{uv} = e^{uv}\frac{d}{dx}(uv) = e^{uv}(u'v+uv')$$

Substituting,

$$\frac{d}{dx}e^{x\tan^{-1}x} = e^{x\tan^{-1}x}\left(\tan^{-1}x + x\left(\frac{1}{1+x^2}\right)\right)$$

18.01 Practice Questions for Exam 1

Solutions will be posted on the 18.01 website.

No books, notes, or calculators will be allowed at the exam.

1. Evaluate each of the following, simplifying where possible; for (b) indicate reasoning. The letters a and k represent constants.

a) $\displaystyle \frac{d}{dt}\left(\frac{3t}{\ln t}\right)\Big|_{e^2}$ b) $\displaystyle \lim_{u\to 0}\frac{3u}{\tan 2u}$ c) $\displaystyle \frac{d^3}{dx^3}\sin kx$ d) $\displaystyle \frac{d}{d\theta}\sqrt[3]{a+k\sin^2\theta}$

2. Derive the formula for $\frac{d}{dx}x^3$ at the point $x = x_0$ directly from the definition of derivative.

3. Find $\displaystyle \lim_{h\to 0}\frac{1-\sqrt[3]{1+h}}{h}$ by relating it to a derivative. (Indicate reasoning.)

4. Sketch the curve $y = \sin^{-1}x$, $-1 \le x \le 1$, and derive the formula for its derivative from that for the derivative of $\sin x$.

5. For the function

$$f(x) = \begin{cases} ax+b, & x > 0 \\ 1-x+x^2, & x \le 0, \end{cases}, \qquad a \text{ and } b \text{ constants,}$$

a) find all values of a and b for which the function will be continuous;

b) find all values of a and b for which the function will be differentiable.

6. For the curve given by the equation

$$x^2y + y^3 + x^2 = 8,$$

find all points on the curve where its tangent line is horizontal.

7. Where does the tangent line to the graph of $y = f(x)$ at the point (x_0, y_0) intersect the x-axis?

8. The volume of a spherical balloon is decreasing at the instantaneous rate of -10 cm^3/sec, at the moment when its radius is 20 cm. At that moment, how rapidly is its radius decreasing?

9. Where are the following functions discontinuous?

a) $\sec x$ b) $\displaystyle \frac{1+x^2}{1-x^2}$ c) $\displaystyle \frac{d}{dx}|x|$

10. A radioactive substance decays according to a law $A = A_0e^{-rt}$, where $A(t)$ is the amount in present at time t, and r is a positive constant.

a) Derive an expression in terms of r for the time it takes for the amount to fall to one-quarter of the initial amount A_0.

b) At the moment when the amount has fallen to $1/4$ the initial amount, how rapidly is the amount falling? (Units: grams, seconds.)

18.01 Practice Questions for Exam 1

Solutions will be posted on the 18.01 website

Problem 1. Evaluate each of the following:

a) $\left.\dfrac{d}{dx}\dfrac{\sqrt{x}}{1+2x}\right|_{x=1}$

b) $\dfrac{d}{du}(u\ln 2u)$ (simplify your answer)

Problem 2. a) Evaluate $\dfrac{d}{dt}\sqrt{1-k\cos^2 t}$, where k is constant.

b) Check your answer to part (a) by showing that if $k=1$, your answer agrees with the derivative calculated by a simpler method.

Problem 3. Derive the formula for $\dfrac{d}{dx}\left(\dfrac{1}{x^2}\right)$ directly from the definition of derivative.

(You will need to transform the difference quotient algebraically before taking the limit.)

Problem 4. Derive the formula for $\dfrac{d}{dx}\sin^{-1}x$ by solving $y=\sin^{-1}x$ for x and using implicit differentiation.

(You may use the known values of $D\sin x$ and $D\cos x$ in your derivation. Your answer must be expressed in terms of x.)

Problem 5. Find all values of the constants a and b for which the function defined by

$$f(x)=\begin{cases} ax+b, & x>1 \\ x^2-3x+2, & x\le 1 \end{cases}$$

will be differentiable.

Problem 6. Evaluate the following, with enough indications to show you are not just guessing:

a) $\displaystyle\lim_{u\to 0}\frac{\tan 2u}{u}$

b) $\displaystyle\lim_{h\to 0}\frac{e^h-1}{h}$ (relate it to the value of a derivative)

Problem 7. A hawk is pursuing a mouse. We choose a coordinate system so the mouse runs along the x-axis in the negative direction, and the hawk is flying over the x-axis, swooping down along the exponential curve $y=e^{kx}$, for some positive constant k. The hawk in flight is always aimed directly at the mouse. It is noon at the equator, and the sun is directly overhead.

When the hawk's shadow on the ground is at the point x_0, where is the mouse?

Sample Test Both

8. $V = \frac{4}{3}\pi r^3$ volume $\Delta = -10\,cm^3/sec$

moment $= 20cm$

$\frac{d}{dt}\, V = \frac{4}{3}\pi r^3$ ← varries w/ tho

$\frac{dv}{dt} = 4\pi r^2 \frac{dr}{dt}$

$-10\,cm^3/sec = 4\pi 20^2 \frac{dr}{dt}$

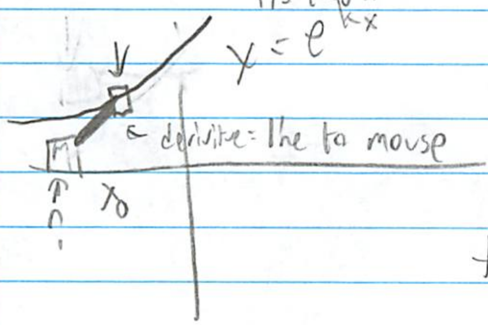
↑ solve for

---

7. Hawk pursuing mouse - always

its flight

$y = e^{kx}$



hawk is x_0

mouse ?

tangent line to $y = e^{kx}$ at x_0

position of mouse is intersection of tangent w/ $y = 0$

$\frac{d}{dx}\, e^{kx}$

$e^{kx} \cdot k$

$k e^{kx_0}$

$y - \underset{e^{kx_0}}{0} = k e^{kx} (x - x_0)$

↑ where hawk is at x_0

direction instantaneous displacement = direction from hawk → mouse

tangent (x_0, e^{kx_0}) with $y = 0$

$\div e^{kx_0}$

Solve for x $k(x - x_0) + 1 \rightarrow x - x_0 = -\frac{1}{k}$ $\boxed{x = x_0 - \frac{1}{k}}$

to another problem

1. $\frac{d}{dt} \frac{3t}{\ln t} \Big|_{e^2}$

$$\frac{3 \cdot \ln t - \frac{1}{t} \cdot 3t}{(\ln t)^2}$$

$$\frac{3\ln t - \frac{3t}{t}}{\ln^2}$$

$$\frac{3\ln t - 3}{\ln t^2} \rightarrow \text{factor} \quad \frac{3(\ln t - 1)}{\ln t^2}$$

where $t \rightarrow e^2$ eval now — almost

$$\frac{3(\ln e^2 - 1)}{(\ln e^2)^2} \rightarrow \frac{3(2-1)}{2^2} \rightarrow \frac{3}{4} \quad 0$$

2. $\lim_{u \to 0} \frac{3u}{\tan 2u} \rightarrow \frac{3u}{\frac{\sin 2u}{\cos 2u}} \quad 3u \cdot \frac{\cos 2u}{\sin 2u} = \frac{3u \cos 2u}{2 \sin u \cos u}$

← seperate → 1

↑ identity double angle

~~$\sin 2u = 2\sin u \cos u$~~

$\cos(0) = 1$

$$\frac{3u \cos 1 u}{2 \sin u \cancel{\cos u}\, 1} \rightarrow \frac{3}{2} \frac{u}{\sin u} \rightarrow \frac{3}{2}$$

$\boxed{\text{so to learn } \cos(0) = 1 \text{ trig identies}}$

$$\boxed{\lim \frac{\sin x}{x} = 1}$$

c $\frac{d^3}{dx^3}$ $\sin kx$

$\cos kx \cdot k$

$k\cos kx$

$k - \sin kx \cdot k$

$-2k\sin(kx)$

carry $-2k\cos kx \cdot k$

$-3k\cos(kx)$

d $\frac{d}{d\theta}$ $\sqrt[3]{a+k\sin^2\theta}$

$(a+k\sin^2\theta)^{2/3}$

did chain rule wrong when converted

$\cancel{\frac{2}{3}\left(0+k\cos^2\theta \cdot 2\right)^{-1/3}}$

$\cancel{\frac{4}{3}\left(k\cos^2\theta\right)^{-1/3}}$ $\rightarrow$ $\cancel{\frac{4}{3\sqrt[3]{k\cos^2\theta}}}$

Chain rule inside

$\frac{2}{3}\left(a+k\sin^2\theta\right)^{-1/3} \cdot 0 + k\,\sin\theta\cos\theta \cdot 2$

when?

$\frac{\cancel{4k\cos^2\theta}\ 2\sin\theta\cos\theta}{3\sqrt{a+k\sin^2\theta}}$

keep forgetting that

$2k\sin\theta \cdot \cos\theta$

3. $\lim_{h \to 0} \frac{1 - \sqrt[3]{1+h}}{h}$ relate to derivitive

$\rightarrow \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

try to write this as

$\downarrow$

$-\frac{\sqrt[3]{1+h} - 1}{h} = \boxed{f(x) = \sqrt[3]{x} \quad x_0 = 1}$

or $f(x) = \sqrt[3]{1+x}, \ x_0 = 0$

$-\sqrt[3]{x}' \Big|_{x=1} = -\frac{1}{3} x^{-2/3} \Big|_{x=1} = -\frac{1}{3}$

$f(x+h) = \sqrt[3]{1+h}$

---

2. $\frac{d}{dx} x^3 \Big|_{x_0}$ $\quad \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$\frac{(x+h)^3 - (x)^3}{h} \rightarrow \frac{(x+h)(x+h)(x+h) - x^3}{h}$

$\cancel{(xx + xh + xx + xh + xh + hh + hx + xh + x \ x + xh + hx + hh)}$

$\frac{3x^2 + \cancel{7xh + 2h^2} - x^3}{h}$ I just this how was going to do

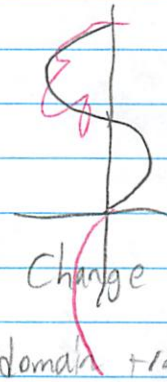
$\frac{\cancel{x^3} + 3x^2h + 3xh^2 + 3h^{\cancel{3}2} - \cancel{x^3}}{h}$

$3x^2 + 3xh + 3h^2$

$\boxed{3x_0^2} + \cancel{3xh} \cdot 0$

4. Sketch normal sin

better



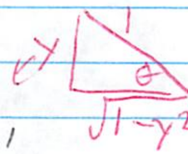
Change y + x

restrict domain + range

$\sin^{-1}(x) = \quad x = \sin(y)$

$1 = \cos y \, y'$

$y' = \frac{1}{\cos y}$



explug in $\frac{1}{\sqrt{1-\sin^2 x}} \Rightarrow \frac{1}{\sqrt{1-x^2}}$

5 $\frac{ax+b}{1-x+x^2}$ $\quad x=0 \quad b=1$

a

$-1+2x$

$a=-1+2(0)$

$a=-1$

6. Where is tangent line horizontal $x^2y+y^3+x^2=8$

$2xy+\cancel{x}y'x^2+3y^2y'+2x=0$

$\frac{yy'x^2+3y^2y'}{yx^2} = -2xy-2x$

??

$2xy+x^2y'+3y^2y'+2x$

slope horiz at $y'=0$

solve ↳ $2x(y+1)=0$

$x=0$ or $y=-1$

↗ when is $x=0$ when is $y=-1$

7. Where does tangent to $y = f(x)$ at $x_0\ y_0$ intersect x axis

$(x, 0)$

$$y - y_0 = f'(x)(x - x_0)$$

will $= 0$ — put here

$$-y_0 = f'(x)(x - x_0)$$

solve for x

$$-y_0 = x f'(x) - f'(x) x_0$$

$$-y_0 + f'(x) x_0 = x f'(x)$$

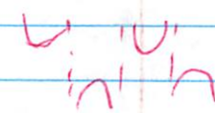
$$\frac{-y_0 + f'(x) x_0}{f'(x)} = x$$

can write as

$$\frac{-y_0}{f'(x)} + x_0 = x$$

close

9. $\sec x = \frac{1}{\cos x}$ at $\frac{\pi}{2}$ drop



graph + show

$$\frac{1 + x^2}{1 - x^2}$$ at $x = \pm 1$

$|x|$ at $x = 0$

10. $A(t) = A_0 e^{-rt}$ A_0 = initial

r is constant > 0

time it takes to go to $\frac{1}{4} A_0$

$A(t)$ $\frac{1}{4} A_0$

$A_0 e^{-rt} = \frac{1}{4} A_0$

$e^{rt} = 4$ reciprocal

ln = ln

$rt = \ln 4$ $t = \frac{\ln 4}{r}$

b Derivitive

at that t

$A'(t)$? $A'(t) = \underbrace{A_0 e^{-rt}}_{\frac{1}{4}A_0} \cdot -r$

chain rule

$\cancel{A_0 e^{-r\frac{\ln 4}{r}} \cdot -r}$

$\frac{1}{4} A_0 \cdot -r$

$\boxed{\frac{-rA_0}{4}}$

18.01 Solutions to practice questions (exam 1)

1 a) $D\frac{3t}{\ln t} = \frac{3\ln t - 3t\cdot\frac{1}{t}}{\ln^2 t} = \frac{3(\ln t - 1)}{\ln^2 t}$

When $t = e^2$, $\ln t = 2\ln e = 2$: $= \frac{3(2-1)}{4} = \frac{3}{4}$

b) $\frac{3u}{\tan 2u} = \frac{3u\cdot\cos 2u}{\sin 2u} = \frac{3u\cos 2u}{2\sin u\cos u}$

$= \frac{3}{2}\cdot\frac{u}{\sin u}\cdot\frac{\cos 2u}{\cos u} \to \frac{3}{2}\cdot 1\cdot 1$

c) $D\sin kx = k\cos kx$

$D^2 \cdots = k^2(-\sin kx)$

$D^3\sin kx = k^3(-\cos kx)$

d) $\frac{d}{d\theta}(a+k\sin^2\theta)^{1/3} = \frac{1}{3}(a+k\sin^2\theta)^{-2/3}\cdot 2k\sin\theta\cos\theta$

$= \frac{2}{3}\frac{k\sin\theta\cos\theta}{(a+k\sin^2\theta)^{2/3}}$

2 $\frac{d}{dx}x^3\Big|_{x_0} = \lim_{\Delta x\to 0}\frac{(x_0+\Delta x)^3 - x_0^3}{\Delta x}$

$= \lim_{\Delta x\to 0}\frac{(\cancel{x_0^3}+3x_0^2\Delta x+3x_0(\Delta x)^2+(\Delta x)^3)}{\Delta x}$

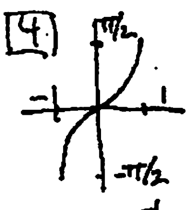
$= 3x_0^2$

3 On the one hand, $\frac{d}{dx}\sqrt[3]{x}\Big|_{x=1} = \frac{1}{3}x^{-2/3}\Big|_1 = \frac{1}{3}$

And: by def'n

$\frac{d}{dx}\sqrt[3]{x}\Big|_1 = \lim_{\Delta x\to 0}\frac{\sqrt[3]{1+\Delta x}-\sqrt[3]{1}}{\Delta x} = \lim_{h\to 0}\frac{\sqrt[3]{1+h}-1}{h}$

$= 1/3$ therefore

4 

$y = \sin^{-1}x$

$y = \sin^{-1}x \iff \sin y = x$

implicit diff'n:

$\cos y\cdot y' = 1$

$y' = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$

(since $\cos y > 0$ for $-\pi/2 \le y < \pi/2$, choose the positive square root.

5 $f(x) = \begin{cases} ax+b, & x>0 \\ 1-x+x^2, & x\le 0\end{cases}$

Continuous $\iff ax+b\big|_0 = 1-x+x^2\big|_0$, or $\boxed{b=1}$

Diff. $\iff$ continuous and slopes are $=$ at 0:

$a = -1+2x\big|_0$ or $\boxed{a=-1}$

6 $x^2y + y^3 + x^2 = 8$ (*)

By implicit diff'n:

$2xy + x^2y' + 3y^2y' + 2x = 0$

Slope horizontal $\iff y' = 0$

$\iff 2x(y+1) = 0$

$\iff x=0$ or $y=-1$

$x=0 \underset{\text{by (*)}}{\Rightarrow} y^3 = 8 \Rightarrow y = 2$ $(0,2)$

$y=-1 \Rightarrow -x^2-1+x^2 = 8 \Rightarrow -1 = 8$ impossible

$\boxed{\therefore \text{ at } (0,2)}$

7 tan line at (x_0, y_0):

$y - y_0 = f'(x_0)(x-x_0)$ at x-intercept, $y=0$

$\therefore -y_0 = f'(x_0)(x-x_0)$

$\therefore x = x_0 - \frac{y_0}{f'(x_0)}$, if $f'(x_0)\ne 0$

8 $V = \frac{4}{3}\pi r^3$ vol of sphere

$\frac{dV}{dt} = 4\pi r^2\cdot\frac{dr}{dt}$, by chain rule

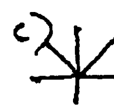
$-10 = 4\pi(20)^2\frac{dr}{dt}$

$\frac{dr}{dt} = \frac{-10}{4\pi(20)^2} = \frac{-1}{160\pi}$ cm/sec

9 a) $\sec x = \frac{1}{\cos x}$; $\cos x = 0$ at $x = \pm\pi/2, \pm 3\pi/2, \ldots$

i.e., at $x = \frac{(2n+1)\pi}{2}$, $n = 0, \pm 1, \ldots$

b) $\frac{1+x^2}{1-x^2} = \frac{1+x^2}{(1-x)(1+x)}$ $\therefore$ at $x=1$, $x=-1$

c) 

no slope at $x=0$

10 a) $A = A_0e^{-rt}$

$\frac{A}{A_0} = \frac{1}{4} = e^{-rt}$ take logs:

$-\ln 4 = -rt$

$\therefore t = \frac{\ln 4}{r}$

b) $\frac{dA}{dt} = A_0e^{-rt}\cdot(-r)$

$= A\cdot(-r)$

$= \frac{1}{4}(-r) = -\frac{r}{4}$

18.01 Solutions for Practice Questions (Exam 1)

1 a) By quotient rule:

$$\frac{d}{dx}\left(\frac{\sqrt{x}}{1+2x}\right) = \frac{\frac{1}{2\sqrt{x}}(1+2x) - \sqrt{x}\cdot 2}{(1+2x)^2}$$

value at $x=1$: $\dfrac{\frac{1}{2}\cdot 3 - 1\cdot 2}{3^2} = -\dfrac{1}{18}$

b) By product rule:

$$\frac{d}{du}(u \ln 2u) = 1\cdot \ln 2u + u\cdot\frac{1}{2u}\cdot 2 = \ln 2u + 1$$

2 a) By two uses of the chain rule:

$$\frac{d}{dt}(1-k\cos^2 t)^{1/2} = \frac{1}{2}(1-k\cos^2 t)^{-1/2}\cdot(-2k\cos t)\cdot(-\sin t)$$

$$= \frac{k\cos t\sin t}{\sqrt{1-k\cos^2 t}} \quad \text{OR} \quad \frac{k}{2}\frac{\sin 2t}{\sqrt{1-k\cos^2 t}}$$

(b) If $k=1$, $\sqrt{1-\cos^2 t} = \sin t$, so the above becomes $= \boxed{\cos t}$ which agrees with

$$\frac{d}{dt}\sqrt{1-\cos^2 t} = \frac{d}{dt}\sin t = \boxed{\cos t}$$

3

$$\frac{d}{dx}\left(\frac{1}{x^2}\right) = \lim_{\Delta x\to 0}\frac{\frac{1}{(x+\Delta x)^2} - \frac{1}{x^2}}{\Delta x}$$

$$= \lim_{\Delta x\to 0}\frac{1}{\Delta x}\left(\frac{x^2-(x^2+2x\Delta x+(\Delta x)^2)}{(x+\Delta x)^2\cdot x^2}\right)$$

$$= \lim_{\Delta x\to 0}\frac{-2x - \Delta x}{(x+\Delta x)^2\cdot x^2} = \frac{-2x}{x^4}$$

$$= \boxed{\frac{-2}{x^3}}$$

4 $y = \sin^{-1}x \Rightarrow x = \sin y$

Differentiating: $1 = \cos y\cdot\dfrac{dy}{dx}$

$$\therefore \frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$$

(Since $-\frac{\pi}{2}\le y\le\frac{\pi}{2}$, $\cos y\ge 0$ so use positive $\sqrt{\ }$)

5 Since $f(x)$ is differentiable, it is also continuous. Thus the two functions must have same value at $x=1$ (in the limit), and the same slope (in the limit):

	$ax+b$		x^2-3x+2
value:	$a+b$	$=$	0
slope:	a	$=$	$2x-3\big\rvert_{x=1} = -1$

$\therefore \boxed{a=-1,\ b=1}$

6 a)

$$\lim_{u\to 0}\frac{\tan 2u}{u} = \lim_{u\to 0}\frac{\sin 2u}{2u}\cdot\frac{2}{\cos 2u} = 1\cdot\frac{2}{1} = 2$$

(another way:

$$= \lim_{u\to 0}\frac{\sin 2u}{u\cdot\cos 2u} = \lim_{u\to 0} 2\frac{\sin u}{u}\,\frac{\cos u}{\cos 2u} = 2\cdot 1\cdot\frac{1}{1} = 2)$$

b)

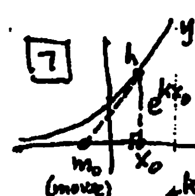
$$\frac{d}{dx}e^x\Big|_{x=0} = \lim_{h\to 0}\frac{e^{0+h}-e^0}{h} = \boxed{\lim_{h\to 0}\frac{e^h-1}{h}}$$

(can use Δx instead of h)

But $\dfrac{d}{dx}e^x\Big|_{x=0} = e^x\Big|_{x=0} = \boxed{1}$

Therefore, the limit is 1.

7



Simplest: calculate slope of tangent line two ways:

$$\frac{e^{kx_0}}{x_0-m_0} = \frac{d}{dx}e^{kx}\Big|_{x_0} = ke^{kx_0}$$

$\therefore x_0 - m_0 = \frac{1}{k}$; $\boxed{m_0 = x_0 - \frac{1}{k}}$

OR Equation of tan. line is

$y - y_0 = \underset{\substack{\text{slope}\\=\text{deriv.}}}{ke^{kx_0}}(x-x_0)$; $\quad y_0 = e^{kx_0}$

The x-intercept m_0 is where $y=0$

$\therefore -e^{kx_0} = ke^{kx_0}(m_0 - x_0)$ ($x=m_0$ there)

so $-\frac{1}{k} = m_0 - x_0$, $\boxed{m_0 = x_0 - \frac{1}{k}}$

18.01: REVIEW FOR EXAM 1

IVAN LOSEV

Contents

1. Computation of derivatives

1.1. **Basic derivatives.** These are to be memorized, although some of them can be deduced from others using derivation rules, see below.

Table 1: Basic derivatives

no	Function $f(x)$	Derivative $f'(x)$
1	x^a	ax^{a-1}
2	$\sin(x)$	$\cos(x)$
3	$\cos(x)$	$-\sin(x)$
4	e^x	e^x
5	$\ln(x)$	$1/x$

A useful generalization of 4th row: $\frac{d}{dx}a^x = \ln(a)a^x$.

1.2. **Derivation rules.** These are roughly divided into two groups:

Differentiation vs algebra.
Sum rule: $(f(x)+g(x))' = f'(x)+g'(x)$.

Product rule: $(f(x)g(x))' = f'(x)g(x)+f(x)g'(x)$.

Quotient rule: $\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x)-f(x)g'(x)}{g(x)^2}$.

A useful special case of the quotient rule: $\left(\frac{1}{g(x)}\right)' = -\frac{g'(x)}{g(x)^2}$.

Problems to practice:

- preexam 1a, problem 1a;
- preexam 1b, problem 1;
- suppl. notes 1E.

Differentiation vs compositions of functions.
Chain rule: $(f(g(x)))' = f'(g(x))g'(x)$. This is a basic rule here.

Inverse rule: $(f^{-1}(x))' = \frac{1}{f'(f^{-1}(x))}$.

Problems to practice:

- preexam 1a, problems 1d,4;
- preexam 1b, problem 2;
- Suppl. Notes 1G;

Implicit differentiation: This is another technique related to the chain rule.

A function $y = y(x)$ is given implictly by the equality $F(x, y) = 0$ (e.g., $x^2 + y^2 - 1 = 0$ or $x^2y - xy^3 + 6 = 0$). Then to find the derivative $y'(x_0)$ we:

a. Differentiate $F(x, y(x))$ using the chain rule. E.g. in the second example we get:

$$2xy + x^2y' - y^3 - x \cdot 3y^2y' = 0.$$

b. Express y' in terms of x and y, e.g.,

$$y' = \frac{2xy - y^3}{3xy^2 - x^2}.$$

Then depending on the situation we do one of the following:

c1) Plug the values of x and y for which we want to compute the derivative. In the previous example for $x_0 = 1, y_0 = 2$ we get $y'(x_0) = -\frac{4}{11}$. We do this, for example, if asked to find the equation of the tangent line to $F(x, y) = 0$ at the point (x_0, y_0) on the graph (i.e., $F(x_0, y_0) = 0$).

c2) Or we can try to express y in terms of x from $F(x, y) = 0$. This will allow us to express y' in terms of x.

We also remark that it is possible that the value of y isn't determined uniquely from the value of x. E.g., for $x^2 + y^2 - 1 = 0$ for any x with $-1 < x < 1$ there are two possible values of y: $y = \sqrt{1 - x^2}$ and $y = -\sqrt{1 - x^2}$.

Problems to practice:

- preexam 1b, problem 4;
- preexam 1a, problem 6 (related to tangent lines);
- Suppl. notes 1G;

2. Computation of limits

2.1. Computation of limits without using derivatives.

Limits vs algebra. Limits are "compatible" with algebraic operations. That is, if $\lim_{x\to x_0} f(x) = A, \lim_{x\to x_0} g(x) = B$, then

1. $\lim_{x\to x_0} f(x) + g(x) = A + B$.
2. $\lim_{x\to x_0} f(x)g(x) = AB$.
3. $\lim_{x\to x_0} \frac{f(x)}{g(x)} = \frac{A}{B}$ provided $B \neq 0$.

The computation of limits $\lim_{x\to x_0} \frac{f(x)}{g(x)}$ with $A = B = 0$ requires additional considerations, for example, one can try to relate such limits to derivatives.

Trigonometric limits.

The most basic limit here: $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$. Most trigonometric limits are deduced from this one with the possible exceptions of limits related to derivatives.

Problems to practice (all limits without derivatives)

- Preexam 1a, problem 6a.
- Preexam 1b, problem 1b.
- Suppl. notes 1D.

2.2. Limits vs derivatives. Limits vs derivatives.

Here two types of problems are possible:

Limits via derivatives. Uses the definition of the derivative: $f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h}$.

Problems to practice:

- Preexam 1a, problem 3.
- Preexam 1b, problem 6b.

Derivatives via limits. These are problems asking to compute the derivative directly from definition.

Problems to practice:

- Preexam 1a, problem 2.
- Preexam 1b, problem 3.
- Wsh 3, problem 1.

3. Functions and their graphs

3.1. Tangent lines.

Case 1. Function is given explicitly: $y = f(x)$. Then the equation of the tangent line to the point $(x_0, y_0), y_0 = f(x_0)$ (we remark that a point has to lie on the curve) is:

$$y = f'(x_0)(x - x_0) + y_0 = f'(x_0)x + (y_0 - x_0 f(x_0)).$$

Case 2. A function $y = y(x)$ is given implicitly by the equation $F(x, y) = 0$ (e.g., $x^2 + y^2 - 1 = 0$). Then differentiate equation to to find the derivative $y'(x_0)$ at the point

(x_0, y_0) of the graph (this derivative can also depend on y_0 but we do not write $y'(x_0, y_0)$ to make the notation simpler). Then the equation of the tangent line is

$$y = y'(x_0)(x - x_0) + y_0 = y'(x_0)x + (y_0 - y'(x_0)x).$$

- Preexam 1a, problems 6,7.
- Preexam 1b, problem 7.
- PSet 1, part II, problem 3.

3.2. Differentiable and continuous functions.

Functions given piecewise. We consider a function $f(x)$ given by $f(x) = \begin{cases} f_1(x), & x < x_0 \\ f_2(x), & x \geqslant x_0 \end{cases}$, where $f_1(x), f_2(x)$ are some differentiable functions. Then

(1) $f(x)$ is continuous if $f_1(x_0) = f_2(x_0)$.
(2) $f(x)$ is differentiable if it is continuous (i.e., $f_1(x_0) = f_2(x_0)$) and $f_1'(x_0) = f_2(x_0)$.

Often, $f_1(x)$ is fixed, and $f_2(x)$ depends on two parameters (say a, b), or vice versa. Then (1) typically leads to one equation on a, b, and (2) leads to two equations.

Problems to practice:

- Preexam 1a, problem 5.
- Preexam 1b, problem 5.

Continuouty/discontinuouty

Here problems ask to check whether given functions are continuous (or differentiable) and if not, where the points of discontinuouty are.

Problems to practice: Preexam 1a, problem 9.

3.3. Drawing graphs. *Problems to practice:*

- Suppl. notes 1D-4 (just some graphing).
- Wsh 1, problem 6 (graphs of inverses).

18.01 EXAM I

Friday, September 25, 2009

Name: Michael Plasmeier

E-mail: theplaz@mit.edu

Recitation Instructor: Briner

Recitation Hour: R05 ?

Instructions: You may not use calculators, notes, textbooks, or personal electronic devices. As a courtesy to other students, please turn off all cell phones. Read each question carefully. Whenever possible, include justification for your reasoning and show your work. Answers without any work or explanation will receive little or no credit. There are 6 questions and a 55 minute time limit on this exam. Good luck.

Issues

plug in point if have it
do product rule
$\frac{d}{dx} \ln(x) = \frac{1}{x} x'$

Question	Score	Maximum
1	3	8
2	2	5
3	0	5
4	2	5
5	0	5
6	2	6
Total	9	34

 Name: Michael Plasmeier

1. Compute the following derivatives. (Simplify your answers when possible.)

(a) $f'(x)$ where $f(x) = \dfrac{x}{1-x^2}$

$$\frac{1(1-x^2) - (-2x \cdot x)}{(1-x^2)^2} \to \frac{1-x^2+2x^2}{(1-x^2)^2} \to$$

think I got it, reduction more

$$\frac{1-x^2}{(1-x^2)^2} + \boxed{\cancel{\frac{2x}{(1-x^2)^2} \to \frac{1}{(1-x^2)} + \frac{2x}{(1-x^2)^2}}}$$

+2/2

? done $\to$ $\dfrac{1-x^2}{(1-x^2)^2}$

(b) $f'(x)$ where $f(x) = \ln(\cos x) - \dfrac{1}{2}\sin^2(x)$

$$\frac{d}{dx} = \ln(\cos x) \cdot -\sin x - \frac{1}{2} \cdot 2\sin(x) \cdot \cos x$$

$$-\cancel{\sin x}\ \ln(\cos x) - \sin x \cos x$$

?, more w/ the ln

did not know $\to$ $\dfrac{1}{x}\ x'$

$$\frac{1}{\cos x} - \sin x - \frac{1}{2} \cdot 2 \sin x \cos x \quad 1/3$$

$$\boxed{-\tan x - \sin x \cos x}$$

start

(c) $f^{(5)}(x)$, the fifth derivative of f, where $f(x) = xe^x$ $\boxed{\text{product rule}}$ $1 \cdot e^x + e^x \cdot x$

$f' = \cancel{x + x e^x = x^2 e^x}$

$f'' = \cancel{x^2 + x e^x = x^3 e^x}$

$f''' = \cancel{3x^3 + x e^x = x^4 e^x}$

$f'''' = \cancel{4x^4 + x e^x = x^5 e^x}$

$f^{(5)} = \cancel{5x^5 + x e^x = x^6 e^x}$

what was I thinking? ? duh

no

① $e^x + xe^x = e^x + xe^x$

② $e^x + e^x + xe^x = 2e^x + xe^x$

expands to

0/3 this always expands each time

③ $2e^x + e^x + xe^x = 3e^x + xe^x$

④ $4e^x + xe^x$

⑤ $\boxed{5e^x + xe^x}$

think got it

go slow + do it right

+ otherwise product rule

 Name: Plasmeier

2. Find the equation of the tangent line to the "astroid" curve defined implicitly by the equation

$$x^{2/3} + y^{2/3} = 4$$

don't distribute ~~$(x^{2/3}+y^{2/3})^3 = 4^3$~~ ← wrong

don't solve for y

at the point $(-\sqrt{27}, 1)$.

Could check to make sure on curve

$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}y' = 0$

$\frac{2}{3}y^{-1/3}y' = -\frac{2}{3}x^{-1/3}$

$y' = \dfrac{-\frac{2}{3}x^{-1/3}}{\frac{2}{3}y^{-1/3}} =$

$\frac{2}{3}\cdot\frac{3}{2} = \frac{6}{6} = 1$

$\frac{1}{2}\cdot\frac{5}{5} = \frac{5}{10}$ ✓

think got it, not sure mistake

$y' = \dfrac{-\sqrt[3]{y}}{\sqrt[3]{x}}$ ~~$\frac{\text{cube } y}{\text{cube } x}$~~ illegal

plug in have point

$y - 1 = -\dfrac{\sqrt{y}}{\sqrt{x}}(x - -\sqrt{27})$

$y = -\dfrac{\sqrt{y}}{\sqrt{x}}(x + \sqrt{27}) + 1$

which is?

-1

$\dfrac{-3^{3/2(-1/2)}}{1}$

$= \dfrac{3^{-1/2}}{1} = \dfrac{1}{\sqrt{3}}$

$\frac{1}{\sqrt{3}}(x + \sqrt{27}) + 1$

$\boxed{\frac{1}{\sqrt{3}}x + 4}$

$\frac{1}{\sqrt{3}}\cdot\sqrt{27} + 1$

$\frac{1}{\sqrt{3}}\cdot\sqrt{3\cdot 9} + 1$

$\frac{1}{\sqrt{3}}\cdot 3\sqrt{3} + 1$

$3 + 1$

4

 Name: Plasmeier

3. A particle is moving along a vertical axis so that its position y (in meters) at time t (in seconds) is given by the equation

$$y(t) = t^3 - 3t + 3, \quad t \geq 0.$$

Determine the total distance traveled by the particle in the first three seconds.

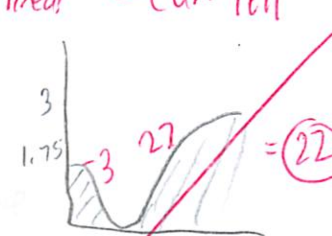
not / linear - can tell if velocity ever = 0 - does at 1

$y(3) = 3^3 - 3(3) + 3$

← can't even cube

$27 - 9 + 3$

21



$= \boxed{22}$

? integration ??

$y(2) = 2^2 - 3(2) + 3$

$4 - 6 + 3$

1

oh was close - I made a mistake cubing

$y(1) = 1^2 - 3(1) + 3$

$1 - 3 + 3$

0

$y(0) = 3$

$+|3 - 0| + |0 - 21| = 24m$

sec 0→1 sec 1→3

22m

$y(.5) = .125 - 1.5 + 3$

$3.125 - 1.5$

1.75

$\frac{d}{dx} = 3t^2 - 3 = v$

$a = 6t$

$\int = \frac{t^4}{4} - \frac{3t^2}{2} + 3t + c$

$\frac{3^4}{4} - \frac{3(3)^2}{2} + 3(3)$

$20.25 + 40.5 + 9$

$69.75\,m$

0.

 Name: Plaseiol

4. State the product rule for the derivative of a pair of differentiable functions f and g using your favorite notation. Then use the DEFINITION of the derivative to prove the product rule. Briefly justify your reasoning at each step.

$$\frac{d}{dx}\left(f(x)\cdot g(x)\right) = \left(\frac{d}{dx} f(x)\right) g(x) + \left(\frac{d}{dx} g(x)\right)\cdot f(x)$$

No, want $(fg)'$ Opps

$$\cancel{f'g'}\ (fg)' = f' \cdot g + g' \cdot f$$

$$\lim_{h\to 0} \frac{f(x+h)\,g(x+h) - f(x)\,g(x)}{h}$$ use limit formula ✓

!

$$\cancel{\frac{f(x+h)^3 + f(x+h)^2 g(x+h) + f(x+h) g(x+h)^2 + g(x+h)^3 / f(x)^2 + 2f(x)g(x) + g(x)^2}{h}}$$ expand

$$\cancel{\boxed{f(x+\cancel{h})\,g(x) + g(x+\cancel{h})\,f(x)}}$$ cancle out

? think remember start not ...

$$\lim_{h\to 0} f(x+h)\,g(x+h) - f(x)\,g(x+h) + f(x)g(x+h) - f(x)\,g(x)$$

$$\lim_{h\to 0}\left(g(x+h)\,\frac{f(x+h)-f(x)}{h} + f(x)\,\frac{g(x+h)-g(x)}{h}\right)$$

g(x) differentiable thus contineous $g(x+h) = g(x)$

$$g(x)\,f'(x) + f(x)\,g'(x)$$

2/5

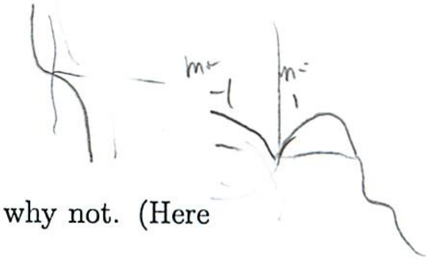
$$\tan = \frac{\sin}{\cos} \quad \frac{\frac{1}{\sqrt{x-1}}}{-\frac{1}{\sqrt{x-1}}} = -1$$

 Name: Plasmeier

0/5

5. Does there exist a set of real numbers a, b and c for which the function

$$f(x) = \begin{cases} \tan^{-1}(x) & x \le 0 \\ ax^2 + bx + c, & 0 < x < 2 \\ x^3 - \frac{1}{4}x^2 + 5, & x \ge 2 \end{cases}$$



is differentiable (i.e. everywhere differentiable)? Explain why or why not. (Here $\tan^{-1}(x)$ denotes the inverse of the tangent function.)

- must be contineous first of all to be differentiable

$\tan^{-1}(0) =$

what angle is tangent 0 $= 0°$

$a(0)^2 + b(0) + c = 0$ must be 0

$\boxed{c = 0}$

$2^3 - \frac{1}{4} 2^2 + 5 =$

dh → $8 - 1 + 5 = 12$ care = here

$a(2)^2 + b(2) + 0 = 12$ not the x value the f(x) value

$4a + 2b = 12$

$2a + b = 6$

derivities must =

~~$\frac{d}{dx} \tan^{-1}(x) = \frac{1}{x-1} = -1$~~

~~$\frac{1}{3} \cdot 2(0) + \frac{1}{3} = 0$~~

~~$\frac{2}{3} + \frac{1}{3} \ne -1$~~

Do not match up - not differentiable at $x=0$

$\frac{d}{dx} \tan^{-1} x \Big|_0 = \frac{d}{dx} ax^2 + bx$

$\boxed{1 = b}$

Must check both to make sure b + a are =

$\frac{d}{dx} ax^2 + bx + c \Big|_2 = \frac{d}{dx} x^3 - \frac{1}{4}x^2 + 5 \Big|_2$

$4a + b = 11$

$\boxed{a = \frac{5}{2}}$

??

Wilda guessed at end

tan⁻¹ did not remember

 Name: Plasmeier

non linear

6. Suppose that f satisfies the equation $f(x+y) = f(x) + f(y) + x^2y + xy^2$ for all real numbers x and y. Suppose further that

$$\lim_{x\to 0} \frac{f(x)}{x} = 1.$$

(a) Find $f(0)$.

$\cancel{f(x)} = f(x) + f(0) + x^2 \cdot 0 + x \cdot 0^2$

$f(0+0) = f(0) + f(0) + 0^2 \cdot 0 + 0 \cdot 0^2$

0 $+2$ score

Since $x \to 0$ and $\lim_{x\to 0} \frac{f(x)}{x} = 1$

$f(x) \to 0$ as $x \to 0$

(b) Find $f'(0)$.

if $f(x) = 0$

$\lim_{x\to 0} \frac{f(x)}{x} =$

$\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = f'(0)$

def of derivitive

$\cancel{f'(0) + f'(0) + 2xy\,y' + x\,2y\,y'}$

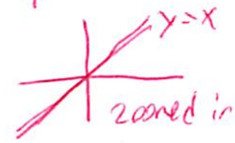
$\cancel{0 + 0 + 2\cdot 0\cdot 0 + 0 \cdot 2 \cdot 0 \cdot 0}$

0 0 score

$f'(x) \to x' = 1$ as $x \to 0$

So $f'(0) = 1$

approaches 0 as $y = x$



(c) Find $f'(x)$.

$\cancel{f'(x) + f'(y) + 2xy + x\,2y\,y'}$

$y' \cancel{= \frac{-f'(x) - f'(y) + 2xy}{2xy}}$

$\boxed{y' = \frac{-f'(x)}{2xy} - \frac{f'(y)}{2xy} + 1}$

$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

rule #1

$\lim_{h\to 0} \frac{f(x) + f(h) + x^2h + xh^2 - f(x)}{h}$

$\lim_{h\to 0} \left(\frac{f(h)}{h} + x^2 + xh\right)$

$1 + x^2 + 0$

$\boxed{x^2 + 1}$

0 score

all the other ones I knew how to start must be the "fun" problem

18.01 EXAM I

Friday, September 25, 2009

Name: OS.

E-mail:

Recitation Instructor:

Recitation Hour:

Instructions: You may not use calculators, notes, textbooks, or personal electronic devices. As a courtesy to other students, please turn off all cell phones. Read each question carefully. Whenever possible, include justification for your reasoning and show your work. Answers without any work or explanation will receive little or no credit. There are 6 questions and a 55 minute time limit on this exam. Good luck.

Question	Score	Maximum
1		8
2		5
3		5
4		5
5		5
6		6
Total		34

 Name: ______________________

1. Compute the following derivatives. (Simplify your answers when possible.)

(a) $f'(x)$ where $f(x) = \dfrac{x}{1-x^2}$

$$f'(x) = \frac{1\,(1-x^2) - (x)(-2x)}{(1-x^2)^2} = \frac{1-x^2+2x^2}{(1-x^2)^2} = \frac{1+x^2}{(1-x^2)^2}$$

(b) $f'(x)$ where $f(x) = \ln(\cos x) - \dfrac{1}{2}\sin^2(x)$

$\frac{1}{u} \cdot u'$ standard

$$f'(x) = \frac{1}{\cos x}(-\sin x) - \frac{1}{2} \cdot 2 \sin x \cos x =$$

$$= \boxed{-\tan x - \sin x \cos x}$$

or

$$-\sin x \left(\frac{1+\cos^2 x}{\cos x}\right)$$

easy in retrospect

(c) $f^{(5)}(x)$, the fifth derivative of f, where $f(x) = xe^x$

$$f'(x) = e^x + xe^x = 1 \cdot e^x + xe^x$$

$$f^{(2)}(x) = e^x + e^x + xe^x = 2e^x + xe^x$$

$$f^{(3)}(x) = 2e^x + e^x + xe^x = 3e^x + xe^x$$

$$f^{(4)}(x) = 4e^x + xe^x$$

$$\boxed{f^{(5)}(x) = 5e^x + xe^x}$$

think got it

induction argument for general case:

$$\left(f^{(k)}(x) = ke^x + xe^x \Rightarrow f^{(k+1)}(x) = ke^x + e^x + xe^x = (k+1)e^x + xe^x\right)$$

$\frac{d}{dx}\ \cancel{x}\ln x = \frac{1}{x} x'$

$1 \cancel{e^x + e^x}$.

$1 \cdot \cancel{e^x + e^x e^x}$

1 $1 \cdot e^x + e^x x$

2 $e^x + e^x + xe^x$

3 $e^x + e^x + e^x + xe^x$

$3e^x + xe^x$

$5e^x + xe^x$ slow + do it right

$\frac{-\sqrt[3]{1}}{+\sqrt[3]{27}}$ $\frac{-1}{\sqrt{3}}$ ← still

$-\frac{1}{\sqrt{3}}(x+\sqrt{27})$

$-\frac{1}{\sqrt{3}}x + 3 + 1$

$\frac{1}{\sqrt{3} \cdot \sqrt{27}}$ ← would not have known that

$\sqrt{3} \cdot 3 \cdot \sqrt{3}$

$3 \cdot 3 + 1$

4

$\sqrt{3} \cdot \sqrt{3} = 3$

 Name: ______________________

2. Find the equation of the tangent line to the "astroid" curve defined implicitly by the equation

$$x^{2/3} + y^{2/3} = 4$$

at the point $(-\sqrt{27}, 1)$.

Check point is on curve:

why?

$$-\sqrt{27} = -3^{3/2}$$

$$(-3^{3/2})^{2/3} + (1)^{2/3} = +3 + 1 = 4 \,//.$$

✓

Use implicit differentiation to get $dy/dx \Big|_{(-\sqrt{27},\,1)}$

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x^{-1/3}}{y^{-1/3}} = -\frac{-3^{3/2(-1/3)}}{1} = \frac{+3^{-1/2}}{1} = \frac{1}{\sqrt{3}}.$$

$$y-1 = \frac{1}{\sqrt{3}}\left(x+\sqrt{27}\right)$$

$$y = \frac{1}{\sqrt{3}}x + \overset{3}{\sqrt{9}} + 1$$

Seems straight forward

$$\boxed{y = \frac{1}{\sqrt{3}}x + 4}$$

 Name: ______________________

3. A particle is moving along a vertical axis so that its position y (in meters) at time t (in seconds) is given by the equation

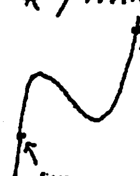
$$y(t) = t^3 - 3t + 3, \quad t \geq 0.$$

Determine the total distance traveled by the particle in the first three seconds.

Compute total dist up and down y-axis
(i.e. total dist, projected onto y-axis).

For this need to find max/min height

(cubics look like this:



, so would want $\overline{I} + I + I$)

(start may be elsewhere — we don't know yet)

So look for min/max.
→ set $f'(x) = 0$.
$y'(t)$

$y'(t) = 3t^2 - 3 (=0) \Rightarrow \quad 3t^2 - 3 = 0 \quad \Rightarrow \quad t^2 - 1 = 0 \quad t^2 = 1$

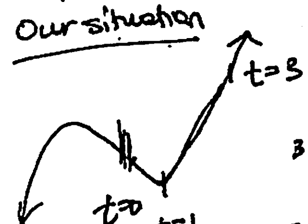
$\boxed{t = \pm 1}$

$y''(t) = 6t$ (2nd der test)

$t = +1$ – max
$t = -1$ – min

so for us, start t is after max but before min.

Our situation



$3 \; I + I \; 21$

$= 2 + 20 = 20$

$y(0) = 3$

$y(1) = 1 - 3 + 3 = 1$ (min)

$y(3) = 3^3 - 9 + 3 = 27 + 3 - 9 = 30 - 9 = 21$

Total dist $= (3 - 1) + (21 - 1) = 2 + 20 = \boxed{22}$

4. State the product rule for the derivative of a pair of differentiable functions f and g using your favorite notation. Then use the DEFINITION of the derivative to prove the product rule. Briefly justify your reasoning at each step.

If f, g are both diff. functions of x,

$$(f \cdot g)'(x) = f'(x)\, g(x) + f(x)\, g'(x) \,..$$

$$(f \cdot g)'(x) = \lim_{h \to 0} \frac{f(x+h)\, g(x+h) - f(x)\, g(x)}{h} =$$

$$= \lim_{h \to 0} \frac{f(x+h)\, g(x+h) - f(x)\, g(x+h) + f(x)\, g(x+h) - f(x)\, g(x)}{h} =$$

$$= \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h}\, g(x+h) + f(x)\, \frac{g(x+h) - g(x)}{h} \right)$$

(f, g are differentiable $\Rightarrow$ continuous $\Rightarrow \lim_{h \to 0} g(x+h) = g(x) \ldots$)

$$= f'(x)\, g(x) + f(x)\, g'(x) \quad //.$$

 Name: ______________________________

5. Does there exist a set of real numbers a, b and c for which the function

$$f(x) = \begin{cases} \tan^{-1}(x) & x \le 0 \\ ax^2 + bx + c, & 0 < x < 2 \\ x^3 - \frac{1}{4}x^2 + 5, & x \ge 2 \end{cases}$$

is differentiable (i.e. everywhere differentiable)? Explain why or why not. (Here $\tan^{-1}(x)$ denotes the inverse of the tangent function.)

First do continuity

$\tan^{-1}(0) = 0 \quad (\tan 0 = \frac{0}{1} = 0). \quad \Rightarrow c = 0.$

@2, $\quad 2^3 - \frac{1}{4}2^2 + 5 = 8 - 1 + 5 = 7 + 5 = 12.$

$\Rightarrow a2^2 + b \cdot 2 = 12 \Rightarrow \underline{2a + b = 6}$

$b = 6 - 2a$ //.

Finally, differentiability (want slopes to match @ $x = 0, 2$).

$f'(x)\Big|_{x=2} = 3x^2 - \frac{1}{2}x\Big|_{x=2} = 3 \cdot 4 - \frac{1}{2} \cdot 2 = 12 - 1 = 11$ //.

So want $\quad 2ax + b\Big|_{x=2} = 2a \cdot 2 + b = 4a + b = 11.$

$b = 6 - 2a$

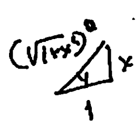
$4a + b = 11$
$2a + b = 6$ $\quad \Rightarrow 2a = 11 - 6 = 5$

$a = 5/2$

$b = 6 - 5 = 1.$

What is derivative of $\quad \frac{5}{2}x^2 + x$

$\tan^{-1}x$? $\quad \frac{1}{x^2+1}$ (also could do trick $\tan y = x$ and take derivative to remember)



$a = 5/2,\ b = 1,\ c = 0$
are our only option so far.
See if it work for diff. @ $x = 0$.

$\tan y = x$

$\frac{-\cos x(\ldots) + \sin^2 x}{\cos^2 y}$

$= \frac{1}{\cos^2 y} = \sec^2 y$

$\sec^2 y \frac{dy}{dx} = 1 \quad \Rightarrow \frac{dy}{dx} = \frac{1}{\sec^2 y} = \cos^2 y$

$= \frac{1}{1+x^2}$

$\frac{dy}{dx}\Big|_{x=0}$ should be $\frac{1}{1+0} = 1.$

$2ax + b\,\Big|_{x=0} = b = 1$ ✓ great!

$\boxed{a = 5/2,\ b = 1,\ c = 0}$

 Name: ____________________

6. Suppose that f satisfies the equation $f(x+y) = f(x) + f(y) + x^2y + xy^2$ for all real numbers x and y. Suppose further that

$$\lim_{x \to 0} \frac{f(x)}{x} = 1.$$

(a) Find $f(0)$.

Since $X \to 0$ and $\lim_{X \to 0} \frac{f(X)}{X} = 1$, $f(x) \to x \Rightarrow 0$. as $x \to 0$. ...

$\Rightarrow \boxed{f(0) = 0}$.

(b) Find $f'(0)$.

$f'(x) \to x' = 1$ as $x \to 0$, so $\boxed{f'(0) = 1.}$

(c) Find $f'(x)$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\cancel{f(x)} + f(h) + x^2h + xh^2 - \cancel{f(x)}}{h}$$

$$= \lim_{h \to 0} \left(\frac{f(h)}{h} + x^2 + xh \right)$$

$$= 1 + x^2 + 0 = \boxed{x^2 + 1} \; //.$$

Lecture 9

Unit 2

Applications of Derivitive

9/29

done

-analized definition $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

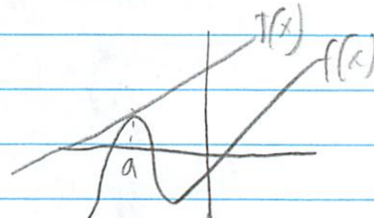
tangent line: to $f(x)$ at $x = a$

$$y = f'(x)(x-a) + f(a)$$

$$T(x) = f(a) + f'(a)(x-a)$$

$T(x)$ is a good approximation to $f(x)$

-only near $x = a$



decent approx of tangent at a

eg Good linear approx to $\sqrt{x}$ at $x = a$

tangent line at a

$$T(x) = \sqrt{a} + \frac{1}{2\sqrt{a}}(x-a)$$

Suppose wanted to compute approx value for $\sqrt{4.05}$

-do tangent at $x = 4 \to \sqrt{4} \to 2$

$$T(x) = 2 + \frac{1}{4}(x-4)$$

good approx of $\sqrt{x}$ near $x = 4$

$f(4.05) \approx f(4.05) = \sqrt{4.05}$
$= 2 + \frac{1}{4}(.05)$

2 questions
how far off of right answer?
↑ answered final week of semester
how can we do better
↑ will answer today

Often easier to do tangent approx at $x=0$

know library of linear approx (tangent lines) at $x=0$

$e^x \approx 1 + \frac{d}{dx_0}\ x-0$ at $x=0$
value there
$1 + x$

$\sin x = 0 + 1(x-0) = x$
↑ value there ↑ value at deriv

$\cos x = 1 + 0(x-0) = 1$

$\ln x =$ can't do not differentable at 0
$\ln(1+x) = \ln(1+x) = 0 + 1(x-0) = x$

$(1+x)^r = 1 + rx$

Trick for harder functions
$\frac{e^{-2x}}{\sqrt{1+x}} \rightarrow$ find $\frac{d}{dx}\Big|_{0=x} = -\frac{5}{12}$
$f(0)=1$

$\approx T(x) = 1 - \frac{5}{12}x$

Tricky way - linear approx of pieces of function

$$f(x) = \frac{e^{-2x}}{\sqrt{1+x}} \approx \frac{1-2x}{1+\frac{1}{2}x} \approx (1-2x) \cdot \overbrace{\text{linear approx } \frac{1}{1+\frac{1}{2}x}}^{1-\frac{1}{2}x}$$

$$= \left(1+\tfrac{1}{2}x\right)^{-1}$$

$$\approx$$

$$\frac{d}{dx}\left(1+\tfrac{1}{2}x\right)^{-1}\Big|_{x=0}$$

$$= -\tfrac{1}{2} \text{ at } x=0$$

$$= \left(1-\tfrac{1}{2}x\right)$$

$$(1-2x)\left(1-\tfrac{1}{2}x\right)$$

$$1-\tfrac{5}{2}x+x^2$$

$$\approx 1-\tfrac{5}{2}x$$

↑1 more linear approx

Can linearize all over the place

* subbing in tangent line

- want to approx tangent line

L_a: act of linearizing at $x=a$

show $L_a\left(\frac{f}{g}\right) = L_a\left(\frac{L_a(f)}{L_a(g)}\right)$

prove that know;

$$L_a\left(f(x)\right)\Big|_{x=a} = f(a)$$

$$L_a\left(f(x)\right)'\Big|_{x=a} = f'(a)$$

the uglier the function is the more you should linearize 1st and take deriv later

---

How we can do better

- linear approx is good because f and its tangent line have same values at $x=a$ same first deriv at $x=a$

- do better w/ quadratics
  $a + bx + cx^2$
  3 choices of constant
  - value - 1st deriv - 2nd deriv

$$f(x) \approx f(a) + f'(a)(x-a) + c(x-a)^2$$

Solve for c by taking 2 derivs on each side

$$f''(a) = \frac{d^2}{dx^2}\left(c \cdot (x-a)^2\right)\Big|_{x=a} = \frac{d}{dx}\left(2c(x-a)\right)\Big|_{x=a} = \boxed{2c}$$

Choose c to $= \frac{f''(a)}{2}$

Conclude $f(x) \approx f(a) + f'(x)(x-a) + \frac{f''(a)}{2}(x-a)$

best quadratic approx when near $x=a$

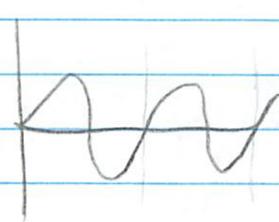
if get best parabola - would be better

* if keep doing that could find function *
- cubic
- 4 th degree
- nth degree

$$f(x) = f(a) + f'(a)(x-a) + f''(a)(x-a)^2 + \frac{f^{(3)}}{3!}(a)(x-a)^3 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

↑ only using derivitives at a

sin(x)



keeps ossilating
cubic polynomial = 3 humps
n-1 humps
so can never perfectly model it

$n \to \infty$ can do
- last week of class

lim polynomials → sin(x)

Lecture 9: Linear and Quadratic Approximations

Unit 2: Applications of Differentiation

Today, we'll be using differentiation to make approximations.

Linear Approximation

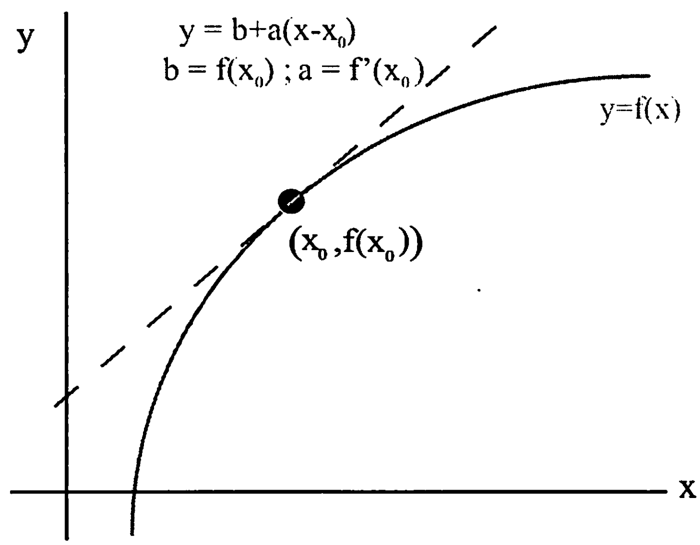


Figure 1: Tangent as a linear approximation to a curve

The tangent line approximates $f(x)$. It gives a good approximation near the tangent point x_0. As you move away from x_0, however, the approximation grows less accurate.

$$\boxed{f(x) \approx f(x_0) + f'(x_0)(x - x_0)}$$ linear

Example 1. $f(x) = \ln x, \quad x_0 = 1 \quad$ (basepoint)

$$\begin{aligned} f(1) &= \ln 1 = 0; \qquad f'(1) = \left.\frac{1}{x}\right|_{x=1} = 1 \\ \ln x &\approx f(1) + f'(1)(x-1) = 0 + 1 \cdot (x-1) = x - 1 \end{aligned}$$

Change the basepoint:

$$x = 1 + u \quad \Longrightarrow \quad u = x - 1$$
$$\ln(1+u) \approx u$$

Basepoint $u_0 = x_0 - 1 = 0$.

Basic list of linear approximations

In this list, we always use base point $x_0 = 0$ and assume that $|x| << 1$.

1. $\sin x \approx x$ (if $x \approx 0$) (see part a of Fig. 2)
2. $\cos x \approx 1$ (if $x \approx 0$) (see part b of Fig. 2)
3. $e^x \approx 1 + x$ (if $x \approx 0$)
4. $\ln(1+x) \approx x$ (if $x \approx 0$)
5. $(1+x)^r \approx 1 + rx$ (if $x \approx 0$)

Proofs

Proof of 1: Take $f(x) = \sin x$, then $f'(x) = \cos x$ and $f(0) = 0$

$$f'(0) = 1, f(x) \approx f(0) + f'(0)(x - 0) = 0 + 1.x$$

So using basepoint $x_0 = 0, f(x) = x$. (The proofs of 2, 3 are similar. We already proved 4 above.)

Proof of 5:

$$\begin{aligned} f(x) &= (1+x)^r; \qquad f(0) = 1 \\ f'(0) &= \frac{d}{dx}(1+x)^r|_{x=0} = r(1+x)^{r-1}|_{x=0} = r \\ f(x) &= f(0) + f'(0)x = 1 + rx \end{aligned}$$

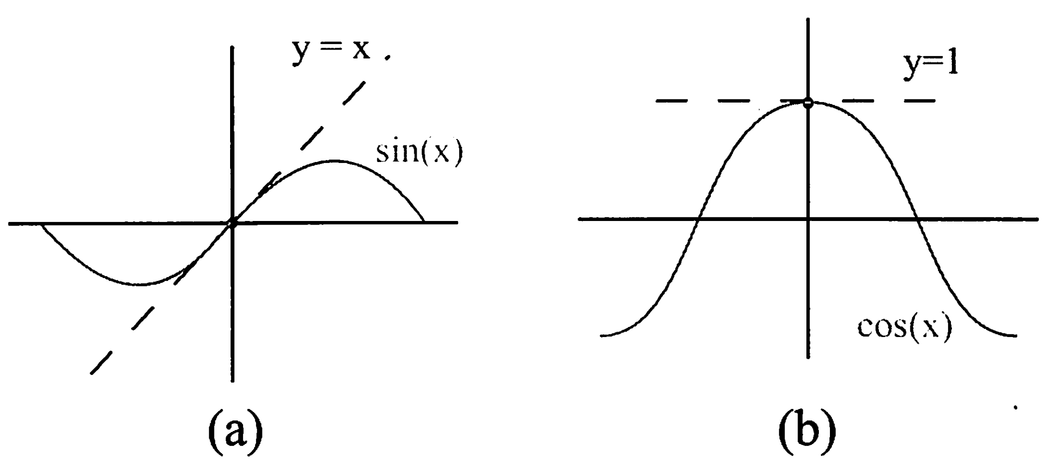


Figure 2: Linear approximation to (a) $\sin x$ (on left) and (b) $\cos x$ (on right). To find them, apply $f(x) \approx f(x_0) + f'(x_0)(x - x_0)$ $(x_0 = 0)$

Example 2. Find the linear approximation of $f(x) = \dfrac{e^{-2x}}{\sqrt{1+x}}$ near $x = 0$.
We could calculate $f'(x)$ and find $f'(0)$. But instead, we will do this by combining basic approximations algebraically.

$$e^{-2x} \approx 1 + (-2x) \qquad (e^u \approx 1 + u, \text{ where } u = -2x)$$

$$\sqrt{1+x} = (1+x)^{1/2} \approx 1 + \frac{1}{2}x$$

Put these two approximations together to get

$$\frac{e^{-2x}}{\sqrt{1+x}} \approx \frac{1-2x}{1+\frac{1}{2}x} \approx (1-2x)(1+\frac{1}{2}x)^{-1}$$

Moreover $(1+\frac{1}{2}x)^{-1} \approx 1 - \frac{1}{2}x$ (using $(1+u)^{-1} \approx 1-u$ with $u = x/2$). Thus [1]

$$\frac{e^{-2x}}{\sqrt{1+x}} \approx (1-2x)(1-\frac{1}{2}x) = 1 - 2x - \frac{1}{2}x + 2(\frac{1}{2})x^2$$

Now, we discard that last x^2 term, because we've already thrown out a number of other x^2 (and higher order) terms in making these approximations. Remember, we're assuming that $|x| << 1$. This means that x^2 is very small, x^3 is even smaller, etc. We can ignore these higher-order terms, because they are very, very small. This yields

$$\frac{e^{-2x}}{\sqrt{1+x}} \approx 1 - 2x - \frac{1}{2}x = 1 - \frac{5}{2}x$$

Because $f(x) \approx 1 - \frac{5}{2}x$, we can deduce $f(0) = 1$ and $f'(0) = \frac{-5}{2}$ directly from our linear approximation, which is quicker in this case than calculating $f'(x)$.

Example 3. $f(x) = (1+2x)^{10}$.

On the first exam, you were asked to calculate $\lim_{x\to 0} \frac{(1+2x)^{10}-1}{x}$. The quickest way to do this with the tools of Unit 1 is as follows.

$$\lim_{x\to 0} \frac{(1+2x)^{10}-1}{x} = \lim_{x\to 0} \frac{f(x)-f(0)}{x} = f'(0) = 20$$

(since $f'(x) = 10(1+2x)^9 \cdot 2 = 20$ at $x = 0$)

Now we can do the same problem a different way, namely, using linear approximation.

$$(1+2x)^{10} \approx 1 + 10(2x) \text{ (Use } (1+u)^r \approx 1 + ru \text{ where } u = 2x \text{ and } r = 10.)$$

Hence,

$$\frac{(1+2x)^{10}-1}{x} \approx \frac{1+20x-1}{x} = 20$$

Example 4: Planet Quirk Let's say I am on Planet Quirk, and that a satellite is whizzing overhead with a velocity v. We want to find the time dilation (a concept from special relativity) that the clock onboard the satellite experiences relative to my wristwatch. We borrow the following equation from special relativity:

$$T' = \frac{T}{\sqrt{1-\frac{v^2}{c^2}}}$$

[1] A shortcut to the two-step process $\frac{1}{\sqrt{1+x}} \approx \frac{1}{1+\frac{x}{2}} \approx 1 - \frac{1}{2}x$ is to write

$$\frac{1}{\sqrt{1+x}} = (1+x)^{-1/2} \approx 1 - \frac{1}{2}x$$

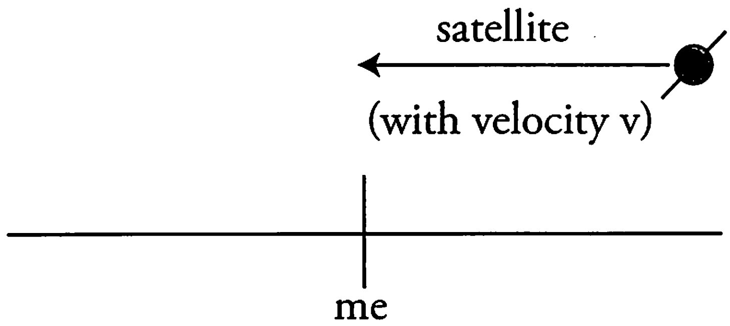


Figure 3: Illustration of Example 4: a satellite with velocity v speeding past "me" on planet Quirk.

Here, T' is the time I measure on my wristwatch, and T is the time measured onboard the satellite.

$$T' = T\left(1 - \frac{v^2}{c^2}\right)^{-1/2} \approx 1 + \frac{1}{2}\left(\frac{v^2}{c^2}\right) \quad \left[(1+u)^4 \approx 1 + ru, \text{ where } u = -\frac{v^2}{c^2}, r = -\frac{1}{2}\right]$$

If $v = 4$ km/s, and the speed of light (c) is 3×10^5 km/s, $\frac{v^2}{c^2} \approx 10^{-10}$. There's hardly any difference between the times measured on the ground and in the satellite. Nevertheless, engineers used this very approximation (along with several other such approximations) to calibrate the radio transmitters on GPS satellites. (The satellites transmit at a slightly offset frequency.)

Quadratic Approximations

These are more complicated. They are only used when higher accuracy is needed.

$$\boxed{f(x) \approx f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 \quad (x \approx x_0)}$$

Geometric picture: A quadratic approximation gives a best-fit parabola to a function. For example, let's consider $f(x) = \cos(x)$ (see Figure 4). If $x_0 = 0$, then $f(0) = \cos(0) = 1$, and

$$\begin{aligned} f'(x) &= -\sin(x) \implies f'(0) = -\sin(0) = 0 \\ f''(x) &= -\cos(x) \implies f''(0) = -\cos(0) = -1 \\ \cos(x) &\approx 1 + 0 \cdot x - \frac{1}{2}x^2 = 1 - \frac{1}{2}x^2 \end{aligned}$$

You are probably wondering where that $\frac{1}{2}$ in front of the x^2 term comes from. The reason it's there is so that this approximation is *exact* for quadratic functions. For instance, consider

$$f(x) = a + bx + cx^2; \qquad f'(x) = b + 2cx; \qquad f''(x) = 2c.$$

Set the base point $x_0 = 0$. Then,

$$\begin{aligned} f(0) &= a + b \cdot 0 + c \cdot 0^2 \implies a = f(0) \\ f'(0) &= b + 2c \cdot 0 = b \implies b = f'(0) \\ f''(0) &= 2c \implies c = \frac{f''(0)}{2} \end{aligned}$$

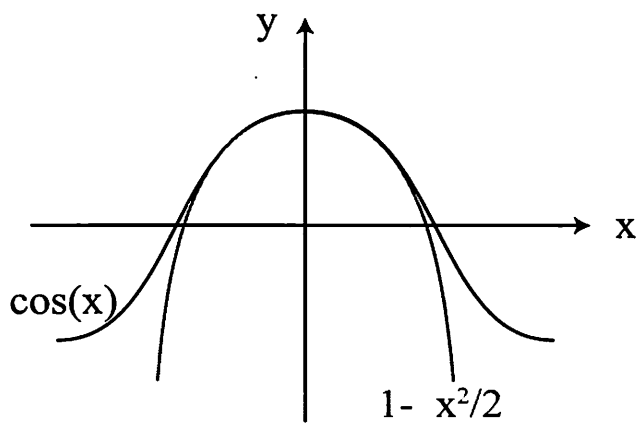


Figure 4: Quadratic approximation to $\cos(x)$.

0.0.1 Basic Quadratic Approximations

:

$$\boxed{f(x) \approx f(0) + f'(0)x + \frac{f''(0)}{2}x^2 \quad (x \approx 0)}$$

1. $\sin x \approx x \quad (\text{if } x \approx 0)$
2. $\cos x \approx 1 - \frac{x^2}{2} \quad (\text{if } x \approx 0)$
3. $e^x \approx 1 + x + \frac{1}{2}x^2 \quad (\text{if } x \approx 0)$
4. $\ln(1+x) \approx x - \frac{1}{2}x^2 \quad (\text{if } x \approx 0)$
5. $(1+x)^r \approx 1 + rx + \frac{r(r-1)}{2}x^2 \quad (\text{if } x \approx 0)$

Proofs: The proof of these is to evaluate $f(0), f'(0), f''(0)$ in each case. We carry out Case 4

$$\begin{aligned}
f(x) &= \ln(1+x) \quad \Longrightarrow \quad f(0) = \ln 1 = 0 \\
f'(x) &= [\ln(1+x)]' = \frac{1}{1+x} \quad \Longrightarrow \quad f'(0) = 1 \\
f''(x) &= \left(\frac{1}{1+x}\right)' = \frac{-1}{(1+x)^2} \quad \Longrightarrow \quad f''(0) = -1
\end{aligned}$$

Let us apply a quadratic approximation to our Planet Quirk example and see where it gives.

$$\left[1 - \frac{v^2}{c^2}\right]^{-1/2} \approx 1 + \frac{1}{2}\frac{v^2}{c^2} + \left[\frac{(\frac{-1}{2})(\frac{-1}{2}-1)}{2}\left(-\frac{v^2}{c^2}\right)^2\right] \qquad \text{Case 5 with } x = \frac{-v^2}{c^2}, r = -\frac{1}{2}$$

Since $\frac{v^2}{c^2} \approx 10^{-10}$, that last term will be of the order $\left(\frac{v^2}{c^2}\right)^2 \approx 10^{-20}$. Not even the best atomic clocks can measure time with this level of precision. Since the quadratic term is so small, we might as well ignore it and stick to the linear approximation in this case.

Example 5. $f(x) = \frac{e^{-2x}}{\sqrt{1+x}}$

Let us find the quadratic approximation of this expression. We can rewrite it as $f(x) = e^{-2x}(1+x)^{-1/2}$. Using the approximation of each factor gives

$$\begin{aligned} f(x) &\approx \left(1 - 2x + \frac{1}{2}(-2x)^2\right)\left(1 - \frac{1}{2}x + \left(\frac{(-\frac{1}{2})(-\frac{1}{2}-1)}{2}\right)x^2\right) \\ f(x) &\approx 1 - 2x - \frac{1}{2}x + (-2)(-\frac{1}{2})x^2 + 2x^2 + \frac{3}{8}x^2 = 1 - \frac{5}{2}x + \frac{27}{8}x^2 \end{aligned}$$

(Note: we drop the x^3 and higher order terms. This is a quadratic approximation, so we don't care about anything higher than x^2.)

Recitation
Linearization

9/30

= linear approximation

- approx function w/ a line at a point
- using tangent line

for function $f(x)$ at $x=a$

$y - f(a) = f'(a)(x-a)$ ← variable

$\boxed{g(x) = f(a) + f'(a)(x-a)}$ } line

$f(a)$: constant at a pt; $f'(a)$: constant at a pt

$g(a) = f(a)$
$g'(a) = f'(a)$
} agrees w/ constant + linear term

Approx at $x=0$
↑ $a=0$

$\sin(x) \approx \sin(0) + \cos(0)(x-0)$
$0 + 1x$

$\sin(0) \approx x$

$\sin\left(\frac{\pi}{64}\right) \approx \frac{\pi}{64}$ ← using it to find $\frac{\pi}{64}$ approx

$$\cos x \approx \cos(0) + -\sin(0)\,(x-0)$$
$$1 + 0x$$
$$1$$

$\cos 2x \approx 1$) use to eval

$$e^x = e^0 + e^0\,(x-0) \qquad e^0 = 1$$
$$1 + x$$

$\sqrt{e} = e^{1/2} = 1 + \frac{1}{2} = \frac{3}{2}$ ← not a good estimation

---

$$\ln(1+x) = x$$
$$(1+x)^r = 1 + rx$$

$$\frac{1}{(1-x)^3} \approx (1-x)^{-3} \approx 1 + (-3)(-x)$$
$$1 + 3x$$

- could also have started from definition

---

linear approx on products + quotients

1. $\ell(fg) = \ell(\ell(f) \cdot \ell(g))$ ← f + g are functions
2. $\ell\left(\frac{f}{g}\right) = \ell\left[\frac{\ell(f)}{\ell(g)}\right]$ linearize twice

ex $\frac{e^{2x}}{\sin(x+1)} = \ell\left[\frac{1+2x}{x+1}\right] = \ell\left((1+2x)(x+1)^{-1}\right)$

$\quad$ already linear

$\ell\left[(1+2x)(1-x)\right]$ ← still not linear convert

$1 + x - 2x^2$

$1 + 1(x-0)$ ↓ linearize

$1 + x$

equation of tangent line at $x=0$ for original function

---

prove

left: $f(0) \cdot g(0) + (fg)'(0)(x-0)$

$f(0) \cdot g(0) + f'(0)g(0) + f(0)g'(0)$

right: $L((f(0) + f'(0)\overset{(x-0)}{x})(g(0) + g'(0)\overset{(x-0)}{x}))$

$L(f(0)g(0) + f'(0)g(0)x + g'(0)f(0)x + f'(0)g'(0)x^2)$

drop quadratic term
- not linear
- only need first 2 terms

$\underbrace{\text{Constant}_1}_{f(0)} + \underbrace{\text{costant}_2}_{f'(0)} x + c_3 x^2$

not needed

Quadratic approx

find $h(x)$ so that for $f(x)$ at $x=a$

$h(a) = f(a)$

$h'(a) = f'(a)$

$h''(a) = f''(a)$

$h(a) = f(a)$
$\leftarrow$ plug in a for x

$$h(x) = f(a) + \cancel{f'(a)(\underset{0}{x-a})} + \cancel{c(\underset{0}{x-a})^2}$$

$h(x) = f(a)$

$\leftarrow$ not eval at a now
deriv $\rightarrow$ then eval

$$h'(x) = 0 + f'(a) + 2c(x-a)$$

$h'(a) = f'(a)$

$$h''(x) = 0 + 0 + 2c$$

$h''(a) = 2c$

but want $h''(a) = f''(a)$

need $c = \frac{f''(a)}{2}$ $\leftarrow$ constant

$$h(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$$

example $\ln(x+1)$ at $x=0$

linear $\rightarrow$ x (look up)

quad $= x + \frac{1}{x+1} \cdot 1 + -1(x+1)^{-2} \cdot 1$

$x - \frac{1}{2}x^2$

$$\boxed{\begin{array}{l} \left.\frac{1}{1+x}\right|_{x=0} = 1 \\ \left.\frac{-1}{(1+x)^2}\right|_{x=0} = -1 \end{array}}$$

$\ln(1.1) \overset{\text{linear}}{\approx} .1$

$\ln(1.1) \overset{\text{quad}}{\approx} .1 - \frac{.1^2}{2} = .095$

$\ln = .0953$
$\uparrow$ calc

Lecture

10/1

PSet 2 due tomorrow

<u>Last time</u> $f(x) \approx T(x)$ ← tangent line to $f(x)$ at $x=a$

near $x=a$

$= f(a)+f'(a)(x-a)$

$f(a)=T(a)$
$f'(a)=T'(a)$ } reason to expect a good approx near a

$$f(x) \approx Q(x) = f(a)+f'(a)(x-a)+\frac{f''(a)}{2}(x-a)^2$$

$f(a)=Q(a)$
$f'(a)=Q''(a)$

<u>ex</u> $f(x)=\sqrt{x}$ at $x=4=$ "a"

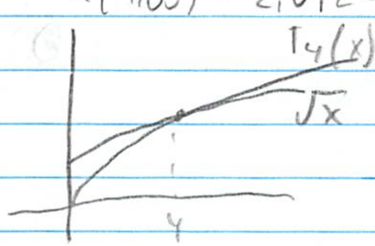
$T_4(x)=2+\frac{1}{4}(x-4)$ ← easy to estimate

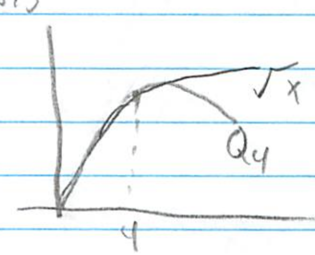
$T_4(4.05)=2.0125$ $f(4.05)=\sqrt{4.05}=2.01246117$

$Q_4(x)=2+\frac{1}{4}(x-4)+\frac{f''(4)}{2}(x-4)^2=$

$= 2+\frac{1}{4}(x-4)\ \frac{1}{64}(x-4)^2$

$Q(4.05)=2.0124609375$ ← what calc does $\sqrt{x}$





$f(16) = \sqrt{16} = 4$

$T_4(16) = 5$

$Q_4(16) = 2.75$ ← get less accurate as move further away

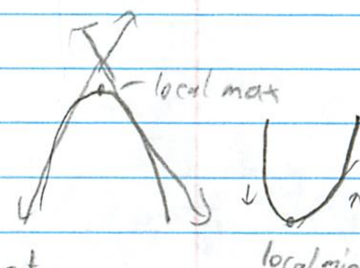
Curve sketching
- using derivitives

$y = f(x)$
- try to sketch

$f'(x) > 0$ slope increasing

$f'(x) < 0$ slope decreasing

$f'(x) = 0$ local max / min
↑ can be or inflection pt



ex $y = 3x - x^3$

$y' = 3 - 3x^2$

$y'' = -6x$ ← can solve for where 0

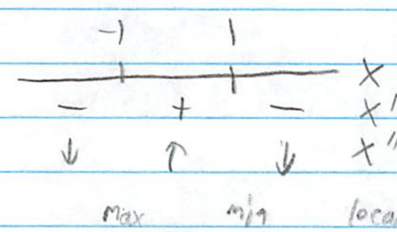
y' $3 - 3x^2 = 0$

$1 - x^2 = 0$

$x = \pm 1$

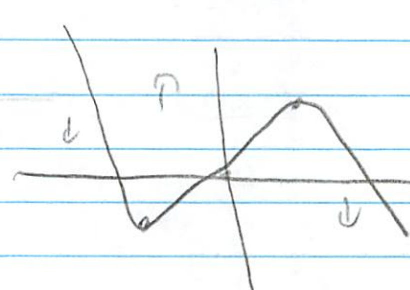
+ −

↑ ↓

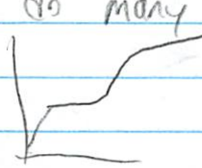


Find zeros

Find some points



function can do many things on an interval



$f(x) = x^4 + 3x^3 + 7x + 2$

$f'(x) = 4x^3 + 9x^2 + 7$

3rd / 4th order solving hard $\quad f'(x) = 0$

5th+ can't do

- but can approximate

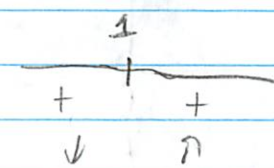
<u>ex2</u>

$y = 3x + 3x^2 + x^3$

$y' = 3 - 6x + 3x^2$

$3(1 - 2x + x^2)$

$3(x-1)^2$



↑ only 1 pt where $\frac{d}{dx} = 0$

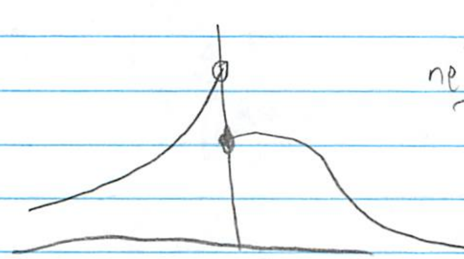
at $x = 1$

so <u>inflection point</u>

<u>ex3</u>

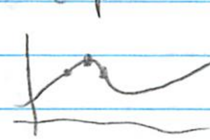
not all functions continuous / differentiable

- annoying extra x values can be local max / local min



<u>neither max nor min</u>

<u>Definition</u> - local max at $x=c$ if $f(x) \leq f(c)$ for all points x in some open interval c



Better strategy for sketching

1. Be aware of places where function may not be differentiable / continuous
   - like abs value, piecewise function, limited domains
     denom $= 0$ $\sqrt{\ }$ log
2. Find the derivitive
3. Solve for where derivitive $= 0$
4. Find max / min - consider all points where $f'(x) = 0$ or f has badness as in step 1.

Theorm if f has a local max or min at pt $x = c$ and $f'(c)$ exists (differentiable at c) than derivitive at $c = 0$

Correlory Step 4 will find all local max/min

$f(c) \geq f(x)$

$f(x)$ in some open interval

call $x = c+h$ for some small h

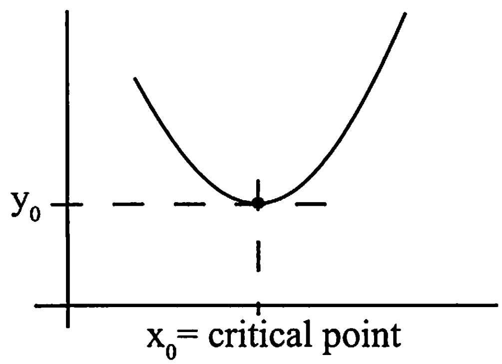
$f(c) \geq f(c+h)$

" do something more that he said

Lecture 10: Curve Sketching

Goal: To draw the graph of f using the behavior of f' and f''. We want the graph to be qualitatively correct, but not necessarily to scale.

Typical Picture: Here, y_0 is the minimum value, and x_0 is the point where that minimum occurs.



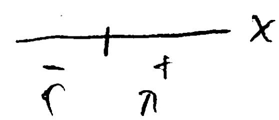
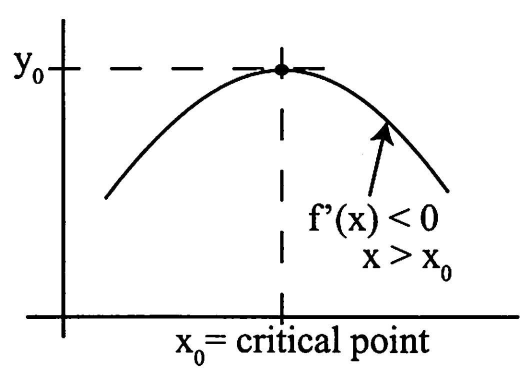


Figure 1: The critical point of a function

Notice that for $x < x_0$, $f'(x) < 0$. In other words, f is decreasing to the left of the critical point. For $x > x_0$, $f'(x) > 0$: f is increasing to the right of the critical point.

Another typical picture: Here, y_0 is the critical (maximum) value, and x_0 is the critical point. f is decreasing on the right side of the critical point, and increasing to the left of x_0.



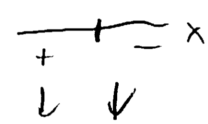


Figure 2: A concave-down graph

Rubric for curve-sketching

1. (Precalc skill) Plot the discontinuities of f — especially the infinite ones!
2. Find the critical points. These are the points at which $f'(x) = 0$ (usually where the slope changes from positive to negative, or vice versa.)
3. (a) Plot the critical points (and critical values), but only if it's relatively easy to do so.
   (b) Decide the sign of $f'(x)$ in between the critical points (if it's not already obvious).
4. (Precalc skill) Find and plot the zeros of f. These are the values of x for which $f(x) = 0$. Only do this if it's relatively easy.
5. (Precalc skill) Determine the behavior at the endpoints (or at $\pm\infty$).

Example 1. $y = 3x - x^3$

1. No discontinuities.
2. $y' = 3 - 3x^2 = 3(1 - x^2)$ so, $y' = 0$ at $x = \pm 1$.
3. (a) At $x = 1$, $y = 3 - 1 = 2$.
   (b) At $x = -1$, $y = -3 + 1 = -2$. Mark these two points on the graph.
4. Find the zeros: $y = 3x - x^3 = x(3 - x^2) = 0$ so the zeros lie at $x = 0, \pm\sqrt{3}$.
5. Behavior of the function as $x \to \pm\infty$.
   As $x \to \infty$, the x^3 term of y dominates, so $y \to -\infty$. Likewise, as $x \to -\infty$, $y \to \infty$.

Putting all of this information together gives us the graph as illustrated in Fig. 3)

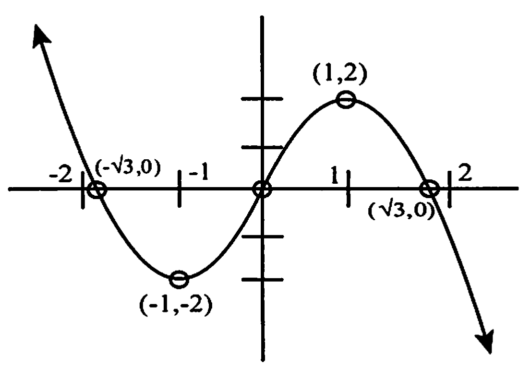


Figure 3: Sketch of the function $y = 3x - x^3$. Note the labeled zeros and critical points

Let us do step 3b (the sign of f') to double-check for consistency.

$$y' = 3 - 3x^2 = 3(1 - x^2)$$

$y' > 0$ when $|x| < 1$; $y' < 0$ when $|x| > 1$. Sure enough, y is increasing between $x = -1$ and $x = 1$, and is decreasing everywhere else.

Example 2. $y = \frac{1}{x}$.
This example illustrates why it's important to find a function's discontinuities before looking at the properties of its derivative. We calculate

$$y' = \frac{-1}{x^2} < 0$$

Warning: The derivative is never positive, so you might think that y is always decreasing, and its graph looks something like that in Fig. 4.

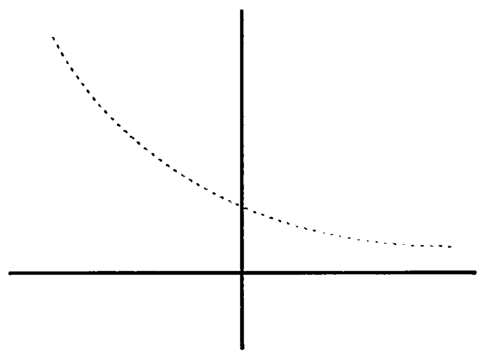


Figure 4: A monotonically decreasing function

But as you probably know, the graph of $\frac{1}{x}$ looks nothing like this! It actually looks like Fig. 5. In fact, $y = \frac{1}{x}$ is decreasing *except* at $x = 0$, where it jumps from $-\infty$ to $+\infty$. This is why we must watch out for discontinuities.

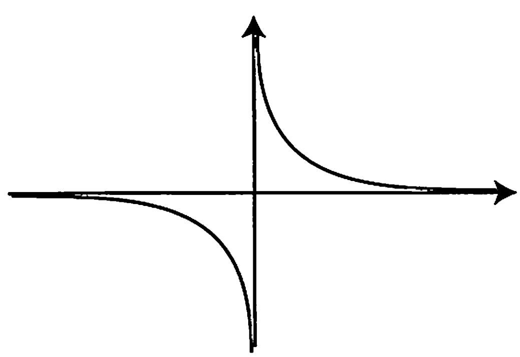


Figure 5: Graph of $y = \frac{1}{x}$.

Example 3. $y = x^3 - 3x^2 + 3x$.

$$y' = 3x^2 - 6x + 3 = 3(x^2 - 2x + 1) = 3(x-1)^2$$

There is a critical point at $x = 1$. $y' > 0$ on both sides of $x = 1$, so y is increasing everywhere. In this case, the sign of y' doesn't change at the critical point, but the graph does level out (see Fig. 6.

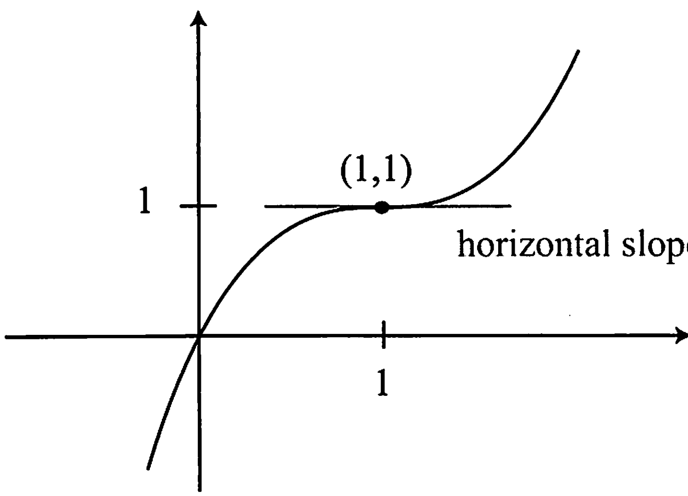


Figure 6: Graph of $y = y = x^3 - 3x^2 + 3x$

Example 4. $y = \dfrac{\ln x}{x}$ (Note: this function is only defined for $x > 0$)

What happens as x decreases towards zero? Let $x = 2^{-n}$. Then,

$$y = \frac{\ln 2^{-n}}{2^{-n}} = (-n \ln 2)2^n \to -\infty \text{ as n} \to \infty$$

In other words, y decreases to $-\infty$ as x approaches zero.

Next, we want to find the critical points.

$$y' = \left(\frac{\ln x}{x}\right)' = \frac{x(\frac{1}{x}) - 1(\ln x)}{x^2} = \frac{1 - \ln x}{x^2}$$

$$y' = 0 \implies 1 - \ln x = 0 \implies \ln x = 1 \implies x = e$$

In other words, the critical point is $x = e$ (from previous page). The critical value is

$$y(x)\,|_{x=e} = \frac{\ln e}{e} = \frac{1}{e}$$

Next, find the zeros of this function:

$$y = 0 \Leftrightarrow \ln x = 0$$

So $y = 0$ when $x = 1$.

What happens as $x \to \infty$? This time, consider $x = 2^{+n}$.

$$y = \frac{\ln 2^n}{2^n} = \frac{n \ln 2}{2^n} \approx \frac{n(0.7)}{2^n}$$

So, $y \to 0$ as $n \to \infty$. Putting all of this together gets us the graph in Fig. 7.

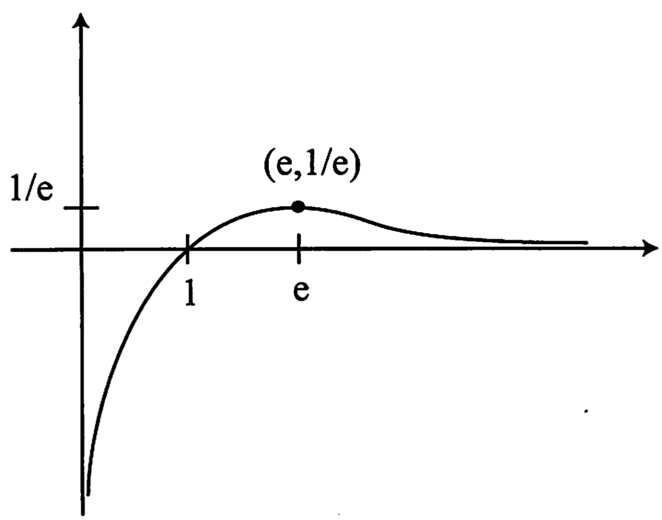


Figure 7: Graph of $y = \frac{\ln x}{x}$

Finally, let's double-check this picture against the information we get from step 3b:

$$y' = \frac{1 - \ln x}{x^2} > 0 \quad \text{for} \quad 0 < x < e$$

Sure enough, the function is increasing between 0 and the critical point.

2nd Derivative Information

When $f'' > 0$, f' is increasing. When $f'' < 0$, f' is decreasing. (See Fig. 8 and Fig. 9)

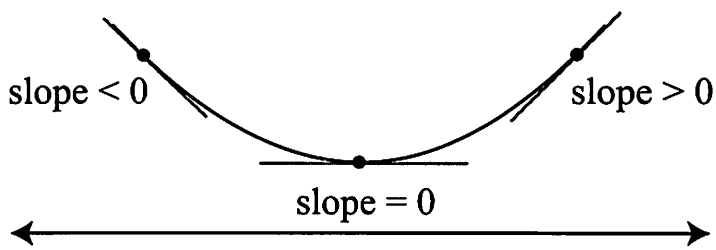


Figure 8: f is convex (concave-up). The slope increases from negative to positive as x increases.

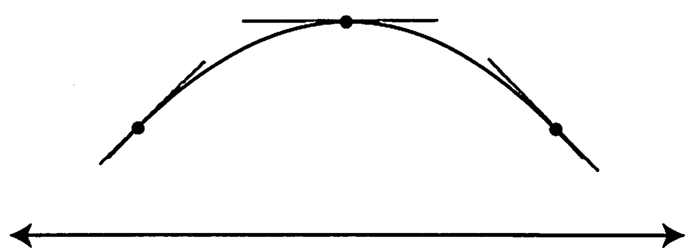


Figure 9: f is concave-down. The slope decreases from positive to negative as x increases.

Therefore, the sign of the second derivative tells us about concavity/convexity of the graph. Thus the second derivative is good for two purposes.

1. Deciding whether a critical point is a maximum or a minimum. This is known as the second derivative test.

$f'(x_0)$	$f''(x_0)$	Critical point is a:
0	negative	maximum
0	positive	minimum

2. Concave/convex "decoration."

The points where $f'' = 0$ are called *inflection points.* Usually, at these points the graph changes from concave up to down, or vice versa. Refer to Fig. 10 to see how this looks on Example 1.

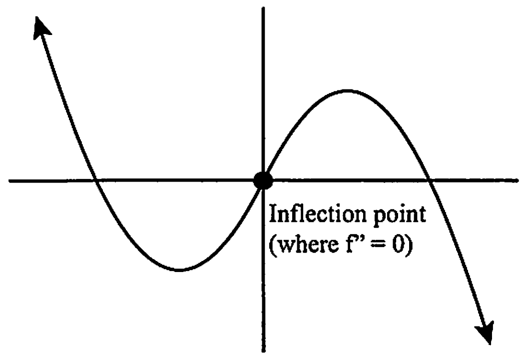


Figure 10: Inflection point: $y = 3x - x^3$, $y'' = -6x = 0$, at $x = 0$.