Michael Plasmeier

$\frac{59}{73}$

18.01 FALL 2009 – Problem Set 2A

Due Friday 10/2/08, 1:45 pm in 2-106

Bunch of little mistakes on 1A Part 2

2A is the first half of Problem Set 2, all of which is due a week after Exam 1 (the second half, 2B, will be issued the day before the exam). Even though it won't be collected until later, you should do 2A before the exam, to prepare for material on the test.

Part I (15 points)

Lecture 5. Fri. Sept. 18 Implicit differentiation; inverse functions and their derivatives.
Read: 3.5, Notes G section 5, 9.5 (bottom p.313 - 315)
Work: 1F-3,5,8ac; 1G-4; 1A-5b; 5A-1abc (just sin, cos, sec); 5A-3f,g,h

Lecture 6. Tues. Sept. 22 Exponentials and logs: def'n, algebra, applications, derivatives.
Read: Notes X (8.2 has some of this), 8.3 to middle p. 267; 8.4 to top p. 271
Work: 1H-1a,b, 2, 3a, 5b; 1I-1c,d,e,f,m; 1I-4a

Lecture 7. Thurs. Sept. 24 Logarithmic differentiation. Hyperbolic functions (which are not on exam). Review.
Read: 9.7 to p. 326 Work: 5A-5abc

Lecture 8. Fri. Sept. 25 **Exam 1** In-class, covering 0-7.

Students not passing the exam will receive an e-mail on Friday evening. Make-up exams are offered Monday-Thursday of the week following at times posted at the web site (see "Exams" section of the course website's home page). We will announce the location of the exam next week in class and on the website.

Part II (32 points)

Directions: Attempt to solve *each part* of each problem yourself. If you collaborate, solutions must be written up independently. It is illegal to consult materials from previous semesters. With each problem is the day it can be done.

0. (not until due date; 2 points) Write the names of all the people you consulted or with whom you collaborated and the resources you used, or say "none" or "no consultation". (See full explanation on PS1).

1. (9/15; 4 pts) Graph the even and odd functions you found in Problem 1, Part II of PS1. Directly below, graph their derivatives. Do this qualitatively using your estimation of the slope. Do not use the formulas for the derivatives (except to check your work if you want). You can use a graphing calculator to check your answer, provided that you mention it in Problem 0. (Note, however, that you may not use books, notes or calculators during tests, so it is unwise to rely on a graphing calculator here.)

2. (9/17; 5 pts = 2 + 3) Compute

a) $(d/dx)\tan^3(x^4)$

b) $(d/dy)(\sin^2 y \cos^2 y)$
(Do this two ways: first use the product rule, then write it as $f(2y)$. Show that the answers agree.)

3. (Fri; 4 pts = 3 +1)

a) The function $\cos^{-1} x$ is the inverse of the $\cos\theta$ on $0 \le \theta \le \pi$. Use implicit differentiation to derive the formula for $(d/dx)\cos^{-1} x$. Pay particular attention to the sign of the square root. (See the book or lecture for the case of the inverse of sine.)

b) Without calculation, explain why $(d/dx)\cos^{-1} x + (d/dx)\sin^{-1} x = 0$

4. (Tues + Thurs; 12pts = 2+2+2+2+2+2) Do 8.2/8ac, 10, 11, 12; 8.4/18,19a.

5. (Thurs.; 2 pts) Compute $(d/dx)x^x$.

6. (Thurs; 3 pts) Derive the formula for $D(u_1 u_2 \cdots u_n)$ from PS1, Part II, ~~7b~~ 8, using logarithmic differentiation.

PSet 2a

Please write double sided, wasting too much paper!! 10/2/09

Part 1

IF-3 Find $\frac{dx}{dx}$ $y = x^{\frac{1}{n}}$ ~~$y' = \frac{1}{n}x^{(\frac{1}{n}-1)}$~~ $\rightarrow$

$$y^n = x$$
$$ny^{n-1}y' = 1$$
$$y' = \frac{1}{ny^{n-1}} = \frac{1}{n}y^{n-1}$$

↑ plug in y

$$\frac{1}{n}\left(x^{\frac{1}{n}}\right)^{-1}$$
$$\frac{1}{n}x^{\frac{1}{n}-1}$$

IF-5 Find points of the curves $\sin x + \sin y = \frac{1}{2}$ w/ horiz tangent lines

$x \rightarrow y + 2\pi$
$x \rightarrow x + 2\pi$

$$\cos x + y'\cos y = 0$$

horiz slope means $y' = 0$

$$\cos x + 0 = 0$$
$$\tau = 0$$

cos is 0 at $\frac{\pi}{2} + k\pi$

get ↑ don't get why needed

$$\sin\left(\frac{\pi}{2} + k\pi\right) = -1^{(k)}$$

1 if k even
-1 if k odd

$x = \frac{\pi}{2} + k\pi$ $\pm 1 + \sin y = \frac{1}{2}$

$$\sin y = \mp 1 + \frac{1}{2}$$
$$\sin y = \frac{3}{2} \text{ no solution}$$

Only where k is even $\sin y = -1 + \frac{1}{2}$

so that $y = -\frac{\pi}{6} + 2n\pi$

$y = -\frac{7\pi}{6} + 2n\pi$

$\left(\frac{\pi}{2} + 2k\pi, -\frac{\pi}{6} + 2n\pi\right) \left(\frac{\pi}{2} + 2k\pi, \frac{7\pi}{6} + 2n\pi\right)$

1F-8c Evaluate w/ implicit differentation

chain rule, product rule?

$c^2 = a^2 + b^2 - 2ab\cos\theta$ $\quad \frac{da}{db} = a'$ $\quad c = \text{constant}$

$0 = 2aa' + 2b - 2\cos\theta(a'b + a)$

~~$2aa' + 2b$~~ distribute??

$a' = \frac{-2b + 2\cos\theta\ a}{2a - 2\cos\theta\ b} = \boxed{\frac{a\cos\theta - b}{a - b\cos\theta}}$

1F-8a $V = \frac{1}{3}\pi r^2 h$ $\quad \frac{dr}{dh}\ r'$ $\quad V = \text{constant}$

$0 = \underbrace{\frac{1}{3}\pi}_{\text{constant}}\underbrace{(2rr'h + r^2)}_{\text{product rule}}$

$0 = \frac{2\pi rr'h}{3} + \frac{r^2\pi}{3}$ write it out

$\cdot 3 \quad \cdot 3 \quad \cdot 3$

$\frac{2\pi rr'h}{2\pi rh} + \frac{r^2\pi}{2\pi rh}$

neg somehow

$r' = \frac{r^2\pi}{2\pi rh} = -\frac{r}{2h}$

1G-4 Formula for nth derivative $y^{(n)}$ of $y = \frac{1}{(x-1)} = 1(x-1)^{-1}$

$y^{(1)} = -1(x-1)^{-2}$

$y^{(2)} = 2(x-1)^{-3}$

$y^{(3)} = -6(x-1)^{-4}$ or $(-1)(-2)(-3)(x-1)^{-4}$

$y^{(4)} = 24(x-1)^{-5}$

$y^n = (-1)^n(n!)(x+1)^{-n-1}$

$n! = n \cdot n \cdot n \cdot n$

if − or not

1A-5b Find the inverse

convert 1st

$$y = x^2 + 2x \rightarrow y = (x+1)^2 - 1$$

$$\cancel{x = y^2 + 2y}$$

$$\cancel{\frac{x}{2}} \quad \cancel{\frac{y^2+2y}{2}}$$

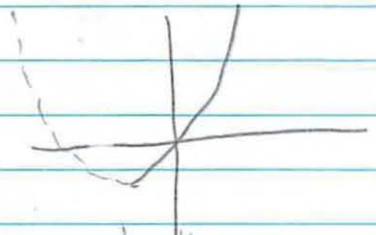
$$\cancel{\frac{y^2}{2} + y = \frac{x}{2}}$$

$$x = (y+1)^2 - 1$$

$$x + 1 = (y+1)^2$$

$$\sqrt{x+1} = y + 1$$

$$\sqrt{x+1} - 1 = y$$



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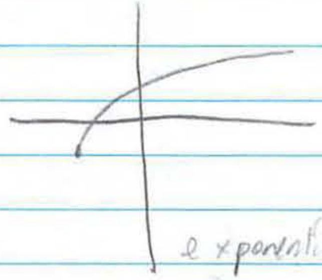


exponential ?

5A-1a Evaluate

$$\tan^{-1}\sqrt{3}$$

↑ angle whose tan $= \sqrt{3}$

$\cancel{\frac{1}{1+(\sqrt{3})^2}}$ $\cancel{\frac{1}{4}}$ ~~not finding derivitive~~

$$\tan(y) = \tan(\tan^{-1}(\sqrt{3}))$$

$$\tan y = \sqrt{3}$$

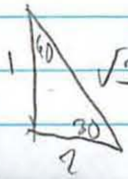
I know tan 60° or $\frac{\pi}{3} = \sqrt{3}$

$$\frac{\sin\frac{\pi}{3}}{\cos\pi/3} = \frac{\sqrt{3}/2}{\frac{1}{2}} = \sqrt{3}$$

1b $\sin^{-1}(\sqrt{3}/2)$

what is $\sin(\sqrt{3}/2)$

60° or $\frac{\pi}{3}$



$5 + 5$ $2 \cdot x$ means for each

$5x + 5x = 10x$

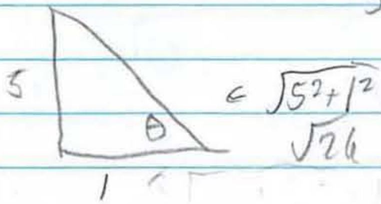
$5 \cdot 5$ $2 \cdot x$ just x by y

$25x$

1c if $\theta = \tan^{-1}(5)$ eval $\sin, \cos, \sec$

$\tan\theta = 5$

$\sin\theta = \frac{5}{\sqrt{26}}$ $\cos\theta = \frac{1}{\sqrt{26}}$ $\sec\theta = \frac{\sqrt{26}}{1}$



$\sec = \frac{hyp}{adj} = \frac{1}{\cos}$

5A-3f Derivitve w/respect to x $\sin^{-1}(a/x)$

$\frac{1}{\sqrt{1-x^2}}$ $\frac{1}{\sqrt{1-(a/x)^2}} \cdot \frac{\overset{0}{a'}x - \overset{1}{x'}a}{x^2}$

$\frac{1}{\sqrt{1-(a/x)^2}} \cdot \frac{-a}{x^2}$ → multiply straight across

~~algebra~~

Breker office hrs

$\frac{-a}{x^2\sqrt{1-\left(\frac{a}{x}\right)^2}}$ $\frac{-a}{x^2\sqrt{1-\frac{a^2}{x^2}}}$ $\frac{-a}{x^2} \cdot \frac{1}{\sqrt{1-\frac{a^2}{x^2}}}$

↓ $\sqrt{x^2\left(1-\frac{a^2}{x^2}\right)}$ then distrib

$\boxed{\frac{-a}{x\sqrt{x^2-a^2}}} \leftarrow \frac{-a\cancel{x}}{x^{\cancel{2}}\sqrt{x^2-a^2}} \leftarrow \sqrt{x^2-a^2}$

3g $\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) \rightarrow \frac{1}{1+\left(\frac{x}{\sqrt{1-x^2}}\right)^2}$ ~~$\cdot \frac{1\sqrt{1-x^2} - \frac{1}{2}(1-x^2)^{\frac{1}{2}-1} \cdot -2x \cdot x}{1-x^2}$~~ ①

all simplifies do it 1st

see next sect

? what do they do

~~or do they do something w/ implicit~~

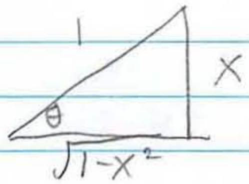
Simpler to figure out

~~$y = \frac{x}{\sqrt{1-x^2}}$ $\frac{dy}{dx} = (1-x^2)^{-3/2}$, $1+y^2 = 1(1-x^2)$~~

~~$\frac{d}{dx}\tan^{-1}y = \frac{dy/dx}{1+y^2} = (1-x^2)^{-3/2}(1-x^2)$~~

3g

Brubaker
office
hrs



$= \sin^{-1}\left(\frac{x}{1}\right) \quad = \sin^{-1}(x)$

$\sin y = x$

$\frac{dx}{dx} = \frac{1}{\sqrt{1-x^2}}$

$$\frac{d}{dx}\left[\arctan(f(x))\right]$$

$$\frac{1}{1+f(x)^2}\cdot f'$$

also watch
+ vs ·
chain rule

$\frac{dy}{dx}\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$

③ⓖ

alt
wrong
solution

$$\frac{1}{1+\left(\frac{x}{\sqrt{1-x^2}}\right)^2}\cdot\frac{1\,(1-x^2)^{1/2}-\frac{1}{2}(1-x^2)^{-1/2}\cdot -2x\cdot x}{1-x^2}$$

simplify numerator ← − + −

$$\frac{1}{1+\frac{x^2}{(1-x^2)}} \text{ expanding } \cdot \quad \sqrt{1-x^2} + \frac{x^2}{\sqrt{1-x2}}$$

$$\frac{\sqrt{1-x^2}\sqrt{1-x^2}+x^2}{\sqrt{1-x^2}}$$

$$\frac{(1-x^2)+x^2}{\sqrt{1-x^2}} = \frac{\frac{1}{\sqrt{1-x^2}}}{1-x^2} = \frac{1}{\sqrt{1-x^2}}\cdot\frac{1}{1-x}$$

numerator

$-\frac{1}{2}$ + -1

$$\frac{1}{1+\frac{x^2}{(1-x^2)}}\cdot\frac{1}{\sqrt{1-x^2}^{\,3}}$$

$$\frac{1}{\frac{1(1-x^2)+x^2}{1-x^2}}$$

same denom

$$\frac{\sqrt{1-x^2}}{\frac{1}{1-x^2}} = \sqrt{1-x^2}\cdot\frac{1-x^2}{1}$$

$$\sqrt{1-x^2}\cdot 1-x^2$$

$$(1-x^2)^{1/2}\cdot(1-x^2)^1$$

$$(1-x^2)^{3/2}$$

$$\sqrt[3]{1-x^2}$$

$$\boxed{\frac{a}{\frac{b}{c}} = a\cdot\frac{c}{b} \qquad \frac{\frac{a}{b}}{c} \quad \frac{a}{b}\cdot\frac{1}{c}}$$

? should be $\frac{1}{\sqrt{1-x^2}}$

3h $\sin^{-1}\sqrt{1-x}$ $\frac{1}{\sqrt{1-(\sqrt{1-x})^2}} \cdot \frac{1}{2}(1-x)^{-1/2} \cdot -1$

? what am I doing seperate

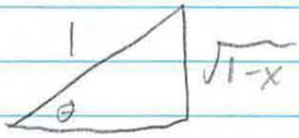
$$-\frac{1}{\sqrt{1-(1-x)}} \cdot \frac{1}{\sqrt{1-x}}$$

$$\frac{1}{\sqrt{x}} \cdot \frac{1}{\sqrt{1-x}}$$

$$\boxed{\frac{-1}{2\sqrt{x(1-x)}}}$$

---

do like the triangle from 3g



Lecture 6

1H-1a Half life λ radioactive decays

$$y = y_0 e^{-kt}$$

(when $y = \frac{y_0}{2}$

a) express λ in terms of k

$$y_0 e^{-kt} = \frac{y_0}{2}$$

$$\frac{1}{y_0}$$

$$\frac{1}{2} = e^{-kt}$$

$$\ln \qquad \ln$$

$$-\ln 2 = -kt$$

$$t = \frac{\ln 2}{k}$$

b) show that λ at t_1 is y_1 then at $t_1 + \lambda$ it will be $\frac{y_1}{2}$ no matter t_1

Well at any point in time + decay lenght you will always be left w/ half

$$\lambda = \frac{-\ln 2}{k} \quad y_0 e^{k(t_1+\lambda)} = y_0 e^{kt_1} \cdot e^{k\lambda}$$

$$= y_1 e^{-\ln 2}$$

^ inverse

$$= y_1 \frac{1}{2}$$

1H-2 If solution w/ lots of hydrogen ions (strong acid) is diluted w/ = amount water - how much does pE change? in terms of original *need a major log/e review/study

$$pH = -\log_{10}[H^+]$$

$$[H^+]_{dill} = \frac{1}{2}[H^+]_{orig}$$

$-\log_{10}$ $-\log_{10}$

$$-\log_{10}[H^+]_{dill} = \log 2 - \log[H^+]_{orig}$$

$\log_a x = y$
$a^y = x$

$$pH_{dil} = pH_{orig} + \underset{\uparrow\ .3}{\log 2}$$

3a Solve for y

$$\ln(y+1) + \ln(y-1) = 2x + \ln x \downarrow \text{multiply by } e$$

simplify ↳ $(y+1)\cdot(y-1) = e^{2x}\cdot x$

$$x^2 - 1 = xe^{2x}$$

$$y = \sqrt{xe^{2x}+1} \quad \text{since } y > 0$$

5b Solve for x -hint put $v = e^x$
solve 1st for v

$$y = e^x + e^{-x} \downarrow \text{abstract w/ } v$$

$$y = v + \frac{1}{v} \searrow \text{multiply everything by } v + \text{subtract}$$

$$v^2 - yv + 1 = 0$$

↓ quadratic formula

$$v = \frac{y \pm \sqrt{y^2-4}}{2} = e^x$$

isolate x

↓ take ln to get x

forget derivite

$$x = \ln\left(\frac{y \pm \sqrt{y^2-4}}{2}\right)$$

1J - 1c Find derivitive e^{-x^2} chain rule

$$e^{-x^2} \cdot -2x \cdot 1$$

$$\boxed{-2x e^{-x^2}}$$ ← real Simple

Simple chain rule

$$\frac{d}{dx} e^{2x} = e^{2x} \cdot 2$$

$$\underbrace{e^{x^2}}_{\text{if you know this}} = e^{x^2} \cdot 2x \cdot 1$$

1d $x \ln x - x$

product rule + chain rule x

product rule

$$1 \ln x + x \frac{1}{x} \cdot 1$$

$$\ln x + 1 - 1$$

$$\ln x$$

1e $\ln(x^2)$

$$\frac{1}{x^2} \cdot 2x = \frac{2x}{x^2} \quad \frac{2}{x}$$

1f $\ln(x)^2$

$$2 \ln x \cdot \frac{1}{x} \cdot 1$$

$$\frac{2 \ln x}{x}$$

1m $\frac{(1-e^x)}{(1+e^x)}$

$$\frac{(-e^x)(1+e^x) - (e^x)(1-e^x)}{(1+e^x)^2}$$

$$\frac{-e^x - (e^x)^2 - e^x + (e^x)^2}{(1+e^x)^2}$$

$$\frac{-2e^x}{(1+e^x)^2}$$ ✓

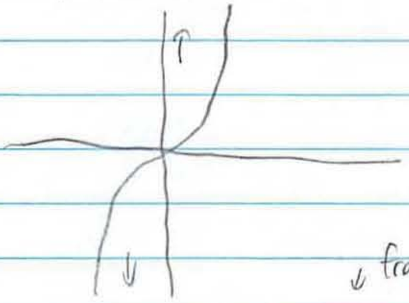
II-4a $\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e$

a) $\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^{3n}$

e^3 as $n \to \infty$

Lecture 7

5A-5a) Sketch $y = \sinh x$



Rhodes office hrs

↓ from book, memorize

$y = \sinh x = \frac{e^x - e^{-x}}{2}$

$y' = \cosh x = \frac{e^x + e^{-x}}{2}$

$y'' = \sinh$

$\lim_{x \to \infty} \sinh = \infty$

$\lim_{x \to -\infty} \sinh = -\infty$

Contineous
- denom never 0
- numerator always defined

will be a ↓ number > 0

← e^x is never 0

when is $\frac{e^x + e^{-x}}{2} = 0$ - never

y' is never 0

- no critical points when 1st deriv = 0

inflection at $x = 0$

- 2nd deriv = 0

slope of y ± 1

$0 = \frac{e^x - e^{-x}}{2}$ ← are none again

$0 = e^x - e^{-x}$

$e^x = e^{-x}$

$\ln(e^x) = \ln(e^{-x})$

$x = -x$

$x = 0$

test for concavity

$f''(-1) = \ominus$ ↓

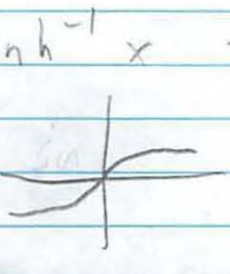
$f''(1) = \oplus$?

sign change

odd function $\frac{e^x}{2}$ $x >> 0$

5b) $\sinh^{-1} x$ + sketch

$x = \sinh y$



Domain whole x axis = not restricted

c Find derivative $\frac{d}{dx} \sinh^{-1} x$

$x = \sinh y$

$1 = \cosh y \, y'$

solve for y'

$y' = \frac{1}{\cosh y}$

plug in y

step

$\frac{1}{\cosh(\sinh^{-1} y)} = \overset{??}{\frac{1}{\pm\sqrt{\sinh^2 y + 1}}}$

↑cosh =cosh² ↑phytagtheorm reorm

$\frac{d}{}$ w/ respect to x

$\frac{d \sinh^{-1} x}{dx} \rightarrow \frac{1}{\pm\sqrt{x^2 + 1}}$

in terms of x

plug in

is it ±?

Function ↑ ⊕

↓ ⊖

is ↑ so ⊕

$* \cosh^2 y - \sinh^2 y = 1$

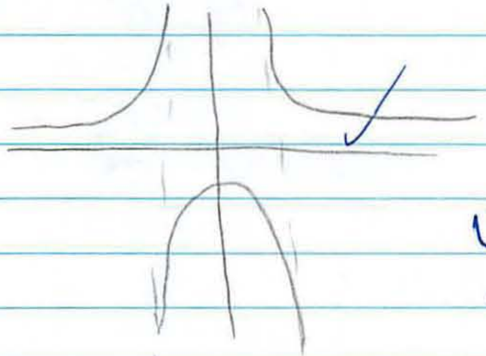
$\left(\frac{e^y + e^{-y}}{2}\right)^2 - \left(\frac{e^y - e^{-y}}{2}\right)^2 \overset{?}{=} 1$

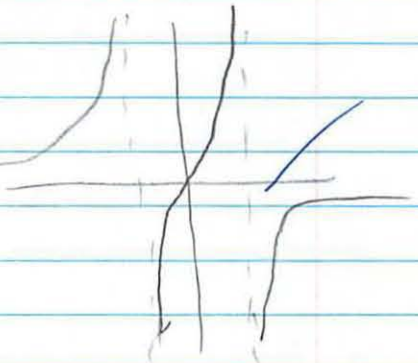
Part 2

0. See sidebar   Sasha Siy ✓ 2/2

1. Graph even + odd functions you found

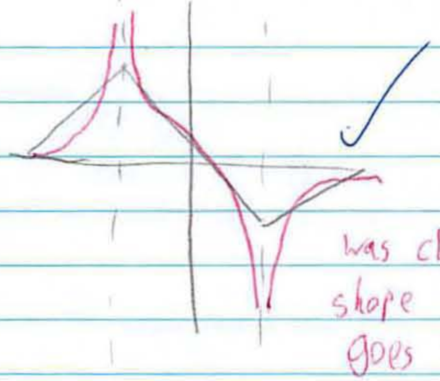
$\frac{1}{2}\left(\frac{x-1}{x+1} + \frac{-(x+1)}{(-x+1)}\right)$ even $\frac{1}{2}\left(\frac{(x-1)}{(x+1)} - \frac{(-x-1)}{(-x+1)}\right)$ odd

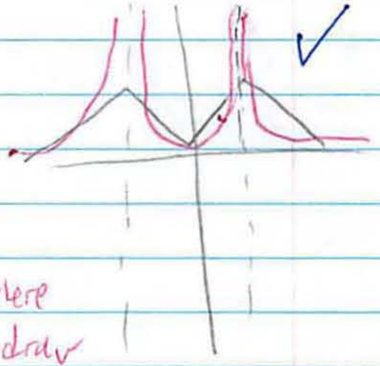




4/4

TI-89+





was close on shape - but where goes to ∞ draw

$\frac{d}{dx}$ calc $\frac{1}{(x+1)^2} - \frac{1}{(x-1)^2}$ $\frac{1}{(x+1)^2} + \frac{1}{(x-1)^2}$

2. $\frac{d}{dx}$ $\tan^3(x^4)$ ← write in back

$(\tan(x^4))^3$

$3(\tan(x^4))^2 \cdot \sec^2(x^4) \cdot 4x^3$

$\boxed{12x^3 \tan^2(x^4) \sec^2(x^4)}$ ← can simplify further

$12x^3 \frac{\sin^2(x^4)}{\cos^2(x^4)} \cdot \frac{1}{\cos^2(x^4)} = 12x^3 \frac{\sin^2(x^4)}{\cos^4(x^4)}$ ✓ 2/2

Char 1/8

$\sin y \cos y = \frac{1}{2}\sin(2y)$

↗ since

b $\sin^2 y \cos^2 y$

~~$(\sin y)^2 (\cos y)^2$~~

~~$2\sin y \cos y \cdot (\cos y)^2 + 2\cos y -\sin y \cdot (\sin y)^2$~~

~~$2\sin y \cos^3 y - 2\cos y \sin^3 y$~~

2 ways?

~~$f(2y)$ what does this mean~~

~~$2y = \sin^2 y \cos^2 y$???~~

I don't get other way

Rhodes office hrs

$\frac{1}{4}\sin^2(2y)$ product rule

$\frac{1}{4}\cdot 2\sin(2y)\cdot\cos(2y)\cdot 2$

$\sin(2y)\cos(2y)$

2/3

rewrite as $f(2y)$

$\sin^2 y \cos^2 y = (\frac{1}{2}\sin 2y)^2$

$\frac{1}{4}\sin^2 2y$

$\frac{d}{dy} = \frac{1}{4}\cdot 2\sin 2y \cos 2y \cdot 2$

$\sin 2y \cos(2y)$

3 $\cos^{-1} x$ Domain $0 \le \theta \le \pi$

$y = \cos^{-1} x$

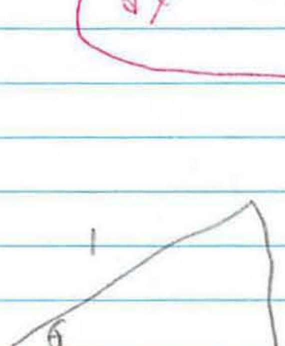
$x = \cos y$

$1 = -\sin y \, y'$

$y' = \frac{1}{-\sin y}$ ← plug in y

$y' = \frac{1}{-\sin(\cos^{-1} x)}$

$\boxed{\frac{1}{\sqrt{1-x^2}}}$ ✓



$\sqrt{1^2-x^2}$

$\sqrt{1-x^2}$

2

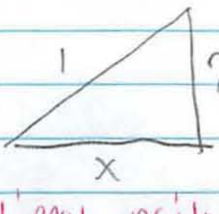
$x^2+y^2=1^2$

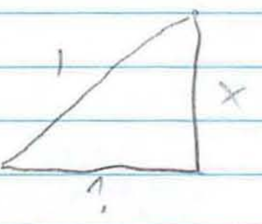
solve for y

$y=\sqrt{1^2-x^2}$

3b $\frac{d}{dy}\cos^{-1} x + \frac{d}{dx}\sin^{-1} x = 0$

1/2





The parts of the triangle cancel each other out

no calculation

at any pt derivative of = magnitude + opposite sign

flipped upside down + righted —

8a. Magnitude M of earthquake on Ricter scale

$0 \to 8.9$

$$M = \frac{2}{3} \log_{10} \frac{E}{E_0} \quad \begin{matrix} \leftarrow \text{energy in killowatt hrs} \\ \leftarrow 7 \times 10^{-3} \end{matrix}$$

a. 2 earthquakes differ by 1 on ricter scale

Ratio larger : smaller $\to 10^{3/2} \simeq 31.62$

$$2 = \frac{2}{3} \log_{10} \frac{E_2}{7 \times 10^{-3}} \to 7$$

$$1 = \frac{2}{3} \log_{10} \frac{E_1}{7 \times 10^{-3}} \to .221$$

$$\frac{7}{.221} = 31.674 \checkmark$$

2/2

c City uses 3×10^5 kWh. How many days electricty for magnitude 6

$$6 = \frac{2}{3} \log_{10} \frac{E_6}{7 \times 10^{-6}} = \frac{7000000 \text{ kWh}}{3 \times 10^5} \checkmark$$

23 and a third days

2/2

10. ~~Show $\log_3 2$ is irrational~~

~~- assume the contrary that $\log_3 2 = \frac{p}{q}$ ← ⊕ integers~~

~~← express in terms of exponents~~

~~$3^x = 2$~~

~~? never get an exponent (x) that is rational to get this~~

~~Can an integral power of 3 = an integral power of 2?~~

next sheet

10

Show $\log_3(2)$ is irrational

Brubaker office hrs

prove $\sqrt{2}$ is irrational

suppose $\sqrt{2} = \frac{a}{b}$

$2 = \frac{a^2}{b^2}$

$2b^2 = a^2$

↑ always # even

$2 | a \rightarrow 2^2 | a^2$

$2 \nmid a \rightarrow 2 \nmid a^2$

odd even

2 ✓

$\sqrt{2}$

---

suppose $\log_3(2) = \frac{a}{b}$ ← a, b integers $b \neq 0$

$\boxed{a^{\log_a x} = x}$

~~$3^{\log_3(2)} = 3^{a/b}$~~

~~$2 = 3^{a/b}$~~

$b \log_3(2) = a$

$\log_3(2^b) = a$

~~$2^b = 3^a$~~ ← only happens if a and b = 0 can't happen if $\frac{a}{b}$

different prime factors

11. Find the flaw in this proof that $\frac{1}{2} < \frac{1}{4}$

multiply both sides of the inequality $1 < 2$ by $\log \frac{1}{2}$ to get

$$1 < 2$$

$$1 \cdot \log \tfrac{1}{2} \; \textcircled{<} \; 2 \cdot \log \tfrac{1}{2} \quad 1 \textcircled{✓}$$

$$\log \tfrac{1}{2} < \log \left(\tfrac{1}{2}\right)^2 \quad 1 \textcircled{✓}$$

$$\log \tfrac{1}{2} < \log \tfrac{1}{4}$$

$$\tfrac{1}{2} < \tfrac{1}{4}$$

2/2

?

reread 8.2

$\log \frac{1}{2}$ is a −

should have flipped sign

12. Prime # integer $p > 1$ no factors except itself and 1

$p = 2, 3, 5, 7, 11, 2^{756,839} - 1$

a. How many digits will it have?

$10^x = 2^{756,839}$ ✓

converted wrong → ~~$\log_{10} 2^{756,839} = x$~~ $x = 756\,839 \log 2$

(calc says ∞ :)) X /2

227831 digits

b. How many pages?

$x / 4600$

50 pages

18 Logarmithic differentation

$$y = \sqrt[3]{(x+1)(x-2)(2x+7)}$$

↑complex

$$\ln y = \frac{1}{3}\left[\ln(x+1) + \ln(x-2) + \ln(2x+7)\right]$$

implicit differentation

$$\frac{1}{y}y' = \frac{1}{3}\left[\frac{1}{x+1}\cdot 1 + \frac{1}{x-2}\cdot 1 + \frac{1}{2x+7}\cdot 2\right]$$

~~$y' = \frac{y}{3x+3} + \frac{y}{3y-6} + \frac{2y}{6x+21}$~~ $\frac{1}{3}\sqrt{(-)}\left(\frac{1}{x+1} + \frac{1}{x-2} + \frac{2}{2x+7}\right)$

↑ wtf?

19a. $\frac{dy}{dx} = \frac{e^x(x^2-1)}{\sqrt{6x-2}}$ X 1

d/ln

~~$\ln y = x\ln(x^2-1)\left(\ln(6x-2)\right)^{-1/2}$~~

~~$\frac{1}{y}y' = 1\cdot\ln(x^2-1) + \frac{1}{x^2-1}\cdot 2x\left(\sqrt{6x-2}\right) - \left(-\frac{1}{2}(6x-2)^{-3/2}\cdot 6\right)\left(x\ln(x^2-1)\right)$~~

~~$y' = y\left(\frac{\ln(x^2-1)\cdot 2x\sqrt{6x-2}}{x^2-1} - \frac{x\ln(x^2-1)}{2\sqrt[3]{6x-2}}\right)$~~

~~sub in y~~

$\log x / \ln(x^2+1) - \frac{\ln(6x-2)}{2}$ $\frac{dy}{dx} = \frac{e^x(x^2-1)}{\sqrt{6x-2}}\cdot\left(1 + \frac{2x}{x^2-1} - \frac{3}{6x-2}\right)$

2??

5 $\frac{d}{dx} \quad x^x$

$\Downarrow$ ln both sides

$\ln y = \ln x^x$ ← rule of differentation

$\ln y = x \ln x$ ← chain rule

$\frac{1}{y} y' = 1 \ln x + \frac{1}{x} x$

$\frac{1}{y} y' = \ln x + 1$ ✓ 2

$y' = y(\ln x + 1)$ ← plug in y

$\boxed{x^x(\ln x + 1)}$ ✓

prob on Final

6. Derive formula $D(u_1 u_2 \ldots u_n)$ from PS 1, Part 2, 7b using logarithmic differentation ← what is that

?? which problem

??

~~$\lim_{x \to 0} \frac{\sin^2 x}{3x^2}$~~

~~$\ln y = \ln(\sin x)^2$~~

~~$2 \ln \sin x$~~

~~$\frac{1}{y} y' = 2 \frac{1}{\sin x} \cdot \cos x$~~

~~$y' = y\left(2 \frac{\cos}{\sin}\right)$~~

~~$y' = 2y \cot$~~

~~$\ln(xy) = \ln(x) + \ln(y)$~~

~~$\ln(u_1 u_2 \ldots u_n) = \ln(u_1) + \ln(u_2) + \ldots u(n)$~~

← answer should be $\frac{1}{3}$

wrong problem →

- take log of both sides

6. Actually # 8 (eabor

Rhodes office hrs

$$D(U_1\ U_2\ \ldots\ U_n)$$

Let $y = U_1\ U_2 \ldots U_n$

$$\ln(y) = \ln(U_1\ U_2\ U_3 \ldots U_n)$$
$$\ln(y) = \ln(U_1) + \ln(U_2') + \ln(U_3) + \cdots + \ln(U_n)$$

$$\frac{1}{y}\,y' = \frac{1}{U_1}U_1' + \frac{1}{U_2}U_2' + \frac{1}{U_3}U_3' + \cdots \frac{1}{U_n}U_n'$$

solve for y'

$$y' = \frac{y}{U_1}U_1' + \frac{y}{U_2}U_2' + \frac{y}{U_3}U_3' + \cdots \frac{1}{U_n}U_n'$$

sub back in for y

$$y' = U_2 U_3 \cdots U_n\ U_1' + U_1 U_3 \cdots U_n\ U_2' + U_1 U_2 U_3 \cdots U_{n-1} U_n'$$

3/3

18.01 FALL 2009 – Problem Set 2B

Due Friday 10/2/09, 1:45 pm in 2-106

2B is the second half of Problem Set 2, due along with the first half 2A on the above date.

Part I (10 points)

Lecture 9. Tues. Sept. 29. Linear and quadratic approximations.
Read: Notes A Work: 2A-1, 3, 6, 11, 12ade

Lecture 10. Thurs. Oct. 1. Curve-sketching.
Read: 4.1, 4.2 Work: 2B-1,2: parts a,e,h only; 2B-4, 6ab, 7ab

Lecture 11. Fri. Oct. 2. Maximum-minimum problems.
Read: 4.3, 4.4 Work: To be assigned on PS3

Part II (16 points)

Directions: Attempt to solve *each part* of each problem yourself. If you collaborate, solutions must be written up independently. Do not consult materials from previous semesters. Next to each problem, we write the day it can be done, according to the lecture schedule.

0. (not until due date; 2 points) Write the names of all the people you consulted or with whom you collaborated and the resources you used, or say "none" or "no consultation." (See full explanation on PS1).

1. (10 points: 2 + 4 + 4) **Golf balls**

The area of a section of a sphere of radius R between two parallel planes that are a distance h apart is [1]

$$\text{area of a spherical section } = 2\pi hR$$

Slice the sphere of radius R by a horizontal plane. The portion of the plane inside the sphere is a disk of radius $r \leq R$. The portion of the spherical surface above the plane is called a *spherical cap*. For example, if the plane passes through the center, then the disk has radius $r = R$, its circumference is the equator, and the spherical cap is the Northern Hemisphere. More generally, a spherical cap is the portion of surface of the Earth north of a given latitude line. The formula above applies to regions between two latitude lines, and, in particular, to spherical caps.

a) Consider a spherical cap defined as the portion of the surface of the sphere above a horizontal plane that slices the sphere at or above its center. Find the area of the cap as a function of R and r. Do this by finding first the formula for the height h of the spherical cap in terms of r and R. (This height is the vertical distance from the horizontal slicing plane to the North Pole.) Then use your formula for h and the formula above for the area of spherical sections.

---

[1] This formula will be derived in Unit 4. Two examples may convince you that it is reasonable. For $h = R$, it gives the area of the hemisphere, $2\pi R^2$. For $h = 2R$ it gives $4\pi R^2$, the area of the whole sphere.

b) Express the formula for the area of a spherical cap in terms of R^2 and r/R. (This is natural because the proportional scaling cr and cR changes the area by the factor c^2.) Then use the linear and quadratic approximations to $(1+x)^{1/2}$ near $x = 0$ to find a good and an even better approximation to the area of the spherical cap, appropriate when the ratio r/R is small. (Hint: What is x?) Simplify your answers as far as possible: the area approximation corresponding to the linear approximation to $(1+x)^{1/2}$ should be very familiar.

c) The following problem appeared on a middle school math contest exam, though we have changed the numbers. Consider a golf ball that is 4.3 centimeters in diameter with 300 hemispherical dimples of diameter 3 millimeters. (Note that this is not a realistic golf ball – in reality, the dimples are much more shallow than full hemispheres.) Find the area of the golf ball rounded to the nearest 1/100 of a square centimeter using the approximation $\pi \approx 3.14$. (The students were given three minutes. We are spending more time on it.) [2]

Under the rules of the contest, an incorrectly rounded answer was counted as wrong with no partial credit, so correct numerical approximation was crucial. Some students objected that they could not figure out the area of portion of the large sphere that is removed when a dimple is inserted. A careless examiner had assumed that the students would use the approximation that the area removed for each dimple was nearly the same as the area of a flat disk. We are going to figure out whether this approximation is adequate or gives the wrong answer according to the rules.

Write down formulas for the surface area of the golf ball for the following three cases listed below. (Put in 300 dimples, but leave r, R, and π as letters.)

i) the approximation pretending that the removed surface is flat (what is the relationship between this and the approximations of part (b)?)

ii) the higher order approximation you derived in part (b)

iii) the exact formula

Finally, evaluate each of the answers for the given values $r = .15$ and $R = 2.15$ centimeters, and find the accuracy of the approximations.

2. (4 points) Draw the graph of $f(x) = 1/(1+x^2)$ and, directly underneath, it the graphs of $f'(x)$ and $f''(x)$. Label critical points and inflection points on the graph of f with their coordinates. Draw vertical lines joining these special points of the graph of f to the corresponding points on the graphs below.

---

[2] Dimples on golf balls are big business, and many patents with various dimple patterns (including those with varying sized dimples) have been issued. Most attempt to arrange the dimples in highly symmetric arrangements patterned on the platonic solids, particularly the icosahedron. See `en.wikipedia.org/wiki/Platonic_solid`.

Part 2B

$f(x_0) + f'(x_0)(x - \overset{\text{usually } 0}{a})$

2A-1 Find $\underset{\text{formula}}{\text{linearization}}$ of $\sqrt{\underset{\text{constant}}{a+b}x}$ at 0 w/ approximation

$$\frac{d}{dx}\sqrt{a+bx} = \frac{1}{2}(a+bx)^{-1/2} \cdot b = \frac{b}{2\sqrt{a+bx}}$$

$$T = \sqrt{a} + \frac{b}{2\sqrt{a}}x$$

algebra $\sqrt{a}\sqrt{1+\frac{bx}{a}} \approx \sqrt{a}\left(1 + \frac{bx}{2a}\right)$

2A-3 $\dfrac{(1+x)^{3/2}}{1+2x}$

$$f' = \frac{\frac{3}{2}(1+x)^{1/2}(1+2x) - ((1+x)^{3/2} \cdot 2)}{(1+2x)^2}$$

$$\frac{\frac{3}{2}\sqrt{1+x} \cdot 1+2x - 2\sqrt[3]{1+x}}{(1+2x)^2}$$

$$\frac{1}{1} + \frac{\frac{3}{2} \cdot 1 - 2}{1^2}x$$ ⇐ dont forget the x!

$1 + -\frac{1}{2}x$

linear + quadratic same

quadratic approx

2a-6 $f(\theta) = \tan\theta$ approx at $\theta \approx 0$

a = pt approx around

$$f(\theta) \approx f(0) + f'(0)\theta + \underbrace{\frac{f''(0)}{2}\theta^2}_{\text{quadratic part}}$$

Rhodes

office hrs

$f(0) = 0$

$f'(0) = \frac{1}{\cos^2\theta} = \frac{1}{1} = 1$

$f''(0) = 2 \cdot 1^2 \cdot 0 = 0$

$f'(\theta) = \sec^2\theta$

$f''(\theta) = 2\sec\theta(\sec\theta\tan\theta)$

$2\sec^2\theta\tan\theta$

$f(\theta) \approx 0 + 1\theta + \frac{0^2}{2}\theta = \theta$

$$f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x-x_0)^2$$

$\uparrow$ ideal gas law

11. $pV^k = C$

$$p = \frac{C}{V^k} = CV^{-k}$$

$V \rightsquigarrow V + \Delta V$

Rhodes Office hrs

$p \rightsquigarrow C(V+\Delta V)^{-k}$ $\downarrow$ goes $\uparrow$ just what you think

$$CV^{-k}\left(1 + \frac{\Delta V}{V}\right)^{-k}$$

$$(1+x)^{-k} \approx 1 - kV$$

$\uparrow x \approx 0$

$\downarrow \approx \quad \uparrow \frac{\Delta V}{V} \approx 0$

stop at office hrs →

$$CV^{-k}\left(1 - k\left(\frac{\Delta V}{V}\right)\right)^{-k}$$

what is that? →

$$\approx \frac{C}{V_0^k}\left(1 - k\frac{\Delta V}{V_0} + \frac{k(k+1)}{2}\left(\frac{\Delta V}{V_0}\right)^2\right)$$

12a Find approx at the point

$\frac{e^x}{1-x}$ quadratic $\leftarrow$

$$y = e^x(1-x)^{-1}$$

$$y' = e^x(1-x)^{-1} + \left(-1(1-x)^{-2} \cdot -1\right)e^x$$

$$\frac{-(x-2)e^x}{(1-x)^2}$$ see back

$y'' =$ see back

$$\frac{1}{1-0} + \left(+\frac{2}{1}\right)x + \frac{5}{2}x^2$$

$$1 + 2x + \frac{5}{2}x^2$$

$$y' = e^x(1-x)^{-1} - 1(x-1)^{-2}\, e^x$$

$$\frac{e^x}{1-x} + \frac{e^x}{(1-x)^2}$$

↑ common denom

$$\frac{e^x(1-x) + e^x}{(1-x)^2}$$

$$\frac{e^x - xe^x + e^x}{(1-y)^2} = \frac{e^x(2-x)}{(1-x)^2} = \frac{-e^x(x-2)}{(x-1)^2}$$

↓ ↑ weird calc answer

$$y'' = \frac{e^x(2-x) + -1 \cdot e^x}{}$$

$$2e^x - xe^x - e^x$$

$$\frac{e^x - xe^x}{(1-x)^2} \qquad \frac{e^x - e^x \cdot e^x x(1-x)^2 - 2(1-x)\cdot -1 (e^x - xe^x)}{(1-x)^2}$$

$$e^x - e^x \cdot xe^x (1-x)^2 + 2(1-x)(e^x - xe^x)$$

↑ could simplify

$$y''(0) = 1 - 1 \cdot 0 + 2 \cdot 1 \cdot (1-0)$$

2 ← should be 5

$$\text{calc } y'' = \frac{-(x^2-4x+5)e^x}{(x-1)^3}$$

12 d $\ln \cos x$ qudratic $x=0$

x chain rule

$y' = \frac{1}{\cos x} \cdot -\sin x = -\tan x$

$y'' = -\sec^2 x$

$0 + \underline{0\,x} + \frac{-1}{2} x^2$

$-\frac{1}{2} x^2$ ✓

12e e^{-x^2} quadradic $x=0$

$y' = e^{-x^2} \cdot -2x = -2xe^{-x^2}$

$y'' = e^{-x^2} \cdot -2x \cdot 2 + -2 \cdot e^{-x^2}$

$(4x^2-2)\,e^{-x^2}$

$1 + 0\,x + \frac{-2}{2} x^2$

$1 - x^2$

Lecture 10

2B-1 Sketch, find intervals, zeros, inflection points

a $y = x^3 - 3x + 1$

$y' = 3x^2 - 3$

$f''(x) = 0$ inflection

$3x^2 - 3 = 0$

$3x^2 = 3$

$x^2 = 1$

$x = \pm 1$

$y'' = 6x$

-1 1

y 3 -1

y' $3(-2)^2-3$ $3(0)^2-3$ $3(2)^2-3$

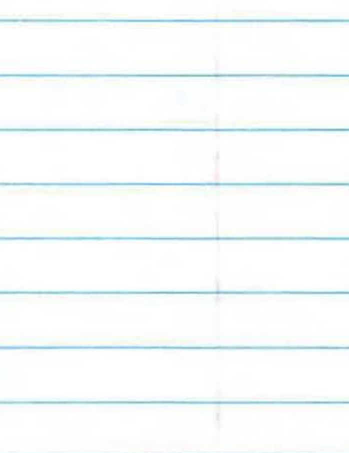
⊕ ⊖ ⊕

y'' $6(-2)$ $6(0)$ $6(2)$

↓ inflection ? should check ↑

$y \to -\infty$ $x \to -\infty$

$y \to \infty$ $x \to \infty$



c

$y = \frac{x}{x+4}$

$x(x+4)^{-1}$

$y' = 1 \cdot (x+4)^{-1} + -1(x+4)^{-2} \cdot 1\, x$

$\frac{1}{x+4} - \frac{x}{(x+4)^2}$

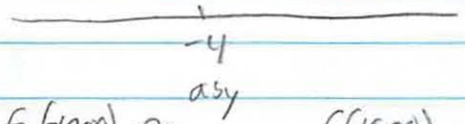
$\frac{x+4 \quad -x}{(x+4)^2} \rightarrow \frac{4}{(x+4)^2} \rightarrow 4(x+4)^{-2}$

$y'' = 0 + -2(x+4)^{-3} \cdot 1 \cdot 4$

$-8(x+4)^{-3}$

$\frac{-8}{\sqrt[3]{x+4}}$ ✓

$\frac{4}{(x+4)^2}$



* no other critical pts

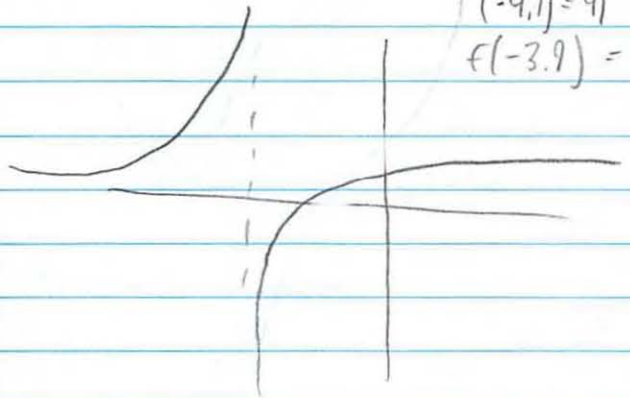
$f(-1000) = \infty$ $\quad$ $f(1000) = \infty$

$f'(-5)$ ⊕ $\quad$ $f'(0) =$ ⊕ $\quad$ $f'(2)$ ⊕

$f''(-5) \uparrow$ $\quad$ $f''(-1) \uparrow$ $\quad$ $f''(2) \uparrow$

$f(-4.1) = 41$

$f(-3.9) = -39$



* forgot to find horiz asymptote

h

$y = e^{-x^2}$

$x^r = (e^{\ln x})$

$= e^{r \ln x}$

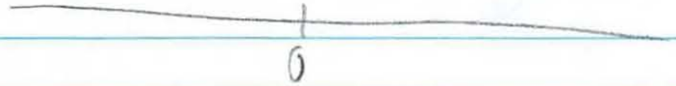
$x^{r\prime} = e^{r \ln x} \frac{r}{x}$

$y' = -2x e^{-x^2}$

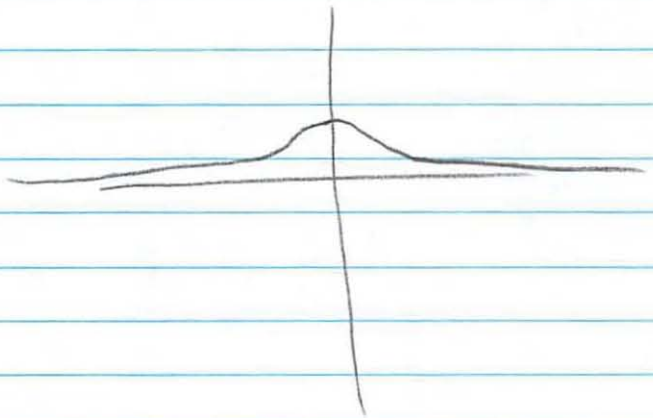
$y'' = (4x^2 - 2) e^{-x^2}$

$-2xe^{-x^2} = 0$

$x = 0$



$f(-1000)$	$f(-1)$	$f(0)$	$f(1)$	$f(1000)$
0	.367	1	.367	0
$f'(-1000)$	$f'(-1)$	$f'(0)$	$f'(1)$	$f'(1000)$
0	.735 ⊕	0	-.7358 ⊖	0
$f''(-1000)$	$f''(-1)$	$f''(0)$	$f''(1)$	$f''(1000)$
0	.735	-2	.7358	0
	↑	↓	↑	



2B-4 Suppose f continuous $0 \le x \le 10$

$f = 0$ at 4, 7, 9

$f'(x) > 0$ ⊕ $0 < x < 5$ $8 < x < 10$

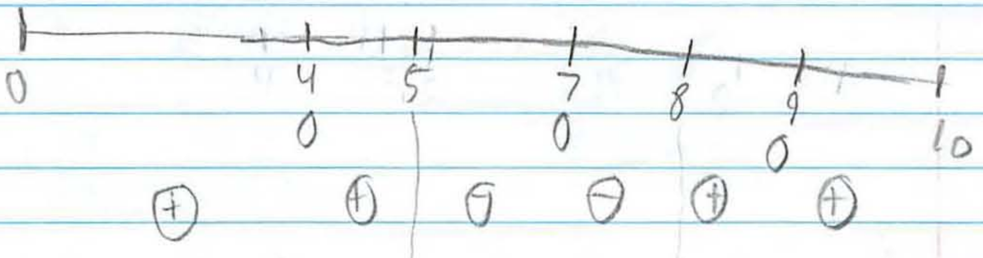
$f'(x) < 0$ ⊖ $5 < x < 8$

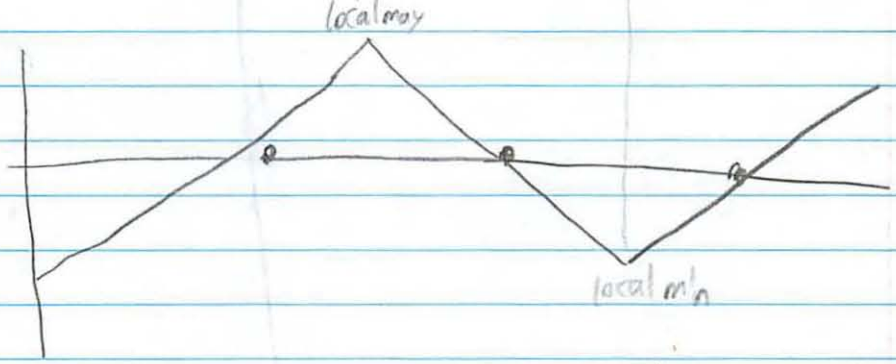
max, min? where?

slopes

⊕ → ⊖ max

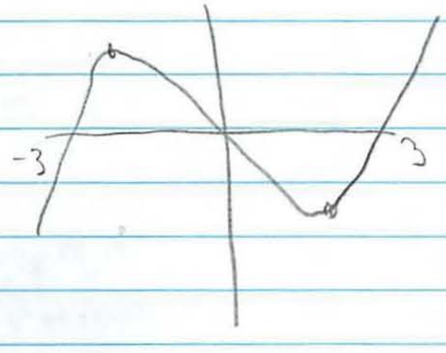
⊖ → ⊕ min





6ab Find cubic polynomial

local max $x = -1$ min $x = 1$



do it that process, want 0s here

$y' = (x+1)(x-1) = x^2 - 1$

$y = \frac{x^3}{3} - x + C$

0

$\downarrow \times 3$

$y = x^3 - 3x$

7a Prove that if f ↑ and has $\frac{d}{dx}$ at a then
$f'(a) > 0$
Hint: ⊕ functions have a limit ≥ 0

$$f'(a) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

if y ⓣ then $\Delta y > 0 \to \Delta x > 0$ so both $\frac{\Delta y}{\Delta x} \geq 0$
$\Delta y < 0 \to \Delta x < 0$

Only proves $><$ inequalities
not $\geq \leq$

7b If conclusion is now $f'(a) > 0$ statement now false
why does it fail + counter example

last step $\frac{\Delta y}{\Delta x} > 0$ does not imply $\lim_{\Delta x \to 0} \frac{\Delta x}{\Delta y} > 0$

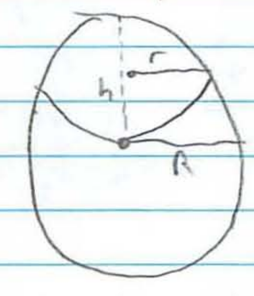
Counter example $f(x) = x^3$ ↑ for all x

but $f'(0) = 0$
↑ inflection pt

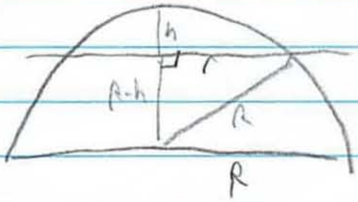
collab?

Part 2 Golf Ball problem

1a.



$$SA_{\substack{sphere\\cap}} = 2\pi R h$$
$$= 2\pi r h$$



$$R^2 = (R-h)^2 + r^2$$
$$(R-h)^2 = R^2 - r^2$$
$$R-h = \pm\sqrt{R^2-r^2}$$
$$R-h = \sqrt{R^2-r^2}$$
$$h = R - \sqrt{R^2-r^2}$$

$$A_s = 2\pi R\left(R - \sqrt{R^2-r^2}\right) \checkmark$$ 2/2

b. $A_s = 2\pi R(R - \sqrt{R^2-r^2}$

$$2\pi R^2 - 2\pi R\sqrt{R^2-r^2}$$
$$2\pi R^2 - 2\pi R\sqrt{R^2\left(1-\frac{r^2}{R^2}\right)}$$
$$2\pi R^2 - 2\pi R\cdot R\sqrt{1-\frac{r^2}{R^2}}$$
$$2\pi R^2 - 2\pi R^2\sqrt{1-\frac{r^2}{R^2}}$$
$$A_s = 2\pi R^2\left(1-\sqrt{1-\left(\frac{r}{R}\right)^2}\right) \checkmark$$

Linearization of A_s at $n=0$, where $w = \left(\frac{r}{R}\right)^2$

$$2\pi R^2 - 2\pi R^2\sqrt{1-\left(\frac{r}{R}\right)^2}$$

at $n \approx 0$ $(1+n)^{1/2} \approx 1 + \frac{1}{2}n$

Let $n = -\left(\frac{r}{R}\right)^2 \rightarrow (1-n)^{1/2} = 1 - \frac{1}{2}n$

$$\sqrt{1-\left(\frac{r}{R}\right)^2} \approx 1 - \frac{1}{2}\left(\frac{r}{R}\right)^2$$

sub in $\sqrt{1-\left(\frac{r}{R}\right)^2}$ into eqation

→

$$A_s = 2\pi R^2 - 2\pi R^2\left(1 - \frac{1}{2}\left(\frac{r}{R}\right)^2\right)$$

$$2\pi R^2 - 2\pi R^2 + 2\pi R^2\left(\frac{1}{2}\frac{r^2}{R^2}\right)$$

$$2\pi R^2 - 2\pi R^2 + \pi r^2$$

$\pi r^2 \rightarrow$ linear approx ✓

Quadratic approx

$$A_s \quad 2\pi R^2 - 2\pi R^2\sqrt{1-\left(\frac{r}{R}\right)^2}$$

at $n \approx 0$ let $n = -\left(\frac{r}{R}\right)^2$

$$(1+n)^{1/2} = 1 + \frac{1}{2}n - \frac{1}{8}n^2$$

So $$\left(1-\left(\frac{r}{R}\right)^2\right)^{1/2} = 1 - \frac{1}{2}\left(\frac{r}{R}\right)^2 - \frac{1}{8}\left(\frac{r}{R}\right)^4$$

Sub into quadratic approx

$$A_s = 2\pi R^2 - 2\pi R^2\left(1 - \frac{1}{2}\frac{r^2}{R^2} - \frac{1}{8}\frac{r^4}{R^4}\right)$$

$$2\pi R^2 - 2\pi R^2 + \pi r^2 + 4\pi\frac{r^4}{R^2}$$

$$\pi r^2 + \frac{4\pi r^4}{R^2}$$ ✓

a) linear approx

b) quadradic approx

Golf C

golf ball (exact)

- cut off  - subtracting SA
- make dimple  - adding SA

- surface area ↑

$$A_{golf} = A_{sphere} - \left(A_{\substack{spherical \\ cap}} + A_{dimple}\right) \cdot \overset{\cdot N}{300}$$

$$4\pi R^2 - (2\pi h R + 2\pi r^2) \cdot 300$$

$$\overset{\uparrow}{R - \sqrt{R^2 - r^2}}$$

iii, $4\pi R^2 - \underbrace{N\left(2\pi R\left(R - \sqrt{R^2 - r^2}\right)\right)}_{\text{linear + quadradic approx}} + N(2\pi r^2)$ ← exact answer

✓ 78.23 cm^2

i surface being flat - best linear oprox

area cap $= \pi r^2$ ⬭ ← just a slice

SA golf ball: $4\pi R^2 - 300(\pi r^2) + 300(2\pi r^2)$

ii Quadratic approx

$$4\pi R^2 - 300\left(\pi r^2 + \frac{4\pi r^4}{R^2}\right) + 300(2\pi r^2)$$

i) using $r = .15$ $R = 2.15$

$$A_s = 4\pi R^2 - 300(\pi r^2) + 300(2\pi r^2)$$

$$4\pi\, 2.15^2 - 300\pi\, .15^2 + 300 \cdot 2\pi\, .15^2$$

$$\underline{79.29 \text{ cm}^2}$$

ii) $$A_s = 4\pi R^2 - 300\left(\pi r^2 + \frac{4\pi r^4}{R^2}\right) + 300(2\pi r^2)$$

$$4\pi \cdot 2.15^2 - 300\left(\pi .15^2 + \frac{4\pi .15^4}{2.15^2}\right) + 300(2\pi .15^2)$$

$$58.0880 - 21.6186 + 42.4115$$

$$\underline{78.88 \text{ cm}^2}$$

iii) $$A_s = 4\pi R^2 - 300(2\pi R(R - \sqrt{R^2 - r^2}) + 300(2\pi r^2)$$

$$4\pi\, 2.15^2 - 300(2\pi \cdot 2.15 \cdot (2.15 - \sqrt{2.15^2 - .15^2}) + 300 \cdot 2\pi\, .15^2$$

$$58.0886 - 21.2316 + 42.4115$$

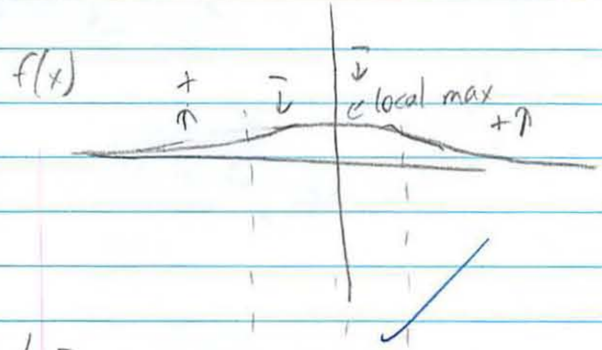
$$\underline{79.28 \text{ cm}^2}$$

accuracy $1 - \left|\frac{\text{exact} - \text{approx}}{\text{exact}}\right| \times 100 = 99.98\%$ linear

99.49% quad

$\frac{4}{4}$

2 $f(x) = \frac{1}{1+x^2} = 1(1+x^2)^{-1}$



$y' =$

$0 + -1(1+x^2)^{-2} \cdot 1 \cdot 2x$

$\frac{-2x}{(1+x^2)^2}$

$\frac{-2x}{(1+x^2)^2} = 0$

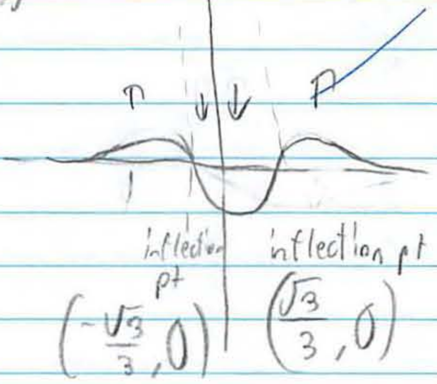
$x = 0$

local max

y''

$-2(1+x^2)^{-2} + 2(1+x^2)^{-3} \cdot 2x \cdot -2x$

$\frac{2(3x^2-1)}{(x^2+1)^3}$



$= 0$ at

$x = \frac{-\sqrt{3}}{3} \quad \frac{\sqrt{3}}{3}$

inflection pt

4/4

Lecture 11

Max/Min Problems

10/2

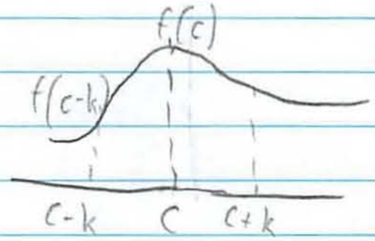
Yesterday: Local Max/Min

If $f(x)$ has a local max/min at $x=c$
and is differentiable at $f'(c)$ then $f'(c)=0$

Proove Assume f has a local max (local min similar)
at $x=C$

By definition $f(c) \geq f(x)$ for all x
in an open interval containing c

$x = c+h$



$f(c) \geq f(c+h)$ so $f(c+h) - f(c) \leq 0$

divide both side by h ↓

if $h \oplus$ $\dfrac{f(c+h) - f(c)}{h} \leq 0$

↓

take limit $\lim_{h \to 0^+} \dfrac{f(c+h) - f(c)}{h} \leq 0$

assumed limit exists

if $h \ominus$ $\dfrac{f(c+h) - f(c)}{h} \geq 0$

$\lim_{h \to 0^-} \dfrac{f(c+h) - f(c)}{h} \geq 0$

Assumed $f'(c)$ exists

so $\lim_{h\to 0} \frac{f(c+h)-f(c)}{h}$ exists

so $\lim_{h\to 0^+} = \lim_{h\to 0^-} = \boxed{0}$

If you want to find local max/min
look where f not differentiable or $f'(x)=0$
for canidates

critical pts

Strategy for curve sketching

- analyze where f not differentiable
  - domain f
  - where f continuous
  - where f differentiable
- compute derivitive + set $f'(x)=0$ + solve
- finish by placing critical pts on numberline + check if local max/min
- horizontal + vertical asymptote
- Symmetry/periodcity

- Intervals between increasing/decreasing
- $f''(x)$ for concavity

ex $y = \frac{x}{\ln x}$

domain $\ln(x)$ is $x > 0$

$\rightarrow$ domain $\frac{x}{\ln x}$ is $x > 0$ except $x = 1$

$$\lim_{x \to 1^+} \frac{x}{\ln(x)} = \frac{1}{\text{Small} +} = +\infty$$

$$\lim_{x \to 1^-} \frac{x}{\ln(x)} \quad \frac{1}{\text{Small} -} = -\infty$$

Horiz asymptote end behavior

limit as $x \to \pm\infty$

$$\lim_{x \to \infty} \frac{x}{\ln x} = \infty$$

---

zero $f'(x) = \left(\frac{x}{\ln}\right)' = x(-1 \ln x)^{-2}\frac{1}{x} + (\ln x)^{-1}$

$$= \frac{-1}{(\ln x)^2} + \frac{1}{\ln x}$$

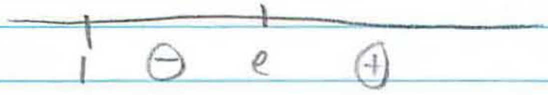
Where = to 0

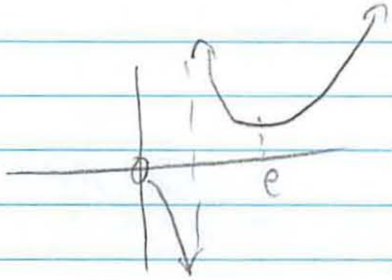
common denom

$$\frac{-1 + \ln x}{(\ln x)^2}$$

Analyse where numerator = 0

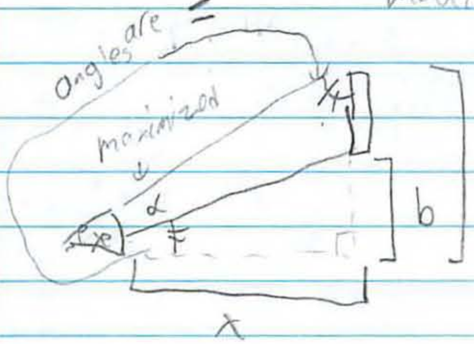
$x = e$





Word Problems

Easy: Picture on a wall
-above eye level
Stand at given distance from picture
-maximize viewing angle



express α as a function of x

$$\alpha = \tan^{-1}\left(\frac{a}{x}\right) - \tan^{-1}\left(\frac{b}{x}\right)$$

$$\frac{d\alpha}{dx} = \dots = 0$$

lone critical pt $\boxed{x = \sqrt{ab}}$

min if under painting
min if far away

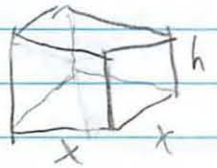
↑ is the max
in between

$$\frac{a}{x_0} = \frac{x_0}{b}$$

Want to build a box

- square base
- no top
- volume V_0 of jelly beans

Want minimal surface area



baddly drawn box →

$SA = x^2 + 4xh$
no top

↑ 2 variable function

$V_0 = x^2 h \Rightarrow \boxed{h = \frac{V_0}{x^2}}$ ← key info

↑ constant

$$x^2 + 4x\frac{V_0}{x^2} = SA(x)$$

↓ find min

find $SA'_{(x)} = 0$

$x_{min} = \sqrt[3]{2V_0}$

$\frac{h}{x} = \frac{V_0}{x^3}$ At $x = x_{min}$ $\frac{h_{min}}{x_{min}} = \frac{V_0}{2V_0} = \frac{1}{2}$

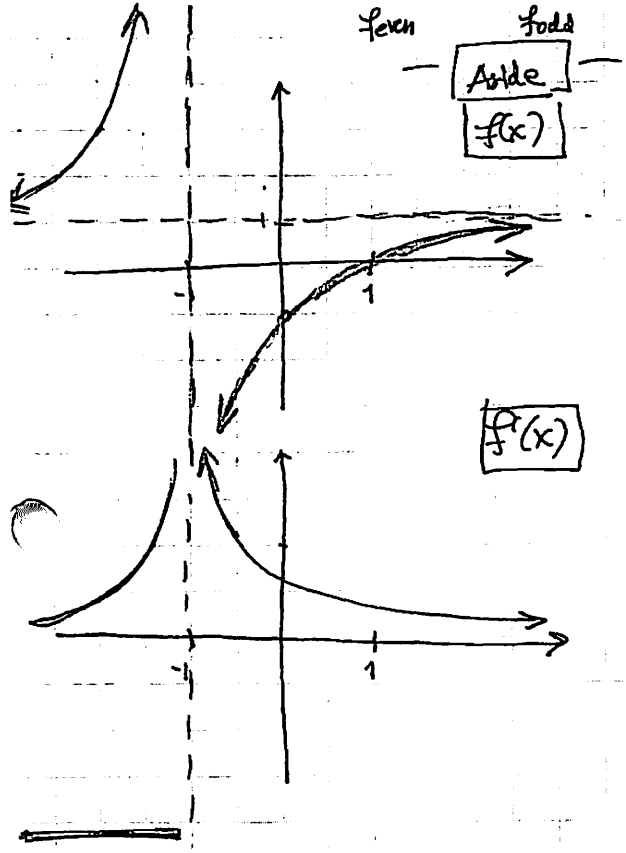
height half base lenght

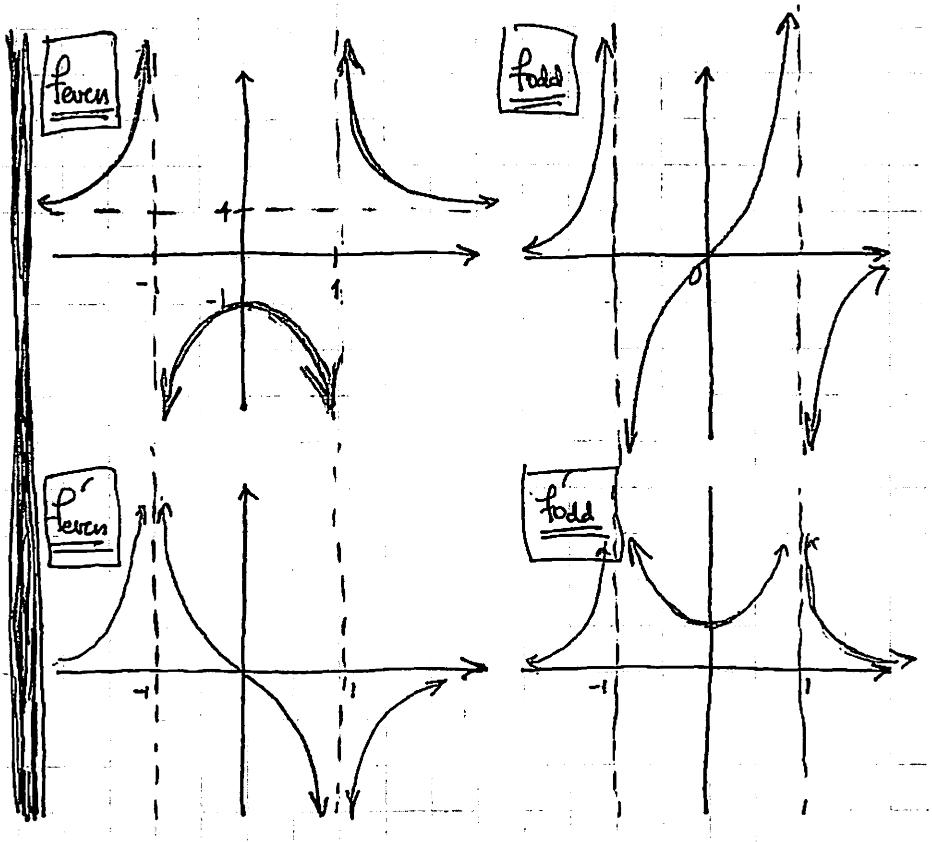
18.01 · Problem Set 2
Fall 2009.

PSet 2A · Part II

Problem 1 . Graph the even and odd functions from PSet 1, Part II, prob. 1. Below, graph derivatives.

$$f(x) = \frac{x-1}{x+1} = \underbrace{\frac{x^2+1}{x^2-1}}_{f_{even}} + \underbrace{\frac{2x}{1-x^2}}_{f_{odd}}$$





Problem 2 Compute

a) $\frac{d}{dx}\tan^3(x^4) = 3\tan^2(x^4)\,(4x^3)\cdot(\sec^2(x^4)) = \boxed{12x^3\tan^2(x^4)\sec^2(x^4)}$

b) $\frac{d}{dy}\sin^2 y\cos^2 y$ (in two ways)

~~p~~ product rule: $(2\sin y\cos y)(\cos^2 y) + \sin^2 y\,(2\cos y(-\sin y)) = 2\sin y\cos y(\cos^2 y - \sin^2 y) = \boxed{\sin 2y\cos 2y}$

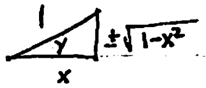
" rewrite as $f(2y)^4$: $\sin^2 y\cos^2 y = \left(\frac{1}{2}\sin 2y\right)^2 = \frac{1}{4}\sin^2 2y$.

same

$\frac{d}{dy} = \frac{1}{4}\cdot 2\sin 2y\cos 2y\cdot 2 = \boxed{\sin 2y\cos 2y}$

Problem 3 a) Derive the formula for $\frac{d}{dx}\cos^{-1}x$.

$y=\cos^{-1}x \Rightarrow \cos y = x$ Implicit diff: $-\sin y\frac{dy}{dx}=1 \Rightarrow \frac{dy}{dx}=-\frac{1}{\sin y}$.



$\sin y = \pm\sqrt{1-x^2}$

need "+"

$$\boxed{\frac{dy}{dx}=\frac{-1}{\sqrt{1-x^2}}}$$

b) Explain why $\frac{d}{dx}\cos^{-1}x+\frac{d}{dx}\sin^{-1}x=0$.

One explanation: graph of $\cos^{-1}x$ is the graph of $\sin^{-1}x$ flipped upside down and shifted

$\Rightarrow$ at any pt have derivatives of equal magnitude and opposite sign.

Problem 4

$\boxed{82\ /8a}$ $M=\frac{2}{3}\log_{10}\frac{E}{E_0}$, $E_0=7\cdot10^{-3}$

$M_2-M_1=\Delta M=1$. Question: $E_2/E_1=?$ $1=M_2-M_1=\frac{2}{3}\log_{10}\frac{E_2/\cancel{E_0}}{E_1/\cancel{E_0}} \Rightarrow$ $\boxed{E_2/E_1=(10^1)^{3/2}}$

$\boxed{8c}$ $M=6 \Rightarrow E=(10^6)^{2/3}E_0 \Rightarrow E=7\cdot10^{-3}\cdot10^9=7\cdot10^6$

$\#\text{days}=E/3\cdot10^5 = \frac{7\cdot10^6}{3\cdot10^5}=\frac{7}{3}10$ $\boxed{\approx 23 \text{ days}}$

$\boxed{10}$ $\log_3 2 = p/q \Rightarrow q\log 2 = p\log 3 \Rightarrow 2^q=3^p$ but $\gcd(2,3)=1$ $\otimes$. (and $\mathbb{Z}$ is UFD).

$\boxed{11}$ $\log 1/2 < 0$. [illegible]

$1<2$, but since $\log 1/2<0$, $1\cdot\log 1/2 > 2\cdot\log 1/2$ . . .

$\boxed{12}$ a) How many digits does $2^{756\,839}-1$ have?

graders – check this!

$10^x=2^{756\,839}$ $x=756\,839\cdot\log 2 = 227\,831{,}24 \approx$ $\boxed{227\,831 \text{ digits}}$

b) How many pages needed to print? $\lfloor x/4600\rfloor = 49.5$ – so, $\boxed{50 \text{ pages}}$

$\boxed{\S 8.4}$ 18) $y=\sqrt[3]{(x+1)(x-2)(2x+7)}$ $\ln y=\frac{1}{3}\left(\ln(x+1)+\ln(x-2)+\ln(2x+7)\right)$

$\Rightarrow \frac{1}{y}\frac{dy}{dx}=\frac{1}{3}\left(\frac{1}{x+1}+\frac{1}{x-2}+\frac{2}{2x+7}\right) \Rightarrow \boxed{\frac{dy}{dx}=\frac{1}{3}\sqrt{(-)}\left(\frac{1}{x+1}+\frac{1}{x-2}+\frac{2}{2x+7}\right)}$

19a) $y=\frac{e^x(x^2-1)}{\sqrt{6x-2}}$ $\ln y = x+\ln(x^2-1)-\frac{\ln(6x-2)}{2}$ $\boxed{\frac{dy}{dx}=\frac{e^x(x^2-1)}{\sqrt{6x-2}}\cdot\left(1+\frac{2x}{x^2-1}+\frac{3}{6x-2}\right)}$

Problem 5 Compute $\frac{d}{dx} x^x$

Use method of log. diff. ~~and implicit diff.~~

$$y = x^x \qquad \ln y = x \ln x \overset{\text{diff}}{\Rightarrow} \frac{1}{y}\frac{dy}{dx} = \ln x + x \cdot \frac{1}{x} = \ln x + 1.$$

$$\frac{dy}{dx} = y \cdot (\ln x + 1) \Rightarrow \boxed{\frac{dy}{dx} = x^x(\ln x + 1)}$$

Problem 6 Derive the formula for $D(u_1 u_2 \cdots u_n)$ from PS1, II, 7b using log. diff.

$$y = u_1 \cdot u_2 \cdot \ldots \cdot u_n = \prod_{i=1}^{n} u_i$$

$$\Rightarrow \ln y = \ln u_1 + \ldots + \ln u_n = \sum_{i=1}^{n} \ln u_i$$

diff: $\frac{1}{y}\frac{dy}{dx} = \sum_{i=1}^{n} \frac{1}{u_i} u_i' \Rightarrow \frac{dy}{dx} = y \sum_{i=1}^{n} \frac{1}{u_i} u_i'$, and, since $y = u_1 u_2 \cdots u_n$

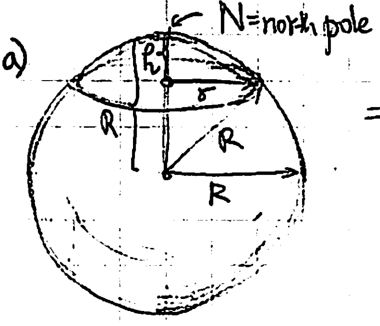
$$\boxed{\frac{dy}{dx} = \sum_{i=1}^{n} u_i' \frac{u_1 u_2 \cdots u_n}{u_i}} \quad //$$

Problem Set 2b

Part II

Problem 1 (Golf balls)

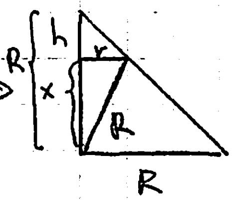
a)



a sphere

top part : spherical cap

$\Rightarrow$



$x := R - h$

(1) $x^2 + r^2 = R^2 \Rightarrow x = \pm\sqrt{R^2 - r^2}$

~~[illegible]~~
(take "+" sign)

$$\Rightarrow h = R - x = R - \sqrt{R^2 - r^2}$$

(2) Area of cap $= A = 2\pi h R$ this was (given in prob statement)

$$\Rightarrow \boxed{A = 2\pi\left(R - \sqrt{R^2 - r^2}\right)R}$$

$\boxed{b)}$ 1) Express formula for area of sph. cap in terms of R^2, r/R.

2) Use linear and 3) quadratic approx. to $\sqrt{1+x}$ near $x=0$ to find a good (2) and even better (3) approx to area.

$$(1)\ A = 2\pi\left(R-\sqrt{R^2-r^2}\right)R = 2\pi\left(R^2 - R\sqrt{R^2-r^2}\right) = 2\pi\left(R^2 - R\sqrt{R^2\left(1-\frac{r^2}{R^2}\right)}\right) =$$

$$= 2\pi\left(R^2 - \underbrace{R\sqrt{R^2}}_{\substack{\| \\ R\cdot R \\ \| \\ R^2}}\sqrt{1-\frac{r^2}{R^2}}\right) = 2\pi\left(R^2\right)\left(1-\sqrt{1-\frac{r^2}{R^2}}\right)$$

$$= \boxed{2\pi\left(R^2\right)\left(1-\sqrt{1-\left(\frac{r}{R}\right)^2}\right)}$$

(2) Linear approx to $(1+x)^{1/2}$ is $1+\frac{1}{2}x$. For us, $x = -\left(\frac{r}{R}\right)^2$. Need to check: as $r/R \to 0$, $-(r/R)^2 \to 0$, so $x \to 0$.

$$\Rightarrow \sqrt{1-\left(\frac{r}{R}\right)^2} \sim 1+\frac{1}{2}\left(-\left(\frac{r}{R}\right)^2\right) = 1-\frac{1}{2}\left(\frac{r}{R}\right)^2$$

Thus $A_{lin} = 2\pi R^2\left(\cancel{1} - \left(\cancel{1} - \frac{1}{2}\left(\frac{r}{R}\right)^2\right)\right) = \cancel{2}\pi\cancel{R^2}\cdot\frac{1}{\cancel{2}}\cdot\left(r/\cancel{R}\right)^2 = \boxed{\pi r^2}$! (Great!)

(if $r \ll R$, spherical cap looks a lot like a flat circle, and area ≈ area of circle w/ given radius r...)

(3) quadratic approx : $(1+x)^{1/2} \approx 1+\frac{1}{2}x - \frac{1}{8}x^2$.

$$A_{quad} = \cancel{2}\pi R^2\left(\cancel{1} - \left(\cancel{1} - \frac{1}{\cancel{2}}(r/R)^2 - \frac{1}{\cancel{8}}(r/R)^4\right)\right) =$$

$$= \pi R^2\left((r/R)^2 + \frac{1}{4}(r/R)^4\right) = \boxed{\pi r^2 + \pi\frac{r^4}{4R^2}}$$

(c) Golf ball w/ removed hemispherical dimples of diam. ~~3 mm.~~ radius r(mm) (300 total).
Find SA w/ $r, R, \pi, 300$. in 3 cases:

(i) approx. pretending removed surface is flat.

$$A_{i)} = 4\pi R^2 - 300\left(\pi r^2\right) + 300\cdot\frac{\overset{2}{\cancel{4}}\pi r^2}{\cancel{2}} = 4\pi R^2 + 300r^2\pi = \boxed{4\pi\left(R^2 + \cancel{100}\ 75r^2\right)}$$

ii) using higher order approx from part 1(b).

$$A_{ii} = 4\pi R^2 - 300\left(\pi r^2 + \frac{\pi r^4}{4R^2}\right) + 600\pi r^2$$

$$= 4\pi R^2 + 300\pi r^2 - 75\pi r^2\left(\frac{r^2}{R^2}\right).$$

iii) the exact formula

$$A\ iii) = 4\pi R^2 - 300\left(2\pi R^2 - 2\pi R^2\sqrt{1-\frac{r^2}{R^2}}\right) + 600\pi r^2$$

$$= 4\pi R^2 - 600\pi R^2\left(1-\sqrt{1-\left(\frac{r}{R}\right)^2}\right) + 600\pi r^2.$$

Evaluate each of the answers for $r=0.15$, $R=2.15$.

i) $4\pi\left((2.15)^2 + 75(0.15)^2\right) = 79.29$

ii) $4\pi(2.15)^2 + 300\pi(0.15)^2 - 75\pi(0.15)^2\left(\frac{0.15}{2.15}\right)^2 = 79.2679939$

iii) $4\pi(2.15)^2 - 600\pi(2.15)^2\left(1-\sqrt{1-\left(\frac{0.15}{2.15}\right)^2}\right) + 600\pi(0.15)^2 = 79.2679309$

graders: check this! Do not trust any numbers.

Prob 2 Draw the graph of $f(x)=\frac{1}{1+x^2}$, and dir. underneath, graphs of $f'(x)$, $f''(x)$.

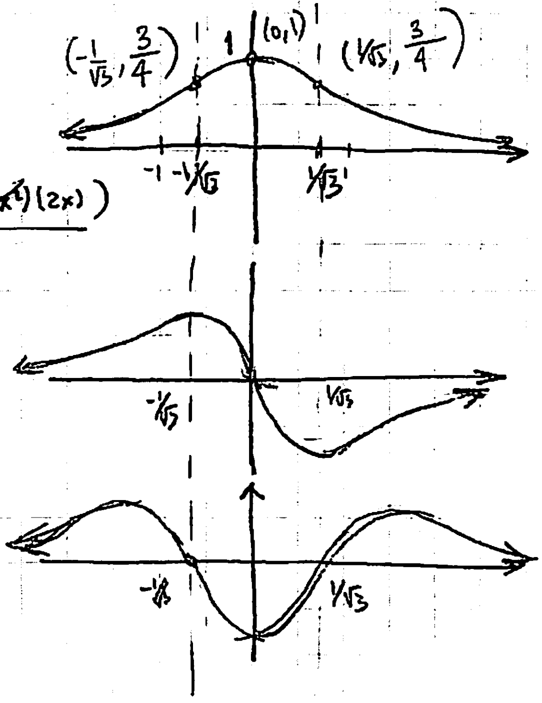
$$f'(x) = \frac{-1}{(1+x^2)^2}(2x) = \frac{-2x}{(1+x^2)^2}$$

$$f''(x) = \frac{(-2)(1+x^2)^{\cancel{2}} - (-2x)(2\cancel{(1+x^2)}(2x))}{(1+x^2)^{\cancel{4}\,3}}$$

$$= \frac{-2(1+x^2) + 8x^2}{(1+x^2)^3}$$

$$= 2\cdot\frac{3x^2-1}{(1+x^2)^3}$$

$f''(x)=0$ at $x=\pm 1/\sqrt{3}$



LOGARITHMIC DIFFERENTIATION

The following problems illustrate the process of logarithmic differentiation. It is a means of differentiating algebraically complicated functions or functions for which the ordinary rules of differentiation do not apply. For example, in the problems that follow, you will be asked to differentiate expressions where a variable is raised to a variable power. An example and two COMMON INCORRECT SOLUTIONS are :

1.) $D\{x^{(2x+3)}\} = (2x+3)x^{(2x+3)-1} = (2x+3)x^{(2x+2)}$

and

2.) $D\{x^{(2x+3)}\} = x^{(2x+3)}(2)\ln x$.

BOTH OF THESE SOLUTIONS ARE WRONG because the ordinary rules of differentiation do not apply. Logarithmic differentiation will provide a way to differentiate a function of this type. It requires deft algebra skills and careful use of the following unpopular, but well-known, properties of logarithms. Though the following properties and methods are true for a logarithm of any base, only the natural logarithm (base e, where e ≈ 2.718281828), ln , will be used in this problem set.

PROPERTIES OF THE NATURAL LOGARITHM

1. $\ln 1 = 0$.
2. $\ln e = 1$.
3. $\ln e^x = x$.
4. $\ln y^x = x \ln y$.
5. $\ln(xy) = \ln x + \ln y$.
6. $\ln\left(\frac{x}{y}\right) = \ln x - \ln y$.

$\frac{d}{dx} \ln(x) = \frac{1}{x} x'$

AVOID THE FOLLOWING LIST OF COMMON MISTAKES

1. $\ln(x+y) = \ln x + \ln y$.
2. $\ln(x-y) = \ln x - \ln y$.

3. $\ln(xy) = \ln x \ \ln y$.

4. $\ln\left(\frac{x}{y}\right) = \frac{\ln x}{\ln y}$.

5. $\frac{\ln x}{\ln y} = \ln x - \ln y$.

The following problems range in difficulty from average to challenging.

- *PROBLEM 1 :* Differentiate $y = x^x$.

  Click HERE to see a detailed solution to problem 1.

- *PROBLEM 2 :* Differentiate $y = x^{(e^x)}$.

  Click HERE to see a detailed solution to problem 2.

- *PROBLEM 3 :* Differentiate $y = (3x^2+5)^{1/x}$

  Click HERE to see a detailed solution to problem 3.

- *PROBLEM 4 :* Differentiate $y = (\sin x)^{x^3}$.

  Click HERE to see a detailed solution to problem 4.

- *PROBLEM 5 :* Differentiate $y = 7x(\cos x)^{x/2}$.

  Click HERE to see a detailed solution to problem 5.

- *PROBLEM 6 :* Differentiate $y = \sqrt{x}^{\sqrt{x}} e^{x^2}$.

  Click HERE to see a detailed solution to problem 6.

- *PROBLEM 7 :* Differentiate $y = x^{\ln x}(\sec x)^{3x}$.

  Click HERE to see a detailed solution to problem 7.

- *PROBLEM 8 :* Differentiate $y = \dfrac{(\ln x)^x}{2^{3x+1}}$.

  Click HERE to see a detailed solution to problem 8.

- *PROBLEM 9 :* Differentiate $y = \dfrac{x^{2x}(x-1)^3}{(3+5x)^4}$.

  Click HERE to see a detailed solution to problem 9.

- *PROBLEM 10 :* Consider the function $f(x) = \dfrac{x^5 e^x (4x+3)}{5^{\ln x}(3-x)^2}$. Find an equation of the line tangent to the graph of *f* at *x*=1 .

Click HERE to see a detailed solution to problem 10.

- *PROBLEM 11 :* Consider the function $f(x) = \pi^2 + 2^x + x^2 + x^{1/x}$. Determine the slope of the line perpendicular to the graph of *f* at *x*=1 .

  Click HERE to see a detailed solution to problem 11.

- *PROBLEM 12 :* Differentiate $y = x^{(x^{(e^4)})}$

  Click HERE to see a detailed solution to problem 12.

Click HERE to return to the original list of various types of calculus problems.

Your comments and suggestions are welcome. Please e-mail any correspondence to Duane Kouba by clicking on the following address :

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- About this document ...

Duane Kouba
1998-06-06

SOLUTIONS TO LOGARITHMIC DIFFERENTIATION

SOLUTION 1 : Because a variable is raised to a variable power in this function, the ordinary rules of differentiation DO NOT APPLY ! The function must first be revised before a derivative can be taken. Begin with

$$y = x^x$$

Apply the natural logarithm to both sides of this equation getting

$$\ln y = \ln x^x$$

$$\ln y = x \ln x \ .$$

Differentiate both sides of this equation. The left-hand side requires the chain rule since y represents a function of x . Use the product rule on the right-hand side. Thus, beginning with

$$\ln y = x \ln x$$

and differentiating, we get

$$\frac{1}{y} y' = x \frac{1}{x} + (1) \ln x$$

$$= 1 + \ln x \ .$$

Multiply both sides of this equation by y, getting Solve for y'

$$y' = y(1 + \ln x) = x^x (1 + \ln x) \ .$$

Click HERE to return to the list of problems.

SOLUTION 2 : Because a variable is raised to a variable power in this function, the ordinary rules of differentiation DO NOT APPLY ! The function must first be revised before a derivative can be taken. Begin with

$$y = x^{(e^x)} \ .$$

Apply the natural logarithm to both sides of this equation getting

$$\ln y = \ln x^{(e^x)}$$

$$= e^x \ln x \ .$$

Differentiate both sides of this equation. The left-hand side requires the chain rule since y represents a function of x . Use the product rule on the right-hand side. Thus, beginning with

$$\ln y = e^x \ln x$$

and differentiating, we get

$$\frac{1}{y}y' = e^x\left\{\frac{1}{x}\right\} + e^x \ln x$$

(Get a common denominator and combine fractions on the right-hand side.)

$$= \frac{e^x}{x} + e^x \ln x\left\{\frac{x}{x}\right\}$$

$$= \frac{e^x}{x} + \frac{xe^x \ln x}{x}$$

$$= \frac{e^x + xe^x \ln x}{x}$$

(Factor out e^x in the numerator.)

$$= \frac{e^x(1 + x\ln x)}{x} \ .$$

Multiply both sides of this equation by y, getting

$$y' = y\frac{e^x(1 + x\ln x)}{x}$$

$$= x^{(e^x)}\frac{e^x(1 + x\ln x)}{x^1}$$

(Combine the powers of x .)

$$= x^{(e^x - 1)}e^x(1 + x\ln x) \ .$$

Click HERE to return to the list of problems.

SOLUTION 3 : Because a variable is raised to a variable power in this function, the ordinary rules of differentiation DO NOT APPLY ! The function must first be revised before a derivative can be taken. Begin with

$$y = (3x^2+5)^{1/x} .$$

Apply the natural logarithm to both sides of this equation getting

$$\ln y = \ln(3x^2+5)^{1/x}$$

$$= (1/x)\ln(3x^2+5)$$

$$= \frac{\ln(3x^2+5)}{x} .$$

Differentiate both sides of this equation. The left-hand side requires the chain rule since y represents a function of x . Use the quotient rule and the chain rule on the right-hand side. Thus, beginning with

$$\ln y = \frac{\ln(3x^2+5)}{x}$$

and differentiating, we get

$$\frac{1}{y}y' = \frac{x\left\{\frac{1}{3x^2+5}\right\}(6x) - \ln(3x^2+5)(1)}{x^2}$$

(Get a common denominator and combine fractions in the numerator.)

$$= \frac{\frac{6x^2}{3x^2+5} - \ln(3x^2+5)\left\{\frac{3x^2+5}{3x^2+5}\right\}}{\frac{x^2}{1}}$$

(Dividing by a fraction is the same as multiplying by its reciprocal.)

$$= \frac{6x^2 - (3x^2+5)\ln(3x^2+5)}{3x^2+5}\frac{1}{x^2}$$

$$= \frac{6x^2 - (3x^2+5)\ln(3x^2+5)}{x^2(3x^2+5)} \ .$$

Multiply both sides of this equation by *y*, getting

$$y' = y\frac{6x^2 - (3x^2+5)\ln(3x^2+5)}{x^2(3x^2+5)}$$

$$= (3x^2+5)^{1/x}\frac{6x^2 - (3x^2+5)\ln(3x^2+5)}{x^2(3x^2+5)^1}$$

(Combine the powers of $(3x^2+5)$.)

$$= \frac{(3x^2+5)^{(1/x-1)}\{6x^2 - (3x^2+5)\ln(3x^2+5)\}}{x^2} \ .$$

Click HERE to return to the list of problems.

SOLUTION 4 : Because a variable is raised to a variable power in this function, the ordinary rules of differentiation DO NOT APPLY ! The function must first be revised before a derivative can be taken. Begin with

$$y = (\sin x)^{x^3} \ .$$

Apply the natural logarithm to both sides of this equation getting

$$\ln y = \ln(\sin x)^{x^3}$$

$$= x^3 \ln(\sin x) \ .$$

Differentiate both sides of this equation. The left-hand side requires the chain rule since *y* represents a function of *x* . Use the product rule and the chain rule on the right-hand side. Thus, beginning with truein $\ln y = x^3 \ln(\sin x)$

and differentiating, we get

$$\frac{1}{y}y' = x^3\left\{\frac{1}{\sin x}\right\}\cos x + (3x^2)\ln(\sin x)$$

(Get a common denominator and combine fractions on the right-hand side.)

$$= \frac{x^3 \cos x}{\sin x} + 3x^2 \ln(\sin x)\left\{\frac{\sin x}{\sin x}\right\}$$

$$= \frac{x^3 \cos x + 3x^2 \sin x \ln(\sin x)}{\sin x} \ .$$

Multiply both sides of this equation by *y*, getting

$$y' = y\frac{x^3 \cos x + 3x^2 \sin x \ln(\sin x)}{\sin x}$$

$$= (\sin x)^{x^3} \frac{x^3 \cos x + 3x^2 \sin x \ln(\sin x)}{(\sin x)^1}$$

(Combine the powers of $(\sin x)$.)

$$= (\sin x)^{(x^3-1)}\left\{x^3 \cos x + 3x^2 \sin x \ln(\sin x)\right\} \ .$$

Click HERE to return to the list of problems.

SOLUTION 5 : Because a variable is raised to a variable power in this function, the ordinary rules of differentiation DO NOT APPLY ! The function must first be revised before a derivative can be taken. Begin with

$$y = 7x(\cos x)^{x/2} \ .$$

Apply the natural logarithm to both sides of this equation and use the algebraic properties of logarithms, getting

$$\ln y = \ln\left((7x)(\cos x)^{x/2}\right)$$

$$= \ln(7x) + \ln(\cos x)^{x/2}$$

$$= \ln(7x) + (x/2)\ln(\cos x) \ .$$

Differentiate both sides of this equation. The left-hand side requires the chain rule since *y* represents a function of *x* . Use the product rule and the chain rule on the right-hand side. Thus, beginning with

$$\ln y = \ln(7x) + (x/2)\ln(\cos x)$$

and differentiating, we get

$$\frac{1}{y}y' = \left\{\frac{1}{7x}\right\}7 + (x/2)\left\{\frac{1}{\cos x}\right\}(-\sin x) + (1/2)\ln(\cos x)$$

$$= \frac{1}{x} - \frac{x\sin x}{2\cos x} + \frac{\ln(\cos x)}{2}$$

(Get a common denominator and combine fractions on the right-hand side.)

$$= \frac{1}{x}\left\{\frac{2\cos x}{2\cos x}\right\} - \frac{x\sin x}{2\cos x}\left\{\frac{x}{x}\right\} + \frac{\ln(\cos x)}{2}\left\{\frac{x\cos x}{x\cos x}\right\}$$

$$= \frac{2\cos x - x^2\sin x + x\cos x\ln(\cos x)}{2x\cos x} .$$

Multiply both sides of this equation by y, getting

$$y' = y\frac{2\cos x - x^2\sin x + x\cos x\ln(\cos x)}{2x\cos x}$$

$$= 7x(\cos x)^{x/2}\frac{2\cos x - x^2\sin x + x\cos x\ln(\cos x)}{2x\cos x}$$

(Divide out a factor of x .)

$$= 7(\cos x)^{x/2}\frac{2\cos x - x^2\sin x + x\cos x\ln(\cos x)}{2(\cos x)^1}$$

(Combine the powers of $(\cos x)$.)

$$= (7/2)(\cos x)^{(x/2-1)}\{2\cos x - x^2\sin x + x\cos x\ln(\cos x)\} .$$

Click HERE to return to the list of problems.

SOLUTION 6 : Because a variable is raised to a variable power in this function, the ordinary rules of differentiation DO NOT APPLY ! The function must first be revised before a derivative can be taken. Begin with

$$y = \sqrt{x}^{\sqrt{x}} e^{x^2} .$$

Apply the natural logarithm to both sides of this equation and use the algebraic properties of logarithms, getting

$$\ln y = \ln \left(\sqrt{x}^{\sqrt{x}} e^{x^2} \right)$$

$$= \ln \left(\sqrt{x}^{\sqrt{x}} \right) + \ln \left(e^{x^2} \right)$$

$$= \sqrt{x} \ln(\sqrt{x}) + x^2 \ln(e)$$

$$= \sqrt{x} \ln(\sqrt{x}) + x^2 (1)$$

$$= \sqrt{x} \ln(\sqrt{x}) + x^2 .$$

Differentiate both sides of this equation. The left-hand side requires the chain rule since *y* represents a function of *x* . Use the product rule and the chain rule on the right-hand side. Thus, beginning with

$$\ln y = \sqrt{x} \ln(\sqrt{x}) + x^2$$

and differentiating, we get

$$\frac{1}{y} y' = \sqrt{x} \left\{ \frac{1}{\sqrt{x}} \right\} (1/2) x^{-1/2} + (1/2) x^{-1/2} \ln(\sqrt{x}) + 2x$$

$$= \frac{1}{2\sqrt{x}} + \frac{\ln(\sqrt{x})}{2\sqrt{x}} + 2x$$

(Get a common denominator and combine fractions on the right-hand side.)

$$= \frac{1}{2\sqrt{x}} + \frac{\ln(\sqrt{x})}{2\sqrt{x}} + 2x \left\{ \frac{2\sqrt{x}}{2\sqrt{x}} \right\}$$

$$= \frac{1 + \ln(\sqrt{x}) + 4x^{1+1/2}}{2\sqrt{x}}$$

$$= \frac{1 + \ln(\sqrt{x}) + 4x^{3/2}}{2\sqrt{x}} .$$

Multiply both sides of this equation by *y*, getting

$$y' = y \frac{1 + \ln(\sqrt{x}) + 4x^{3/2}}{2\sqrt{x}}$$

$$= \sqrt{x}^{\sqrt{x}} e^{x^2} \frac{1 + \ln(\sqrt{x}) + 4x^{3/2}}{2\sqrt{x}^1}$$

(Combine the powers of $\sqrt{x}$.)

$$= (1/2) \sqrt{x}^{(\sqrt{x}-1)} e^{x^2} \{ 1 + \ln(\sqrt{x}) + 4x^{3/2} \} \ .$$

Click HERE to return to the list of problems.

SOLUTION 7 : Because a variable is raised to a variable power in this function, the ordinary rules of differentiation DO NOT APPLY ! The function must first be revised before a derivative can be taken. Begin with

$$y = x^{\ln x} (\sec x)^{3x} \ .$$

Apply the natural logarithm to both sides of this equation and use the algebraic properties of logarithms, getting

$$\ln y = \ln \left(x^{\ln x} (\sec x)^{3x} \right)$$

$$= \ln x^{(\ln x)} + \ln (\sec x)^{3x}$$

$$= (\ln x)(\ln x) + 3x \ln(\sec x)$$

$$= (\ln x)^2 + 3x \ln(\sec x) \ .$$

Differentiate both sides of this equation. The left-hand side requires the chain rule since *y* represents a function of *x* . Use the product rule and the chain rule on the right-hand side. Thus, beginning with

$$\ln y = (\ln x)^2 + (3x) \ln(\sec x)$$

and differentiating, we get

$$\frac{1}{y} y' = 2(\ln x) \left\{ \frac{1}{x} \right\} + 3x \left\{ \frac{1}{\sec x} \right\} (\sec x \tan x) + (3) \ln(\sec x)$$

(Divide out a factor of $\sec x$.)

$$= \frac{2\ln x}{x} + 3x\tan x + 3\ln(\sec x)$$

(Get a common denominator and combine fractions on the right-hand side.)

$$= \frac{2\ln x}{x} + 3x\tan x\left\{\frac{x}{x}\right\} + 3\ln(\sec x)\left\{\frac{x}{x}\right\}$$

$$= \frac{2\ln x + 3x^2\tan x + 3x\ln(\sec x)}{x} .$$

Multiply both sides of this equation by y, getting

$$y' = y\frac{2\ln x + 3x^2\tan x + 3x\ln(\sec x)}{x}$$

$$= x^{\ln x}(\sec x)^{3x}\frac{2\ln x + 3x^2\tan x + 3x\ln(\sec x)}{x^1}$$

(Combine the powers of x .)

$$= x^{(\ln x - 1)}(\sec x)^{3x}\{2\ln x + 3x^2\tan x + 3x\ln(\sec x)\}$$

Click HERE to return to the list of problems.

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- About this document ...

Duane Kouba
1998-06-06

Lecture 11: Max/Min Problems

Example 1. $y = \frac{\ln x}{x}$ (same function as in last lecture)

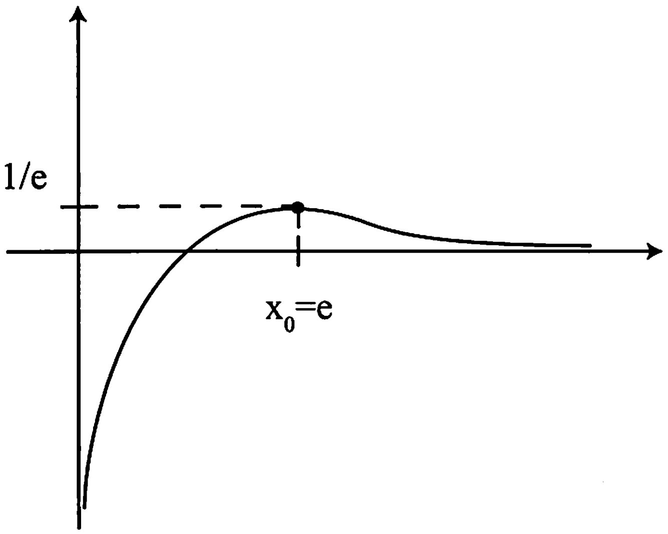


Figure 1: Graph of $y = \frac{\ln x}{x}$.

- What is the maximum value? Answer: $y = \frac{1}{e}$.
- Where (or at what point) is the maximum achieved? Answer: $x = e$. (See Fig. 1).)

Beware: Some people will ask "What is the maximum?". The answer is *not* e. You will get so used to finding the critical point $x = e$, the main calculus step, that you will forget to find the maximum value $y = \frac{1}{e}$. Both the critical point $x = e$ and critical value $y = \frac{1}{e}$ are important. Together, they form the point of the graph $(e, \frac{1}{e})$ where it turns around.

Example 2. Find the max and the min of the function in Fig. 2

Answer: If you've already graphed the function, it's obvious where the maximum and minimum values are. The point is to find the maximum and minimum without sketching the whole graph.

Idea: Look for the max and min among the critical points and endpoints.You can see from Fig. 2 that we only need to compare the heights or y-values corresponding to endpoints and critical points. (Watch out for discontinuities!)

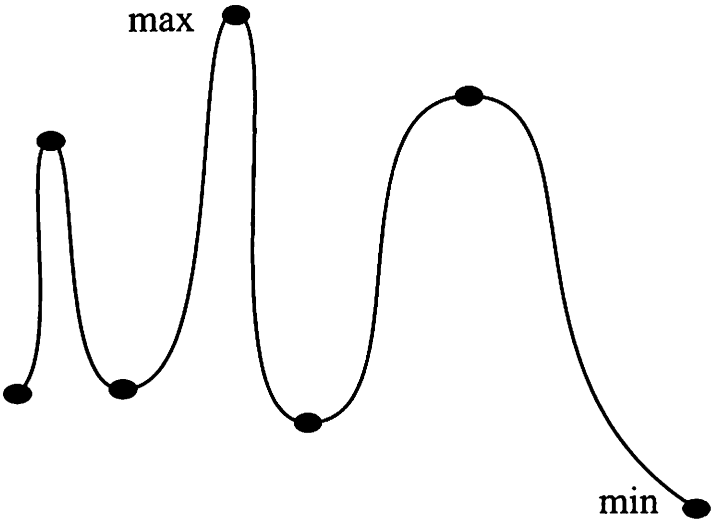


Figure 2: Search for max and min among critical points and endpoints

Example 3. Find the open-topped can with the least surface area enclosing a fixed volume, V.

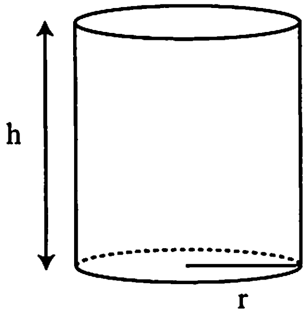


Figure 3: Open-topped can.

1. Draw the picture.
2. Figure out what variables to use. (In this case, r, h, V and surface area, S.)
3. Figure out what the constraints are in the problem, and express them using a formula. In this example, the constraint is

   $$V = \pi r^2 h = \text{constant}$$

   We're also looking for the surface area. So we need the formula for that, too:

   $$S = \pi r^2 + (2\pi r)h$$

   Now, in symbols, the problem is to minimize S with V constant.

4. Use the constraint equation to express everything in terms of r (and the constant V).

$$h = \frac{V}{2\pi r}; \quad S = \pi r^2 + (2\pi r)\left(\frac{V}{\pi r^2}\right)$$

5. Find the critical points (solve $dS/dr = 0$), as well as the endpoints. S will achieve its max and min at one of these places.

$$\frac{dS}{dr} = 2\pi r - \frac{2V}{r^2} = 0 \implies \pi r^3 - V = 0 \implies r^3 = \frac{V}{\pi} \implies r = \left(\frac{V}{\pi}\right)^{1/3}$$

We're not done yet. We've still got to evaluate S at the endpoints: $r = 0$ and "$r = \infty$".

$$S = \pi r^2 + \frac{2V}{r}, \quad 0 \le r < \infty$$

As $r \to 0$, the second term, $\frac{2}{r}$, goes to infinity, so $S \to \infty$. As $r \to \infty$, the first term πr^2 goes to infinity, so $S \to \infty$. Since $S = +\infty$ at each end, the minimum is achieved at the critical point $r = (V/\pi)^{1/3}$, not at either endpoint.

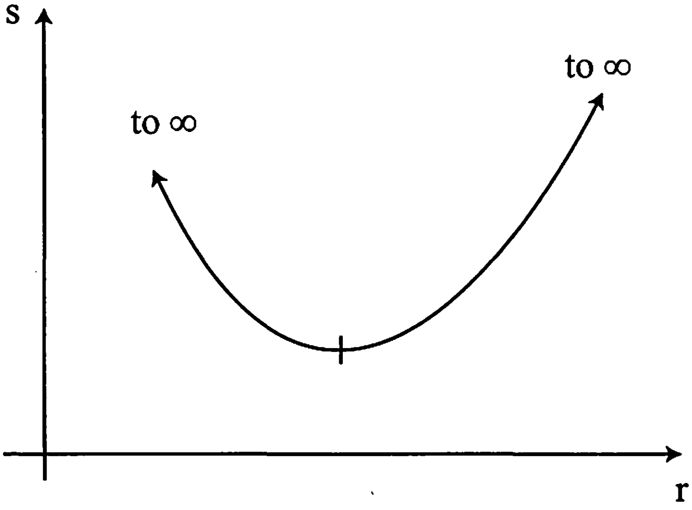


Figure 4: Graph of S

We're <u>still</u> not done. We want to find the minimum value of the surface area, S, and the values of h.

$$r = \left(\frac{V}{\pi}\right)^{1/3}; \quad h = \frac{V}{\pi r^2} = \frac{V}{\pi\left(\frac{V}{\pi}\right)^{2/3}} = \frac{V}{\pi}\left(\frac{V}{\pi}\right)^{-2/3} = \left(\frac{V}{\pi}\right)^{1/3}$$

$$S = \pi r^2 + 2\frac{V}{r} = \pi\left(\frac{V}{\pi}\right)^{2/3} + 2V\left(\frac{V}{\pi}\right)^{1/3} = 3\pi^{-1/3}V^{2/3}$$

Finally, another, often better, way of answering that question is to find the proportions of the can. In other words, what is $\frac{h}{r}$? Answer: $\frac{h}{r} = \frac{(V/\pi)^{1/3}}{(V/\pi)^{1/3}} = 1$.

Example 4. Consider a wire of length 1, cut into two pieces. Bend each piece into a square. We want to figure out where to cut the wire in order to enclose as much area in the two squares as possible.

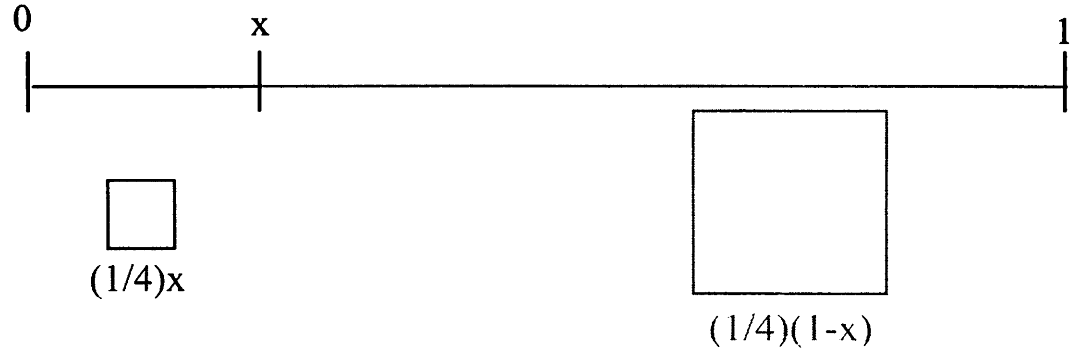


Figure 5: Illustration for Example 5.

The first square will have sides of length $\frac{x}{4}$. Its area will be $\frac{x^2}{16}$. The second square will have sides of length $\frac{1-x}{4}$. Its area will be $\left(\frac{1-x}{4}\right)^2$. The total area is then

$$\begin{aligned} A &= \left(\frac{x}{4}\right)^2 + \left(\frac{1-x}{4}\right)^2 \\ A' &= \frac{2x}{16} + \frac{2(1-x)}{16}(-1) = \frac{x}{8} - \frac{1}{8} + \frac{x}{8} = 0 \implies 2x - 1 = 0 \implies x = \frac{1}{2} \end{aligned}$$

So, one extreme value of the area is

$$A = \left(\frac{\frac{1}{2}}{4}\right)^2 + \left(\frac{\frac{1}{2}}{4}\right)^2 = \frac{1}{32}$$

We're not done yet, though. We still need to check the endpoints! At $x = 0$,

$$A = 0^2 + \left(\frac{1-0}{4}\right)^2 = \frac{1}{16}$$

At $x = 1$,

$$A = \left(\frac{1}{4}\right)^2 + 0^2 = \frac{1}{16}$$

By checking the endpoints in Fig. 6, we see that the *minimum* area was achieved at $x = \frac{1}{2}$. The maximum area is not achieved in $0 < x < 1$, but it is achieved at $x = 0$ or 1. The maximum corresponds to using the whole length of wire for one square.

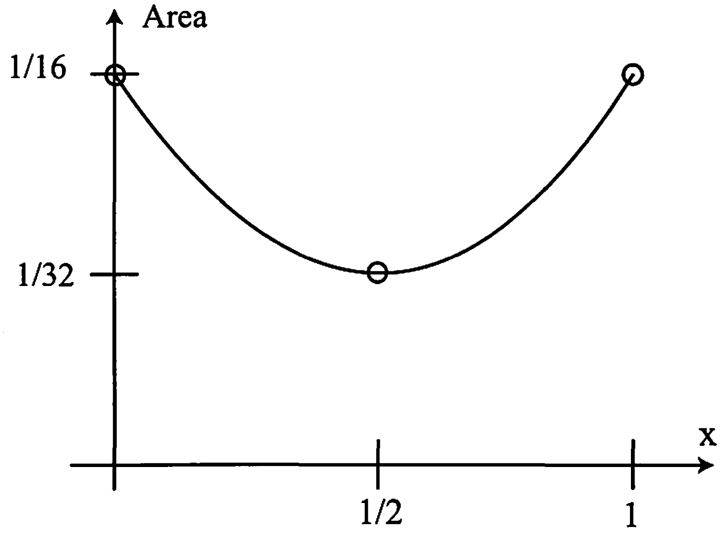


Figure 6: Graph of the area function.

Moral: Don't forget endpoints. If you only look at critical points you may find the worst answer, rather than the best one.

Recetation
Curve Sketching

10/5

$r(x) = \frac{p(x)}{q(x)}$ = factor top + bottom

graph

- asymptotes (horiz, vertical, slant)
- determine 0s
- end limit behavior ($x \to \pm\infty$)
- concavity increasing/decreasing (1st deriv = 0) + concavity 2nd deriv = 0

What are slant asymptotes? +horiz
- at limit behavior
- as x becomes really large
- look at leading terms

horiz asy $\frac{-2x}{x} = -2$

↑ coefficent of leading terms =

$\frac{1-2x^2}{1+x} \to \frac{-2x^2}{x} \to -2x$ slant asymptote

$\frac{1-2x^3}{1+x} \to \frac{-2x^3}{x} \to -2x^2$ original function behaves like a -quadratic as $x \to \pm\infty$

$\frac{1-2x}{1+x^2} \to \frac{-2x}{x^2} \to n \quad \frac{-2x}{x^2} \to \frac{-2}{x} \to 0$

degree bigger

horiz asy at $x = 0$

4 possibilities of end limit behavior
of $\frac{p(x)}{q(x)}$ quotients of polynomials

- x-axis (y=0)
- constant nonzero horiz line (y=c ≠ 0)
- y= ax slanted line
- higher degree polynomial

$\boxed{\text{Find by comparing leading degree}}$

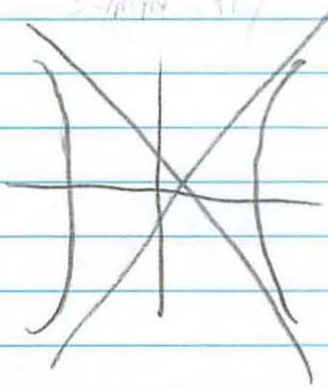
degrees of

$\frac{p}{q}$ ← bigger

$\frac{p}{q}$) equal

$\frac{p}{q}$ ← bigger by 1 degree

$\frac{p}{q}$ ← bigger by more than 1 degree



← slant asymptotes

~~Symmetry~~ Optimization

- max or min something based on constrain
- hardest part is knowing that it is one

10/5

Worksheet 10/05: Curve Sketching. Optimization.

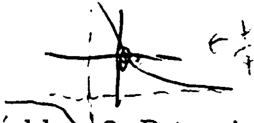
18.01 Fall 2009

Problem 1. Graph the function

$$\frac{1-2x}{1+x}$$

Label all asymptotes (vertical, horizontal, and slant) that may exist. Determine intervals of concavity.

$X=-1$ vert asy $\frac{-2x}{x} = -2 = y$ horiz asy $X=\frac{1}{2}$ ← is a 0



↑ leading term

1st deriv $\frac{-3}{(1+x)^2}$ 2nd deriv $\frac{6}{(1+x)^3}$

$f'(x)$ is <0 everywhere ⊖ no min/max

(↓) | (↑) -1 ← changes concavity w/o having to be zero

Problem 2. Determine concavity of the following functions:

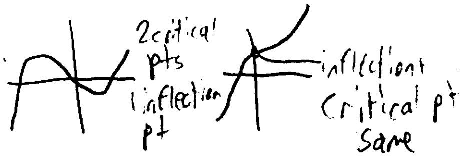
a) $y = ax^2 + bx + c$, (you will have three cases here)

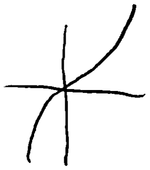
parabolas always have same concavity

↓ ↑ could be a line $(a=0)$ no/flat

based on a

b) $y = x^3 - x^{2} + 3$, $y = x^3 + 3$, $y = x^3 + x + 3$



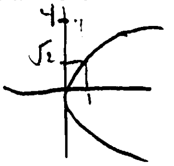


← no critical pt only inflection pt

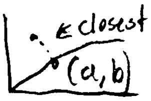
c) $y = x^4 - x^2 + 3$, $y = x^4 + 3$, $y = x^4 + x^2 + 3$

Problem 3. Find the point on the parabola $y^2 = 2x$ which is closest to the point $(1, 4)$. Would your answer change if x were limited to the interval $[-1, 1]$? $y=\sqrt{2x}$

↑ why can't take 1,4 - not on curve $(1,\sqrt{2})$ is



← not the closet pt



← closest (min distance) $D=\sqrt{(b-4)^2+(a-1)^2}$ → $x=\frac{y^2}{2}$ $\sqrt{(b-4)^2+(\frac{b^2}{2}-1)^2}$

use constrant to get rid of 1 variable

minimizing d is same as d^2

* trick * b/c $d \geq 0$ and $-^2$ is monotono increasing when $- \geq 0$

$d^{2\prime} = 2(\frac{y^2}{2}-1)(\frac{2y}{4})+2(y-4)$

$y^3-2y+2y-8$

$y^3-8=0$

$y=2$

$x=2$ $(2,2)$

Sanity check

Problem 4. Given a piece of wire with unit length, make a cut to form one piece into a square and one piece into an equilateral triangle. Where should you cut to form wire figures enclosing the most area?

$-3(1+x)^{-2}$ $\quad$ $\underline{-2(1+x)^{-3} \cdot 1 \cdot -3}$

$$\frac{6}{(1+x)^3}$$

Lecture 12
Related Rates

10/6

Mon: Holiday
Tue: Monday schedule - back to back Recitation w/ Wed

Thur + Fri: Antiderivitives + Differential Equation
-not on exam 2 (2 weeks from today)

So far: Linear/Quadratic approx
Curve Sketching
Optimization Problems (strategies)
Today: Related Rates
Thur: Newton's Method (roots)
Fri: Mean Value Theorm (hard)

all on exam 2

Curve Sketching - don't need every strategy on problem

1. Domain/continuity/differentiability
2. asymptotes
3. Symetry/periodic Functions
4. compute local maxima
   - set $f'(x) = 0$
   - consider critical pts
   - solve for $f'(x) = 0$
5. Concavity w/ $f''(x) = 0$

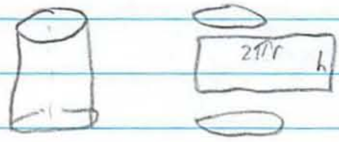
$\lim_{x\to\infty}$ ← still useful $\lim_{x\to\infty}(e^{-x} + x) = 0 + \infty = \infty$

checking for horiz asy/end behavior

Optimization

2 variables to optimize

minimum amt surface area cylinder
w/ fixed amt of volume to some constraint equation



$V_0 = 100$

$SA(rh) = 2\pi rh + 2\pi^2$

$100 = \pi r^2 h$ ← constraint equation

$h = \frac{100}{\pi r^2}$

$2\pi r\left(\frac{100}{\pi r^2}\right) + 2\pi^2$

$SA = \frac{200}{r} + 2\pi^2$

Can compute derivitive with respect to r

$SA(r)' = \frac{dSA(r)}{dr} = 4\pi r - \frac{200}{r^2} = \frac{4\pi r^3 - 200}{r^2}$

? how know local min is global

$r = \sqrt[3]{\frac{200}{4\pi}} = \sqrt[3]{\frac{50}{\pi}} = \boxed{\sqrt[3]{\frac{V_0}{2\pi}}}$ ← critical pt - min?

? local min
how do you know global min
? did not here what he said

sub in for r_{min}

$\boxed{h = 2r}$ ← how did you get that

just know →

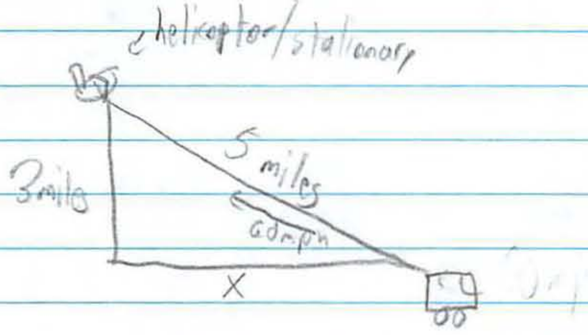
$h = \frac{V_0}{\pi r_{min}^2} \geq 2r_{min}$

$\frac{V_0}{\pi \sqrt[3]{\frac{V_0}{2\pi}}^{2}}$

Related Rate

Can check speed by aircraft radar

Speeding = 65 mph



Given Rate Rate of change hyp $= \frac{dh}{dt}$ ← diff in hyp ← as a function of time $= 60$ mph

Want Rate Want to know ground speed $\frac{dx}{dt}$

Relate h and x

$$H(t) = \sqrt{3^2 + x(t)^2}$$

$$H(t)^2 = 3^2 + x(t)^2$$

$$2H\frac{dH}{dt} = 2x\frac{dx}{dt}$$

↓ $\frac{dH}{dt}$

want $\frac{dx}{dt}$

$$2 \cdot 3 \cdot 60 = 2 \cdot 4 \frac{dx}{dt}$$

$$\frac{dx}{dt} = \frac{5}{4}(-60) \text{ mph} = -75 \text{ mph}$$ speeding

Lecture 12: Related Rates

Example 1. Police are 30 feet from the side of the road. Their radar sees your car approaching at 80 feet per second when your car is 50 feet away from the radar gun. The speed limit is 65 miles per hour (which translates to 95 feet per second). Are you speeding?

First, draw a diagram of the setup (as in Fig. 1):

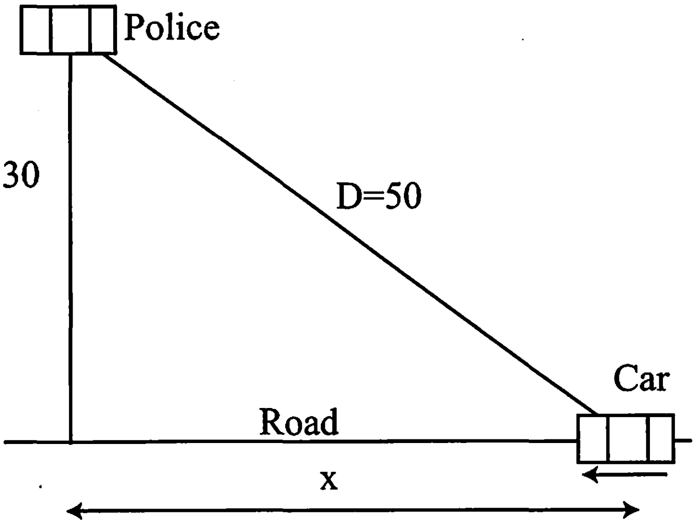


Figure 1: Illustration of example 1: triangle with the police, the car, the road, D and x labelled.

Next, give the variables names. The important thing to figure out is which variables are changing.

At $D = 50$, $x = 40$. (We know this because it's a 3-4-5 right triangle.) In addition, $\frac{dD}{dt} = D' = -80$. D' is negative because the car is moving in the $-x$ direction. Don't plug in the value for D yet! D is changing, and it depends on x.

The Pythagorean theorem says

$$30^2 + x^2 = D^2$$

Differentiate this equation with respect to time (implicit differentiation:

$$\frac{d}{dt}\left(30^2 + x^2 = D^2\right) \implies 2xx' = 2DD' \implies x' = \frac{2DD'}{2x}$$

Now, plug in the instantaneous numerical values:

$$x' = \frac{50}{40}(-80) = -100\frac{\text{feet}}{\text{s}}$$

This exceeds the speed limit of 95 feet per second; you are, in fact, speeding.

There is another, longer, way of solving this problem. Start with

$$D = \sqrt{30^2 + x^2} = (30^2 + x^2)^{1/2}$$

$$\frac{d}{dt}D = \frac{1}{2}(30^2 + x^2)^{-1/2}(2x\frac{dx}{dt})$$

Plug in the values:

$$-80 = \frac{1}{2}(30^2 + 40^2)^{-1/2}(2)(40)\frac{dx}{dt}$$

and solve to find

$$\frac{dx}{dt} = -100\frac{\text{feet}}{\text{s}}$$

(A third strategy is to differentiate $x = \sqrt{D^2 - 30^2}$). It is easiest to differentiate the equation in its simplest algebraic form $30^2 + x^2 = D^2$, our first approach.

The general strategy for these types of problems is:

1. Draw a picture. Set up variables and equations.
2. Take derivatives.
3. Plug in the given values. Don't plug the values in until *after* taking the derivatives.

Example 2. Consider a conical tank. Its radius at the top is 4 feet, and it's 10 feet high. It's being filled with water at the rate of 2 cubic feet per minute. How fast is the water level rising when it is 5 feet high?

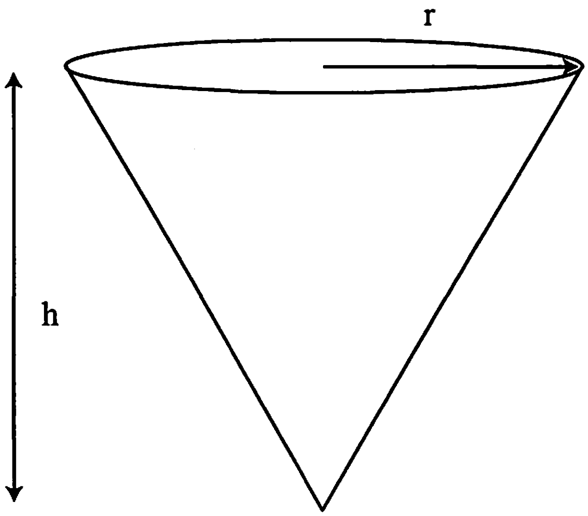


Figure 2: Illustration of example 2: inverted cone water tank.

From Fig. 2), the volume of the tank is given by

$$V = \frac{1}{3}\pi r^2 h$$

The key here is to draw the two-dimensional cross-section. We use the letters r and h to represent the variable radius and height of the water at any level. We can find the relationship between r and h from Fig. 3) using similar triangles.

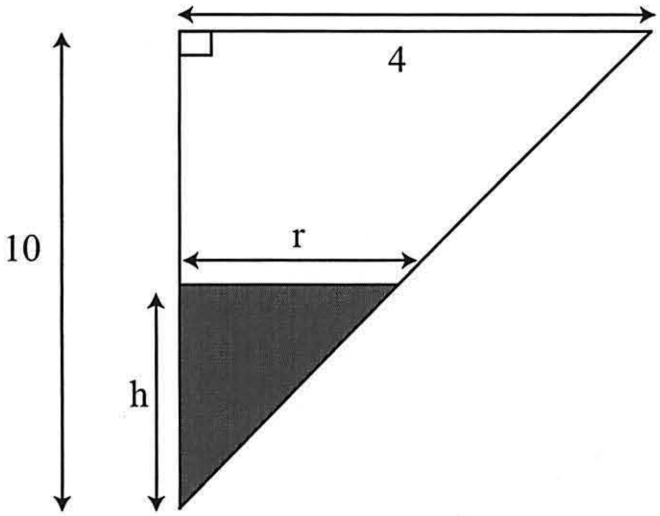


Figure 3: Relating r and h.

From Fig. 3), we see that

$$\frac{r}{h} = \frac{4}{10}$$

or, in other words,

$$r = \frac{2}{5}h$$

Plug this expression for r back into V to get

$$V = \frac{1}{3}\pi\left(\frac{2}{5}h\right)^2 h = \frac{4}{3(25)}\pi h^3$$

$$\frac{dV}{dt} = V' = \frac{4}{25}\pi h^2 h'$$

Now, plug in the numbers ($\frac{dV}{dt} = 2$, $h = 5$):

$$2 = \left(\frac{4}{25}\right)\pi(5)^2 h'$$

$$h' = \frac{1}{2\pi}$$

Related rates also arise on Problem Set 3 (Fig. 4). There's a part II margin of error problem involving a satellite, where you're asked to find $\frac{\Delta L}{\Delta h}$.

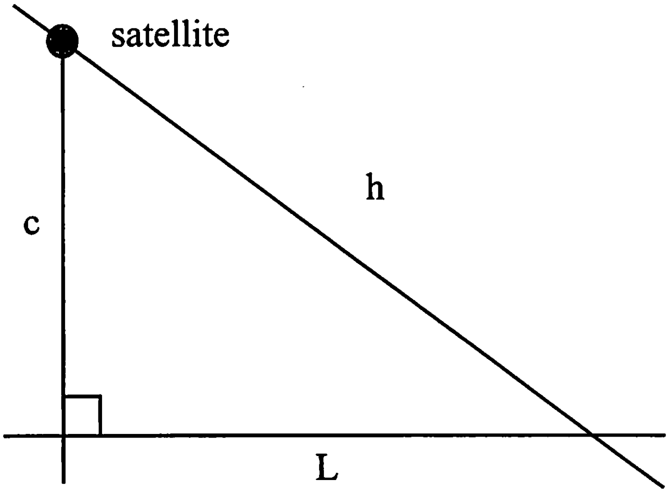


Figure 4: Illustration of the satellite problem.

$$\begin{aligned} L^2 + c^2 &= h^2 \\ 2LL' &= 2hh' \\ \text{Hence, } \frac{\Delta L}{\Delta h} \approx \frac{L'}{h'} &= \frac{h}{L} \end{aligned}$$

There is also a parabolic mirror problem based on similar ideas (Fig. 5).

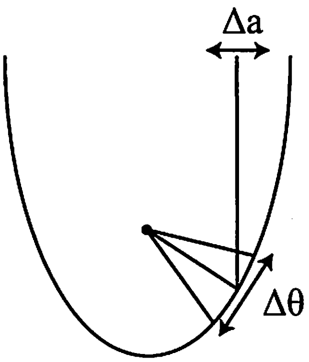
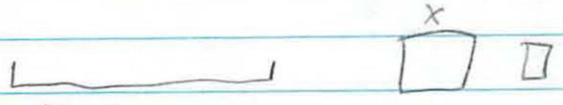


Figure 5: Illustration of the parabolic mirror problem.

Here, you want to find either $\frac{\Delta a}{\Delta \theta}$ or $\frac{\Delta \theta}{\Delta a}$. This type of sensitivity of measurement problem matters in every measurement problem, for instance predicting whether asteroids will hit Earth.

Recitation

10/7

1. 

Maximize area

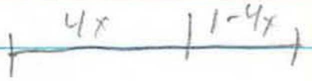
Gien f(x) find min/max f(x)

Step 1: Find variables
side of 1st square x

Step2: produce function to be maximized

$$A = x^2 + \left(\frac{1-4x}{4}\right)^2$$

↑ small square lenght



$$A = x^2 + \left(\frac{1}{4} x\right)^2$$

Step 3 endpoints/
interval of possible
values of variable $0 \le x \le \frac{1}{4}$

↑ must be (+) ↑ if only make 1 square

Step 4 Maximize $f(x) = x^2 + \left(\frac{1}{4} - x\right)^2$ for $0 \le x \le \frac{1}{4}$

$2x - 2\left(\frac{1}{4} - x\right) \cdot -1$

$4x\left(\frac{1}{4} - x\right)$

~~$x - 4x^2$ when 0~~

$2x + 2x - \frac{1}{2}$

$4x - \frac{1}{2}$ when $= 0$

at $x = \frac{1}{8}$

Step 5 A: Compare values at critical pts + end points

x	$\frac{1}{8}$	0	$\frac{1}{4}$
f(x)	$\frac{1}{32}$	$\frac{1}{16}$	$\frac{1}{16}$
		↑max	↑max

Option B: check double derivative
at critical pt

$x''(\frac{t}{4})$ if $\oplus$ if $\ominus$

concave ↑ concave ↓

local min local min

* not correct in all situations like this one
b/c local not global

Related Rates

given 2 quantities
depend on time
know rate of change for one
Want rate of change for other

Worksheet #3

Step 1 What are the 2 quantities?

$v(t)$ - 1st quantity w/ known rate of change
$p(t)$ - 2nd quantity - need to find rate

Step 2 - Gather info + understand what need to find

$v'(t) = -10^{-4}\ m^3/min$ (compression) $v(t_0) = .6 \cdot 10^{-3}\ m^3$
$p(0) = 1.2 \cdot 10^4\ \frac{N}{m^2}$ $v(t)\,p(t) = \text{constant}$
$v(0) = 10^{-3}\ m^3$ Find $p'(t_0)$? Boyle's law

Step 3 Find relation

$$v(t)\,p(t) = c$$

Step 4 Differenitate relation

*Express $p'(t_0)$ in terms of known valves

$$v'(t)\,p(t) + p'(t)\,v(t)$$

↓ just specify at t_0

$$p'(t_0) = -\frac{p(t_0)\,v'(t_0)}{V(t_0)}$$

←don't know but can compute using

$$p'(t_0) = -\frac{c\,v'(t_0)}{v(t_0)}$$

$v(0)\,p(0)$

$10^{-3}\,m^3 \cdot 1.2 \cdot 10^4 \frac{N}{m^2}$

$12\ Nm$

$$p'(t_0) \quad -\frac{(12\ Nm)\left(-10^{-4}\frac{m^3}{min}\right)}{.6 \cdot 10^{-3}\ m^3}$$

* make sure did not miss much

Worksheet 10/07: Related Rates.

18.01 Fall 2009

Problem 1. If $y = x^3 + 2x$, and $dx/dt = 5$, find dy/dt when $x = 2$.

Problem 2. You have two bowls. One of them is hemispherical. The other is conical, with a base radius equal to the height. Both bowls have the same base diameter of 30 cm.

You begin to pour water into both of them at the same rate of $1\,\mathrm{cm}^3/\mathrm{s}$. How fast is the level in each changing after 5 s?

Problem 3. Boyle's law for gases states that, at a constant temperature, the product of the pressure p and volume V of a gas is constant. $pV =$

You have a liter container of gas that is at a pressure of $1.2 * 10^4\,\mathrm{N}/\mathrm{m}^2$. You begin compressing it, keeping the temperature constant, and changing the volume at an even rate of $10^{-4}\,\mathrm{m}^3/\mathrm{min}$.

At what rate is the pressure changing when the volume is at 0.6 liters?

(1 liter = 0.001 cubic meters) Convert to cubic metter

See problem

Problem 4. A (spherical) snowball melts at a rate proportional to its surface area. Show that its radius decreases at a constant rate.

Lecture 13
Newton's Method

10/8/09

Pset 3 due tmo
Pset 4 due as usual
exam 2 Oct 20

Newton's Method - to approximate zeros of function

$f(x) = 0$ $\quad F(x) = 7$

$f(x) - 7 = 0$

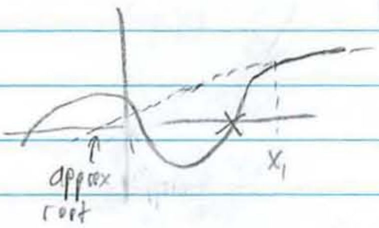
More generally, where 2 functions intersect

solve $f(x) = g(x)$

$f(x) - g(x) = 0$

key idea Use tangent lines

$f(x) = x^5 - 3x^2 + 7$



Guess a pt x_1
Find the tangent line approx

Instead of trying to find root of $f(x)$
find the root of a tangent line to $f(x)$ at $x = x$

Tangent $y - f(x_1) = f'(x_1)(x - x_1)$

Tangent has root of x ; $0 - f(x_1) = f'(x)(x - x_1)$

approx root $x = x_1 - \dfrac{f(x_1)}{f'(x_1)}$

↑ call this x_2, repeat

Tangent at x_2 $y - f(x_2) = f'(x_2)(x - x_2)$

Set $y = 0$ solve for x

$$x = x_2 - \frac{f(x_2)}{f'(x_2)}$$

tangent will get closer + closer to root

will converge to root

?, did in HS w/ complicated table

① Pick an inital x_1

② Compute $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

③ take x_3 compute $x_2 - \frac{f(x)}{f'(x)}$

④ $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

Newton's Original Example

HS - meet more often

- more hw

-more qu, less hard

semester much longer

$f(y) = x^3 - 2x - 5$

find root

$f(0) = -5$

$f(1) = -6$

$f(2) = -1$ ← seems like a good place to start

$f(3) = 16$

$x_1 = 2$

$x_2 = x_1 - \frac{f(y)}{f'(x)} = 2 - \frac{(2)^3 - 2(2) - 5}{3(2)^2 - 2} = 2 - \frac{-1}{10} = 2.1$

$x_3 = 2.1 - \frac{(2.1)^3 - 2(2.1) - 5}{3(2.1)^2 - 2} = \ldots$

$x_n = 2.0946\ldots$

Suggestive

When first $\overset{n}{\wedge}$ digits don't change after several consecutive steps of Newton's method, it gets correct to n digits

$f'(x) = 3x^2 - 2$
$f'(x) = 0$ when $x = \pm\sqrt{\frac{2}{3}}$

← guess before hand makes it easier ✓

Quick curve sketching tells you how many roots to expect

Often want to pick close to ⊕ root in order for Newton's Method to converge to ⊕ root

---

Find approx to $\sqrt[6]{2}$
need equation w/ $\sqrt[6]{2}$ as a root

$f(x) = x^6 - 2$

For initial guess pick close ⊕ root

$x_1 = 1 - \frac{-2}{6} = \frac{7}{6}$

$x_2 = \frac{7}{6}$
$x_3 = 1,126$
$\Big\}$ doing it
$x_4 = 1,2249707$
$x_5 = 1,2246205$
$x_6 - 1,2246205$
$\Big)$ both same so exceeding calculator so 6 is exceeding

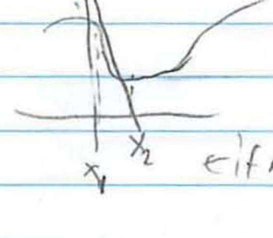
$\underbrace{}_{\text{Calc output}}$

a million things all barralling towards completion

Possible Failures

What if pt → deriv = 0

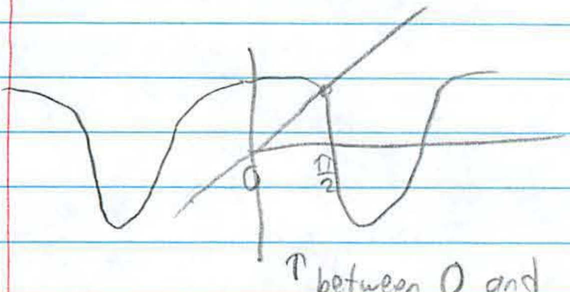
⊛ Make sure to pick pt close to root



← if run into that – pick new pt + start over

Don't have to be polynomials

Find where x and cos(x) intersect



$f(x) = \cos x - x$
Find root of $f(x)$

↑ between 0 and $\frac{\pi}{2}$

$x_1 = 1$ (radian)
$X_2 = 1 - \frac{\cos 1 - 1}{-\sin 1 - 1} \approx .75036$

$x_3 \approx .7391\ldots$
$x_4 \approx .73908\ldots$
$x_5 \approx .73908\ldots$

Better your 1st choice the less steps you must do

Related Rate Given 1 rate, asked to find another rate

Volume ↑ 5 ft^3/min $\left(\frac{dV}{dt}\right)$
Find how fast h ↑ $\left(\frac{dh}{dt}\right)$

Need equation relating V + h → $V = \frac{1}{3}\pi r^2 h$
$V(t) = \frac{1}{3}\pi r(t)^2 h(t)$
↙

$$\frac{dV}{dt} = \frac{1}{3}\pi \left(r(t)^2 \cdot \underset{\frac{dh}{dt}}{h'(t)} + 2\underset{\frac{dr}{dt}}{r'(t)} \cdot h(t) \right)$$

Plug in quantities known + solve

Lecture 13: Newton's Method and Other Applications

Newton's Method

Finding the 0s

Newton's method is a powerful tool for solving equations of the form $f(x) = 0$.

Example 1. $f(x) = x^2 - 3$. In other words, solve $x^2 - 3 = 0$. We already know that the solution to this is $x = \sqrt{3}$. Newton's method, gives a good numerical approximation to the answer. The method uses tangent lines (see Fig. 1).

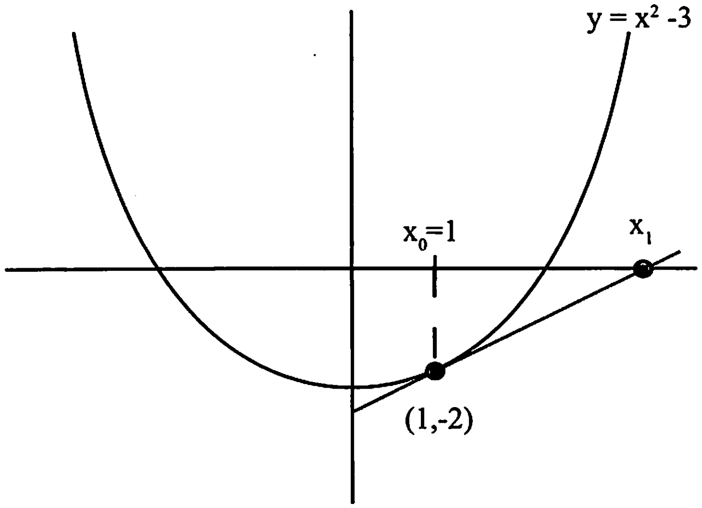


Figure 1: Illustration of Newton's Method, Example 1.

The goal is to find where the graph crosses the x-axis. We start with a guess of $x_0 = 1$. Plugging that back into the equation for y, we get $y_0 = 1^2 - 3 = -2$, which isn't very close to 0.

Our next guess is x_1, where the tangent line to the function at x_0 crosses the x-axis. The equation for the tangent line is:

$$y - y_0 = m(x - x_0)$$

When the tangent line intercepts the x-axis, $y = 0$, so

$$\begin{aligned} -y_0 &= m(x_1 - x_0) \\ -\frac{y_0}{m} &= x_1 - x_0 \\ x_1 &= x_0 - \frac{y_0}{m} \end{aligned}$$

solve for x_1

Remember: m is the slope of the tangent line to $y = f(x)$ at the point (x_0, y_0).

In terms of f:

$$\begin{aligned} y_0 &= f(x_0) \\ m &= f'(x_0) \end{aligned}$$

Therefore,

$$\boxed{x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}}$$

YUP

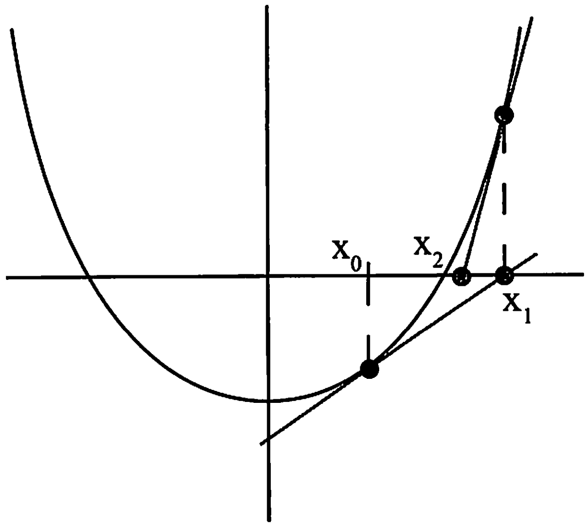


Figure 2: Illustration of Newton's Method, Example 1.

In our example, $f(x) = x^2 - 3, f'(x) = 2x$. Thus,

$$\begin{aligned} x_1 &= x_0 - \frac{(x_0^2 - 3)}{2x} = x_0 - \frac{1}{2}x_0 + \frac{3}{2x_0} \\ x_1 &= \frac{1}{2}x_0 + \frac{3}{2x_0} \end{aligned}$$

The main idea is to repeat (iterate) this process:

$$\begin{aligned} x_2 &= \frac{1}{2}x_1 + \frac{3}{2x_1} \\ x_3 &= \frac{1}{2}x_2 + \frac{3}{2x_2} \end{aligned}$$

and so on. The procedure approximates $\sqrt{3}$ extremely well.

x	y	accuracy: $\lvert y-\sqrt{3}\rvert$
x_0	1	
x_1	2	3×10^{-1}
x_2	$\frac{7}{4}$	2×10^{-2}
x_3	$\frac{7}{8}+\frac{6}{7}$	10^{-4}
x_4	$\frac{18{,}817}{10{,}864}$	3×10^{-9}

Notice that the number of digits of accuracy doubles with each iteration.

Summary

Newton's Method is illustrated in Fig. 3 and can be summarized as follows:

$$\boxed{x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}}$$

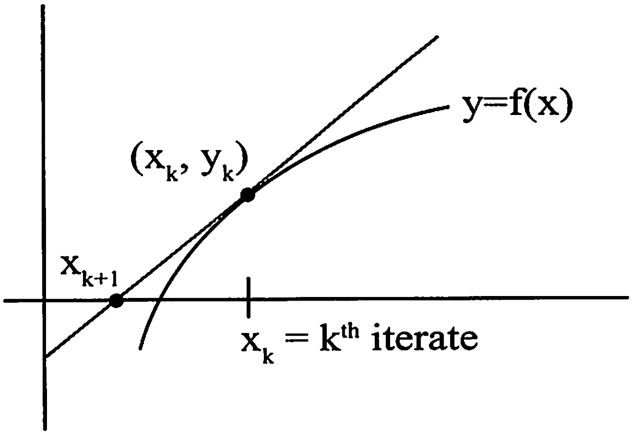


Figure 3: Illustration of Newton's Method.

Example 1 considered the particular case of

$$\begin{aligned} f(x) &= x^2 - 3 \\ x_{k+1} &= x_k - \frac{f(x_k)}{f'(x_k)} = \ldots = \frac{1}{2}x_k + \frac{3}{2x_k} \end{aligned}$$

Now, we define

$$\overline{x} = \lim_{k\to\infty} x_k \quad (x_k \to \overline{x} \text{ as } k \to \infty)$$

To evaluate $\overline{x}$ in Example 1, take the limit as $k \to \infty$ in the equation

$$x_{k+1} = \frac{1}{2}x_k + \frac{3}{2x_k}$$

This yields

$$\bar{x} = \frac{1}{2}\bar{x} + \frac{3}{2\bar{x}} \implies \bar{x} - \frac{1}{2}\bar{x} = \frac{3}{2\bar{x}} \implies \frac{1}{2}\bar{x} = \frac{3}{2\bar{x}} \implies \bar{x}^2 = 3$$

which is just what we hoped: $\bar{x} = \sqrt{3}$.

Warning 1. Newton's Method can find an unexpected root.
Example: if you take $x_0 = -1$, then $x_k \to -\sqrt{3}$ instead of $+\sqrt{3}$. This convergence to an unexpected root is illustrated in Fig. 4

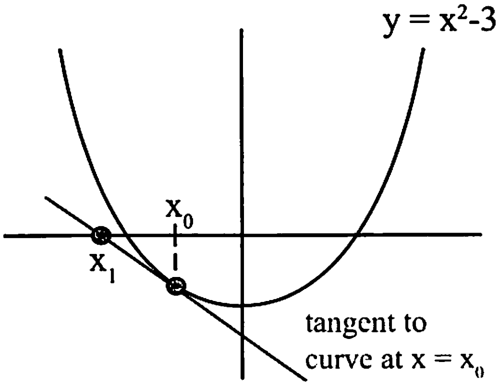


Figure 4: Newton's method converging to an unexpected root.

Warning 2. Newton's Method can fail completely.
This failure is illustrated in Fig. 5. In this case, $x_2 = x_0$, $x_3 = x_1$, and so forth. It repeats in a cycle, and never converges to a single value.

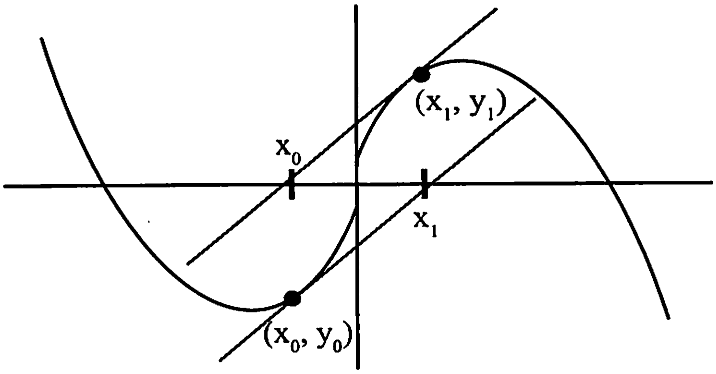


Figure 5: Newton's method converging to an unexpected root.

Ring on a String

Consider a ring on a string [1] held fixed at two ends at $(0,0)$ and (a,b) (see Fig. 6). The ring is free to slide to any point. Find the position (x,y) of the string.

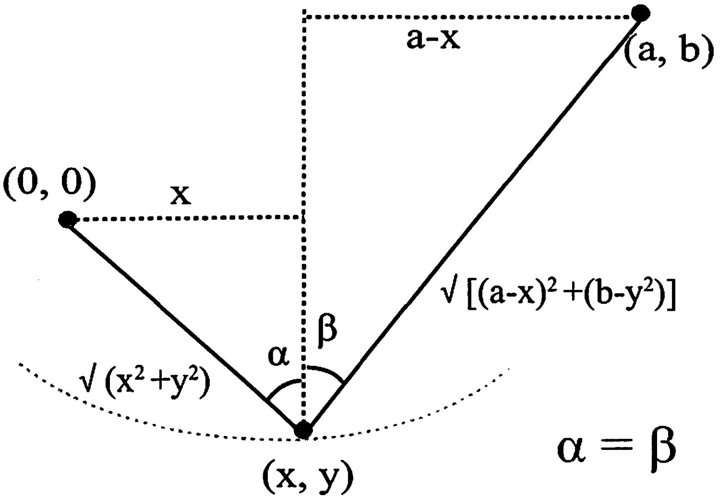


Figure 6: Illustration of the Ring on a String problem.

Physical Principle The ring settles at the lowest height (lowest potential energy), so the problem is to minimize y subject to the constraint that (x,y) is on the string.

Constraint The length L of the string is fixed:

$$\sqrt{x^2+y^2}+\sqrt{(x-a)^2+(y-b)^2}=L$$

The function $y=y(x)$ is determined implicitly by the constraint equation above. We traced the constraint curve (possible positions of the ring) on the blackboard. This curve is an ellipse with foci at $(0,0)$ and (a,b), but knowing that the curve is an ellipse does not help us find the lowest point.

Experiments with the hanging ring show that the lowest point is somewhere in the middle. Since the ends of the constraint curve are higher than the middle, the lowest point is a critical point (a point where $y'(x)=0$). In class we also gave a physical demonstration of this by drawing the horizontal tangent at the lowest point.

To find the critical point, differentiate the constraint equation implicitly with respect to x,

$$\frac{x+yy'}{\sqrt{x^2+y^2}}+\frac{x-a+(y-b)y'}{\sqrt{(x-a)^2+(y-b)^2}}=0$$

Since $y'=0$ a the critical point, the equation can be rewritten as

$$\frac{x}{\sqrt{x^2+y^2}}=\frac{a-x}{\sqrt{(x-a)^2+(y-b)^2}}$$

---

[1] ©1999 and ©2007 David Jerison

From Fig. 6, we see that the last equation can be interpreted geometrically as saying that

$$\sin\alpha = \sin\beta$$

where α and β are the angles the left and right portions of the string make with the vertical.

Physical and geometric conclusions

The angles α and β are equal. Using vectors to compute the force exerted by gravity on the two halves of the string, one finds that there is *equal tension* in the two halves of the string - a physical equilibrium. (From another point of view, the equal angle property expresses a geometric property of ellipses: Suppose that the ellipse is a mirror. A ray of light from the focus $(0,0)$ reflects off the mirror according to the rule angle of incidence equals angle of reflection, and therefore the ray goes directly to the other focus at (a,b).)

Formulae for x and y

We did not yet find the location of (x,y). We will now show that

$$x = \frac{a}{2}\left(1 - \frac{b}{\sqrt{L^2-a^2}}\right), \quad y = \frac{1}{2}\left(b - \sqrt{L^2-a^2}\right)$$

Because $\alpha = \beta$,

$$x = \sqrt{x^2+y^2}\sin\alpha; \quad a - x = \sqrt{(x-a)^2+(y-b)^2}\sin\alpha$$

Adding these two equations,

$$a = \left(\sqrt{x^2+y^2} + \sqrt{(x-a)^2+(y-b)^2}\right)\sin\alpha = L\sin\alpha \implies \sin\alpha = \frac{a}{L}$$

The equations for the vertical legs of the right triangles are (note that $y < 0$):

$$-y = \sqrt{x^2+y^2}\cos\alpha; \quad b - y = \sqrt{(x-a)^2+(y-b)^2}\cos\beta$$

Adding these two equations, and using $\alpha = \beta$,

$$b - 2y = \left(\sqrt{x^2+y^2} + \sqrt{(x-a)^2+(y-b)^2}\right)\cos\alpha = L\cos\alpha \implies y = \frac{1}{2}(b - L\cos\alpha)$$

Use the relation $\sin\alpha = \frac{a}{L}$ to write $L\cos\alpha = L\sqrt{1-\sin^2\alpha} = \sqrt{L^2-a^2}$. Then the formula for y is

$$y = \frac{1}{2}\left(b - \sqrt{L^2-a^2}\right)$$

Finally, to find the formula for x, use the similar right triangles

$$\tan\alpha = \frac{x}{-y} = \frac{a-x}{b-y} \implies x(b-y) = (-y)(a-x) \implies (b-2y)x = -ay$$

Therefore,

$$x = \frac{-ay}{b-2y} = \frac{a}{2}\left(1 - \frac{b}{\sqrt{L^2-a^2}}\right)$$

Thus we have formulae for x and y in terms of a, b and L.

I omitted the derivation of the formulae for x and y in lecture because it is long and because we got all of our physical intuition and understanding out of the problem from the balance condition that was the immediate consequence of the critical point computation.

Final Remark. In 18.02, you will learn to treat constrained max/min problems in any number of variables using a method called Lagrange multipliers.

Skipped Lecture 10/9/08

Lecture 14: Mean Value Theorem and Inequalities

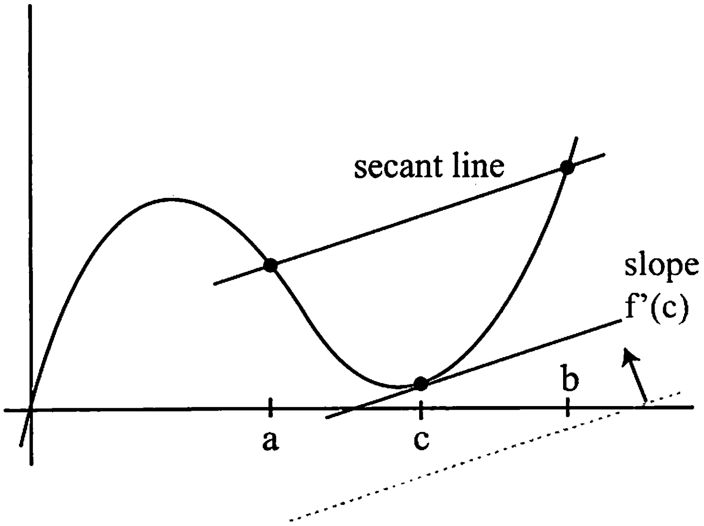
Mean-Value Theorem

The Mean-Value Theorem (MVT) is the underpinning of calculus. It says:

If f is differentiable on $a < x < b$, and continuous on $a \leq x \leq b$, then

$$\frac{f(b)-f(a)}{b-a} = f'(c) \quad \text{(for some } c,\ a < c < b)$$

Here, $\frac{f(b)-f(a)}{b-a}$ is the slope of a secant line, while $f'(c)$ is the slope of a tangent line.



between 2 pts

Figure 1: Illustration of the Mean Value Theorem.

Geometric Proof: Take (dotted) lines parallel to the secant line, as in Fig. 1 and shift them up from below the graph until one of them first touches the graph. Alternatively, one may have to start with a dotted line above the graph and move it down until it touches.

If the function isn't differentiable, this approach goes wrong. For instance, it breaks down for the function $f(x) = |x|$. The dotted line always touches the graph first at $x = 0$, no matter what its slope is, and $f'(0)$ is undefined (see Fig. 2).

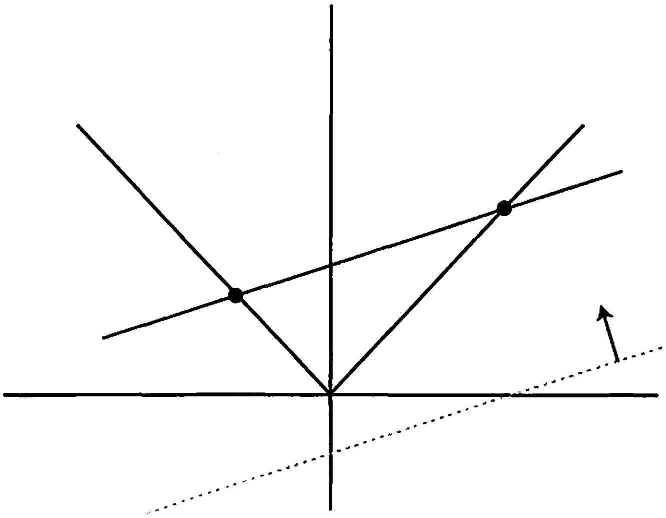


Figure 2: Graph of $y = |x|$, with secant line. (MVT goes wrong.)

Interpretation of the Mean Value Theorem

You travel from Boston to Chicago (which we'll assume is a 1,000 mile trip) in exactly 3 hours. At some time in between the two cities, you must have been going at exactly $\frac{1000}{3}$ mph.

$f(t)$ = position, measured as the distance from Boston.

$$f(3) = 1000, \quad f(0) = 0, \quad a = 0, \text{ and b} = 3.$$

$$\frac{1000}{3} = \frac{f(b) - f(a)}{3} = f'(c)$$

where $f'(c)$ is your speed at some time, c.

Versions of the Mean Value Theorem

There is a second way of writing the MVT:

2nd way

$$\begin{aligned} f(b) - f(a) &= f'(c)(b-a) \\ f(b) &= f(a) + f'(c)(b-a) \quad \text{(for some } c, a < c < b) \end{aligned}$$

There is also a third way of writing the MVT: change the name of b to x.

$$\boxed{f(x) = f(a) + f'(c)(x-a) \quad \text{for some } c, a < c < x}$$

3rd way

The theorem does not say what c is. It depends on f, a, and x.

This version of the MVT should be compared with linear approximation (see Fig. 3).

$$f(x) \approx f(a) + f'(a)(x-a) \quad x \text{ near } a$$

The tangent line in the linear approximation has a definite slope $f'(a)$. by contrast formula is an exact formula. It conceals its lack of specificity in the slope $f'(c)$, which could be the slope of f at any point between a and x.

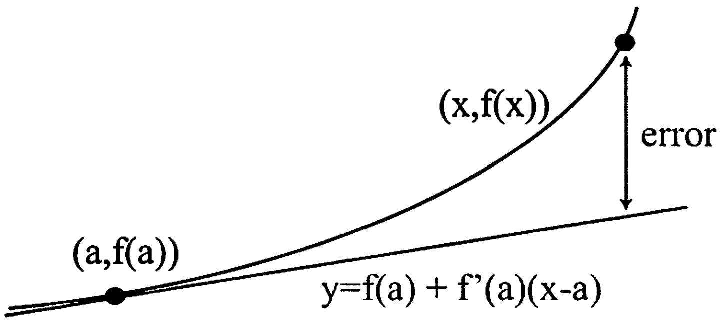


Figure 3: **MVT vs. Linear Approximation.**

Uses of the Mean Value Theorem.

Key conclusions: (The conclusions from the MVT are theoretical)

1. If $f'(x) > 0$, then f is increasing.
2. If $f'(x) < 0$, then f is decreasing.
3. If $f'(x) = 0$ all x, then f is constant.

Definition of increasing/decreasing:
Increasing means $a < b \Rightarrow f(a) < f(b)$. Decreasing means $a < b \implies f(a) < f(b)$.

Proofs:
Proof of 1:

$$\begin{aligned} a &< b \\ f(b) &= f(a) + f'(c)(b-a) \end{aligned}$$

Because $f'(c)$ and $(b-a)$ are both positive,

$$f(b) = f(a) + f'(c)(b-a) > f(a)$$

(The proof of 2 is omitted because it is similar to the proof of 1)

Proof of 3:

$$f(b) = f(a) + f'(c)(b-a) = f(a) + 0(b-a) = f(a)$$

Conclusions 1,2, and 3 seem obvious, but let me persuade you that they are not. Think back to the definition of the derivative. It involves infinitesimals. It's not a sure thing that these infinitesimals have anything to do with the non-infinitesimal behavior of the function.

Inequalities

The fundamental property $f' > 0 \implies f$ is increasing can be used to deduce many other inequalities.

Example. e^x

1. $e^x > 0$
2. $e^x > 1$ for $x > 0$
3. $e^x > 1 + x$

Proofs. We will take property 1 ($e^x > 0$) for granted. Proofs of the other two properties follow:

Proof of 2: Define $f_1(x) = e^x - 1$. Then, $f_1(0) = e^0 - 1 = 0$, and $f_1'(x) = e^x > 0$. (This last assertion is from step 1). Hence, $f_1(x)$ is increasing, so $f(x) > f(0)$ for $x > 0$. That is:

$$e^x > 1 \text{ for } x > 0$$

.

Proof of 3: Let $f_2(x) = e^x - (1 + x)$.

$$f_2'(x) = e^x - 1 = f_1(x) > 0 \quad (\text{if } x > 0).$$

Hence, $f_2(x) > 0$ for $x > 0$. In other words,

$$e^x > 1 + x$$

Similarly, $e^x > 1 + x + \frac{x^2}{2}$ (proved using $f_3(x) = e^x - (1 + x + \frac{x^2}{2})$). One can keep on going: $e^x > 1 + x + \frac{x^2}{2} + \frac{x^3}{3!}$ for $x > 0$. Eventually, it turns out that

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots \quad (\text{an infinite sum})$$

prove w/ 3 simple cases

We will be discussing this when we get to Taylor series near the end of the course.

last 18 min Roll's Theorn – to prove Mean vale theorm

to get $g(x) = L(x) - f(x)$ ← work

find that $L'(c) - f'(c)$

L'

$$\frac{f(b) - f(a)}{b - a}$$

Use Mean value to prove function ↑ when MVT +

or something like that

Take $f(x) = \ln(x)$

Apply MVT on $(1, x)$

There exists a c such that $f'(c) = \dfrac{f(x) - f(1)}{x-1}$

$$\frac{1}{c} = \frac{\ln(x) - \ln(1)}{x-1}$$

MVT $\rightarrow \frac{1}{c} \cdot (x-1) = \ln x$ for some $c\,(1,x)$

$(x-1) >$ ↑

<1 when $c \in (1,x)$

$(x-1) > \ln x$ for $x > 1$

↑ hard to prove inequality

use MVT to prove

* test question

Recitation

10/13

Test next Tue
- covers up to Lecture 15

optimization
- need constraint + optimizing equation ← relates 2 of 3 variables
- starts w/ 3 variables
- reduce to 2 w/ constraint and plug in
- differentiate
- set = 0
- solve for variable

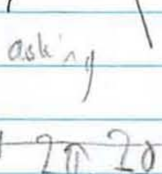
Maximize volume cylinder
w/ SA 40π

$$SA = 2\pi r h + 2\pi r^2$$
$$40\pi = 2\pi r h + 2\pi r^2$$
$$20 = rh + r^2 \quad \leftarrow \text{constraint equation}$$
$$h = \frac{20 - r^2}{r}$$

$$V = \pi r^2 h$$
$$\pi r^2 \left(\frac{20 - r^2}{r}\right)$$
$$\pi r(20 - r^2)$$
$$\pi(\overset{\leftarrow \text{constant}}{20}r - r^3) \quad \leftarrow \text{reduce}$$

$$\frac{dV}{dr} = \pi(20 - 3r^2) \quad \leftarrow \text{differentiate}$$

$$\frac{dV}{dr} = 0 \text{ at } r = \pm\sqrt{\frac{20}{3}}$$



Find h ← what is problem asking to maximize volume

~~$h = 40\pi = 2\pi\sqrt{\frac{20}{3}}\,h + 2\pi\frac{20}{3}$~~ ← don't really need

$$V = \pi\left(20\frac{\sqrt{20}}{\sqrt{3}} - \left(\frac{\sqrt{20}}{\sqrt{3}}\right)^3\right. \quad \leftarrow \text{plug in to original}$$

Related Rate

- problem will say to look at
- get equation
- implicitly differenitate $\frac{d}{dt}$
- Hard part: finding things given in problem (might have to solve for to find)

- signs should be anticipate
↓
ie Cylinder V fixed
as $r \uparrow$ $h \downarrow$
can ballpark as well

- don't need time
- just rate things are changing

snowball melting
- rate proportional SA
show radis ↓ at constant rate

know that → melting → Volume is ↓

$$V = \frac{4}{3}\pi r^3$$

$$V' = \frac{4}{3}\pi \cdot 3r^2$$

$$V' = 4\pi r^2 r'$$

$$k \cdot 4\pi r^2 = \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$k = \frac{dr}{dt}$$

-k for snowball to melt

Not change in SA
↑ fixed surface area

radio $\frac{dV}{dt} = k \cdot SA$

$SA = 4\pi r^2$
↑ just the derivative

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$SA = 4\pi r^2$$

$$d(SA) = 8\pi r \frac{dr}{dt}$$

$$\frac{\frac{dV}{dt}}{\frac{dSA}{dt}} = \frac{r}{2}$$

↖ always up by radius

not proportional

Can't be dSA

Mean Value Theorm

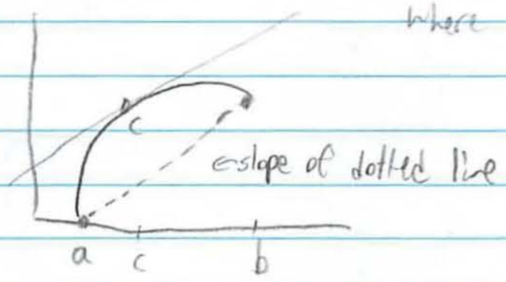
if f is contineous (on [a,b]) + differentiable on (a,b)

$\Rightarrow$
↑implies

there is a c is in (a, b) so that

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

↑ does not tell about where c is



finding c is easy if know f(x), b, a
- can be more than 1

Gets interesting when look at [a, x]

$$\frac{f(x)-f(a)}{x-a} = f'(c)$$

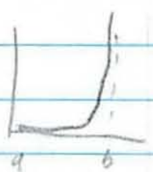
$$f(x) = f(a) + f'(c)(x-a)$$

↑ looks like linearization
but this is exact

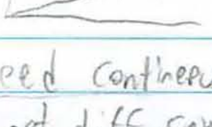
In class: show $x-1 \geq \ln x$
for $x \geq 1$

Counter example + problems if only have some of conditions

F not contineous on all [a, b]



not contineous at b



So need contineous on [a,b]

f not diff somewhere

Look at $f(x) = x - 1 - \ln x$ is > 0

Show $f(x) \geq 0$

$[1, \overset{\infty}{x}]$

$f(x) = f(1) + f'(c)(x-1)$

in interval
don't know

$\in$ = all #

$\in (1, \overset{\infty}{x})$

$f(1) = 0$

$f'(x) = 1 - \frac{1}{x}$

all x in between

$f'(c) = \in (0, 1)$

$f(x) = 0 + \text{pos \#}\,(x-1) > 0$ for $x > 1$

18.01 FALL 2009 – Problem Set 3

Due Friday 10/09/09, 1:45 pm in 2-106

Part I (10 points)

Lecture 11. Fri. Oct. 2. Maximum-minimum problems.
Read: 4.3, 4.4 Work: 2C-1, 2, 4, 10, 13.

Lecture 12. Tue. Oct. 6 Related rate problems.
Read: 4.5 Work: 2E-2, 3, 5, 7

Lecture 13. Thu. Oct. 8 Newton's method.
Read: 4.6, (4.7 is optional) Work: 2F-1

Lecture 14. Fri. Oct. 9 Mean-value theorem. Inequalities.
Read: 2.6 to middle p. 77, Notes MVT Work: assigned on PS4

Part II (31 points + 10 Bonus)

Directions: Attempt to solve *each part* of each problem yourself. If you collaborate, solutions must be written up independently. It is illegal to consult materials from previous semesters. With each problem is the day it can be done.

0. (not until due date; 3 pts) Write the names of all the people you consulted or with whom you collaborated and the resources you used, or say "none" or "no consultation". (See full explanation on PS1).

1. (Friday, 6pts: 3 + 3) Work the following problems in Simmons' text:

a) 4.3/28 (Hint: Use as variable the distance x from the foot of the ladder to the house. Check endpoints.)

b) 4.4/28

2. (Friday, 9 pts: 3 + 3 + 3) Sketch the following functions. Your pictures should indicate asymptotes, local max and min values, intervals on which the function is increasing/decreasing and points of inflection (that is, points where the concavity changes). All this might be hard to write on the graph itself, so it would be better to include a small table with this information next to the graph.

a) $y = \dfrac{2x^2}{x^2 - 1}$

b) $y = 2\cos x + \sin 2x$

c) $y = \ln(4 - x^2)$

3. (Friday, 5 pts: 3 + 2)

a) Find the area of the largest rectangle that can be inscribed in a semicircle of radius r. In solving the problem, center the semicircle above the x-axis and express the area of the rectangle in terms of the coordinates (x, y) on the semicircle.

b) Do the problem again, but now express the coordinates in terms of an angle measure θ whose coordinates $(r\cos\theta, r\sin\theta)$ give one corner of the rectangle. (This should be easier than part (a), and shows that the way we represent the optimization problem using functions can affect the difficulty.)

4. (Thursday, 8 pts: 3 + 3 + 2) **Newton's method.**

a) Compute the cube root of 9 to 6 significant figures using Newton's method. Give the general formula, and list numerical values, starting with $x_0 = 2$. At what iteration k does the method surpass the accuracy of your calculator or computer? (Display your answers to the accuracy of your calculator or computer.)

b) For each step x_k, $k = 0, 1, \ldots$, say whether the value is i) larger or smaller than $9^{1/3}$; ii) larger or smaller than the preceding value x_{k-1}. Illustrate on the graph of $x^3 - 9$ why this is so.

c) Find a quadratic approximation to $9^{1/3}$, and estimate the difference between the quadratic approximation and the exact answer. (Hint: To get a reasonable quadratic approximation, use the fact that 8 and 9 are reasonably close.)

5. (Bonus! 10 pts: 2 + 3 + 2 + 2 + 1) **Return of the Astroid.**[1]

a) Show that every tangent line to the curve $x^{2/3} + y^{2/3} = 1$ in first quadrant has the property that portion of the line in the first quadrant has length 1. (Use implicit differentiation; this is the same as problem 45 page 114 of text.)

b) Next we reverse the logic, deriving the equation for the astroid in part (a), assuming it is a curve with the above property.

Think of the first quadrant of the xy-plane as representing the region to the right of a wall with the ground as the positive x-axis and the wall as the positive y-axis. A unit length ladder is placed vertically against the wall. The bottom of the ladder is at $x = 0$ and slides to the right along the x-axis until the ladder is horizontal. At the same time, the top of the ladder is dragged down the y-axis ending at the origin $(0, 0)$. We are going to describe the region swept out by this motion. In more picturesque language, this would be the blurry region in a photograph of the ladder's motion if the eye of the camera is open during the entire sliding process.

a) Suppose that L_1 is the line segment from $(0, y_1)$ to $(x_1, 0)$ and L_2 is the line segment from $(0, y_2)$ to $(x_2, 0)$. Find the formula for the point of intersection (x_3, y_3) of the two line segments. Don't expect the formula to be simple: It must involve all four parameters x_1, x_2, y_1, and y_2. But simplify as much as possible!

It's important to make sure you have the right formulas before proceeding further. You can doublecheck your formulas in four ways. (This is optional.)

i) If $y_2 = 0$, then $x_3 = x_1$.

ii) When the x's and y's are interchanged the formulas should be the same. What transformation of the plane does the exchange of x and y represent?

iii) It is impossible to find x_3 and y_3 if the lines are parallel, so the denominator in the formula must be zero when L_1 and L_2 have the same slope.

iv) Rescaling all variables by a factor c leaves the formula unchanged, so the numerator of the formula for x_3 and y_3 should have degree (in all variables) one greater than the denominator.

b) Write the equation involving x_2 and y_2 that expresses the property that ladder L_2 has length

---

[1] Bonus problems are completely optional. Your bonus points are recorded in a separate column in Stellar to avoid affecting the PSet average.

one. We will suppose that L_1 represents the ladder at a fixed position, and L_2 tends to L_1. Thus

$$x_2 = x_1 + \Delta x; \quad y_2 = y_1 + \Delta y$$

Use implicit differentiation (related rates) to find

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

(Express the limit as a function of the fixed values x_1 and y_1.)

c) Substitute $x_2 = x_1 + \Delta x$ and $y_2 = y_1 + \Delta y$ into the formula in part (a) for x_3 and use part (b) to compute

$$X = \lim_{x_2 \to x_1} x_3 = \lim_{\Delta x \to 0} x_3$$

Simplify as much as possible. Deduce, by symmetry alone, the formula for

$$Y = \lim_{x_2 \to x_1} y_3$$

d) Show that $X^{2/3} + Y^{2/3} = 1$. (The limit point (X, Y) that you found in part (c) is expressed as a function of x_1 and y_1. This is the unique point of the ladder L_1 that is also part of the boundary curve of the region swept out by the family of ladders.)

Michael Plasmeier

PSet 3

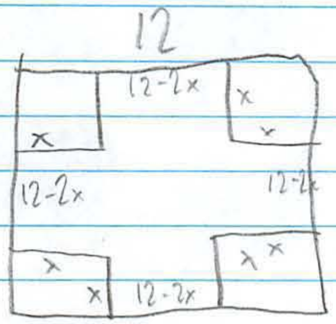
$\frac{30}{41}$ not corrected

+1 bonus

10/9

Lecture 11

2C-1 Cut 4 identical squares + Fold sides from 12 × 12 cardboard to max w/o top. Size square to max volume



find x to max A-one side

$V = x(12-2x)^2$ $x \cdot (12-2x)$

↑ think about

endpoints

$0 \le x \le \cancel{12}\ 6$

↑ 2x

~~$v' = 2(12-2x) \cdot -2$~~
~~$-4(12-2x)$~~
~~$-48+8x$~~
~~$8x = 48$~~
~~$x = 6$~~

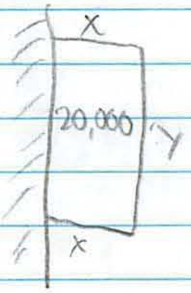
$(12-2x)^2 + x \cdot 2 \cdot (12-2x)(-2)$
$(12-2x)(12-2x-4x)$
$(12-2x)(12-6x)$
$x' = 0$ at $x = 2$
$x = 6$

x	0	2	6	~~12~~
f(x)	0	128	0	~~1728~~

↑ local minimum x

at x = 2

2C-2 Barnyard enclosure 20,000 sq ft fencing on 3 sides



L $x + x + y = \min$

$x \cdot y = \text{area} = 20{,}000$

Solve for y

$y = \frac{20{,}000}{x}$

$2x + \frac{20{,}000}{x} = \min L$ $= 20000x^{-1}$

$\min L' = 2 + -20{,}000x^{-2}$

$2 - \frac{20{,}000}{x^2}$

use less paper please!!

$$0 = 2 - \frac{20000}{x^2}$$

$$-2x^2 = -20{,}000$$

$$x^2 = 10{,}000$$

$\boxed{x = 100}$ ← critical pt

1C-4 Post Service

$l + 4s = 188$ in ← constraint

largest volume

$s \cdot s \cdot l = \max V$

sub in

$s \cdot s \cdot (108 - 4s) = V$

$s^2 (108 - 4s)$

$V' = 2s(108 - 4s) + (-4) \cdot s^2$

$216s - 8s^2 - 4s^2$

$216s - 12s^2$

$0 = 216s - 12s^2$

$s = 0$

$s = 18$

getting easy now

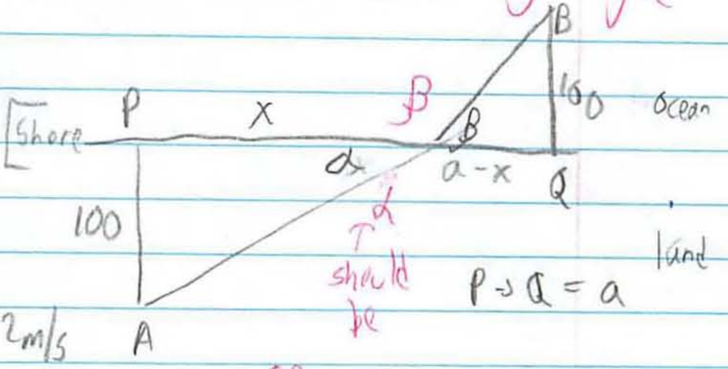
	low		max
x	0	18	27
f(x)	0	11664	0

sides should be 18 in

heigh = 36 in

Book used wrong angle

2C-10 Swimmer at point A
closest to pt P on shore
Plat form at B



Can run 5m/s swim 2m/s

Or use cos w/ those angles

Show that: $\frac{\overset{\cos}{\cancel{\sin}}\,\alpha}{\underset{\cos}{\cancel{\sin}}\,\beta} = \frac{5}{2}$ ∈ snell's law
least time
is

$$T = \sqrt{\frac{100^2+x^2}{5}} \frac{m}{s} + \sqrt{\frac{100^2+(a-x)^2}{2}}$$

as $x \to \infty$, $T \to \infty$ - want min value

$$T' = \frac{x}{5\sqrt{100^2+x^2}} - \frac{(a-x)}{2\sqrt{100^2+(a-x)^2}} = \frac{\overset{\cos}{\cancel{\sin}}\,\alpha}{5} - \frac{\overset{\cos}{\cancel{\sin}}\,\beta}{2}$$

$$\frac{\overset{\cos}{\cancel{\sin}}\,\alpha}{5} - \frac{\overset{\cos}{\cancel{\sin}}\,\beta}{2} = 0 \qquad \frac{\overset{\cos}{\cancel{\sin}}\,\alpha}{5} = \frac{\overset{\cos}{\cancel{\sin}}\,\beta}{2} \qquad \frac{\cancel{\sin}\,\alpha}{\cos\,\cancel{\sin}\,\beta} = \frac{5}{2}$$

?? algebra

2C-13 Airline will fill 100 seats aircraft at $200
for every $5 ↑ ↓2 seats max revenue
x = # changes

$(100 - 2x)(200 + 5x)$ = max profit

profit' = $-2(200 + 5x) + 5(100 - 2x)$
$-400 - 10x + 500 - 10x$
$100 - 20x = 0$
$-20x = -100$
$x = 5 \cdot 5$ = price $225

way diff than book

at 90 seats

2E-13
Answer

p = price in \$ ← other way around
people = $100 + \left(\frac{2}{5}\right)(200-p)$

$R = p\left(100 + \frac{2}{5}\right)(200-p)$
$p\left(180 - \frac{2}{5}p\right)$

from
book

at $p = 0$ and $p = 450$ no \$
↑ free ↑ too expensive

$R' = \left(180 - \frac{2}{5}p\right) - \frac{2}{5}p$
$180 - \frac{4}{5}p = 0$
$p = \frac{5}{4}(180)$
225 ← same I said

b Power plant

x killowatt hrs at cost $10 - \frac{x}{10^5}$ cents for $0 \leq x \leq 8 \times 10^5$

Consumers use $10^5(10 - p/2)$ killowats
↑ [illegible]

Max profit

profit = revenue − cost

$\overbrace{\text{(price·amt}}^{\text{rev}}\text{ - cost·amt)}$

$10^5(10 - p/2)$ ~~$= (10 - x/10^5)$~~ why? ↓ $\left(p - 10 + \left(10 - \frac{p}{2}\right)\right)$

$P = 10^5(10 - p/2)(p - 10) + (10 - p/2))$
$10^5(10 - p/2)(p/2)$
$(10^5/4)\,p(20 - p)$

$\frac{dP}{dp} = (10^5/2)(10 - p)$
P ← profit, p ↑ price

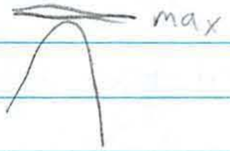
$\left(\frac{10^5}{2}\right)\left(10-p\right) = 0$

$p = 10$

use $= 10^5\left(10 - 10/2\right)$

500,000 killowatt hours

↗within acceptible range



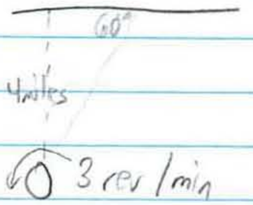
profit $10^5\left(10-10/2\right)\left(2-10\right)+\left(10-\frac{10}{2}\right)$

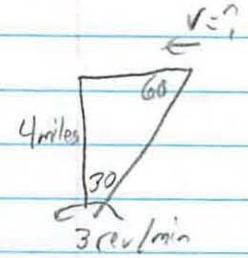
2.5×10^6 cents

$ 250,000

Lecture 12

2E-2 Becon 4 miles off shore





How fast is light moving on shore
when beam = 60° with shore line

$$\tan\theta = \frac{x}{4} \qquad \frac{d\theta}{dt} = 3\cdot 2\pi = 6\pi$$

↓ differentiate implicitly

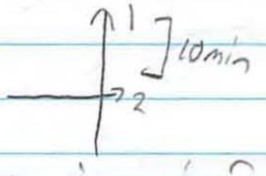
$$\sec^2\theta \, \frac{d\theta}{dt} = \frac{1}{4}\frac{dx}{dt}$$

$\frac{\pi}{6}$

$$\frac{4}{3}\cdot 6\pi = \frac{1}{4}\frac{dx}{dt}$$

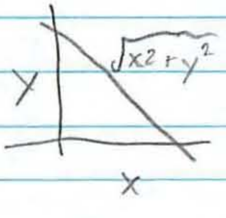
$$\frac{dx}{dt} = 32 \text{ miles/min}$$

2E-3 2 boats one north
30 mph one east



← can find distance
5 miles

at what rate is the distance increasing?



$x = 10 \qquad x' = 30$

$y = 15 \qquad y' = 30$

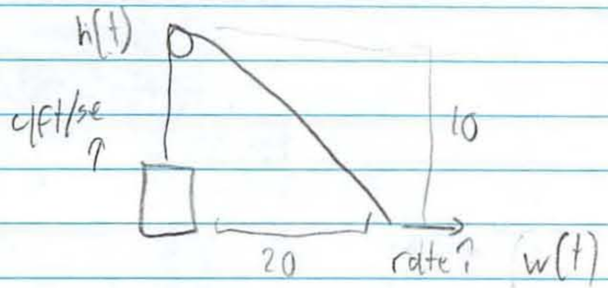
?? how find that moving for 30 min

$$\left((x^2+y^2)^{1/2}\right)' = \frac{1}{2}(2xx' + 2yy')(x^2+y^2)^{-1/2}$$

$$\frac{(10\cdot 30 + 15\cdot 30)}{\sqrt{10^2+15^2}}$$

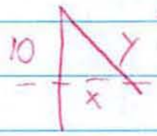
$$\boxed{\frac{150}{\sqrt{13}} \text{ mph}}$$

2F-5. Person walks away from pully w/ rope on it
rope held 10 ft below pully
other end rising 4ft/sec
what rate is person walking when 20 ft from pully



$h'(t)$ = know = 4ft/sec
$w'(t)$ want to know
$w(0) = 20$ ft
$h(0) = 10$ ft ↙ also ' = 4ft/sec
$h(t)^2 + w(t)^2 =$ rope lenght^2 ← would have never thought of

book different



$x^2 + 10^2 = z^2$ $z' = 4$

↘ deriv

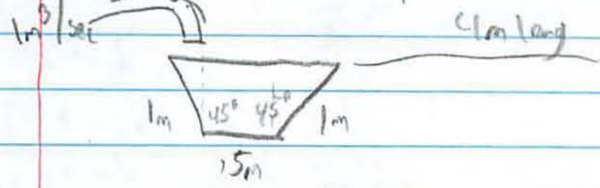
$2xx' = 2yy'$

$z^2 = 20^2 + 10^2 = 500$

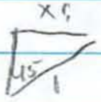
↙ almost had just need to do

$x' = \frac{(zz')}{x}$ $\frac{4\sqrt{500}}{20} = 2\sqrt{5}$

2E-7 Trough filled w/ water at 1 m^3/sec



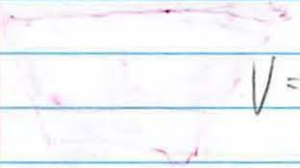
rate water ↑
when $h = \frac{1}{2}$ m



$\sin 45 = \frac{x}{1}$
$x = .8509$

$A = \left(\frac{1}{2} bh\right) \cdot 2 + 5h$
$\frac{1}{2} \cdot .8509 \cdot .8509 \cdot 2 + .5 \cdot .8509$
$V = 1.149 \cdot 4$
$V = 4.597 m^3$

Rhodes

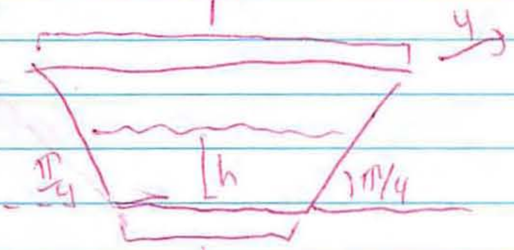


$V = 4\left(h^2 + \frac{h}{2}\right)$ $V' = 1$
Find h' at $h = 1$

$1 = V' = 8hh' + 2h'$
$8 \cdot \frac{1}{2} \cdot h' + 2h'$
$1 = 6h'$
$h' = \frac{1}{6}$ m/sec height increase of water

want $\frac{dh}{dt}$ when $h = 0.5$

Stuff constant ~ use #



Given $\frac{dV}{dt} = 1 m^3$

Find $V(h) = 4 A_{water}$

$4h \cdot \left(\frac{1}{2} + h\right)$

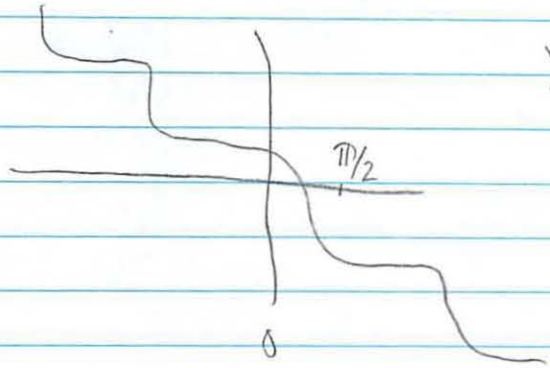
Do $\frac{dV}{dt} = 4hh' + \left(\frac{1}{2} + h\right)(4h')$

Sub in when $h = .5$
$1 = 4 \cdot .5h' + \left(\frac{1}{2} + \frac{1}{2}\right)(4h')$

Solve for h'

2F-1 a) Graph $y = \cos x - x$

Show using y' that there is 1 root



$y' = -\sin x - 1$

$-\sin x - 1 \leq 0$

so y is decreasing

thus there can only be 1 root

if $x_1 < x_2$

then $f(x_1) < f(x_2)$

$0 < x < \frac{\pi}{2}$

b) Use Newton's Method to find root

$x_1 = \frac{\pi}{2}$

$$x_2 = \frac{\pi}{2} - \frac{\cos\frac{\pi}{2} - \frac{\pi}{2}}{-\sin\frac{\pi}{2} - 1} = \frac{\pi}{4}$$

$x_3 = .7395361$

$x_4 = .7390852$

$x_5 = .7390851$ ← 3 decimal place lock down

$x_0 = .7390851$

c) stablize method

$x_1 = \frac{\pi}{2}$

$x_2 = \cos\frac{\pi}{2} = 0$

$x_3 = \cos 0 = 1$

$x_4 = \cos 1 = .5403023$

$x_5 = .8575532$

$x_6 = .6542898$

$x_7 = .7934804$

$x_8 = .7013688$

$x_9 = .7639597$

$x_{10} = .7221024$

$x_{11} = .7504178$

$x_{12} = .731404$

$x_{13} = .7442374$

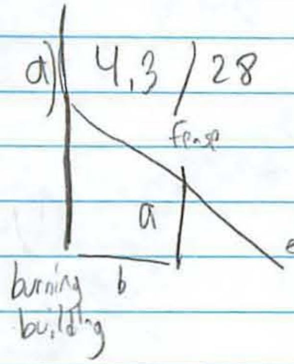
$x_{14} = .7356047$

$x_{15} = .7414251$

Part 2

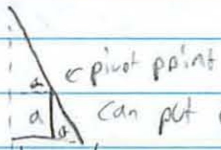
0. See side bar + kimberly + Cole 3

1. a) 4,3 / 28 Fireman's problem

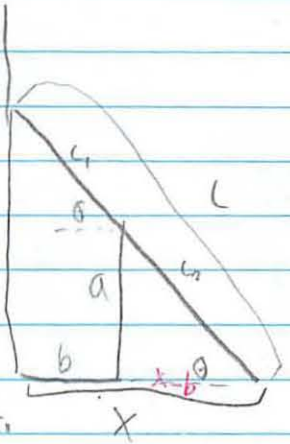


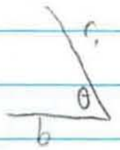
← shortest ladder

Rhodes office hrs



← pivot point can put θ in





$\cos\theta \frac{b}{L}$

$\cos\theta L = b$

$L = \frac{b}{\cos\theta}$

$\theta = \cos^{-1}\frac{b}{L_1}$

$L = \frac{b}{\cos\theta} + \frac{a}{\sin\theta}$ a + b fixed

min $b\cos\theta^{-1}$

$\frac{dL}{d\theta} = -1 b \cos\theta^{-2} - \sin\theta + -1 a \sin\theta^{-2} \cos\theta$

$0 = \frac{\sin\theta b}{\cos\theta^{2}} - \frac{a\cos\theta}{\sin\theta^{2}}$

; what now

-use x as distance ladder → house

Check end points

w/ θ is harder

Office hrs

$\frac{L}{x} = \frac{\sqrt{a^2+(x-b)^2}}{x-b}$ ← lenght w/ respect to x

$L = x\frac{\sqrt{a^2+(x-b)^2}}{x-b}$

$L'(x) = \frac{(x-b)\left(\sqrt{a^2+(x-b)^2} + x\left(\frac{1}{2}\right)\left(a^2+(x-b)^2\right)^{-\frac{1}{2}}(2x-b)\right) - x\sqrt{a^2+(x-b)^2}}{(x-b)^2}$

Rhodes do w/ respect to x - ecliser

cancle
↓ $*(x-b)^2$
multiply sq rt at

$$\underbrace{(\cancel{x}-b)(a^2+(x-b)^2] + x(x-b) - x}_{\text{cancles out if you distribute}}\cancel{[a^2+(x-b)^2]} = 0$$

office hrs

$$-ba^2 - b(x-b)^2 + x(x-b) = 0$$

$$-ba^2 + (x-b)^3 = 0$$

$$x - b = (a^2 b)^{1/3}$$

$$x = b + (a^2 b)^{1/3}$$

↑ minimum

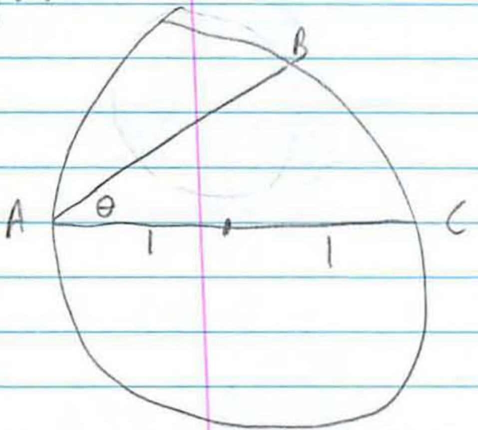
(deriv = 0)

$$L = \sqrt{\frac{a^2 + [b + (a^2 b)^{1/3}]^2}{(b + (a^2 b)^{1/3}) - b}} \cdot \left(b + (a^2 b)^{1/3}\right)$$

$$\sqrt{\frac{a^2 + (b + (a^2 b)^{1/3})^2}{(a^2 b)^{1/3}}} \cdot b + (a^2 b)^{1/3}$$ X

2

b 4.4/28



walk 6 mph
row 3 mph
At what angle should he row?
minimize time

$\boxed{0 \le \theta \le \frac{\pi}{2}}$
row whole way, walk

$= \frac{\pi}{2}$

2

Rhodes
office hrs

$T = (2\pi R)6 + 3(2R)$

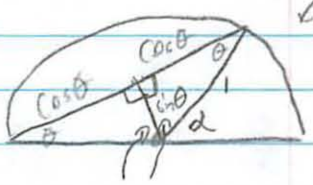
$T(\theta) = T_{row}(\theta) + T_{walk}(\theta)$
w respect to θ

$\frac{d_{row}(\theta)}{3} + \frac{d_{walk}(\theta)}{6}$

$\frac{2\cos\theta}{3} + \frac{2\theta}{6}$

$\frac{2\cos\theta}{3} + \frac{\theta}{3}$

bisect



$\frac{\pi}{2} - \theta$

$\alpha + \pi - 2\theta = \pi$

$\alpha = 2\theta$

$T'(\theta) = -\frac{2\sin\theta}{3} + \frac{1}{3}$

$\sin\theta = \frac{1}{2}$

$\frac{\pi}{6}$

now must check

$0 \quad \frac{\pi}{6} \quad \frac{\pi}{2}$

$\frac{2}{3}$ (+) $\frac{\sqrt{3}}{3} + \frac{\pi}{18}$ (−) $\frac{\pi}{6}$ ← fastest to only walk around lake

← max!

Sketch

2. $y = \frac{2x^2}{x^2-1}$ horiz asy = 2

$\uparrow \pm 1$ vertical asy

$$f' = \frac{2 \cdot 2x(x^2-1) - 2x(2x^2)}{(x^2-1)^2}$$

$$\frac{4x^3 - 4x - 4x^3}{(x^2-1)^2} = \frac{-4x}{(x^2-1)^2} = 0 \quad ✓$$

$x = 0$

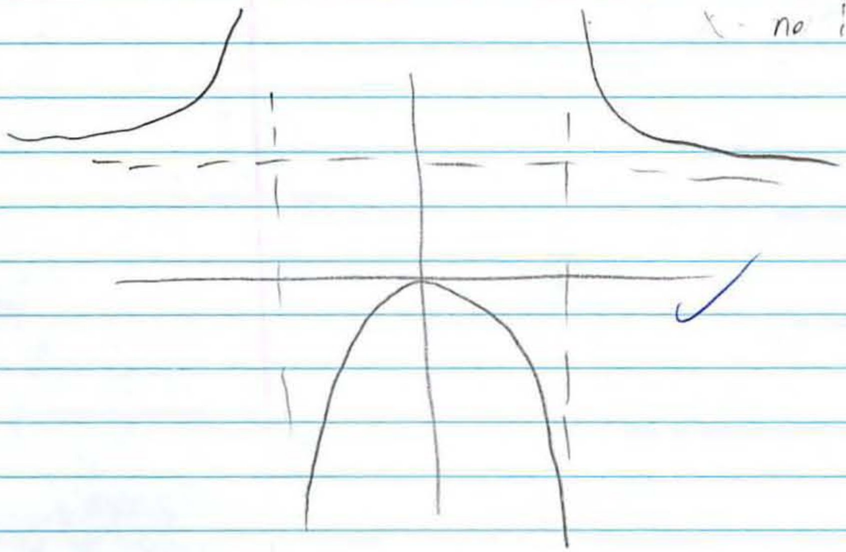
$-\infty \quad -1 \quad 0 \quad \infty$

$2 \quad ⊕ \quad \infty \,|\, -\infty \quad ⊕ \quad 0 \quad ⊖ \quad -\infty \,|\, \infty \quad ⊖ \quad 2$

$$f'' = \frac{-4(x^2-1)^2 - 2(x^2-1) \cdot 2x \cdot -4x}{(x^2-1)^4}$$

$$\frac{-4(x^2-1)^{\cancel{2}} + 16x\cancel{(x^2-1)}}{(x^2-1)^{\cancel{4}3}} = \frac{4(3x^2+1)}{(x^2-1)^3} = 0$$

∴ no inflection pt ✗



✓

1.5

b) $2\cos x + \sin 2x$ $(-210, 150^{\circ})$

?

no asymptotes

$f' = -2\sin x + \cos(2x) \cdot 2$

$2\cos(2x) - 2\sin x$

$0 = -90^{\circ}$

30°

$= 2(\cos^2 x - \sin^2 x) - 2\sin x$

$2\cos^2 x - 2\sin^2 x - 2\sin x$

$2(1-\sin^2 x) - 2\sin^2 x - 2\sin x$

$2 - 2\sin^2 x - 2\sin^2 x - 2\sin x$

$2 - 4\sin^2 x - 2\sin x = 0$

$-4\sin^2 x - 2\sin x = 2$

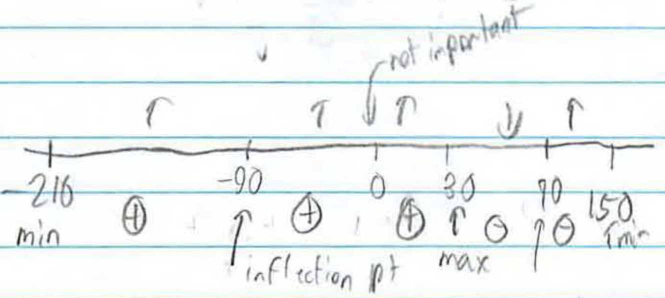
$4\sin^2 x + 2\sin x = -2$

$2\sin^2 x + \sin x = -1$

$(2\sin x - 1)(\sin x + 1) = 0$

$\sin^{-1}(\frac{1}{2})$ $\sin^{-1}(-1)$

30° 90°



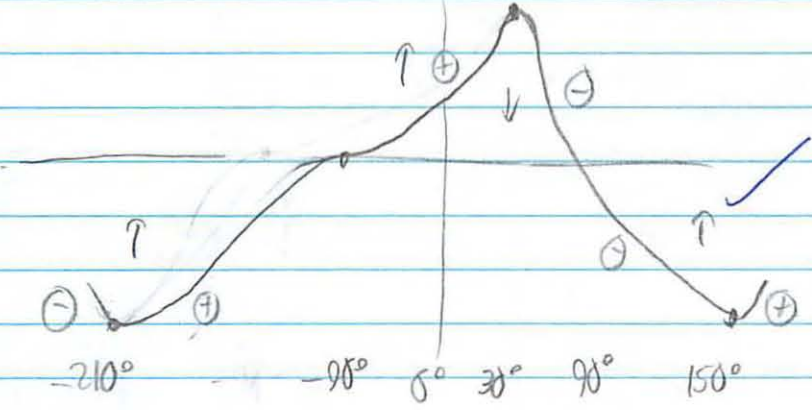
product rule + chain rule

$f'' = -2 \cdot \sin(2x) \cdot 2 - 2\cos x$

$-4\sin(2x) - 2\cos x$

$-90^{\circ}, -14.47^{\circ}$

1.5



c) $y = \ln(4-x^2)$ $(\lim -2 < x < \lim 2$

$f' \; \frac{1}{4-x^2} \cdot -2x = \frac{-2x}{4-x^2} = -0$ ✓

$x=0$

-2 0 2

\+ -

? °

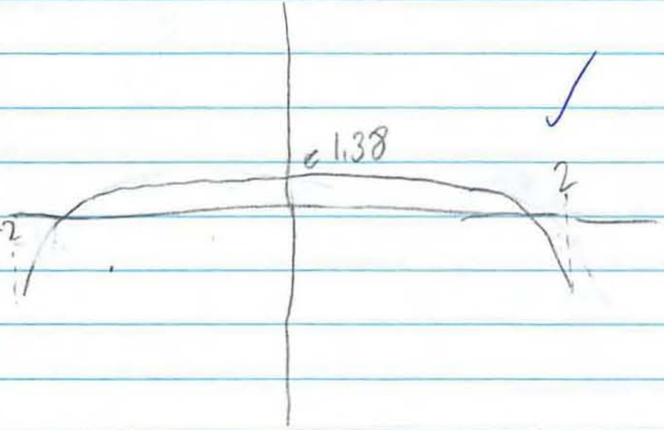
$f'' = \frac{-2(4-x^2) - (-2x)(-2x)}{(4-x^2)^2}$ foil

$\frac{-2(4-x^2) - (4x^2)}{(4-x^2)^2} = \frac{-8+2x^2-4x^2}{(4-x^2)^2} = \frac{-2x^2-8}{(4-x^2)^2}$

$\frac{-2(x^2+4)}{(x^2-4)^2}$ ← factor

$=0$

no solutions



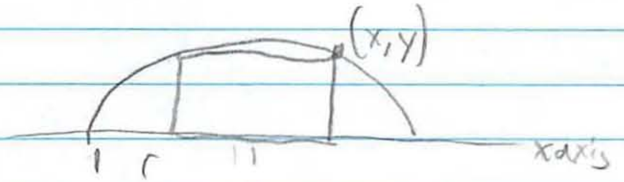
✓

1

3a. Find area of largest rectangle that can be inscribed in semic circle radius r

can parameterize in diff ways

Rhodes

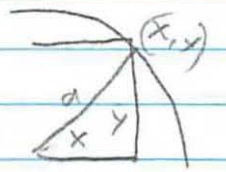


$x^2 + y^2 = a^2$

$y = \sqrt{a^2 - x^2}$

$(a^2 - x^2)^{1/2}$

$A = 2x(a^2 - x^2)^{1/2}$



x lies in interval $0 < x < a$

$$\frac{dA}{dx} = 2x\left[\frac{1}{2}(r^2 - x^2)^{-1/2} - 2x\right] + \sqrt{r^2 - x^2}\,2$$

$$-2x\left(\frac{x}{\sqrt{r^2 - x^2}}\right) + 2\sqrt{r^2 - x^2}$$

$$\frac{-2x^2}{\sqrt{r^2 - x^2}} + 2\sqrt{r^2 - x^2} = 0$$

$$-2x^2 + 2(r^2 - x^2) = 0$$

$$-2x^2 + 2r^2 - 2x^2 = 0$$

$$-4x^2 = -2r^2$$

$$x^2 = \frac{r^2}{2}$$

$$x = \sqrt{\frac{r}{2}} = \frac{1}{2}\sqrt{2r}$$

$$y = \sqrt{r^2 - \frac{r^2}{2}} = \sqrt{\frac{r^2}{2}} = \frac{1}{2}2\sqrt{r}$$

$x = \frac{1}{2}\sqrt{2r}$ $\quad y = \frac{1}{2}2\sqrt{r}$

$\therefore$ Area $= 2\left(\frac{1}{2}\sqrt{r}\,r\right)\left(\frac{1}{2}\sqrt{2r}\right) = r^2$

2

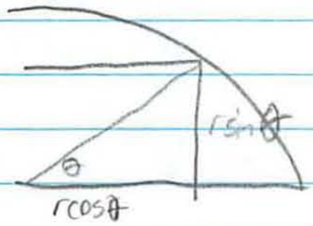
polar coords

3b in terms of $\theta \rightarrow (r\cos\theta, r\sin\theta)$

express x, y in terms of sin + cos

$Area = 2xy$

$A = 2r^2 \cos\theta \sin\theta$



$dA = 2r^2(\cos^2\theta - \sin^2\theta)d\theta$

$dA = 2r^2 \cos 2\theta d\theta$

$\frac{dA}{d\theta} = 2r^2 \cos 2\theta$

$0 = 2r^2 \cos 2\theta$

$0 = \cos 2\theta$

$\cos^{-1} 0 = 2\theta$

$90 = 2\theta$

$\theta = 45$

$2(r\sin 45)(r\cos 45)$

$2r^2\left(\frac{1}{\sqrt{2}}\right)^2$

$\cancel{2}r^2\left(\frac{1}{\cancel{2}}\right)$

$\boxed{r^2}$

✓

2

4. Newton's Method

Find cubic root of 9 to 6 sig fig

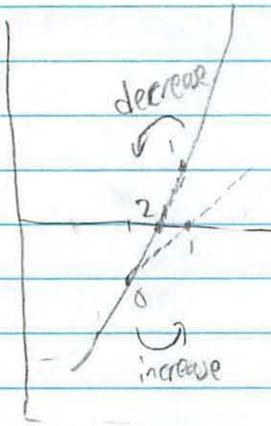
$$x_n = x_{n-1} - \frac{f(x)}{f'(x)} \qquad f(x) = x^3 - 9 \qquad f'(x) = 3x^2$$

	vs $3^{2/3}$	vs x_{k-1}
$x_0 = 2$	- smaller	
$x_1 = 2 - \frac{2^3 - 9}{3(2)^2} = 2.0833333333$	- bigger	increase
$x_2 = 2.08008888889$	- bigger	decrease
$x_3 = 2.08008382306$ ← 6 sig figs	- bigga	decrease
$x_4 = 2.08008382305$	- same	decrease
$x_5 = 2.08008382305$ ← surpassed calc	- same	same

3

actual result $= 3^{2/3} \approx 2.08008382305$ ✓

b. see Sidebar



and the points get too small to draw but same idea i.e. if it overshoots its smaller/bigger flip - it is always trying to get back to x_{real} - if it is bigger than x_{real} it will be smaller than the previous one

1

table w/ signs

c $9^{1/3}$ quadratic approximation $f'' = 6x$

~~$f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2$~~

~~$(9^{1/3})^3 - 9 + 3(9^{1/3})^2(x - 9^{1/3}) + \frac{6(9^{1/3})}{2}(x - 9^{1/3})^2$~~

~~$f(x) = 9 - 9 + 3 \cdot 3 \cdot 3^{1/3}(x - 9^{1/3}) + 3(9^{1/3})(x^2 - 2 \cdot 3^{2/3}x + 2 \cdot 3^{1/3})$~~

~~$9 \cdot 3^{1/3}x - 27 + 3 \cdot 9^{1/3}x^2 - 18 \cdot 3^{1/3}x + 18 \cdot 3^{1/3}$~~

c) Quadratic approximation — 8 close to 9

↙ $\sqrt[3]{8} = 2$

$f(x) = x^3 - 9$ — $f(2) = -1$

$f'(x) = 3x^2$ — $f'(2) = 12$

$f''(x) = 6x$ — $f''(2) = 12$

$$f(x) = -1 + 12(x-2) + 6(x-2)^2$$
$$-1 + 12x - 24 + 6(x^2 - 4x + 4)$$
$$-1 + 12x - 24 + 6x^2 - 24x + 24$$
$$6x^2 - 12x - 1$$

set = to 0

$$0 = 6x^2 - 12x - 1$$
$$x = \frac{12 \pm \sqrt{144 - 4(-1)(6)}}{12}$$
$$x = \frac{12 \pm \sqrt{144 + 24}}{12}$$
$$x = \frac{12 \pm \sqrt{168}}{12}$$
$$x = 2.08012345$$

1

estimate between difference

-.000039626884

difference

Bonus

5. Return of Astroid

a. $x^{2/3} + y^{2/3} = 1$

$x^{2/3} + y^{2/3} = 1$ = hypocycloid
of 4 cusps

tangent at (x_0, y_0) = $x_0^{-1/3} + y_0^{-1/3} y = a^{2/3}$

Segment cut from the tangent by the axis has constant lenght a with ends sliding along axis always touches curve

1

18.01 · Fall 2009.
Problem Set 3

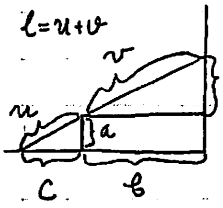
Part II

Problem 1 (3+3) pts

[Fireman's problem] a fence a ft high is b ft away from a burning bld. (high)
Find the length of the shortest ladder that will reach from ground to bld.

a) 4.3 /28

(alternative solution) Label triangle sides u, v, c, d



Use similar triangles to get rid of c, d:

$$\frac{d}{v}=\frac{u}{a}=\frac{u+v}{d+a}\Rightarrow d=\frac{av}{u} \quad \Big\| \quad \frac{u}{c}=\frac{v}{b}\Rightarrow c=\frac{bu}{v}.$$

Constraint: $u^2-a^2=c^2=\left(\frac{bu}{v}\right)^2 \Rightarrow bu/v=\sqrt{u^2-a^2} \Rightarrow v=\frac{bu}{\sqrt{u^2-a^2}}$ (*).

We want to minimize $l=u+v$ subject to above constraint. Thus, we want to minimize $l=u+\frac{bu}{\sqrt{u^2+a^2}}$. Take derivatives: and set = 0. $\frac{dl}{du}=1+\frac{b\sqrt{u^2+a^2}-\frac{1}{2}2u(u^2-a^2)^{-1/2}bu}{u^2-a^2}=0.$

$\Rightarrow b(u^2-a^2)-bu^2=-(u^2-a^2)^{3/2} \Rightarrow ba^2=(u^2-a^2)^{3/2} \Rightarrow u=\sqrt{(ba^2)^{2/3}+a^2}$.

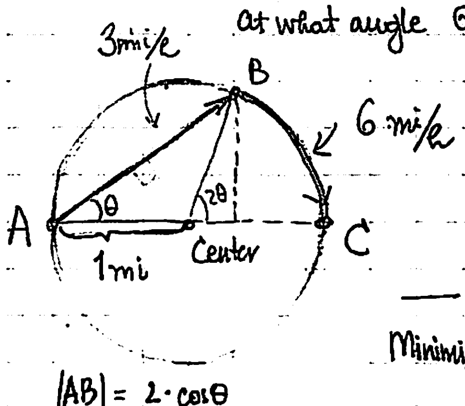
Plug this into constraint equation (*), obtain $v=\sqrt{b^2+(b^2a)^{2/3}}$ (should check this is a min, but I'm skipping that step)

$$\Rightarrow l_{min}=\sqrt{a^2+(ba^2)^{2/3}}+\sqrt{b^2+(b^2a)^{2/3}}=b^{1/3}\left(a^{2/3}+b^{2/3}\right)\left(1+\frac{a^2}{a^{4/3}b^{2/3}}\right)^{1/2}$$

$$=\boxed{\left(a^{2/3}+b^{2/3}\right)^{3/2}}$$ 4

b) 4.4/28

A man at pt A on the shore of a circular lake of radius 1 mi. wants to reach the opposite point C as soon as possible. He can walk 6 mi/h and row 3 mi/h. At what angle θ to the diameter AC should he row?



$$\text{Time}=T=\frac{1}{3}\underbrace{\left(2\cos\theta\right)}_{|AB|}+\frac{1}{6}\underbrace{(2\theta)}_{|\text{arc}(BC)|}$$

$$=\frac{2}{3}\cos\theta+\frac{1}{3}\theta. \quad \text{(defined for } 0\le\theta\le\tfrac{\pi}{2})$$

Minimize $T=T(\theta)$. Find critical pts by taking derivative, set = 0.

$|AB|=2\cdot\cos\theta$

$$\frac{dT}{d\theta}=-\frac{2}{3}\sin\theta+\frac{1}{3}=0 \Rightarrow \sin\theta=\tfrac{1}{2} \Rightarrow \theta=\pi/6.$$

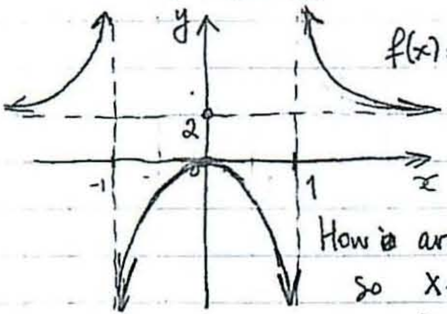
We are working on a bounded domain. So have to compare values at critical pts and boundaries

Test values @ $\theta=0,\frac{\pi}{6},\frac{\pi}{2}$: $T(0)=\frac{2}{3}$ $\underbrace{T(\pi/2)=\pi/6}_{\text{minimum!}}$ $T(\pi/6)=\frac{\sqrt{3}}{3}+\frac{\pi}{18}$

So $\boxed{\text{shortest time is by only walking } (\theta=\pi/2)}$ + 4 (!).

Problem 2 Sketch the following functions. Indicate asymptotes, max/min, intervals of increasing/decreasing and points of inflection.

a) $y = \frac{2x^2}{x^2-1}.$ notice: even symmetry



$f(x) = \frac{2x^2}{x^2-1} = \frac{2x^2}{(x+1)(x-1)}$ $\Rightarrow$ zero @ $x=0$; vertical asympt. $x=+1$, $x=-1$; horizontal asymp. $y=2$.

$\left(\frac{2x^2}{x^2} \to 2\right)$

How ~~is~~ are the v. asymptotes approached? $f(x)$ $+$ $-$ $+$; -1 $+1$

so $x \to -1^-$ $y \to +\infty$ $x \to 1^-$ $y \to -\infty$

$x \to -1^+$ $y \to -\infty$ $x \to 1^+$ $y \to +\infty$

critical pts:

$$f'(x) = \frac{4x(x^2-1) - 2x^2(2x)}{(x^2-1)^2} = \frac{\cancel{4x^3} - 4x - \cancel{4x^3}}{(x^2-1)^2} = \frac{-4x}{(x^2-1)^2}$$

$f'(x) = 0$ at $x=0$. $\Rightarrow$ one critical pt : $(0,0)$

function increasing: $f'(x) > 0$ for $x<0$ (it will turn out from $f''(x)$ that this is a maximum.)

decreasing $x>0$

inflection pts
$$f''(x) = \frac{-4(x^2-1)^{\cancel{2}} + 4x(2)(\cancel{x^2-1})(2x)}{(x^2-1)^{\cancel{4}3}} = 4\,\frac{-x^2+1+4x^2}{(x^2-1)^3} = 4\cdot\frac{3x^2+1}{(x^2-1)^3}$$

$(f''(0) < 0)$ so $(0,0)$ is max.

$f''(x)$ changes sign at ± 1. So these are pts of inflection (the v. asymptotes ~~are~~ are where concavity changes)

b) $y = 2\cos x + \sin 2x = f(x)$

Function is bounded, periodic. Has a period of 2π. It is mainly a large cosine wave with a sine wave of half the period "threaded through". (added on top) we need calculus to see how this threading actually looks.

critical pts:

$f'(x) = -2\sin x + 2\cos 2x = 0$ $\cos 2x - \sin x = 0 \Rightarrow 1 - 2\sin^2 x - \sin x = 0$

$\Rightarrow 2\sin^2 x + \sin x - 1 = 0$

$(2\sin x - 1)(\sin x + 1) = 0$

$\sin x = +1/2$ or $\sin x = -1$

$\boxed{x = \pi/6 + 2k\pi}$ this will be a max

$\boxed{x = \frac{5\pi}{6} + 2k\pi}$ min

$\boxed{x = +\frac{3\pi}{2} + 2k\pi.}$ Saddle point

concavity:

$f''(x) = -2\cos x - 4\sin 2x = 0$

~~cos x = 2 sin~~

$\cos x + 2\sin 2x = 0$ $\cos x + 4\cos x \sin x = 0$

$1 + 4\sin x = 0$ or $\cos x = 0$

$\sin x = -1/4.$ $\boxed{x = \pi/2 + k\pi}$

$\boxed{x = \arcsin(-1/4) + 2k\pi}$

inflection points.

between π and $\frac{7\pi}{6}$, $-\alpha$

between $\frac{11\pi}{6}$ and $2\pi - \beta$

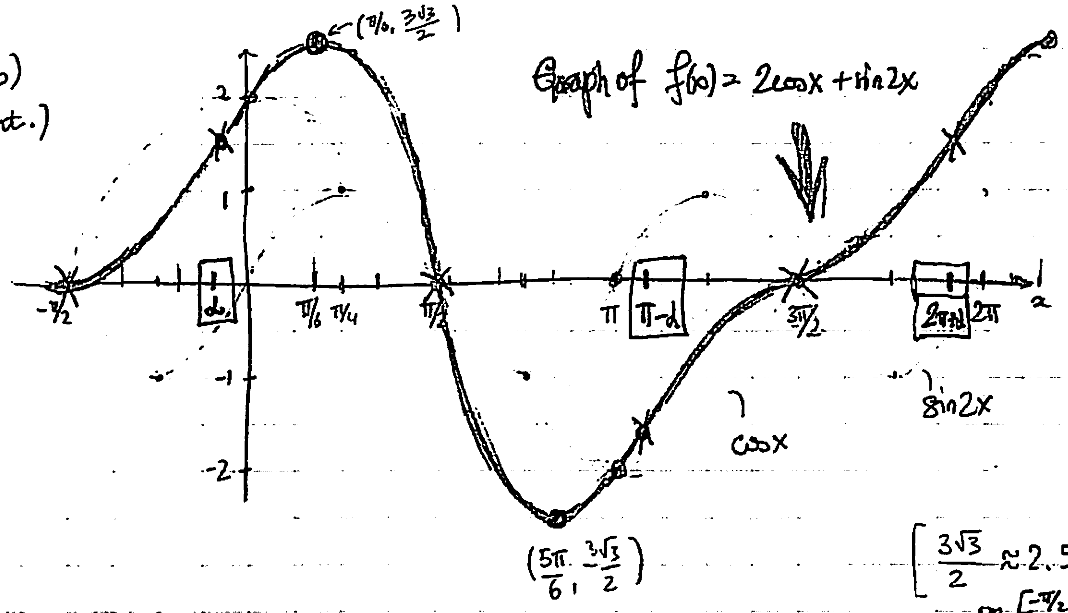
$\boxed{\text{no asymptotes}}$

zeroes: $2\cos x + 2\sin x \cos x = 0$ $\cos x = 0$

$\Rightarrow \sin x = -1 \Rightarrow \boxed{x = \frac{\pi}{2} + k\pi}$ zeroes - same as for cosine.

2b)
(cont.)

Graph of $f(x) = 2\cos x + \sin 2x$



$\left[\frac{3\sqrt{3}}{2} \approx 2.598\right]$
on $[-\pi/2, 0]$

$\left[\arcsin(-1/4) \approx -0.2527\right.$
$\left. -\pi/6 \approx -0.5236\right]$

Inflection points (changes of concavity): (marked w/ X)

@ $x = \arcsin(-1/4) + 2k\pi$;
→ $-0.2527 + 2k\pi$ call this "d" $+2k\pi$ — d is boxed above
→ $(\pi - d) + 2k\pi$. — $\pi - d$ is —"—

↙ intervals of inc./decreasing obvious from graph...

and
@ $x = \frac{\pi}{2} + k\pi$

[Graph is periodic, w/ period 2π].

c) $y = \ln(4 - x^2)$ — defined for $4 - x^2 > 0 \Rightarrow \underline{-2 < x < 2}$

symmetric: even symmetry.

ln is monotone increasing, $4 - x^2$ looks like ⌒ ⇒ max @ $x = 0$ ($\ln 4$) and decreases from there.

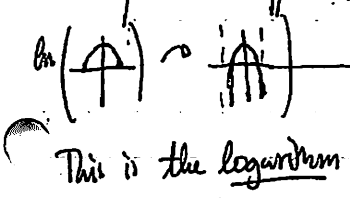
vertical asymptotes: $x = \pm 2$. $\left.\begin{matrix} x \to -2^+ \\ x \to +2^- \end{matrix}\right\} \Rightarrow y \to -\infty$

$f(x) = \ln(4 - x^2)$ $f'(x) = \frac{1}{4 - x^2}(-2x)$ $f'(x) = 0$ @ $x = 0$. this will turn out to be a max.

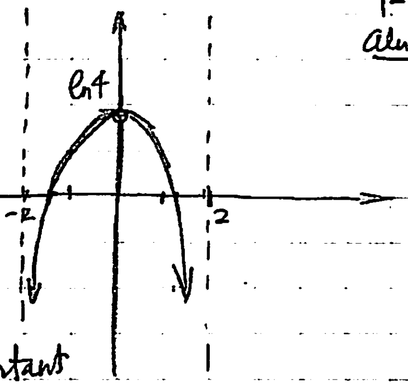
$$f''(x) = \frac{(-2)(4 - x^2) + 2x(-2x)}{(4 - x^2)^2}$$

changes sign when numerator changes sign
@ (divide by -2)
$4 - x^2 + 2x^2 = 0 \Rightarrow 4 + x^2 = 0$ never changes sign
Always concave down.

(Note what happened ln(x) takes



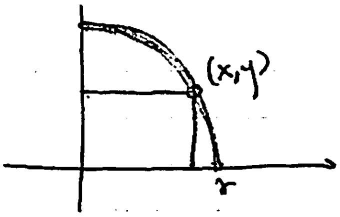
This is the logarithm map — very important [illegible])



Zeroes: $\ln(4 - x^2) = 0$
$4 - x^2 = e$
$x^2 = 4 - e$.
$x = \pm\sqrt{4 - e}$.
(whatever that may be...)
(Between 1, 2)

Problem 3

a) Find the area of the largest rect. that can be inscribed in semicircle of rad. r.



Sufficient to examine quarter circle,

area will be half that inscribed in half-circle.
call this "A")
(total area = 2xy, but whatever)

$A = xy.$, $x^2+y^2=r^2.$

maximize A, A>0, ⇒ same as maximizing A^2. (this is better for our constraint)

This problem would probably be more painful if did not pass to A^2...

$A^2 = x^2y^2$ $y^2 = r^2 - x^2$

$f(x) = x^2(r^2-x^2) = x^2r^2 - x^4.$ $f'(x) = 2xr^2 - 4x^3$

$x=0$ or $r^2 = 2x^2$, $x>0$

$x = r\frac{\sqrt{2}}{2}.$

$\Rightarrow$ area will be $2 \cdot (r\frac{\sqrt{2}}{2})(r\frac{\sqrt{2}}{2})$
$= 2 \cdot (\frac{1}{2})(r^2) = r^2 /\!/.$

Kind of interesting!

b) repeat w/ polar coords.

$(x,y) = (r\cos\theta, r\sin\theta). \Rightarrow$ maximize $g(\theta) = xy = r\cos\theta\, r\sin\theta = r^2 \frac{1}{2} \sin 2\theta.$

$g'(\theta) = r^2 \frac{1}{2} \cdot 2 \cos 2\theta.$ $g'(\theta) = 0$ at $\cos 2\theta = 0 \Rightarrow \boxed{\theta = \pi/4.}$
on $0 \le \theta \le \pi/2.$

$\theta = \pi/4$ gives [illegible] exactly $x,y = (\frac{r\sqrt{2}}{2}, \frac{r\sqrt{2}}{2})$, from above.

Problem 4 [Newton's [illegible] method]

a) Compute $\sqrt[3]{9}$ to 6 sig figs. using Newton's method, starting w/ $x_0 = 2$

$f(x) = x^3 - 9$ $f'(x) = 3x^2$ $x_0 = 2$ $x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = x_k - \frac{x_k^3 - 9}{3x_k^2} = \frac{2}{3}x_k + \frac{3}{x_k^2}$

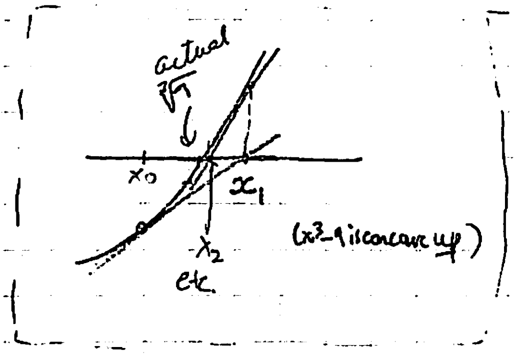
x_0	2.00000
x_1	2.083333
x_2	2.080089
x_3	2.080084
x_4	2.080084

⟵⟶ $\boxed{@ x_3: \sqrt[3]{9} \approx 2.080084}$

(#4, cont.)

b) For each step x_k, $k=0,1,\ldots$ say whether i) value is $\gtrless 9^{1/3}$, ii) $\lessgtr x_{k-1}$.
Illustrate on graph why this is so.

K	$\lessgtr \sqrt[3]{9}$	$x_k \lessgtr x_{k-1}$
0	<	.
1	>	>
2	>	<
3	>	<
4	>	..



Comparison of

c) Find a quadratic approx. to $9^{1/3}$, estimate the difference between quadratic approx. ~~[illegible]~~ and the exact ~~[illegible]~~ answer.

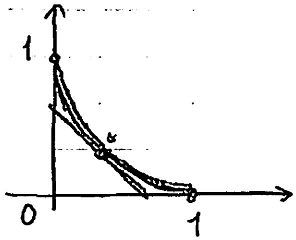
$$\sqrt[3]{9} = \sqrt[3]{8\left(1+\frac{1}{8}\right)} = 2\sqrt[3]{1+\frac{1}{8}} \approx 2\left(1+\frac{1}{3}\frac{1}{8}-\frac{1}{9}\left(\frac{1}{8}\right)^2\right) = \frac{599}{288}$$

$$\left(\sqrt[3]{1+x} \approx 1+\frac{1}{3}x-\frac{1}{9}x^2\right).$$

~~[illegible]~~)
~~[illegible]~~

Problem 5 [Astroid] [Extra credit]

a) Show that every tangent line to $x^{2/3}+y^{2/3}=1$ in the first quadrant has the property that the portion of the line in the first quadrant has length 1.



$$x^{2/3}+y^{2/3}=1 \Rightarrow \tfrac{2}{3}x^{-1/3}+\tfrac{2}{3}y^{-1/3}\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=-\frac{x^{-1/3}}{y^{-1/3}}=-\left(\frac{y}{x}\right)^{1/3}.$$

Tangent line at (a,b) on curve is

$$Y=-\left(\frac{b}{a}\right)^{1/3}x+c\ ,\ \text{where } c=y\text{-intercept}.$$

Solve for c using fact that tangent line goes through (a,b)

$b=-\left(\frac{b}{a}\right)^{1/3}a+c$

$b+b^{1/3}a^{2/3}=c$

$b^{1/3}\underbrace{(\cancel{b^{2/3}+a^{2/3}})}=c$

$\Rightarrow \boxed{c=b^{1/3}} \leftarrow$

WTS: $(y\text{-int})^2+(x\text{-int})^2=1$ (of tangent line).

$y\text{-int}=c$, x-int can be solved for: , $\left(\frac{b}{a}\right)^{1/3}x=c \quad x=c\,\frac{a^{1/3}}{b^{1/3}}.$

so, WTS: $c^2+c^2\left(\frac{a}{b}\right)^{2/3}=1 \Rightarrow c^2\left(\frac{\cancel{b^{2/3}+a^{2/3}}}{b^{2/3}}\right)=1$ so same as showing $c^2=b^{2/3}$

this is exactly what we need! $c=b^{1/3} \Rightarrow c^2=b^{2/3} \Rightarrow c^2+c^2\left(\frac{a}{b}\right)^{2/3}=1$ //.

b) Derive equation for the astroid.

- Find formula for intersection (x_3,y_3) ~~for~~ of line segments $\overset{L_1}{(0,y_1)-(x_1,0)}$, $\overset{L_2}{(0,y_2)-(x_2,0)}$

$$y_3=y_1y_2\,\frac{(x_2-x_1)}{x_2y_1-x_1y_2} \qquad x_3=x_1x_2\,\frac{y_2-y_1}{y_2x_1-y_1x_2}$$

(solve two equations simultaneously) Do the mental check (i) — (iv)

- Write the eq. involving x_2,y_2 that expresses that L_2 has length 1. $x_2=x_1+\Delta x$

$x_2^2+y_2^2=1$ Find $\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}$ as function of x_1,y_1. $y_2=y_1+\Delta x$.

~~[illegible]~~ $2x_2+2y_2\frac{dy_2}{dx_2}=0.$

$$\Rightarrow dy_2/dx_2=-x_2/y_2.$$

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=dy_2/dx_2=\lim_{\Delta x\to 0}-\frac{x_2}{y_2}=\lim_{\substack{\Delta x\to 0\\(\Rightarrow \Delta y\to 0)}}-\frac{x_1+\Delta x}{y_1+\Delta y}=\boxed{-\frac{x_1}{y_1}}.$$

- Substitute $x_2=x_1+\Delta x$, $y_2=y_1+\Delta y$ to compute $X=\lim_{x_2\to x_1}x_3=\lim_{\Delta x\to 0}x_3$

$$X=\lim_{\Delta x\to 0}x_3=\lim_{\Delta x\to 0}\frac{x_1^2\Delta y}{x_1\Delta y-y_1\Delta x}+\lim_{\Delta x\to 0}\cancel{\left(\frac{x_1\Delta y}{x_1\Delta y+y_1\Delta x}\right)\Delta x}=\lim_{\Delta x\to 0}\frac{x_1^2\,(\Delta y/\Delta x)}{x_1(\Delta y/\Delta x)-y_1}=\frac{x_1^2\,(-x_1/y_1)}{x_1(-x_1/y_1)-y_1}=\frac{x_1^3}{\underbrace{x_1^2+y_1^2}_{=1}}=x_1^3$$

Symmetrically, $Y=y_1^3$.

- Show $X^{2/3}+Y^{2/3}=1$. $(x_1^3)^{2/3}+(y_1^3)^{2/3}=x_1^2+y_1^2=1$ //.

Lecture 15
Differentials + Antiderivitives

10/15

David Jerison - Substitute

antidifferiation - integration
- harder
- not on test

<u>differentials + antiderivitives</u>

What is a differential?
- new notation
- encodes derivitives

$$y = f(x)$$
$$\underbrace{dy}_{\text{differential}} = \underbrace{f'(x)\,dx}$$

$$\frac{dy}{dx} = f'(x)$$

treat as ratio

same thing

Uses
1. Linear approximation
   - different notation

$$\frac{dy}{dx} \approx \frac{\Delta y}{\Delta x}$$

Good test q✓

$63^{1/3}$

Method 1 $f(x) = x^{1/3}$
- use a basepoint (64) $[64^{1/3} = 4]$

$$f(x) \approx f(a) + f'(a)(x-a)$$
$$a = 64$$
$$x = 63$$

$$f(a) + f'(a)(x-a)$$

$$63^{1/3} \approx 4 + \frac{1}{48}(-1)$$

$$\boxed{3.98}$$

Method 2 - algebraic

Use $(1+x)^r \approx (1+rx) \quad (x \approx 0)$

- important linear approx

$$63^{1/3} = (64-1)^{1/3} = 64^{1/3}\left(1-\frac{1}{64}\right)^{1/3}$$

$$4\left(1-\frac{1}{64}\right)^{1/3}$$

$x = -\frac{1}{64}$ ← close to 0 ✓

$r = 1/3$

$$\approx 4\left(1+\frac{1}{3}\left(-\frac{1}{64}\right)\right)$$

$$4 + \frac{-1}{3 \cdot 16} =$$

$$4 - \frac{1}{48}$$ (same)

3.18

Method 1 w/ Differential Notation

$$dy = f'(x)\ dx$$

$$d\left(x^{1/3}\right) = \frac{1}{3}x^{-2/3}\ dx$$

no letter a →

$$\boxed{\Delta x^{1/3} \approx \frac{1}{3}^{-2/3} \Delta x}$$

$$d\ dy = \frac{1}{48}(-1) = -\frac{1}{48}$$

did not follow well

$y = x^{1/3}$

$dy = \frac{1}{3}x^{-2/3}dx = \frac{1}{3}(64)^{-2/3}$

$x + dx = 63$ $\quad dy = \frac{1}{48}dx$

$x = 64$

$$63^{1/3} = y + dy \approx 4 - \frac{1}{48} \text{ as before}$$

Antiderivitives

New notation → $F(x) = \int f(x)\, dx$ ↓integral ↙differential

$$F'(x) = f(x)$$

$$dF = f(x)\, dx$$

F is antiderivitive of f

examples $\int \sin x \, dx = -\cos x + C$

↑indefinite integral → ↑ must have C don't know which function it's

2. $\int x^r \, dx = \frac{x^{r+1}}{r+1} + C \quad r \neq -1$

3. $\int x^{-1} \, dx = (\ln|x|) + C$

↑ remember special case

4. $\int \sec^2 x \, dx = \tan x$

5. $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}(x) + C$

↑arcsin

6. $\int \frac{dx}{1+x^2} = \tan^{-1}(x) + C$

just read backwards
but can be much tricker

Proove #3

$x > 0$ already known $\frac{d}{dx} \ln x = \frac{1}{x}$

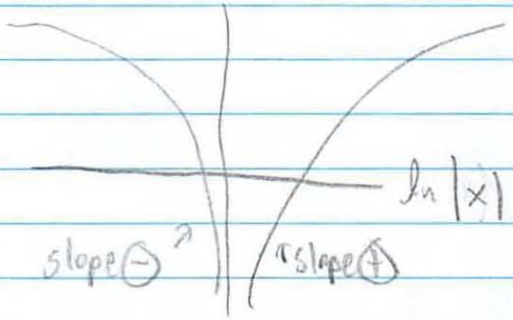
$x < 0 \quad \frac{d}{dx} \ln|x| = \frac{d}{dx} \ln(-x)$

Use chain rule to substitute

$$\boxed{\begin{array}{l} u = -x \\ \frac{du}{dx} = -1 \end{array}} \quad \frac{d}{dx} \ln u = \frac{du}{dx} \frac{d}{du} \ln u =$$

$$-1 \cdot \frac{1}{-x}$$

$$\boxed{\frac{1}{x}}$$



✱ most important thing in Calculus

- how knew not more solutions ?

Uniqueness up to a constant of the antiderivitive

- Suppose 2 functions

$$\begin{array}{l} F' = f \\ G' = f \end{array} \rightarrow (F - G)' = f - f = 0$$

- But function w/ deriv 0 is <u>constant</u> !!

$F - G = c$

$F(x) = G(x) + C$

↑ This is prooved w/ MVT

Only use MVT to know that Function w/ deriv = 0 is constant

If $H' = 0$, H is constant - Proove

$$\frac{H(b) - H(a)}{b - a} = H'(c) = 0 \quad \underline{MVT}$$

Can only be 0 if $H(b) = H(a)$
true for all a and b
So H is constant

Method of Substitution

1. $\int x^3 (x^4 + 2)^5 \, dx$

$u = x^4 + 2$
$du = 4x^3 \, dx$ ↖write it

$\int u^5 \frac{1}{4} \, du$

$\frac{1}{4} \int u^5 \, du$

$\frac{1}{4} \cdot \frac{1}{6} u^6 + c$

$\frac{1}{24} (x^4 + 2)^6 + C$

Ex 2 $\int \sin x \cos x \, dx$

$u = \sin x$
$du = \cos x \, dx$

$\int u \, du$

$\frac{1}{2} u^2 + C$

$\frac{1}{2} \sin^2 x + C$

Ex 2 Alt $\int \sin x \cos x \, dx$

$u = \cos x$
$du = -\sin x \, dx$

$-\int u \, du$

$-\frac{u^2}{2} + C$

$-\frac{\cos^2 x}{2} + C$

2 answers to problem

$F(x) = \frac{1}{2} \sin^2 x \qquad G(x) = -\frac{1}{2} \cos^2 x$

$F - G = \frac{1}{2}$ costant

So are same family
- different c constant
-

Lecture 15: Differentials and Antiderivatives

Differentials

← oh saw that in physics
thought did in calc

New notation:

$$\boxed{dy = f'(x)dx} \quad (y = f(x))$$

Both dy and $f'(x)dx$ are called *differentials*. You can think of

$$\frac{dy}{dx} = f'(x)$$

as a quotient of differentials. One way this is used is for linear approximations.

$$\frac{\Delta y}{\Delta x} \approx \frac{dy}{dx}$$

Example 1. Approximate $65^{1/3}$

Method 1 (review of linear approximation method)

$$\begin{aligned} f(x) &= x^{1/3} \\ f'(x) &= \frac{1}{3}x^{-2/3} \\ f(x) &\approx f(a) + f'(a)(x-a) \quad \leftarrow \text{linear approx -} \\ x^{1/3} &\approx a^{1/3} + \frac{1}{3}a^{-2/3}(x-a) \end{aligned}$$

A good base point is $a = 64$, because $64^{1/3} = 4$.

Let $x = 65$. ↑ Close + easy

$$65^{1/3} = 64^{1/3} + \frac{1}{3}64^{-2/3}(65-64) = 4 + \frac{1}{3}\left(\frac{1}{16}\right)(1) = 4 + \frac{1}{48} \approx 4.02$$

Similarly,

$$(64.1)^{1/3} \approx 4 + \frac{1}{480}$$

Method 2 (review)

$$65^{1/3} = (64+1)^{1/3} = [64(1+\frac{1}{64})]^{1/3} = 64^{1/3}[1+\frac{1}{64}]^{1/3} = 4\left[1+\frac{1}{64}\right]^{1/3}$$

Next, use the approximation $(1+x)^r \approx 1 + rx$ with $r = \frac{1}{3}$ and $x = \frac{1}{64}$.

$$65^{1/3} \approx 4(1 + \frac{1}{3}(\frac{1}{64})) = 4 + \frac{1}{48}$$

This is the same result that we got from Method 1.

Method 3 (with differential notation)

$$\begin{aligned} y &= x^{1/3}|_{x=64} = 4 \\ dy &= \frac{1}{3}x^{-2/3}dx|_{x=64} = \frac{1}{3}\left(\frac{1}{16}\right)dx = \frac{1}{48}dx \end{aligned}$$

We want $dx = 1$, since $(x + dx) = 65$. $dy = \frac{1}{48}$ when $dx = 1$.

$$(65)^{1/3} = 4 + \frac{1}{48}$$

What underlies all three of these methods is

$$\begin{aligned} y &= x^{1/3} \\ \frac{dy}{dx} &= \frac{1}{3}x^{-2/3}|_{x=64} \end{aligned}$$

Anti-derivatives

$F(x) = \int f(x)dx$ means that F is the antiderivative of f.

integral

↑ why not just call it that?

Other ways of saying this are:

$$F'(x) = f(x) \quad \text{or,} \quad dF = f(x)dx$$

Examples:

1. $\int \sin x dx = -\cos x + c$ where c is any constant.
2. $\int x^n dx = \frac{x^{n+1}}{n+1} + c$ for $n \neq -1$.
3. $\int \frac{dx}{x} = \ln|x| + c$ (This takes care of the exceptional case $n = -1$ in 2.)
4. $\int \sec^2 x dx = \tan x + c$
5. $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + c$ (where $\sin^{-1} x$ denotes "inverse sin" or arcsin, and not $\frac{1}{\sin x}$)
6. $\int \frac{dx}{1+x^2} = \tan^{-1}(x) + c$

Proof of Property 2: The absolute value $|x|$ gives the correct answer for both positive and negative x. We will double check this now for the case $x < 0$:

$$\begin{aligned} \ln|x| &= \ln(-x) \\ \frac{d}{dx}\ln(-x) &= \left(\frac{d}{du}\ln(u)\right)\frac{du}{dx} \quad \text{where } u = -x. \\ \frac{d}{dx}\ln(-x) &= \frac{1}{u}(-1) = \frac{1}{-x}(-1) = \frac{1}{x} \end{aligned}$$

Uniqueness of the antiderivative up to an additive constant.

If $F'(x) = f(x)$, and $G'(x) = f(x)$, then $G(x) = F(x) + c$ for some constant factor c.

Proof:

$$(G - F)' = f - f = 0$$

Recall that we proved as a corollary of the Mean Value Theorem that if a function has a derivative zero then it is constant. Hence $G(x) - F(x) = c$ (for some constant c). That is, $G(x) = F(x) + c$.

Method of substitution.

Example 1. $\int x^3(x^4+2)^5 dx$

Substitution:

$$u = x^4 + 2, \quad du = 4x^3 dx, \quad (x^4+2)^5 = u^5, \quad x^3 dx = \frac{1}{4} du$$

Hence,

$$\int x^3(x^4+2)^5 dx = \frac{1}{4}\int u^5 du = \frac{u^6}{4(6)} = \frac{u^6}{24} + c = \frac{1}{24}(x^4+2)^6 + c$$

Example 2. $\int \frac{x}{\sqrt{1+x^2}} dx$

Another way to find an anti-derivative is "advanced guessing." First write

$$\int \frac{x}{\sqrt{1+x^2}} dx = \int x(1+x^2)^{-1/2} dx$$

Guess: $(1+x^2)^{1/2}$. Check this.

$$\frac{d}{dx}(1+x^2)^{1/2} = \frac{1}{2}(1+x^2)^{-1/2}(2x) = x(1+x^2)^{-1/2}$$

Therefore,

$$\int x(1+x^2)^{-1/2} dx = (1+x^2)^{1/2} + c$$

Example 3. $\int e^{6x} dx$

Guess: e^{6x}. Check this:

$$\frac{d}{dx} e^{6x} = 6e^{6x}$$

Therefore,

$$\int e^{6x} dx = \frac{1}{6} e^{6x} + c$$

Example 4. $\int xe^{-x^2}dx$

Guess: e^{-x^2} Again, take the derivative to check:

$$\frac{d}{dx}e^{-x^2} = (-2x)(e^{-x^2})$$

Therefore,

$$\int xe^{-x^2}dx = -\frac{1}{2}e^{-x^2} + c$$

Example 5. $\int \sin x \cos x dx = \frac{1}{2}\sin^2 x + c$

Another, equally acceptable answer is

$$\int \sin x \cos x dx = -\frac{1}{2}\cos^2 x + c$$

This seems like a contradiction, so let's check our answers:

$$\frac{d}{dx}\sin^2 x = (2\sin x)(\cos x)$$

and

$$\frac{d}{dx}\cos^2 x = (2\cos x)(-\sin x)$$

So both of these are correct. Here's how we resolve this apparent paradox: the difference between the two answers is a constant.

$$\frac{1}{2}\sin^2 x - (-\frac{1}{2}\cos^2 x) = \frac{1}{2}(\sin^2 x + \cos^2 x) = \frac{1}{2}$$

So,

$$\frac{1}{2}\sin^2 x - \frac{1}{2} = \frac{1}{2}(\sin^2 x - 1) = \frac{1}{2}(-\cos^2 x) = -\frac{1}{2}\cos^2 x$$

The two answers are, in fact, equivalent. The constant c is shifted by $\frac{1}{2}$ from one answer to the other.

Example 6. $\int \frac{dx}{x \ln x}$ (We will assume $x > 0$.)

Let $u = \ln x$. This means $du = \frac{1}{x}dx$. Substitute these into the integral to get

$$\int \frac{dx}{x\ln x} = \int \frac{1}{u}du = \ln u + c = \ln(\ln(x)) + c$$

16

Ordinary Differential Equations

10/16

Topposet Partial (in 18.02 - more variables)

Here w/ respect to 1 thing at time

- very important

Ex

$$\frac{dy}{dx} = f(x)$$

$y = \int f(x)\,dx$ - can calculate (next few units)

- trick is to get it in the right form

Ex2

$$\underbrace{\left(\frac{d}{dx} + x\right)}_{\text{differential operator}} y = 0$$

- unknown → $y = y(x)$

specifically the annililation operator

- killed y off (set to 0)

Technique

Seperation of variables

$$\frac{dy}{dx} + xy = 0$$

↓ ?? heard correctly?

interesting step

$$\frac{dy}{dx} = -xy$$

↓ lose $y = 0$

Must reintroduce 0

rate of change depends on y and time

- change speed based on pos

Seperation

y on oe side
treat derivitive as fraction
* seperate x and y
-always works

$$\frac{dy}{y} = -x\,dx$$

Anti differntiate
-both sides

$$\int \frac{dy}{y} = \int -x\,dx$$

$\ln|y|$ ← know from yesterday

$$\ln|y| = -\frac{x^2}{2} + C$$

← additive constant

↑ don't need 2nd c -redudent

$$\ln|y| + C_1 = -\frac{x^2}{2} + C_2$$

$$\ln|y| = -\frac{x^2}{2} + \underbrace{C_2 - C_1}_{C}$$

$$|y| = e^{-\frac{x^2}{2} + C}$$

↑ exponeitate

comes up almost every time

$$|y| = e^{-\frac{x^2}{2}} e^{C}$$

rewrite

$$y = \pm e^{C} e^{-\frac{x^2}{2}}$$

$$\boxed{y = a e^{-\frac{x^2}{2}}}$$

$$\left(a = \pm e^{C}\right)$$

← $+$

↑ a is ⊕ or ⊖

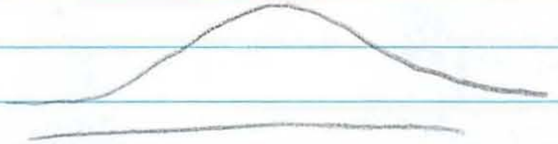
a can be $a > 0$
$a > 0$
$a = 0$

typical / usually - when exponential in setup

↑ "trivial solution"

c often converted to multiplicitive constant
normally just consider ⊕ then fix it later
don't miss case $a = 0$

$e^{-\frac{x^2}{2}}$



in real life

probability distribution of particle

$$\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx = ?$$

General seperation of variable general method

$$\frac{dy}{dx} = f(x)\, g(x)$$

$$\frac{dy}{g(y)} = f(x)dx$$

$$\int \frac{dy}{g(y)} = \int f(x)\, dx$$

subtile when abstract

$$H'(y) = \frac{1}{g(y)}$$

↓ reducing complexity

$$F'(x) = f(x)$$

Implicit equation for y

$$H(y) = F(x) + c$$

$$y = H^{-1}(F(x) + c)$$

$g(y) = y$ matching pattern in $H^{-} =$ exponential

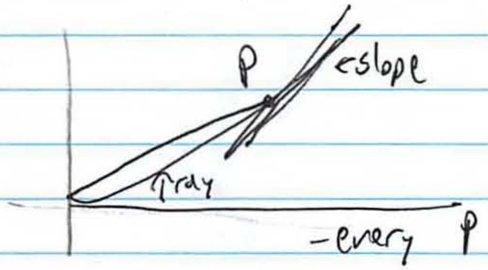
$f(x) = -x$

$H(y) = \ln y$ or $\ln|y|$

$F(x) = -\frac{x^2}{2}$

Lots of algebra not much calculus

Geometric equation



slope = twice slope of ray from origin at P

what is the curve?

How to encode differential equation?

$P = x, y$ $\quad \underset{\text{of ray}}{\text{slope}} = \frac{y}{x}$ $\quad \Big) \quad \underset{\uparrow \text{slope}}{\frac{dy}{dx}} = 2\frac{y}{x}$

$$\int \frac{dy}{y} = \int \frac{2dx}{x}$$

$$\int \frac{dy}{y} = \int \frac{2dx}{x}$$

$\underset{(y>0)}{\ln y} = \underset{(x>0)}{2\ln x} + c$ ← forgetting abs value go back later + add in

exponential trick

$y = e^{2\ln x + c}$ $\qquad e^{2\ln x} = x^2$ $(x>0)$

$y = e^c \cdot e^{2\ln x}$

$y = e^c x^2$

$y = ax^2$ $\qquad x>0, y>0, a>0$

Check when x is 0, ⊖

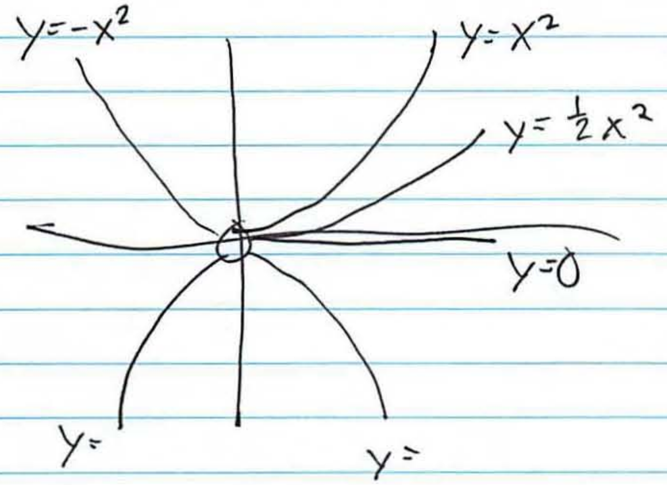
$y < 0,\ x < 0 \qquad a > 0$

< 0

$= 0$

not $x = 0$

since x in denom in original equation



~~peices only in a quadrant~~

family of solutions a different

1st order often have families

~~when~~

rate of ~~#~~ change + one more thing on harder problems

2nd Geometric Example

Find curves perpendicular to the parabola

- another differential equation

$$\text{parabola} = \frac{dy}{dx} = \frac{2y}{x} \qquad \perp \qquad = -\frac{x}{2y} = \frac{dy}{dx}$$

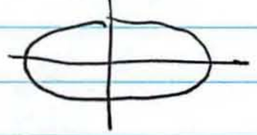
$$2y\,dy = -x\,dx$$

$$\int 2y\,dy = \int -x\,dx$$

$$y^2 = -\frac{x^2}{2} + C$$

-2 ways - implicit $\frac{x^2}{2} + y^2 = C$

ellipse



* hit perpendicularly

not all possible C

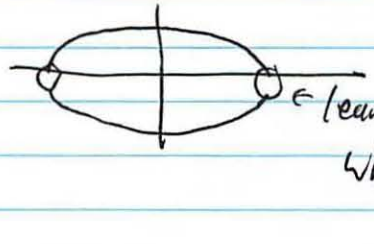
$\underline{C > 0}$!

other way $y = \pm\sqrt{C - \frac{x^2}{2}}$

$y = y(x)$

- y is only defined when $\frac{x^2}{2} < C$

$|x| < \sqrt{2C}$



← leave hole where deriv = 0

) really have 2 functions

Trick from yesterday

Shortcut in method of substitution

"advanced guessing"

$$\int \frac{x}{\sqrt{1+x^2}}\, dx = \int x\left(1+x^2\right)^{-1/2} dx$$

$u = 1+x^2$

$\downarrow$

guessing is to know what will happen
if really familiar

$$\frac{d}{dx}\left(1+x^2\right)^{1/2}$$

$$\frac{1}{2}\left(1+x^2\right)^{-1/2} \underset{\checkmark}{2x}$$ ← check w/ differentiation

Constant could still be off

$$\int e^{6x}\, dx$$

Slow $u = 6x$ $du = 6\,dx$ ...

fast guess e^{6x}

$$\frac{d}{dx} e^{6x}$$

$6e^{6x}$ ← know off by 6 so $\cdot \frac{1}{6}$

$$\boxed{\frac{1}{6}e^{6x} + C}$$

Fri

18.03

Lecture 16: Differential Equations and Separation of Variables

Ordinary Differential Equations (ODEs)

Example 1. $\frac{dy}{dx} = f(x)$

solve for y

Solution: $y = \int f(x)dx$. We consider these types of equations as solved.

distribute

Example 2. $\left(\frac{d}{dx} + x\right) y = 0$ $=0$ $\left(\text{or } \frac{dy}{dx} + xy = 0\right)$

($\left(\frac{d}{dx} + x\right)$ is known in quantum mechanics as the *annihilation operator.*) goes to 0

Besides integration, we have only one method of solving this so far, namely, substitution. Solving for $\frac{dy}{dx}$ gives:

$$\frac{dy}{dx} = -xy$$

The key step is to *separate variables.*

$$\frac{dy}{y} = -xdx$$

Note that all y-dependence is on the left and all x-dependence is on the right.

Next, take the antiderivative of both sides:

integrate

$$\begin{aligned} \int \frac{dy}{y} &= -\int xdx \\ \ln|y| &= -\frac{x^2}{2} + c \quad \text{(only need one constant } c\text{)} \\ |y| &= e^c e^{-x^2/2} \quad \text{(exponentiate)} \\ y &= ae^{-x^2/2} \quad (a = \pm e^c) \end{aligned}$$

Solve for y

Despite the fact that $e^c \neq 0, a = 0$ is possible along with all $a \neq 0$, depending on the initial conditions. For instance, if $y(0) = 1$, then $y = e^{-x^2/2}$. If $y(0) = a$, then $y = ae^{-x^2/2}$ (See Fig. 1).

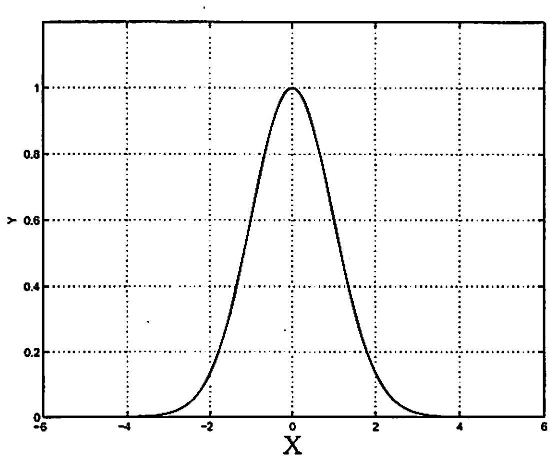


Figure 1: Graph of $y = e^{-\frac{x^2}{2}}$.

In general:

$$\begin{aligned} \frac{dy}{dx} &= f(x)g(y) \\ \frac{dy}{g(y)} &= f(x)dx \quad \text{which we can write as} \\ h(y)dy &= f(x)dx \text{ where } h(y) = \frac{1}{g(y)}. \end{aligned}$$

?

Now, we get an implicit formula for y:

$$H(y) = F(x) + c \quad (H(y) = \int h(y)dy; \quad F(x) = \int f(x)dx)$$

where $H' = h$, $F' = f$, and

$$y = H^{-1}(F(x) + c)$$

(H^{-1} is the inverse function.)

In the previous example:

$$\begin{aligned} f(x) &= x; \quad F(x) = \frac{-x^2}{2}; \\ g(y) &= y; \quad h(y) = \frac{1}{g(y)} = \frac{1}{y}, \quad H(y) = \ln|y| \end{aligned}$$

Example 3 (Geometric Example). $\frac{dy}{dx} = 2\left(\frac{y}{x}\right)$.
Find a graph such that the slope of the tangent line is twice the slope of the ray from $(0,0)$ to (x,y) seen in Fig. 2.

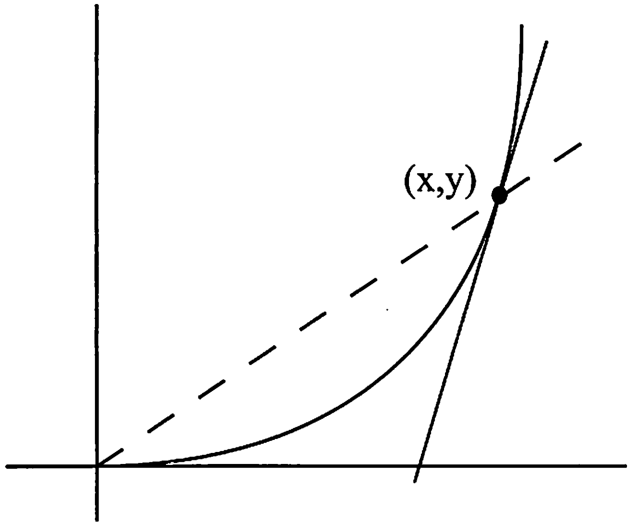


Figure 2: The slope of the tangent line (red) is twice the slope of the ray from the origin to the point (x,y).

$$\begin{aligned}
\frac{dy}{y} &= \frac{2dx}{x} \quad \text{(separate variables)} \\
\ln|y| &= 2\ln|x| + c \quad \text{(antiderivative)} \\
|y| &= e^c x^2 \quad \text{(exponentiate; remember, } e^{2\ln|x|} = x^2 \text{)}
\end{aligned}$$

Thus,

$$y = ax^2$$

Again, $a < 0$, $a > 0$ and $a = 0$ are all acceptable. Possible solutions include, for example,

$$\begin{aligned}
y &= x^2 \quad (a = 1) \\
y &= 2x^2 \quad (a = 2) \\
y &= -x^2 \quad (a = -1) \\
y &= 0x^2 = 0 \quad (a = 0) \\
y &= -2y^2 \quad (a = -2) \\
y &= 100x^2 \quad (a = 100)
\end{aligned}$$

Example 4. Find the curves that are perpendicular to the parabolas in Example 3.
We know that their slopes,

$$\frac{dy}{dx} = \frac{-1}{\text{slope of parabola}} = \frac{-x}{2y}$$

Separate variables:

$$ydy = \frac{-x}{2}dx$$

Take the antiderivative:

$$\frac{y^2}{2} = -\frac{x^2}{4} + c \quad \Longrightarrow \quad \frac{x^2}{4} + \frac{y^2}{2} = c$$

which is an equation for a family of ellipses. For these ellipses, the ratio of the x-semi-major axis to the y-semi-minor axis is $\sqrt{2}$ (see Fig. 3).

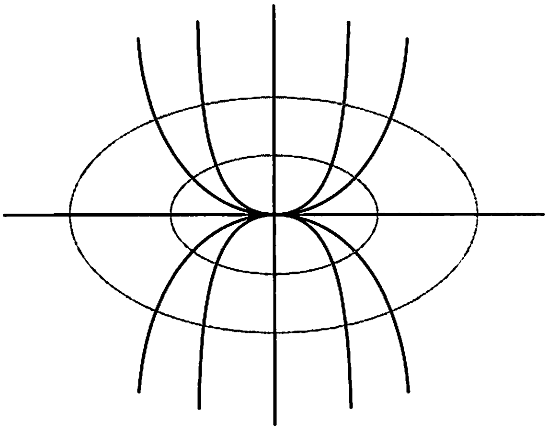


Figure 3: The ellipses are perpendicular to the parabolas.

Separation of variables leads to implicit formulas for y, but in this case you can solve for y.

$$y = \pm\sqrt{2\left(c - \frac{x^2}{4}\right)}$$

Exam Review

Exam 2 will be harder than exam 1 — be warned! Here's a list of topics that exam 2 will cover:

1. Linear and/or quadratic approximations
2. Sketches of $y = f(x)$
3. Maximum/minimum problems.
4. Related rates.
5. Antiderivatives. Separation of variables.
6. Mean value theorem.

More detailed notes on all of these topics are provided in the Exam 2 review sheet.

18.01 FALL 2009 – Problem Set 4

Due Friday 10/16/09, 1:45 pm in 2-106

Part I (10 points)

Lecture 14. Fri. Oct. 9 Mean-value theorem.
Read: 2.6 up to p. 79, Notes MVT Work: 2G-1b, 2b, 5, 6

Recitations held on Tuesday, October 13

Lecture 15. Thurs. Oct. 15 Differentials and antiderivatives.
Read: 5.2, 5.3 Work: 3A-1de, 2acegik, 3aceg

Lecture 16. Fri. Oct. 16 Differential equations; separating variables.
Read: 5.4, 8.5 Work assigned on the next problem set.

Exam 2. Tues. Oct. 20 **Exam 2** Covers Lectures 8–16. Rooms to be announced.

Part II (21 points)

Directions: Attempt to solve *each part* of each problem yourself. If you collaborate, solutions must be written up independently. Beside each problem is the date on which corresponding material in class is covered.

0. (not until due date; 3 pts) Write the names of all the people you consulted or with whom you collaborated and the resources you used, or say "none" or "no consultation". (See full explanation on PSet1).

1. (Friday, 3 pts: 1 + 2)

(a) Does there exist a function f such that $f(0) = -1$, $f(2) = 4$, and $f'(x) \leq 2$ for all real x? Explain.

(b) A number a is called a "fixed point for f" if $f(a) = a$. Prove that a function $f(x)$ such that $f'(x) \neq 1$ for all real x can have at most one fixed point.

2. (Friday, 10pts: 2 + 2 + 2 + 2 + 2))

a) Use the mean value property to show that if $f(0) = 0$ and $f'(x) \geq 0$, then $f(x) \geq 0$ for all $x \geq 0$.

b) Deduce from part (a) that $\ln(1 + x) \leq x$ for $x \geq 0$. Hint: Use $f(x) = x - \ln(1 + x)$.

c) Use the same method as in (b) to show $\ln(1+x) \geq x - x^2/2$ and $\ln(1+x) \leq x - x^2/2 + x^3/3$ for $x \geq 0$.

d) Find the pattern in (b) and (c) and make a general conjecture.

e) Show that $\ln(1 + x) \leq x$ for $-1 < x \leq 0$. (Use the change of variable $u = -x$.)

3. (Thursday, 2 pts) Simmons 5.3/68

4. (Thursday, 2 pts) Find a function f such that $f'(x) = x^3$ and the line $x + y = 0$ is tangent to the graph of f.

Pset 4

$\frac{30}{31}$ - Cool

Michael Plasmeier

10/16

Part 1

* I scan in my work so I can't write 2x ✓

26-1b Find all point c using Mean Value Theorm

$$\ln(x) \text{ on } [1,2] \qquad \frac{f(b)-f(a)}{b-a} = f'(c)$$

$$\frac{\ln(2)-\ln(1)}{2-1} = \ln(2)$$

$$\ln(2) = f'(c) \qquad f' = \frac{1}{x}$$

$$\ln(2) = \frac{1}{c}$$

$$c\ln(2) = 1$$

$$c = \frac{1}{\ln(2)}$$

2b $\sqrt{1+x} < 1 + \frac{x}{2}$ if $x > 0$

$$f(x) = f(a) + f'(c)(x-0)$$

↗ like linearization

$$f(x) = \sqrt{1+x}$$

$$f'(x) = \tfrac{1}{2}(1+x)^{-1/2}$$

$$f'(c) = \tfrac{1}{2}(1+c)^{-1/2}$$

↙ pick a value

taking $a=0$ $\quad f(a) = 1$

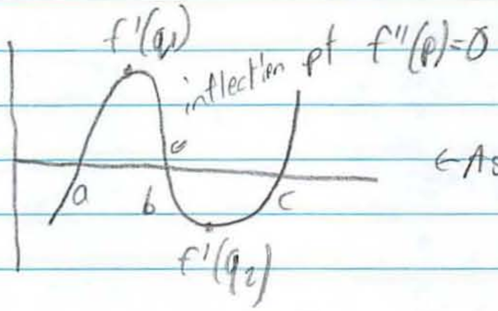
$$f(x) = \sqrt{1+x} = 1 + \tfrac{1}{2}(1+c)^{-1/2}(x-0)$$

$$\left(1 + \frac{1}{2\sqrt{1+c}}\, x < 1 + \frac{x}{2}\right.$$

↗ for $x > 0$

denominator >2

5. Suppose $f''(x)$ exists on interval I and $f(x)$ has a zero at 3 distinct points $a < b < c$ on I
show that there is a point p on $[a, c]$
where $f''(p) = 0$



← Assumed $f(x) = 0$ at a, b, c

↑ Rolles theorm to q_1 + q_2
so point $f''(p) = 0$

also between a + c

6. Use form 2 prove that on $[a, b]$
$f'(x) > 0 \rightarrow f(x)$ increasing
$f'(x) = 0 \rightarrow f(x)$ constant

$$a \leq x_1 < x_2 \leq b$$

$$f(x_2) = f(x_1) + f'(c)(x_2 - x_1) \quad \text{where } x_1 < c < x_2$$

$f'(x) > 0$ on $[a, b]$ $\qquad x_2 - x_1 > 0$
so $f'(c)$ is > 0

so $f(x_2) > f(x_1)$
$f(x)$ increasing

b) $f(x) = f(a) + f'(c)(x-a) \quad a < c < x$
Since $f'(c) = 0$, $f(x) = f(a)$ for $a \leq x \leq b$
so $f(x)$ continuous $[a, b]$

Lecture 15

3A-1d

Find differentials

$$d\left(\underset{\uparrow\text{chain}}{e^{3x}}\ \underset{\cdot\,\text{prod}}{\sin x}\right)$$

$$\left(3e^{3x}\sin x + \cos x\, e^{3x}\right)dx$$

↓ ?: why write this

$dy = f'(x)\,dx$

guess that is "differential"

e) Express dy in terms of x and dx if $\sqrt{x}+\sqrt{y}=1$

$$0 = \tfrac{1}{2}x^{-1/2}x' + \tfrac{1}{2}y^{-1/2}y'$$

$$-\tfrac{1}{2}x^{-1/2}x' = \tfrac{1}{2}y^{-1/2}y'$$

$$\frac{-\frac{1}{2}x^{-1/2}x'}{\frac{1}{2}y^{-1/2}} = y'$$

$$-\frac{\sqrt{y}}{x}x' = y'$$

$$-\frac{\sqrt{y}}{x}\,dx = dy$$

↓ they take it further * Plug in y *

$$\frac{1-\sqrt{x}}{\sqrt{x}}\,dx$$

$$\left(1-\frac{1}{\sqrt{x}}\right)dx$$ ↓ split up

2a Find indefinite integrals

HS spent 1 month integration here introduced in hw

$$\int\left(2x^4+3x^2+x+8\right)dx$$

$$\frac{2x^5}{5}+\frac{3x^3}{3}+\frac{x^2}{2}+8x+C$$

$$\tfrac{2}{5}x^5+x^3+\tfrac{1}{2}x^2+8x+C$$ ✓

c $\int \sqrt{8+9x}\ dx$

? what does this mean

- w/ respect to x - the default

$\int (8+9x)^{1/2}$ → use u → $u = 8+9x$ $u^{1/2}dx$

$\cancel{\frac{(8+9x)^{3/2}}{3/2}}$ $\cancel{\frac{u^{3/2}}{3/2}}$ deriv ↳ $du = 9dx$

$\int u^{1/2}\ \underbrace{1/9\ du}_{dx}$ $= \frac{1}{9}\frac{2}{3}u^{3/2}$

$\boxed{\frac{27}{2}(8+9x)^{1/2}+c}$

or guess + ✓

Guess $(8+9x)^{3/2} = \frac{3}{2}(9)(8+9x)^{1/2}$

$\frac{27}{2}(8+9x)^{1/2}$

∘ Guess by $\frac{2}{27} = \frac{2}{27}(8+9x)^{3/2}+c$

? why?

e $\int \frac{x}{\sqrt{8-2x^2}}\ dx$ $u = 8-2x^2$ ↲ d/dx

$du = -4x\,dx$

↲ solve for dx

$\int u^{1/2}\left(-\frac{1}{4}\right)du$

$-\frac{1}{4}\cdot\frac{2}{3}u^{3/2}+c$

$-\frac{1}{6}(8-2x^2)^{3/2}+c$

g $\int 7x^4 e^{x5}$

$7\int x^4 \cdot e^{x5}$

$7\cdot\frac{x^5}{5}\cdot\frac{e^{x5}}{x^5}+C$ → $\frac{7}{5}e^{x^5}+C$

↙? right step?

j $\int \frac{dx}{3x+2} \to \frac{1}{3x+2}\,dx$ $\quad u = 3x+2$

$\downarrow u^{-1}$ $\quad du = 3\,dx$

$dx = \frac{1}{3}du$

~~$\frac{u^0}{0} \cdot \frac{1}{3}du \to \frac{1}{3}(3x+2)\cdot 0$~~

$\boxed{0}$

↓write before integral

$\int \frac{dx}{3x+2} = \int \frac{1/3\,du}{u} = \frac{1}{3}\ln(u) + C$

↷ why ln?

↓plug u in

$\boxed{\frac{1}{3}\ln(3x+2) + C}$

k $\int \frac{x}{x+5}\,dx$ $\left(\text{Write } \frac{x}{x+5} = 1 + \ldots\right)$

↓reverse

$\int 1 - \frac{5}{x-5}\,dx = x - 5\ln(x+5) + C$

3a Compute indefinite integral

$\int \sin(5x)\,dx$ $\quad u = 5x$

$du = 5\,dx$

↓

~~$\cos(5x)\cdot 5$~~

↓??

$\int \sin(u)\cdot 5\,du = \cos u \cdot -\frac{1}{5} + C$

$\boxed{-\frac{1}{5}\cos(5x) + C}$

c $\int \cos^2 x \sin x \, dx$

?

$\frac{\cos^3 x}{3}$ $-\cos x$

$-\frac{1}{3} \cos^3 x + C$

e $\int \sec^2 \left(\frac{x}{5}\right) dx$

?

$5 \tan\left(\frac{x}{5}\right) + C$

g $\int \sec^9 x \tan x \, dx$

$u = \sec x$

$du = \sec x \tan x \, dx$

↓ remember trying to go backwards

$\int (\sec x)^8 \underbrace{\sec x \tan x}_{\text{break out}} \, dx$ convert to sec

$\frac{1}{9} \sec^9 x + C$ ↙ add back in

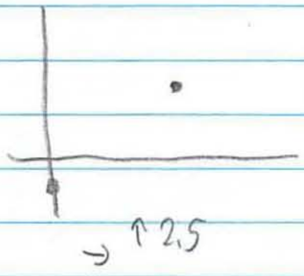
$\overline{10}$

Part 2

0. See side bar + Kimberly ✓

1. Does exist a function f such that

$f(0) = -1$
$f(2) = 4$
$f'(x) \leq 2$ for all real x



← deriv must be less than x

way of writing out nice

(Use Mean Value Theorm
if function exists along points - Mean Value = $f'(c)$

$$f'(c) = \frac{f(2) - f(0)}{2 - 0} = \frac{4 - -1}{2} = \frac{5}{2}$$

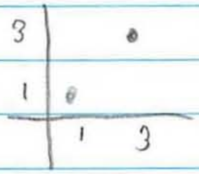
✓

$f'(c) = \frac{5}{2}$
where is deriv $\frac{5}{2}$
↑ anyway > 2 so
function does not exist

what I thought

¿i?

b. A number a is "fixed point for f" if $f(a) = a$
Prove that a function $f(x)$ such that $f'(x) \neq 1$
For all real x can have at most 1 such fixed pt



← Slope would have to be 1 between the 2
← and it can't be

Could do MLT to show

$$\frac{3-1}{3-1} = f'(c)$$

✓

$1 = f'(c)$ ← but $\neq 1$ so can not exist

2. Use MVT to show that if $f(0) = 0$ and $f'(x) > 0$
a.
then
$f(x) \geq 0$ for all $x \geq 0$

$$\frac{5-0}{4-0} = \frac{5}{4}$$

✓

↑ ↑always 0
always $\frac{+}{+} = +$ So always $\frac{⊕}{⊕} - 0$ = will always = ⊕

↑$f'(x) = \frac{f(x_0)}{x_0}$ ⇒ $f(x_0) = x_0 f'(x)$

b. deduce from a that $\ln(1+x) \leq x$ for $x \geq 0$
Hint use $f(x) = x - \ln(1+x)$

$$0 \geq x - \frac{x^2}{2} - \ln(1+x)$$

let $f(x) = x - \frac{x^2}{2} - \ln(1+x)$

✓

want $f(x) \leq 0$

$f(0) = 0 - \frac{0^2}{2} - \ln(1+0)$

0

$f(0) = 0$

$f'(x) = 1 - x - \frac{1}{1+x}$ differmilate

$x \geq 0$ so, $f'(x) \leq 0$ since $1 - x - \frac{1}{1+x}$ will always be < 0

So have $f(0) = 0$ $f'(x) \leq 0$
so from part a $f(x) \leq 0$ since for $f(x) \geq 0$
$f(0) = 0$ and $f'(x) \geq 0$
which is **not** true for this function

$f(x) \leq 0$ $\ln(1+x) \geq x - \frac{x^2}{2}$ ✓
is true for
$x \geq 0$

C. Use same method in b to show $\ln(1+x) \geq x - \frac{x^2}{2}$
and $\ln(1+x) \leq x - \frac{x^2}{2} + \frac{x^3}{3}$ for $x \geq 0$

$0 \leq x - \frac{x^2}{2} + \frac{x^3}{3} - \ln(1+x)$

Let $f(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ln(1+x)$

$f(0) = 0 - 0 + 0 - 0 = 0$ ✓

$f'(x) = 1 - x + x^2 - \frac{1}{1+x}$ $x \geq 0$

$f'(x) \geq 0$

Use part a and fact that $f(0)=0$ and $f'(x) \geq 0$ to conclude

$$f(x) \geq 0 \quad \text{so} \quad \ln(1+x) \leq x - \frac{x^2}{2} + \frac{x^3}{3} \quad \text{for } x \geq 0$$

d. Find general conjectur

$$\ln(1+x) \geq x - \frac{x^2}{2}$$
$$\ln(1+x) \leq x - \frac{x^2}{2} + \frac{x^3}{3} \text{ en}$$

$$\ln(1+x) \underset{\substack{\uparrow \\ \text{if n is} \\ \text{even}}}{\geq} \underset{\substack{\uparrow \\ \text{if n} \\ \text{is odd}}}{\leq} x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \ldots + \ldots \frac{x^n}{n}$$

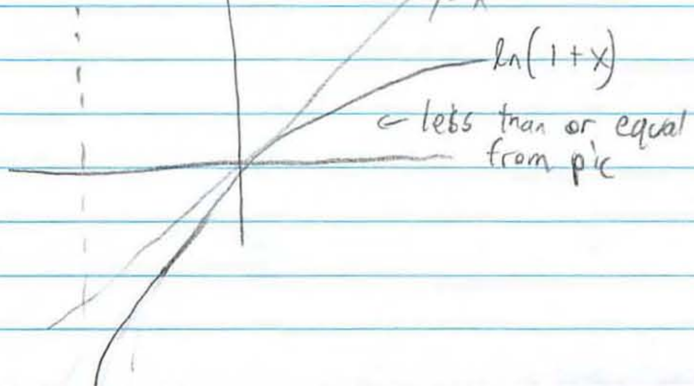
✓

???

e. Show that $\ln(1+x) \leq x$ for $-1 \leq x \leq 0$ (use change of variable $u=-x$)

??

$f(x) = x - \ln(1+x)$



← less than or equal from pic

use substitution of $u = -x$

$f(u) = -u - \ln(1-u)$ for $0 \le u < 1$

show that $f(u) > 0$ for $u \le 1$

$f(0) = -0 - \ln(0) = 0$

$f'(u) = -1 + \frac{1}{1-u} \geq 0$ for $0 \le u < 1$

since $f(0) = 0$ $f'(u) \geq 0$ then
$f(u) \geq 0$ for $0 \le u < 1$

due to part a
$(f(x_0) = x_0 f'(x))$

means that $\ln(1+x) \le x$ for $-1 < x \le 0$

✓

3. Simmons 5.3. #68

Show that both integrals are correct + Explain

$$\int \frac{dx}{(1-x)^2} = \frac{1}{1-x} \qquad \int \frac{dx}{(1-x)^2} = \frac{x}{1-x}$$

$$\frac{d}{dx}\,\frac{1}{1-x} = \frac{d}{dx}\,1(1-x)^{-1}$$

$$= -1(1-x)^{-2} \cdot -1$$

$$\frac{1}{(1-x)^2} \quad ✓$$

easy ⊣ [illegible]

$$\frac{d}{dx}\,\frac{x}{1-x} = \frac{d}{dx}\,x(1-x)^{-1}$$

$$1 \cdot (1-x)^{-1} + -1(1-x)^{-2} \cdot -x$$

$$\frac{1}{1-x} + \frac{-x}{(1-x)^2}$$

r don't forget -

$$\frac{(1-x)}{(1-x)^2} + \frac{\cdot x}{(1-x)^2}$$

$$\frac{\cancel{1-x}+x}{(1-x)^2} = \frac{x}{(1-x)}$$

4. Find a function f such that $f'(x) = x^3$
and the line $x + y = 0$ is tangent
to graph of f

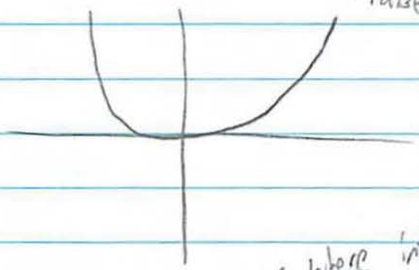
$x + y = 0$

$y = 0 - x$

$y = -x$ is tangent

$\int x^3 = \frac{x^4}{4} + C$ so that $y = -x$ tangent

raise it up (+)



where intersection

$\frac{x^4}{4} + C = -x$

$\frac{x^4}{4} + x = -C$

slopes

$f'(x) = x^3$ $\quad y' = -1$

where =

$x^3 = -1$

$x = -1$ $\quad y = 1$

plug in

find intersection
where is C there

$+1 = \frac{1}{4} + C$

$C = \cancel{\frac{15}{4}}\ \frac{3}{4}$

$$\boxed{f(x) = \frac{x^4}{4} + \frac{3}{4}}$$

Close to perfect

Problem Set 4
18.01 Fall 2009

Part II

Problem 1

(a) Does there exist a function f s.t. $f(0)=-1$, $f(2)=4$, and $f'(x)\leq 2$, $\forall x\in\mathbb{R}$? Explain.

No, there does not.
If such a function f existed, then by the Mean Value Theorem there would exist a c in $(0,2)$ s.t. $f'(c)=\frac{f(2)-f(0)}{2-0}=\frac{4-(-1)}{2}=\frac{5}{2}=2.5>2$, contradicting $f'(x)\leq 2$ for all real x.

(b) Prove that a function $f(x)$ s.t. $f'(x)\neq 1$ for all real x must have at most one fixed pt.

Suppose f - diff. function w/ $f'(x)\neq 1$ and two fixed points $a\neq b$.
Then by Mean Value Theorem $\exists c\in((\min(a,b)),(\max(a,b)))$ s.t. $f'(c)=\frac{b-a}{b-a}=1$, $\times$.

Problem 2

a) Use the MVT to show that if $f(0)=0$ and $f'(x)\geq 0$, then $f(x)\geq 0$, $\forall x\geq 0$.

Again assuming f is differentiable for all $x\geq 0$. Suppose $\exists \overset{x>0}{\cancel{x\geq 0}}$ s.t. $f(x)<0$.
By MVT, $\exists c\in(0,x)$ s.t. $f'(c)=\frac{f(x)-f(0)}{x-0}=\frac{f(x)-0}{x}=\frac{\overset{<0}{f(x)}}{\underset{>0}{x}}\ \underline{<0}$ $\times$.

b) Deduce from (a) that $\ln(1+x)\leq x$ for $x\geq 0$.

Let $f(x)=x-\ln(1+x)$. $\Rightarrow f'(x)=1-\frac{1}{1+x}\geq 0$ for $x\geq 0$ since $1+x\geq 1$. $\Rightarrow f'(x)\geq 0$.
Since $f'(x)\geq 0$ for all $x\geq 0$, by (2a), $f(x)\geq 0$, $\forall x\geq 0$, giving us $x-\ln(1+x)\geq 0\Rightarrow$
(and since $f(0)=0$) $x\geq \ln(1+x)$ //

c) Using the same method, show that $\overset{1.}{\ln}(1+x)\geq x-x^2/2$ and $\overset{2.}{\ln}(1+x)\leq x-x^2/2+x^3/3$.
1. Show $\ln(1+x)\geq x-x^2/2$
Let $g(x)=\ln(1+x)-x+x^2/2$ $(g(0)=0\,//)$. $g'(x)=\frac{1}{1+x}-1+x$. WTS $g'(x)\geq 0$ for all $x\geq 0$.
$x^2\geq 0\Rightarrow -x^2\leq 0$ $\quad 1-x^2\leq 1\Rightarrow (1-x)\underset{\underset{0\text{ for }x\geq 0}{\vee}}{(1+x)}\leq 1\Rightarrow 1-x\leq\frac{1}{1+x}$ //.

2. Show $\ln(1+x)\leq x-x^2/2+x^3/3$. $h(x)=x-x^2/2+x^3/3-\ln(1+x)$ $(h(0)=0)$ //. WTS $h'(x)\geq 0$ $\forall x\geq 0$.
Sufficient to show $(1+x)(1-x+x^2)-1\overset{?}{\geq}0$ $(x\geq 0)$, i.e. $\cancel{1}\cancel{-x}\cancel{+x^2}+\cancel{x}-x^2+x^3\cancel{-1}=x^3\geq 0$
true since for $x\geq 0$ //

(Problem 2, cont.)

d) Find the pattern in (b) and (c) and make a general conjecture.

General conjecture: for K - a pos. integer,

$$\ln(1+x) \geq \sum_{i=1}^{2k} (-1)^{i+1} \frac{x^i}{i} \quad ; \quad \ln(1+x) \leq \sum_{i=1}^{2k+1} (-1)^{i+1} \frac{x^i}{i} \; . \; //$$

e) Show $\ln(1+x) \leq x$ for $-1 < x \leq 0$.

Let $u = -x$. Sufficient to show $\ln(1-u) \leq -u$ for $0 \leq u < 1$.

Let $h(u) = -u - \ln(1-u)$ Bound $u \in [0,1)$.
$(h(0) = 0)$.

$h'(u) = -1 + \frac{1}{1-u}$. $1-u \in (0,1]$, so $\frac{1}{1-u} \geq 1$. $\Rightarrow -1 + \frac{1}{1-u} \geq -1+1 = 0$ //.

Problem 3 Simmons 5.3/68.

Show that both of the following integrals are correct. Explain $\int \frac{dx}{(1-x)^2} = \begin{cases} 1/1-x \\ x/1-x. \end{cases}$

Compute derivatives of both. $\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{-(-1)}{(1-x)^2} = \frac{1}{(1-x)^2}$ ✓ $\frac{d}{dx}\left(\frac{x}{1-x}\right) = \frac{(1-x) - x(-1)}{(1-x)^2} = \frac{1}{(1-x)^2}$ //.

Explanation : $\frac{x}{1-x} - \frac{1}{1-x} = \frac{x-1}{1-x} = -1$, so antiderivatives differ by a constant...

Problem 4 Find a function f s.t. $f'(x) = x^3$ and $(x+y) = 0$ is tangent to the graph of f.

$f(x) = \frac{x^4}{4} + \text{const.}$ $\quad \underset{\frac{dy}{dx}}{\overset{\parallel}{f'(x)}} = x^3$ $\quad 1 + \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -1$.

Want derivatives to match at some common point.

$-1 = \frac{dy}{dx} = x^3 \Rightarrow x^3 = -1 \Rightarrow x = -1$.

$-1 + y = 0 \Rightarrow \boxed{y = 1}$

So want $f(x) = \frac{x^4}{4} + c$ to go through $(-1, 1)$. Solve for c :

$$1 = \frac{(-1)^4}{4} + c \Rightarrow c = 1 - \frac{1}{4} = \frac{3}{4}.$$

$$\Rightarrow \boxed{f(x) = \frac{x^4}{4} + \frac{3}{4}} \quad //.$$

18.01: REVIEW FOR EXAM 2

IVAN LOSEV

CONTENTS

1. APPROXIMATION

Approximation of $f(x)$ at (near) $x = x_0$
Linear: $f(x) \approx f(x_0) + f'(x_0)(x - x_0)$.
Quadratic: $f(x) \approx f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2$.

Standard approximations:
The following approximations **are at** $x = 0$.

at

Table 1: Basic approximations at $x = 0$

Function $f(x)$	Linear appr.	Quadratic appr.
e^x	$1 + x$	$1 + x + \frac{1}{2}x^2$
$\sin(x)$	x	x
$\cos(x)$	1	$1 - \frac{1}{2}x^2$
$\ln(1 + x)$	x	$x - \frac{1}{2}x^2$
$(1 + x)^r$	$1 + rx$	$1 + rx + \frac{r(r-1)}{2}x^2$

) memorize

$\frac{1}{1-x} \approx x+1 = e^x$

Problems to practice[1]**:**

- Practice questions, Problem 7.
- Practice exam, Problem 5a.
- Exam, Problem 1.

Remark. Let us explain a recipe for getting the formulas for approximation. For instance, the linear (or, the other name, 1st order) approximation $f(x_0) + f'(x_0)(x - x_0)$ (denote it by $l(x)$) for $f(x)$ at $x = x_0$ is the only linear function such that $l(x_0) = f(x_0), l'(x_0) = f'(x_0)$.

---

[1] All problems are taken from exam 2 materials on OCW 06

Similarly, the quadratic (2nd order) approximation (say, $q(x)$) is the only quadratic function such that $q(x_0) = f(x_0)$, $q'(x_0) = f'(x_0)$, $q''(x_0) = f''(x_0)$.

2. GRAPHING

Graphing of a function includes:

1. Finding critical points ($f'(x) = 0$), intervals, where a function is increasing ($f'(x) > 0$) and decreasing ($f'(x) < 0$).
2. Finding inflection points ($f''(x) = 0$) and intervals, where a function is concave up ($f''(x) > 0$) and down ($f''(x) < 0$).
3. Finding vertical asymptotes, if any. These are vertical lines $x = a$, where $f(x)$ is undefined at $x = a$ and, moreover, one or both of single sided limits $\lim_{x\to a^+} f(x), \lim_{x\to a^-} f(x)$ is $+\infty$ of $-\infty$.
4. Finding horizontal asymptotes. This are horizontal lines $y = b$ such that one (or two) of limits $\lim_{x\to+\infty} f(x), \lim_{x\to-\infty} f(x)$ exists and equals b.
5. Finding slant asymptotes (a more advanced stuff). This are lines $y = ax + b$ such that $\lim_{x\to+\infty}(f(x) - ax - b) = 0$ or $\lim_{x\to-\infty}(f(x) - ax - b) = 0$. The first equality, for example, means that the graph of $y = f(x)$ approaches $y = ax + b$ as $x \to +\infty$ (example: look at $f(x) = x + 1 + \frac{1}{x}$; here $y = x + 1$ is a slant asymptote).
6. Symmetries (even/odd) and/or periodicity.
7. Finally, it is useful to find a few points on the graph, in particular, to find zeroes of a function.

Problems to practice:

- Practice questions: problems 1,2.
- Practice exam: problem 1.
- Exam: problem 1.

Remark. Sometimes a problem explicitly says which steps it wants from you.

3. MAX-MIN PROBLEMS

Max-min problems can appear in two different forms: "explicit" and "implicit".

"Explicit": Here we are given a function $f(x)$ and asked to find a point of min/max (and/or compute min/max value) on a "closed" interval $[a, b]$ (meaning that $a \leqslant x \leqslant b$) or on an "open" interval (a, b) (meaning that $a < x < b$, here it is possible that, for instance, $a = +\infty$) etc.

"Implicit": Here we don't have any function from the beginning. Instead we need to min/max some geometric, physical etc. quantity. The usual way to deal with such problems is to reduce them to explicit ones and then solve the latter.

How to reduce an "implicit problem" to an "explicit" one.

One can proceed in the following three steps that are illustrated below by an example.

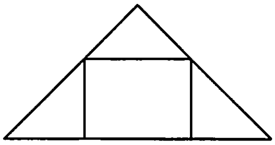
Step 1. Pick a variable.

Step 2. Write down a function (of that variable) to be maximized/minimized.

Step 3. Determine the domain, where the function should be maximized/minimized, i.e., all possible values of the variable.

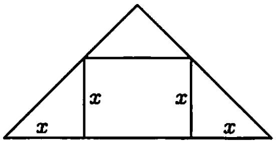
Example. *Suppl. notes, 2C-8 b): Find the dimensions of the rectangle of largest area inscribed in the right triangle so that one side of the rectangle is parallel to the hypotenuse. We assume for simplicity that the catheti of the triangle are equal.*

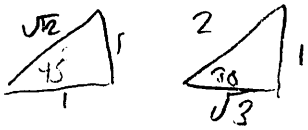
Being geometric, this problem has "Step 0": draw a picture



Step 1. We asked to find dimensions. Therefore it is natural to take one of the dimensions of the rectangle as a variable. It is a bit more convenient to denote the vertical dimension by x. Also we need the notation for one of dimensions of the triangle: denote the length of cathetus by a (this is <u>not</u> a variable, it is fixed). constant

Step 2. We need to express the area of the rectangle in terms of x. The area equals to the product of the vertical and horizontal dimensions, so we need to express the horizontal dimension in terms of x. To do this look at the following picture:





Since the length of the hypotenuse is $\sqrt{2}a$, we see that the horizontal dimension of the rectangle is $\sqrt{2}a - 2x$. So the area equals $x(\sqrt{2}a - 2x) = \sqrt{2}ax - 2x^2$.

Step 3. We need to understand the domain of possible values of x. Clearly, $x \geqslant 0$ (for $x = 0$ the rectangle "degenerates" to a horizontal line). Also x cannot be bigger than the height of the triangle, which is $\sqrt{2}a/2$ (if the equality holds, then the rectangle degenerates to a vertical line). So $x \in [0, \sqrt{2}a/2]$.

So we arrive to the following problem: *find* $x \in [0, \sqrt{2}a/2]$ maximizing $f(x) = \sqrt{2}ax - 2x^2$.

How to solve an "explicit" problem.

Suppose we want to maximize/minimize a function $f(x)$ on the interval $[a, b]$.

Step 1. Find all critical points of $f(x)$ lying on (a, b).

Step 2. Here we have several options. The most straightforward one is as follows: we know that the maximal/minimal value is achieved either in a critical point of in an endpoint. So one can compare values in these points and choose the maximal/minimal one.

Sometimes, however, this computation is not practical (we have many critical points, or it is difficult to compare values). Then one can solve try to solve the problem based on the computation of the second derivative. There are several observations to be made:

(1) if $f''(x) > 0$ for a critical point x, then x is a point of <u>local</u> minimum. Therefore x cannot be a point of global maximum. In some situations, x is not a point of global minimum (even if it is a unique critical point of local minimum – try to produce a picture, one should have at least one point of local maximum).

(2) If there is a unique critical point (as it happens in many geometric problems), then the situation is much simpler. Namely, let x be a point of local minimum, and there are no other critical points on the interval in consideration. Then x is the point of global minimum, and the maximum is achieved in one of the critical points. The situation when x is a point of local maximum is similar[2].

Example. Return to the problem we considered: maximize the function $f(x) = \sqrt{2}ax - 2x^2$ for $x \in [0, \sqrt{2}a/2]$.

---

[2] What happens when x is critical but neither local minimum nor maximum?

Step 1. Find critical points: $f'(x) = \sqrt{2}a - 4x$. So there is a unique critical point $x = \sqrt{2}a/4$.

Step 2. The value of $f(x)$ at both end points is zero. So $x = \sqrt{2}a/4$ does maximize the function. The maximal value of $f(x)$ is $f(\sqrt{2}a/4) = \sqrt{2}a\sqrt{2}a/4 - 2(\sqrt{2}a/4)^2 = a^2/4$. Also we can argue as in observation (2) above: $f''(x) = -4$, so $\sqrt{2}a/4$ is a point of local maximum. Being a unique critical point, it is the point of global maximum.

Problems to practice:

- Practice questions: Problems 3 and 4.
- Practice exam: Problem 2.
- Exam: Problem 3.

Remark (what happens if we are dealing with the open interval (a, b) instead of the closed interval $[a, b]$). In this case we replace the values $f(a), f(b)$ with limits $\lim_{x\to a} f(x), \lim_{x\to b} f(x)$. However, if it happens that one of this values is, say, smaller than all values at critical points, then the function $f(x)$ does not achieve a minimum on (a, b). For instance, there is no x maximizing $f(x) = x^2$ on the interval (-1,1). Indeed, for any $x \in (-1, 1)$ there is another point $y \in (-1, 1)$ with $f(y) > f(x)$ (just take y with $|y| > |x|$).

4. RELATED RATES

In a related rates problem we have two quantities depending on time but subject to a relation that does not depend on time. One needs to express the rate of change of one quantity (at a specified moment) in terms of that for the other.

In many (if not most) cases it is reasonable to approach a related rates problem using the following steps[3].

Step 1. Introduce (notation for) quantities. If you have more than 2 possible quantities (e.g., dimensions of a right triangle) you can use the following rule: your first quantity will be that with known rate of change, say $x(t)$, while your second quantity will be one whose rate of change you want to determine, say $y(t)$.

Step 2. Read carefully the statement to gather all information contained there (this may be tricky). In many situations this information will involve an unknown moment of time t_0. Also write down what you need to find.

Step 3. Write down a relation btw. $x(t), y(t)$. In many problems this will have a geometric origin. In other situations (say, physical) this relation can be (intrinsically) a part of the statement.

Step 4. Differentiate the relation. Express $y'(t)$ in terms of everything else.

Step 5. Plug all information you have. In most cases, the answer should be a number. If you do not get a number (your answer involves some unknown values), it may mean that you missed something on step 2.

Example.

A 200 foot tree is falling in the forest; the sun is directly overhead. At the moment when the tree makes an angle of 30° with the horizontal, its shadow is lengthening at the rate of 50 feet/sec. How fast is the angle changing at that moment?

Solution.

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[3] please note that this recipe does not always give the easiest solution, see, for example, 2E-3 from suppl. notes

Step 1. The quantity with known rate of change is the position of the projection of the tree at the moment t, say $x(t)$. The quantity, whose rate of change we want to find is the angle, say $\theta(t)$.

Step 2. We know the height of the tree: 200 feet. We also know that at the certain moment t_0, the angle $\theta(t_0) = \pi/6 = 30°$ and $x'(t_0) = 50$(feet/sec). What we want to find is $\theta'(t_0)$.

Step 3. The relation comes from geometry: $x(t)$ is the cathetus in the right triangle, whose hypotenuse is 200 and the angle between the cathetus and the hypotenuse is $\theta(t)$. So $x(t) = 200 \cos \theta(t)$.

Step 4. $x'(t) = -200 \sin(\theta(t))\theta'(t)$ and therefore $\theta'(t) = -\frac{x'(t)}{200 \sin(\theta(t))}$.

Step 5. Plugging $\theta(t_0) = \pi/6, x'(t_0) = 50$, we get $\theta'(t_0) = -\frac{50}{200 \cdot 0.5} = -0.5$(rad/sec). Please note the unit.

Problems to practice:

- Practice questions: problems 4 and 5.
- Practice exam: problem 4.
- Exam: problem 4 (partly).

5. NEWTON METHOD

This is a method to find approximately solutions of the equation $f(x) = 0$, where $f(x)$ is a differentiable function. This method produces a solution by iterations. More precisely, we construct a sequence x_n of points. Under favorable circumstances, this sequence approaches the actual solution x.

We start by picking a point x_0 (in practice, this point should be "close" to a solution we expect to find). The next point x_1 is obtained by the formula: $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$ (draw the tangent to $y = f(x)$ at the point $(x_0, f(x_0))$, then x_1 is the point of intersection of this tangent with the x-axis). In general, having constructed a point x_n, we set

$$x_{n+1} := x_n - \frac{f(x_n)}{f'(x_n)}.$$

Remark. The natural question here is: what are "favorable circumstances"? There are several possible answers. One of them is as follows:

Suppose that $f(x)$ has a zero (otherwise, there is nothing to find), and that $f(x)$ is increasing and concave up (decreasing and concave up, etc., also works). Then Newton method always works no matter which x_0 you choose (however, you still would like to choose a point close to the actual solution to make the method work faster). Moreover, if you draw a picture, you can notice the following pattern: x_1 is always bigger, than the actual solution, and each x_{n+1} lies between x_n and the actual solution.

6. MEAN VALUE THEOREM

The theorem asserts the following:

- if $f(x)$ is a differentiable function and $a < b$ are numbers, then there is a point c with $a < c < b$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

6.1. **MVT and inequalities.** A typical problem here is to show that $f(x) \geqslant 0$ for all $x \geqslant a$, where a is some fixed number (variant: $f(x) < 0$ for $x > a$; or $f(x) \geqslant 0$ for $x \leqslant a$, etc.).

Usually these problems are solved as follows (we consider the problem with $f(x) < 0$ for $x > a$):

It is often happens that $f(a) = 0$ (although $f(a) < 0$ is also OK). We need to show that $f(b) < 0$ for all $b > a$. This will follow if we show that $f(b) - f(a) < 0$ for all $b > a$, or, equivalently, that $\frac{f(b)-f(a)}{b-a} < 0$. By MVT, $\frac{f(b)-f(a)}{b-a} = f'(c)$ for some $c \in (a, b)$. So our problem reduces to checking $f'(x) < 0$ for all $x > a$. This is often doable.

Useful exercise. Make sure you can modify the argument in the previous paragraph to make it work for the problem like: show that $f(x) > 0$ for all $x < a$.

Example.

Show that $\sin(x) < x$ *for all* $x > 0$.

Rewrite the inequality in interest as $\sin(x) - x < 0$. Our $f(x)$ is $\sin(x) - x$. We need to make sure that $f'(x) < 0$ for all $x > 0$. But $f'(x) = \cos(x) - 1$. Since $\cos(x) \leqslant 1$ for all x, we see that $f'(x) \leqslant 0$. It follows that $f(x) < 0$ for all $x > 0$, and we are done[4].

Problems to practice:

- Practice questions: problem 8.
- Practice exam: problem 6, part a.

6.2. **Other applications.** We also have some other applications of MVT, see Practice exam, problem 6b, or Exam, Problem 6.

Rolle's theorm

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[4] There is a little mistake in this argument (on exam, this may be worth 1 point, especially if a grader has headache/ problems with digestion, etc.) Actually, we have proved that $f'(x) \leqslant 0$, although in the general argument above we should have had $f'(x) < 0$. However, since $f'(x) = 0$ only in isolated points ($x = 2\pi k$, where k is integer) this does not matter.

18.01 UNIT 2 REVIEW; Fall 2007

The central theme of Unit 2 is that knowledge of f' (and sometimes f'') tells us something about f itself. This is even true of our first topic, approximation. For instance, knowing that $f(x) = e^x$ satisfies $f(0) = 1$ and $f'(0) = 1$, we can say

$$e^x \approx 1 + x \quad \text{provided} \quad x \approx 0$$

The linear function $1 + x$ is much simpler than e^x, so $f(0)$ and $f'(0)$ give us a (very) simplified picture of our function, useful only near near 0. For more detail, use the quadratic approximation,

$$e^x \approx 1 + x + x^2/2 \quad \text{provided} \quad x \approx 0$$

(still only works well near 0)

The second and third practice exams are actual tests from previous years. The exam this year is similar to the one from 2006 posted at our site. It has 6 questions covering the following topics. (No Newton's method, but there is a seventh, extra credit problem.)

1. Linear and/or quadratic approximations
2. Sketch a graph $y = f(x)$
3. Max/min
4. Related rates
5. Find antiderivatives and solve a differential equation by separating variables
6. Mean value theorem.

Remarks.

1. Recall that linear [and quadratic] approximation is

$$f(x) \approx f(a) + f'(a)(x - a) \; [+(f''(a)/2)(x - a)^2]$$

2. You should expect to graph a function $y = f(x)$, where $f(x)$ is a rational function (ratio of polynomials).

Warnings:

a) When asked to label the critical point on the graph, find and mark the point (a, b). In lecture we called $x = a$ the critical point and $y = b$ the critical value, and this is what is used in 18.02, and elsewhere. But for this exam (and this is just an inconsistency in language that you will have to tolerate) the words "critical point" refer to the point on the graph (a, b), not the number a and the point on the x-axis. The same applies to inflection points.

b) $y = 1/(x - 1)$ is decreasing on the intervals $-\infty < x < 1$ and $1 < x < \infty$, but it is **not decreasing** on the interval $-\infty < x < \infty$. Draw the graph to see.

You cannot just use the fact that $y' = -1/(x - 1)^2 < 0$ because there is a point in the middle at which y is not differentiable — and not even continuous. So the mean value theorem does not apply.

c) Similarly, $y = 1/(x - 1)^2$ is concave up on $-\infty < x < 1$ and $1 < x < \infty$, but it is **not concave up** on the interval $-\infty < x < \infty$. Here $y'' = 6/(x - 1)^4 > 0$, but there is a singularity in the middle. Plot the graph yourself to see.

3. The mean value theorem says that if f is differentiable, then for some c, $a < c < x$,

$$f(x) = f(a) + f'(c)(x - a)$$

It is used as follows. Suppose that $m < f'(c) < M$ on the interval $a < c < x$, then

$$f(x) = f(a) + f'(c)(x - a) < f(a) + M(x - a)$$

Similarly,

$$f(x) = f(a) + f'(c)(x - a) > f(a) + m(x - a)$$

Put another way, if $\Delta f = f(x) - f(a)$ and $\Delta x = x - a$, and $m < f'(c) < M$ for $a < c < x$, then

$$m\Delta x < \Delta f < M\Delta x$$

More consequences of the mean value theorem.

A function f is called increasing (also called strictly increasing) if $x > a$ implies $f(x) > f(a)$. The reasoning above with $m = 0$ shows that if $f' > 0$, then f is increasing. Similarly if $f' < 0$, then f is decreasing. We use these facts every time we sketch a graph of a function or find a maximum or minimum.

A similar discussion works when the inequality is not strict. If $m \leq f'(c) \leq M$ for $a < c < x$, then

$$f(a) + m(x - a) \leq f(x) \leq f(a) + M(x - a)$$

A function is called nondecreasing if $x > a$ implies $f(x) \geq f(a)$. If $f' \geq 0$, then the inequality above shows that f is nondecreasing. Conversely, if the function is nondecreasing and differentiable, then $f' \geq 0$. Similarly, differentiable functions are nonincreasing if and only if they satisfy $f' \leq 0$.

Key corollary to the mean value theorem: $f' = g'$ implies $f - g$ is constant.

In Unit 2, we have found that information about f' gives information about f. In particular, knowing a starting value for a function and its rate of change determines the function. A seemingly obvious example is that if $f' = 0$ for all x, then f is constant. If this were not true, then the mathematical notion of derivative would fail to coincide with our intuitive notion of what rate of change and cause and effect mean.

But this fundamental fact needs a proof. Derivatives are instantaneous quantities, obtained as limits. It is the mean value theorem that allows us to pass in rigorous mathematical fashion from the infinitesimal to the practical, human scale. Here is the proof. If $f' = 0$, then one can take $m = M = 0$ in the inequalities above, and conclude that $f(x) = f(a)$. In other words, f is constant. As an immediate consequence, if $f' = g'$, then f and g differ by a constant. (Apply the previous argument to the function $f - g$, whose derivative is 0.) This basic fact will lead us shortly to what is known as the fundamental theorem of calculus.

Practice Problems

(1) What are the largest and smallest possible values taken of the product of three distinct numbers, spaced such that the middle number is distance one away from the other two, if the middle number is in the interval $[-2, 2]$?

(2) By sketching the graph of $y = f(x) = x^3 - 3x - 5$, show that it has only one real root.

(3) How many solutions to the equation $\sqrt{x} = x$? Why?

(4) Find the triangle of smallest area in the half-plane to the right of the $y-$axis whose three sides respectively are segments of the x-axis, the line y=x, and a line through $(2, 1)$.

(5) Calculate $\sqrt[3]{10}$ to six decimal places of accuracy.

(6) A boat is being pulled into a dock by means of a rope with one end tied to the bow of the boat and the other end passing through a ring attached to the dock at a point five ft higher than the bow of the boat. If the rope is being pulled in at a rate of 4 ft/s, how fast is the boat moving through the water when 13 ft of rope are out?

(7) A cylindrical tank without a top is to have a specified volume. If the cost of the material used for the bottom is three times the cost of that used for the curved lateral part, find the ratio of the height to the diameter of the base for which the cost is the least.

(8) Show that Newton's method applied to the function $f(x) = \sqrt[3]{x}$ leads to $x_2 = -2x_1$ and is therefore useless for finding where $f(x) = 0$. Sketch the situation.

(9) Draw a reasonably good sketch of $y = \sqrt{x}$ and mark the point on the graph that seems the closest to $(1/2, 0)$. Then calculate the coordinates of this point.

(10) A point moves around the circle $x^2 + y^2 = a^2$ in such a way that $\frac{dx}{dt} = -y$. Find $\frac{dy}{dt}$ and decide whether the direction of the motion is clockwise or counterclockwise.

(11) What is the smallest value of the constant a for which the inequality $ax + 1/x \geq 2\sqrt{2}$ is always true?

1) Optimization w/ x = middle #

$P = x(x-1)(x+1)$

$P' = (x-1)(x+1) + x(x+1) + x(x-1)$

$= x^2-1 + x^2+x + x^2-x$

$= 3x^2-1$

$P'=0$ iff $3x^2=1$

$x = \pm\sqrt{1/3} = \frac{\sqrt{3}}{3}$

Check: $x=-2$

$(-3)(-2)(-1) = -6$

$x = -\frac{1}{\sqrt{3}}$

$\left(\frac{-\sqrt{3}-3}{3}\right)\left(-\frac{\sqrt{3}}{3}\right)\left(\frac{-\sqrt{3}+3}{3}\right) = \left(\frac{3+3\sqrt{3}}{9}\right)\left(\frac{3-\sqrt{3}}{3}\right) = \frac{9+9\sqrt{3}-3\sqrt{3}-9}{27} = \frac{6\sqrt{3}}{27}$

$x = \frac{1}{\sqrt{3}}$

$\left(\frac{\sqrt{3}-3}{3}\right)\left(\frac{\sqrt{3}}{3}\right)\left(\frac{\sqrt{3}+3}{3}\right) = \left(\frac{3-3\sqrt{3}}{9}\right)\left(\frac{3+\sqrt{3}}{3}\right) = \frac{9-9\sqrt{3}+3\sqrt{3}-9}{27} = \frac{-6\sqrt{3}}{27}$

$x=2$

$1 \cdot 2 \cdot 3 = 6$

Largest value is 6, smallest is -6.

2) $f'(x) = 3x^2-3 = 3(x^2-1)$

$f'=0$ @ $x=\pm 1$

$f''(x) = 6x$

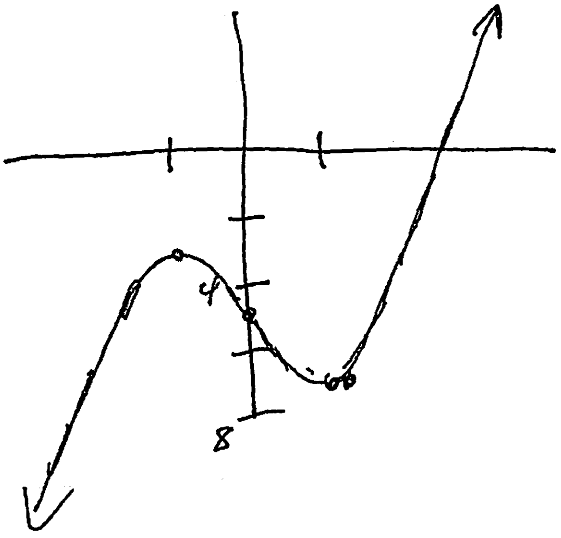
$f''=0$ @ $x=0$

$f(1) = -7$

$f(-1) = -3$

$f \to \infty$ as $x \to \infty$

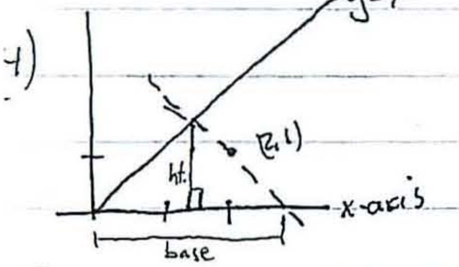
$f \to -\infty$ as $x \to -\infty$



(3) Consider $f(x)=\sqrt{x}-x$. We know $f(0)=0$ & $f(1)=0$. Area there any other roots? That's the question. Note f is continuous on $[0,\infty)$ & differentiable on $(0,\infty)$. Rolle's Th'm tells us if a & b are roots of f (i.e. $f(b)=f(a)=0$), then there is a $c\in(a,b)$ with $f'(c)=0$.

Now, $f'(x)=\frac{1}{2\sqrt{x}}-1$ & $f'(x)=0$ only at $x=\frac{1}{4}$. This is enough to justify only two roots. Suppose $a\neq 0$, $a\neq 1$ was another root of f. If $0<a<1$, then by Rolle's Th'm there exist two values for which $f'=0$. So can't happen. If $a>1$, Rolle's implies there exists a root of f' for $x>1$. Not true either. Therefore $x=0$ & $x=1$ are the only roots of f.

4)



Trying to minimize Area $=\frac{1}{2}bh$. We only have control over the slope of the dashed line. (That's what Area should depend upon.

Base = x-int value of dashed line. Eq. for line is $y-1=m(x-2)$. x-int at $x=2-\frac{1}{m}$.

Ht = y-value of intersection of $y=x$ & $y-1=m(x-2)$

$$y-1=m(y-2) \Rightarrow y=\cancel{\frac{(1-2m)}{(m-1)}}\ \frac{1-2m}{1-m}$$

$$\text{Area}=\frac{1}{2}\left(\frac{2m-1}{m}\right)\left(\frac{1-2m}{1-m}\right)=\frac{1}{2}\,\frac{(2m-1)^2}{m^2-m}$$

$$\frac{d}{dm}(\text{Area})=\frac{1}{2}\,\frac{(m^2-m)(2)(2m-1)(2)-(2m-1)^2(2m-1)}{(m^2-m)^2}=\frac{(2m-1)\left[4m^2-4m-(2m^2-1)^2\right]}{2(m^2-m)^2}$$

$$=\frac{(2m-1)\left[4m^2-4m-4m^2+4m-1\right]}{2(m^2-m)^2}=\frac{-(2m-1)}{2(m^2-m)^2}$$

$\frac{d}{dm}(\text{Area})=0$ if $m=\frac{1}{2}$.

So the line is $y-1=\frac{1}{2}(x-2)$ &

the Area is 0. Yikes! In fact, check "endpts". Note for $0<m<1$, Area is negative. For $m>1$, Area decays asymptotically to 4. Thus, we want the value for area as $m\to\infty$. i.e. we was the right triangle.

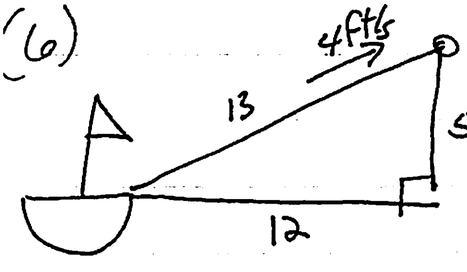
(5) Use Newton's Method on $f(x)=x^3-10$.

First guess should be near 2 as $2^3=8$ & $3^3=27$.

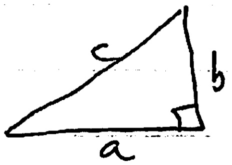
$x_1=2$

$x_2=2-\frac{-2}{3\cdot 2^2}=2+\frac{1}{6}$ . . .

(6)



Use related rates on



$a^2+b^2=c^2$

$2a\frac{da}{dt}+2b\frac{db}{dt}=2c\frac{dc}{dt}$

$\frac{db}{dt}=0$ as 5 is a constant (fixed) value

$a\frac{da}{dt}=c\frac{dc}{dt}$. $a=12, c=13, \frac{dc}{dt}=4 \Rightarrow$

$\frac{da}{dt}=\frac{13\cdot 4}{12}=\frac{13}{3}$ $\boxed{\frac{13}{3}\text{ ft/s}}$

7) This is an optimization problem. Let

$c=$ cost per square unit (a fixed value, not a variable)

Then

$\text{Cost}=2\pi rh\cdot c+\pi r^2\cdot 3c.$

Constraint is the fixed volume $V=\pi r^2h$. So $h=\frac{V}{\pi r^2}$

$\text{Cost}=\frac{2\pi rV}{\pi r^2}\cdot c+\pi r^2\cdot 3c=\frac{2V}{r}\cdot c+\pi r^2\cdot 3c$ Take the derivative $\left(\frac{d}{dr}\right)$

$\text{Cost}'=-\frac{2V}{r^2}\cdot c+6\pi rc$ This equals zero when $\frac{2V}{r^2}=6\pi r$ i.e.

where does this come from?

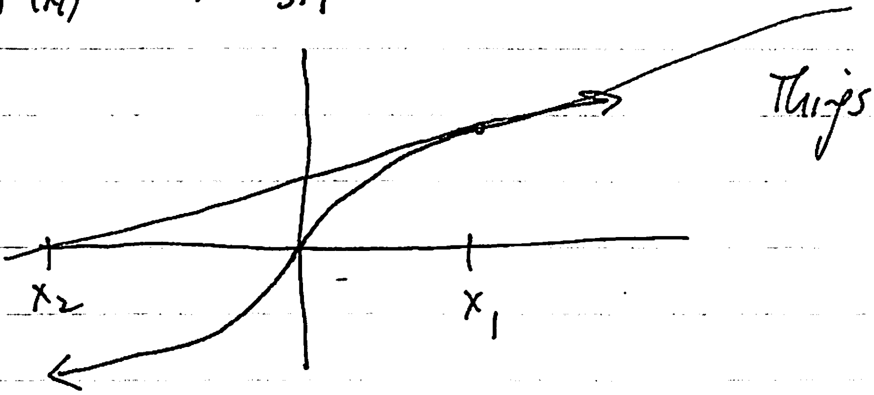
$r^3=\frac{2V}{6\pi}=\frac{V}{3\pi}$ Now $V=\pi r^2h$ so $r^3=\frac{\pi r^2h}{3\pi}$. And thus

$r=\frac{h}{3}$. So $d=2r=\frac{2h}{3}$. $\boxed{\text{Thus } \frac{h}{d}=\frac{3}{2}.}$

(8) $f(x) = x^{1/3}$

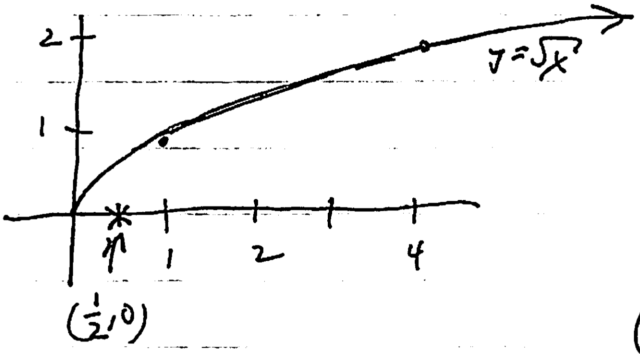
$f'(x) = \frac{1}{3}x^{-2/3}$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = x_1 - \frac{x_1^{1/3}}{\frac{1}{3}x_1^{-2/3}} = x_1 - 3x_1 = -2x_1$$



Things get worse instead of better!

(9)



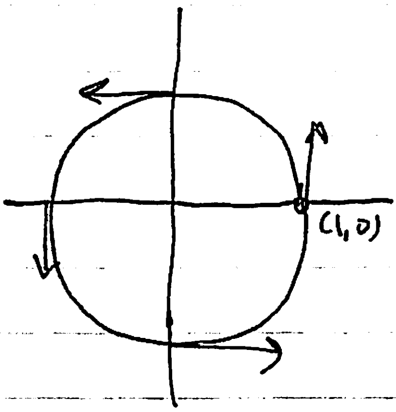
Optimize distance between curve $y=\sqrt{x}$ and the pt $(\frac{1}{2},0)$. (Minimize)

$$d^2 = (y-0)^2 + (x-\tfrac{1}{2})^2$$
$$= (x^{1/2})^2 + (x-\tfrac{1}{2})^2 = x + (x-\tfrac{1}{2})^2$$
$$(d^2)' = 1 + 2(x-\tfrac{1}{2})$$
$$= 1 + 2x - 1 = 2x$$

Distance is minimized @ $x=0$, so want the point $(0,0)$!

(10) Related Rates: Note a^2 is a constant, so $\frac{d}{dt}(a^2)=0$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$
$$\cancel{2}x(-y) = -\cancel{2}y\frac{dy}{dt}$$
$$-xy = -y\frac{dy}{dt}$$
$$\boxed{\frac{dy}{dt} = x}$$



counterclockwise

@ (1,0)

$\frac{dx}{dt}=0$ a $\frac{dy}{dt}=1$, so moving UP

@ (0,1)

$\frac{dx}{dt}=-1$, $\frac{dy}{dt}=0$ so moving left

(11) It looks like a mean value th'm problem, but that makes things harder instead of easier.

Note: We only consider $x>0$, as mentioned.

Let $f(x) = ax + \frac{1}{x} - 2\sqrt{2}$. We want to show $f(x) > 0$ for $x > 0$.
Note that $a > 0$, ~~as~~ is necessary. This is because for x large, ax is large & $\frac{1}{x}$ is small. If $a < 0$, then $ax + \frac{1}{x}$ very negative. So $a > 0$.

(1) For $x < \frac{1}{2\sqrt{2}}$, $f(x) > 0$. $ax > 0$, $\frac{1}{x} > 2\sqrt{2}$. Together we have $f(x) > 0$.

(2) For $x > \frac{2\sqrt{2}}{a}$, $f(x) > 0$. $ax > 2\sqrt{2}$, $\frac{1}{x} > 0$. Together we have $f(x) > 0$.

So, $f(x)$ is positive as $x \to 0^+$ & as $x \to \infty$. Thus, if we can choose a so that $f(x) > 0$ at its minimum, then we're done.

Look @ $f'(x) = a - \frac{1}{x^2}$. Thus f has a min when $a = \frac{1}{x^2}$.
So, for what values of x will the minimum be ≥ 0?
Substitute $a = \frac{1}{x^2}$ into $f(x)$:

$$\frac{1}{x^2} \cdot x + \frac{1}{x} - 2\sqrt{2} = \frac{1}{x} + \frac{1}{x} - 2\sqrt{2} = \frac{2}{x} - 2\sqrt{2}$$

This is ≥ 0 if $x \leq \frac{1}{\sqrt{2}}$.

Thus $a = \frac{1}{\left(\frac{1}{\sqrt{2}}\right)^2} = 2$ is the minimum possible value for a.

18.01 Practice Questions for Exam 2

Solutions will be posted on the 18.01 website. No books, notes, calculators. Show work.

1. For the function $3x^5 - 5x^3 + 1$, sketch the graph over a suitable interval showing all the local maximum and minimum points on the graph, the points of inflection, and the approximate location of its zeros (show on which intervals of the form $[n, n+1]$, (n is an integer) they occur. Show work, or indicate reasoning.

2. Sketch the graph of $4x^2 - \frac{1}{x}$ over an interval showing its interesting features – local maxima and minima, points of inflection, zeros, asymptotes.

3. A line of negative slope through $(1, 2)$ cuts off a triangle in the first quadrant. For which such line will the triangle have least area? (Use its slope m as the independent variable. Show that you get a minimum.)

4. The bottom of the legs of a three-legged table are the vertices of an isosceles triangle with sides 5, 5, and 6. The legs are to be braced at the bottom by three wires in the shape of a Y. What is the minimum length of wire needed? Show it is a minimum.

5. A 200 foot tree is falling in the forest; the sun is directly overhead. At the moment when the tree makes an angle of 30^o with the horizontal, its shadow is lengthening at the rate of 50 feet/sec. How fast is the angle changing at that moment?

6. A container in the shape of a right circular cone with vertex angle a right angle is partially filled with water.

a) Suppose water is added at the rate of 3 cu.cm./sec. How fast is the water level rising when the height $h = 2$cm.?

b) Suppose instead no water is added, but water is being lost by evaporation. Show the level falls at a constant rate. (You will have to make a reasonable physical assumption about the rate of water loss—state it clearly.)

7. How should the parameter λ be chosen so that $f(x) = \dfrac{e^{-\lambda x}}{1 + 2\sin x}$ remain as close to 1 as possible, when $x \approx 0$? Using this value of λ, estimate $f(.1)$ to two decimal places.

8. State the Mean-value Theorem, and use it to prove that

a) if $f(x)$ is differentiable and $f'(x) > 0$ for all x, then $f(x)$ is an increasing function;

b) $e^x > 1 + x$ for all $x > 0$.

9. Evaluate: $a)\ \displaystyle\int \frac{dx}{(3x+2)^2}$ $\qquad b)\ \displaystyle\int \sin 2x \sin x\, dx$ $\qquad c)\ \displaystyle\int \frac{\ln^2 x}{x}\, dx$

10. Find the function $y(x)$ satisfying $\dfrac{dy}{dx} = xy^2 + x, \quad y(0) = 1.$

11. The rate at which a body heats up by conduction is proportional to the difference between its temperature T and the temperature T_e of its surroundings. A fish at room temperature (20^o) is cooked by putting it into boiling water at 100^o . After 5 minutes its temperature has risen to 30^o. How long will it take to be done (60^o) ?

a) Set up a differential equation for the fish temperature $T(t)$. (Call the constant of proportionality k.)

b) Find the solution $T(t)$ satisfying the given data, then use it to answer the question.

18.01 Practice Questions for Exam 2 Solutions. Fall 2006

1) $f(x) = 3x^5 - 5x^3 + 1$

$f'(x) = 15x^4 - 15x^2$

$f''(x) = 60x^3 - 30x$

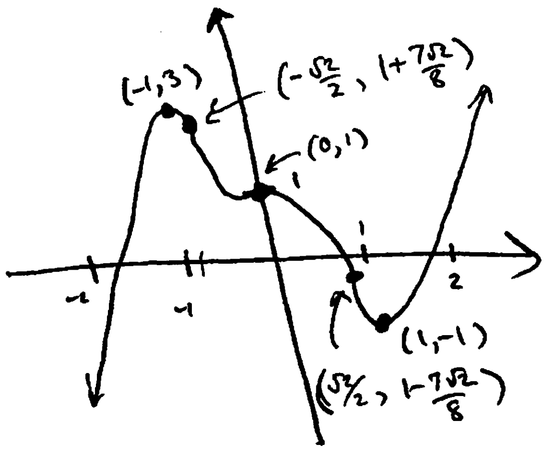
$f'(x) = 0 \quad x = 0, \pm 1$

$f''(x) = 0 \quad x = 0, \pm \sqrt{2}/2$

$f(x) \rightarrow -\infty \quad x \rightarrow -\infty$

$f(x) \rightarrow +\infty \quad x \rightarrow +\infty$

x	f(x)	f'(x)	f''(x)	
-2	$-55<0$	\	\	\
-1	$3>0$	0	-30	loc. max.
$-\sqrt{2}/2$	$1+\frac{7\sqrt{2}}{8}$	/	0	inflection
0	1	0	0	inflection
$\sqrt{2}/2$	$1-\frac{7\sqrt{2}}{8}$	\	0	inflection
1	-1	0	30	loc. min.
2	$57>0$	\	\	\



There is an x, $-2<x<-1$, since $f(-2)<0$ and $f(-1)>0$.
There is an x, $1<x<2$, since $f(1)<0$, $f(2)>0$.
There is a x, $0<x<1$, since $f(0)>0$, $f(1)<0$.

---

2) $f(x) = 4x^2 - \frac{1}{x}$ $\quad f(x)=0, \; x = 1/\sqrt[3]{4}$

$f'(x) = 8x + 1/x^2 \quad f'(x)=0, \quad x = -1/2$

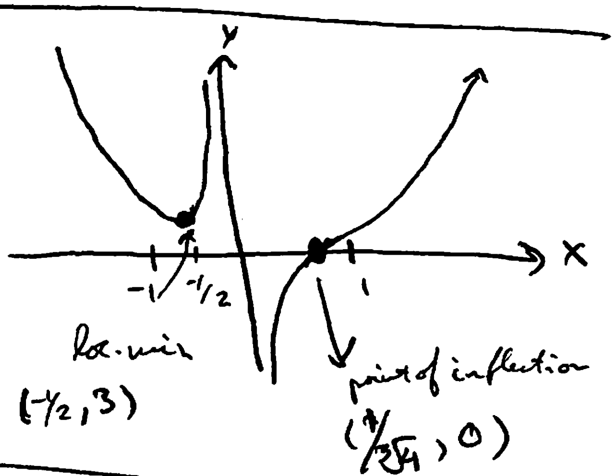
$f''(x) = 8 - 2/x^3 \quad f''(x)=0, \quad x = 1/\sqrt[3]{4}$

as $x \rightarrow \infty$, $f(x) \rightarrow \infty$

as $x \rightarrow -\infty$, $f(x) \rightarrow \infty$

asymptote at $x=0$

$f''(-1/2)>0$



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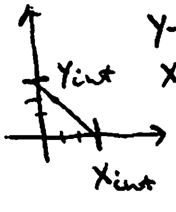
3)



$y - 2 = m(x-1)$

$X_{int} = -2/m + 1$

$Y_{int} = 2 - m$

$A = \frac{1}{2} X_{int} \cdot Y_{int} = 2 - \frac{2}{m} - m/2$

$-\infty < m < 0$

$\frac{dA}{dm} = -1/2 + 2/m^2$

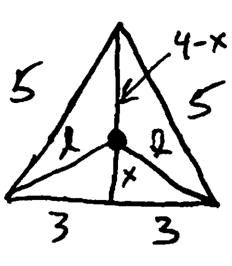
$\frac{dA}{dm} = 0$ at $m = -2$

$\left.\frac{d^2A}{dm^2}\right|_{m=-2} = \left.-4/m^3\right|_{m=-2} = 1/2 > 0$ so $m=-2$, $A=4$ is a $\overset{\text{local}}{\text{min}}$.

$A \rightarrow \infty$ as $m \rightarrow 0$ or as $m \rightarrow -\infty$. So $m=-2$, $A=4$ is the global min.

---

4)



$L = 2\ell + 4 - x = 2\sqrt{9+x^2} + 4 - x$

$0 \le x \le 4$

$L(0) = 10$

$L(4) = 10$

$\frac{dL}{dx} = \frac{2x}{\sqrt{9+x^2}} - 1$

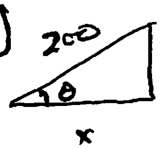
$\frac{dL}{dx} = 0$ at $x = \sqrt{3}$

$L(\sqrt{3}) = 3\sqrt{3} + 4 \approx 9.2$

$L(\sqrt{3}) \approx 9.2 < 10$

Since L at the endpoints is larger than at the unique interior crit. pt., this unique crit. pt. is a min.

5)

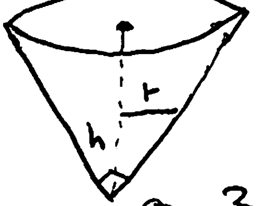


$\cos\theta = \frac{x}{200}$

$-\sin\theta \frac{d\theta}{dt} = \frac{1}{200}\frac{dx}{dt}$

$\frac{dx}{dt}\Big|_{\theta=\pi/6} = 50 \qquad \frac{d\theta}{dt}\Big|_{\theta=\pi/6} = \frac{1}{200}\cdot 50\cdot(-2) = -1/2\ \frac{\text{rad}}{\text{sec}}$

6)



$r = h$ since this is a right circular cone.

$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi h^3$

ⓐ $3 = \frac{dV}{dt} = \pi h^2 \frac{dh}{dt}, \quad \frac{dh}{dt}\Big|_{h=2} = \frac{3}{4\pi}$

ⓑ Assume the rate of evaporation is proportional to the surface area, i.e. $\frac{dV}{dt} = C\pi r^2 = C\pi h^2$ (C is some negative constant).

$C\pi h^2 = \frac{dV}{dt} = \pi h^2 \frac{dh}{dt}$ So $\boxed{\frac{dh}{dt} = \text{Const}}$

7) $f(x) = e^{-\lambda x}(1+2\sin x)^{-1}$

$\approx (1-\lambda x)(1+2x)^{-1}$

$\approx (1-\lambda x)(1-2x)$

$= 1-(2+\lambda)x+2\lambda x^2$

So f is const. to first order if $\boxed{\lambda = -2}$

To estimate

$f(x) = e^{2x}(1+2\sin x)^{-1}$

we use 2nd order approx.

$f(x) = e^{2x}(1+2\sin x)^{-1}$

$\approx \left(1+2x+\frac{(2x)^2}{2}\right)(1+2x)^{-1}$

$\approx (1+2x+2x^2)(1-2x+2x^2)$

$= 1+2x^2 + \dots$

So $f(.1) \approx 1+2(.1)^2 = \boxed{1.02}$

8) From M.V.T.: for any $a < b$ there is a c $a < c < b$ s.t.

ⓐ $f(b) - f(a) = f'(c)(b-a) > 0$ since $f'(c) > 0$, $b-a > 0$.

so $f(b) > f(a)$, i.e. f is increasing.

ⓑ For $x > 0$ $\frac{e^x - e^0}{x-0} = \frac{d}{dx}(e^x)\Big|_{x=c} = e^c$ for $0 < c < x$

or $e^x = 1 + e^c x$. For $0 < c$, $e^c > 1$,

so $e^x = 1 + e^c x > 1 + x$.

9) a) $\int \frac{dx}{(3x+2)^2} \underset{\substack{u=3x+2\\ du=3dx}}{=} \frac{1}{3}\int \frac{du}{u^2} = -\frac{1}{3}\frac{1}{u} + C$

$= -\frac{1}{3}\frac{1}{3x+2} + C$

b) $\int \sin(2x)\sin x\,dx = \int 2\sin^2 x\cos x\,dx$ $\quad u = \sin x,\ du = \cos x\,dx$

$= 2\int u^2 du = \frac{2}{3}u^3 + C = \frac{2}{3}\sin^3 x + C$

c) $\int \frac{(\ln x)^2}{x}dx \underset{\substack{u=\ln x\\ du=dx/x}}{=} \int u^2 du = \frac{1}{3}u^3 + C$

$= \frac{1}{3}(\ln x)^3 + C$

10) $\frac{dy}{dx} = x(y^2+1) \quad \frac{dy}{y^2+1} = x\,dx \quad \tan^{-1}y = \frac{x^2}{2} + C$

$y = \tan\left(\frac{x^2}{2} + C\right) \qquad 1 = \tan C, \ C = \pi/4$

$y = \tan\left(x^2/2 + \pi/4\right)$.

11) $\frac{dT}{dt} = k(T - T_e) \quad \frac{dT}{T-T_e} = k\,dt \quad \ln(T-T_e) = kt + C$

$T = T_e + Ae^{kt}$, (let $A = e^C$).

$20 = T(0) = 100 + A$ so $A = -80$

$30 = T(5) = 100 - 80e^{5k}$ $k = \frac{1}{5}\ln 7/8$

$60 = T(t_{done})$

$= 100 - 80e^{kt_{done}}$

$t_{done} = \frac{5\ln 1/2}{\ln 7/8} \approx 26$

18.01 Practice Exam 2

Problem 1. (20) Find the local maxima and minima and points of inflection of

$$2x^3 + 3x^2 - 12x + 1\ .$$

Then use this data to sketch its graph on the given axes, showing also where it is convex (concave up) or concave (down). (Note that different scales are used on the two axes.)

Problem 2. (20) A new junk food — NoKarb PopKorn — is to be sold in large cylindrical metal cans with a removable plastic lid instead of a metal top. The metal side and bottom will be of uniform thickness, and the volume is fixed to be 64π cubic inches.

What base radius r and height h for the can will require the least amount of metal?

Show work, and include an argument to show your values for r and h really give a minimum.

Problem 3. (15) Evaluate the following indefinite integrals:

a) $\int e^{-3x} dx$

b) $\int \cos^2 x \sin x\, dx$

c) $\int \frac{x\, dx}{\sqrt{1-x^2}}$

Problem 4. (15)

A searchlight L is 100 meters from a prison wall. It is rotating at a constant rate of one revolution every 8 minutes. (How many radians/minute is that?)

Martha, an escaping prisoner, is running along the wall trying to keep just ahead of the beam of light. At the moment when the searchlight angle θ is 60 degrees, how fast does she have to run?

Problem 5. (15: 5, 10)

a) What value for the constant c will make the function $e^{-x}\sqrt{1+cx}$ approximately constant, for values of x near 0? (Show work.)

b) Find the solution $x(t)$ to the differential equation $\frac{dx}{dt} = 2t\sqrt{1-x^2}$ which also satisfies the condition $x(0) = 1$.

Problem 6. (15: 8,7)

a) Use the Mean-value Theorem to show that $\ln(1+x) < x$, if $x > 0$.

(You do not have to state the theorem.)

b) Let c be any constant. Show that the function $f(x) = x^3 + x + c$ cannot have two zeros.

(Use the Mean-value theorem, or some other argument.)

18.01 Practice Exam 2 Fall 2006

Problem 1. $f(x) = 2x^3+3x^2-12x+1$

$f'(x) = 6x^2+6x-12$ $\quad f'(x)=0 \quad x=1,-2$

$f''(x) = 12x+6$ $\quad f''(x)=0 \quad x=-1/2$

$f''(1)=18>0$ so $(1,-6)$ is a loc. min

$f''(-2)=-18<0$ so $(-2,21)$ is a loc. max

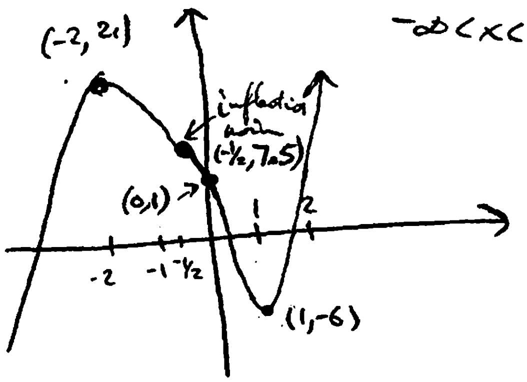
$x\to\infty \quad f(x)\to\infty$

$x\to-\infty \quad f(x)\to-\infty$

concave up: $-1/2<x<\infty$

concave down:

$-\infty<x<-1/2$



Problem 2. $V=\pi r^2 h = 64\pi$

$r^2h=64 \quad h=64/r^2$

$A = 2\pi r h + \pi r^2$

$= \frac{128\pi}{r} + \pi r^2$

$0<r<\infty$

$\frac{dA}{dr} = -\frac{128\pi}{r^2} + 2\pi r \qquad \frac{dA}{dr}=0,\ r=4$ (so $h=4$)

$A(r=4) = 48\pi$

$\frac{d^2A}{dr^2} = \frac{256\pi}{r^3}+2\pi>0$ at $r=4$

so $r=4, h=4, A=48\pi$ is the minimum,

since as $r\to0$ or as $r\to\infty$, $A\to\infty$.

Problem 3. a) $\int e^{-3x}dx = -\frac{1}{3}e^{-3x}+C$

b) $\int \cos^2 x \sin x\, dx = -\int u^2 du$

$u=\cos x$

$du=-\sin x\, dx$

$= -\frac{1}{3}u^2+C$

$= -\frac{1}{3}\cos^3 x + C$

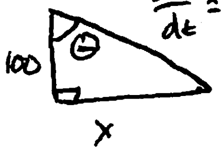
c) $\int \frac{x\,dx}{\sqrt{1-x^2}} = -\frac{1}{2}\int\frac{du}{\sqrt{u}} = -\frac{1}{2}\cdot 2u^{1/2}+C$

$u=1-x^2$

$du=-2x\,dx$

$= -\sqrt{1-x^2}+C$

Problem 4.



$\frac{d\theta}{dt} = \frac{\pi}{4}$ rad/min $\qquad \tan\theta = x/100 \qquad 100\sec^2\theta \frac{d\theta}{dt} = \frac{dx}{dt}$

$\left.\frac{dx}{dt}\right|_{\theta=\pi/3} = 100\cdot4\cdot\frac{\pi}{4} = \boxed{100\pi} \approx 314$ m/min $=18$ km/h

---

Problem 5. a) $e^{-x}\sqrt{1+cx} \approx (1-x)(1+\frac{1}{2}cx) = 1+(\frac{c}{2}-1)x-\frac{1}{2}cx^2$. so f is const to first order if $c=2$.

b) $\frac{dx}{\sqrt{1-x^2}} = 2t\,dt$

$\arcsin x = t^2+c$

$x=\sin(t^2+c)$

$1=x(0)=\sin c$

$c=\pi/2$

$x=\sin(t^2+\frac{\pi}{2})$

Problem 6. ⓐ $\frac{\ln(1+x)-\ln(1+0)}{x-0} = \left.\frac{d}{dx}(\ln(1+x))\right|_{x=c}$

$\ln(1+x) = \frac{1}{1+c}x < x$, if $x>0$. $\quad 0<c<x$

ⓑ Suppose $f(a)=f(b)=0, a\neq b$. So there is $a<d<b$ s.t. $f'(d)=0$. f can't

$f'(d)=3d^2+1>0$ for any d. No two zeros.

Recitation

10/19

quad aprox - get 1 term
- don't know how far off

Newton's method - know if off
- not guarenteed to work (x^3)

$\sin(\pi + \frac{1}{100})$

$a = \pi$

$x = \frac{1}{100}$ after found expression

~ no quadratic part ~ $\sin(\pi) = 0$

Stuff to prove
- MVT statement - hypotheses
  - 2 counter examples
- deriv ⊕ = function ↑

deriv ≠ 1, no fixed pt (from H/W)

know shorthand tricks
- pratic tests

Find c so that $\frac{\sin(x-c)}{\sqrt{1-2x}}$ is approx near $x=0$
- figure out what it looks like
- what does c do

Look at linear approx

$(x-c)(1-2x)^{-1/2}$

$(x-c)(1 - \frac{1}{2} \cdot -2x)$

$(x-c)(1+x)$

know

$\sin x \approx x$ at 0

$(1+x)^r \approx 1 + rx$

+ the rest should memorize

only at $x=0$

$x - c + x^2 - cx$ ← shortcut method too hard

$-c + (1-c)x + x^2$

↑drop to linearize

$-c + (1-c)x$

↑ if $c = 1$, linear term disappears

= approx constant

#don't plug in $x = 0$

↑ so value ≈ -1

MVT

- one function bigger - if defined
- en hw $\ln(x)$ bound by poloynomials

#11 on last worksheet

has asymptote

MVT sastisfied - continuesd closed
- diff. on open

$f(x)$ s.t.

$f(0) = 0$ Find bounds $f(8)$

↑upper + lower

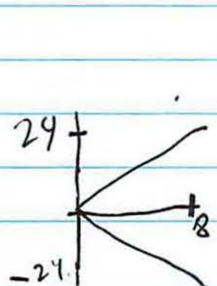
means MVT problem → $|f'(x)| \le 3$

$$\frac{f(b) - f(a)}{b - c} = f'(c)$$

$$\frac{f(8)}{8} = f'(c) = |f(8)| = 8|\underset{\uparrow 3}{f'(c)}| \le 24$$

$8 \cdot 3 \le 24$

↑ biggest can be



$-24 \le y \le 24$

One function bigger than other
Move over
Use MVT form 2
- always for inequality
- want right endpoint undefined

↑ looks like linear approx = exact however

Form 1
- bounds given on derivative

↙ can differ from line

Linear approx ≈ $f(x) \approx f(a) + f'(a)(x-a)$ x near a

MVT $f(x) = f(a) + \underset{\substack{\uparrow \\ \text{but if differs this} \\ \text{will change a lot}}}{f'(c)}(x-a)$ $c \in (\underset{\substack{\uparrow \\ a}}{[illegible]}, b)$

$f(x) = x - \frac{1}{x}$ ~~for~~

> 0 for ~~[illegible]~~ $x > 1$

$f'(x) = 1 - \frac{1}{x^2} > 0$ for $x > 1$

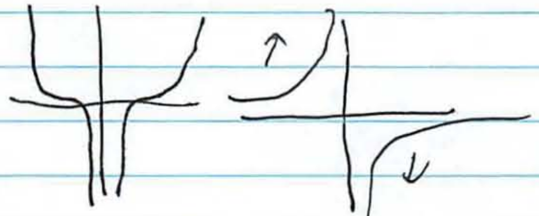
$\underset{\substack{\uparrow \\ 0}}{f(a)} + \underset{\substack{\uparrow \\ \oplus}}{f'(c)}\underset{\substack{\uparrow \\ \oplus}}{(x-a)}$

$1 - \frac{1}{c^2} > 0$ ↑want
$1 > \frac{1}{c^2}$
$c^2 > 1$
$c > 1$ ↓ is that true

7PM - 4-153 or on site

can have change of concavity at asymptote

-yeah



$f(x) = \frac{1}{x}$

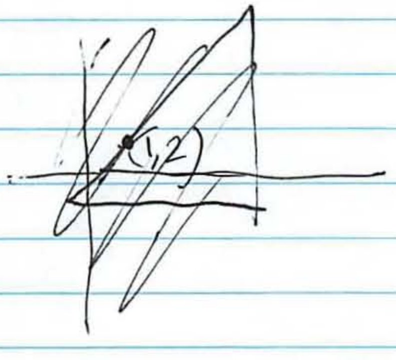
$f'(x) = \frac{1}{x^2}$

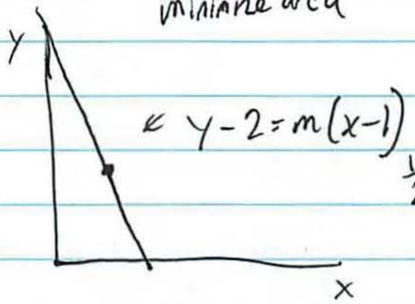
? must look here too $f''(x) = \frac{+2x}{x^3}$

- vert asy
- where $f''(x) = 0$

Triangle in 1st quad

minimize area





y x int

$\frac{1}{2}$ b·h

deriv

= 0 ~~±20~~

solve

find 2 pts

know which is right

$\frac{dA}{dm}$

$-\frac{2}{m} + 1$ = base

$2 - \overset{\cancel{m}}{m}$ = height

set 1 = 0

What can varry?
How does it influence?

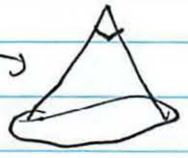
Office Hrs

10/19

related rate

right circular cone * know vol cones / cyl

upside down →



vertex 4

water added 3 cm³/sec ↑ volume

how fast water ↑

$V = \frac{1}{3}\pi r^2 h$

↑ ↑ changes

$\frac{dV}{dt} = \frac{1}{3}\pi \cdot 2 r r' h + \frac{1}{3}\pi r^2 h'$

↑ product rule - implicit differentiation

$3 = \frac{2}{3}\pi r h r' + \frac{1}{3}\pi r^2 h'$

$3 = \quad 2 \quad\quad 1$

→ redo

$V = \frac{1}{3}\pi h^3$

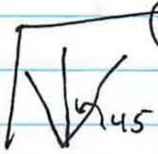
$\frac{dV}{dt} = \frac{1}{3} \cdot 3\pi h h'$

$3 = \pi \, 2 h'$

$h' = \frac{3}{2\pi}$

↓ w/ info

go back + find this constraint



$\circledast$

$r = h$

$r' = h'$

height + radius = 45,45-90

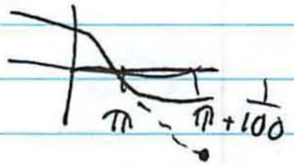
---

approx $\sin\left(\pi + \frac{1}{100}\right)$

$f(a) + f'(a)(x - a)$ tangent $(a, f(a))$

- figure what is at π

$a = \pi$ $\quad f' = \cos(x)$



under estimate

$0 + -1\left(\pi + \frac{1}{100} - \pi\right)$

$-1 \cdot \frac{1}{100}$

$-\frac{1}{100}$ = ~~value at~~

$\left(\pi + \frac{1}{100}, -\frac{1}{100}\right)$ ← approx

---

asy

$\frac{2x^2}{3x^2}$ = horiz asy $\frac{2}{3} = x$

$\frac{2x^2}{3x} \rightarrow \frac{2}{3}x$

do polynomial division

$0^{x^2}_{+}$ $\frac{2x}{3x^2}$ = horiz asy at 0

Key Equations

Circle

Area

Sphere

$SA - 4\pi r^2$

$V = \frac{4}{3}\pi r^3$

Cone

$SA = \pi r s + \pi r^2$

$V = \frac{1}{3}\pi r^2 h$

cylinder

$SA = 2\pi r^2 + 2\pi r h$) can calc

$V = \pi r^2 h$

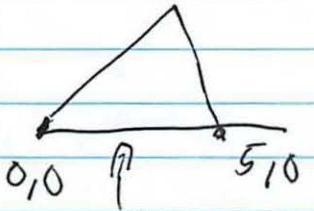
Rectangle

obvious

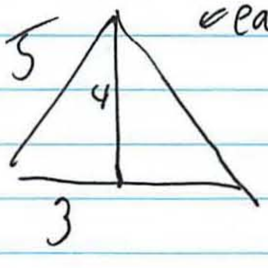
Test Review
Session

10/19

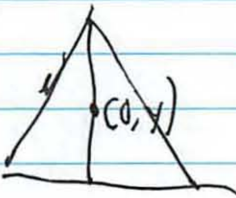
minimize = optimization problem



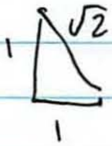
↑ put the origin here



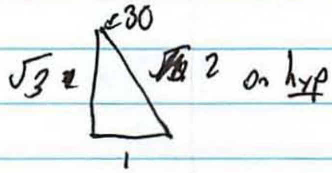
← easy now



lenght $= 2\sqrt{9+y^2} + \sqrt{(4-y)^2}$



↑ don't just minimize 1 thing



$\frac{d\,\text{lenght}}{d\cancel{x}}$ ↑ w/ respect to y - what is it a function of?

$y = \sqrt{3}$

lenght find ⇒ $2\sqrt{18} + 4 - \sqrt{3}$

↑ must be < endpoint length

Compare to end points

$V = \frac{1}{3}\pi r^3 h$

did in office hrs

$\downarrow \frac{dh}{dt}$

Show evaporation -height is constant

$$\frac{dh}{dt} = \frac{dV}{dt} \cdot \frac{1}{\pi h^2}$$

Show $\frac{dV}{dt} = \text{constant} \cdot h^2$

↑ water evaporates proportional to h^2

↑ which is SA of water πh^2

↑ DV

$$f(x) = \frac{e^{-\delta x}}{1+2\sin x}$$

$e^x = 1+x$

$e^{-\delta x} = 1-\delta x$

want $f(x) = 1$ near $x=0$

$1+2\sin x \approx 1+2x$

$$f(x) \approx \frac{1-\delta x}{1+2x} = (1-\delta x)(1+2x)^{-1}$$

e $(1+x)^r$

$(1-\delta x)(1-2x)$ multiply out

$1-\delta x-2x+2\delta x^2$

↑ linearize by drop quadratic term

$$\approx 1+x\underbrace{(-\delta-2)}_{\uparrow\text{ must be }0}$$

So when $\delta = -2$

$f(.1) \approx 1$ Linear
$f(.1) \approx 1 - .04 = .96$ quadratic

$e^x > 1 + x$ for all $x > 0$
show its > 0

$f(x) = e^x - 1 - x$

$f(x) = f(a) + f'(c)(x-a)$ $c \in (a,x)$
$f(x) = f(0) + f'(c) = x$
$f(x) = 0 + \uparrow ?$

$f'(x) = e^x - 1$
since $e^x > 1$ for $x > 0$
$e^x - 1 > 0$ for $x > 0$

$f(x) = 0 + \uparrow + \uparrow > 0$
$\uparrow$
$f'(c) > 0$ for $c > 0$

Use MVT form 2 for inequality

MVT if condition on derivitive

Rolle's Theorm
~~Roll's theorm~~

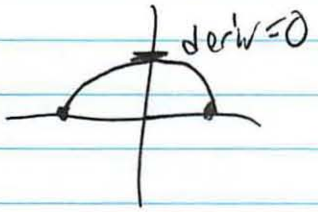
- only 1 place where deriv
- 2 end points where function $= 0$

$f(0) = 0$ $f'(\frac{1}{2}) = 0$

$f(1) = 0$

↑only place where deriv $= 0$

if does not cross x axis



*Only changes direction once

* review

On exam suppose $f(w) = 0$

↑ $0 < w < \frac{1}{2}$

then rolle's theorm implies c $0 \le c < w$

where $f'(c) = 0$

This is a contradiction

By symatry from $\frac{1}{2} \to 1$

HW w/ fixed point

- show that if 2 it contridicts

↑good way to prove

Prove if d is ⊕ function ↑

increasing function

2 x values → $a < b$ $f(a) < f(b)$ ← would never think to write like that

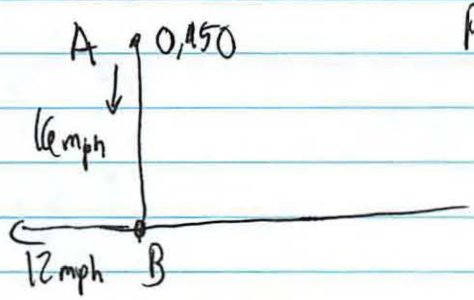
$f(x)$ differetiable $f'(x) > 0$ for all x

~~2 x values~~ Consider closed interval $[a, b]$ + apply MVT

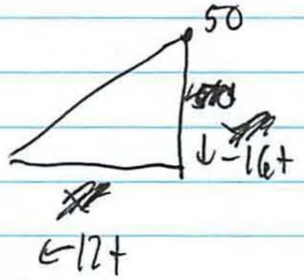
Then $f(b) - f(a) = \oplus$ quantity $\underset{\oplus > 0}{f'(c)}\,\underset{b>a \;\oplus}{(b-a)}$

$f(b) - f(a)$
must be ⊕

Starting proof
1. Definitions
   - positive → $b - a$ → $f(b) - f(a)$
2. Use that
3. Then they tell you to use MVT



Pythagorean theorm problem



both changing linearly

can measure distance in time

$x = 12t$
$y = 50 - 16t$

↑can write it in
1 variable

] should really know to
set up
just think creativly
about it

do actual derivitive

$$d = \sqrt{\overset{\downarrow\text{chain rule}}{(12t)^2 + (50-16t)^2}}$$

$$2dd' = 288t + 2(50-16t)(-16)$$

$$-1600 + 800t$$

at $t = ?$ = to 0
solve for d

Why did I
lose pts on #3 ?

18.01 EXAM II

Tuesday, Oct. 20, 2009

Name: Michael Plasmeier

E-mail: theplaz@mit.edu

Recitation Instructor (Circle One): [✓ C. Breiner] / I. Losev / X. Ma / S. Ramakrishnan / B. Rhoades

Recitation Hour: 1 PM

Instructions: You may not use calculators, notes, textbooks, or personal electronic devices. As a courtesy to other students, please turn off all cell phones. Read each question carefully. Whenever possible, include justification for your reasoning and show your work. Answers without any work or explanation will receive little or no credit. There are 6 questions and a 55 minute time limit on this exam. Good luck.

Question	Score	Maximum
1	1	5
2	5	5
3	5	6
4	2	5
5	4.5	5
6	1	5
Total	19	31

← disaster

← id. values @ min / max

← copy error + more?

– did not remember / explain more

Basic area and volume formulas:

Volume of a Cone: $\frac{1}{3}\pi r^2 h$
Volume of a Sphere: $\frac{4}{3}\pi r^3$
Surface Area of a Sphere: $4\pi r^2$

thanks

πr^2

$2\pi r$

1. (a) Give a general expression for the quadratic approximation to a twice differentiable function $f(x)$ at $x = a$.

1

a-x
x-a

$$f(a) + \underline{f'(a)(a-x)} + \frac{f''(a)}{2}\overset{x-a}{\cancel{(a-x)}}^2$$

x-a

I did flip it :(

grrr

(b) Use your answer from part (a) to give an approximate value for ln(1.2), where ln(x) is the natural log function.

0

$x = 0$

$$\ln = \frac{1}{1+x}$$

did I do "?

$$\cancel{\frac{1}{1+1.2} + \frac{1}{1.2}(1.2-0) + \frac{\frac{1}{1.2}}{2}(1.2-0)^2}$$

:(I should have mem better focus more disaster

$$(\ln(x))' = \frac{1}{x}$$

$$(\ln(x))'' - \frac{1}{x^2}$$

$$\ln(1.2) = \ln(1) + \ln'(1)(1.2-1) + \frac{\ln''(1)}{2}(1.2-1)^2$$

$$\ln(1.2) = 0 + .2 - \frac{1}{2}(.2)^2$$

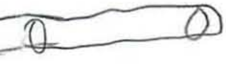
$$.2 - \frac{.04}{2}$$

$$\boxed{.18}$$

Question 2 of 6, Page 3 of 7 Name: ______________________

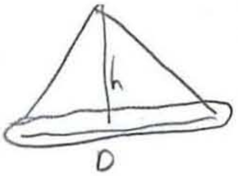
related rate

2. Salt is poured from a conveyer belt at a rate of 30 ft^3/min, forming a conical pile with a circular base whose height and diameter of base are always equal. How fast is the height of the pile increasing when the pile is 10 ft. high?



$\downarrow 30\,ft^3/min = \frac{dV}{dt}$

$\frac{dh}{dt} = ?$



$h = D = 2r$

$r = \frac{1}{2}h$

$h = 10$

$V = \frac{1}{3}\pi r^2 h$

$V = \frac{1}{3}\pi r^2 \cdot 2r$

$V = \frac{2}{3}\pi r^3$

$\frac{dV}{dt} = \frac{2}{3}\pi \cdot 3r^2 r'$

$\frac{dV}{dt} = 2\pi r^2 r'$

$\frac{dV}{dt} = \frac{1}{2}h^2 \pi \frac{dh}{dt}$

$30 = \frac{1}{2}10^2 \underset{100\pi}{\pi} \frac{dh}{dt}$

$\frac{dh}{dt} = \frac{30}{50\pi} = \frac{3}{50\pi}$ ft/min

i figure out - why different??

$2r^2$

$2(\frac{1}{2}h)^2$

$2\frac{h^2}{4}$

$\frac{1}{2}h^2$

2r or h

$V = \frac{1}{3}\pi(\frac{1}{2}h)^2 h$

$V = \frac{1}{12}\pi h^3$

$\frac{dV}{dt} = \frac{1}{12} \cdot 3\pi h^2 \frac{dh}{dt}$

$\frac{dV}{dt} = \frac{1}{4}\pi h^2 \frac{dh}{dt}$

$30 = \frac{1}{4}\pi \cdot 10^2 \frac{dh}{dt}$

$\frac{30}{25\pi} = \frac{6}{5\pi}$ ft/min

$(\frac{1}{2}h)^2$

$(\frac{h}{2})^2$

$\frac{h^2}{4}$ $\frac{1}{4}h^2$

$\frac{1}{4} \cdot \frac{1}{3} = \frac{1}{12}$ $\frac{3}{12} = \frac{1}{4}$

? picked right one

$\frac{5}{5}$

bingo

$f(0) = 0$
$f(1) = -2$
$f(-1) = 2$

3. Draw a careful picture of the graph of the function

$$f(x) = x - 3x^{1/3}.$$

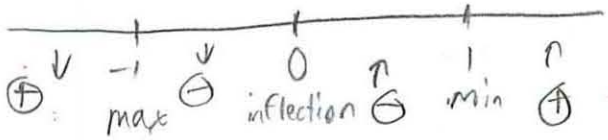
Be sure to indicate the coordinates of any local maxima and minima, the intervals on which the function is increasing and decreasing, and asymptotes (if any of these features occur). Computing inflection points may help you draw an accurate picture, but is not necessary.

$f'(x) = 1 - 3\cdot\frac{1}{3}x^{-2/3} = 1 - x^{-2/3} = 1 - \frac{1}{\sqrt[3]{x^2}}$

$f''(x) = --\frac{2}{3}x^{-5/3} = \frac{2}{3}x^{-5/3} = \frac{2}{3\sqrt[3]{x^5}}$

$1 - x^{2/3} = 0$ +1
$-x^{-2/3} = 1$
$\frac{1}{\sqrt[3]{x^2}} = 1$
$1^3 = (\sqrt[3]{x^2})^3$
$1 = x^2$
$\sqrt{\ } \quad \sqrt{\ }$
$x = \pm 1$ +1

$\frac{2}{3\sqrt[3]{x^5}} = 0$
$0 = \sqrt[3]{x^5}$
$\sqrt[3]{\ } \quad \sqrt[3]{\ }$
$0 = x^5$
$0 = x$



+1

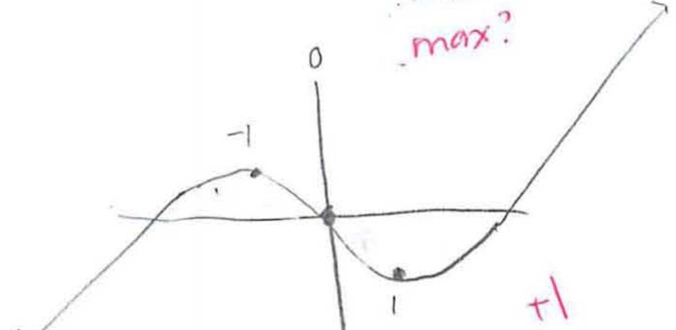
⊕⊖ = direction slope
↑↓ = concavity

increasing $(\infty, -1)(1, \infty)$
decreasing $(-1, 1)$ +1

$\frac{2}{3\sqrt[3]{1^5}} = \frac{2}{3}$ ↑ ⋃ min
$\frac{2}{3}\sqrt[3]{-1^5} = \frac{2}{-3} =$ ↓ ⋂ max

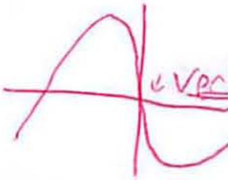
tilla got

min?
max?



+1

5/6


vertical tangent line at $x = 0$

this time

4. A metal storage tank is to be made in the shape of a cylinder with a circular base and a hemispherical top. Find the dimensions of the tank which require the least amount of metal used to hold a fixed constant volume V.

$$SA = \underset{\text{side}}{2\pi r h} + \underset{\text{bottom}}{\pi r^2} + \underset{\text{top half}}{\frac{1}{2}\cdot 4\pi r^2} = 2\pi r h + \pi r^2 + 2\pi r^2$$

$$2\pi r h + 3\pi r^2$$ ✓

$$V = \underset{\text{main}}{\pi r^2 h} + 0 + \frac{1}{2}\frac{4}{3}\pi r^3$$

✓ $\pi r^2 h + \frac{4}{6}\pi r^{3}$ copy error :(

$$V = \pi r^2 h + \frac{2}{3}\pi r^2$$

$$\pi r^2 h = V - \frac{2}{3}\pi r^2$$

watch signs

$$h = \frac{V}{\pi r^2} - \frac{2\cancel{\pi r^2}}{3\cancel{\pi r^2}}$$

$$h = \frac{V}{\pi r^2} - \frac{2}{3} = \text{fixed}$$

$$SA = 2\pi r\left(\frac{V}{\pi r^2} - \frac{2}{3}\right) + 3\pi r^2$$

$$\frac{2\cancel{\pi}\cancel{r}V}{\cancel{\pi}r^{\cancel{2}}} - \frac{4\pi r}{3} + 3\pi r^2$$

$$\frac{2V}{r}\leftarrow\text{fixed} - \frac{4}{3}\pi r + 3\pi r^2$$

$$2Vr^{-1} - \frac{4}{3}\pi r + 3\pi r^2$$

$$\frac{dSA}{dr} = 2V\,{-1}r^{-2} - \frac{4}{3}\pi + 3\pi 2r$$

$$\frac{-2V}{r^2} - \frac{4}{3}\pi + 6\pi r$$

$$0 = \frac{2V}{r^2} - \frac{4}{3}\pi + 6\pi r$$

solve for V

$$\frac{2V}{r^2} = \frac{4}{3}\pi - 6\pi r$$

$$2V = \frac{4}{3}\pi r^2 - 6\pi r^3$$

$$\boxed{V = \frac{2}{3}\pi r^2 - 3\pi r^3}$$

Constant

$$r^3\left(\frac{2}{3}\pi - 3\pi\right) = V$$

$$r = \sqrt[3]{\frac{V + \frac{2}{3\pi}}{3\pi}}$$

[illegible] rush

seems really wrong - where did I go off track

want dimensions

what is r?

②

$V = \pi\,{}^2 h + \frac{2}{3}\pi r^2$

$SA = \pi r^2 + 2\pi r^2 + 2\pi r h$

sub h into SA

. . .

$SA = 3\pi r^2 + 2\pi r \left(\frac{V - \frac{2}{3}\pi r^3}{\pi r^2} \right) \to \frac{2V}{r} - \frac{4}{3}\pi r^2$

$\frac{d}{dr} = 6\pi r - \frac{2V}{r^2} - \frac{8}{3}\pi r$

$\frac{10}{3}\pi r = \frac{2V}{r^2}$ at a critical pt

↑ sub V back in to get in terms r + h

r = h at critical pt

↑ why? what causes this

no endpoints

2nd deriv - check sign <u>have to justify</u>

really focused

now

- happy w/ grade

like this

5. Explain why Newton's method eventually fails when finding zeroes of $f(x) = x^3 - 3x + 7$ with a starting value $x_1 = 2$. from pratice - kinda

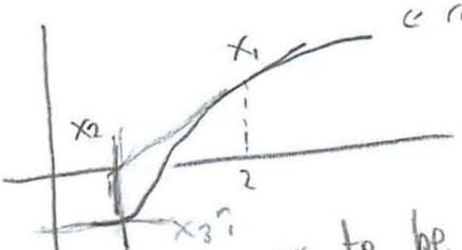
~~the estimations do not get closer to 0~~

$$f'(x) = 3x^2 - 3$$

$$x_1 = 2$$

$$x_2 = 2 - \frac{x^3-3x+7}{3x^2-3} = \frac{2^3-3(2)+7}{3(2)^2-3} = \frac{8-6+7}{12-3} = \frac{9}{9} = 1$$ ✓

$$x_3 = 1 - \frac{(1)^3-3(1)+7}{3(1)^2-3} = \frac{1-3+7}{3-3} = \frac{5}{0}$$ ← divide by 0 error



↑ slope happens to be 0 at x_3
which means newton's method will fail
where will you go next
nowhere because you flattened at ✓
never intercepts x-axis

-.5 ← needed the words

looking for tangent line intercept x axis

got this I hope

mvt??

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

6. Prove that

$$\overset{\oplus\; y_1}{\sqrt{1+x}} < \overset{\oplus\; y_2}{1+\frac{1}{2}x}, \quad \text{if } x > 0. \quad x\oplus$$

for it to be less than

X	Y_1	Y_2
1	$\sqrt{2}$	$\frac{3}{2}$
2	$\sqrt{3}$	2
3	$\sqrt{4}$	$\frac{5}{2}$
4	$\sqrt{5}$	3

$$(1+x)^{1/2}$$

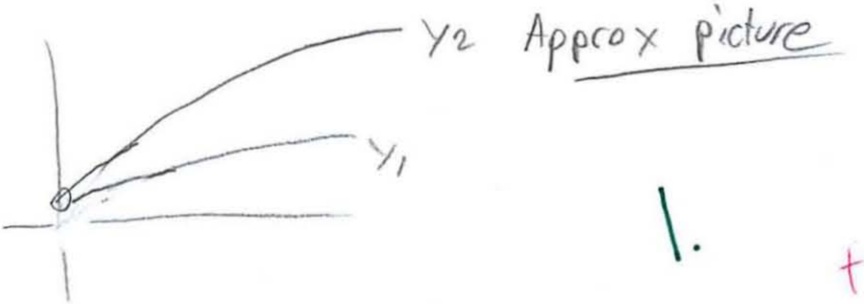
$$y_1' = \frac{1}{2}(1+x)^{-1/2}\cdot 1 \qquad y_2' = \frac{1}{2}$$

$$\frac{1}{2(1+x)^2}$$

denominator gets larger as $x \to \infty$ $(x>0)$
means slope gets smaller $(m \to 0)$
but sill positive as $x \to \infty$
$x > 0$

So $y_1 < y_2$ at the start
and y_1 slope will approch 0 as $x \to \infty$ $(x>0)$
where as y_2 will grow at a constant rate further increasing it's lead over y_1

did not really use proper termonolgy



similar to pset problem

1.

Opps + in class forgot

Look at $f(x) = 1 + \frac{1}{2}x - \sqrt{1+x} \quad [0,w]$

$$f(w) = f(0) + f'(c)(w-0) \qquad c \in (0,w)$$

$$= 0 + \underbrace{f'(c)}_{\oplus} - w$$

$$f'(x) = \frac{1}{2} - \frac{1}{2\sqrt{1+x}}$$

$$= \frac{1}{2}\left(1 - \frac{1}{\sqrt{1+x}}\right) \leftarrow \text{why is this } \oplus$$

$\left[\text{for } x > 0 \ \sqrt{1+x} > 1 \ \text{ so } \ 1 > \frac{1}{\sqrt{1+x}}\right]$

So other part $\oplus$ as well

18.01 EXAM II

Tuesday, Oct. 20, 2009

Name: ______________________________

E-mail: ______________________________

Recitation Instructor (Circle One): C. Breiner / I. Losev / X. Ma / S. Ramakrishnan / B. Rhoades

Recitation Hour: ________________________

Instructions: You may not use calculators, notes, textbooks, or personal electronic devices. As a courtesy to other students, please turn off all cell phones. Read each question carefully. Whenever possible, include justification for your reasoning and show your work. Answers without any work or explanation will receive little or no credit. There are 6 questions and a 55 minute time limit on this exam. Good luck.

Question	Score	Maximum
1		5
2		5
3		6
4		5
5		5
6		5
Total		31

Basic area and volume formulas:

Volume of a Cone: $\frac{1}{3}\pi r^2 h$
Volume of a Sphere: $\frac{4}{3}\pi r^3$
Surface Area of a Sphere: $4\pi r^2$

1. (a) Give a general expression for the quadratic approximation to a twice differentiable function $f(x)$ at $x = a$.

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$$

(b) Use your answer from part (a) to give an approximate value for $\ln(1.2)$, where $\ln(x)$ is the natural log function.

$$\ln(1.2) = \ln(1) + \ln'(1)(1.2-1) + \frac{\ln''(1)}{2}(1.2-1)^2$$

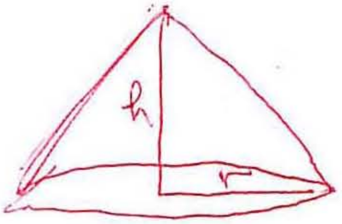
$$\left((\ln(x))' = \frac{1}{x} \quad (\ln(x))'' = \frac{-1}{x^2} \right)$$

$$\Rightarrow \ln(1.2) = 0 + (0.2) - \frac{1}{2}(0.2)^2$$

$$= 0.2 - 0.04/2$$

$$= 0.18$$

2. Salt is poured from a conveyer belt at a rate of 30 ft^3/min, forming a conical pile with a circular base whose height and diameter of base are always equal. How fast is the height of the pile increasing when the pile is 10 ft. high?



$$h = 2r$$

$$V = \frac{1}{3}\pi r^2 h$$

$$= \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h$$

$$= \frac{1}{12}\pi h^3$$

$$\frac{dV}{dt} = \frac{1}{4}\pi h^2 \frac{dh}{dt}$$

$$\Rightarrow \frac{dh}{dt} = \frac{dV}{dt}\left(\frac{4}{\pi h^2}\right)$$

$$= (30)\left(\frac{4}{10^2\pi}\right) = \frac{120}{100\pi}$$

$$= \frac{12}{10\pi}\ \frac{ft}{min}$$

3. Draw a careful picture of the graph of the function

$$f(x) = x - 3x^{1/3}.$$

Be sure to indicate the coordinates of any local maxima and minima, the intervals on which the function is increasing and decreasing, and asymptotes (if any of these features occur). Computing inflection points may help you draw an accurate picture, but is not necessary.

$f(x) = 0$ $\quad$ $x - 3x^{1/3} = 0$ $\quad$ $x^{1/3}(x^{2/3} - 3) = 0$

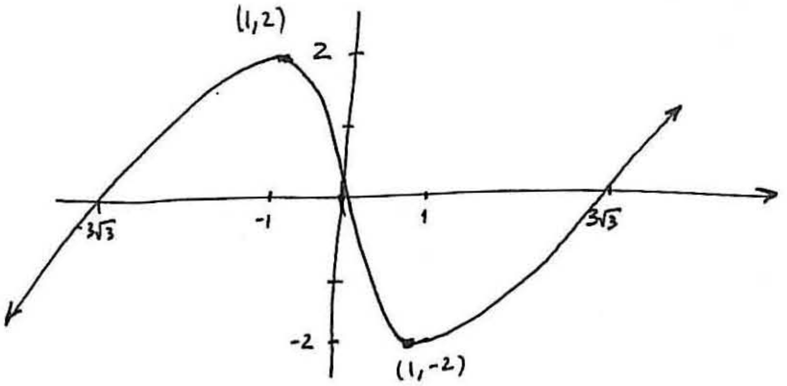
$x = 0, +3\sqrt{3}, -3\sqrt{3}$.

$f'(x) = 1 - 3\cdot\frac{1}{3}x^{-2/3} = 1 - x^{-2/3}$: critical pts at ± 1. $\Rightarrow$ at $\overset{\text{min}}{(1,-2)}$ $\overset{\text{max}}{(-1,+2)}$

Vertical tangent line at $x = 0$

inc. dec. inc.

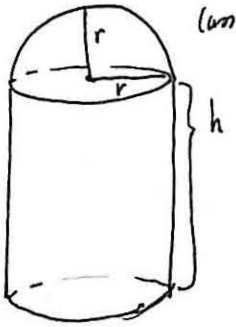
-1 1



4. A metal storage tank is to be made in the shape of a cylinder with a circular base and a hemispherical top. Find the dimensions of the tank which require the least amount of metal used to hold a fixed constant volume V.

const $= V = \frac{1}{2}\left(\frac{4}{3}\pi r^3\right) + (\pi r^2)h. = \frac{2}{3}\pi r^3 + \pi r^2 h$

(assuming) amount of metal used is prop. to surface area → minimize SA.



$$SA = \frac{1}{\cancel{2}}\left(\overset{2}{\cancel{4}}\pi r^2\right) + (2\pi r)h + \pi r^2 =$$

$$= 3\pi r^2 + 2\pi r h.$$

$$h = \frac{V - \frac{2}{3}\pi r^3}{\pi r^2}$$

$$SA = 3\pi r^2 + 2\pi r\left(\frac{V - \frac{2}{3}\pi r^3}{\pi r^2}\right) = 3\pi r^2 + \frac{2V}{r} - \frac{4}{3}\pi r^2$$

$$= \frac{5}{3}\pi r^2 + \frac{2V}{r}.$$

$$\frac{d}{dr}SA = \frac{10}{3}\pi r + (-1)\frac{2V}{r^2}$$ set $=0.$, assume $r \neq 0$.

$$^{5}\frac{\cancel{10}}{3}\pi r = \frac{\cancel{2}V}{r^2} \qquad V = \frac{5}{3}\pi r^3 \Rightarrow r = \sqrt[3]{\frac{3V}{5\pi}}$$

$$\Downarrow$$

$$h = \frac{5/3\,\pi r^3 - 2/3\,\pi r^3}{\pi r^2} = \frac{\pi r^3}{\pi r^2} = r \Rightarrow \boxed{h = r = \sqrt[3]{\frac{3V}{5\pi}}}$$

This is a critical pt. should show it is indeed a min.

Check boundaries: $r=0$, $h=0$.

$r=0 \Rightarrow SA=0$

$h=0 \Rightarrow SA = 3\pi r^2$, $V = \frac{2}{3}\pi r^3 \Rightarrow r = \sqrt[3]{\frac{3V}{2\pi}}$

$SA = 3\pi\left(\frac{3V}{2\pi}\right)^{2/3}$ when $h=0$.

when $h=r$, $h = r = \sqrt[3]{\frac{3V}{5\pi}}$, so

$SA = 5\pi\left(\frac{3V}{5\pi}\right)^{2/3}$

Comparing $5^{1/3}$ to $\frac{3}{2^{2/3}}$. cube both sides.

$5 < \frac{27}{4}$, so SA at critical pt < SA at bdry pt

b/c $20 < 27$

$\Rightarrow$ min attained at $\boxed{h = r = \sqrt[3]{\frac{3V}{5\pi}}}$

5. Explain why Newton's method eventually fails when finding zeroes of $f(x) = x^3 - 3x + 7$ with a starting value $x_1 = 2$.

$f'(x) = 3x^2 - 3$

For $x_1 = 2$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2 - \frac{f(2)}{f'(2)} = 2 - \frac{9}{9} = 1$$

* $f(x_2) = f(1) = 1 - 3 + 7 = 5$

$\underline{f'(x_2) = f'(1) = 3 - 3 = 0}$

The problem is that the slope of the tangent line at $x=1$ is horizontal. Thus, it does not intersect the x-axis & we cannot continue Newton's method.

6. Prove that

$$\sqrt{1+x} < 1 + \frac{1}{2}x, \quad \text{if } x > 0.$$

Set $g(x) = 1 + \frac{1}{2} - \sqrt{1+x}.$ WTS $g(x) > 0$ for $x > 0$.

$g(0) = 1 + 0 - 1 = 0.$ Sufficient to show $g'(x) > 0$ for all $x > 0$.

$$g'(x) = 0 + \frac{1}{2} - \frac{1}{2}\,\frac{1}{\sqrt{1+x}} = \frac{1}{2} - \frac{1}{2\sqrt{1+x}}$$

For $x > 0$, $\sqrt{1+x} > \sqrt{1} = 1.$, so

$$\frac{1}{\sqrt{1+x}} < \frac{1}{1} = 1.,\ \text{and}$$

$$-\frac{1}{2\sqrt{1+x}} > -\frac{1}{2}\cdot 1 = -\frac{1}{2}$$

$$\Rightarrow$$

$$\underbrace{\frac{1}{2} - \frac{1}{2\sqrt{1+x}}}_{g(x)} > \frac{1}{2} - \frac{1}{2} = 0 //$$

showed $g(x) > 0$ for $x > 0$. //.