Recitation

10/22

Last 2 days

Antiderivatives

$\int \sin x \, dx$

$-\cos(x + 2\pi) = -\cos x$

↑periodicity

$\int \frac{1}{(1-x)^2} dx = \frac{1}{1-x}$

$= \frac{x}{1-x}$ subtract to see differ by constant 1

$\left(\frac{x}{1-x} + 1 = \frac{x}{1-x} + \frac{1-x}{1-x} = \frac{1}{1-x} \right)$

focusing' paying attention today!

Strategy
- Substitution
  - only one we know

$\int \frac{(\ln x)^3}{x} dx$ "recognition" function in here & derivative

I know $\frac{d}{dx} \ln(x) = \frac{1}{x}$ → $\frac{u^3}{x} dx$ & get rid of

$\frac{du}{dx} = \frac{1}{x}$ ∴ ⇒ $du = \frac{dx}{x}$ ← some funkyness

substitute $\int u^3 du = \frac{u^4}{4} + C \xrightarrow{\text{replace}} \frac{(\ln x)^4}{4} + C$

$\int \underline{sec^2 x} \underbrace{\tan x \quad dx}_{} \frac{dx}{du}$

$u = \tan x$
$du = sec^2 x \, dx$

$\int u\,du \rightarrow \frac{u^2}{2} + c \rightarrow \frac{\tan^2 x}{2} + c$

or

$u = sec\, x$ $\qquad \frac{sec^2 x}{2} + c$
$du = sec\, x \tan x \, dx$

$\cancel{\int \frac{x}{\sqrt{1-x^2}} dx}$

Differential Equations

Given rate of change

① $\frac{dy}{dx} = x^2 y + xy$ ← y depends on x ← what is relationship

② $y(0) = 7$ ← give value

③ Find $y(x)$

-did seperation of variables

$\frac{dy}{y} = (x^2 + x)\,dx$

$\ln y = \frac{1}{3}x^3 + \frac{1}{2}x^2 + c$

$y = e^{1/3 x^3 + 1/2 x^2 + c}$

find constant w/ $y(0)=7$

when take derivitive it should be back

Seem to know concepts - just do it
+ pratice w/o looking !!!
+ where don't know solution

Constraint $V = \pi r^2 h \rightarrow h = \frac{V}{\pi r^2}$

fixed

optimize cost $\quad cost = 2\pi r h c + \pi r^2 \cdot 3c$

~~solve for h~~ $\frac{2\pi r V}{\pi r^2} c + \pi r^2 \cdot 3c$

$\frac{2V}{r} \cdot c + 3\pi r^2 c$

constant

$$\frac{d\,cost}{dr} = c\left(-\frac{2V}{r^2} + 6\pi r\right)$$

$$0 = c\left(-\frac{2V}{r^2} + 6\pi r\right)$$

$$\frac{0}{c} = 0$$

$$0 = -\frac{2V}{r^2} + 6\pi r$$

$$r^3 = \frac{V}{3\pi} \xrightarrow{\text{hard}} r = \frac{V^{1/3}}{(3\pi)^{1/3}}$$

$\downarrow$ easy

$$r^3 = \frac{\pi r^2 h}{3\pi} = \frac{r^2 h}{3} \rightarrow r = \frac{h}{3}$$

$$d = \frac{2h}{3}$$

end point when $h = 0$ and $r \rightarrow \infty$

$r = 0 \qquad h \rightarrow \infty$

Lecture 18
Definite Integral

10/22

Pset 5 due Fri
- 5a + 5b

Part 3 starts today

Antiderivitives + Differental Equations
- exam 3 + Final

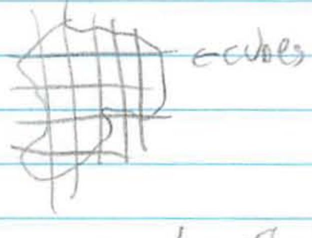
Today: Begin integral Calculus

Central Problem: How to compute area under a curve

Area of region in 2D space = # of 1x1 blocks can place in a space

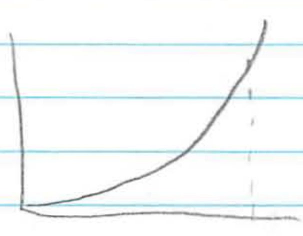
Great for rectangles + triangles + polygons - by dividing into triangles

Why is the area of a circle πr^2
- need some dynamic process to approximate area

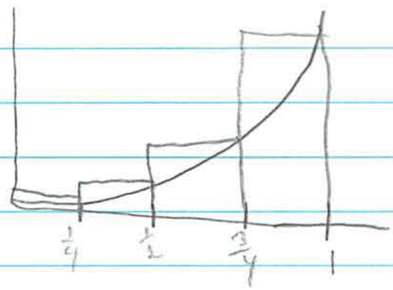


the smaller the cube, the more accurate the #

Same process to find area under a curve for a function $y = f x$ between 2 points $x = a$, $x = b$



use rectangles



Σ the area of the rectangles

- right side
  - over estimate

Approx area $= R_4 =$

$$\left(\tfrac{1}{4}\right)\cdot\left(\tfrac{1}{4}\right)^2 + \left(\tfrac{1}{4}\right)\left(\tfrac{1}{2}\right)^2 + \tfrac{1}{4} + \left(\tfrac{3}{4}\right)^2 + \tfrac{1}{4} \cdot 1^2$$

$= {}^{15}/_{32} \approx$ approx area under $x^2\,[0,1]$

*The smaller the rectangle - the better

$R_{58} = .3434$
$R_{100} = .3383$
$R_{1000} = .3338$

need a general expression for r_n

$$R_n = \underbrace{\left(\frac{1}{n}\right)}_{\text{base always}}\cdot\left(\frac{1}{n}\right)^2 + \left(\frac{1}{n}\right)\left(\frac{2}{n}\right)^2 + \left(\frac{1}{n}\right)\left(\frac{3}{n}\right)^2 + \dots \left(\frac{1}{n}\right)\left(\frac{n}{n}\right)^2$$

$$= \frac{1}{n^3}\left(1^2 + 2^2 + 3^2 + \dots n^2\right)$$

$$= \frac{1}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}$$

for 2 #

$\frac{n(n+1)}{2}$ sum of first n #

can prove by induction

must guess ans then prove its right

sum first n^3 coming soon

↓ amazingly good → definition

$\lim_{n \to \infty} R_n$ = Area under $f(x) = x^2$ from $x=0$ to $x=1$

$= \frac{3n^2 + \cdots}{6n^3}$

$\lim_{n \to \infty} \frac{3n^3 + \cdots}{6n^3} \cdot \frac{1/n^3}{1/n^3}$

$\frac{2 + \frac{k}{n} + \cdots}{6} = \boxed{2/6}$

↑ the exact answer

Could use left-hand endpoints
- Want same answer as right
  L_n = area approx using left hand endpoints
- But won't = R_n exactly, but approaches as $n \to \infty$

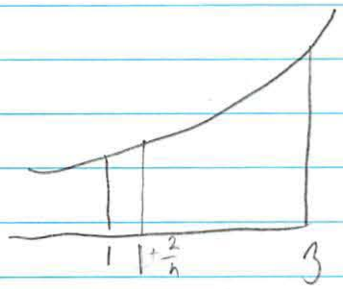
$L_{1000} = .3328$

Check $\lim_{n \to \infty} L_n = \lim_{n \to \infty} R_n$ = area

Why use left or right endpoints?
Want same answer for any point inside each interval

What changes if we ask for R_n w/ endpoints $x=1$ $x=3$

distance between $3 - 1 = 2$
width $\frac{2}{n}$

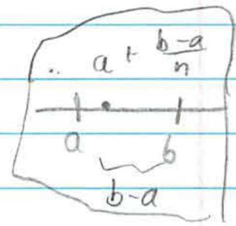


↑ each chunk $1 + \frac{2}{n}$

$$\left(\frac{2}{n}\right)\left(1+\frac{2}{n}\right)^2 + \cdots \left(\frac{2}{n}\right)\underbrace{\left(1+\frac{2n}{n}\right)}_{3 \text{ check}}^2$$

For any contineocs $f(x)$ from $x = a$ $x = b$

$$R_n = \underset{\text{width}}{\left(\frac{b-a}{n}\right)}\left[f\left(a + \frac{b-a}{n}\right) + f\left(a + \frac{2(b-a)}{n}\right) + \cdots + f\underbrace{\left(a + \frac{n(b-a)}{n}\right)}_{b} \right]$$



$$\boxed{\text{Area} \overset{\text{def}}{=} \lim_{n \to \infty} R_n \quad \text{under } f \text{ from } x = a \text{ to } x = b}$$

Can also write

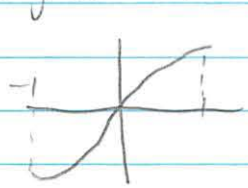
$$\boxed{\frac{b-a}{n} = \Delta x}$$

$$\boxed{f\left(a + \frac{b-a}{n}\right) = f(a + \Delta x)}$$

dB

do I really want to be sillicon valley insidor too

* signed area



= total = 0

area for $-1 \to 1$ under odd function = boring

Tomorrow: area problem related to antiderivitives

$$\lim_{n \to \infty} R_n \overset{\text{sum}}{\underset{\text{notation}}{=}} \lim_{n\to\infty} \sum_{i=1}^{n} \overset{\text{base}}{\Delta x} \cdot \overset{\text{height}}{f(x_i)}$$

more fancy way to represent
base · height
-hard to compute stuff

$$= \int_a^b f(x)\,dx$$

just notation → definite integral
-short hand for counting rectangles
as $\lim_{n \to \infty}$

Lecture 18: Definite Integrals

Integrals are used to calculate cumulative totals, averages, areas.

Area under a curve: (See Figure 1.)

1. Divide region into rectangles
2. Add up area of rectangles
3. Take limit as rectangles become thin

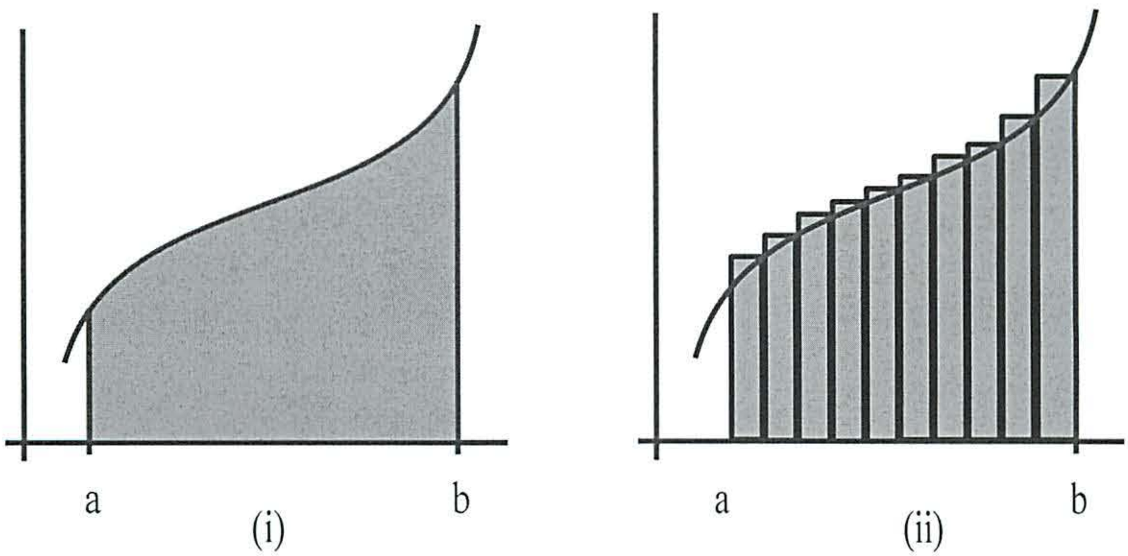


Figure 1: (i) Area under a curve; (ii) sum of areas under rectangles

Example 1. $f(x) = x^2$, $a = 0$, b arbitrary

1. Divide into n intervals
   Length $b/n =$ base of rectangle
2. Heights:
   - 1^{st}: $x = \frac{b}{n}$, height $= \left(\frac{b}{n}\right)^2$
   - 2^{nd}: $x = \frac{2b}{n}$, height $= \left(\frac{2b}{n}\right)^2$

Sum of areas of rectangles:

$$\left(\frac{b}{n}\right)\left(\frac{b}{n}\right)^2 + \left(\frac{b}{n}\right)\left(\frac{2b}{n}\right)^2 + \left(\frac{b}{n}\right)\left(\frac{3b}{n}\right)^2 + \cdots + \left(\frac{b}{n}\right)\left(\frac{nb}{n}\right)^2 = \frac{b^3}{n^3}(1^2 + 2^2 + 3^2 + \cdots + n^2)$$

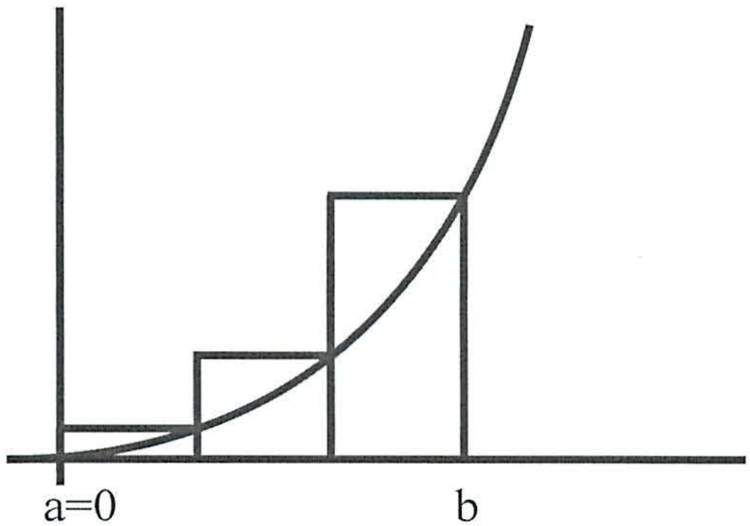


Figure 2: Area under $f(x) = x^2$ above $[0, b]$.

We will now estimate the sum using some 3-dimensional geometry.

Consider the staircase pyramid as pictured in Figure 3.

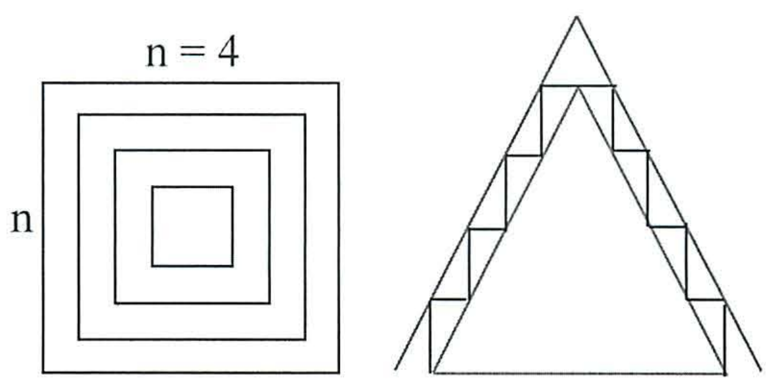


Figure 3: Staircase pyramid: left(top view) and right (side view)

1^{st} level: $n \times n$ bottom, represents volume n^2.
2^{nd} level: $(n-1) \times (n-1)$, represents volumne $(n-1)^2)$, etc.
Hence, the total volume of the staircase pyramid is $n^2 + (n-1)^2 + \cdots + 1$.

Next, the volume of the pyramid is greater than the volume of the inner prism:

$$1^2 + 2^2 + \cdots + n^2 > \frac{1}{3}(\text{base})(\text{height}) = \frac{1}{3}n^2 \cdot n = \frac{1}{3}n^3$$

and less than the volume of the outer prism:

$$1^2 + 2^2 + \cdots + n^2 < \frac{1}{3}(n+1)^2(n+1) = \frac{1}{3}(n+1)^3$$

In all,

$$\frac{1}{3} = \frac{\frac{1}{3}n^3}{n^3} < \frac{1^2 + 2^2 + \cdots + n^2}{n^3} < \frac{1}{3}\frac{(n+1)^3}{n^3}$$

Therefore,

$$\lim_{n\to\infty} \frac{b^3}{n^3}(1^2 + 2^2 + 3^2 + \cdots + n^2) = \frac{1}{3}b^3,$$

and the area under x^2 from 0 to b is $\frac{b^3}{3}$.

Example 2. $f(x) = x$; area under x above $[0, b]$. Reasoning similar to Example 1, but easier, gives a sum of areas:

$$\frac{b^2}{n^2}(1 + 2 + 3 + \cdots + n) \to \frac{1}{2}b^2 \quad (\text{as } n \to \infty)$$

This is the area of the triangle in Figure 4.

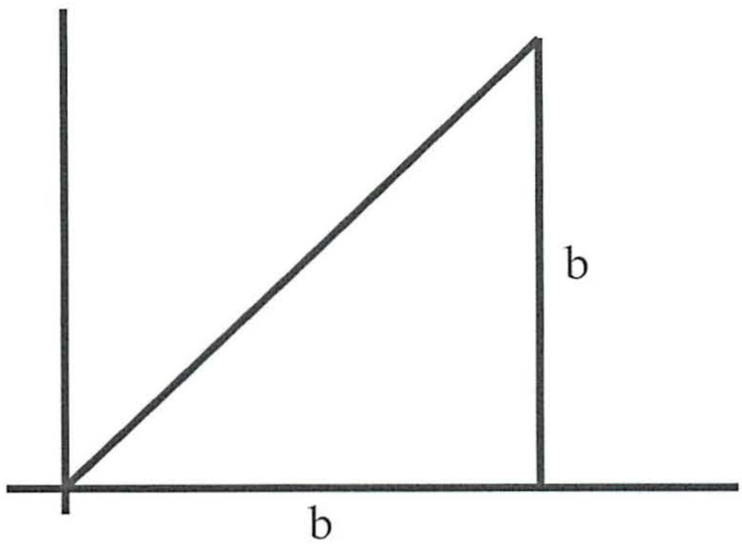


Figure 4: Area under $f(x) = x$ above $[0, b]$.

Pattern:

$$\frac{d}{db}\left(\frac{b^3}{3}\right) = b^2$$

$$\frac{d}{db}\left(\frac{b^2}{2}\right) = b$$

The area $A(b)$ under $f(x)$ should satisfy $A'(b) = f(b)$.

General Picture

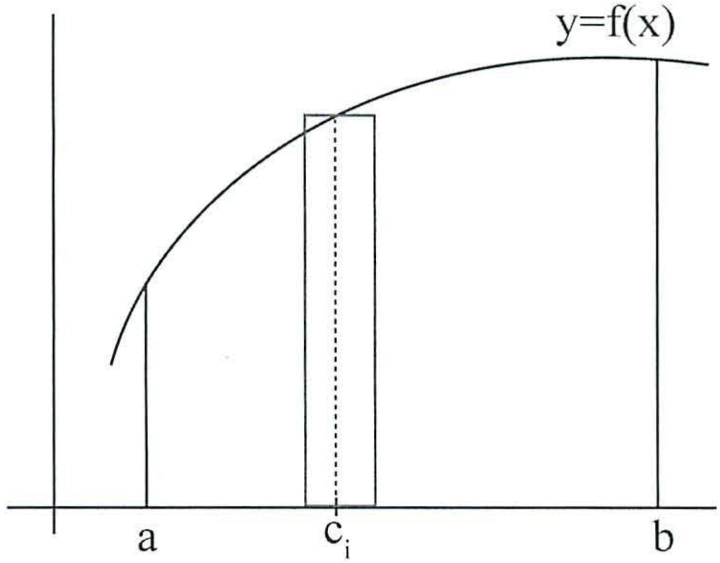


Figure 5: One rectangle from a Riemann Sum

- Divide into n equal pieces of length $= \Delta x = \dfrac{b-a}{n}$
- Pick any c_i in the interval; use $f(c_i)$ as the height of the rectangle
- Sum of areas: $f(c_1)\Delta x + f(c_2)\Delta x + \cdots + f(c_n)\Delta x$

In summation notation: $\displaystyle\sum_{i=1}^{n} f(c_i)\Delta x \quad \leftarrow$ called a *Riemann sum.*

Definition:

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x = \int_a^b f(x)dx \quad \leftarrow \text{ called a } \textit{definite integral}$$

This definite integral represents the area under the curve $y = f(x)$ above $[a, b]$.

Example 3. (Integrals applied to quantity besides area.) Student borrows from parents.
P = principal in dollars, t = time in years, r = interest rate (e.g., 6 % is $r = 0.06$/year).
After time t, you owe $P(1 + rt) = P + Prt$

The integral can be used to represent the total amount borrowed as follows. Consider a function $f(t)$, the "borrowing function" in dollars per year. For instance, if you borrow \$ 1000 /month, then $f(t) = 12,000$/year. Allow f to vary over time.

Say $\Delta t = 1/12$ year $= 1$ month.

$$t_i = i/12 \quad i = 1, \cdots, 12.$$

$f(t_i)$ is the borrowing rate during the i^{th} month so the amount borrowed is $f(t_i)\Delta t$. The total is

$$\sum_{i=1}^{12} f(t_i)\Delta t.$$

In the limit as $\Delta t \to 0$, we have

$$\int_0^1 f(t)dt$$

which represents the total borrowed in one year in dollars per year.

The integral can also be used to represent the total amount owed. The amount owed depends on the interest rate. You owe

$$f(t_i)(1 + r(1 - t_i))\Delta t$$

for the amount borrowed at time t_i. The total owed for borrowing at the end of the year is

$$\int_0^1 f(t)(1 + r(1 - t))dt$$

19 First Fundimental Theorm of Calculus

10/23

PSet 5 Big

One of the biggest ideas of last 1000 years

Yesterday = $\int_0^1 x^2 dx$ definite integral

$\uparrow \lim_{n \to \infty} R_n$ where R_n = Riemann sum using right hand end points defined as

$$R_n = \underbrace{\frac{1}{n}}^{\text{base}} \left(\left(\frac{1}{n}\right)^2 + \left(\frac{2}{n}\right)^2 + \ldots + \left(\frac{n}{n}\right)^2 \right)$$

- sum of areas of rectangles under the curve $y = x^2$ from $x = 0$ to $1 = \boxed{\frac{1}{3}}$

$\int_0^b x^2 \, dx$ ← not much worse

$$\lim_{n \to \infty} = \left(\frac{b}{n}\right)\left(0 + \left(\frac{b}{n}\right)^2 + \left(\frac{2b}{n}\right)^2 + \ldots + \left(\frac{bn}{n}\right)^2\right)$$

$= b^3 \cdot$ old sum (above)

$= \boxed{b^3/3}$

Antiderivitives

$\int x^2 \, dx$ "take the antiderivitive of x^2 with respect to x"

$\boxed{\frac{1}{3}x^3}$

* 2 big different meanings *

Definite integral computed for x^k w/ $k \leq q$
by Barrow.
↑ Newton's Teacher

Not so hard for limit def for other basic functions

e^x $\cos x$ $\sin x$
↑ pset   ↑ book

Few other properties definite integral

(0) When we write $\int_a^b f(x)\,dx$ assumbe $a<b$

But Riemann sum definition makes sense for any pair of a,b

When $a<b$: Δx base of rectangle $\frac{b-a}{n}$

Reverses roles of a,b

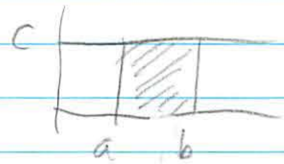
$$\frac{b-a}{n} \text{ becomes } \frac{a-b}{n} = -\frac{(b-a)}{n}$$

$$\boxed{\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx}$$

(0.1) $\boxed{\int_a^a f(x)\,dx = 0}$

① $\int_a^b c \, dx = c(b-a)$ easy just rectangle

c ↑ constant



② $\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

③ Same for subtraction

④ $\int_a^b c\, f(x)\,dx = c\int_a^b f(x)$ for any c

⑤ $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

Slightly more advanced

-suppose $f(x) \geq 0$ $(a \leq b)$

⑥ $\int_a^b f(x) \geq 0$

⑦ If $f(x) \geq g(x)$ $(a \leq b)$

then ⑥ w/ $(f-g) \rightarrow \int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$

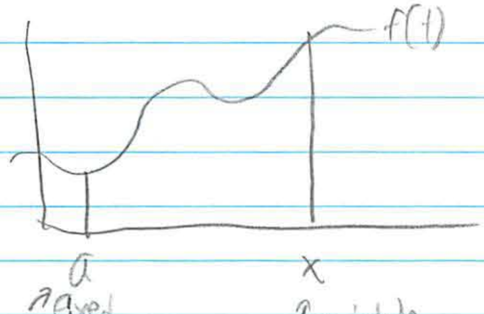
⑧ if m is minimum $f(x)$ on $[a,b] \leq f(x) \leq M$

max $[a,b]$

then $m(b-a) \leq \int_a^b f(x) \leq M(b-a)$

rectangle in between

Create a function out of definite integral



function of x: $\int_a^x f(t)\,dt$

eg $f(t) = \sqrt{1+t^2}$ $\qquad a=5$

$$\int_5^x \sqrt{1+t^2}\,dt$$

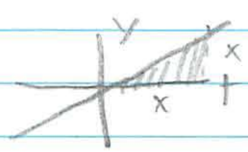
- give an x
- get a # back from 5 to x under curve = g(x)

$x \to g(x)$ = function

In example from before

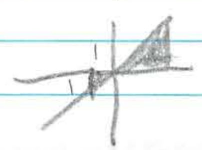
$$\int_0^x t^2\,dt = \frac{x^3}{3}$$

$\int_0^x t\,dt$



area right triangle
$a = \frac{1}{2}x^2$

$\int_{-1}^x t\,dt$
$x > 0$



$\int_{-1}^x t\,dt = \frac{1}{2}x^2 - \frac{1}{2}$

Suspect $\int_a^x f(t)\,dt$ is antiderivitive $f(x)$

↑ area problem connected to derivitive

- only determined up to a constant

Much more elegant

- take derivitive of above

$$\frac{d}{dx}\Big(\underbrace{\int_a^x f(t)\,dt}_{g(x)}\Big) = f(x)$$

↑ f contineous

$\boxed{\text{* Fundemental Theorm of Calculus [First]}}$

consider:

$$\lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

compare to $f(x)$

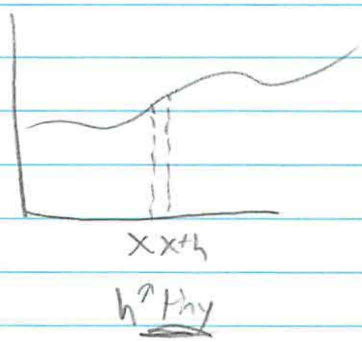
↓ plug in

$$= \lim_{h \to 0} \frac{\int_a^{x+h} f(x)\,dt - \int_a^{x} f(x)\,dt}{h}$$

$$= \frac{\int_x^{x+h} f(t)\,dt}{h}$$

↓ width = h

↓ height = roughly $f(x)$



$\sim f(x) \cdot h$

area of rectangle

gets better as $h \to 0$

$$\boxed{\approx \lim_{h \to 0} \frac{f(x) \cdot h}{h}}$$

Great intuition
Not satisfied
No wiggly lines in proof

$f(t)$ is contineous $[x, x+h]$

Then $f(t)$ attains a maximum (say at u_h) and minimum (at v_h) in $[x, x+h]$

h is changing
- might affect min + max

$$f(v_h) \cdot h \le \int_x^{x+h} f(t)\,dt \le f(u_h) \cdot h$$

depends on h

depends on h

Take lim as $h \to 0$

$$\underbrace{\lim_{h \to 0} f(v_h)}_{= f(x)} \le \underbrace{\lim_{h \to 0} \frac{\int_x^{x+h} f(t)\,dt}{h}}_{= g'(x)} \le \underbrace{\lim_{h \to 0} f(u_h)}_{= f(x) \text{ by similar}}$$

minimum on
interval $[x, x+h]$
as $h \to 0$
$v_h \to h$ and f is contineous
$f(v_h) \to f(x)$

shrinking
interval

By squeeze theorem middle limit $= f(x)$!

$$\frac{d}{dx}\left(\int_0^x \sqrt{1+t^2}\,dt\right) = \sqrt{1+x^2}$$

① Next week to prove part 2 Fund Theorm
When ? - if F is anti deriv of f
② then $\int_a^b f(t)\,dt = F(b) - F(a)$

"super interesting"

(2) Some functions defined implicitly via integrals

Fresnel P-set on Pset 5b

- integral on trig function

Lecture 19: First Fundamental Theorem of Calculus

Fundamental Theorem of Calculus (FTC 1)

If $f(x)$ is continuous and $F'(x) = f(x)$, then
$$\int_a^b f(x)dx = F(b) - F(a)$$

Notation: $F(x)\Big|_a^b = F(x)\Big|_{x=a}^{x=b} = F(b) - F(a)$

Example 1. $F(x) = \frac{x^3}{3}$, $F'(x) = x^2$; $\int_a^b x^2 dx = \frac{x^3}{3}\Big|_a^b = \frac{b^3}{3} - \frac{a^3}{3}$

Example 2. Area under one hump of $\sin x$ (See Figure 1.)

$$\int_0^\pi \sin x\, dx = -\cos x\Big|_0^\pi = -\cos\pi - (-\cos 0) = -(-1) - (-1) = 2$$

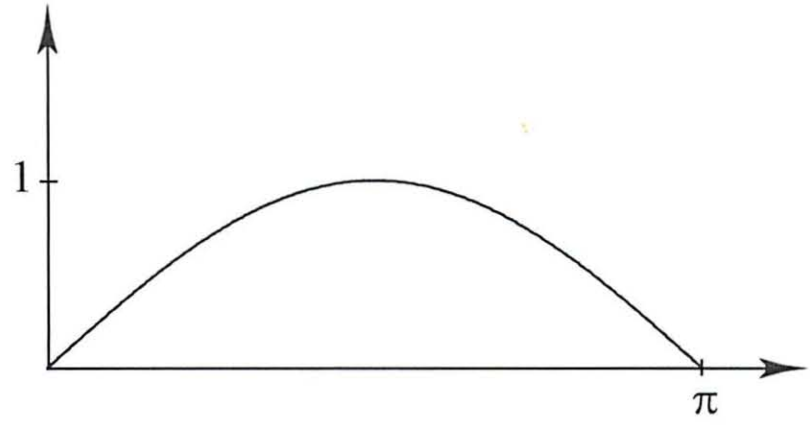


Figure 1: Graph of $f(x) = \sin x$ for $0 \le x \le \pi$.

Example 3. $\int_0^1 x^5 dx = \frac{x^6}{6}\Big|_0^1 = \frac{1}{6} - 0 = \frac{1}{6}$

Intuitive Interpretation of FTC:

$x(t)$ is a position; $v(t) = x'(t) = \dfrac{dx}{dt}$ is the speed or rate of change of x.

$$\int_a^b v(t)dt = x(b) - x(a) \qquad \text{(FTC 1)}$$

R.H.S. is how far $x(t)$ went from time $t = a$ to time $t = b$ (difference between two odometer readings). L.H.S. represents speedometer readings.

$$\sum_{i=1}^{n} v(t_i)\Delta t \quad \text{approximates the sum of distances traveled over times } \Delta t$$

The approximation above is accurate if $v(t)$ is close to $v(t_i)$ on the i^{th} interval. The interpretation of $x(t)$ as an odometer reading is no longer valid if v changes sign. Imagine a round trip so that $x(b) - x(a) = 0$. Then the positive and negative velocities $v(t)$ cancel each other, whereas an odometer would measure the total distance not the net distance traveled.

Example 4. $\displaystyle\int_0^{2\pi} \sin x\,dx = -\cos x\Big|_0^{2\pi} = -\cos 2\pi - (-\cos 0) = 0.$
The integral represents the sum of areas under the curve, above the x-axis minus the areas below the x-axis. (See Figure 2.)

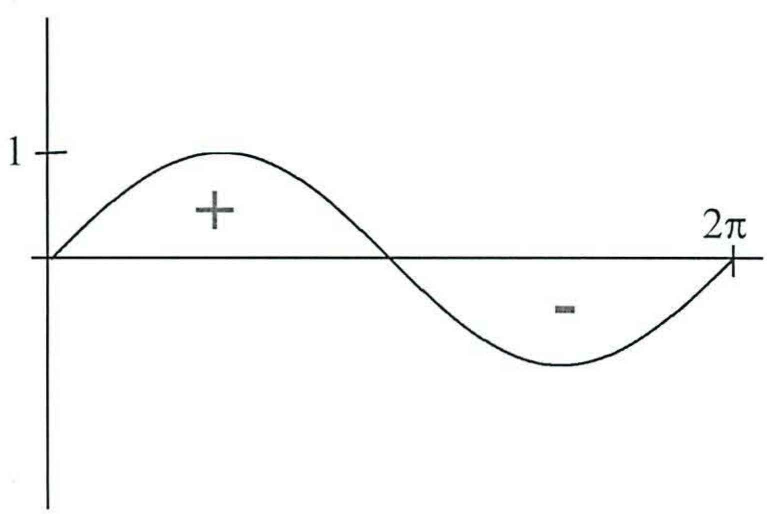


Figure 2: Graph of $f(x) = \sin x$ for $0 \le x \le 2\pi$.

Integrals have an important additive property (See Figure 3.)

$$\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$$

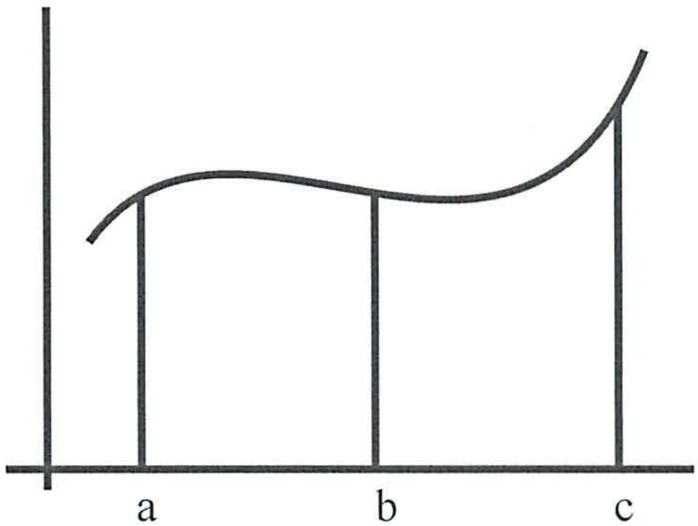


Figure 3: Illustration of the additive property of integrals

New Definition:

$$\int_b^a f(x)dx = -\int_a^b f(x)dx$$

This definition is used so that the fundamental theorem is valid no matter if $a < b$ or $b < a$. It also makes it so that the additive property works for a, b, c in any order, not just the one pictured in Figure 3.

Estimation:

If $f(x) \leq g(x)$, then $\int_a^b f(x)dx \leq \int_a^b g(x)dx$ (only if $a < b$)

Example 5. Estimation of e^x
Since $1 \leq e^x$ for $x \geq 0$,

$$\int_0^1 1dx \leq \int_0^1 e^x dx$$

$$\int_0^1 e^x dx = e^x \Big|_0^1 = e^1 - e^0 = e - 1$$

Thus $1 \leq e - 1$, or $e \geq 2$.

Example 6. We showed earlier that $1 + x \leq e^x$. It follows that

$$\int_0^1 (1+x)dx \leq \int_0^1 e^x dx = e - 1$$

$$\int_0^1 (1+x)dx = \left(x + \frac{x^2}{2}\right)\Big|_0^1 = \frac{3}{2}$$

Hence, $\frac{3}{2} \leq e - 1$,or, $e \geq \frac{5}{2}$.

Change of Variable:

If $f(x) = g(u(x))$, then we write $du = u'(x)dx$ and

$$\int g(u)du = \int g(u(x))u'(x)dx = \int f(x)u'(x)dx \qquad \text{(indefinite integrals)}$$

For definite integrals:

$$\int_{x_1}^{x_2} f(x)u'(x)dx = \int_{u_1}^{u_2} g(u)du \quad \text{where } u_1 = u(x_1),\ u_2 = u(x_2)$$

Example 7. $\int_1^2 \left(x^3+2\right)^4 x^2 dx$

$$\text{Let } u = x^3 + 2. \quad \text{Then } du = 3x^2 dx \implies x^2 dx = \frac{du}{3};$$

$$x_1 = 1, x_2 = 2 \implies u_1 = 1^3 + 2 = 3,\ u_2 = 2^3 + 2 = 10, \text{ and}$$

$$\int_1^2 \left(x^3+2\right)^4 x^2 dx = \int_3^{10} u^4 \frac{du}{3} = \frac{u^5}{15}\Big|_3^{10} = \frac{10^5 - 3^5}{15}$$

office hrs
3-4

Recitation

10/26

$\int f(x)\,dx$ find $F(x)$ so that $F'(x) = f(x)$

↑ sometimes can not find $F(x)$
the antiderivative
- exists - but
not an easy form

$\int_a^b f(x)\,dx$ means signed area
$\cancel{=}$

Next hr
- ML
- give cable + paper
- scan
- call MEAP

Riemann sum - take partition of interval $[a,b]$
look at rectangles associated to that partition
- width $= \frac{b-a}{n}$
- heigh can choose left, right, upper, lower

← highest pt
does not have
to be right or left

- grow together
as $n \to \infty$

$J(x) = \begin{cases} 0 & \text{if } x \text{ rational} \\ 1 & \text{if } x \text{ irrational} \end{cases}$

<u>Weird example</u>
pathlogically discontinuous

$\int_0^1 J(x)\,dx = \begin{cases} 0 \\ 1 \end{cases}$ ← so Riemann does not work

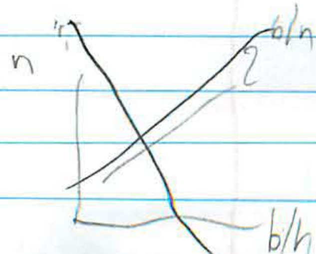
difference between $= \frac{1}{n}$ ← as $n \to \infty$ goes to 0

<u>Example</u>

evaluate $\lim_{n \to \infty} \frac{b}{n}\left(2^{b/n} + 2^{2b/n} + \dots + 2^{nb/n}\right)$

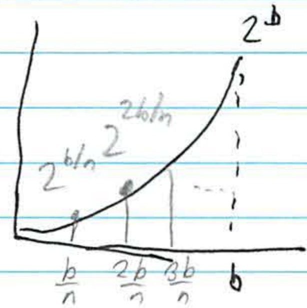
- doing a shorter way on wed
- longer way today

n factored out with
n figure out each height



$$\int_0^b 2^x \, dx$$

∴ Have trouble finding pieces - study more



↑ pieces getting smaller as $n \nearrow \infty$
<u>Not</u> pushing b out

Can write n in parentheses

* geometric - increasing by multiplicitive sum

$$\sum_{k=0}^{n-1} a \cdot r^k$$

a = 1st term
r = ratio

↑ messy

↓ cleaner

$$a\left(\frac{1-r^n}{1-r}\right)$$

← n is some fixed #
← look like summation
↑ did for k^2

* when $n \to \infty$ no longer approx

$$\lim_{n \to \infty} \frac{b}{n} \cdot \overset{a}{2^{b/n}} \left(\frac{1-2^{\frac{bn}{n}}}{1-2^{b/n}}\right)$$

$$\lim_{n \to \infty} \frac{b}{n} \cdot 2^{b/n} \left(\frac{1-2^b}{1-2^{b/n}}\right)$$

← fixed

↑ goes to $2^0 = 1$

don't need to consider

headed towards $\infty \cdot 0$
- no clue immediatly

Examine $\lim_{n \to \infty} \frac{n}{b}\left(1 - 2^{b/n}\right)$ let $\frac{b}{n} = h$

$$\lim_{\substack{h \to 0 \\ \frac{b}{\infty} = 0}} \frac{1}{h}\left(1 - 2^{h}\right) = \lim_{h \to 0} \frac{-\left(2^{h} - 1\right)}{h}$$

$$= -\lim_{h \to 0} \frac{2^{h+0} - 2^{0}}{h}$$

deriv 2^x at $x = 0$

3 weeks ago →

$$\frac{d}{dx}\left(2^{x}\right)\Big|_{x=0} = 2^{x} \ln 2 \Big|_{x=0}$$
$$= \ln 2$$

Back to function

$$\frac{1 - 2^{b}}{-\ln 2} = \frac{2^{b} - 1}{\ln 2}$$

1st Fundemental Theorm Of Calculus

if $f(x)$ - something else happens
height rectangle

Define $g(x) = \int_{0}^{x} f(t)\, dt$

then $g'(x) = f(x)$

is a theorm

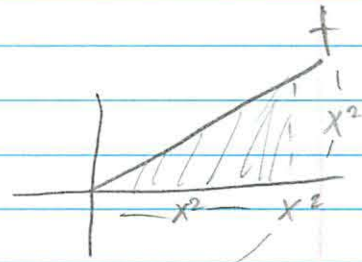
"Definitional thing"

$G(x) = \int f(x)\, dx$
if
$G'(x) = f(x)$

- find antiderivative

Exalme

$$g(x) = \int_0^{x^2} t\,dt$$



bound related to what you pick

x = bound changes

varrying only bound

$$g(x) = \frac{x^4}{2}$$

$$g'(x) = 2x^3$$ ← something more comple

* Can not only use 1st fundemental theorm
prove next the

Lecture 20
2nd Fundemental Theorm

10/27

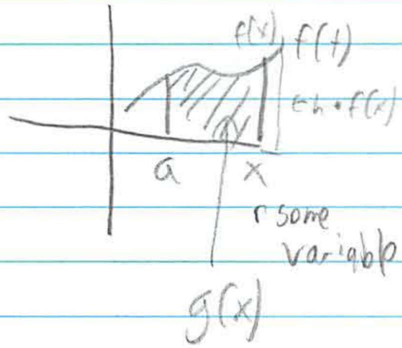
yesterday

why does switch from t to x? does that means anything

$f \to$ continuous

$$\frac{d}{dx}\underbrace{\int_a^x f(t)\,dt}_{g(x)} = f(x)$$

$g(x)$ is an antiderivitive of $f(x)$



$$\frac{g(x+h)-g(x)}{h} \approx f(x)$$

Becomes exact as $h \to 0$

Did examples in recitation

$$\frac{d}{dx}\int_1^{x^2} \sin t\,dt$$

$x^2 \leftarrow$ now a function of x

$u(x) = x^2$

$$\int_a^{u^2} \sin(t)\,dt$$

must use chain rule
Use #1 FCT

$$\boxed{\frac{d}{du}\int_a^u \sin t\,dt = \sin u}$$

$\downarrow$

$$\frac{dg}{du} \cdot \frac{du}{dx} = \sin(u(x)) \cdot u'(x)$$

$$= \sin x^2 \, 2x$$

Try $\frac{d}{dx}\int_0^{x^2} t\,dt$ ← think did in recitation

area triangle

x and t just 2 different variables

$$\frac{x^4}{2}$$ $\boxed{\text{not the same}}$

$$\frac{d}{dx} = \frac{4x^3}{2} \to 2x^3$$

Important functions defined this way

$$\int_1^x \frac{1}{t}\,dt = \ln(x)$$

← the definition of

$$\frac{d}{dx}(\ln x) = \frac{d}{dx}\int_1^x \frac{1}{t}\,dt = \frac{1}{x}$$

↑ just like old definition for $\ln(x)$ as inverse function e^x

$$\ln(1) = \int_1^1 \frac{1}{t}\,dt = 0$$

2 definitions are the same

Prove properties of $\ln(x)$ in new way

$$\frac{d}{dx} \ln(\underset{\uparrow \text{constant}}{a}x) = \underbrace{\frac{d}{dx} \int_1^{ax} \frac{1}{t}\,dt = \frac{1}{ax}}_{\text{chain rule}} \cdot \underset{\uparrow \text{derivitive of } ax}{a} = \frac{1}{x}$$

$\ln(ax)$ and $\ln(x)$ are both antiderivitives of $\frac{1}{x}$

$$\ln(ax) = \ln(x) + C$$

- used MVT to prove

take $x = 1$

$$\ln(a) = \underset{\uparrow 0}{\ln(1)} + \underset{\uparrow \text{so is } \ln(a)}{C}$$

$$\ln(ax) = \ln(x) + \ln(a)$$

- proved a property of log
- can prove others through similar process

People were obsessed w/ functions obtained through integration

$$\int_1^x \frac{1}{\sqrt{1-t^2}}\,dt \rightarrow \sin^{-1}(x)$$

$$\text{erf}(x) = 2\int_0^x e^{-t^2/2}\,dt \leftarrow \text{gaussian distribution}$$

$$\ln(x) = \int_2^x \frac{dt}{\ln 2} \leftarrow \text{count \# prime \#} \leq x \text{ for large } x$$

- has some errors

$si(x) = \int_0^x \frac{\sin t}{t}\, dt$ signal processing

$Fres(x) = \int_0^x \sin(t^2)\, dt$ optics

tonight
red pen
MP
Prelab
Study Chem
Suma 2nd

- no nice way to understand
- must call it a new way

FTC Part 1 used to Prove FCT Part 2

f continuous on $[a, b]$
then $\int_a^b f(t)\, dt = F(b) - F(a)$

where F is any antiderivite

eg $\int_0^1 x^3 dx$ - before Rieman sums

anti deriv
$\frac{1}{4} x^4$

$= \frac{1}{4}(1)^4 - \frac{1}{4}(0)^4$

no more Rieeman sums
if can find easy antiderivite
↑ oh so see how integration + anti deriv diff.

only need to know antideriv at endpoints

to compute area $a \to b$

Prove #2 w/ FCT 1

*perfect exam qu

$g(x) = \int_a^x f(t)\,dt$ is an antiderivive

if F is also antiderivitive of f then differ by a constant

← only value we know $\int_a^a = \int_1^1 = 0$

$g(a) = 0 = F(a) + C$

↑ $C = -F(a)$ ← can solve for c

$g(x) = F(x) - F(a)$

- in particular $g(b) = \int_a^b f(t)\,dt = F(b) - F(a)$

↑ FTC part 2

$F(x) \Big|_{x=a}^{x=b}$ shorthand notation

eg

$$\int_1^4 x(x^2-3)^3$$

↑ substitution

$u(x) = x^2 - 3$

$u'(x) = \frac{du}{dx} = 2x \; dx$ ↑ write to remind

$\int u^3$ $\quad du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$\int \frac{1}{2}u^3\,du$

- when we change variable, need to change bounds

$u(x) = x^2 - 3$

$x = 1 \quad u(x) = -2 \qquad x = 4 \quad u(x) = 13$

$$\int_{-2}^{13} \frac{1}{2} u^3 \, du$$

-change bounds
-forget about old problem

↑ now do this problem

$$= \frac{1}{2}\left(\frac{1}{4}u^4\right)\Big|_{-2}^{13}$$

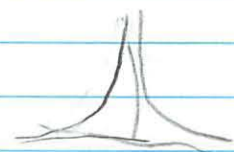
$$= \frac{1}{8}(13)^4 - \frac{1}{8}(-2)^4$$

then solve

Other way to solve

- don't do it
  - just change bounds

$$\int_{-2}^{3} \frac{1}{x^2}\, dx$$



$$-\frac{1}{x}\Big|_{-2}^{3}$$

Wrong - not continuous at $x=0$
- and always positive

$$-\frac{1}{3} - \left(\frac{-1}{-2}\right) = \frac{-5}{6}$$

Can't use FCT

Tomorrow: Lots more pratice definite integrals

Lecture 20: Second Fundamental Theorem

Recall: First Fundamental Theorem of Calculus (FTC 1)

$$\boxed{\text{If } f \text{ is continuous and } F' = f, \text{ then } \int_a^b f(x)dx = F(b) - F(a)}$$

We can also write that as

$$\int_a^b f(x)dx = \int f(x)dx\Big|_{x=a}^{x=b}$$

Do all continuous functions have antiderivatives? Yes. However...
What about a function like this?

$$\int e^{-x^2}dx = ??$$

Yes, this antiderivative exists. No, it's not a function we've met before: it's a new function.

The new function is defined as an integral:

$$F(x) = \int_0^x e^{-t^2}dt$$

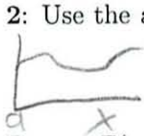
It will have the property that $F'(x) = e^{-x^2}$.

Other new functions include antiderivatives of $e^{-x^2}, x^{1/2}e^{-x^2}, \frac{\sin x}{x}, \sin(x^2), \cos(x^2), \ldots$

Second Fundamental Theorem of Calculus (FTC 2)

$$\boxed{\text{If } F(x) = \int_a^x f(t)dt \text{ and } f \text{ is continuous, then } F'(x) = f(x)}$$

Geometric Proof of FTC 2: Use the area interpretation: $F(x)$ equals the area under the curve between a and x.



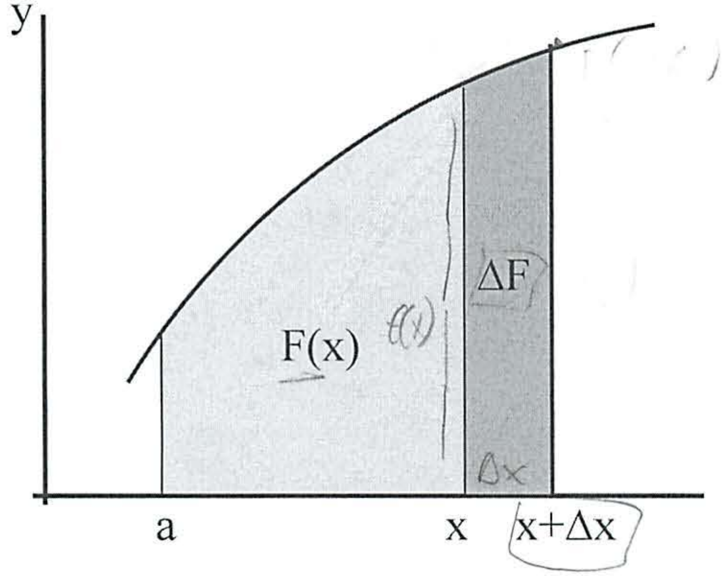
$$\begin{aligned}
\Delta F &= F(x+\Delta x) - F(x) \\
\Delta F &\approx (\text{base})(\text{height}) \approx (\Delta x)f(x) \qquad \text{(See Figure 1.)} \\
\frac{\Delta F}{\Delta x} &\approx f(x) \\
\text{Hence } \boxed{\lim_{\Delta x \to 0}\frac{\Delta F}{\Delta x} = f(x)}
\end{aligned}$$

← ~~part before it~~ , ~~diff b/w~~ 2 lines

Chart next page

But, by the definition of the derivative:

$$\lim_{\Delta x \to 0}\frac{\Delta F}{\Delta x} = F'(x)$$ ← so what does that prove?



f = point
F = area

Figure 1: Geometric Proof of FTC 2.

Therefore,

$$F'(x) = f(x)$$ I thought we assumed that earlier

Another way to prove FTC 2 is as follows:

dark – lighter

$$\begin{aligned}\frac{\Delta F}{\Delta x} &= \frac{1}{\Delta x}\left[\int_a^{x+\Delta x} f(t)dt - \int_a^x f(t)dt\right] \\ &= \frac{1}{\Delta x}\int_x^{x+\Delta x} f(t)dt \quad \text{(which is the "average value" of } f \text{ on the interval } x \le t \le x+\Delta x.)\end{aligned}$$

As the length Δx of the interval tends to 0, this average tends to $f(x)$.

Proof of FTC 1 (using FTC 2)

Start with $F' = f$ (we assume that f is continuous). Next, define $G(x) = \int_a^x f(t)dt$. By FTC2, $G'(x) = f(x)$. Therefore, $(F-G)' = F' - G' = f - f = 0$. Thus, $F - G =$ constant. (Recall we used the Mean Value Theorem to show this).

Hence, $F(x) = G(x) + c$. Finally since $G(a) = 0$,

$$\int_a^b f(t)dt = G(b) = G(b) - G(a) = [F(b) - c] - [F(a) - c] = F(b) - F(a)$$

which is FTC 1.

Remark. In the preceding proof G was a definite integral and F could be any antiderivative. Let us illustrate with the example $f(x) = \sin x$. Taking $a = 0$ in the proof of FTC 1,

$$G(x) = \int_0^x \cos t\, dt = \sin t\Big|_0^x = \sin x \quad \text{and } G(0) = 0.$$

don't forget to integrate :)

If, for example, $F(x) = \sin x + 21$. Then $F'(x) = \cos x$ and

$$\int_a^b \sin x \, dx = F(b) - F(a) = (\sin b + 21) - (\sin a + 21) = \sin b - \sin a$$

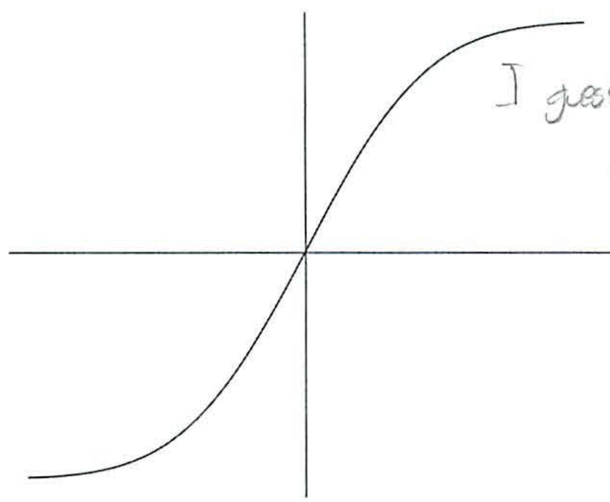
Every function of the form $F(x) = G(x) + c$ works in FTC 1.

Examples of "new" functions

The *error function*, which is often used in statistics and probability, is defined as

$$\begin{aligned} \operatorname{erf}(x) &= \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \\ \text{and } \lim_{x\to\infty} \operatorname{erf}(x) &= 1 \quad \text{(See Figure 2)} \end{aligned}$$

the erf



I guess how important an error is

Figure 2: Graph of the error function.

Another "new" function of this type, called the *logarithmic integral*, is defined as

$$\operatorname{Li}(x) = \int_2^x \frac{dt}{\ln t}$$

This function gives the approximate number of prime numbers less than x. A common encryption technique involves encoding sensitive information like your bank account number so that it can be sent over an insecure communication channel. The message can only be decoded using a secret prime number. To know how safe the secret is, a cryptographer needs to know roughly how many 200-digit primes there are. You can find out by estimating the following integral:

$$\int_{10^{200}}^{10^{201}} \frac{dt}{\ln t}$$

We know that

$$\ln 10^{200} = 200 \ln(10) \approx 200(2.3) = 460 \quad \text{and} \quad \ln 10^{201} = 201 \ln(10) \approx 462$$

We will approximate to one significant figure: $\ln t \approx 500$ for $200 \le t \le 10^{201}$.

With all of that in mind, the number of 200-digit primes is roughly [1]

$$\int_{10^{200}}^{10^{201}} \frac{dt}{\ln t} \approx \int_{10^{200}}^{10^{201}} \frac{dt}{500} = \frac{1}{500}\left(10^{201} - 10^{200}\right) \approx \frac{9 \cdot 10^{200}}{500} \approx 10^{198}$$

There are LOTS of 200-digit primes. The odds of some hacker finding the 200-digit prime required to break into your bank account number are very very slim.

Another set of "new" functions are the Fresnel functions, which arise in optics:

$$C(x) = \int_0^x \cos(t^2)dt$$
$$S(x) = \int_0^x \sin(t^2)dt$$

Bessel functions often arise in problems with circular symmetry:

$$J_0(x) = \frac{1}{2\pi}\int_0^\pi \cos(x\sin\theta)d\theta$$

On the homework, you are asked to find $C'(x)$. That's easy!

$$C'(x) = \cos(x^2)$$

We will use FTC 2 to discuss the function $L(x) = \int_1^x \frac{dt}{t}$ from first principles next lecture.

[1] The middle equality in this approximation is a very basic and useful fact

$$\int_a^b c\,dx = c(b-a)$$

Think of this as finding the area of a rectangle with base $(b-a)$ and height c. In the computation above, $a = 10^{200}, b = 10^{201}, c = \frac{1}{500}$

Lecture 21: Applications to Logarithms and Geometry

Application of FTC 2 to Logarithms

The integral definition of functions like $C(x)$, $S(x)$ of Fresnel makes them nearly as easy to use as elementary functions. It is possible to draw their graphs and tabulate values. You are asked to carry out an example or two of this on your problem set. To get used to using definite integrals and FTC2, we will discuss in detail the simplest integral that gives rise to a relatively new function, namely the logarithm.

Recall that

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c$$

except when $n = -1$. It follows that the antiderivative of $1/x$ is not a power, but something else. So let us define a function $L(x)$ by

$$L(x) = \int_1^x \frac{dt}{t}$$

(This function turns out to be the logarithm. But recall that our approach to the logarithm was fairly involved. We first analyzed a^x, and then defined the number e, and finally defined the logarithm as the inverse function to e^x. The direct approach using this integral formula will be easier.)

All the basic properties of $L(x)$ follow directly from its definition. Note that $L(x)$ is defined for $0 < x < \infty$. (We will not extend the definition past $x = 0$ because $1/t$ is infinite at $t = 0$.) Next, the fundamental theorem of calculus (FTC2) implies

$$L'(x) = \frac{1}{x}$$

Also, because we have started the integration with lower limit 1, we see that

$$L(1) = \int_1^1 \frac{dt}{t} = 0$$

Thus L is increasing and crosses the x-axis at $x = 1$: $L(x) < 0$ for $0 < x < 1$ and $L(x) > 0$ for $x > 1$. Differentiating a second time,

$$L''(x) = -1/x^2$$

It follows that L is concave down.

The key property of $L(x)$ (showing that it is, indeed, a logarithm) is that it converts multiplication into addition:

Claim 1. $L(ab) = L(a) + L(b)$

Proof: By definition of $L(ab)$ and $L(a)$,

$$L(ab) = \int_1^{ab} \frac{dt}{t} = \int_1^a \frac{dt}{t} + \int_a^{ab} \frac{dt}{t} = L(a) + \int_a^{ab} \frac{dt}{t}$$

To handle $\int_a^{ab} \frac{dt}{t}$, make the substitution $t = au$. Then

$$dt = adu; \quad a < t < ab \implies 1 < u < b$$

Therefore,

$$\int_a^{ab} \frac{dt}{t} = \int_{u=1}^{u=b} \frac{adu}{au} = \int_1^b \frac{du}{u} = L(b)$$

This confirms $L(ab) = L(a) + L(b)$.

Two more properties, the end values, complete the general picture of the graph.

Claim 2. $L(x) \to \infty$ as $x \to \infty$.

Proof: It suffices to show that $L(2^n) \to \infty$ as $n \to \infty$, because the fact that L is increasing fills in all the values in between the powers of 2.

$$\begin{aligned} L(2^n) &= L(2 \cdot 2^{n-1}) = L(2) + L(2^{n-1}) \\ &= L(2) + L(2) + L(2^{n-2}) = L(2) + L(2) + \cdots + L(2) \quad (n \text{ times}) \end{aligned}$$

Consequently, $L(2^n) = nL(2) \to \infty$ as $n \to \infty$. (In more familiar notation, $\ln 2^n = n \ln 2$.)

Claim 3. $L(x) \to -\infty$ as $x \to 0^+$.

Proof: $0 = L(1) = L\left(x \cdot \frac{1}{x}\right) = L(x) + L(1/x) \implies L(x) = -L(1/x)$. As $x \to 0^+$, $1/x \to +\infty$, so Claim 2 implies $L(1/x) \to \infty$. Hence

$$L(x) = -L(1/x) \to -\infty, \quad \text{as } x \to 0^+$$

Thus $L(x)$, defined on $0 < x < \infty$ increases from $-\infty$ to ∞, crossing the x-axis at $x = 1$. It is concave down and its graph can be drawn as in Fig. 1.

This provides an alternative to our previous approach to the exponential and log functions. Starting from $L(x)$, we can *define* the log function by $\ln x = L(x)$, *define* e as the number such that $L(e) = 1$, *define* e^x as the inverse function of $L(x)$, and *define* $a^x = e^{xL(a)}$.

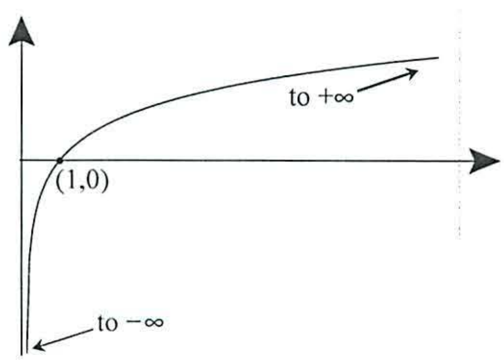


Figure 1: Graph of $y = \ln(x)$.

Application of FTCs to Geometry (Volumes and Areas)

1. Areas between two curves

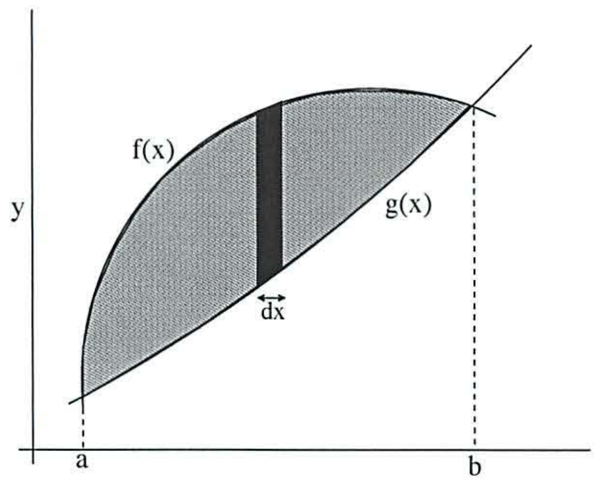


Figure 2: Finding the area between two curves.

Refer to Figure 2. Find the crossing points a and b. The area, A, between the curves is

$$A = \int_a^b (f(x) - g(x))\, dx$$

Example 1. Find the area in the region between $x = y^2$ and $y = x - 2$.

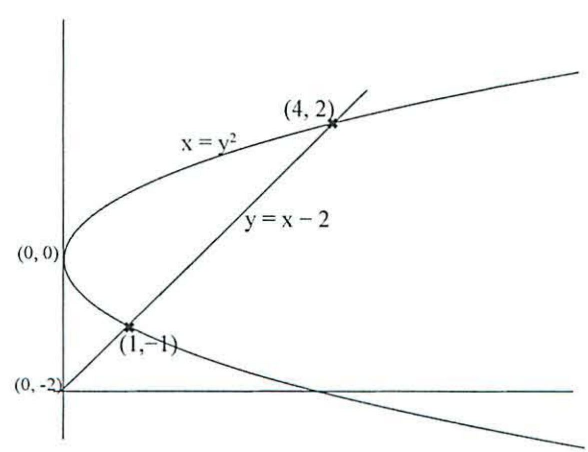


Figure 3: The intersection of $x = y^2$ and $y = x - 2$.

First, graph these functions and find the crossing points (see Figure 3).

$$\begin{aligned} y+2 &= x = y^2 \\ y^2 - y - 2 &= 0 \\ (y-2)(y+1) &= 0 \end{aligned}$$

Crossing points at $y = -1, 2$. Plug these back in to find the associated x values, $x = 1$ and $x = 4$. Thus the curves meet at $(1, -1)$ and $(4, 2)$ (see Figure 3).

There are two ways of finding the area between these two curves, a hard way and an easy way.

Hard Way: Vertical Slices
If we slice the region between the two curves vertically, we need to consider two different regions.

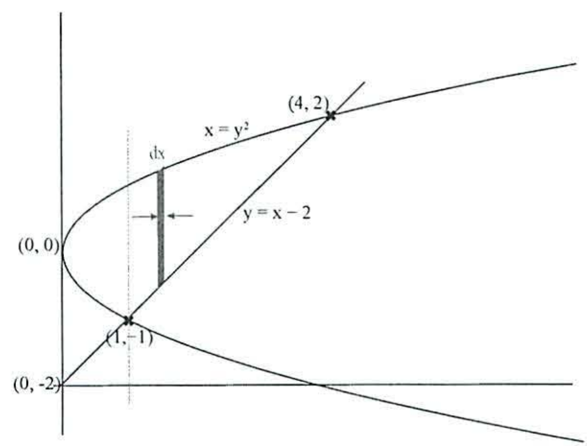


Figure 4: The intersection of $x = y^2$ and $y = x - 2$.

Where $x > 1$, the region's lower bound is the straight line. For $x < 1$, however, the region's lower bound is the lower half of the sideways parabola. We find the area, A, between the two curves by integrating the difference between the top curve and the bottom curve in each region:

$$A = \int_0^1 \{\sqrt{x} - (-\sqrt{x})\}\, dx + \int_1^4 \{\sqrt{x} - (x-2)\}\, dx = \int (y_{top} - y_{bottom})\, dx$$

break into parts

Easy Way: Horizontal Slices
Here, instead of subtracting the bottom curve from the top curve, we subtract the right curve from the left one.

$$A = \int (x_{\text{left}} - x_{\text{right}})\, dy = \int_{y=-1}^{y=2} [(y+2) - y^2]\, dx = \left(\frac{y^2}{2} + 2y + \frac{-y^3}{3}\right)\Big|_{-1}^{2} = \frac{4}{2} + 4 - \frac{8}{3} - \left(\frac{1}{2} - 2 + \frac{1}{3}\right) = \frac{9}{2}$$

↑, for some have to change equation - not worth it, right

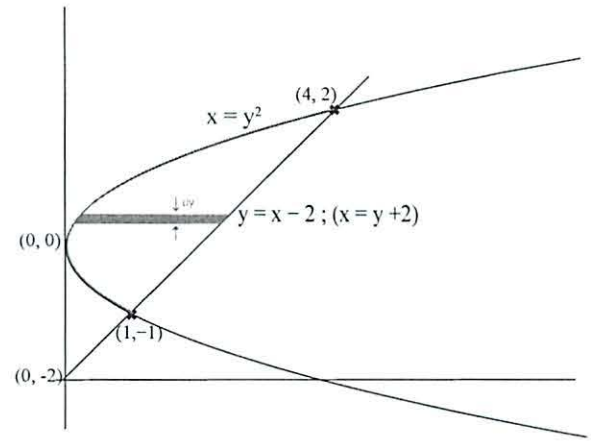


Figure 5: The intersection of $x = y^2$ and $y = x - 2$.

2. Volumes of solids of revolution

Rotate $f(x)$ about the x-axis, coming out of the page, to get:

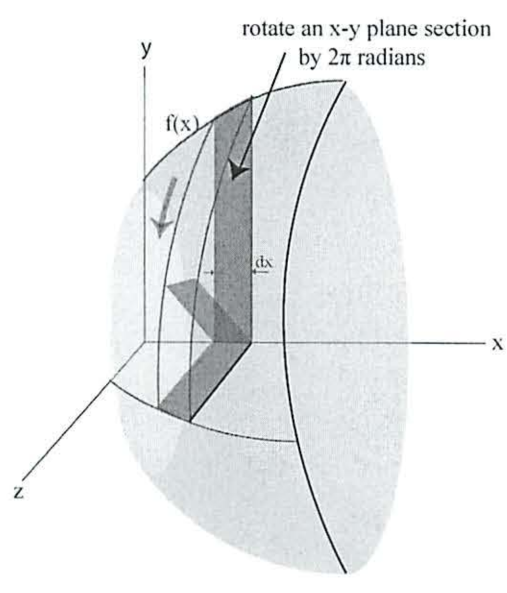


Figure 6: A solid of revolution: the purple slice is rotated by $\pi/4$ and $\pi/2$.

We want to figure out the volume of a "slice" of that solid. We can approximate each slice as a *disk* with width dx, radius y, and a cross-sectional area of πy^2. The volume of one slice is then:

$$dV = \pi y^2 dx \quad \text{(for a solid of revolution around the } x\text{-axis)}$$

Integrate with respect to x to find the total volume of the solid of revolution.

Example 2. Find the volume of a ball of radius a.

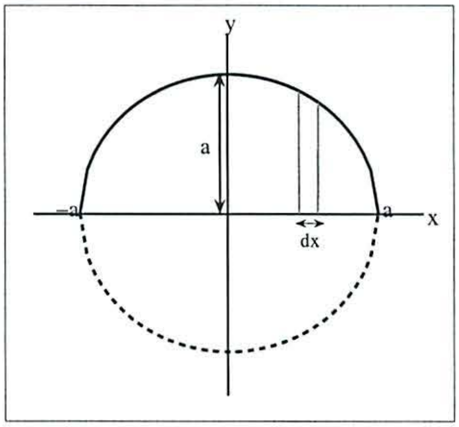


Figure 7: A ball of radius a

The equation for the upper half of the circle is

$$y = \sqrt{a^2 - x^2}.$$

If we spin the upper part of the curve about the x-axis, we get a ball of radius a. Notice that x ranges from $-a$ to $+a$. Putting all this together, we find

$$V = \int \pi y^2 dx = \int_{x=-a}^{x=a} \pi(a^2 - x^2)dx = \left(\pi a^2 x - \frac{\pi x^3}{3}\right)\Big|_{-a}^{a} = \frac{2}{3}\pi a^3 - \left(-\frac{2}{3}\pi a^3\right) = \frac{4}{3}\pi a^3$$

One can often exploit symmetry to further simplify these types of problems. In the problem above, for example, notice that the curve is symmetric about the y-axis. Therefore,

$$V = \int_{-a}^{a} \pi(a^2 - x^2)dx = 2\int_0^a \pi(a^2 - x^2)dx = 2\left(\pi a^2 x - \frac{x^3}{3}\right)\Big|_0^a$$

(The savings is that zero is an easier lower limit to work with than $-a$.) We get the same answer:

$$V = 2\left(\pi a^2 x - \frac{x^3}{3}\right)\Big|_0^a = 2\left(\pi a^3 - \frac{\pi}{3}a^3\right) = \frac{4}{3}\pi a^3$$

Skipped Lecture

Lecture 22: Volumes by Disks and Shells

Disks and Shells

We will illustrate the 2 methods of finding volume through an example.

Example 1. A witch's cauldron

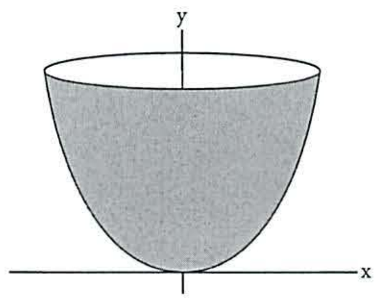


Figure 1: $y = x^2$ rotated around the y-axis.

Method 1: Disks

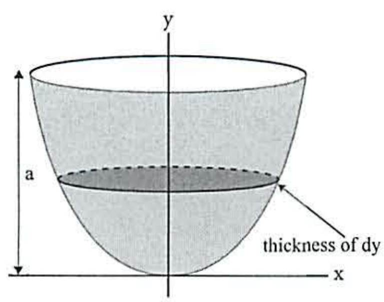


Figure 2: Volume by Disks for the Witch's Cauldron problem.

The area of the disk in Figure 2 is πx^2. The disk has thickness dy and volume $dV = \pi x^2 dy$. The volume V of the cauldron is

$$V = \int_0^a \pi x^2\, dy \quad (\text{substitute} \quad y = x^2)$$

$$V = \int_0^a \pi y\, dy = \pi \frac{y^2}{2}\Big|_0^a = \frac{\pi a^2}{2}$$

If $a = 1$ meter, then $V = \frac{\pi}{2}a^2$ gives

$$V = \frac{\pi}{2}\, m^3 = \frac{\pi}{2}(100\ \text{cm})^3 = \frac{\pi}{2}10^6\ \text{cm}^3 \approx 1600\ \text{liters} \quad \text{(a huge cauldron)}$$

Warning about units.

If $a = 100\,\text{cm}$, then

$$V = \frac{\pi}{2}(100)^2 = \frac{\pi}{2}10^4\ \text{cm}^3 = \frac{\pi}{2}10 \sim 16\ \text{liters}$$

But 100cm = 1m. Why is this answer different? The resolution of this paradox is hiding in the equation.

$$y = x^2$$

At the top, $100 = x^2 \implies x = 10\,\text{cm}$. So the second cauldron looks like Figure 3. By contrast, when

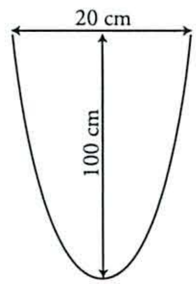


Figure 3: The skinny cauldron.

$a = 1\,\text{m}$, the top is ten times wider: $1 = x^2$ or $x = 1\,\text{m}$. Our equation, $y = x^2$, is not scale-invariant. The shape described depends on the units used.

Method 2: Shells

This really should be called the cylinder method.

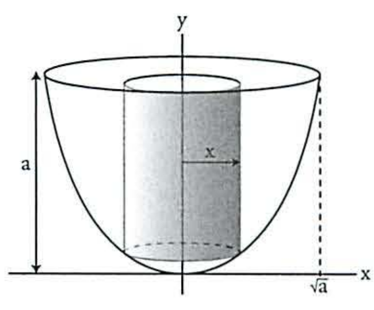


Figure 4: x = radius of cylinder. Thickness of cylinder $= dx$. Height of cylinder $= a - y = a - x^2$.

The thin shell/cylinder has height $a - x^2$, circumference $2\pi x$, and thickness dx.

$$\begin{aligned} dV &= \overset{\text{height}}{(a - x^2)}\overset{\text{circ}}{(2\pi x)}\overset{\text{dx}}{dx} \\ V &= \int_{x=0}^{x=\sqrt{a}} (a - x^2)(2\pi x)dx = 2\pi \int_0^{\sqrt{a}} (ax - x^3)dx \\ &= 2\pi \left(a\frac{x^2}{2} - \frac{x^4}{4} \right)\Big|_0^{\sqrt{a}} = 2\pi \left(\frac{a^2}{2} - \frac{a^2}{4} \right) = 2\pi \left(\frac{a^2}{4} \right) = \frac{\pi a^2}{2} \quad \text{(same as before)} \end{aligned}$$

Example 2. The boiling cauldron
Now, let's fill this cauldron with water, and light a fire under it to get the water to boil (at 100°C). Let's say it's a cold day: the temperature of the air outside the cauldron is 0°C. How much energy does it take to boil this water, i.e. to raise the water's temperature from 0°C to 100°C? Assume the

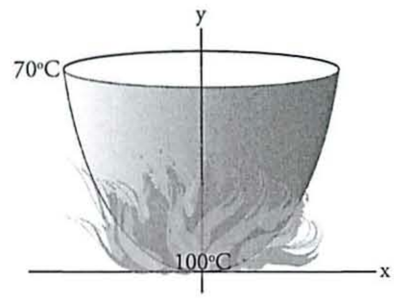


Figure 5: The boiling cauldron ($y = a = 1$ meter.)

temperature decreases linearly between the top and the bottom ($y = 0$) of the cauldron:

$$T = 100 - 30y \quad \text{(degrees Celsius)}$$

Use the method of disks, because the water's temperature is constant over each horizontal disk. The total heat required is

$$\begin{aligned} H &= \int_0^1 T(\pi x^2)dy \quad \text{(units are (degree)(cubic meters))} \\ &= \int_0^1 (100 - 30y)(\pi y)dy \\ &= \pi \int_0^1 (100y - 30y^2)dy = \pi(50y^2 - 10y^3)\Big|_0^1 = 40\pi \text{ (deg.)m}^3 \end{aligned}$$

How many calories is that?

$$\#\text{ of calories} = \frac{1\,\text{cal}}{\text{cm}^3 \cdot \text{deg}}(40\pi)\left(\frac{100\,\text{cm}}{1\,\text{m}}\right)^3 = (40\pi)(10^6)\,\text{cal} = 125 \times 10^3\,\text{kcal}$$

There are about 250 kcals in a candy bar, so there are about

$$\#\text{ of calories} = \left(\frac{1}{2}\text{ candy bar}\right) \times 10^3 \approx 500\,\text{candy bars}$$

So, it takes about 500 candy bars' worth of energy to boil the water.

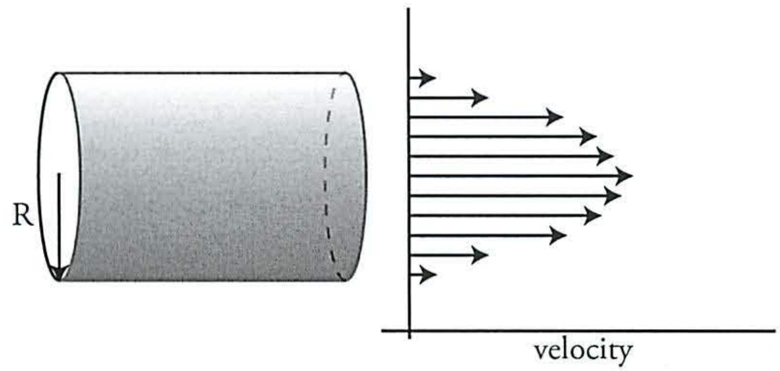


Figure 6: Flow is faster in the center of the pipe. It slows– "sticks"– at the edges (i.e. the inner surface of the pipe.)

Example 3. Pipe flow
Poiseuille was the first person to study fluid flow in pipes (arteries, capillaries). He figured out the velocity profile for fluid flowing in pipes is:

$$v = c(R^2 - r^2)$$
$$v = \text{speed} = \frac{\text{distance}}{\text{time}}$$

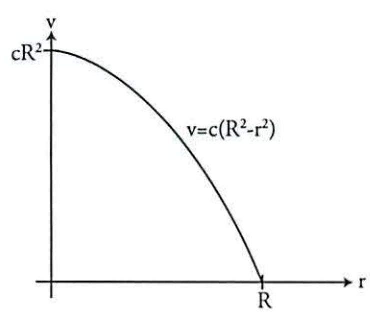


Figure 7: The velocity of fluid flow vs. distance from the center of a pipe of radius R.

The flow through the "annulus" (a.k.a ring) is (area of ring)(flow rate)

area of ring $= 2\pi r dr$ (See Fig. 8: circumference $2\pi r$, thickness dr)

v is analogous to the height of the shell.

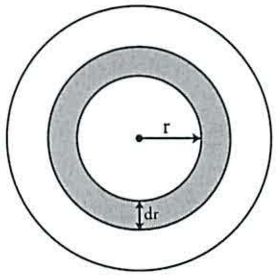


Figure 8: Cross-section of the pipe.

$$\begin{aligned}
\text{total flow through pipe} &= \int_0^R v(2\pi r dr) = c\int_0^R (R^2 - r^2)2\pi r dr \\
&= 2\pi c \int_0^R (R^2 r - r^3)dr = 2\pi c\left(\frac{R^2 r^2}{2} - \frac{r^4}{4}\right)\Big|_0^R \\
\text{flow through pipe} &= \frac{\pi}{2}cR^4
\end{aligned}$$

Notice that the flow is proportional to R^4. This means there's a big advantage to having thick pipes.

Example 4. Dart board

You aim for the center of the board, but your aim's not always perfect. Your number of hits, N, at radius r is proportional to e^{-r^2}.

$$N = ce^{-r^2}$$

This looks like:

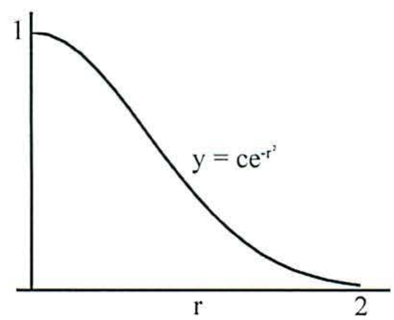


Figure 9: This graph shows how likely you are to hit the dart board at some distance r from its center.

The number of hits within a given ring with $r_1 < r < r_2$ is

$$c\int_{r_1}^{r_2} e^{-r^2}(2\pi r dr)$$

We will examine this problem more in the next lecture.

Michael Plasmeier

13 pages

18.01 FALL 2009 – Problem Set 5A

$\frac{71}{73}$ $\frac{69}{71}$

Due Friday 10/30/09, 1:45 pm in 2-106

This is part A of problem set 5. The second portion of the problem set will be available on the 18.01 website on Friday, Oct. 23.

Part I (10 points)

Lecture 16. Fri. Oct. 16 Differential equations; separating variables.
Read: 5.4, 8.5 Work: 3F-1cd, 2ae, 4bcd, 8b

Lecture 18. Thurs. Oct. 22 Definite integrals and Riemann sums.
Read: 6.3 through formula (4), 6.4, 6.5,
Work: 3B-2ab, 3b, 4a, 5, 4J-1 (just set up the integral. don't have to evaluate)

Lecture 19. Fri. Oct. 23 First fundamental theorem of calculus.
Read: 6.6, 6.7 to top of p. 215 Work: Assigned on part B of the problem set.

Part II (13 points)

Directions: Attempt to solve *each part* of each problem yourself. If you collaborate, solutions must be written up independently. Beside each problem is the date on which corresponding material in class is covered.

0. (not until due date; 3 pts) Write the names of all the people you consulted or with whom you collaborated and the resources you used, or say "none" or "no consultation". (See full explanation on PS1).

1. (Friday, 3 pts) From the supplementary notes: 3F-5abc

2. (Friday, 2 pts) Famous Investment Bank (FIB for short) begins with 676 employees. Strong government regulations on executive pay and bonuses cause FIB's workforce population P to lose employees at a rate of $-\sqrt{P}$ people per week. When will FIB no longer have any employees?

3. (Tuesday, 5 pts) Calculate

$$\int_0^1 e^x \, dx$$

using lower Riemann sums. In the course of computing this, you'll need to sum a geometric series to get a workable formula for the Riemann sum.

Michael Plasmeier

PSet 5A

10/16

Lecture 16

3F-1c Solve $\frac{dy}{dx} = \frac{3}{\sqrt{y}}$

$dy\, dx^{-1} = 3\, y^{-1/2}$

$\leftarrow \cdot \frac{1}{y^{-1/2}} \quad \cdot \frac{1}{dx^{-1}}$

$dy\, y^{1/2} = 3dx$

$\int$ integrate

$\frac{2}{3} y^{3/2} = 3x + c$

solve for y

$y = \left(\frac{9x}{2+a}\right)^{2/3}$

$\nwarrow a = \frac{3}{2} c$

↑? how to find

d $\frac{dy}{dx} = xy^2$

$\cdot \frac{1}{y^2} \cdot dx \quad \cdot \frac{1}{y^2} \cdot dx$

$\frac{dy}{y^2} = x\,dx$ ✓ ↷ do both sides (did) * but it was -2

$\rightarrow -y^{-1} = \frac{x^2}{2} + c$ Only mistake

$\cancel{\frac{y^3}{3} = x + c}$

$\cancel{y^3 = 3(x+c)}$

$\cancel{y = \sqrt[3]{3(x+c)}}$

$y = \frac{-1}{\frac{x^2}{2} + c}$

2a Solve + eval at positio

$\frac{dy}{dx} = 4xy$ $y(1) = 3$ Find $y(3)$

$dy \cdot dx^{-1} = 4xy$

$dy\, y^{-1} = 4x\,dx$ $\int \frac{1}{y} \rightarrow \ln y$ special case

$\cancel{\frac{y^0}{0} = \frac{4x^2}{2}}$ $\ln y = 2x^2 + c$

$y(1) = 3$

$\ln 3 = 2(1)^2 + c$

$\ln 3 = 2 + c$

$c = \ln 3 - 2$

$\ln y = 2x^2 + \ln(3) - 2$

e $\frac{dy}{dx} = e^x$ $y(3) = 0$ Find $y(0) \to 1$ ~~$y = e^3$~~

$y = -\ln 4$ at $x = 0$

Defined $-\infty < x < 4$

$e^{-y}\,dy = dx$

$-y^{-1} = x + c$

$-1^{-1} = 0 + c$ $c = -1$

$-y^{-1} = x - 1$

$-1/y = x - 1$

$y = \frac{1}{1-x}$ What values of x is y defined

$y(1/2) = 2$? why these

$y(-1) = -1/2$

$y(1) = $ und

4b Newton's Law of cooling

$$\frac{d\overset{\text{temp}}{T}}{d\underset{\text{time}}{t}} = k\,(T_e - T)$$

surrounding

constant proportionality

properties

surface area

$$\frac{dT}{dt} = k(T_e - T)$$

b. Find formula for T if inital tem $t=0$ is T_0

- seperate variables

$$(T-T_e)^{-1} dt = -k\,dt$$

c, ask

- integrate

$$\ln(T-T_e) = -kt + c$$

$\int y^{-1} \overset{?}{=} \ln y$

what is ??

$$T - T_e = \pm e^c e^{-kt} = A e^{-kt}$$

$$T(0) = T_0 \rightarrow A = T_0 - T_e$$

$$T = T_e + (T_0 - T_e) e^{-kt}$$

c Show that $T \to T_e$ $t \to \infty$

- yeah makes sense since temp will assume temp of surrounding area

$$\cancel{T(\infty) = T_e \rightarrow A = T_e - T_e = 1}$$

$$\cancel{T = T_e + (1) e^{-kt}}$$

$$\cancel{T = T_e + e^{-kt}}$$

apperents not

Since $k > 0$ $\underline{e^{-kt} \to 0}$

$$T = T_e (T_0 - T_e) e^{-kt} \rightarrow T_e \text{ as } t \to \infty$$

~~That is just the above solution~~ ??

d Suppose ingot leaves at temp 680 °C in 40°C room
Cools 200° in 8 hrs
How long 680 → 50°C ?

$$\frac{dT}{dt} = k(T_e - T)$$

$$\frac{200}{8} = k(\cancel{680}^{40} - 200)$$

$25 = k\, \cancel{480}\ -160$

$k = .\cancel{052}\ 5/32$ ← really small

$$\frac{630}{dt} = \cancel{.052}^{5/32}(\cancel{680}^{40} - 50)$$

1.56

$630 = \cancel{32.81}\, dt$

$dt = \cancel{19.2}$ hrs ← wrong Oh $\underline{T_e}$ is room temp 40

-103 ← even worse

$200 - 40 = 680 - 40\ e^{-8k}$ Use other equation

$e^{-8k} = \frac{160}{640} = \frac{1}{4}$

$-8k = -\ln 4$

$50 - 40 = 640 e^{-kt}$

$e^{-kt} = \frac{1}{64}$

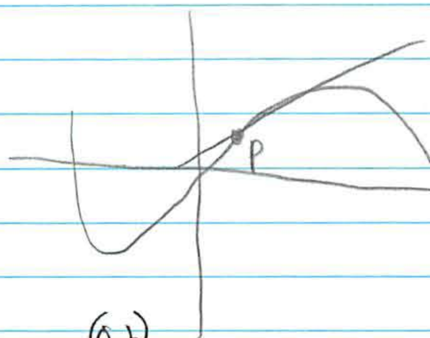
$-kt = -3 \ln 4$

Sole symaltaneous for t

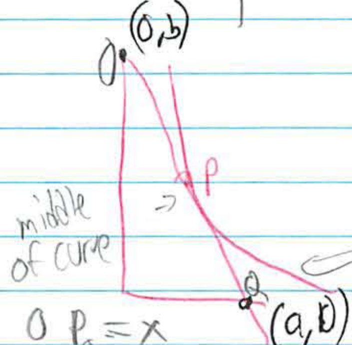
$\frac{t}{8} = 3$

$t = 24$

8b Find all plane curves in 1st quad such that for every point P on curve, P bisects part of the tangent line at P that lies in first quad



? bisect
- divide 2 = parts



P bisects tangent
P_0 bisects OQ (euclidean geometry)

Office Hrs

$OP_0 = x$
$P_0Q = x$

$p(x, y)$
$p\left(\frac{a}{2}, \frac{b}{2}\right)$

$a = 2x$
$b = 2b$

$\frac{dy}{dx} = \frac{-b}{a}$

$\frac{dy}{dx} = \frac{-2y}{2x}$

? find a function + solve

→ see next

$y' = -\frac{y}{x} \quad \frac{dy}{y} = -\frac{dx}{x}$

$\int$

$\ln y = -\ln x + c$

$y = \frac{1}{x} e^c = \frac{c}{x}$

$y = \frac{c}{x} \quad c > 0$

? original curve
- any y will work

$$\frac{dx}{dx} = \frac{-2x}{2x}$$

$$dy\, dx^{-1} = -2y(2x)^{-1}$$

$$\frac{dy}{-2y} = \cancel{2x}\,\frac{dx}{2x}$$ ← get all on 1side

$\int$ integrate $\int$

move constant out →

$$-\frac{1}{2}\int\frac{dy}{y} = \frac{1}{2}\int\frac{dx}{x}$$

$$-\frac{1}{2}\ln(y) = \frac{1}{2}\ln(x) + C$$

$\cdot -2 \qquad \cdot -2$

$$\ln(y) = -\ln(x) + C$$

✱ know log rules

solve for y

← log rule can move − here

$$\ln(y) = \ln x^{-1} + C$$

Office Hrs

$$e^{\ln y} = e^{\ln x^{-1}} \cdot e^{C}$$ ← multiply

$$y = x^{-1} e^{C}$$

$$y = \frac{e^{C}}{x}$$ ← e^{C} must be (+) — always is

→ can say here $\frac{C}{x}$

$7C = C$

Log Rules Review

$\ln(1) = 0$

$\ln e = 1$

$\ln e^{x} = x$ ← so $-1\ln x = \ln x^{-1}$

$\ln y^{x} = x \ln y$

$\ln(xy) = \ln x + \ln y$

$\ln\left(\frac{x}{y}\right) = \ln x - \ln y$

Lecture 18

3B-2a Find sum notiation

e but signs switch

$3 - 5 + 7 - 9 + 11 - 13$

$\underset{2}{\curvearrowright} \quad \underset{2}{\curvearrowright} \quad \underset{2}{\curvearrowright}$

limited → $\sum_{n=1}^{6} (-1)^{n+1} (2n+1)$

↑ sign switch piece

↑ +2 incrementing section

b $1 + \frac{1}{4} + \frac{1}{9} + \ldots \frac{1}{n^2}$

$\sum_{n=1}^{\infty} \frac{1}{n^2}$

3b Upper, lower, left, right Riemann sum 4 intervals

$\int_{-1}^{3} x^2 \, dx$

left) $1 \cdot 1 + 0 \cdot 1 + 1 \cdot 1 + 4 \cdot 1$

6

right) $0 \cdot 1 + 1 \cdot 1 + 4 \cdot 1 + 9 \cdot 1$

14

upper) $1 \cdot 1 + 1 \cdot 1 + 4 \cdot 1 + 9 \cdot 1$

15

? what is the values at the highest of square

lower) $0 + 0 + 1 \cdot 1 + 4 \cdot 1$

5

4a Calc diff upper + lower Riemann sum

$$\int_0^b x^2\,dx$$

- upper sum = right
- lower sum = left

upper $\frac{b}{n}\left(f(x_1)+\ldots+f(x_n)\right]$

lower $\frac{b}{n}\left[f(0)+f(x_1)+\ldots f(x_{n-1})\right]$

diff $\frac{b}{n}\left[\underset{b}{f(x_n)}-\underset{0}{f(x_0)}\right]$

$$\frac{b}{n}\left(f(b)-f(0)\right)$$

$$\frac{b}{n}\left[b^2-0^2\right]$$

$$\frac{b}{n}\cdot b^2=\frac{b^3}{n}$$

Will ↓ 0 as $n\to\infty$

since does not matter if do left or right

4J-1 k units of energy to lift $1m^3$ water ↑ 1m
How much E to pump circle hole 1m diameter 100m deep

- hard since circle
- and takes x energy to lift water x meters down

$A = \pi r^2 = \pi .5^2 = .7853$ m of water in a flat level

Divide water into thickness Δy
Energy to raise $\frac{\pi}{4} k y_i \Delta y$
Add Energies to lift + disk

$$E = \lim_{n \to \infty} \sum_{1}^{n} \frac{\pi}{4} k y_i \Delta y$$

$\pi .5^2$

100m → $\int_{0}^{100} \frac{\pi}{4} k y \, dy$

↓ integrate

→ definite integral

$$\frac{\pi k}{4} \frac{y^2}{2} \Big]_0^{100} = \frac{k \pi 10^4}{8}$$

Ans from book
- don't have to integrate
- looked fairly messy

Part 2

0. See sidebar + Sasha + Kimberly ✓ 3

1. 3F-5a Air pressure

$\frac{dp}{dh} = -(.13)p$ where h is altitude from sea level in km

a) At sea level pressure is $1 \ kg/cm^2$
Solve for pressure at top Mt. Everst (10 km)

Officehrs

$\frac{dp}{p} = -.13 \ dh$

Oh duh sea level = $h = 0$

$\ln p = -.13h + c$

$p = e^{-.13h} \cdot e^{c}$

$1 \ kg/cm^2 = e^{-.13 \cdot 0} e^{c}$

Brier needs to be km

$1 \times 10^{-10} \ kg/km^2$

$e^{c} = 1 \times 10^{-10}$ ← use that

~~$c = 0$~~ ← should not be 0 - tiny ff

$c = -23.02$ ← keep e^{c} it in km

km

~~Should work in km~~

cm: $p = e^{-.13(10)}$

km: $e^{-.13(10)} \cdot 1 \times 10^{-10}$

$p = .2725 \ kg/cm^2$ ⇒ $2.72 \times 10^{-11} \ kg/km^2$

get same ans same

↑ I think I was right - Briner wrong

b Green Building $100 \ m = .1 \ km$

$p = e^{-.13(.1)} e^{.13}$

$p = .9870$

b Use linear approx

-near $x=0$

$f(x) = f(a) + f'(a)(x-a)$

$a=0$

$x=-.13h$

↑pick

Office Hrs

— help figure out

$p = e^{-.13h}$

↑use $x=0$ as nearby value

$1 + 1(\cancel{\ } -.13h - 0)$ ← do linear approx

$1 + -.13h$

$1 - .13h$

$1 - .13(.1)$

$1 - .013$

$.987$ ← same ans -nice ← same thing ≈1 ← but don't want to know

Percent drop

$\frac{.013}{1} = 1.3\%$

c Use linear approx $\Delta p = p'(0)\Delta h$ ← .1

Compute $p'(0)$ directly from differential equation to find diff differential pressure from bottom → top Green Building - gives answer w/o knowing solution to differential equation. Compare to b. What does linear approx $p'(0)\Delta h$ give for pressure at top Mt. Everest?

Office hrs

Sasha

↑better ans

$\Delta p \approx p'(0)\Delta h \approx -.13e^{-.13h}(\Delta h)\big|_{h=0}$

$\Delta p = 1$ ~~kg~~ kg/cm^3 for Δh → change to killometers

$\Delta p =$ ~~.0000+~~ $kg/km^2 \cdot \Delta h$

$.0000000001\ kg/km^2 \cdot h$

↑square the ant

.1

$-.13 \cdot .1 = -.013\ kg/cm^3$

~~.10~~ $-.13 \cdot .0000000001$

$-.000000000013$ kg/km inital change in pressure

3

change → do inital - change = final pressure

10

$-.13 \cdot .000000001$ $-.13 \cdot 10 = -1.3\ kg/cm3$

$.00000000013$ kg/km inital change in pressure

2. FIB 676 employees
lose $-\sqrt{p}$ people /week
When have 0 employees?

$\frac{dP}{dt} = -\sqrt{P}$ rate of change

time in week

← p as function of t
then plug in inital p
find when p=0

$dP\, dt^{-1} = -P^{1/2}$

$-P^{-1/2}$

$\frac{dP}{-\sqrt{P}} = dt$ ✓

$\int \quad \int$

$\boxed{\int dt \to \int 1 dt \to t}$

$\frac{\sqrt{P}}{1/2} = 2\sqrt{P} - t + c = \cancel{t}$

when p=0

$\cancel{676 - 2\sqrt{p} = t}$ ← don't just add it

$\cancel{676 = 2\sqrt{p} =}$

$\cancel{-2\sqrt{p} = -676}$

$\cancel{\sqrt{p} = 338}$

$2\sqrt{p} = -t + c$

↑when $t = 0$ $p = 676$

* how find c

$2\sqrt{676} = c$

$c = 52$

2 Flaws
knowing $\int dt = t$
solving for c

$2\sqrt{p} = -t + 52$

$-t = 2\sqrt{p} - 52$

$t = -2\sqrt{p} + 52$

$t = 52$ 52 weeks

3. Calculate $\int_0^1 e^x \, dx$

lower Riemann sum
- need a geometric series

lower = left $\left(e^0 \cdot \frac{1}{4}\right) + \left(e^{1/4} \cdot \frac{1}{4}\right) + \left(e^{1/2} \cdot \frac{1}{4}\right) + \left(e^{3/4} \cdot \frac{1}{4}\right)$

but I guess they want as $n \to \infty$ not $=4$

generic $\lim_{n \to \infty} \sum_{i=1}^{n} f(c_i)\,\Delta x$ $\quad \Delta x = \frac{b-a}{n}$

c_i = interval inside
$f(c_i)$ = the height
$f(c_1)\Delta x + f(c_2)\Delta x + \dots + f(c_n)\Delta x$

$$\lim_{n \to \infty} \sum_{i=1}^{n} e^{c_i} \Delta x$$

$$\lim \; R_1 + R_2 + R_n$$
$$\lim \; \frac{1}{n}(e^0) + \frac{1}{n}(e^{1/n}) + \frac{1}{n}(e^{2/n}) + \left(\frac{1}{n}\right)\left(e^{\frac{n-1}{n}}\right)$$
$$\lim \; \frac{1}{n}\left[e^0 + e^{1/n} + e^{2/n} + e^{\frac{n-1}{n}}\right]$$

$$\lim_{n \to \infty} \frac{1}{n}\left[\frac{e^n - 1}{e - 1}\right]$$ ← from recitation

geometric series $\left[\frac{1-q^n}{1-q}\right]$ $\boxed{\text{flips due to } q > 1}$

$$\int_0^1 e^x\,dx = \lim_{n\to\infty}$$ - Rieman sums w/ n rectangle

$$= \lim_{n\to\infty}\left(R_1 + \ldots + R_n\right)$$

Brendon
offic
Hrs

$$= \lim_{n\to\infty}\left(\frac{1}{n}\cdot 1 + \frac{1}{n}e^{\frac{1}{n}} + \frac{1}{n}e^{\frac{2}{n}} + \ldots \frac{1}{n}e^{\frac{n-1}{n}}\right)$$

$$= \lim_{n\to\infty}\frac{1}{n}\left(\sum_{k=0}^{n-1}\left(e^{1/n}\right)^k\right)$$

$$= \lim_{n\to\infty}\frac{1}{n}\left(\frac{1-(e^{1/n})^n}{1-e^{1/n}}\right)$$

$$= \lim_{n\to\infty}\frac{1}{n}\left(\frac{1-e}{1-e^{1/n}}\right)$$

$$1 + q + q^2 + \ldots + q^{n-1} = \frac{1-q^n}{1-q}$$ ← geometric series

↑ can getting it by multiplying through

$$(1-e)\lim_{n\to\infty}\frac{1}{n(1-e^{1/n})} = (*)$$

could use ← L'Hopital

$$(*) = (1-e)\lim_{x\to 0}\frac{1}{\left(\frac{1}{x}\right)(1-e^x)}$$

$$(1-e)\lim\frac{x}{1-e^x}$$ ← limit = quotient of limit

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$$

$$\frac{1-e}{\lim_{x\to 0^+}\frac{1-e^x}{x}} = \frac{1-e}{\lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0}} = \frac{1-e}{(-e^x)'\big|_{x=0}} = e-1$$

13

$$f(x) = -e^x$$
$$f(0) = -1$$

Michael Plasmeier

22 pages
10 today

18.01 FALL 2009 – Problem Set 5B

Due Friday 10/30/09, 1:45 pm in 2-106

This is part B of problem set 5. The first portion of the problem set is available on the 18.01 website.

Part I (15 points)

Lecture 19. Friday, Oct. 23 First fundamental theorem of calculus.
Read: 6.6, 6.7 to top of p. 215 Work: 3C-1, 2a, 3a, 5a; 3E-6bc; 4J-2

Lecture 20. Tuesday, Oct. 27 Second fundamental theorem. Definition of log.
Read: Notes PI, p.2 [eqn.(7) and example]; Notes FT.
Work: 3E-1, 3a; 3D-1, 5, 7ab, 8a; 3E-2ac

Lecture 21. Thursday, Oct. 29 Areas between curves. Volumes by slicing.
Read: 7.1, 7.2, 7.3 Work: 4A-1a, 2, 4; 4B-1de, 6, 7

Lecture 22. Friday, Oct. 30 Volumes by disks and shells.
Read: 7.4 Work: Assigned on the next problem set

Part II (35 points)

Directions: Attempt to solve *each part* of each problem yourself. If you collaborate, solutions must be written up independently. Beside each problem is the date on which corresponding material in class is covered.

0. (~~not until due date; 3 pts (included in total from 5A)~~) Write the names of all the people you consulted or with whom you collaborated and the resources you used, or say "none" or "no consultation". (See full explanation on PS1).

1. (Friday, 7 pts) a) Simmons 6.3/8

b) Make a conjecture about a general formula for the sum of the first n r-th powers,

$$\sum_{k=1}^{n} k^r.$$

You may not be able to conjecture an exact formula, but you should be able to describe (at least) some features of the formula, e.g. leading terms. (Hint: Use a Riemann sum interpretation of this quantity to guide you in your conjecture.)

2. (Friday/Tuesday, 3 pts) Use an integral to estimate the sum

$$\sum_{i=1}^{10{,}000} \sqrt{i}$$

3. (Friday, 4 pts) Compute

$$\lim_{x \to 3} \left(\frac{x^2}{x-3} \int_3^x \frac{\sin t}{t} dt \right).$$

4. (Friday, 16 pts: 2 + 2 + 4 + 2 + 6) Consider the function $f(x) = \int_0^x \cos(t^2)dt$. There is no expression for $f(x)$ in terms of standard elementary functions. It is known as a Fresnel integral, along with the corresponding sine integral, and appears in everything from optics (its original use) to highway design.

a) Draw a rough sketch of $\cos(t^2)$, showing the first positive and negative zeros. What does the curve look like at $t = 0$? Is the function even or odd?

b) List the critical points of $f(x)$ in the entire range $-\infty < x < \infty$. Which critical points are local maxima and which ones are local minima?

c) Sketch the graph of f on the interval $-2 \le x \le 2$, with labels for the critical points **and inflection points**. (The drawing should be qualitatively correct, but just estimate the values of f at the labelled points.)

d) Estimate $f(0.1)$ to six decimal places.

e) Fresnel integrals are sometimes expressed using different scaling of the variables. We investigate this in the following three parts.

i) Let $g(x) = \int_0^x \cos((\pi/2)u^2)du$. Make a change of variables to show that $f(x) = c_1 g(c_2 x)$ for some constants c_1 and c_2. Why did we choose the factor $\pi/2$?

ii) Let $h(x) = \int_0^x \frac{\cos v}{\sqrt{v}} dv$. (This integral is called *improper* because $1/\sqrt{v}$ is infinite[1] at $v = 0$.) Make a different change of variable to show that $f(x) = c\,h(x^2)$ for some constant c (assume that $x > 0$).

iii) Let $k(x) = \sqrt{x} \int_0^1 \cos(xt^2)dt$, $x > 0$. Use the change of variable $z = xt^2$ and part (ii) to find the relationship between the functions k and f. Hint: Which quantities are variable and which are constant?

5. (Tuesday, 5 pts: 2 + 3)

a) Do 7.3/22.

b) Find the volume of the region in 3-space with $x > 0$, $y > 0$ and $z > 0$ given by

$$z^2/2 < x + y < z$$

Hint: First find the area of the horizontal cross-sections.

[1] Although the integrand is infinite, the area under the curve is finite. The function h is continuous, $h(0) = 0$, but its graph has infinite slope at $x = 0$.

5B

Lecture 19 First fundemental theorem of calculus

3C-1 Find area under graph $\frac{1}{\sqrt{x-2}}$ $3 \leq x \leq 6$

$\int_3^6 \frac{1}{\sqrt{x-2}}$ ← need to substitute.

$\int_3^6 (x-2)^{-1/2}$

↓

$\frac{(x-2)^{1/2}}{\frac{1}{2}} \Big|_3^6$

$2(x-2)^{1/2} \Big|_3^6$

$2(6-2)^{1/2} - 2(3-2)^{1/2}$

$4 \quad -2$

$2 \quad \theta$

2a Calculate

$\int_0^2 \sqrt{3x+5}\, dx$

$\int_0^2 (3x+5)^{1/2}$

$\frac{(3x+5)^{3/2}}{\frac{3}{2}} \Big|_0^2$

$\frac{2}{3}(3(2)+5)^{3/2} - \frac{2}{3}(0+5)^{3/2}$

$\frac{22\sqrt{11}}{3} - \frac{10\sqrt{5}}{3}$ $\frac{2}{9}\left(11^{3/2} - 5^{3/2}\right)$

16.868

3a Calculate

$$\int_1^2 \frac{x\,dx}{x^2+1}$$

? what happens xdx

$\frac{1}{2}$ somehow?

$$\frac{1}{2}\ln(x^2+1)\Big|_1^2$$

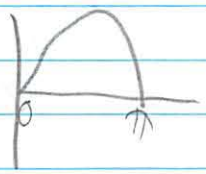
$$\frac{1}{2}\ln(2^2+1) - \frac{1}{2}\ln(1^2+1)$$

$$.5\ln(5) - .5\ln(2)$$

$$.45814 \quad \frac{1}{2}\ln\left(\frac{5}{2}\right)$$

same

5a Find the area under 1 arc sin x



$$\int_0^{\pi} \sin x$$

$$-\cos x \Big|_0^{\pi}$$

$$-\cos\pi + \cos(0)$$

$$1 + 1$$

$$2$$

3E-6b Compare integral w/ integral easier to evaluate establish estimation

$$\int_0^{\pi} \sin^2 x\,dx < 2$$

? don't really get why

$0 < \sin x < 1$ on $(0,1)$

$\sin^2 x < \sin x$ $(0,\pi)$

$$\int_0^{\pi} \sin^2 x\,dx < \int_0^{\pi} \sin x\,dx$$

$$-\cos x\Big|_0^{\pi} = -(-1-1) = 2$$

6c

$$\int_{10}^{20} \sqrt{x^2+1}\, dx > 150$$

$$\int_{10}^{20} \sqrt{x^2+1}\, dx > \int_{10}^{20} \sqrt{x^2}\, dx$$

? where from → $\frac{x^2}{2}\Big|_{10}^{20} = \frac{1}{2}(400-100)$

150

4J-2 The amount x (in grams) radioactive material decays exponentially over time (min)

$x = x_0 e^{-kt}$ $\qquad x_0$ = inital amt $t = 0$

1 gram produces r units radiation / min
How much R produced over 1 hr by x_0 grams of material?

t in min

$r = x_0 e^{-k \cdot 1}$ ← guess solve for k
$R = x_0 e^{-k \cdot 60}$ ← plug in

$\ln r = \ln x_0 \cdot -k$

$k = -\frac{\ln r}{\ln x_0}$

$R = \cancel{x_0} \frac{r}{\cancel{x_0}} \cdot e^{60}$

$R = r \cdot e^{60}$

→ Book solution

Book Solution

$t_i \quad i = 1, \ldots, n$

$x_0 e^{-kt_i}$ grams

$r x_0^{-kt_i} \Delta t$ units over interval $[t_i, t_i + \Delta t$

$$R = \lim_{n \to \infty} \sum_{1}^{n} r x_0 e^{-kt} \Delta t$$

$$R = \int_0^{60} r x_0 e^{-kt} \, dt$$

$$R = r x_0 \frac{e^{-kt}}{-k} \Big|_0^{60}$$

$$R = \frac{r x_0}{k}\left(1 - e^{-60k}\right)$$

Lecture 20
3E-1

Prove directly from definition

$$L(x) = \int_1^x \frac{dt}{t} \quad \text{that} \quad L\left(\frac{1}{a}\right) = -L(a)$$

by making a change of variables in the definite integral

$$\int_1^{\frac{1}{a}} \frac{1}{t}\,dt \rightarrow \int_1^{\frac{1}{a}} \ln t$$

$$\ln \frac{1}{a} - \ln 1$$
$$\ln \frac{1}{a} - 0$$

$$-\int_1^a \frac{1}{t}\,dt \quad -\ln(a) - \ln(1)$$
$$-\ln(a)$$

$$\underbrace{\ln\left(\frac{1}{a}\right) = -\ln(a)}$$

are equal property of logs

got it!

$$\ln(a^{-1}) = -1\ln(a)$$

3a Evaluate w/ substitution (change limits too)

$$\int_1^e \frac{\sqrt{\ln x}}{x}\,dx \qquad u = \ln x$$

$$u = \ln x$$
$$du = \frac{1}{x}\,dx$$

know this

$$\rightarrow \int_{0 = \ln(1)}^{1 = \ln(e)} \sqrt{u}\,du = \frac{2}{3}u^{3/2}\Big|_0^1 = \frac{2}{3}(1)^{3/2} - \frac{2}{3}(0)^{3/2}$$

$\frac{2}{3}$ ✓

3D-1 Prove that $\int_0^x \frac{dt}{\sqrt{t^2+a^2}} = \ln\left(x+\sqrt{x^2+a^2}\right) - \ln(a)$

$a>0 \quad x>0$

differentiate

$\frac{d}{dx}\int_0^x \frac{dt}{\sqrt{t^2+a^2}}$ $\qquad$ $\frac{d}{dx}\ln\left(x+\sqrt{x^2+a^2} - \ln(a)\right)$

$\frac{1}{\sqrt{x^2+a^2}} - \frac{1}{\sqrt{0^2+a^2}} + C$ $\qquad$ $\frac{1+\frac{x}{\sqrt{a^2+x^2}}}{x+\sqrt{a^2+x^2}}$

$\frac{1}{\sqrt{a^2+x^2}} + C = \frac{1}{\sqrt{a^2+x^2}}$

$C = \frac{1}{\sqrt{x^2+a^2}} - \frac{1}{\sqrt{x^2+a^2}} = 0$ ewell $x=0,\ C=0$

$\cancel{\int_0^0 \frac{dt}{\sqrt{t^2+a^2}}} + 0 = \ln\left(0+\sqrt{0^2+a^2}\right) - \ln(a)$

0

$0 = \ln(a) - \ln(a)$

$0=0$

So equal

b. For what c is $\int_c^x \frac{dt}{\sqrt{t^2+a^2}} = \ln\left(x+\sqrt{x^2+a^2}\right)$

$x=C$ so $0 = \ln\left(c+\sqrt{c^2+a^2}\right)$ solve for c

$e^0 = c + \sqrt{c^2+a^2}$

$(1-c)^2 = c^2+a^2$

$c^2 = (1-c)^2 - a^2$

$c = 1 - c - a^2$

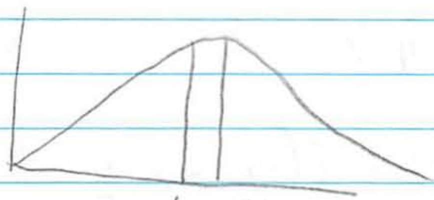
$+c \qquad +c$

$2c = 1-a^2$

$c = \frac{1-a^2}{2}$

5. Evaluate $\lim_{\Delta x \to 0} \frac{1}{\Delta x} \int_1^{1+\Delta x} \frac{t}{\sqrt{1+t^4}}$

a) area under curve
- Riemann sum



1 1+Δx

$$\lim_{\Delta x \to 0} \frac{\text{area}(\Delta x \cdot f(1))}{\text{width}(\Delta x)} = \frac{f(1)\Delta x}{\Delta x} = f(1) = \frac{1}{\sqrt{1+1^4}} = \frac{1}{\sqrt{2}}$$

b. Relation to $F'(1)$ where $F(x) = \int_0^x \frac{t}{\sqrt{1+t^4}}\,dt$

$$\frac{d}{dx} \int_0^x \frac{t}{\sqrt{1+t^4}}$$

$$\frac{x}{\sqrt{1+x^4}} - \frac{0}{\sqrt{1+0^4}} = \frac{d}{dx} \frac{x}{\sqrt{1+x4}} = \frac{d}{dx} x(1+x^4)^{-1}$$

$$1(1+x^4)^{-1} + -1(1+x^4)^{-2} \cdot 4x^3 \cdot x$$

$$\frac{1}{(1+x^4)} - \frac{4x^4}{(1+x^4)^2}$$ Wrong approch

$$\frac{(1+x^4) - 4x^4}{(1+x^4)^2}$$

$$\frac{1-3x^4}{(1+x^4)^2}$$

*Use this form

Book $F'(1) = \lim_{\Delta x \to 0} \frac{F(1+\Delta x) - F(1)}{\Delta x}$

$= \lim_{\Delta x \to 0} \frac{1}{\Delta x} \int_{1}^{1+\Delta x} f(t)\, dt$

FTC #2

$F'(1) = f(1) = \frac{1}{\sqrt{2}}$ ← what does that prove ??

7a Evaluate $F'(x)$ if $F(x) =$

$$\int_0^{x^2} \sqrt{u} \sin u \, du$$

$$\frac{d}{dx} \int_0^{x^2} \sqrt{u} \sin u \, du$$

just plug in for u

$$\sqrt{x^2} \sin x^2 \frac{dx^2}{x}$$ ← ??? differentiate $2x$

$$2x^2 \sin(x^2)$$

b $$\int_0^{\sin x} \frac{dt}{\sqrt{1-t^2}}$$

$$\frac{d}{dx} \int_0^{\sin x} \frac{1}{\sqrt{1-t^2}} \, dt$$

$$\frac{1}{\sqrt{1-\sin^2 x}} \cdot \cos x$$ ← differentiate $\cos x$

$$1$$

$$\boxed{\int_0^{\sin x} \frac{dt}{1-t^2} = x}$$

8a Let $f(x)$ be continuous
Find $f(\frac{\pi}{2})$

$$\int_0^{x} f(t) \, dt = 2x(\sin x + 1)$$

$$\frac{d}{dx} \int_0^{x} f(t) \, dt = \frac{d}{dx} 2x(\sin x + 1)$$

$f(x)\,dx = 2(\sin x+1) + (\cos x)2x$

$f(\frac{\pi}{2}) = 2(\sin\frac{\pi}{2}+1) + \cos\frac{\pi}{2} \cdot 2\frac{\pi}{2}$

$2(1+1) + 0 \cdot \pi$

4 V

3E-2. The function defined by $E(x) = \frac{1}{\sqrt{2\pi}}\int_0^x e^{-u^2/2}$

used in probability + statistics (similar in importance to sin cos in trig)

a) Express $E(x)$ in terms of the function of example 5

Example 5 → $F(x) = \int_0^x e^{-t^2}\,dt$

by making a change in variable,

$\lim_{x\to\infty} F(x) = \frac{\sqrt{\pi}}{2}$ What is $\lim_{x\to\infty} E(x)$?

$-t^2 = \frac{-u^2}{2}$

$u = t\sqrt{2}$

$du = \sqrt{2}\,dt$

$\frac{1}{\sqrt{2\pi}}\int_0^{\frac{x}{\sqrt{2}}} e^{-t^2}\sqrt{2}\,dt$

$\frac{1}{\sqrt{\pi}}\int_0^{x/\sqrt{2}} e^{-t^2}\,dt$

$E(x) = \frac{1}{\sqrt{\pi}} F\left(\frac{x}{\sqrt{2}}\right)$

$\lim E(x) = \frac{1}{\pi} \cdot \frac{\sqrt{\pi}}{2} = \frac{1}{2}$

b Evaluate $\lim_{N \to \infty} \frac{1}{\sqrt{2\pi}} \int_{-N}^{N} e^{-v^2/2} dv = ?$

$\lim_{x \to -\infty} E(x) = ?$

? integrand is even ← ? what does this mean?

$\frac{1}{\sqrt{2\pi}} \int_{-N}^{N} e^{-v^2/2} dv$

2x of this region $\frac{2}{2\pi} \int_{0}^{N} e^{-v^2/2} dv$

$2E(N) \to 1$ as $N \to \infty$

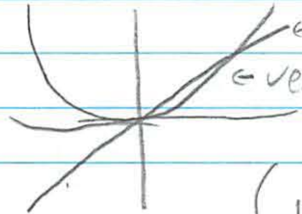
$\lim_{x \to -\infty} E(x) = -\frac{1}{2}$ because $E(x)$ is odd

x value as ← $-\infty$

Lecture 21

4A-1a Find the area between the curves

$y = 2x^2$
$y = 3x - 1$



$(.5, .5)$ $(1, 2)$

$$\int_{.5}^{1} 3x - 1 - 2x^2 \, dx$$

~~$3(1) - 1 - 2(1)^2 - 3(.5) - 1 - 2(.5)^2$~~

~~$1 - 3 \mid \rightarrow 3$~~

Integrate 1st

$$\frac{3x^2}{2} - x - \frac{2x^3}{3} + C$$

$$\frac{3}{2}x^2 - x - \frac{2}{3}x^3$$

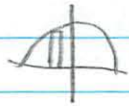
$$\frac{3}{2}(1)^2 - (1) - \frac{2}{3}(1)^3 - \frac{3}{2}(.5)^2 - (.5) - \frac{2}{3}(.5)^3$$

$\frac{1}{24}$

2. Find the area under the curve $y = 1 - x^2$ in 2 ways ← dx, dy

← can do horizontally or vertically
dx dy

horiz



$$\int_{-1}^{1} (1-x^2)\,dx = 2\int_0^1 (1-x^2)\,dx$$

$$2\cdot\left(x - \frac{x^3}{3}\right)\Big|_0^1$$

$$2(1) - \frac{2(1)^3}{3} = \frac{4}{3}$$

vert

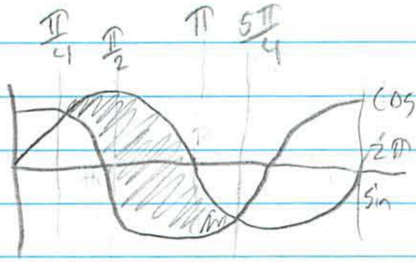
$$\int_0^1 2\sqrt{1-y}\,dy$$

$x = \pm\sqrt{1-y}$ $\qquad \frac{4}{3}(1-y)^{3/2}\Big|_0^1$

$$\frac{4}{3}\left(1-(1)\right)^{3/2}$$

$$\frac{4}{3}$$

4. Find the area between $y = \sin x$ from 1 crossing to next
$y = \cos x$



$$\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (\sin x - \cos x)\,dx$$

$$-\cos x - \sin x \Big|_{\frac{\pi}{4}}^{\frac{5\pi}{4}}$$

$$-\cos\left(\frac{5\pi}{4}\right) - \sin\left(\frac{5\pi}{4}\right) - \cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{4}\right)$$

$$-\frac{\sqrt{2}}{2} - -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} - -\frac{\sqrt{2}}{2} = -\frac{2\sqrt{2}}{2} = 2\sqrt{2}$$

guess added wrong

4B-1 Find the volume of the solid of revolution generated by rotating the regions bounded by the curve around the x-axis

I don't get how this is done

d) $y = x$
$y = 0$
$x = a$

$$\int_0^a \pi x^2 \, dx = \pi \frac{a^3}{3}$$

where does this come from?

e) $y = 2x - x^2$
$y = 0$

$$\int_0^2 \pi (2x - x^2)^2 \, dx$$

$$\int_0^2 \pi (4x^2 - 4x^3 + x^4) \, dx$$

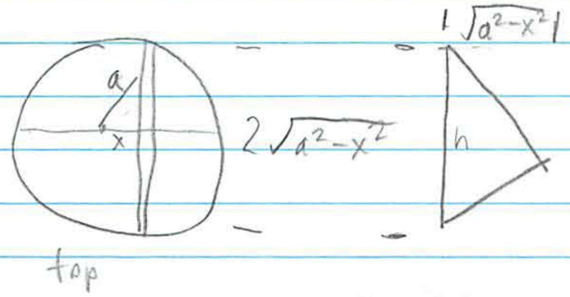
$$\pi \left(\frac{4x^3}{3} - \frac{4x^4}{4} + x^5 \right) \Big|_0^2$$

$$\frac{4}{3}\pi x^3 - \pi x^4 + \pi x^5 \Big|_0^2$$

$$\frac{4}{3}\pi (2)^3 - \pi (2)^4 + \pi (2)^5$$

$$\frac{16\pi}{15}$$

6. Base of a solid is the disk $x^2 + y^2 \leq a^2$
Planes perpendicular to xy plane and perpendicular to x-axis slice the solid into isoceles right triangles. Hypotenuse is where plane meets disk. Volume of solid?



area $= \frac{h^2}{4}$

endpoints $y = \pm\sqrt{a^2 - x^2}$

$h = 2\sqrt{a^2 - x^2}$

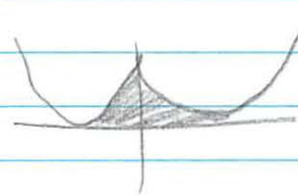
So volume $= \sum \frac{h^2}{4}$

$\int_{-a}^{a} \frac{h^2}{4}\, dx$

$\int_{-a}^{a} a^2 - x^2\, dx = \frac{4a^3}{3}$

7. Tower is constructed with a square base
Square horiz cross section

Viewed from any direction perpendicular to a side
tower base $y = 0$
profile line $y = (x-1)^2$
$y = (x+1)^2$



Volume?

Solve for x: $x = 1 \pm \sqrt{y}$ ← which represents end points?
$x = -1 \pm \sqrt{y}$ ←
(0,1) must be on both curves
$x = 1 - \sqrt{y}$ $x = -1 + \sqrt{y}$
$0 = 1 - \sqrt{1}$ ✓ $0 = -1 + \sqrt{1}$ ✓

$V = \int_0^1 \left(2(1-\sqrt{y})\right)^2 dy$

$4\int_0^1 (1 - 2\sqrt{y} + y)\, dy$

$4(1 - 2\sqrt{1} + 1) - 4(1 - 2\sqrt{0} + 0)$

$4(1 - 2 + 1)$

~~0 − 4~~

$\frac{2}{3}$ where did I screw up?

Part 2

0. See Week 1

1. Simmons G.3/8

$$\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + \dots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$$

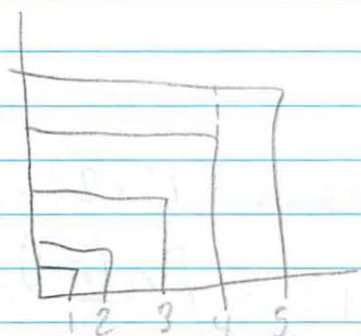
Riemans Sum

Geometric proof of formula (4)
- begin at O
- lay off sucessive segments lenght 1, 2, 3, etc, n up to point A.

Do same on a line OB perpendicular to OA

$$OA = OB = 1 + 2 + \dots + n$$

$$= \frac{n(n+1)}{2}$$

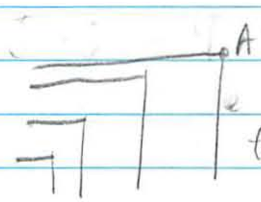


Area $S = \left[\frac{n(n+1)}{2}\right]^2$

= sum L shaped regions
$= L_1 + L_2 + \quad L_n$

↑ lenght diff increasing by 1

Show $L_n = n^3$ so that $S = 1^3 + 2^3 + \dots + n^3$ and proove 4.



← I see
follow instructions

but does not really
makes sense
- why cubed

- why diagonal lines?

→

Area $= \left[\dfrac{n(n+1)}{2}\right]^2$ ①

Area square side $n-1$ $\left[\dfrac{(n-1)(n)}{2}\right]^2$ ②

$l_n =$ ① $-$ ②

$$\left(\frac{n(n+1)}{2}\right)^2 - \left(\frac{(n-1)(n)}{2}\right)^2$$

$$\frac{n^2}{4}\left[(n+1)^2 - (n-1)^2\right.$$

$$\frac{n^2}{4}\left[n^2+2n+1-n^2+2n-1\right]$$

$$\frac{n^2}{4} 4n$$

$$l_n = n^3$$

$L =$ ① $-$ ② $\in$ sums $1^3 + 2^3 + 3^3 + \ldots + n^3$

$$\sum_{n=1}^{n} t^3 = \left[\frac{n(n+1)}{2}\right]^2$$

b Make a conjecture about general formula for sum of the first n rth powers

$$\sum_{k=1}^{n} k^r$$ ← base changing
↑ r is constant
↑ got backward

asking for leading term

i did in recitation 10/26

- describe
- could use Riemann sum
- look at geometric again

$\int_0^n x^r\,dx$ ← under estimate

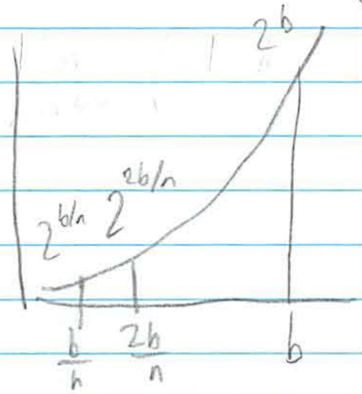
-estimating rieman sum w/ integral

Office Hrs Brendan

$$a\left(\frac{1-r^n}{1-r}\right) \Leftarrow \sum_{k=0}^{n-1} a \cdot r^k$$

~~geometric~~ sum

↑ k ↑ ratio



$$\lim_{n\to\infty} \frac{b}{n} \cdot k^{b/n}\left(\frac{1-k^{\frac{bn}{n}}}{1-k^{b/n}}\right)$$

$$\lim_{n\to\infty} \frac{b}{n} \cdot k^{b/n}\left(\frac{1-k^{b}}{1-k^{b/n}}\right)$$

↑ ∞ ↑ 0

$$= \frac{x^{r+1}}{r+1}\Big|_0^n = \frac{n^{r+1}}{r+1}$$

very general
have lots of sums
can approx as
riemann sum

7

2. Use Integral to Estimate Sum

$$\sum_{i=1}^{10,000} \sqrt{i}$$

$$\int_1^{10,000} \sqrt{i}$$

$$\frac{i^{3/2}}{\frac{3}{2}}\Bigg|_1^{10,000} = \frac{2}{3} i^{3/2} = \frac{2}{3}(10,000)^{3/2} - \frac{2}{3}(1)^{3/2}$$

$$\frac{2000000}{3} - \frac{2}{3}$$

$$666,666$$

3 Compute $\lim_{x \to 3} \left(\frac{x^2}{x-3} \int_3^x \frac{\sin t}{t} \, dt \right)$

$\lim_{x \to 3} \frac{x^2}{x-3}$ $\lim_{x \to 3} \int_3^x \frac{\sin t}{t} \, dt$

$-\ln(t)\cos(t) \Big|_3^x$

? Brendon office hrs

use def derivitive
- some function at some point

$-\ln(x)\cos(x) + \ln(3)\cos(3)$

~~will approach 0~~ ; not useful

$= \lim_{x \to 3} \frac{f(x) - f(3)}{x-3} = f'(3)$

aiming for

take deriv of $f(x)$ evaluate at $x = 3$

$f(x) = x^2 \int_3^x \frac{\sin t}{t} \, dt$

$\int_3^3 = 0$

$f(3) = x^2 \cdot 0$

0

$f'(x) = 2x \int_3^x \frac{\sin t}{t} \, dt + x^2 \frac{\sin x}{x}$

First FCT
- whole point of it is bound is variable

? final ans

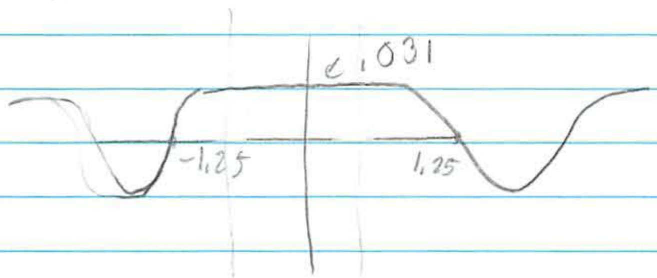
$f'(3) = 2(3) \cdot 0 + 3^2 \left(\frac{\sin 3}{3}\right)$

$9 \frac{\sin 3}{3}$ ✓

4.a Consider $\int_0^x \cos(t^2)\,dt$

no expression for $f(x)$ w/ ordinary functions
Fresnel integral -optics

a) Rough sketch $\cos(t^2)$



even

b) Critical Pts $-\infty < x < \infty$ max/min?

$0 = \cos(t^2)$

Local Max $-2.5066, 1$
Local Min $= -1.7725, -1$ $\in f'(x)$
Local Max $= 0, 1$
Local Min $= 1.7725, 1$
Local Max $= 2.5066, 1$

Guess it repeats
Should get a pattern
Not fixed interval

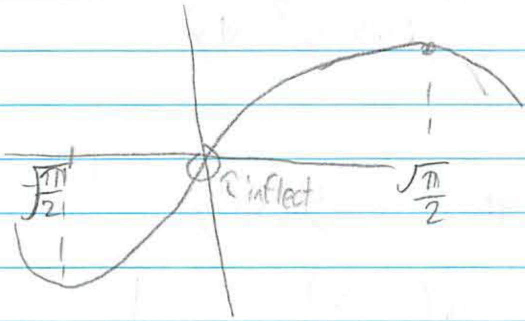
$\sqrt{\frac{\pi}{2} \pm 2\pi n}$ = local max n = integer
$\sqrt{\frac{3\pi}{2} \pm 2\pi n}$ = local min

inflection pts $-2\sin(t) = 0$
$t = 0, \pi, \ldots$

c. Sketch f

- Memory error on calc

Brubaker office hr



d $f(.1) = \int_0^{0.1} \cos(t^2)\,dt = .1$

taylor series could do lin/quad but don't know how accurate
* cubic, quartic, quintic approx

e: $f(x) = \int_0^n \cos(t^2)$

$g(x) = \int_0^n \cos\left(\frac{\pi}{2} u^2\right) du$

u substitution

Brendon's office hrs

want $f(x) = c_1 g(c_2 x)$

want to make arg of $g(x)$ look like arg of $f(y)$

Brubaker office hrs

$t = \sqrt{\frac{\pi}{2}}\, u$

when $u=0$ $t=0$

$u=x$ $t=\sqrt{\frac{\pi}{2}}\, n$

$dt = \sqrt{\frac{\pi}{2}}\, du$

$g(x) = \int_0^{\sqrt{\frac{\pi}{2}}} \cos(t^2) \sqrt{\frac{2}{\pi}}\, dt$

$\rightarrow$ over

$\frac{\pi}{2} u^2 = y^2$

$y = \sqrt{\frac{\pi}{2} u}$

$g(x) = \sqrt{\frac{2}{\pi}} \int_0^{\sqrt{\frac{2}{\pi}}} \cos y^2 \, dy$

$c = \sqrt{\frac{2}{\pi}} \qquad g(c_1 x)$

$f(x) = c_1 \, g(c_2 x)$ ✓

$f(x) = \sqrt{\frac{2}{\pi}} \, g\left(\sqrt{\frac{\pi}{2}} x\right)$

$$g(x) = \sqrt{\frac{2}{\pi}} \int_0^{\sqrt{\pi/2}\,x} \cos(t^2)\,dt$$

eii $h(x) = \int_0^x \frac{\cos v}{\sqrt{v}}\,dv$ want $f(n) = ch(x^2)$ $x > 0$

let $v = t^2$ $v = 0$ $t = 0$

$\frac{dv}{dt} = 2t$ $v = x$ $t = \sqrt{n^3}$

$\frac{1}{2} t\,dv = dt$

$$h(x) = \int_0^x \frac{\cos t^2}{\cancel{t}} \cdot 2\cancel{t}\,dt$$

$$\int_0^{\sqrt{n^3}} 2 \cos(t)^2\,dt$$

*one function in terms of other

$$h(x) = \underset{c}{\overset{2}{}} \underbrace{\int_0^{\sqrt{n}^2} \cos(t)^2\,dt}_{n(x^2)}$$

eiii $k(n) = \sqrt{n} \int_0^1 \cos(nt^2)\,dt$ $n > 0$

$t < 0$

Brubaker office hrs

upper limit $t = 1$

$z = n$

Lower $t = 0$ $z = 0$

$k(x) = \sqrt{x} \int_0^1 \cos(xt^2)\,dt$

$x > 0$

changes of variables to relate

- work w/ constants
- have what is inside

$z = nt^2$

$dz = 2nt\,dt$

$z = xt^2$

$dz = 2x\,dt$

$k(x) = \frac{1}{2}\int_0^x \frac{\cos z}{z^2}\,dz$

$$k(n) = \sqrt{n} \int_0^x \frac{\cos z}{2n\sqrt{\frac{z}{n}}}\, dt$$

$$\frac{\sqrt{n}}{2x} \int_0^n \frac{\cos z}{\frac{\sqrt{z}}{\sqrt{n}}}\, dt$$

$$\frac{\sqrt{n}\cdot\sqrt{n}}{2n} \int_0^n \frac{\cos z}{\sqrt{z}}\, dt$$

$$\frac{n}{2n} \int_0^n \frac{\cos z}{\sqrt{z}}\, dt$$

$$\frac{1}{0} \int_0^n \frac{\cos z}{\sqrt{z}}\, dt$$

but $h(x) = \int_0^x \frac{\cos v}{\sqrt{v}}\, dv$ where v is any variable

$k(x) = \frac{1}{2} h(x)$

from part b $k(x^2) = \frac{1}{2} h(x^2)$

$f(x) = h(x^2)$

$k(x) = \frac{h(x)}{2}$

$f(x) = ch(x^2)$

$k(x^2) = \frac{h(x^2)}{2}$

$f(x) = \frac{1}{2} h(x^2)$

$2kx^2$

5. 7.3 #22

Water evaporates from an open bowl
- unspecified shape
- rate proportional to area water surface

$$\frac{dV}{dt} = -cA(h)$$

V = volume
$A(h)$ = area at depth h
c = constant

a) Show that $\frac{dh}{dt} = -c$ so that water level drops at constant rate regardless of shape of bowl

height changes constant rate

$$V = \int_0^h A(x)\,dx$$

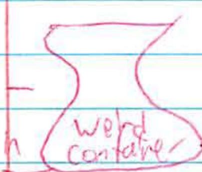
$V = 0 \Rightarrow h = 0$

volume

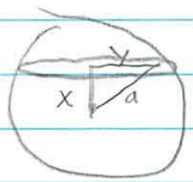
area under curve

similar to today's recitation
- no

Find area of thin cross section



office hrs Brandon



area $= \pi y^2$

$\pi(a^2 - x^2)$

decreases

decreases

area of surface area $\rightarrow A(h)$

more SA = more evaporating
less SA = less evaporating
) same dh

starting $h' \; -c$ ← when is $h = 0$

h = rate of the

$t = \frac{h'}{c} = 0$ therefore $V = 0$

constant Δ

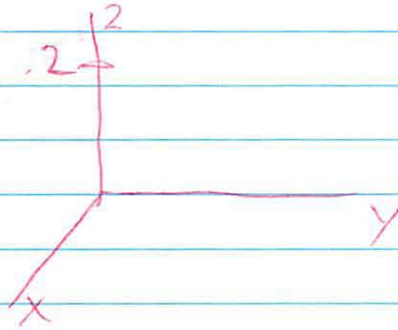
$5L - 2L/hr = 2.5$ hrs

$\frac{c}{2}$

5b Find volume of region in 3-space $x > 0$, $y > 0$, $z > 0$

$$\frac{z^2}{2} < x + y < 2$$

office hrs Brendon



trying to graph

when is $\frac{z^2}{2} = z$

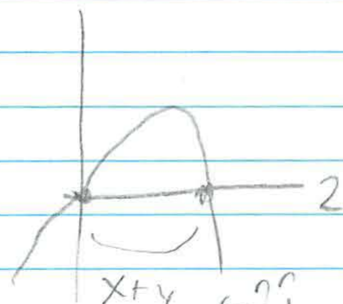
$z\left(\frac{z}{2} - 1\right) = 0$

$\boxed{z = 0,\ z = 2}$

true? at $z = \sqrt{2}$

top must be atleast < 2

$z - \frac{z^2}{2}$

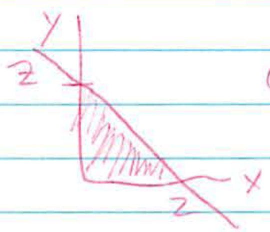


answer? -2

need area of cross section

For fixed z $\quad 0 < z < 2$

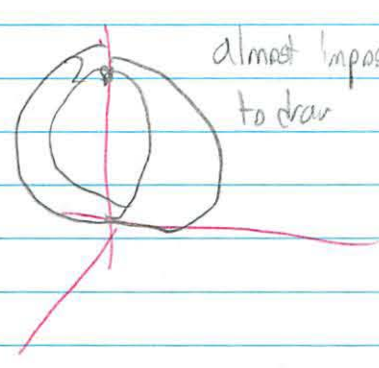
$\frac{z^2}{2} < x + y < z$



← need area

$y = z - x$

$y < z^2 - 2$

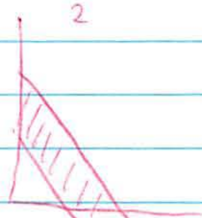


almost impossible to draw

changes w/ value of z

← so many trapizodes

$\frac{z^2}{2}$



Find where in both

$x + y > 2$

$A(z) = \frac{1}{2} z^2 - \frac{1}{2}\left(\frac{z^2}{2}\right)^2$

$\frac{1}{2} z^2 - \frac{1}{8} z^4$

take integral

PSet 5
18.01 · Fall 2009.

5A · Part II

Problem 1 · 3F-5abc

air pressure satisfies the diff. eq. $dp/dh = -(0.13)p$, h = alt. from sea level, km.

a) At sea level $p_0 = 1$ kg/cm². Solve the eq. and Find the pressure at the top of Mt. Everest (10 km).

$p > 0$

$$\frac{1}{p}\,dp = -0.13\,dh \Rightarrow \ln p = -0.13h + c \quad @\ h=0,\ p=1 \Rightarrow c = \ln(1) = 0.$$

$$\boxed{p = e^{-0.13h},\ @h=10,\ p = e^{-1.3}}$$

b) Find the diff in pressure between the top and bottom of the Green building (pretend $\Delta h = 100$ m $= 0.1$ km).
Compute numerical val. using a calc. Use the lin. approx to e^x near $x=0$ to estimate % drop in pressure

$$p_{top} = e^{-0.13\cdot 0.1} = e^{-0.013} \quad p_{bottom} = e^{-0.13\cdot 0} = 1. \quad \boxed{\Delta p} = p_{top} - p_{bottom} \approx 0.987084 - 1 \approx \boxed{-0.012916}\ (\text{kg/cm}^2)$$

linear approx is $(e^x)\big|_{x=0} + (e^x)'\big|_{x=0}(x-0) = 1 + x$.

and to $e^{-0.13}$ is $1 + (-0.13)(x) \Rightarrow$ Estimated difference is $(1 - 0.13\cdot 0.1) - 1 = \boxed{-0.013}$ (kg/cm²)

which is $\boxed{\approx 1.3\% \text{ out of pressure at sea level.}}$

c) Use the linear approx. $\Delta p \approx p'(0)\Delta h$, and compute $p'(0)$ directly to find drop in pressure for Green building. What does the lin. approx. give for the top of Mt. Everest?

$$\Delta p \approx p'(0)\,\Delta h = (-0.13)\cdot 0.1 \approx -0.013.\ //$$

Mt. Everest: $\Delta p \approx (-0.13)\cdot 10 = -1.3$ impossible! drop in pressure to a negative value!
($\Rightarrow$ pressure at top would be $1 - 1.3 = -0.3$ ⊗)

Problem 2 FIB begins w/ 676 employees. FIB's workforce population P loses employees at a rate of $-\sqrt{P}$ people per week. When will FIB no longer have any employees?

$$\frac{dP}{dt} = -\sqrt{P} \qquad P^{-1/2}\,dP = -1\,dt \Rightarrow 2P^{1/2} = -t + c' \qquad P = \left(-\tfrac{1}{2}t + c\right)^2$$

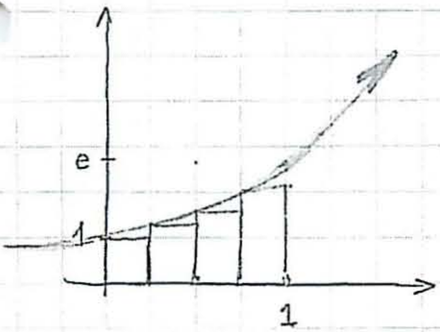
(take $c > 0$)

Find c: $P(0) = 676 \Rightarrow c^2 = 676 \Rightarrow c = \sqrt{676} \Rightarrow c = 2\cdot 13 = 26.$ ($676 = 4\cdot 13^2$)

$$P = \left(-\tfrac{1}{2}t + 26\right)^2 \Rightarrow P = 0 \text{ when } \tfrac{1}{2}t = 26 \Rightarrow \boxed{t = 52 \text{ weeks}}$$

. . .

Problem 3 Calculate $\int_0^1 e^x \, dx$ using lower Riemann sums.



Since e^x is monotone increasing, lower Riemann sums are left Riemann sums.
Label endpts of n equal int. $x_0, x_1, \dots, x_n$.

n^{th} L.RS.: $S_n = \sum_{i=1}^{n} \text{width}(n) \cdot \text{height}(n) =$

$$= \sum_{i=1}^{n} \frac{1}{n} f(x_{i-1}) = \sum_{i=1}^{n} \frac{1}{n} e^{i-1/n} =$$

$$\overset{\text{reindex (reuse notation)}}{=} \sum_{i=0}^{n-1} \frac{1}{n} e^{i/n} = \frac{1}{n} \sum_{i=0}^{n-1} \underbrace{e^{i/n}}_{(e^{1/n})^i} = \frac{1}{n} \frac{1-e^{n-1+1/n}}{1-e^{1/n}} = \frac{1}{n} \frac{1-e}{1-e^{1/n}}$$

$$\Rightarrow \int_0^1 e^x \, dx = \lim_{n \to \infty} (1-e)\left(\frac{1}{n}\right)\left(\frac{1}{1-e^{1/n}}\right) =$$

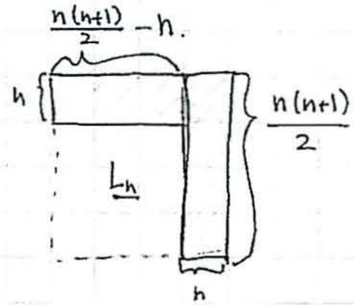
$$= \lim_{n \to \infty} (1-e) \frac{1/n}{1-e^{1/n}} \underset{\text{reuse notation}}{=} (1-e) \lim_{n \to 0} \frac{n}{1-e^{n}} = (1-e) \lim_{h \to 0} \frac{h}{1-e^h}$$

$$\overset{L'H.}{=} (1-e) \underbrace{\lim_{h \to 0} \frac{1}{-e^h}}_{=-1} = \boxed{e-1}.$$

$\boxed{\text{5B-Part II}}$

Problem 1

a) Simmons 6.3/8. Use a special division of a square to complete proof of $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$.



$$\text{"Area"}(L_n) = n\left(\frac{n(n+1)}{2} - n\right) + n\left(\frac{n(n+1)}{2}\right)$$

$$= n^2\left(\frac{n+1}{\cancel{2}}\right)(\cancel{2}) - n^2 = n^2\left(n+\cancel{1}-\cancel{1}\right) = n^3 \; //.$$

proof/

This completes the argument in the statement of the problem that

$$1^3 + \dots + n^3 = L_1 + \dots + L_n = \text{Area of square of side } \frac{n(n+1)}{2} = \left(\frac{n(n+1)}{2}\right)^2.$$

b) Make a conjecture about a general formula for $\sum_{k=1}^{n} k^r$

The general formula is $\sum_{k=1}^{n} k^r = \sum_{l=1}^{r+1} \binom{r+1}{l} \frac{B_{r+1-l}}{r+1} (n+1)^l$ (*), where B_m is the m^{th} Bernoulli number

(look up "Faulhaber's formula" or "Sum of consecutive powers" for more info.)

You can get some information about this formula also by looking at the integral the sum is approximating. $\sum_{k=1}^{n} k^r$ is a right Riemann sum for $\int^n x^r \, dr$, which equals $\frac{n^{r+1}}{r+1}$.
Can argue that the leading term in a formula for (*) should be $\frac{n^{r+1}}{r+1}$ // ...

Problem 2 Use an integral to estimate the sum $\sum_{i=1}^{10000} \sqrt{i}$

$$\sum_{i=1}^{10000} \sqrt{i} \approx \int_0^{10000} \sqrt{i}\, di = \frac{i^{3/2}}{3/2}\Big|_0^{10^4} = \frac{2}{3}\left(10^4\right)^{3/2} = \boxed{\frac{2}{3}\cdot 10^6 \approx 666\,667}$$

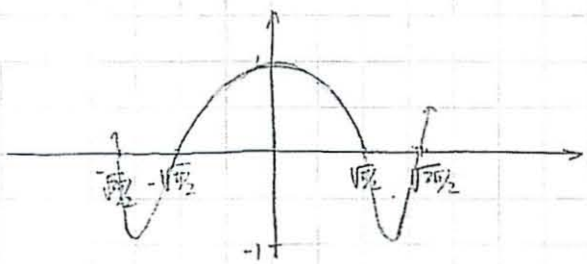
Problem 3 Compute $\lim_{x\to 3} \frac{x^2}{x-3}\int_3^x \frac{\sin t}{t}\,dt$

$$\lim_{x\to 3} \frac{x^2 \int_3^x \frac{\sin t}{t}\,dt}{x-3} \overset{\text{L'Hopital's}}{=} \lim_{x\to 3} \frac{2x\int_3^x \frac{\sin t}{t}\,dt + x^{\cancel{2}}\frac{\sin x}{\cancel{x}}}{1} = 0 + \lim_{x\to 3} x\sin x = \boxed{3\sin 3}.$$

Problem 4 [Fresnel integrals] Consider $f(x) = \int_0^x \cos(t^2)\,dt$. There is no expression for $f(x)$ in terms of standard elementary functions.

a) Draw a rough sketch of $\cos(t^2)$, showing the first positive and negative zeroes. What does the curve look like at $t=0$? Is the function even or odd?

$\cos(-t)^2 = \cos t^2 \Rightarrow$ function is even.



b) List the critical points for $f(x)$ in the entire range $-\infty < x < \infty$.
Which critical points are local maxima and which are local minima?

$f'(x) = \cos(x^2) \qquad f'(x)=0 \Rightarrow \cos(x^2)=0 \Rightarrow x^2 = \frac{\pi}{2} + \pi k \Rightarrow$ $\boxed{\text{Critical pts.: } x = \pm\sqrt{\frac{\pi}{2}+\pi k},\ k\ge 0}$

$\boxed{\text{maxima}}$: $\begin{cases} +\sqrt{\frac{\pi}{2} + 2k\pi} \\ -\sqrt{\frac{\pi}{2} + (2k+1)\pi} \end{cases}$ $\quad \boxed{\text{minima}}$: $\begin{cases} +\sqrt{\frac{\pi}{2} + (2k+1)\pi} \\ -\sqrt{\pi/2 + 2k\pi}. \end{cases}$

$\boxed{k \ge 0, \text{int.}}$

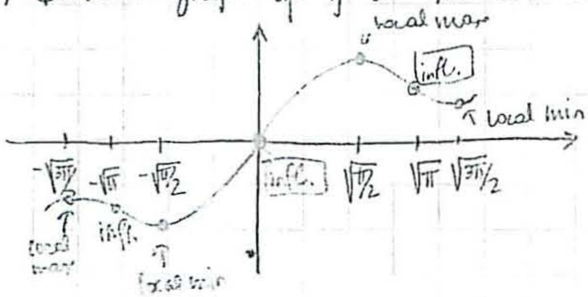
(from graph above, deduce which are local max/min)

c) Sketch a graph of f on the interval $-2 \le x \le 2$, labelling critical pts and inflection pts.

Inflection pts: $f''(x)$ changes sign.

$f''(x) = (\cos x^2)' = -2x\sin(x^2) \qquad f''(x)=0$ when $x=0$

or $\sin(x^2)=0 \Rightarrow \boxed{x = \pm\sqrt{\pi k}}$



d) Estimate $f(0.1)$ to six decimal places.

[Note: lin. or quadratic approx. is also ok here (in terms of grading)]

Estimate $f(0.1)$ "area under $\cos(t^2)$ from $t=0$ to $t=0.1$" using Riemann sums.

$$\sum_{i=1}^{n} \cos(t_i^2)\,\Delta t_i = \sum_{i=1}^{n} \cos\left(\frac{0.1\,i}{n}\right)^2 \cdot \left(\frac{0.1}{n}\right) =: S_n .$$

How many subdiv. of $[0, 0.1]$ are necessary to get 6 sig fig accuracy? (i.e. an error $< 10^{-6}$ or 10^{-7}?)

$$10^{-7} = \frac{0.1}{n}(\cos 0 - \cos 0.1^2) \Rightarrow n \approx 5 \cdot 10^{-6} \cdot 10^{7} = 50.$$

For $n=50$, above sum is 0.0999989.

e) Scaling of variables and Fresnel integrals.

i) Let $g(x) = \int_0^x \cos\left(\frac{\pi}{2}u^2\right) du$. Show that $f(x) = c_1\, g(c_2 x)$. Why choose $\frac{\pi}{2}$?

$f(x) = \int_0^x \cos t^2\, dt$. Let $v = \sqrt{\frac{\pi}{2}}\, u$ $\quad dv = \sqrt{\frac{\pi}{2}}\, du \quad 0 \le u \le x \Rightarrow 0 \le v \le \sqrt{\frac{\pi}{2}}\, x$.

$$\boxed{g(x)} = \int_0^{\sqrt{\pi/2}\,x} (\cos v^2) \sqrt{\frac{2}{\pi}}\, dv = \sqrt{\frac{2}{\pi}} \int_0^{\sqrt{\frac{\pi}{2}}\,x} \cos v^2\, dv = \boxed{\sqrt{\frac{2}{\pi}}\, f\left(\sqrt{\frac{\pi}{2}}\, x\right)}$$

$\frac{\pi}{2}u^2 = \frac{\pi}{2} + k\pi \Rightarrow u^2 = 1 + 2k$

Here chose $\pi/2$ to get nice looking critical pts ($g(x)$ will have critical pts at $x = \pm\sqrt{1+2k}$, k-nonneg int.)

ii) Let $h(x) = \int_0^x \frac{\cos v}{\sqrt{v}}\, dv$. Show $f(x) = c\,h(x^2)$ for some const. c. (assume $x > 0$)

Let $t = \sqrt{v} \Rightarrow dt = +\frac{1}{2\sqrt{v}}\, dv \qquad 0 \le t \le x \Rightarrow 0 \le v \le x^2$

$$\boxed{f(x)} = \int_0^x \cos t^2\, dt = \int_0^{x^2} \cos v \left(+\frac{1}{2\sqrt{v}}\, dv\right) = +\frac{1}{2} \int_0^{x^2} \frac{\cos v}{\sqrt{v}}\, dv = \boxed{+\frac{1}{2}\, h(x^2)}$$

iii) Let $k(x) = \sqrt{x} \int_0^1 \cos(x t^2)\, dt$, $x > 0$. Use the change of var. $z = xt^2$ and part (ii) to find the rel. between the functions k and f.

$z = xt^2 \quad dz = 2xt\, dt \Rightarrow dt = \frac{1}{2}\frac{1}{x^{1/2} z^{1/2}}\, dz . \qquad 0 \le t \le 1 \Rightarrow 0 \le z \le x.$

$z = \frac{(xt)^2}{x} \Rightarrow \sqrt{xz} = xt \qquad dt = \frac{dz}{2xt} = \frac{dz}{2x\sqrt{z/x}}$

$$k(x) = \sqrt{x} \int_0^x \cos z\, \frac{1}{2\sqrt{x}\sqrt{z}}\, dz = \frac{\sqrt{x}}{2\sqrt{x}} \int_0^x \frac{\cos z}{\sqrt{z}}\, dz = \frac{\cancel{\sqrt{x}}}{2\cancel{\sqrt{x}}}\, h(x) = \frac{1}{2}\, h(x)$$

but $h(x) = +2 f(\sqrt{x})$ $\quad$ so $k(x) = \frac{1}{2}(+2 f(\sqrt{x})) = + f(\sqrt{x}) \Rightarrow \boxed{k(x) = f(\sqrt{x})}$

Problem 5

a) 7.3/22

Given: $\underset{\text{rate of evaporation.}}{\frac{dV}{dt}} = -\underset{\text{pos. const.}}{c}\; \underset{\text{depth.}}{A(h)}$, volume of water in bowl, area of surface

Know: $V = \int_0^h A(x)\,dx$.

- Show that $dh/dt = -c$

$$\frac{dV}{dt} = \frac{d}{dt}\int_0^h A(x)\,dx = A(h)\frac{dh}{dt}$$. But also have $\frac{dV}{dt} = -c\,A(h)$

$$\Rightarrow \cancel{A(h)}\frac{dh}{dt} = -c\,\cancel{A(h)} \quad \Rightarrow \boxed{\frac{dh}{dt} = -c}.$$

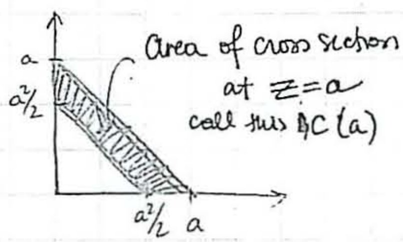
- If $h = h_0$ @ $t=0$, when will the bowl be empty?

Determine formula for $h = h(t)$ $\quad h = \int \frac{dh}{dt}\,dt = \int -c\,dt \Rightarrow h = -ct + \underset{\text{a const}}{C_0} \quad C_0 = h(0) = h_0.$

$$\Rightarrow h = -ct + h_0 \quad \Rightarrow \text{bowl will be empty at } \boxed{t = \frac{h_0}{c}}.$$

b) Find the volume of the region in 3 space w/ $x, y, z > 0$ and $z^2/2 < x+y < z$.

Looking down, get the following horizontal cross-sections:



$$\text{Area } @\ z=a = C(a) = \frac{1}{2}a^2 - \frac{1}{2}\left(a^2/2\right)^2 = \frac{1}{2}a^2 - \frac{1}{8}a^4$$

$$\Rightarrow \text{Volume} = \int_{z=0}^{2} C(z)\,dz =$$

$$= \int_0^2 \frac{1}{2}z^2 - \frac{1}{8}z^4\,dz = \frac{1}{2}z^3/3 - \frac{1}{8}\frac{z^5}{5}\Big|_0^2 =$$

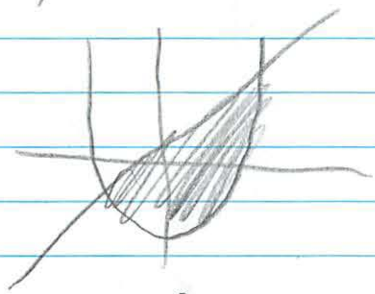
$$= \frac{2^3}{2\cdot 3} - \frac{2^5}{2^3 \cdot 5} = \frac{2^2}{3} - \frac{2^2}{5} = 2^2\left(\frac{5-3}{15}\right)$$

$$= \frac{2^2(2)}{15} = \boxed{\frac{8}{15}}.$$

Recitation
Volume + Area

11/2

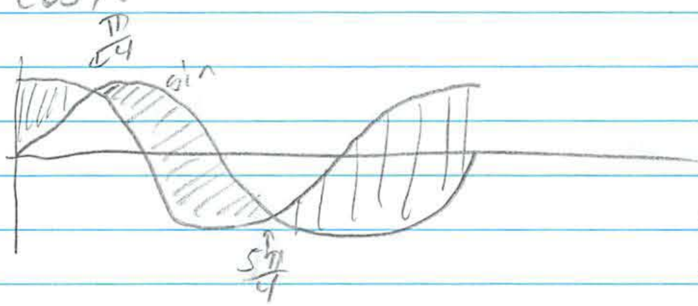
Area between
$y = x$
$y = x^2 - 6$



(1) Find where intersect
$x = x^2 - 6$ factor, set = to 0
$0 = x^2 - x - 6$
quad formula if desperate
$x = 3, -2$

$\int_{-2}^{3} f(x) - g(x)$ = area between f + g when
$f \geq g$

Area between
$\sin x$ on $0 \leq x \leq 2\pi$
$\cos x$



can't just do one
b/c it switches

$\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \sin x - \cos x = \cos x + \sin x \Big|_{\frac{\pi}{4}}^{\frac{5\pi}{4}}$

→ don't forget to integrate

$\sin \frac{5\pi}{4} + \cos \frac{5\pi}{4} - \left(\sin \frac{\pi}{4} + \cos \frac{\pi}{4}\right)$
$\sqrt{2} - 1$

$\int_{\frac{5\pi}{4}}^{\frac{8\pi}{4}} \cos x - \sin x = \sin x - \cos x \Big|$

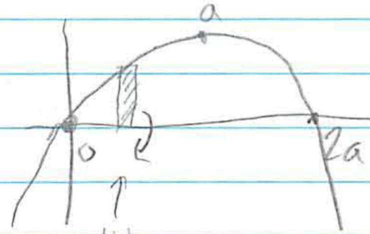
→ don't forget to integrate

$\cos \frac{8\pi}{4} - \sin \frac{8\pi}{4} - \left(\cos \frac{5\pi}{4} - \sin \frac{5\pi}{4}\right)$

Find the volume of the shape made by rotating region between $y=0$

$y = 2ax - x^2$ ← quadratic opening down

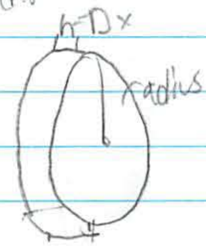
around the x axis



rotate

would be sharp point since deriv does not go to 0

disk method

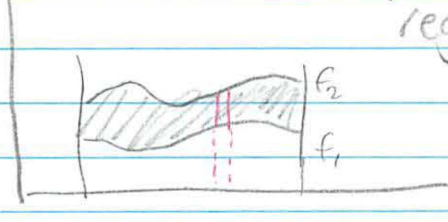


$$\sum \overset{\pi r^2 h}{\pi f(x)^2 \Delta x} \quad \underline{|\Delta x| \to 0} \quad \int \pi f(x)^2 dx$$

must do $\boxed{\int f_2^2 - f_1^2}$ not $\cancel{\int (f_2 - f_1)^2}$

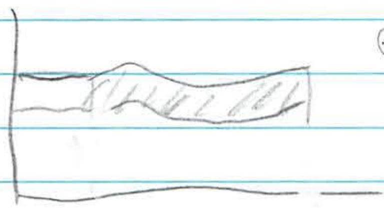
Example w/ disks

Find volume made by rotating region between f_2 and f_1 around x axis



would make a pipe

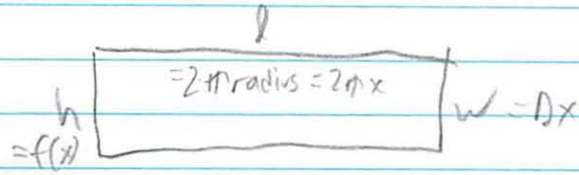
shell method



← donut

around y axis

can cut open + find volume



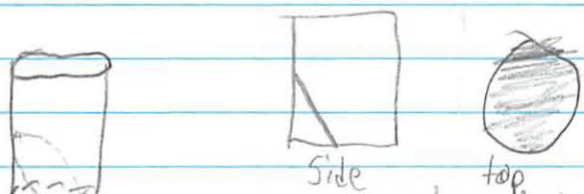
$$2\pi x f(x) \Delta x \rightarrow 2\pi \int_0^{2a} f(x) \cdot x \cdot dx$$

↓ plug in x

$$2\pi \int_0^{2a} (2ax - x^2) \, x \, dx$$

$$2\pi \int_0^{2a} 2ax^2 - x^3 \, dx$$

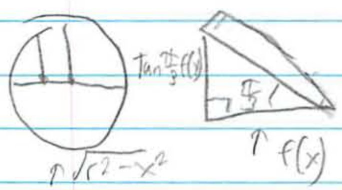
Find the volume



not rotational problem

↑ plane in cylinder volume below

slicing w/ perpendicular planes - makes triangle



base $= f(x)$
height $= \sqrt{3} f(x)$
area $= \frac{1}{2} bh$
$\sqrt{3} f(x)^2$
volume $= \frac{\sqrt{3} f(x)^2}{2} \Delta x$

$$2 \int_0^r \frac{\sqrt{3}}{2} (r^2 - x^2) \, dx \qquad r = \text{constant}$$

Briver Office Hrs

11/2

w/ Sacha

$\int 7x^4 e^{x^5}$ ← $5x^4$ ← identify as u

$u = x^5$
$du = 5x^4\,dx$ ← $\frac{1}{5}du = x^4dx$

$\int 7\underbrace{x^4}_{\frac{1}{5}du} e^u\,dx$

$\int \frac{7}{5} e^u\,du$

$\frac{7}{5}e^u$ plug x back in
$\frac{7}{5}e^{x^5} + C$

$\int x^3 (1-12x^4)^{1/8}\,dx$

$u = 1-12x^4$
$du = -48x^3\,dx$ ← $-\frac{1}{48}du = x^3dx$

$\int -\frac{1}{48}du\; u^{1/8}$

$-\frac{1}{48} \frac{u^{9/8}}{\frac{9}{8}}$

$-\frac{1}{48}\cdot\frac{8}{9}(1-12x^4)^{9/8}$

$\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$

$u = \sqrt{x}$ $(x)^{1/2}$ $\frac{1}{2}x^{-1/2}$

$du = \frac{1}{2\sqrt{x}}\,dx$

$\int \frac{e^u}{u}\,dx$

$dx = 2\sqrt{x}\,du$

* $\boxed{\frac{dx}{\sqrt{x}} = 2du}$ * ← do this or it would cancle out

$\int e^u \, 2du$

$2\int e^u \, du$

$2e^u$

$2e^{\sqrt{x}}$

other way

$\int \frac{e^u}{\cancel{u}} 2\sqrt{x}$

*note ↑ is equal and would cross out

$\int \frac{dx}{x \ln x} \quad \frac{1}{x \ln x} dx$

need u and du to be helpful

~~$u = \frac{1}{x}$~~

~~$du = \ln x \, dx$~~ wrong

~~$dx = \frac{du}{\ln x} = \frac{1}{\ln x} du$~~

~~$\int u \frac{1}{\ln x} dx$~~

~~$\int u$~~

$u = \ln x$

$du = \frac{1}{x} dx$

$dx = x \, du$

$\int \frac{1}{\cancel{x} u} \cancel{x} \, du$

$\int \frac{du}{u} \rightarrow \ln u = \ln(\ln x)$

can check w/ derivative (use chain rule)

$\int \sin x \cos x \, dx$

$u = \sin x$

$du = \cos x \, dx$

$dx = \frac{du}{\cos x}$

$\int u \cos x \, dx$

$\int u \, du$

$\frac{u^2}{2} \rightarrow \frac{\sin^2 x}{2} + C$ ✓ just like any other one

$\int \tan^6 x \sec^2 x \, dx$

$u = \tan x$

$du = \sec^2 x \, dx$

$dx = \frac{du}{\sec^2 x}$

$\int u^6 \, du$

$\frac{u^7}{7}$ $\frac{\tan^7 x}{7} + C$

Lecture 23

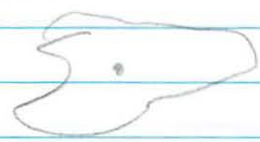
Work, Average Value, Probability   11/3

Pset due fri
Exam next tue
Next wed off

today 4 more applications of integration

Rieeman sums → integrals in new ways

Center of Mass



← 3D shape (lamina)

← should balance on COM

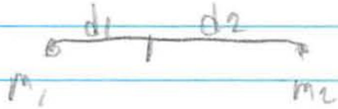
in physics consider all mass as concentrated to COM

to start look at COM in 1 dimension

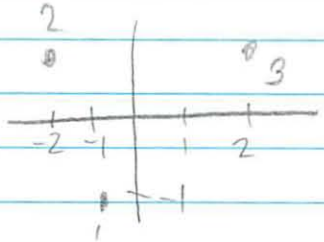
m_1 m_2 ← masses on end, rod neglible

Where should put fulcrum?

$$m_1 d_1 = m_2 d_2$$



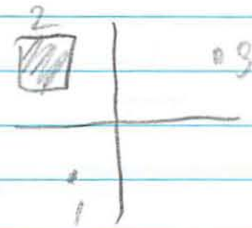
Can show that if have endpoints on a line same principle holds $\bar{x}$ ← center of mass $= \dfrac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i}$



add up Σx Σy

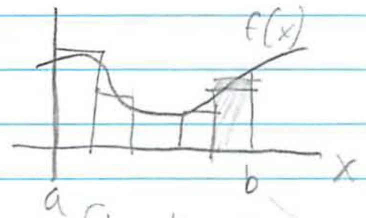
$$\frac{-1m_1 - 2m_2 + 2m_3}{m_1 + m_2 + m_3} = \bar{x}$$

and for $\bar{y}$



find center of square and just use that

What about



density ρ constant

What is $\bar{x}$?

Use symmetry principle
center of mass of rectangle (w/ respect to x) should lie on axis of symmetry
- ie on midpoint

Mass of rectangle = density · volume
[w/ laminae - ignore 3rd dimension
so mass = density · area]

$$\bar{x} \approx \rho \cdot f(x_i)\Delta x \approx \frac{\sum_{i=1}^{n} (\rho f(x_i) \Delta x) x_i}{\sum_{i=1}^{n} \rho f(x)} \begin{matrix} \leftarrow m_i x_i \\ \\ \leftarrow m_i \end{matrix}$$

definite integral

$$\bar{x}_{def} = \frac{\int_a^b \rho f(x) x \, dx}{\int_a^b \rho f(x) \, dx} \begin{matrix} \leftarrow \text{moment } (m_y) \\ \leftarrow \text{mass } (m) \end{matrix}$$

With respect to y

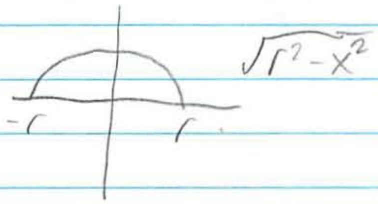
- on axis of symmetry again

$$\bar{y} \approx \frac{\sum_{i=1}^{n} \rho_i f(x_i) \Delta x \, \frac{f(x_i)}{2}}{M} \qquad M = \int_a^b \rho f(x)\, dx$$

$$\bar{y} = \frac{\int_a^b \rho \frac{f(x)^2}{2} dx}{\int_a^b \rho f(x)\, dx}$$

Find center of mass of a semicircle of radius r

- constant density



What is $\bar{x}$?

$= 0$ symmetric

← or could compute the integral

$\bar{y}$?

- much harder

$$\int_{-r}^{r} \frac{\rho}{2} \left(\sqrt{r^2-x^2}\right)^2 dx$$

$$\int_0^r \rho(r^2-x^2)\, dx$$

$$\rho\left(r^2 x - \frac{1}{3}x^3\right)\Big|_{x=0}^{x=r}$$

Divide by mass $= \rho \cdot \text{area}$

$= \rho \cdot \frac{\pi r^2}{2}$

$$\bar{y} = \frac{4r}{3\pi}$$

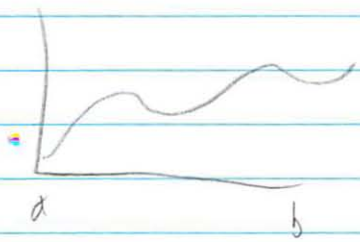
Theorm of Pappos for volume of revolution

Volume of solid of revolution = Area revolving $\cdot$ distance traveled by center of mass in a circle around line of rotation

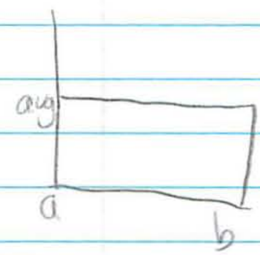
$$\text{area} = \frac{\pi r^2}{2} \qquad \text{distance} = \left(\frac{4r}{3\pi}\right) 2\pi$$

Volume $\quad \frac{4r}{3\pi} \cdot 2\pi \cdot \frac{\pi r^2}{2} = \boxed{\frac{4}{3}\pi r^3}$

Average Value of Function



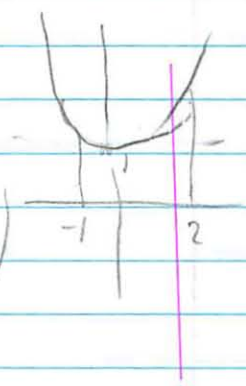
where is avg =



Should have same value

Compute avg value of $1 + x^2$ on $[-1, 2]$

$$\text{Avg vale} = \int_{-1}^{2} \frac{(1+x^2)\,dx}{3}$$



$$\frac{1}{3}\left(x + \frac{1}{3}x^3\right)$$

2

Finite set of #

Avg $y_1 + y_2 + \dots + y_n$

$$f_{Avg} \approx \frac{f(x_1) + f(x_2) + \dots f(x_n)}{n}$$

$$\approx \frac{\sum_{i=1}^{n} f(x_i)}{n} \qquad n = \frac{b-a}{\Delta x}$$

$$\approx \frac{\sum_{i=1}^{n} f(x_i)}{\frac{b-a}{\Delta x}}$$

$$\frac{\sum_{i=1}^{n} f(x_i)\Delta x}{b-a}$$

$$\frac{1}{b-a}\int_a^b f(x)\,dx$$

Called calculus b/c can calculate things

Lecture 23: Work, Average Value, Probability

Note did center of mass - not in these notes

Application of Integration to Average Value

You already know how to take the average of a set of discrete numbers:

$$\frac{a_1 + a_2}{2} \text{ or } \frac{a_1 + a_2 + a_3}{3}$$

Now, we want to find the average of a continuum.

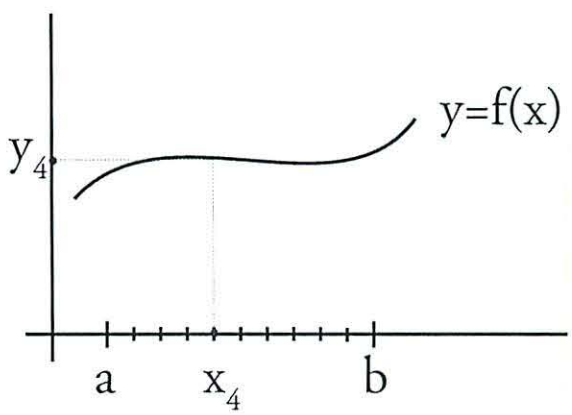


Figure 1: Discrete approximation to $y = f(x)$ on $a \leq x \leq b$.

$$\text{Average } \approx \frac{y_1 + y_2 + ... + y_n}{n}$$

where

$$a = x_0 < x_1 < \cdots x_n = b$$

$$y_0 = f(x_0),\ y_1 = f(x_1),\ \ldots y_n = f(x_n)$$

and

$$n(\Delta x) = b - a \quad \Longleftrightarrow \quad \Delta x = \frac{b-a}{n}$$

and

The limit of the Riemann Sums is

$$\lim_{n\to\infty} (y_1 + \cdots + y_n)\frac{b-a}{n} = \int_a^b f(x)\, dx$$

← riemann sum/it integrate

Divide by $b - a$ to get the continuous average

$$\lim_{n\to\infty} \frac{y_1 + \cdots + y_n}{n} = \frac{1}{b-a}\int_a^b f(x)\, dx$$

divide b − a

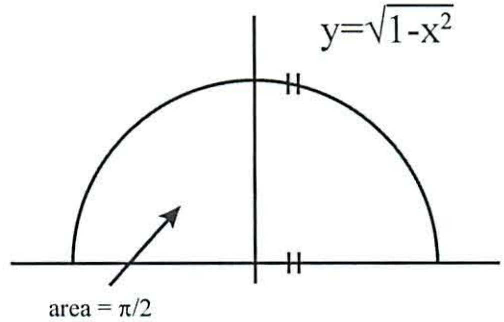


Figure 2: Average height of the semicircle.

Example 1. Find the average of $y = \sqrt{1-x^2}$ on the interval $-1 \le x \le 1$. (See Figure 2)

← split up if doing real life

$$\text{Average height} = \frac{1}{2}\int_{-1}^{1} \sqrt{1-x^2}dx = \frac{1}{2}\left(\frac{\pi}{2}\right) = \frac{\pi}{4}$$

Example 2. The average of a constant is the same constant

$$\frac{1}{b-a}\int_a^b 53\, dx = 53$$

Example 3. Find the average height y on a semicircle, with respect to *arclength.* (Use $d\theta$ not dx. See Figure 3)

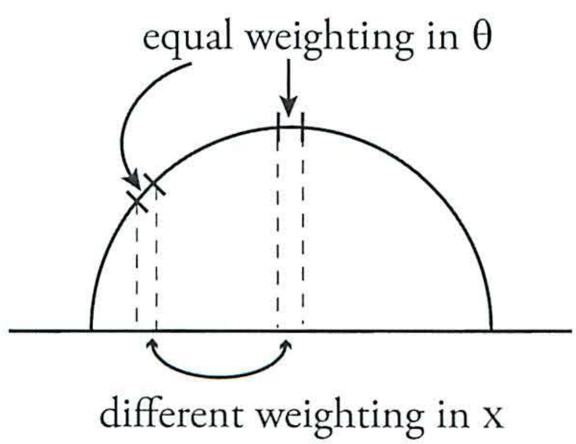


Figure 3: Different weighted averages.

$$\begin{aligned} y &= \sin\theta \\ \text{Average } = \frac{1}{\pi}\int_0^{\pi}\sin\theta\,d\theta = \frac{1}{\pi}(-\cos\theta)\Big|_0^{\pi} &= \frac{1}{\pi}(-\cos\pi - (-\cos 0)) = \frac{2}{\pi}\end{aligned}$$

Example 4. Find the average temperature of water in the witches cauldron from last lecture. (See Figure 4).

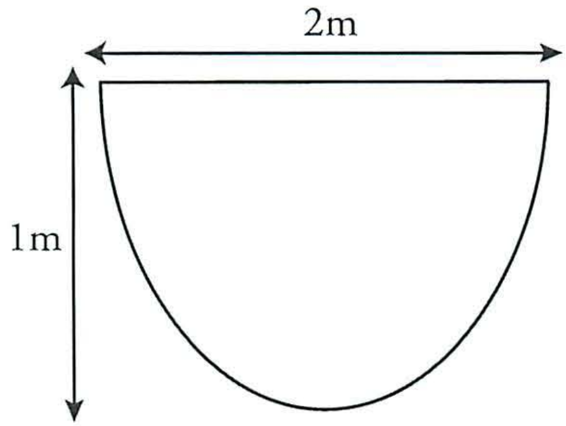


Figure 4: $y = x^2$, rotated about the y-axis.

First, recall how to find the volume of the solid of revolution by disks.

$$V = \int_0^1 (\pi x^2)\,dy = \int_0^1 \pi y\,dy = \frac{\pi y^2}{2}\Big|_0^1 = \frac{\pi}{2}$$

Recall that $T(y) = 100 - 30y$ and $(T(0) = 100^o; T(1) = 70^o)$. The average temperature per unit volume is computed by giving an importance or "weighting" $w(y) = \pi y$ to the disk at height y.

$$\frac{\int_0^1 T(y)w(y)\,dy}{\int_0^1 w(y)\,dy}$$

The numerator is

$$\int_0^1 T\pi y\,dy = \pi\int_0^1 (100-30y)y dy = \pi(500y^2 - 10y^3)\Big|_0^1 = 40\pi$$

Thus the average temperature is:

$$\frac{40\pi}{\pi/2} = 80^o C$$

Compare this with the average taken with respect to height y:

$$\frac{1}{1}\int_0^1 T\,dy = \int_0^1 (100-30y)dy = (100y - 15y^2)\Big|_0^1 = 85^o C$$

T is linear. Largest $T = 100^o C$, smallest $T = 70^o C$, and the average of the two is

$$\frac{70+100}{2} = 85$$

The answer 85^o is consistent with the ordinary average. The weighted average (integration with respect to $\pi y\, dy$) is lower (80^o) because there is more water at cooler temperatures in the upper parts of the cauldron.

Dart board, revisited

probability

Last time, we said that the accuracy of your aim at a dart board follows a "normal distribution":

$$ce^{-r^2}$$ ← given

Now, let's pretend someone – say, your little brother – foolishly decides to stand close to the dart board. What is the chance that he'll get hit by a stray dart?

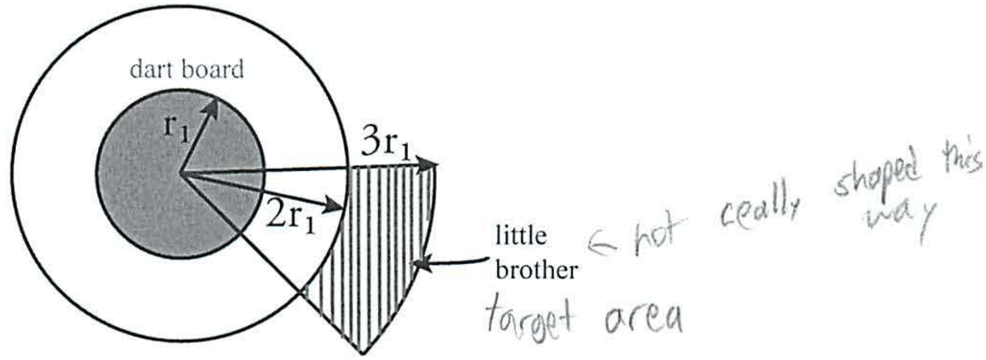


Figure 5: Shaded section is $2r_i < r < 3r_1$ between 3 and 5 o'clock.

To make our calculations easier, let's approximate your brother as a sector (the shaded region in Fig. 5). Your brother doesn't quite stand in front of the dart board. Let us say he stands at a distance r from the center where $2r_1 \leq r < 3r_1$ and r_1 is the radius of the dart board. Note that your brother doesn't surround the dart board. Let us say he covers the region between 3 o'clock and 5 o'clock, or $\frac{1}{6}$ of a ring.

Remember that

$$\boxed{\text{probability} = \frac{\text{part}}{\text{whole}}}$$ ← *

$$\text{Probability} = \frac{-e^{-9r_1^2} + e^{-4r_1^2}}{6}$$

Let's assume that the person throwing the darts hits the dartboard ring $0 \le r \le r_1$ about half the time. (Based on personal experience with 7-year-olds, this is realistic.)

$$P(0 \le r \le r_1) = \frac{1}{2} = \int_0^{r_1} 2e^{-r^2} r\,dr = -e^{-r_1^2} + 1 \implies e^{-r_1{}^2} = \frac{1}{2}$$

r_1 to calibrate

$$e^{-r_1^2} = \frac{1}{2}$$

$$e^{-9r_1^2} = \left(e^{-r_1^2}\right)^9 = \left(\frac{1}{2}\right)^9 \approx 0$$

$$e^{-4r_1^2} = \left(e^{-r_1^2}\right)^4 = \left(\frac{1}{2}\right)^4 = \frac{1}{16}$$

So, the probability that a stray dart will strike your little brother is

hits bro

$$\left(\frac{1}{16}\right)\left(\frac{1}{6}\right) \approx \frac{1}{100}$$

In other words, there's about a 1% chance he'll get hit with each dart thrown.

will even hit

Volume by Slices: An Important Example

Compute $Q = \int_{-\infty}^{\infty} e^{-x^2}\, dx$

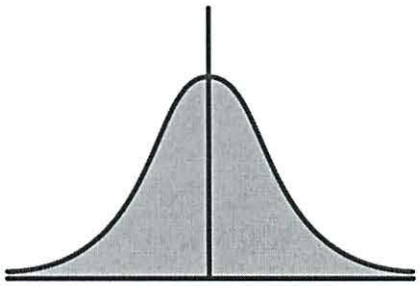


Figure 8: $Q =$ Area under curve $e^{(-x^2)}$.

This is one of the most important integrals in all of calculus. It is especially important in probability and statistics. It's an improper integral, but don't let those ∞'s scare you. In this integral, they're actually easier to work with than finite numbers would be.

To find Q, we will first find a volume of revolution, namely,

$$V = \text{volume under } e^{-r^2} \quad (r = \sqrt{x^2 + y^2})$$

We find this volume by the method of shells, which leads to the same integral as in the last problem. The shell or cylinder under e^{-r^2} at radius r has circumference $2\pi r$, thickness dr; (see Figure 9). Therefore $dV = e^{-r^2} 2\pi r dr$. In the range $0 \le r \le R$,

$$\int_0^R e^{-r^2} 2\pi r\, dr = -\pi e^{-r^2}\Big|_0^R = -\pi e^{-R^2} + \pi$$

When $R \to \infty, e^{-R^2} \to 0$,

$$V = \int_0^\infty e^{-r^2} 2\pi r\, dr = \pi \qquad \text{(same as in the darts problem)}$$

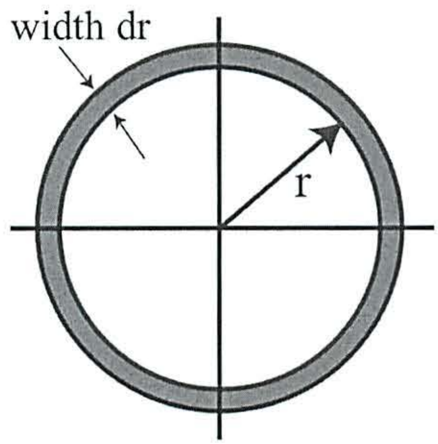


Figure 9: Area of annulus or ring, $(2\pi r)dr$.

Next, we will find V by a second method, the method of slices. Slice the solid along a plane where y is fixed. (See Figure 10). Call $A(y)$ the cross-sectional area. Since the thickness is dy (see Figure 11),

$$V = \int_{-\infty}^{\infty} A(y)\,dy$$

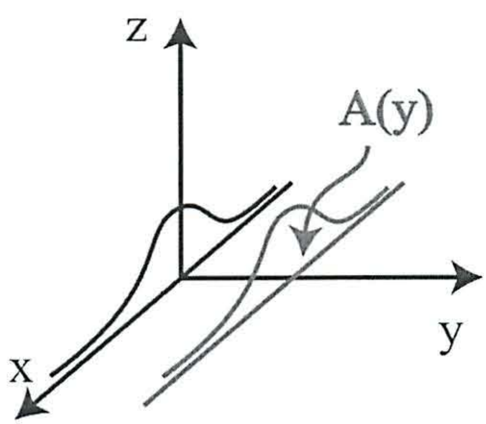


Figure 10: Slice $A(y)$.

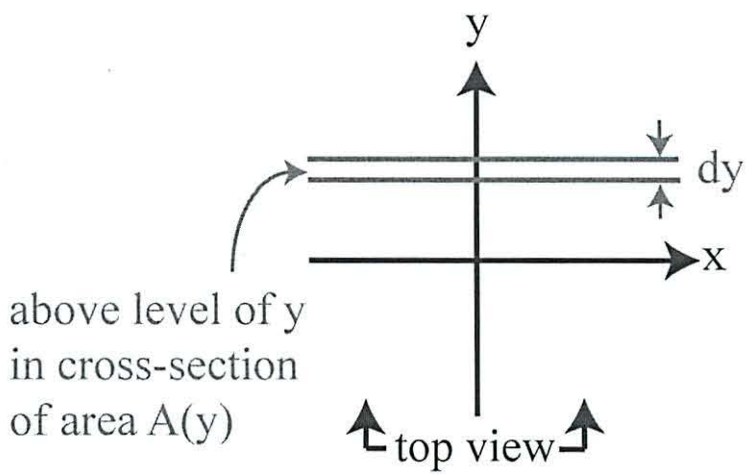


Figure 11: Top view of $A(y)$ slice.

To compute $A(y)$, note that it is an integral (with respect to dx)

$$A(y) = \int_{-\infty}^{\infty} e^{-r^2}\,dx = \int_{-\infty}^{\infty} e^{-x^2-y^2}\,dx = e^{-y^2}\int_{-\infty}^{\infty} e^{-x^2}\,dx = e^{-y^2}Q$$

Here, we have used $r^2 = x^2 + y^2$ and

$$e^{-x^2-y^2} = e^{-x^2}e^{-y^2}$$

and the fact that y is a constant in the $A(y)$ slice (see Figure 12). In other words,

$$\int_{-\infty}^{\infty} ce^{-x^2}\,dx = c\int_{-\infty}^{\infty} e^{-x^2}\,dx \quad \text{with} \quad c = e^{-y^2}$$

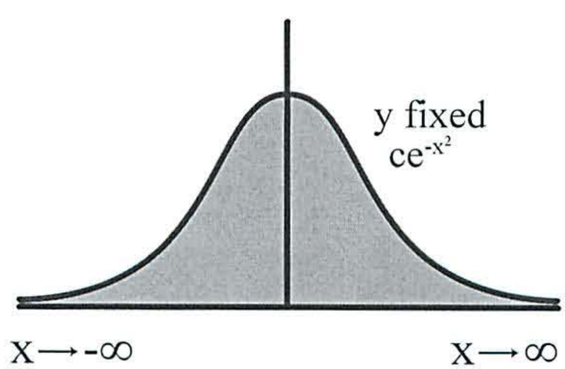


Figure 12: Side view of $A(y)$ slice.

It follows that

$$V = \int_{-\infty}^{\infty} A(y)\,dy = \int_{-\infty}^{\infty} e^{-y^2} Q\,dy = Q\int_{-\infty}^{\infty} e^{-y^2}\,dy = Q^2$$

Indeed,

$$Q = \int_{-\infty}^{\infty} e^{-x^2}\,dx = \int_{-\infty}^{\infty} e^{-y^2}\,dy$$

because the name of the variable does not matter. To conclude the calculation read the equation backwards:

$$\pi = V = Q^2 \implies \boxed{Q = \sqrt{\pi}}$$

We can rewrite $Q = \sqrt{\pi}$ as

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-x^2}\,dx = 1$$

An equivalent rescaled version of this formula (replacing x with $x/\sqrt{2}\sigma$)is used:

$$\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-x^2/2\sigma^2}\,dx = 1$$

This formula is central to probability and statistics. The probability distribution $\frac{1}{\sqrt{2\pi}\sigma}e^{-x^2/2\sigma^2}$ on $-\infty < x < \infty$ is known as the normal distribution, and $\sigma > 0$ is its standard deviation.

Recitation

11/4

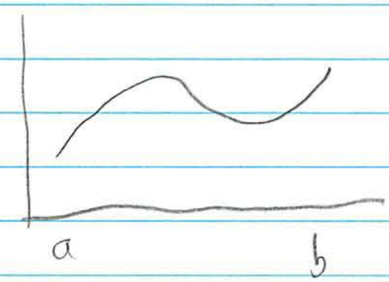
Work

Force : something traveling in a straight line

$F = ma = kg\, m/s^2 = N$

If F is constant
$W = F \cdot d$ $(N \cdot m)$

Suppose $F = F(x)$



(Consider F constant for small intervals)

$$\sum_{i=1}^{n} F(x_i) \cdot \Delta x$$

$\downarrow n \to \infty$

$$\int_a^b F(x)\,dx$$

Hooke's Law

$F(x) = kx$

Force needed to hold a spring x units beyond resting lenght - varries directly w/ x

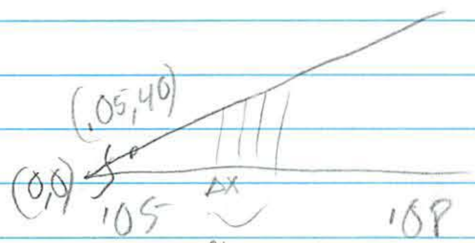
$\to$

Suppose lenght = 10 cm
40 N to hold at 15 cm

Work to pull string 15 → 18 cm

W = Fd
need to find F as a function F = kx

40N = k · .15 − .10 m
k = 800



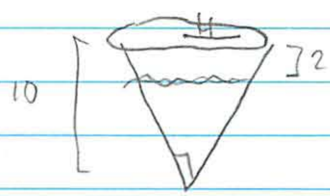
Riemann Sum

So can set up as integral

$$\int_{.05}^{.08} F(x)\,dx$$

Δx goes to 0

Cone w/ height 10 m
radius 4m
sitting on tip
Water to 8m
How much work to empty



W = F·d
F = m·a

have to move the bottom water further
d changes w/ h

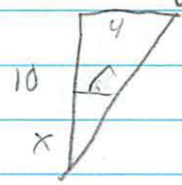
Force changes as well
Since mass of slice changes

$F = ma = 9.8 \cdot \text{mass}$

x = area $\cdot \Delta x$ $\qquad A = \pi r^2$

r changes

use similar triangles



$\frac{4}{10} = \frac{r}{x}$

$r = \frac{4x}{10} = \frac{2}{5}x$

$V = \pi\left(\frac{2}{5}x\right)^2 \Delta x$

distance to travel $10 - x$

mass = 1000 Volume

$\underbrace{9.8\pi \cdot 160}_{\text{constants}} \int_8^0 \underbrace{x^2\,dx}_{\text{force}} \; \underbrace{(10-x)}_{\text{distance}}$

8 since starting at top

Average Value

$\boxed{\Delta x = \frac{b-a}{n}}$

$\frac{\sum_{i=1}^{n} f(x_i)}{n}$ as $n \to \infty$ $\quad = \frac{1}{b-a}\sum_{i=1}^{n} f(x)\,\Delta x$

$n \to \infty$

$\frac{1}{b-a}\int_a^b f(x)\,dx$

all odd functions $\int_{-a}^{a}$ → avg = 0

Let $\frac{1}{2}\sin\left(\frac{2\pi}{5}t\right)$ represent the rate of intake of air into lung

Assume no air at $t=0$

Find the avg volume in the lungs over 1 breathing cycle (period $\frac{2\pi}{\frac{2\pi}{5}} = 5$ sec)

↑ rate of change in volume
directly differential equation

$$\frac{dV}{dt} = \frac{1}{2}\sin\left(\frac{2\pi}{5}t\right)$$

can figure w/ trial error

Guess

$$\cancel{v(t) = -\frac{1}{2}\cos\left(\frac{2\pi}{5}\right)t}$$ wrong - is chain rule
kill off chain rule result

$$\frac{-\frac{1}{2}\cos\left(\frac{2\pi}{5}\right)t}{\frac{2\pi}{5}}$$

$$C = \frac{5\pi}{4}$$

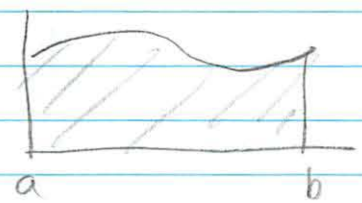
Avg value of $V = \frac{1}{5}\int_0^5 \underbrace{-\frac{5\pi}{4}\cos\left(\frac{2\pi}{5}t\right)}_{\text{goes to } 0} + \frac{5}{4\pi}\,dt$

$= \boxed{5/4\pi}$

Center of Mass

- confusion on how to find
- rotational issue

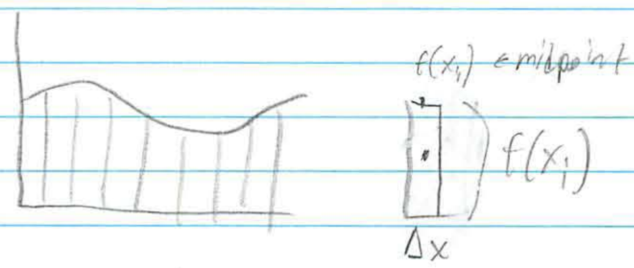
- one where based at 0
- but also where diff between 2 things



$\bar{x} = \frac{\sum mass_i \cdot x_i}{total\ mass}$

if multiple items - take com of each piece

divide up into increments

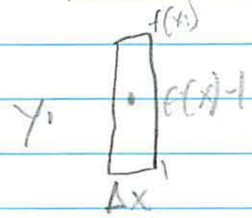


$x_i \cdot f(x_i) \cdot \Delta x$

- if based at $y=0 \rightarrow f(x_i)$
- if based at $y=1 \rightarrow f(x_i) - 1$

$\int_0^b x(f(x)-1)\,dx = \bar{x}$

For $\bar{y}$ = area still the same



← at midpoint of rectangle

$$y_i = \frac{f(x_i) - 1}{2} + 1$$

↑ from 0

Lecture 24
Numerical Integration

11/5

Exam 3 Tue in Walker Gym
Review sessions soon

yesterday
Applications of integration

- Moments (COM)
- Average value

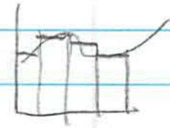
← not on exam ← easy
Today: Work + probability
and numeric integration

"Numeric Integration"

idea - What happens if can't find antiderivative
for definite integral (no FTC to save you)?
What if we want a computer to find integral

Can approximate

$$\int_a^b f(x)\,dx \quad \sum_{i=1}^{n} f(x_i)\,\Delta x \quad \leftarrow \frac{b-a}{n}$$

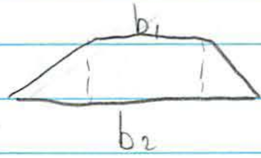


right or left hand endpoint
or can pick midpoint
matters for finite

What if wanted to use other shapes
- only ones we can easily find area of

← connect lines

$$\int_a^b f(x)\,dx \approx \sum \text{area of trapezoids}$$



$$\frac{b_1 + b_2}{2} h$$

check w/ rectangle ✓

$$h = \Delta x \left[\frac{f(x_1) + f(x_2)}{2} + \frac{f(x_2) + f(x_3)}{2} + \frac{f(x_3) + f(x_4)}{2} \right]$$

$$h = \frac{\Delta x}{2} \left[f(x_1) + \underbrace{2(x_2) + 2(x_3)}_{\text{in middle}} + f(x_4) \right]$$

$$h = \frac{\Delta x}{2} \left[f(x_1) + 2f(x_2) + 2(x_3) + \dots + f(x_{n+1}) \right]$$

- answer for error

Given differential function f, suppose

$$|f''(x)| < k \quad \text{on } x \in (a,b)$$

Error in midpoint rule

$$|E_m| \le \frac{k(b-a)^3}{24n^2}$$

← can prove for any differential function

n number of pieces

$$\underset{\text{trapizoidal}}{|E_T|} \le \frac{k(b-a)}{12n^2}$$

← diff is $\frac{1}{2}$
not much better in most cases

Moral: Trapizoes are no better than midpoint right angles

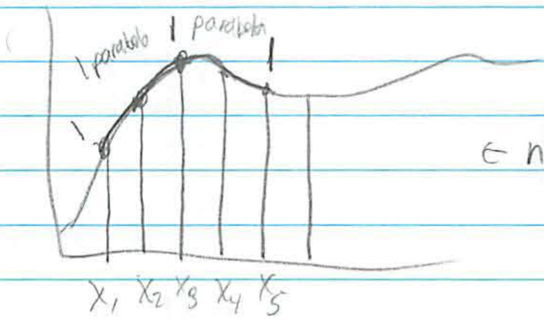
↓ to improve trapizodal rules

Take the best parabola you can think of

Try parabolas $Ax^2 + Bx + C$

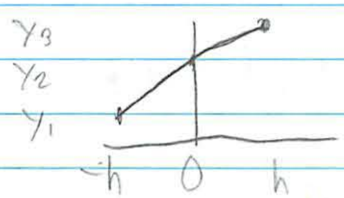
↑ 3 degrees of freedom

Declare 3 points



← n must be even so get 3 points each time

Goal Find area under parabola



↑ makes life easy

x_1 x_2 x_3

$$Ax^2 + Bx + C$$

$$\int_{-h}^{h} (Ax^2 + Bx + C)\,dx$$

↓

$$\frac{A}{3}x^3 + \frac{B}{2}x^2 + Cx \Big|_{-h}^{h}$$

$$\frac{2Ah^3}{3} + \underbrace{\frac{B}{2}(h^2) - (-h^2)}_{0} + 2Ch$$

$$\frac{2Ah^3}{3} + 2Ch$$ ← realize area as linear comb. y_1, y_2, y_3

$$y_1 = f(-h) = Ah^2 - Bh + C \qquad y_2 = 0 = A(0)^2 - B(0) + C = C \qquad y_3 = f(h) = Ah^2 + Bh + C$$

↓ need 6C

$$\frac{1}{3}(2Ah^3 + 6Ch) = \boxed{\frac{h}{3}(y_1 + 4y_2 + y_3)}$$

I_n approx to integral

$$\int_a^b f(x)\,dx = \frac{h}{3}(y_1 + 4y_2 + y_3) + \frac{h}{3}(y_3 + 4y_4 + y_5)$$

$$\frac{h}{3}(y_1 + 4y_2 + 2y_3 + 4y_4 + y_5)$$

Simpson's Rule

$f(x)$ diff $f^{(4)}(x) < k$ on $x \in (a,b)$

then $|E_{simp}| \le \dfrac{k(b-a)^5}{180n^4}$ ← much lower err

Applying is easy

$\int_1^2 \frac{1}{x}\,dx = \ln 2$ ← can use simpson's method

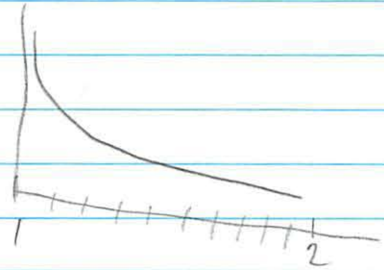
$\approx S_{10}$ ← 10 pieces

(Only do short things on test)

don't recode on test

know

1, 4, 2, 4, 2, 4, 1



$S_{10} = \frac{h}{3}\big(f(1) + 4f(1.1) + 2f(1.2) + 4f(1.3) + 2f(1.4) + \ldots + 4f(1.9) + f(2)\big)$

$\frac{1}{30}\left(\frac{1}{1} + 4\frac{1}{1.1} + \ldots\right)$

$\approx .693150\ldots$

Calc $\ln(2) \approx .693147$

Other Applications

Probability — How to do when your answers are sampled from a contineous set of real #

* Always ask for probabilty that contineus variable lies between a + b

Probability Density $f(x)$ has the property that
$$p(a \le X \le b) = \int_a^b f(x)\,dx$$

$$P(-\infty \le X \le \infty) = 1$$ ← know this

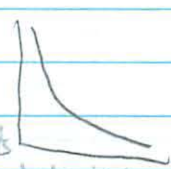
$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

↓means

$$\lim_{N \to \infty} \int_{-N}^{N} f(x)\,dx$$

Suppose wanted to model waiting time on phone call for tech support

- exponential model



$$f(t) = A e^{-ct}$$ ← constants to be determined

said $\int_{\infty}^{\infty} f(t)\,dt = 1$

in our example = $\int_0^\infty A e^{-ct} dt = 1$
no −

Solve this integral in P-set, in order for it to
= 1 then $a = c$
next time how to determine

Lecture 24: Numerical Integration

Numerical Integration

We use numerical integration to find the definite integrals of expressions that look like:

$$\int_a^b \text{(a big mess)}$$

We also resort to numerical integration when an integral has no elementary antiderivative. For instance, there is no formula for

$$\int_0^x \cos(t^2)dt \quad \text{or} \quad \int_0^3 e^{-x^2}dx$$

Numerical integration yields numbers rather than analytical expressions.

We'll talk about three techniques for numerical integration: Riemann sums, the trapezoidal rule, and Simpson's rule.

1. Riemann Sum

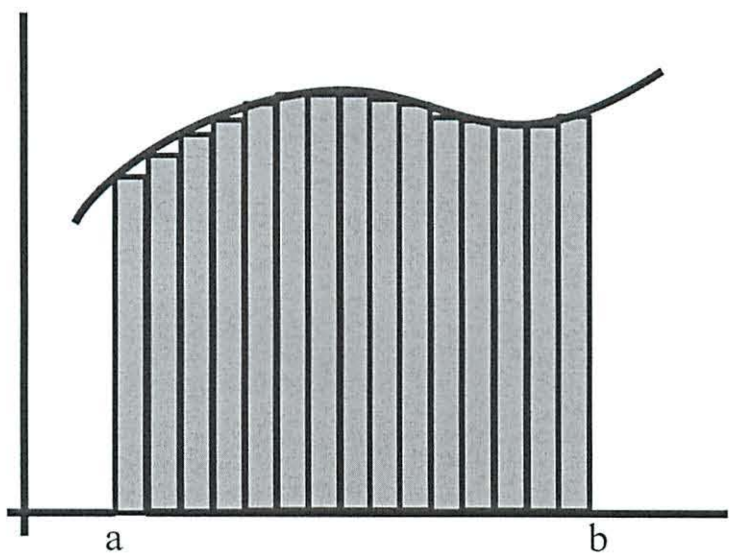


Figure 1: Riemann sum with left endpoints: $(y_0 + y_1 + \ldots + y_{n-1})\Delta x$

Here,

$$x_i - x_{i-1} = \Delta x$$

$$(\text{or, } x_i = x_{i-1} + \Delta x)$$

$$a = x_0 < x_1 < x_2 < \ldots < x_n = b$$
$$y_0 = f(x_0),\ y_1 = f(x_1), \ldots y_n = f(x_n)$$

2. Trapezoidal Rule

The trapezoidal rule divides up the area under the function into trapezoids, rather than rectangles. The area of a trapezoid is the height times the average of the parallel bases:

$$\text{Area} = \text{height}\left(\frac{\text{base 1} + \text{base 2}}{2}\right) = \left(\frac{y_3 + y_4}{2}\right)\Delta x \quad \text{(See Figure 2)}$$

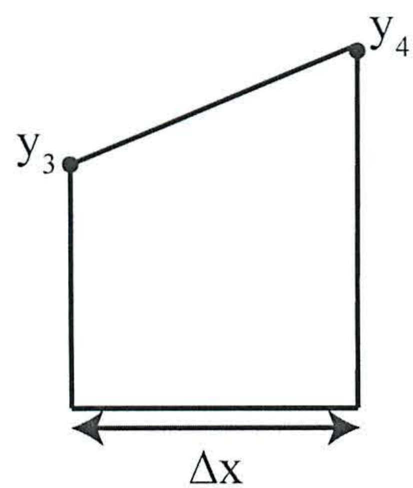


Figure 2: Area $= \left(\frac{y_3 + y_4}{2}\right)\Delta x$

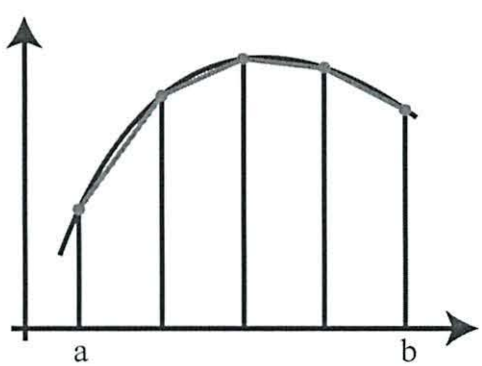


Figure 3: Trapezoidal rule = sum of areas of trapezoids.

$$\begin{aligned}\text{Total Trapezoidal Area} &= \Delta x\left(\frac{y_0 + y_1}{2} + \frac{y_1 + y_2}{2} + \frac{y_2 + y_3}{2} + \ldots + \frac{y_{n-1} + y_n}{2}\right)\\ &= \Delta x\left(\frac{y_0}{2} + y_1 + y_2 + \ldots + y_{n-1} + \frac{y_n}{2}\right)\end{aligned}$$

Note: The trapezoidal rule gives a more symmetric treatment of the two ends (a and b) than a Riemann sum does — the average of left and right Riemann sums.

3. Simpson's Rule

This approach often yields much more accurate results than the trapezoidal rule does. Here, we match quadratics (i.e. parabolas), instead of straight or slanted lines, to the graph. This approach requires an even number of intervals.

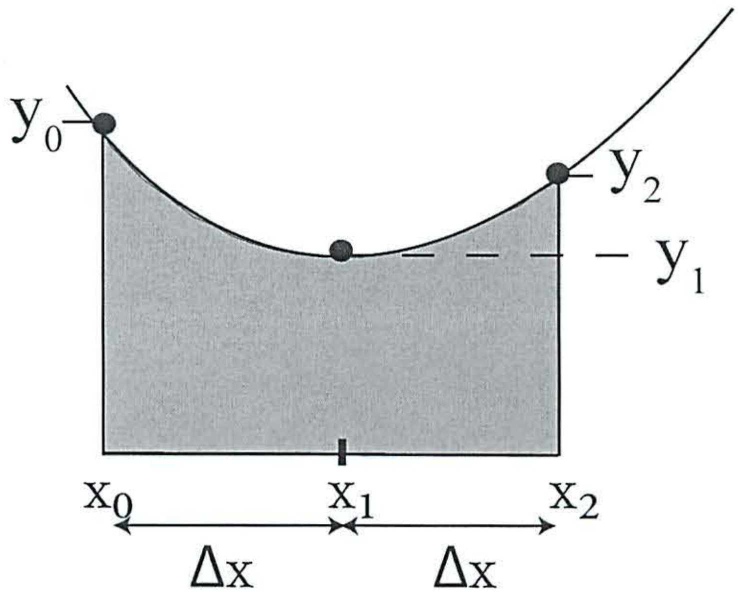


Figure 4: Area under a parabola.

b-a

$$\text{Area under parabola} = (\text{base})(\text{weighted average height}) = (2\Delta x)\left(\frac{y_0 + 4y_1 + y_2}{6}\right)$$

Simpson's rule for n intervals (n must be even!)

$$\text{Area} = (2\Delta x)\left(\frac{1}{6}\right)[(y_0 + 4y_1 + y_2) + (y_2 + 4y_3 + y_4) + (y_4 + 4y_5 + y_6) + \cdots + (y_{n-2} + 4y_{n-1} + y_n)]$$

Notice the following pattern in the coefficients:

$$\begin{array}{ccccccc} 1 & 4 & 1 & & & & \\ & & 1 & 4 & 1 & & \\ & & & & 1 & 4 & 1 \\ \hline 1 & 4 & 2 & 4 & 2 & 4 & 1 \end{array}$$

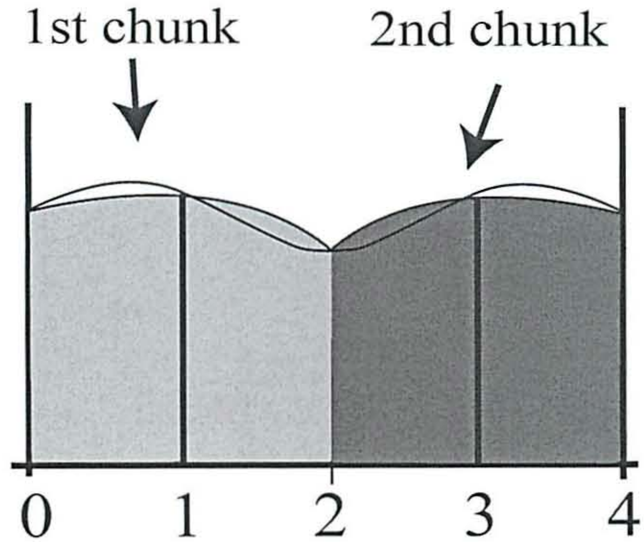


Figure 5: Area given by Simpson's rule for four intervals

Simpson's rule:

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \ldots + 4y_{n-3} + 2y_{n-2} + 4y_{n-1} + y_n)$$

The pattern of coefficients in parentheses is:

$$\begin{array}{ccccccccc} & & 1 & 4 & 1 & & & = & \text{sum } 6 \\ & 1 & 4 & 2 & 4 & 1 & & = & \text{sum } 12 \\ 1 & 4 & 2 & 4 & 2 & 4 & 1 & = & \text{sum } 18 \end{array}$$

To double check – plug in $f(x) = 1$ (n even!).

$$\frac{\Delta x}{3}(1 + 4 + 2 + 4 + 2 + \cdots + 2 + 4 + 1) = \frac{\Delta x}{3}\left(1 + 1 + 4\left(\frac{n}{2}\right) + 2\left(\frac{n}{2} - 1\right)\right) = n\Delta x \quad (n \text{ even})$$

Example 1. Evaluate $\int_0^1 \frac{1}{1+x^2}\,dx$ using two methods (trapezoidal and Simpson's) of numerical integration.

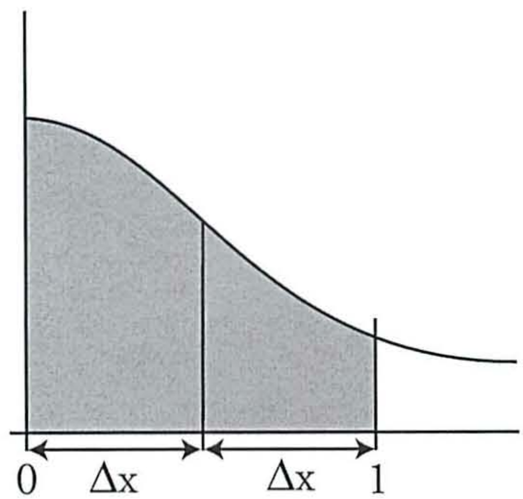


Figure 6: Area under $\frac{1}{(1+x^2)}$ above $[0,1]$.

x	$1/(1+x^2)$
0	1
$\frac{1}{2}$	$\frac{4}{5}$
1	$\frac{1}{2}$

By the trapezoidal rule:

$$\Delta x\left(\frac{1}{2}y_0 + y_1 + \frac{1}{2}y_2\right) = \frac{1}{2}\left(\frac{1}{2}(1) + \frac{4}{5} + \frac{1}{2}\left(\frac{1}{2}\right)\right) = \frac{1}{2}\left(\frac{1}{2} + \frac{4}{5} + \frac{1}{4}\right) = 0.775$$

By Simpson's rule:

$$\frac{\Delta x}{3}\left(y_0 + 4y_1 + y_2\right) = \frac{1/2}{3}\left(1 + 4\left(\frac{4}{5} + \frac{1}{2}\right)\right) = 0.78333...$$

Exact answer:

$$\int_0^1 \frac{1}{1+x^2}\,dx = \tan^{-1} x\Big|_0^1 = \tan^{-1} 1 - \tan^{-1} 0 = \frac{\pi}{4} - 0 = \frac{\pi}{4} \approx 0.785$$

Roughly speaking, the error, | Simpson's − Exact |, has order of magnitude $(\Delta x)^4$.

Exam 3 Review + Improper Integrals
Lecture 25

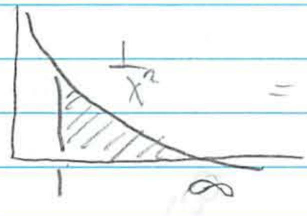
11/6

But some review

Improper integral (bound ∞)

$$\int_0^\infty A e^{-ct}\, dt \quad \leftarrow \text{def} = \lim_{n \to \infty} \int_0^n A e^{-ct}\, dt$$

if you want integral $= 1$, choose $A = c$
strange that producing an answer



$$= \lim_{n \to \infty} \int_1^n \frac{1}{x^2}\, dx$$

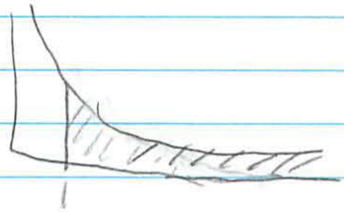
$$\left. \frac{-1}{x} \right|_1^n$$

$$\lim_{n \to \infty} \left(1 - \frac{1}{n}\right) = 1$$

As $n \to \infty$ $A(n) \to 1$
As n grows larger and larger than the area from $1 \to n$ gets arbiterally close to 1

More perplexing

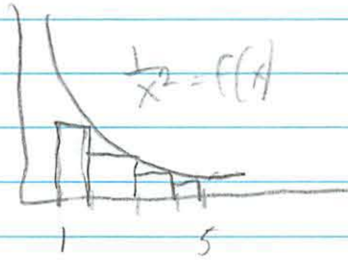
$f(x) = \frac{1}{x}$ $\quad \int_1^\infty \frac{1}{x}\, dx$



$$\lim_{n \to \infty} (\ln n)$$

Blows up
- limit does not exist ("limit $= \infty$")

Think about it in terms of Riemann Sums



$1 > \int_1^n \frac{1}{x^2}\,dx >$ right hand endpoints $\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{n^2}$

for any value of n

$$2 > \frac{1}{1^2} + \frac{1}{2^2} + \dots \frac{1}{n^2}$$

Define

$$\sum_{i=1}^{\infty} \frac{1}{i^2} = \lim_{n \to \infty} \sum_{i=1}^{n} \frac{1}{i^2} < 2$$

$\boxed{\frac{\pi^2}{6}}$ ← mind blowing Euler

Compare

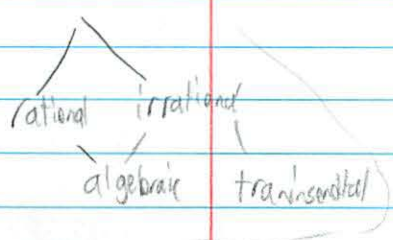
$$\sum_{i=1}^{\infty} \frac{1}{i} > \int_1^n \frac{1}{x}\,dx$$ ← blows up as $n \to \infty$

↑ blowing up as $n \to \infty$ worse

$$\sum_{i=1}^{\infty} \frac{1}{i^3} = ??$$ some weird # — not rational

can compute

1978 Apery



For $\int_{-\infty}^{\infty} =$ def $= \int_{-\infty}^{b} f(x)\,dx + \int_{b}^{\infty} f(x)\,dx$

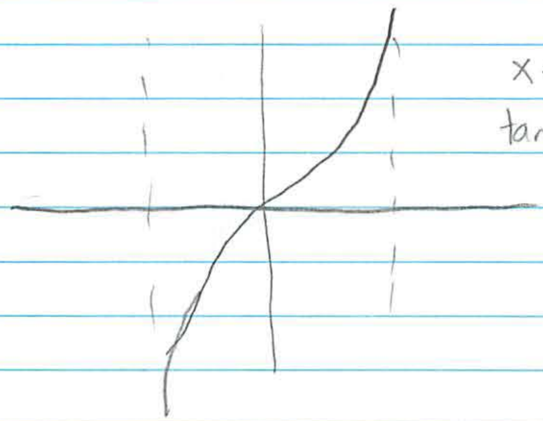
$$\lim_{n \to -\infty} \int_n^b f(x)\,dx + \lim_{m \to \infty} \int_b^m f(x)\,dx$$

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{17} \frac{1}{1+x^2}\,dx + \int_{17}^{\infty} \frac{1}{1+x^2}\,dx$$

Antiderivitive $\frac{1}{1+x^2} = \tan^{-1}(x)$

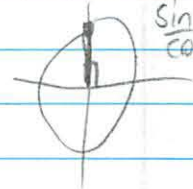
$$\lim_{n\to-\infty} \tan^{-1}(17) - \tan^{-1}(n) + \lim_{m\to\infty} \tan^{-1}(m) - \tan^{-1}(17)$$

$$= \tan^{-1}(17) - \lim_{n\to-\infty} \tan^{-1}(n) + \lim_{m\to\infty} \tan^{-1}(m) - \tan^{-1}(17)$$



$x \to \frac{\pi}{2}$

$\tan x \to \infty$



$\frac{\sin}{\cos} \to 0$

$$= -\left(-\frac{\pi}{2}\right) + \frac{\pi}{2} = \boxed{\pi}$$

Yesterday: waiting times on phone w/ exponential function

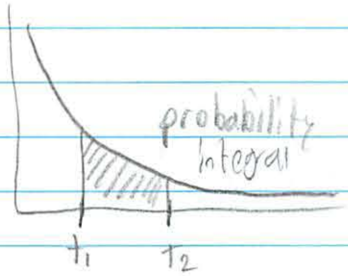
Probability density: $ce^{-ct} = f(t)$

$$\int_0^\infty f(t)\,dt = 1$$

Find c in terms of something observable

Avg wait time?

N callers $\to$ How many have q_V ans between t_1 + t_2



# of callers that get helped between t_1 + t_2

$= (t_2 - t_1) f(t_1) \cdot$ # callers $\cdot t_1$

$\Delta t \quad f(t_1) \; n \, t_1$

$$\frac{\sum_{i=1}^{n} \Delta t \, f(t_1) N}{N} \approx \text{Avg wait time}$$

$$\sum_{i=1}^{n} \Delta t \, f(t_1)$$

$\int_0^{\infty} f(t) \, t \, dt$ as $n \to \infty$ = moment w/ respect to t

$$\bar{t} = \frac{\int_0^{\infty} t \cdot f(t) \, dt}{\int_0^{\infty} f(t) \, dt}$$ cmdus $= 1$

Now can solve for c

- if have μ (avg wait time) $= \frac{1}{\mu}$

Exam 3 Topics

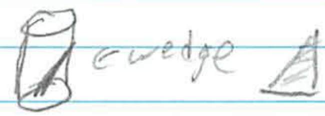
Differential equation (last unit)

- Separation of variables

most important thing in unit → Rieeman sum definite definite integral

- Fundemental Theorm of Calculus - both parts
  - - Part 1 → like Fresnel P-set dv
  - Part 2 #1 - Product rule
- Solids of revolution (pick disk, washer, shells)

add up
- Cross sectional area
  - regular shapes



Applications
- avg value
- com — Pappus
- probability
  - given density function
- improper integral
  - proving probability integrates to 1

Numerical Integration
- midpoint
- trapizodal
- Simpsons

Lecture 25: Exam 3 Review

Integration

1. Evaluate definite integrals. Substitution, first fundamental theorem of calculus (FTC 1), (and hints?)

2. FTC 2:
$$\frac{d}{dx}\int_a^x f(t)\,dt = f(t)$$
If $F(x) = \displaystyle\int_a^x f(t)\,dt$, find the graph of F, estimate F, and change variables.

3. Riemann sums; trapezoidal and Simpson's rules.

4. Areas, volumes.

5. Other cumulative sums: average value, probability, work, etc.

There are two types of volume problems:

1. solids of revolution

2. other (do by slices)

In these problems, there will be something you can draw in 2D, to be able to see what's going on in that one plane.

In solid of revolution problems, the solid is formed by revolution around the x-axis or the y-axis. You will have to decide how to chop up the solid: into shells or disks. Put another way, you must decide whether to integrate with dx or dy. After making that choice, the rest of the procedure is systematically determined. For example, consider a shape rotated *around the y-axis.*

- *Shells*: height $y_2 - y_1$, circumference $2\pi x$, thickness dx
- *Disks (washers)*: area πx^2 (or $\pi x_2^2 - \pi x_1^2$), thickness dy; integrate dy.

Work

$$\text{Work} = \text{Force} \cdot \text{Distance}$$

We need to use an integral if the force is variable.

Example 1: Pendulum. See Figure 1
Consider a pendulum of length L, with mass m at angle θ. The vertical force of gravity is mg ($g =$ gravitational coefficient on Earth's surface)

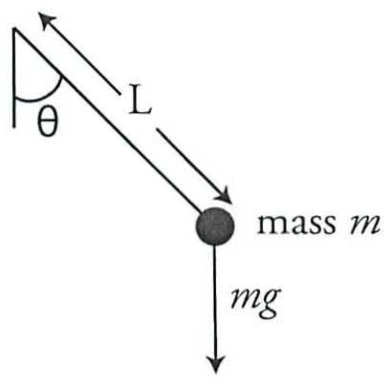


Figure 1: Pendulum.

In Figure 2, we find the component of gravitational force acting along the pendulum's path $F = mg\sin\theta$.

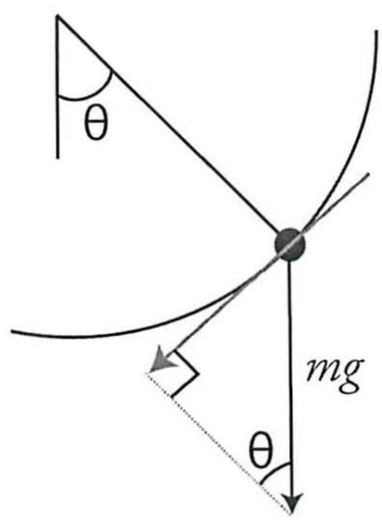


Figure 2: $F = mg\sin\theta$ (force tangent to path of motion).

Is it possible to build a perpetual motion machine? Let's think about a simple pendulum, and how much work gravity performs in pulling the pendulum from θ_0 to the bottom of the pendulum's arc.

Notice that F varies. That's why we have to use an integral for this problem.

$$W = \int_0^{\theta_0} (\text{Force}) \cdot (\text{Distance}) = \int_0^{\theta_0} (mg \sin\theta)(L\, d\theta)$$

$$W = -Lmg\cos\theta\Big|_0^{\theta_0} = -Lmg(\cos\theta_0 - 1) = mg\,[L(1 - \cos\theta_0)]$$

In Figure 3, we see that the work performed by gravity moving the pendulum down a distance $L(1 - \cos\theta)$ is the same as if it went straight down.

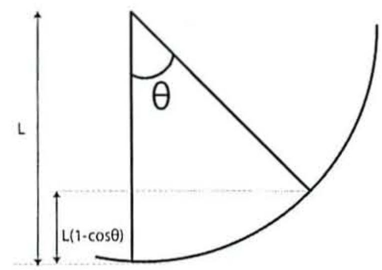


Figure 3: Effect of gravity on a pendulum.

In other words, the amount of work required depends only on how far down the pendulum goes. It doesn't matter what path it takes to get there. So, there's no free (energy) lunch, no perpetual motion machine.

Michael Plasmeier

39.5 / 48

18.01 Problem Set 6 – Fall 2009

Due FRIDAY 11/06/09, 1:45 pm in 2-106

Part I (15 points)

Lecture 22. Friday, Oct. 30 Volumes by disks and shells.
Read: 7.4 Work: 4B-2 for parts e,g from 4B-1, 5; 4C-2,3; 4J-3

Lecture 23. Tuesday, Nov. 3 Work; average value; centroids.
Read:. 7.7, to middle p. 247 Notes AV.
Work: Simmons 249/5, 6, 15 (solutions to be posted); 4D-2, 3, 5

Lecture 24. Thursday, Nov. 5 Probability; Numerical Integration.
Read: 10.9 Work: 3G-1ad, 4

Lecture 25. Friday, Nov. 6 Improper integrals, exam review.
Read: Work: Assigned on the next problem set.

EXAM 3. Tuesday, Nov. 10, covering lectures 16–25 (including differential equations)

Part II (33 points)

Directions: Attempt to solve *each part* of each problem yourself. If you collaborate, solutions must be written up independently. It is illegal to consult materials from previous semesters. With each problem is the day it can be done.

0. (not until due date; 3 pts) Write the names of all the people you consulted or with whom you collaborated and the resources you used, or say "none" or "no consultation". (See full explanation on PS1).

1. (Lec. 22, 3 pts) Find the volume of the solid obtained by rotating the region between the curves $y = x^2 - 3x + 2$ and $y = 0$ about the y-axis.

2. (Lec. 21/22, 5 pts) Give a formula for the volume of a donut. In mathematics, this shape is more commonly referred to as a "torus." When setting up your integral, use R to denote the distance from the center of the hole in the donut to the center of a circular cross-section of the donut (where the creme filling is located). Also, use r to denote the radius of the circular cross-section. Papus? - check ans only

3. (Lec. 23, 8 pts: 2 + 3 + 3)

a) Find the centroid (i.e. center of mass) of the lamina bounded by the curves $y = \sin x$, $y = \cos x$, $x = 0$ and $x = \pi/4$.

b) Use the Theorem of Pappus to find the volume of a cone with height h and radius r. (The statement of the (First) Theorem of Pappus is on p. 391 in Simmons.)

c) Prove that the centroid of any triangle is located at the point of intersection of the medians. (Recall that the medians of the triangle are the lines connecting vertices to midpoints of the opposite side of the triangle. The medians are known to intersect each other at a distance 2/3 of the way along each median from the vertex to the opposite side.) Hint: Place vertices in the coordinate plane at $(a, 0)$, $(0, b)$ and $(c, 0)$.

center of mass

4. (Lec. 23, 4 pts)

The Mean Value Theorem for Integrals states that if f is a continuous function on $[a, b]$, then there exists a number c in $[a, b]$ such that

$$\int_a^b f(x)\,dx = f(c)(b-a).$$

Prove the Mean Value Theorem for Integrals by applying the MVT for derivatives to the function

$$F(x) = \int_a^x f(t)\,dt.$$

5. (Lec. 24, 6 pts: 1 + 2 + 1 + 2)

a) What is the probability that $x^2 < y$ if (x, y) is chosen from the unit square $0 \le x \le 1$, $0 \le y \le 1$ with probability equal to the area.

b) What is the probability that $x^2 < y$ if (x, y) is chosen from the square $0 \le x \le 2$, $0 \le y \le 2$ with probability **proportional** to the area. (Probability = Part/Whole).

c) Evaluate

$$W = \int_0^\infty e^{-at}dt = \lim_{N\to\infty} \int_0^N e^{-at}dt$$

This is known as an improper integral because it represents the area of an unbounded region. We are using the letter W to signify "whole."

The probability that a radioactive particle will decay some time in the interval $0 \le t \le T$ is

$$P([0,T]) = \frac{\text{PART}}{\text{WHOLE}} = \frac{1}{W}\int_0^T e^{-at}dt$$

Note that $P([0,\infty)) = 1 = 100\%$.

d) The half-life is the time T for which $P([0,T]) = 1/2$. Find the value of a and W for which the half-life is $T = 1$. Suppose that a radioactive particle has a half-life of 1 second. What is the probability that it survives to time $t = 1$, but decays some time during the interval $1 \le t \le 2$? (Give an integral formula, and use a calculator to get an approximate numerical answer.)

6. (Lec. 24, 4pts) Use a calculator to make a table of values of the integrand and find approximations to the Fresnel integral $\int_0^a \cos(t^2)dt$ for $a = \sqrt{\pi/2}$, using Simpson's rule with four and eight intervals. (The exact answer to five decimal places is 1.22505. Record your approximations to six decimal places to compare.)

Michael Plasmeier

PSet 6

11/6/09

Part 1

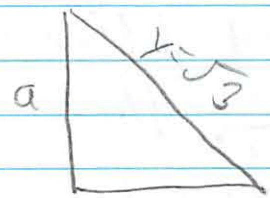
Lecture 22

4B-2 Find the volume of the solid of revolution around y axis

which parts?

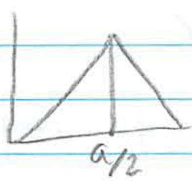
?

4B-5 Find the volume of the solid obtained by revolving an equalateral triangle of side length a around a side



- find com
- rotate around Papus

they draw



$$2\int_0^{a/2} \pi y^2 \, dx$$

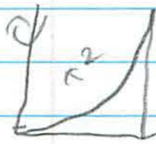
$$2\int_0^{a/2} \pi (\sqrt{3}x)^2 \, dx$$

$$\frac{\pi a^3}{4}$$

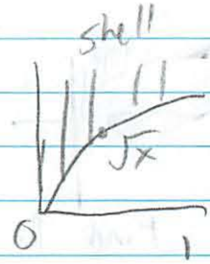
4C-2 $0 \leq y \leq x^2$ revolved around x axis

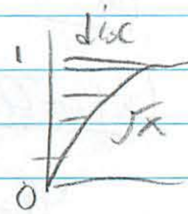
$x \leq 1$

$$\int_0^1 2\pi x y \, dx = \int_0^1 2\pi x^3 \, dx = \frac{\pi}{2}$$



4C-3 $\sqrt{x} \le y \le 1$ by shell
$x \ge 0$ and dish washer





Shell $\int_0^1 2\pi x(1-y)\,dx$

$\int_0^1 2\pi x(1-\sqrt{x})\,dx$

$\frac{\pi}{5}$

Dish $\int_0^1 \pi x^2\,dy = \int_0^1 \pi y^4\,dy = \frac{\pi}{5}$

area · distance traveled by com

4J-3 A very shallow circular reflecting pool D - R
chemical released in center
hors of symmetrical diffusion

at pt $r = \frac{k}{1+r^2}$ g/m^3

How much A released?

Volume = $2\pi r_i D\Delta r$

n concentric shells thickness = Δr

So amt chem in ith shell

$\frac{k}{1+r_i^2} 2\pi r_i D\Delta r$

↑amt · volume

add shells

$A = \lim_{n\to\infty} \sum_1^n \frac{k}{1+r_i^2} 2\pi r_i D\Delta r$

$\int_0^R 2\pi k D \frac{r}{1+r^2}\,dr$

$$\pi k D \ln(1+r^2) \Big|_0^R$$

$\pi k D \ln(1+R)^2$ grams initally added

Lecture 23

Simmons p249 #5

Bucket weighing 5 lbs is loaded w/ 60 lbs sand. Hole + leaks 1/3 gone when 10 ft up. Work?

$W = Fd$ Weight (height)

↑weight (mass · gravity)

$W(0) = 5 + 60 = 65$

$W(10) = 65 - (\frac{1}{3} \cdot 60) = 45$

rate $= \frac{-60}{3} \cdot \frac{1}{10} = -2$ $\quad w(h) = 65 - 2h$

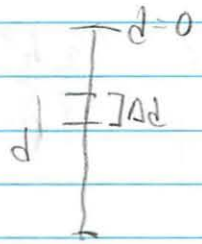
$$\text{Work} = \int_0^{10} w(h)\,dh = \int_0^{10} 65 - 2h\,dh$$

$$65h - 2h^2 \Big|_0^{10}$$

$$65(10) - 2(10)^2$$

$$550 \text{ lbs} \cdot \text{ft}$$

6. Cable 100ft long weighs 4 lbs/ft hanging from windlass ←? How much work to wind it?



sum each Δm piece to find total change

$$W = \underbrace{4\,\Delta d}_{\text{weight}} \cdot \underbrace{d}_{\text{distance}}$$

$$W = \int_0^{100} 4d\,\Delta d$$

$$\frac{4d^2}{2} \Big|_0^{100} = 2(100)^2 = 20{,}000 \text{ ft lbs}$$

easy

15. 2 Particles M m How much work to move 2x apart

a units apart

$$dW = F\,dr = G\frac{Mm}{r^2}\,dr$$

$$W = \int F\,dr = \int G\frac{Mm}{r^2}\,dr = -G\frac{Mm}{r}\Big|_a^{2a}$$

$$-GMm\frac{1}{2a} - \left(-GMm\frac{1}{a}\right) = \boxed{G\frac{Mm}{2a}}$$

4D-2 ~~A heavy metal 2 lbs pail has 100 lbs paint~~
~~Pulled up 30 ft~~
~~Leaks at stedy rate 30ft 1/5 gone~~
~~Work?~~

$$\cancel{W(0) = 2 + 100}$$
$$\cancel{W(30) = 102 - (1/5 \cdot 100) = 82 \text{ lbs}}$$

$$\cancel{-\frac{100}{5} \cdot \frac{1}{30} = -\frac{2}{3}} \quad \cancel{W(h) = 102 - 2/3h}$$

$$\cancel{\text{Work } \int_0^{100} 102 - 2/3h = 102 - \frac{2/3 h^2}{2}\Big|_0^{100}}$$

$$\cancel{102 - \frac{1}{3}(100)^2 =}$$

? wrong problem

4D-2 Show avg value $\frac{1}{x}$ over $[a, 2a]$ is $\frac{c}{a}$ ← constant independent of a $(a > 0)$

$$\frac{1}{a}\int_a^{2a}\frac{dx}{x} = \frac{1}{a}\ln x\Big|_a^{2a} = \frac{1}{a}\ln(2a) - \ln(a) = \frac{1}{a}\ln\left(\frac{2a}{a}\right) = \frac{\ln 2}{a}$$

4D-3 Point moving on x axis $x = s(t)$
Over $[a, b]$ avg value $v(t)$ = same
as avg velocity over this integral

$s(t)$ = distance function $\rightarrow v(t) = s'(t)$

$$\frac{1}{b-a}\int_a^b s'(t)\,dt$$

$\downarrow$ use FTC #1

$$\frac{1}{b-a}\left(s(b) - s(a)\right)$$ ← avg value over interval
- easy, just the def.

4D-5 If the avg value of $f(t)$ is between 0 and x
given by $g(x)$ - give $f(x)$ in terms of $g(x)$

$$g(x) = \frac{1}{x}\int_0^x f(t)\,dt$$

Express $f(x)$ in terms of $g(x)$
- So multiply through by x → 2nd Fund. Theorm

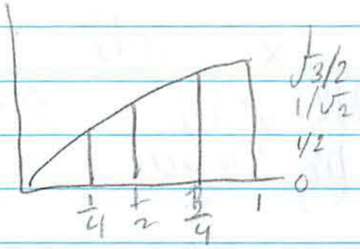
$$\int_0^x f(t)\,dt = x g(x) \rightarrow f(x) = g(x) + x g'(x)$$

Lecture 24

3G-1a Find approx w/ Riemann sum left endpoint
trapizodal rule
Simpson's Rule

$$\int_0^1 \sqrt{x}\,dx \quad (= 2/3)$$

Rules
- Left Riemann $(\Delta x)(y_0 + y_1 + y_2 + y_3)$ of 4
- Trapizodal $(\Delta x)\left(\frac{1}{2}y_0 + y_1 + y_2 + y_3 + \frac{1}{2}y_4\right)$
- Simpson's $\left(\frac{\Delta x}{3}\right)(y_0 + 4y_1 + 2y_2 + 4y_3 + y_4$



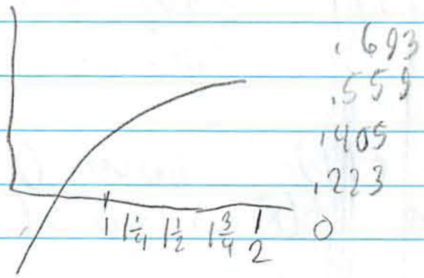
Left sum $\left(\frac{1}{4}\right)\left(0 + \frac{1}{2} + \frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2}\right) \approx .518$

Trapizodal $\left(\frac{1}{4}\right)\left(\frac{1}{2}\cdot 0 + \frac{1}{2} + \frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2} + \frac{1}{2}\cdot 1\right) \approx .643$

Simpson $\frac{1/4}{3}\left(0 + 2\frac{1}{2} + 4\frac{1}{\sqrt{2}} + 2\frac{\sqrt{3}}{2} + 1\right) \approx .657$

exact remember = .6666

d. $\int_1^2 \frac{dx}{x} = \ln 2$



Left sum $\left(\frac{1}{4}\right)(0 + .223 + .405 + .559) = .76$

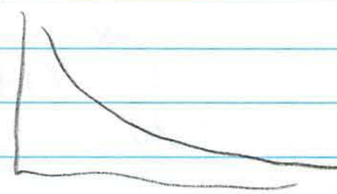
Trapiz $\left(\frac{1}{4}\right)\left(\frac{1}{2}\cdot 0 + .223 + .405 + .559 + \frac{1}{2}.693\right) \approx .697$

Simpson $\frac{.25}{3}(0 + 2\cdot.223 + 4\cdot.405 + 2\cdot.559 + .693) \approx .69325$

$\ln(2) \approx .69315$

4. Use the trapizodal rule to estimate $\sqrt{1} + \sqrt{2} + \ldots + \sqrt{10000}$
Too high or low?

$S = \frac{1}{1} + \frac{1}{2} + \ldots \frac{1}{n}$



$\int_1^n \frac{dx}{x} \approx \frac{1}{2}\cdot\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots \frac{1}{2}\cdot\frac{1}{n} = S - \frac{1}{2} - \frac{1}{2n}$

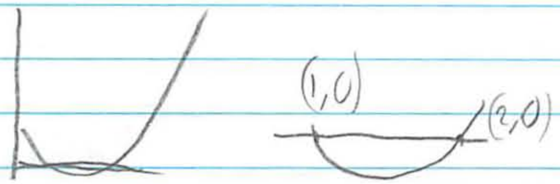
$\int_1^n \frac{dx}{x} = \ln n \quad S \approx \ln n + \frac{1}{2} + \frac{1}{2n} \leftarrow$ so our estimate was too low

1/5

Part 2

0. See sidebar 3

1. Find the volume of a solid obtained by rotating the region between the curves $y = x^2 - 3x + 2$
$y = 0$ about y-axis.



$$\int_1^2 x^2 - 3x + 2 - 0 \, dx$$

$$\frac{x^3}{3} - 3\frac{x^2}{2} + 2x \Big|_1^2$$

$$\frac{(2)^3}{3} - \frac{3(2)^2}{2} + 2(2) - \frac{(1)^3}{3} - \frac{3(1)^2}{2} + 2(1)$$

$$8/3 - 6 + 4 - 1/3 - 3/2 + 2$$

$$\frac{5}{6}$$

= 1 slice now ↷ 360°

$\cancel{V = \int_0^{2\pi} \frac{5}{6} x}$ not changing $\quad \frac{5}{6} \cdot 2\pi \cdot 1 = \frac{5\pi}{3}$ ✗

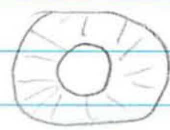
↙ distance traveled

= Area

? unsure here

back part travels more ?!

-and in from center by 1

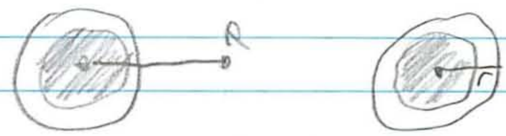


1.5

=
4.5

2. Give the formula for the volume of a donut / torus

R = distance from center of donut to center of cross section area



textbook

torus: $(x-R)^2 + y^2 = r^2 \quad (0 < r < R)$

$V = \pi r^2 \cdot 2\pi R$ Papus circle · distance traveled

$2\pi^2 r^2 R$ ✓

hint from office hrs use - u substitution - works out) but why would you need to - where would you integrate

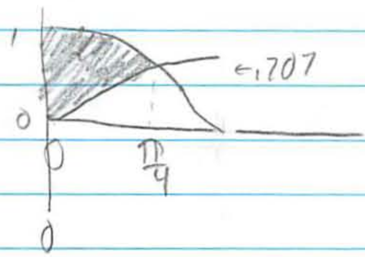
5

3. Find the centroid (center of mass) of the lamina bounded by the curves $y = \sin x$

$y = \cos x$

$x = 0$

$x = \pi/4$



$\Sigma x \rightarrow \int_0^{\pi/4} \cos x - \sin x \, dx$

$\sin x + \cos x$

$\left(\sin \frac{\pi}{4} + \cos \frac{\pi}{4}\right) - \left(\sin 0 + \cos 0\right)$

$\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) - (0 + 1)$

$\frac{\sqrt{2} - 1}{2}$ to find middle

$\Sigma y \rightarrow \int_0^{.707} \sin^{-1} y - 0 + \int_{.707}^{1} \cos^{-1} y - 0$

$y = \sin x$

$x = \sin^{-1} y$

$x = \cos^{-1} y$

?

$y \sin^{-1} y + \sqrt{1 - y^2} \Big|_0^{.707} + y \cos^{-1}(y) - \sqrt{1 - y^2} \Big|_{.707}^{1}$

$.126238 + .308411$

$.2854$

$\left(\frac{\sqrt{2} - 1}{2}, .2854\right)$ X

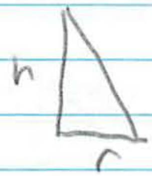
$?.20711$

1

b Use Theorm of Pappus to find volume of cone w/ height h and radius r

- radius of cone changes with height

- or use theorm
  ← right triangle rotated



$\frac{1}{3}\pi r^2 h$

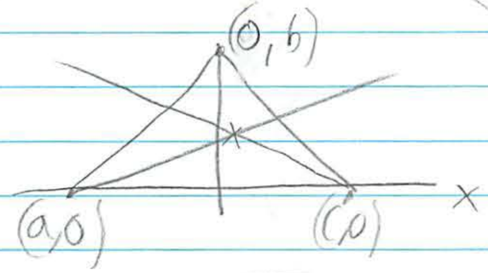
so $\frac{1}{2} r h \cdot \cancel{2\pi}$ distance traveled by centroid $\frac{1}{3} r \cdot 2\pi$

$\cancel{V = \pi r h}$

$V = \frac{1}{3}\pi r^2 h$ ✓

/3

c) Prove that the centroid of any triangle is located at pt intersection of medians ($d = 2/3$)



Break up integral and do piecewise

Break up coords + find CoM Where lines intersect

Brendan office hrs

want $(\bar{x}, \bar{y})$ $\quad \bar{x} = \dfrac{m_x}{m}$

$m_x = \int_a^c f(x)\, x\, dx$

$\boxed{f(x) = \begin{cases} a \le x \le 0 \\ 0 \le x \le c \end{cases} \Big) \text{ 2 parts}}$ $\qquad \bar{x} = \dfrac{\int_a^c f(x) x\, dx}{\int_a^c f(x)\, dx}$

$m = \int_a^c f(x)\, dx$

for horiz - break up into horiz slices

he likes this better $\rightarrow$ $\quad m_y = \int_0^b g(y)\, y\, dy$ $\qquad \bar{y} = \dfrac{\int_0^b g(y)\, y\, dy}{\int_0^b g(y)\, dy}$

but have to find $g(y)$ - will work on 3D

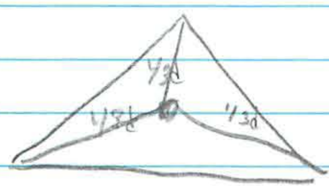
or ??

Find intersection pt of the 3 medians w/ distance formula

$(a, 0) \to (c-0,\ 0-b)$ $\quad$ length $= 2/3$ $\quad / 2 = \frac{1}{3}$

$(0, b) \to (a-c,\ 0-0)$ $\quad l = 2/3$ $\quad / 2 = \frac{1}{3}$

$(c, 0) \to (a-0,\ 0-b)$ $\quad l = 2/3$ $\quad / 2 = \frac{1}{3}$



? then what

1.5

?

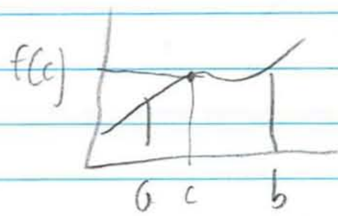
Brandon
Office
H/s

4. The MVT for Integrals; says that if f is contineous on $[a,b]$

there exists c w/ $a < c < b$ so that

$$\int_a^b f(x)\,dx = f(c)(b-a)$$

average value $\Rightarrow f(c)$ is = to avg value
application



Going to define new function

$$F(x) = \int_a^x f(t)\,dt$$

MVT derivitives
There is a c
$a < c < b$

So that $\dfrac{F(b) - F(a)}{b-a} = F'(c) = f(c)$

1st FTC ($\int_a^b f(b) - f(a)$)

So $F(b) - F(a) = f(c)(b-a)$

$$\int_a^b f(t)\,dt - \cancel{\int_a^a f(t)\,dt} \to 0$$

didn't solve
don't totally get

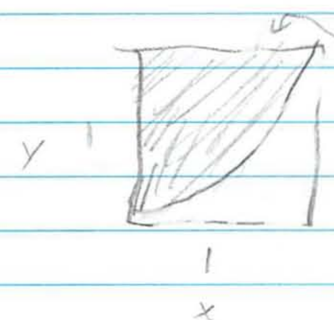
$$\boxed{\int_a^b f(t)\,dt}$$

= 4

Brendan office hrs

5a What is probability that $x^2 < y$ if (x, y) is chosen from unit square $0 \le x \le 1$, $0 \le y \le 1$

w/ probability = to the area



i can use assume uniform - in b proportional to area

If uniform just probility here

So need to add the area inside curve

$P(-\infty, \infty) = 1$ ← do I need or just compare part / whole

$$\int_0^1 1 - x^2 \, dx$$

$$x - \frac{x^3}{3} \Big|_0^1$$

$$1 - \frac{1^3}{3} - \left(0 - \frac{0^3}{3}\right) \quad / \quad = 1$$

$\frac{2}{3}$ ← 66% chance that $x^2 < y$

b What is probability that $x^2 < y$ if (x, y) chosen from $0 \le x \le 2$ $0 \le y \le 2$ with probability proportional to area

?? what does this mean?

?

①

c Evaluate $\overset{\text{whole}}{W} = \int_0^\infty e^{-at}\,dt$ ← improper integral because area of unbound region

$= \lim_{N\to\infty} e^{-at}\,dt$

what should I be evaluating?

from $0 \le t \le T$ Note $P[0, \infty] = 1 = 100\%$

d. $P([0,T]) = \frac{1}{w} \int_0^T e^{-at}\, dt = \frac{\text{part}}{\text{whole}}$

$\frac{1}{2} = \frac{1}{w} \int_0^1 e^{-at}\, dt$ use 1/2

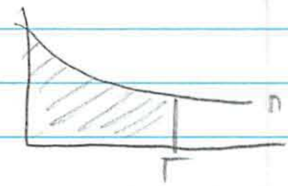
is the probability need to find a w

why

$\frac{1}{2} = \frac{1}{w} \left(\frac{-1}{a}\right) e^{-at} \Big|_0^1$

Brendan office hrs

$\frac{1}{2} = \frac{-1}{wa} \left(e^{-a} - 1\right)$



propability functions add to 1

use 1

$\frac{1}{w} \int_0^\infty e^{-at}\, dt = 1$

other equation for half life

very large - power

$\frac{1}{w} \left(-\frac{1}{a} e^{-at}\right) \Big|_0^\infty$

$1 = \frac{1}{aw}$ ← 2 equations w/ 2 unknowns (w + n)

$a = \frac{1}{w}$ $w = \frac{1}{a}$

$\frac{1}{2} = \frac{-a}{a} \left(e^{-a} - 1\right) \Big|_0^1$

$\frac{1}{2} = -e^{-a} - 1$

$\frac{1}{2} = ae - 1$

$\frac{3}{2} = ae$

$a = \frac{3}{2e}$

X

$1 = \frac{1}{\frac{3}{2e} w}$

$\frac{2e}{3} = w$

0.5

d2 Surives to $t=1$ but decays $1 \le t \le 2$

- find w
- integrate $1 \to 2$
- use calc to find because don't know e easily

$$\text{prob} = \frac{1}{W} \int_1^2 e^{-at}\, dt$$

$$\frac{1}{W}\left(\frac{-1}{a}\right) e^{-at} \Big|_1^2$$

$$\frac{3}{2e}\left(-\frac{2e}{3}\right) e^{-\left(\frac{3}{2e}\right)\cdot 2} - \left[\frac{3}{2e}\left(-\frac{2e}{3}\right) e^{-\frac{3}{2e}\cdot 1}\right]$$

$$-1 \cdot .0339 + .01695$$

$$= .01695$$

1.6% ← too small Wrong

6. Use a calculator to make a table of values of the integrand and find aproximations to Fresnel integral

$$\int_0^a \cos(t^2)\,dt \qquad a=\sqrt{\pi/2}$$

Using Simpson's rule for 4 / 8 Intervals

$$\int_0^{\sqrt{\pi/2}} \cos(t^2)\,dt \qquad n=\sqrt{\pi/2}$$

$$\frac{\frac{\sqrt{\pi/2}}{4}}{3}\left[f(0)+4f\left(\frac{n}{4}\right)+2f\left(\frac{2n}{4}\right)+4f\left(\frac{3n}{4}\right)+f(n)\right]$$

$$\frac{\sqrt{\pi/2}}{12}\left(1+4\cdot .99518+2\cdot .92388+4\cdot .6343+0\right)$$

$$.97818$$

GT wrong ans

* calc has diff offical ans

$$\frac{\frac{\sqrt{\pi/2}}{8}}{3}\left[f(0)+4f\left(\frac{n}{8}\right)+2f\left(\frac{2n}{8}\right)+4f\left(\frac{3n}{8}\right)+2f\left(\frac{4n}{8}\right)+4f\left(\frac{5n}{8}\right)+2f\left(\frac{6n}{8}\right)+4f\left(\frac{7n}{8}\right)+f(n)\right]$$

$$\frac{\frac{\sqrt{\pi/2}}{8}}{3}\left[1+4\cdot .9997+2\cdot .99518+4\cdot .9757+2\cdot .92388+4\cdot .81758+2\cdot .63439+4\cdot .3549+0\right]$$

$$18.718$$

$$.9775025$$

4

Calc says exact ans $= .9774514$

Simmons p. 249

· P set 6 . Part I · 18.01 · Fall 2009.

5 a bucket weighing 5 lb when empty is loaded w/ 60 lb of sand. Unfortunately there is a hole in the bucket, and sand leaks out uniformly at such a rate that a third of the sand has been lost when the bucket has been lifted 10 ft. Find the work done in lifting the bucket this distance.

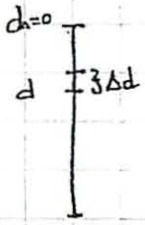
Assume leaks uniformly wrt height lifted. Work = Force · distance.
denote Weight = Weight @ height h = W(h). $\quad$ Force $\overset{||}{}$ weight of sand

$W(0) = W_0 = 5+60 = 65.$ $\quad$ losing sand at rate $-\frac{60}{3}\cdot\frac{1}{10} = -2$ (lb/ft)

⇒ know weight as function of height: $W(h) = 65 - 2h.$

$$\text{Work} = \int_0^{10} W(h)\,dh = \int_0^{10} 65-2h\,dh = 65h - h^2\Big|_0^{10} = 65\cdot 10 - 10^2 = 650 - 100 = \boxed{550 \text{ (lb-ft)}}$$

#6 A cable 100 ft long that weighs 4 lb/ft is hanging from a windlass. How much work is done in winding it up?



Work done to wind entire wire = Σ work done to lift each infinitesimally small piece of the wire.

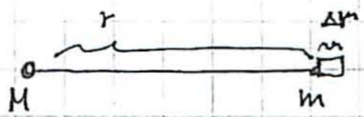
~~WELL~~ work done to lift piece d away from top = $\underbrace{4\cdot\Delta d}_{\text{weight}}\cdot\underbrace{d}_{\text{distance}}$

⇒ need to Σ $4d\Delta d$ from d = 0 to 100, take $\Delta d \to 0$ ⇒ can set up integral

$$W = \int_0^{100} 4d\, d(d) = 2d^2\Big|_0^{100} = 2(10^2)^2 = \boxed{20{,}000 \text{ (ft-lbs)}}$$

Sorry for poor choice of notation

#7 If two particles of matter of masses M and m are a units apart, how much work must be done to move them twice as far apart?



Move m away from M from r=a to r=2a.

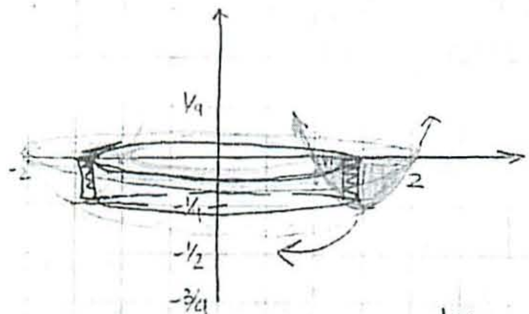
as in example 2, $\quad dW = F\,dr = G\frac{Mm}{r^2}\,dr$

$$\text{Work} = W = \int_{r=a}^{2a} dW = \int_{r=a}^{2a} G\frac{Mm}{r^2}\,dr = GMm\left(-\frac{1}{r}\right)\Big|_a^{2a} = GMm\left(\frac{1}{a} - \frac{1}{2a}\right) = \boxed{\frac{GMm}{2a}}$$

18.01 · Fall 2009
PSet 6 · Part II.

(1)

Problem 1 Find the volume of the solid obtained by rotating the region between $y=x^2-3x+2$ and $y=0$ about the y-axis



$x^2-3x+2=(x-2)(x-1) \Rightarrow$ zeroes @ $x=1,2$, min @ $x=\frac{3}{2}$

Compute volume by shells.
Vol of one piece $=(2\pi r)\cdot(\Delta\text{width})\cdot(\text{height})=(2\pi x)(\Delta x)(y)$
and $1\le x\le 2$. $\Rightarrow$ gives us total vol as integral
(will take abs. value of answer!)

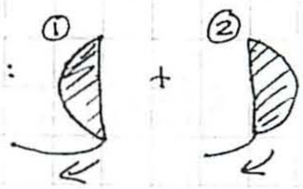
$$-\text{Vol}=\int_1^2 (2\pi x)(y)\,dx=\int_1^2 (2\pi x)(x^2-3x+2)\,dx=$$

$$=2\pi\int_1^2 x^3-3x^2+2x\,dx=2\pi\left(\frac{x^4}{4}-x^3+x^2\right)\Big|_1^2=2\pi\left(\cancel{4}-\cancel{8}+\cancel{4}-\frac{1}{4}+\cancel{1}-\cancel{1}\right)$$
$$=2\pi\left(-\frac{1}{4}\right)=-\frac{\pi}{2}$$

$$\Rightarrow \text{Volume}=-\left(-\frac{\pi}{2}\right)=\boxed{\frac{\pi}{2}}.$$

Problem 2 Give a formula for the volume of a torus. Use R to denote distance from center of the hole to the center of a circular cross-section. Use r to denote radius of circular cross section.

Use shells, divide volume into two integrals: ① + ②

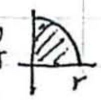


$y=\sqrt{r^2-x^2}$

$$V=\underset{②}{\int_0^r} 2\pi(R+x)\,2\sqrt{r^2-x^2}\,dx+\underset{①}{\int_0^r} 2\pi(R-x)\,2\sqrt{r^2-x^2}\,dx$$

$$=\int_0^r 4\pi(2R)\sqrt{r^2-x^2}\,dx=8\pi R\int_0^r\sqrt{r^2+x^2}\,dx$$

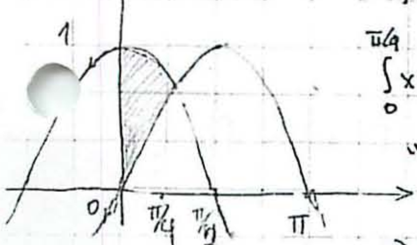
$$=8\pi R\left(\frac{1}{4}\pi r^2\right)=\boxed{2\pi^2r^2R}$$



‖ Area of quarter-circle of radius r

Problem 3

a) Find the centroid of the lamina bounded by the curves $y=\sin x$, $y=\cos x$, $x=0$, and $x=\frac{\pi}{4}$



$f(x)=\cos x-\sin x$. $\text{area}=\int_0^{\pi/4}\cos x-\sin x\,dx=\sin x+\cos x\Big|_0^{\pi/4}=\frac{\sqrt{2}}{2}\cdot 2-1=\sqrt{2}-1.$ //.

integrals sort of given in class

$$\int_0^{\pi/4}xf(x)\,dx=\int_0^{\pi/4}x\cos x-x\sin x\,dx=-\sin x+x\cos x+x\sin x+\cos x\Big|_0^{\pi/4}=-1+\frac{\pi\sqrt{2}}{4}$$

trick

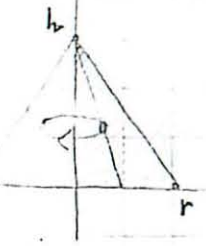
$$\text{“}\int y\,g(y)\,dy\text{”}=\int_0^{\pi/4}\frac{1}{2}(\cos^2x-\sin^2x)\,dx=\int_0^{\pi/4}\frac{1}{2}\cos 2x\,dx=\frac{1}{4}\sin 2x\Big|_0^{\pi/4}=\frac{1}{4}\ //$$

$$\Rightarrow \text{Centroid is at } (\bar{x},\bar{y})=\left(\frac{-1+\frac{\pi\sqrt{2}}{4}}{\sqrt{2}-1},\frac{1/4}{\sqrt{2}-1}\right)=\boxed{\left(\frac{-4+\pi\sqrt{2}}{4\sqrt{2}-4},\frac{1}{4\sqrt{2}-4}\right)}\ //$$

(Prob3, cont.)

(b) Use the Theorem of Pappus to find the volume of a cone w/ height h and radius r.

Theorem of Pappus: Vol= Area (Region) · distance travelled by centroid.



pt of intersection of medians

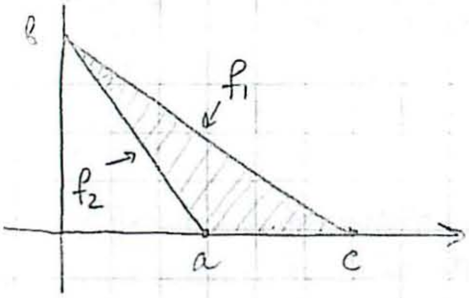
From part (c), centroid is at $\left(\frac{2}{3}\frac{r}{2}, h-\frac{2}{3}h\right) = \left(\frac{r}{3}, \frac{h}{3}\right)$ //

dist travelled = $(2\pi(r/3))$ area = $(1/2)hr$

$\Rightarrow$ Vol $= \left(\frac{1}{2}hr\right)\cdot\left(2\pi\frac{r}{3}\right) = \boxed{\frac{\pi h r^2}{3}}$ //

(c) Prove that the centroid of any triangle is located at the point of intersection of the medians.

Brute-force compute location of centroid of an arbitrary triangle (which can be placed on xy axes w/ vertices at $(0,b)$; $(a,0)$; $(c,0)$) in terms of a, b, c. Then use the hint to show the answer is the desired one.



Area $= \frac{1}{2}(c-a)\cdot b = \frac{b(c-a)}{2}$ // $\begin{cases} f_1 = b - x(b/c) \\ f_2 = b - x(b/a)\end{cases}$

need to divide into two intervals

"$\int x f(x)dx$" $= \int_0^a x(f_1-f_2)\,dx + \int_a^c x f_1\,dx =$

$= \int_0^a x\left(-x\frac{b}{c} + x\frac{b}{a}\right)dx + \int_a^c xb - x^2\frac{b}{c}\,dx =$

$= b\left[\left(\frac{1}{a}-\frac{1}{c}\right)\frac{x^3}{3}\Big|_0^a + \left(\frac{x^2}{2} - \frac{x^3}{3c}\right)\Big|_a^c\right] = b\left[\frac{a^3}{3}\left(\frac{1}{a}-\frac{1}{c}\right) + \frac{c^2}{2} - \frac{c^3}{3c} - \frac{a^2}{2} + \frac{a^3}{3c}\right] =$

$= b/6\,[c^2 - a^2]$ //

"$\int y\, g(y)\,dy$" $= \int_0^b y(g_1-g_2)\,dy = \int_0^b y\left(c-a+y\left(\frac{a}{b}-\frac{c}{b}\right)\right)dy = (c-a)\frac{y^2}{2} + \frac{1}{b}(a-c)\frac{y^3}{3}\Big|_0^b =$

$\begin{cases} g_1 = c - y\frac{c}{b} \\ g_2 = a - y\frac{a}{b}\end{cases}$ $= \frac{b^2}{2}(c-a) + \frac{a-c}{b}\,\frac{b^3}{3} = \frac{b^2}{6}\left(3(c-a) - 2(c-a)\right) = \frac{b^2}{6}(c-a)$ //

~~Thus centroid is co~~ Centroid is at $\boxed{\left(\frac{c+a}{3}, \frac{b}{3}\right)}$ (divide by area $= \frac{b(c-a)}{2}$ from above)

We are told in the hint that the pt of intersection of the medians is at $\left(\frac{2}{3}\frac{a+c}{2}, \frac{1}{3}b\right) = \left(\frac{c+a}{3}, \frac{b}{3}\right)$. Thus, centroid is at pt of intersection of the medians //.

Problem 4 Prove the Mean Value Theorem for Integrals.

Let $F(x) = \int_a^x f(t)\,dt$. By the MVT for ~~derivative~~ derivatives, $\exists c \in (a,b)$ s.t.

$F(b) - F(a) = F'(c)(b-a)$. We know $F(a)=0$ and $F'(c) = f(c)$.

Thus, $\int_a^b f(t)\,dt = F(b) - F(a) = f(c)(b-a)$ //.

Problem 5

a) What is the prob. that $x^2 < y$ if (x,y) is chosen from $0 \le x \le 1$, $0 \le y \le 1$ w/ probability = area?

Total Area = 1. $\int_0^1 x^2\,dx = \frac{x^3}{3}\Big|_0^1 = 1/3$ Prob= $\frac{\text{Area}(y > x^2)}{\text{Total Area}} = \frac{\text{Area above } y = x^2}{\text{Total area}} = \frac{1 - 1/3}{1} = \boxed{\frac{2}{3}}$

b) What is prob $x^2 < y$ if (x,y) is chosen from $0 \le x \le 2$, $0 \le y \le 2$?

$\cancel{\int_0^2 x^2\,dx = \frac{x^3}{3}\Big|_0^2 = \frac{8}{3}.}$ ~~Total area = 4.~~ ~~Probability~~ $\cancel{= \frac{4 - 8/3}{4} = \frac{4/3}{4} = \boxed{\frac{1}{3}}}$ WRONG! See bottom

c) Evaluate $W = \int_0^\infty e^{-at}\,dt = \lim_{N \to \infty} \int_0^N e^{-at}\,dt$

$\lim_{N \to \infty} \int_0^N e^{-at}\,dt = \lim_{N \to \infty} -\frac{1}{a} e^{-at}\Big|_0^N = \lim_{N \to \infty} \left(\frac{1}{a} - \cancel{\frac{1}{a e^{Na}}}^{\,0}\right) = \boxed{\frac{1}{a}}$

d) Find the value of a, W for which half-life $T = 1$

What is then the prob that a particle survives to $t = 1$, but decays some time between $1 \le t \le 2$?

$\frac{1}{1/a} \int_0^1 e^{-at}\,dt = \frac{1}{2} \Rightarrow \frac{\frac{1}{a} - \frac{1}{a e^a}}{1/a} = 1/2 \Rightarrow 1 - e^{-a} = \frac{1}{2} \Rightarrow \boxed{a = \ln 2}$

$e^{-a} = 1/2$

($W = 1/a$ from above)

$= 1/\ln 2$

$P([1,2]) = \ln 2 \int_1^2 e^{-\ln 2\, t}\,dt = e^{-\ln 2} - e^{-2\ln 2} = \frac{1}{2} - \frac{1}{4} = \boxed{\frac{1}{4}}$ //.

Problem 6 Find approximations to the Fresnel integral $\int_0^a \cos(t^2)\,dt$ for $a = \sqrt{\pi/2}$,

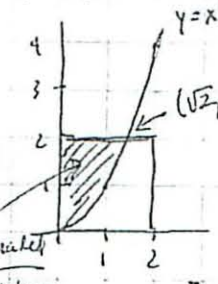
using Simpson's rule w/ four and eight intervals.

4 intervals: $\Delta x = \frac{\sqrt{\pi/2}}{4}$ $x_0 = 0,\ x_1,\ x_2,\ x_3,\ x_4 = \frac{\sqrt{\pi/2}}{4}$

$SR_4 = \frac{\Delta x_4}{3}\left(\cos x_0^2 + 4\cos x_1^2 + 2\cos x_2^2 + 4\cos x_3^2 + \cos x_4^2\right) = 0.97829$ //

$SR_8 = \frac{\Delta x_8}{3}\left(\cos x_0^2 + \dots + \cos x_8^2\right) = 0.977503$ //.

→ Fix to 5(b):



interested in ...

Prob = $\frac{\text{shaded}}{\text{total area}}$

So, Prob = $\frac{\int_0^{\sqrt{2}} 2 - x^2\,dx}{\text{total area}} = \frac{2x - \frac{x^3}{3}\Big|_0^{\sqrt{2}}}{4} = \frac{2\sqrt{2} - \frac{2\sqrt{2}}{3}}{4} = \frac{4\sqrt{2}/3}{4} =$

$= \boxed{\frac{\sqrt{2}}{3}}$ //

18.01: REVIEW FOR EXAM 3

IVAN LOSEV

Contents

1. Differential equations

Review

The equation here can be written in the form $y' = \frac{f(x)}{g(y)}$. Rewrite the equation as $g(y)dy = f(x)dx$. Let $F(x), G(y)$ be antiderivatives of $f(x), g(y)$. Then we get $F(x) - G(y) = C$, where C is some constant. So we can express y in terms of x. The expression will still include an unknown constant C. This constant is usually recovered from knowing the value of $y(x)$ at some fixed point a.

↑ indefinite integral

Problem to practice (from exam 2)

- Practice questions for exam 2, problem 10.
- Practice exam 2, problem 5b.
- Exam 2, problem 5b.

2. Definite integrals vs Riemann sums

can just do it — don't have to memorize

2.1. Riemann sums from definite integrals. Let $f(x)$ be a function defined on an interval $[a, b]$. Pick an integer n and divide $[a, b]$ into n equal intervals (each of length $\frac{b-a}{n}$). These intervals will be $[a, a+\frac{b-a}{n}], [a+\frac{b-a}{n}, a+2\frac{b-a}{n}], \ldots, [a+(i-1)\frac{b-a}{n}, a+i\frac{b-a}{n}], \ldots, [a+(n-1)\frac{b-a}{n}, b]$. The interval number i has end-points $a + (i-1)\frac{b-a}{n}, a + i\frac{b-a}{n}$. This is because there are $i - 1$ intervals of length $\frac{b-a}{n}$ before the left-end point and i such intervals before the right end-point.

Pick points $x_1, \ldots, x_n$, where x_i lies on the interval number i, so that $a + (i-1)\frac{b-a}{n} \leqslant x_i \leqslant a + i\frac{b-a}{n}$. Then

$$\frac{b-a}{n}\sum_{i=1}^{n} f(x_i) = \frac{b-a}{n}(f(x_1) + f(x_2) + \ldots + f(x_n))$$

is a Riemann sum for the integral $\int_a^b f(x)dx$. If $f(x)$ is continuous, then (by definition) the integral $\int_a^b f(x)dx$ is the limit of its Riemann sums as $n \to +\infty$.

There are some distinguished Riemann sums:

- Left R.S.: here x_i is the left end-point of the i-th interval, i.e., $x_i := a + (i-1)\frac{b-a}{n}$.
- Right R.S.: here x_i is the right end-point of the i-th interval, i.e., $x_i := a + i\frac{b-a}{n}$.
- Upper R.S.: here $f(x_i)$ is the maximal value of $f(x)$ on the i-th interval.
- Lower R.S.: here $f(x_i)$ is the minimal value of $f(x)$ on the i-th interval.

In particular, the integral $\int_a^b f(x)dx$ is always $\geqslant$ any of its lower R.S., but is always $\leqslant$ any of its upper Riemann sum.

- Practice questions, problem 3.
- Exam, problem 2a.

2.2. **Definite integrals from Riemann sums.** The goal here is to recover integrals from their Riemann sums. A typical problem is to compute a certain limit (or to express it as an integral).

A guide to recognizing Riemann sums.

A problem here looks like the following: Compute the limit

$$\lim_{n\to\infty} \frac{d}{n}(F_1 + \ldots + F_n).$$

Here d will be some number, and $F_1, \ldots, F_n$ will be values of an appropriate function $f(x)$ at some points $x_1, \ldots, x_n$ (in many cases a function should be clear from the formula).

This limit is calculated by relating it to the integral $\int_a^b f(x)dx$. Here:

- f is a function mentioned above.
- We mostly have left or right Riemann sums in these problems. In the case of a left sum, the smallest argument x_1 is the same for all n, it will coincide with the left end point a. To determine the right end-point b one can consider the largest argument x_n. It will depend on n but will approach b as $n \to +\infty$. The points $x_1, \ldots, x_n$ will form an arithmetic series with difference $\frac{b-a}{n}$.

  In the case of a right sum, the largest argument x_n is the same for all n. This common value will be b. To determine a compute the limit of the smallest argument x_1. The points $x_1, \ldots, x_n$ form an arithmetic series with difference $\frac{b-a}{n}$.

  In general, a is still the limit of x_1, and b is the limit of x_n.

2.3. **Computation of an integral from definition.** The strategy here is as follows:

1. Write down the required Riemann sum (if no requirement is made, left or right should be used).
2. Compute the Riemann sum for given n.
3. Compute the limit as $n \to \infty$.

- Practice questions, problem 2.

3. Fundamental theorems of calculus

* Study this way more *

3.1. Statements. 1st (form of) FTC: If $f(x)$ is continuous, then $\frac{d}{dx}\int_a^x f(t)dt = f(x)$ (please note that the variable of integration is different from x).

i or d or diff from

2nd (form of) FTC: If F(x) is antiderivative of f(x), then $\int_a^b f(t)dt = F(b) - F(a)$.

FTC (especially in its second form) is a powerful tool to compute definite integrals.

$f(x) = \frac{d}{dx} F(x)$

3.2. Integration via substitution. Recall that if $u = u(x)$, then $\int f(u(x))u'(x)dx = \int f(u)du$ (follows from the chain rule). Then one can computes $\int f(u)du$ and plugs $u = u(x)$ in the result.

When we compute definite integrals we can make substitution both for the integrand and for the bounds: $\int_a^b f(u(x))u'(x)dx = \int_{u(a)}^{u(b)} f(u)du$.

This is convenient because we do not need to remember which substitution we made, we can just compute the integral on the right-hand side.

Problems to practice:

- Practice questions, problem 1.
- Exam 3, problem 1, problem 5d.

did w/ Sasha
better be able to do

3.3. Derivatives of integrals. Problems here are about computation of derivatives like $\frac{d}{dx}\int_{a(x)}^{b(x)} f(t)dt$ (we remark that the function we differentiate is a function of x not of t). In easier problems $a(x)$ will be constant and $b(x)$ will be x, then the answer is just $f(x)$ (by the 1st FTC).

In more difficult problems both $a(x)$ and $b(x)$ do depend on x. One way to do such problems is, first, to compute the integral explicitly and then to differentiate it. However, this requires too much work. Actually, there is a general formula that is proved by using the 2nd FTC and the chain rule:

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(t)dt = f(b(x))b'(x) - f(a(x))a'(x).$$

?? can always write like that

Problems to practice.

- Practice questions, problem 4b), problem 5.
- Practice exam: problem 3 a),b).
- Exam, problem 5 a),b).

4. Applications: areas, volumes etc.

4.1. Areas. To find an area of some region one needs

- to place it to a coordinate plane (if it is not there from the beginning).
- Compute the length $l(x)$ the cross-section of the region by the vertical line corresponding to x.
- Determine bounds $a < b$. Usually they are specified in the statement of the problem or are such that the region is enclosed btw. $x = a$ and $x = b$.

For instance, a possible problem is to determine the area between the graphs $y = f(x), y = g(x)$. The simplest possibility here is that the graphs have only two points of intersection: $x = a$ and $x = b$ with $a < b$. If $f(x) \geqslant g(x)$ for $x \in (a, b)$, then the area is given by $A = \int_a^b f(x)dx$. For three points of intersection ($a < b < c$) the formula becomes more complicated: $A = \int_a^b |f(x) - g(x)|dx + \int_b^c |f(x) - g(x)|dx$, etc.

Concepts seem simple
can I do the math?

Sometimes, it is more convenient to use horizontal lines to form the cross-section. The formula for the area will look line $\int_a^b l(y)dy$. The basic principle to choose between the two choices of coordinates is that one needs to use cross-sections producing easier/more explicit integrals.

4.2. **General volumes.** To find a volume of some region one needs

- to place it to a coordinate space (if it is not there from the beginning).
- Compute the area $A(x)$ of the cross-section of the solid by a plane consisting of all points with given x-coordinate (equal to x). Instead of x one can take either of the other two coordinates.
- Determine bounds $a < b$ (with the same reasoning as for areas). rotate

Here problems of two type are possible. Either the solid is given by some inequalities (like in PSet 5b, problem 5) or it is given geometrically (and not placed into the coordinate space). In the first case, cross-sections will be again given by inequalities (with, say, x fixed) and bounds a, b will be such that the inequalities have no solutions for $x < a$ and $x > b$. In the second case one needs to choose direction of cross-sections so that they become as simple geometrically as possible.

4.3. **Volumes with rotational symmetries.** When a solid has rotational symmetry (= obtained by revolving some flat region about a line) there are special methods of computing its volume: disks (washers) and shells.

A disk method is a general method described above, where one takes cross-sections perpendicular to axis of rotation. In this case a cross-section is a disk or, more generally, a "washer" (a disk where a smaller disk with the same center is excluded).

where theorm of paps

A shell method is different. We cut the solid into thin cylindrical shells. The area of the corresponding cross-section is $2\pi r h(r)$, where r is its radius (=distance from the cross-section to the axis of rotation) and $h(r)$ is the height of the corresponding shell.

For instance, consider the following problem. Consider the region given by $a \leqslant x \leqslant b, g(x) \leqslant y \leqslant f(x)$ (we assume that $0 \leqslant a$ and $0 \leqslant g(x)$ for $x \in [a, b]$). Then we rotate the region about one of the coordinate axis.

If we rotate it about x-axis, it is better to use the disk method because it produces the integral very easy. Namely, cross-section is a washer with smaller radius $g(x)$ and larger radius $f(x)$. So the volume will be $\int_a^b \pi(f(x)^2 - g(x)^2)dx$.

If we rotate our region about y-axis, it is better to use the shell method. The height of the shell at position x is $f(x) - g(x)$. So the volume is $\int_a^b 2\pi x(f(x) - g(x))dx$.

Another trick that can be used to compute volumes is the Pappus theorem, see the section on centroids.

Problems to practice:

- Practice questions: Problems 6,7.
- Practice exam: Problem 2.
- Exam: Problem 3.

I just use that - is that wrong?

4.4. **Average value.** The average value of the function $f(x)$ on an interval $[a, b]$ is $\frac{1}{b-a}\int_a^b f(x)dx$.

Problems to practice:

- Practice questions, problem 8.
- Practice exam, problem 5.
- Exam 4b (although this is not a typical problem here).

Center of Mass
what day?

What again? moment of inertia

4.5. **Centroids.** Given a region on the plain, its center of mass is the point with coordinates $(m_y/m, m_x/m)$, where m is the area of the region and m_y, m_x are momenta about y- and x-axis computed as follows. Let $h(x)$ ("height") be the length of the cross-section of the region by the vertical line with x-coordinate equal to x. Similarly, let $w(y)$ ("width") be the length of the cross-section of the region by the vertical line with y-coordinate equal to y. Then

$$m_y = \int_a^b xh(x)dx, m_x = \int_c^d yw(y)dy, m = \int_a^b h(x)dx = \int_c^d w(y)dy.$$

? picture

The center of mass (centroid) is easier to determine when a region has some symmetry. For instance, if it is symmetric about a point, then its centroid is just this point. If a region is symmetric about some line, then its centroid lies on this line.

Centroids have a nice application to computation of volumes (the 1st Pappus theorem). Namely, given a region on a plane and some line such that the region lies entirely on one side from the line, consider the solid obtained by revolution of the region about the line. Its volume V is equal to $2\pi Ar$, where A is the area of the region and r is the distance from the centroid to the axis of rotation. This is especially helpful when we know the centroid without computing integrals, i.e., when the region is symmetric about a point, or when we just know the distance r. This happens when the region is symmetric about the line that is parallel to the axis of revolution (the centroid just lies on that line).

? Just what we did in recitation today?

Also the Pappus theorem may be useful when the line of revolution is not parallel to coordinate axis.

Remark. Sometimes computing m_y, m_x by the formulas given above is not practical. There are alternative formulas for m_y and m_x. They can be deduced from the Pappus theorem. Namely, the volume of the solid obtained by rotating the region about x-axis is $2\pi m_x$ (the computation using the shell method). On the other hand, we can compute the same volume using the disk method, which will give a different (sometimes easier) integral.

For example, suppose that our region is given by $a \leqslant x \leqslant b, 0 \leqslant y \leqslant f(x)$. Then we have $m_y = \int_a^b xf(x)dx, m_x = \int_a^b \frac{f(x)^2}{2}dx$.

4.6. **Probability.** Let x be some quantity. We want to determine how often x takes values btw. a and b, or, in other words, the *probability* of the event $x \in [a,b]$. In some cases to answer this question we need to know a probability distribution $f(x)$.

A *probability distribution* $f(x)$ is a function with $f(x) \geqslant 0$ and $\int_{-\infty}^{\infty} f(x)dx = 1$. The probability of the event $x \in [a,b]$ with distribution function $f(x)$ is $\int_a^b f(x)dx$. Sometimes, the domain of all possible values of x is not the whole $(-\infty, +\infty)$, for instance, x may take only positive values. Then we consider $f(x)$ with $\int_0^\infty f(x)dx$.

Now suppose we want to determine the average value of some quantity y depending on x (the term for this average value is "expectation" or "mathematical expectation"). Then it is given by the integral $\int_{-\infty}^{\infty} y(x)f(x)dx$. If x takes values only on $[A,B]$, then the integral becomes $\int_A^B y(x)f(x)dx$.

Just set it up + integrate

Other problems on probability are possible, compare with PSet6, problem 5.

Problems to practice: Exam, problem 4.

5. Numerical integration

This includes methods for computing approximate values of integrals. All methods we have are ramifications of Riemann sums. More precisely, we approximate an integral on a small interval using some values of a function on this interval. Then, for large intervals, we break them into n small intervals, make an approximation on each small interval, and then add all n approximations.

Basic formulas (approximations on a single interval).

Trapezoidal rule. $\int_a^b f(x)dx \approx (b-a)\frac{f(a)+f(b)}{2}$ (when $f(a), f(b) > 0$ this is just the area of the trapeze with vertices $(a, f(a)), (a, 0), (b, 0), (b, f(b))$).

memorize

Simpson rule. $\int_a^b f(x)dx \approx (b-a)(\frac{1}{6}f(a) + \frac{4}{6}f(\frac{a+b}{2}) + \frac{1}{6}f(b))$. Geometrical meaning of this formula is the following. Draw a parabola through the points $(a, f(a)), (\frac{a+b}{2}, f(\frac{a+b}{2})), (b, f(b))$. If $f(a), f(\frac{a+b}{2}), f(b) > 0$, then the r.h.s. of the previous approximate equality is the area under this parabola, for $a \leqslant x \leqslant b$.

In other words, the coefficients of $f(a), f(\frac{a+b}{2}), f(b)$ are chosen such that the approximation is exact for $1, x, x^2$ (as it happens, due to symmetry, the formula also gives an exact value for $f(x) = x^3$).

General formulas.

Trapezoidal rule. Divide an interval $[a, b]$ into n small intervals (of length $\frac{b-a}{n}$). Let $x_i, i = 0, 1, \ldots, n$, be end-points, $x_i = a + i\frac{b-a}{n}$. Set $y_i = f(x_i)$. Then the approximation is given by

$$\int_a^b f(x)dx \approx \frac{b-a}{n}\sum_{i=0}^{n-1}\frac{y_i + y_{i+1}}{2} = \frac{b-a}{n}\left(\frac{1}{2}y_0 + y_1 + \ldots + y_{n-1} + \frac{1}{2}y_n\right).$$

Simpson rule. Divide an interval $[a, b]$ into $2n$ small intervals (of length $\frac{b-a}{2n}$). Let $x_i, i = 0, 1, \ldots, 2n$, be end-points, $x_i = a + i\frac{b-a}{2n}$. Set $y_i = f(x_i)$. Then the approximation is given by

$$\begin{aligned}\int_a^b f(x)dx \approx& \frac{b-a}{n}\sum_{i=0}^{n-1}\frac{y_{2i} + 4y_{2i+1} + y_{2i+2}}{6}\\ &= \frac{b-a}{n}(\frac{1}{6}y_0 + \frac{4}{6}y_1 + \frac{2}{6}y_2 + \frac{4}{6}y_3 + \frac{2}{6}y_4 + \ldots + \frac{2}{6}y_{2n-2} + \frac{4}{6}y_{2n-1} + \frac{1}{6}y_{2n}).\end{aligned}$$

Problems to practice.

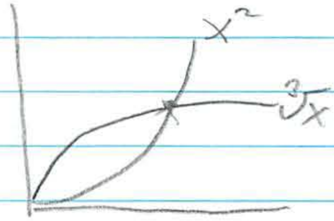
- Practice question, problem 9.
- Practice exam, problem 5.
- Exam, problem 2b,c.

Recitation

Review Exam 3 11/9

Center of Mass

$\frac{x^2}{\sqrt[3]{x}}$ $0 \le x \le 1$



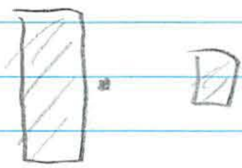
Center of Mass - rectangle

$\bar{x}$ = midpoint ← average
$\bar{y}$ = midpoint ← average

$\boxed{\bar{x} = \frac{\text{midpoint} \cdot \text{mass}}{\text{mass}}}$ moment

bunch of rectangles

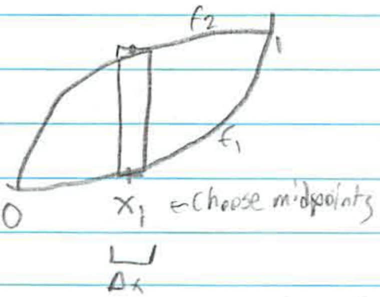
weighted average



(2) $\bar{x} = \frac{\sum x_i \, m_i}{\sum m_i}$ for all coms

(1) find the com of each rectangle

* think about like Riemann Sum



Area $= \Delta x \, (f_2(x_i) - f_1(x_i))$

(x_i, y_i) rectangle $= \left(x_i, \frac{f_2 + f_1}{2}\right)$

avg of y values

$$\bar{x} = \lim_{n \to \infty} \frac{\sum x_i \, (f_2(x_i) - f_1(x_i)) \Delta x}{\sum [f_2(x_i) - f_1(x_i)] \Delta x}$$

evaluating 1 function at a bunch of pts

weighted avg of x value each strip →

$$= \frac{\int_0^1 x \, (f_2(x) - f(x)) \, dx}{\int_0^1 f_2(x) - f_1(x) \, dx}$$

Harder one

- Rule does not apply everywhere

$$\bar{y} = \frac{\lim_{n\to\infty} \sum \overbrace{\left(\frac{f_2(x_i)+f_1(x_i)}{2}\right)}^{y_i} \cdot \overbrace{\left(f_2(x_i) - f_1(x_i)\right)\Delta x}^{\text{area}}}{\int_0^1 (f_2 - f_1)\,dx}$$

$$= \frac{\int_0^1 \frac{f_2 + f_1}{2}(f_2 - f_1)\,dx}{\int_0^1 (f_2 - f_1)\,dx}$$ ← if $f_1 = 0$ get $\frac{f_2}{2}$ - semicircle

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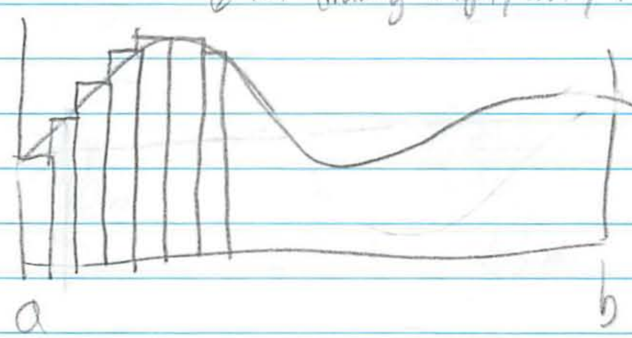
Riemann Sum

← not drawing right/left/top consistently



$$\sum_{i=1}^{n} \frac{b-a}{n} \cdot f(x_i) = \sum_{i=1}^{n} \overbrace{\frac{b-a}{n}}^{\text{width}} \cdot \overbrace{f\left(a + \frac{i(b-a)}{n}\right)}^{\text{height}}$$

Note $x_0 = a$

$x_1 = a + \frac{b-a}{n}$

$x_i = a + i\left(\frac{b-a}{n}\right)$

↑ left ↑ right ↑ moving up this # $\frac{b-a}{n}$

Evaluate

$$\frac{\lim_{n\to\infty} \sum_{i=1}^{n} \sin\left(\frac{ib}{n}\right)}{n}$$

*key - make it look like integral

$a=0$ i is index $\rightarrow \frac{b}{n} - a \rightarrow$ goes to b

$b=b$ i starts at 0

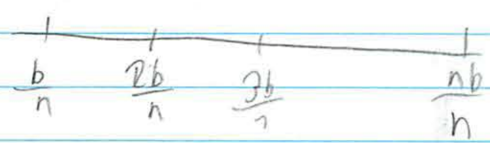
where is $\Delta x \Rightarrow$ should be $\frac{b}{n}$ have $\frac{1}{n}$

Almost $\int_0^b \sin x \, dx$ but should have $\frac{b}{n}$

$$= \lim_{n\to\infty} b \sum_{i=1}^{n} \sin x\left(\frac{ib}{n}\right)$$

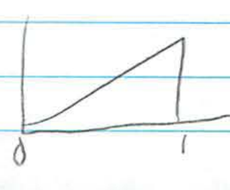
So say

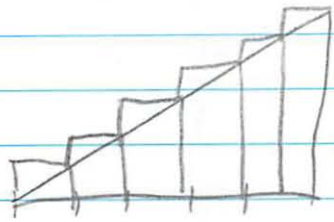
$$\frac{1}{b}\int_0^b \sin x\,dx$$



$\int_0^1 x\,dx$ by upper Riemann Sum

- Same as right hand





$$\sum_{i=1}^{n} \frac{1}{n} f\left(\frac{i}{n}\right) =$$

$$\frac{b-a}{n} = \frac{1}{n} = \Delta x \qquad \sum_{i=1}^{n} \frac{1}{n} \cdot \frac{i}{n}$$

$$a + i\Delta x = \frac{i}{n} \qquad = \int_0^1 x\,dx = \lim_{n\to\infty} \frac{1}{n^2} \sum_{i=1}^{n} i$$

can pull it out

$$= \lim_{n\to\infty} \frac{1}{n^2} \cdot \frac{n(n+1)}{2}$$

$$= \frac{1}{2}$$

can't do if don't know antiderivative
- no closed form of expression

$$P(0,T) = \int_0^T f(t)\,dt = \frac{1}{2}$$

when $T = \frac{1}{2}$ life

$$\int_0^\infty f(t)\,dt = 1 \quad \leftarrow \text{infinity} = 1$$

Improper

$\int_0^\infty$ ← 1 or both bounds ∞

or asymptote is somewhere

$$\int_1^{\infty} \frac{1}{x^2}\,dx = \lim_{N\to\infty} \int_1^{N} \frac{1}{x^2}\,dx$$

$$= \lim_{N\to\infty} -\frac{1}{x}\Big|_1^{N} = \lim_{N\to\infty} -\underbrace{\frac{1}{N}}_{0} - (-1)$$

$$\int_3^{7} \frac{1}{x-3}\,dx = \lim_{N\to 3^+} \int_N^{7} \frac{1}{x-3}\,dx =$$

$$\lim_{N\to 3^+} \ln(x-3)\Big|_N^{7} = \ln(4) - \lim_{N\to 3^+} \ln(n-3)$$

Pretend where bound is not bad
Push it in some
Then find bound as it gets bad

Review Session

Ivan

1/9

$$\int_0^x t^2 e^{-t^2}\,dt$$

$u = t^2$ 2 differentiate

$du = 2t\,dt$

$\frac{1}{2}du = t\,dt$

break up into nice pieces

$$e^{-t^2}\,\frac{t}{2}\,\underbrace{2t\,dt}_{2t\,dt}$$

Complex but should be able to do

$t = \sqrt{u}$

$$\int_{0^2}^{x^2} \frac{\sqrt{u}}{2} e^{-u}\,\underbrace{du}_{2t\,dt}$$

$$F(x) = \frac{1}{2}\int_0^{x^2} \sqrt{u}\, e^{-u}\,du$$

$$\int_0^9 \sqrt{u}\, e^{-u}\,du$$

$$2\sqrt{3}$$

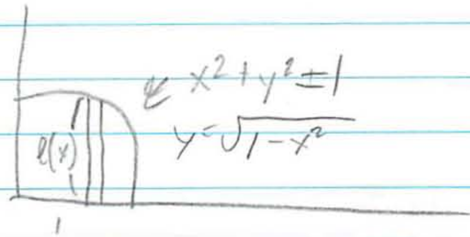
$$F(x) = \frac{x^3}{3} = \int_0^x t^2 e^{-t^2}\,dt \qquad \left[\begin{array}{l} e^{-t^2} \le 1 \\ t^2 e^{-t^2} \le t^2 \end{array}\right.$$

^ estimate ingrant by something

$$\int_0^x t^2\,dt = \frac{x^3}{3}$$

Center of Mass of Semicircle 

- need to place in coordinate plane



$(\bar{x}, \bar{y}) = \left(\frac{m_x}{m}, \frac{m_y}{m}\right)$ ← momentum about axis

$l(x) = \sqrt{1-x^2}$

know area $= \frac{\pi}{4}$

Can't find intergral w/ our simple techniques

$m_y = \int_0^1 x\, l(x)\, dx$

$\int_0^1 x\sqrt{1-x^2}\, dx$

↑ substitution

$u = 1 - x^2$

$du = -2x\, dx$

$x\, dx = -\frac{1}{2}\, du$ ← replace w/

don't forget $+C$ as well

$\int_0^1 \sqrt{u}\; -\frac{1}{2}\, du$

↓ actually integrate

$-\frac{1}{2}\frac{u^{3/2}}{\frac{3}{2}}\Big]_1^0$ ← do substitution for bounds

$\boxed{= \frac{1}{3}}$ → $\bar{x} = \frac{\frac{1}{3}}{\pi/4} = \frac{4}{3\pi}$

$\bar{y} = \frac{4}{3\pi}$

since symetrical on $y = x$ so lies on angle of symetry $\bar{x} = \bar{y}$

But let's compute $y(x)$ integral

$$m_x = \int_0^1 \frac{\ell(x)^2}{2}\,dx$$

$$\int \frac{1-x^2}{2} = \frac{1}{2} - \frac{1}{6} = \frac{1}{3}$$

relation momenta + volume

→ get solid of revolution

Shell formula

$$\boxed{V = 2\pi m_x}$$

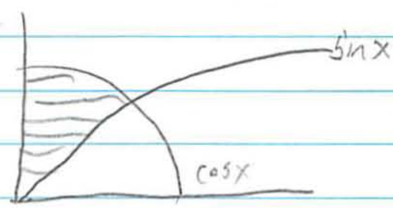
↑ theory of papus

$$V = \int_c^d 2\pi\, y\, w(y)\,dy \quad \leftarrow \text{shell method}$$

$$m_x = \int_c^d y\, w(y)\,dy$$

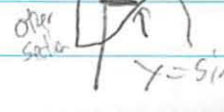
Can also do w/ disks

on p-set →



$$m_y = \int_0^{\pi/4} x\,(\cos x - \sin x)\,dx$$

$$m_x = \int_0^{\frac{\sqrt{2}}{2}} \sin^{-1}(y)\,dy + \int_{\frac{\sqrt{2}}{2}}^1 \cos^{-1}(y)\,dy$$

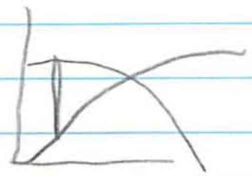


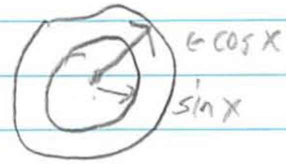
$y = \sin x \to x = \sin^{-1}(y)$

Don't know how to compute yet

so use ~~Vertical cross section~~ to find volume

← disc method





$$\text{Volume} = \int_0^{\pi/4} \pi\left((\cos x)^2 - (\sin x^2)\right) dx$$

Need momentum
So / 2π

$$m_x = \int_0^{\pi/4} \frac{1}{2}\left(\cos^2 x\right) - \left(\sin^2 x\right) dx$$

On exam all integrals computed by substitution

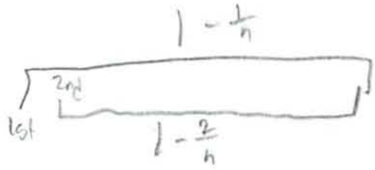
Bank + Interest Q

Amount A_0 → after t years → $A(t)$
Make continuous desposit $k\Delta t$ over Δt

Find definite integral for end 1st year $(t=1)$

- use Rieman Sums
- Break interval $(0,1)$ into n pieces
- interest added at end of interval

1st deposit $= \frac{k}{n}$



After $1 - \frac{1}{n}$ years:

$$\frac{k}{n} e^{(1 - \frac{1}{n}) r}$$

But are making deposits $\frac{k}{n}$

- after 2nd deposit $(1 - \frac{2}{n}$ years$)$

$$\frac{k}{n} e^{(1 - \frac{2}{n}) r}$$

Deposit N_i becomes $(1 - \frac{i}{n})$

⋮

Deposit N_n becomes ← end of year

$$\frac{k}{n} N_n$$

Total $\sum_{i=1}^{n} \frac{k}{n} e^{(1 - \frac{i}{n}) r}$

? riemann sum of integral ?

my thought → $\cancel{\int_0^1 \frac{k}{n} e^{(1 - \frac{i}{n}) r} \, di}$

know how to use → $\int_0^1 k e^{(1-t) r} \, dt$ (get rid of n)

Right Riemann Sum $\frac{1}{n} \sum_{i=1}^{n} k e^{(1 - \frac{i}{n}) r}$

Simpsons's Rule

-from sample exam

Have data measured time to time

$x(1) = 3$

$x(8) = 2$

$x(15) = 0$

$x(22) = 1$

$x(29) = 3$

Need to find average value of $x(t)$ between $x(1)$ $x(29)$

$$\underbrace{\frac{1}{28}}_{b-a} \int_1^{29} x(t)\, dt$$

Problem becomes estimate integral using Simpson's rule w/ 5 points

$$\frac{\Delta x}{3} \left(y_0 + 4y_1 + 2y_2 + 4y_3 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n \right)$$

length of single interval

$y_0 \quad y_1 \quad y_2 \quad y_3$

Δx

$\frac{b-a}{2n}$

Can write simpler

$$\int_a^b f(x)\, dx \approx (b-a) \frac{1}{6} \left(f(a) + 4f\left(\frac{a+b}{2}\right) + 2f\left(\frac{a+b}{2}\right) + 4\left(\frac{a+b}{2}\right) + f(b) \right)$$

$\cancel{\text{or}}$ $2\Delta x\left(\frac{1}{6}y_0 + \frac{4}{6}y_1 + \frac{2}{6}y_2 + \ldots\right)$

$$\frac{\Delta x}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + y_4\right)$$

$\uparrow \Delta x = 7 = b-a$

$$\frac{7}{3}\left(3 + 4\cdot 2 + 2\cdot 0 + 4\cdot 1 + 3\right) = 42$$

$$\approx \int_1^{29} x(t)\,dt$$ * need avg value

$$\frac{42}{28} = \frac{3}{2} = 1.5$$

$\uparrow b-a$

$$\int_0^{\sqrt{\pi}} \cos\left(x^2\right)dx =$$ Simpson's rule w/ 8 intervals

$$\Delta x = \frac{\sqrt{\pi}}{8}$$

$$\frac{\frac{\sqrt{\pi}}{8}}{3}\Big(\cos(0^2) + 4\left(\cos\frac{\pi}{64}\right) + 2\cos\left(\frac{4\pi}{64}\right) + 4\cos\left(\frac{9\pi}{64}\right) + 2\cos\left(\frac{16\pi}{64}\right) + 4\cos\left(\frac{25\pi}{64}\right) + 2\cos\left(\frac{36\pi}{64}\right) + 4\cos\left(\frac{49\pi}{64}\right) + \cos\pi\Big)$$

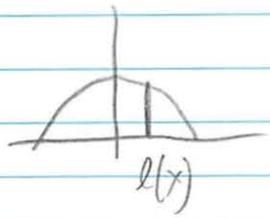
$0 \le y \le \cos(x^2)$ $\quad -\frac{\sqrt{\pi}}{2} \le x \le \frac{\sqrt{\pi}}{2}$

– volume of revolution about y axis

– symetric around y axis

– relove 1 half

Just use shells → not Papus



$$V = \int_0^{\sqrt{\pi}/2} 2\pi x \cos(x^2)\, dx$$

- u substitue
- integrate

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \left(\frac{2i}{n}\right)^2 = \int_a^b f(x)\, dx$$

$$f(x) = x^2$$

$$\overset{\circ}{0} \quad \overset{\times}{\tfrac{2}{n}} \quad \overset{\times}{\tfrac{4}{n}} \quad \overset{\times}{\tfrac{6}{n}} \quad \overset{\times}{\tfrac{2n}{n}} \leftarrow b = 2$$

$\uparrow a = 0$

$$\frac{1}{2} \int_0^2 x^2\, dx$$

$\uparrow$ to compensate that did not $\cdot$ by width in limit

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18.01 Practice Questions for Exam 3 – Fall 2006

1. Evaluate $\quad a) \displaystyle\int_0^1 \frac{x\,dx}{\sqrt{1+3x^2}} \qquad b) \int_{\pi/3}^{\pi/2} \cos^3 x \sin 2x\,dx$

2. Evaluate $\displaystyle\int_0^1 x\,dx$ directly from its definition as the limit of a sum.

Use upper sums (circumscribed rectangles). You can use the formula $\displaystyle\sum_1^n i = \frac{1}{2}n(n+1)$.

3. A bank gives interest at the rate r, compounded continuously, so that an amount A_0 deposited grows after t years to an amount $A(t) = A_0 e^{rt}$.

You make a daily deposit at the constant annual rate k; in other words, over the time period Δt you deposit $k\Delta t$ dollars. Set up a definite integral (give reasoning) which tells how much is in your account at the end of one year. (Do not evaluate the integral.)

4. Consider the function defined by $F(x) = \displaystyle\int_0^x \sqrt{3+\sin t}\,dt$. Without attempting to find an explicit formula for $F(x)$,

a) (5) show that $F(1) \le 2$;

b) (5) determine whether $F(x)$ is convex ("concave up") or concave ("concave down") on the interval $0 < x < 1$; show work or give reasoning;

c) (10) give in terms of values of $F(x)$ the value of $\displaystyle\int_1^2 \sqrt{3+\sin 2t}\,dt$.

5. If $\displaystyle\int_0^x f(t)\,dt = e^{2x}\cos x + c$, find the value of the constant c and the function $f(t)$.

6. A glass vase has the shape of the solid obtained by rotating about the y-axis the area in the first quadrant lying over the x-interval $[0, a]$ and under the graph of $y = \sqrt{x}$. By slicing it horizontally, determine how much glass it contains.

7. A right circular cone has height 5 and base radius 1; it is over-filled with ice cream, in the usual way. Place the cone so its vertex is at the origin, and its axis lies along the positive y-axis, and take the cross-section containing the x-axis. The top of this cross-section is a piece of the parabola $y = 6 - x^2$. (The whole filled ice-cream cone is gotten by rotating this cross-section about the y-axis.)

What is the volume of the ice cream? (Suggestion: use cylindrical shells.)

8. Rectangles are inscribed as shown in the quarter-circle of radius a, with the point x being chosen randomly on the interval $[0, a]$. Find the average value of their area.

9. Find the approximate value given for the integral below by the trapezoidal rule and also by Simpson's rule, taking $n = 2$ (i.e., dividing the interval of integration into two equal subintervals):

$$\int_0^{\pi/2} \sin^6 x\,dx$$

Other possible problems: Volumes by vertical slicing 4B, Work problems (P.Set 5)

EXAM 3 PRACTICE : SOLUTIONS : FALL 2006

$\boxed{1}$ a) $\displaystyle\int_0^1 \frac{x\,dx}{\sqrt{1+3x^2}} = \frac{1}{3}(1+3x^2)^{1/2}\Big]_0^1 = \frac{1}{3}(2-1) = 1/3$

b) $\displaystyle\int_{\pi/3}^{\pi/2} \cos^3 x \sin 2x\,dx = \int_{\pi/3}^{\pi/2} 2\cos^4 x \sin x\,dx$

$\displaystyle = -\frac{2}{5}\cos^5 x\Big]_{\pi/3}^{\pi/2} = -\frac{2}{5}\left[0-\left(\frac{1}{2}\right)^5\right]$

$\displaystyle = \frac{1}{5\cdot 16}$

$\boxed{2}$

Upper sum is sum of circums. rectangles

$\displaystyle = \sum_{i=1}^{n}\left(\frac{i}{n}\right)\frac{1}{n}$
height width

$\displaystyle = \frac{1}{n^2}\sum_1^n i = \frac{n^2+n}{2n^2} = \frac{1}{2}+\frac{1}{2n}$

$\displaystyle \lim_{n\to\infty} = \int_0^1 x\,dx = \frac{1}{2}$

$\boxed{3}$ $\underset{0}{\vert}\ t_1\ t_2\ \dots\ \underset{1}{\vert}$ Divide t-interval into n equal subintervals, length Δt.

(*) $k\Delta t$ deposited in the ith time interval grows by year-end to $k\Delta t\, e^{r(1-t_i)}$

Total at year-end $\approx \sum_1^n k e^{r(1-t_i)}\Delta t$

As $n\to\infty$, this $\to \int_0^1 k e^{r(1-t)}dt$

[or can make argument using dt instead of Δt: replace Δt by dt in (*), then pass directly to the last line]

$\boxed{4}$ a) on $[0,1]$, $\sin t < 1$

So $\displaystyle\int_0^1 \sqrt{3+\sin t}\,dt < \int_0^1 \sqrt{4}\,dt = 2.$

b) $F'(x) = \sqrt{3+\sin x}$ (2nd F.T.)

$\displaystyle F''(x) = \frac{\cos x}{2\sqrt{3+\sin x}} > 0$ on $[0,1]$

$\therefore$ convex (concave up).

c) Set $u = 2t$, so $du = 2\,dt$

$\displaystyle\int_1^2 \sqrt{3+\sin 2t}\,dt = \int_2^4 \sqrt{3+\sin u}\cdot\frac{1}{2}du$

$\displaystyle = \frac{1}{2}F(u)\Big]_2^4 = \frac{1}{2}\left[F(4)-F(2)\right]$

$\boxed{5}$ $\displaystyle\int_0^0 f(t)\,dt = 0 = e^{2\cdot 0}\cos 0 + c$

$\therefore c = -1$

By 2nd F.T.:

$f(x) = D(e^{2x}\cos x - 1)$

$= 2e^{2x}\cos x - e^{2x}\sin x.$

$\boxed{6}$ horizontal slice →

$\displaystyle\int_0^{\sqrt{a}} (\pi a^2 - \pi x^2)\,dy = \pi\int_0^{\sqrt{a}} (a^2 - y^4)\,dy$

$\displaystyle = \pi\left(a^2 y - \frac{y^5}{5}\right)\Big]_0^{\sqrt{a}} = \pi\left[a^{5/2} - \frac{1}{5}a^{5/2}\right]$

$\displaystyle = \pi\cdot\frac{4}{5}a^{5/2}$

$\boxed{7}$

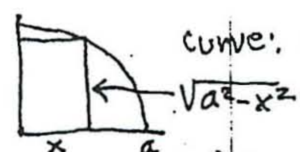
The strip rotated has volume $2\pi x(6-x^2-5x)\,dx$

$\displaystyle \text{volume} = 2\pi\int_0^1 x(6-x^2-5x)\,dx$

$\displaystyle = 2\pi\left[3x^2 - \frac{1}{4}x^4 - \frac{5}{3}x^3\right]_0^1$

$\displaystyle = 2\pi\left[3 - 1/4 - 5/3\right] = 2\pi\cdot\frac{13}{12}$

$\boxed{8}$



curve: $y = \sqrt{a^2-x^2}$

Area $= x\sqrt{a^2-x^2}$

$\displaystyle \text{Average area} = \frac{1}{a}\int_0^a x\sqrt{a^2-x^2}\,dx$

$\displaystyle = \frac{1}{a}(a^2-x^2)^{3/2}\left(-\frac{1}{3}\right)\Big]_0^a = \frac{1}{3}(a^2)^{3/2} = \frac{a^2}{3}$

$\boxed{9}$

$x =$	0	$\pi/4$	$\pi/2$
$\sin x$	0	$1/\sqrt{2}$	1
$\sin^6 x$	0	$1/8$	1

$\Delta x = \pi/4$

$\boxed{\text{TRAPEZOIDAL}}$

$\displaystyle \approx \frac{\pi}{4}\left(\frac{1}{2}\cdot 0 + \frac{1}{8} + \frac{1}{2}\cdot 1\right)$

$\displaystyle = \frac{\pi}{4}\cdot\frac{5}{8} = \frac{5\pi}{32}$

$\boxed{\text{SIMPSON}}$

$\displaystyle\int_0^{\pi/2} \sin^6 x\,dx$

$\displaystyle \approx \frac{\pi/4}{3}\left(0 + 4\cdot\frac{1}{8} + 1\right)$

$\displaystyle = \frac{\pi}{12}\left(\frac{3}{2}\right) = \frac{\pi}{8}$

18.01 Practice Exam 3

Problem 1.

a) (10) Derive the trigonometric formula $\cos 2x = 1 - 2\sin^2 x$ and use it to evaluate $\int \sin^2 x dx$.

b) (10) Differentiate $x \ln x$, and use your answer to evalute $\int_1^e \ln x \, dx$.

Problem 2. (15) K-mart is selling at half-price its left-over Great Pumpkins— thin orange plastic shells filled with half-price Halloween candy.

A Great Pumpkin has the shape of the curve $x^2 + y^4 = 1$, rotated about the vertical axis, i.e., the y-axis. This curve is symmetric about the x-axis and the y-axis — it looks something like a circle, but somewhat flatter at the top and bottom.

Using units in feet, how many cubic feet of candy will it take to fill a Great Pumpkin? Give the exact answer, then tell if 5 cubic feet will be enough.

Problem 3. (20: 3,7,5,5) The function $F(x) = \int_0^x t^2 e^{-t^2} dt$ is not elementary; it comes up in calculating the standard deviation of The Curve of normal distribution. (In the following, (a) and (b) go together, but (c) and (d) are both independent questions.)

a) Find $F'(x)$.

b) Find the critical point(s) of $F(x)$, and determine their type(s) by studying the sign of $F'(x)$ when x is near a critical point.

c) Express $\int_0^9 \sqrt{u} e^{-u} du$ in terms of values of $F(x)$.

d) Estimate $F(x)$ by showing that $F(x) \leq \frac{x^3}{3}$, if $x > 0$.

Problem 4. (15: 12, 3) The end portion of a boneless AllSoy SmartHam of length a has approximately the shape of the region under the curve $y = \sqrt{x}$, $0 \leq x \leq a$, rotated about the x-axis.

a) When it is sliced vertically into thin slices, what is the average area of a slice?

b) Where on the SmartHam is there a slice having this average area (i.e., how far from the tip)?

Problem 5. (15: 7,8) You only have time to look at the newspaper on Sunday, but the first thing you turn to is the baseball statistics from Saturday's game, to see how many hits your favorite ball-player Pepe LeMoko got. In September (which started on a Saturday) he had a slump in the middle, but came out of it. His record on the five successive Saturdays was

Day:	1	8	15	22	29
No. hits:	3	2	0	1	3

Suppose there was a game every day; estimate the total number of hits he got during those 29 games by using

avg valp

a) the trapezoidal rule

review session

b) Simpson's rule

Problem 6. (15) Rain falls for 10 hours on a little garden pool, increasing from a drizzle to a downpour, then tapering off to a drizzle again. The rate of raining is given by

$$r(t) = t^2(10-t)^2 \quad \text{cm/hr}.$$

.

This being the city, the rain is polluted with acid; at the start of the rain ($t = 0$), it contains 2 nanograms/cu.cm. of acid, but this decreases linearly to only 1 nanogram/cu.cm. by the end of the rain.

Assume the pool has an area of one square meter.

Set up, but **do not evaluate** a definite integral which tells how many nanograms of acid are in the pool at the end of the rain. Give brief reasoning, either by dividing up the time interval into small subintervals, or by using infinitesimal time intervals.

18.01 Practice Exam 3 Sol'ns Fall 2006

$\boxed{1}$ a) $\cos 2x = \cos^2 x - \sin^2 x$
$= (1-\sin^2 x) - \sin^2 x$
$= 1 - 2\sin^2 x$

$$\int \sin^2 x\,dx = \int \frac{(1-\cos 2x)}{2}\,dx = \frac{x}{2} - \frac{\sin 2x}{4} + c$$

b) $D(x\ln x) = \ln x + x\cdot\frac{1}{x} = \ln x + 1$

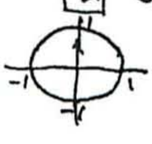
$\therefore$ by the fundamental theorem,

$$x\ln x\Big]_1^e = \int_1^e \ln x\,dx + \int_1^e 1\,dx$$

$$e\cdot 1 - 0 = \int_1^e \ln x\,dx + e - 1$$

$$\therefore \int_1^e \ln x\,dx = 1.$$

$\boxed{2}$ By horizontal slices,
(calculate vol. of top half & double it)



$$= \int_0^1 \pi x^2\,dy = \pi\int_0^1 (1-y^4)\,dy$$

$$= \pi\left(y - \frac{y^5}{5}\right)\Big|_0^1 = \pi\cdot\frac{4}{5}$$

By cylindrical shells: $y = (1-x^2)^{1/4}$

$$= \int_0^1 2\pi x\cdot(1-x^2)^{1/4}$$

$$= -\frac{4\pi}{5}(1-x^2)^{5/4}\Big]_0^1 = \frac{4\pi}{5}$$

$\therefore$ Volume is $\frac{8\pi}{5} \approx \frac{8\cdot(3.14)}{5} > \frac{25}{5}$

5 cubic feet is not enough.

$\boxed{3}$ a) $F(x) = \int_0^x t^2 e^{-t^2}\,dt$; $F'(x) = x^2 e^{-x^2}$
(second fund thm)

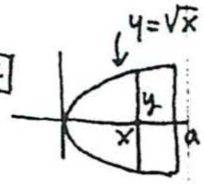
b) $F' = 0$ when $x = 0$; otherwise $F'(x) > 0$. Thus F is increasing, so $x = 0$ is a point of horiz. inflection (not a max or min)

c) $u = t^2$; $du = 2t\,dt$ $\int_0^9 \sqrt{u}\,e^{-u}\,du = \int_0^3 t\,e^{-t^2}\cdot 2t\,dt$
$= 2\cdot F(3)$

d) $e^{-t^2} \le 1$

$$\therefore \int_0^x t^2 e^{-t^2}\,dt \le \int_0^x t^2\,dt = \frac{x^3}{3}.$$

$\boxed{4}$



Area of slice at x is $\pi y^2 = \pi x$

Average area of slices $= \frac{1}{a}\int_0^a \pi x\,d$
$= \frac{1}{a}\,\pi\frac{x^2}{2}\Big]_0^a$

Therefore

average area $= \frac{\pi a}{2}$

which is the area of the slice at $x_0 = a/2$ (halfway)

$\pi\cdot(\sqrt{x_0})^2 = \frac{\pi a}{2} \Rightarrow x_0 = a/2$

$\boxed{5}$

1	8	15	22	29
3	2	0	1	3

a) by trapezoidal rule:

Total # hits $\approx \left(\frac{3}{2} + 2 + 0 + 1 + \frac{3}{2}\right)7 = 6\cdot 7 = \boxed{42}$

b) by Simpson's rule:

Total # hits $\approx \left(\frac{3 + 4\cdot 2 + 2\cdot 0 + 4\cdot 1 + 3}{6}\right)\cdot 14$
$= \frac{18}{6}\cdot 14 = \boxed{42}$

$\boxed{6}$ In an infinitesimal time interval dt at time t,

concentration over time
slope $= -\frac{1}{10}$

$C = 2 - \frac{1}{10}t$

flow rate of water at time t into pool (cc/hour) $= t^2(10-t)^2\cdot 10^4$

pool surface is $(100\text{ cm})^2$

$\therefore$ amt entering from time t to $t+dt$

(*) $= t^2(10-t)^2\cdot 10^4\cdot\left(2 - \frac{t}{10}\right)\cdot d$

Total amt $= 10^4\int_0^{10} t^2(10-t)^2\cdot\left(2-\frac{t}{10}\right)dt$ (**)
nanograms

For Δt calculation:
replace dt by Δt in (*)
write (**) as $\sum_1^n 10^4 t_i^2(10-t_i)^2(2 - t_i/10)\Delta$ → integral given.
and pass to limit as $n\to\infty$

(1) Calc Exam 3 Review 11/9/07

$$\int_0^1 \frac{x}{\sqrt{1+3x^2}}\,dx$$

$u = 1+3x^2$

$du = 6x\,dx$

$x\,dx = \frac{1}{6}du$

$$\int_0^4 \frac{1}{6\sqrt{u}}\,du$$

So don't convert bounds — thought how to why?

$$\frac{1}{6}\int_0^4 u^{-1/2}\,du \qquad \frac{1}{3}(1+3x^2)^{1/2}$$

$$\frac{1}{6}\int_0^4 \frac{u^{1/2}}{\frac{1}{2}}\,du$$

$$\frac{1}{6}\,2\sqrt{1+3x^2}\,\Big|_0^4$$

$$\frac{1}{3}\sqrt{1+3x^2}\,\Big|_0^1$$

$$\frac{1}{3}\sqrt{1+3(4)^2} - \left(\frac{1}{3}\sqrt{1+3(1)^2}\right)$$

$$\frac{7}{3} - \frac{2}{3} = \frac{5}{3} \qquad \frac{1}{3}$$

16
9
48

(b)

$$\int_{\pi/3}^{\pi/2} \cos^3 x \ \sin 2x \ dx$$

? how to get started ← $u = \sin x$ $2\sin x$
$du = \cos x$ $2\cos x$

So just identify something likely & do it

$\int 2\cos^4 x \sin x \, dx$ ← I do again

$-\frac{2}{5} \cos^5 x \Big|_{\pi/3}^{\pi/2} = \frac{1}{5 \cdot 16}$

?

$\cancel{-\sin^5 x / 5}$ ← would it not be sin
$\int \cos = \sin x$

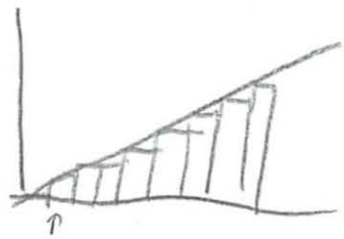
but if do on calc
$\int \cos^4 x = \frac{\sin x \cos^3 x}{4} + \ldots$
weirdness

but $\cos^4 x \sin x$ comes out right — why
weirdness — ask what deriv of this = this

$d \; -\frac{\cos^5 x}{5}$ $\;-\sin^4 x$ ← weird - never saw before

(2) $\int_0^1 x\,dx$ from <u>def</u> as limit sum

~~Rien~~ ereman sum



$\uparrow$

$\frac{1-0}{n}$

$\frac{1}{10}\ \frac{2}{10}\ \frac{3}{10}$

10 pieces

$$\sum_{i=1}^{10} \frac{1}{10} f\left(\frac{i}{10}\right)$$

$\frac{i}{10}$

$\frac{1}{10}^2$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{b-a}{n} f\left(\frac{b-a}{n}\right) + \cancel{2\left(\frac{b-a}{n}\right)^2} + 3\left(\frac{b-a}{n}\right)^2$$

$\uparrow b-a = 1-0 = 1$ Wrong

Since width alway $\frac{1}{n}$

$$\sum_{i=1}^{n} \frac{i}{n}\cdot\frac{1}{n}$$

simplify

$$\frac{1}{n^2}\sum_{1}^{n} i = \frac{n^2+n}{2n^2} = \frac{1}{2} + \frac{1}{2n} = \frac{1}{2}$$

pay attention

$\uparrow$ where is this from

- oh there was a 2nd line to problem and part was given

(3) Did in Ivan's review

- key is knowing time $= 1 - \frac{i}{n}$ ← # deposit

  ↑ year ↑ # intervals

- just have to think + come up w/

$$\sum_{i=1}^{n} \frac{k}{n} e^{(1-\frac{i}{n})r}$$

- and turn into integral get rid of n

$$\int_0^1 k e^{(1-t)r}\, dt$$

Simpson

$$\boxed{\frac{\Delta x}{3}\left(y_1 + 4y_2 + 2y_3 + 4y_4 + 2y_5 + 4y_6 + y_7\right)}$$

(4) $\int_0^x \sqrt{3+\sin t}\, dt$ $\quad F(1) \le 2$

a) - FTC #1 shows what you really know

$\int_0^1$? what more

know $\sin t < 1$ always ← but isn't that finding an explicit formula for

thus $\sqrt{4}\, dt = 2$

↑ ~~well did not find~~ dt

~~$\int \sqrt{4}\, dt$~~ no did not 1st eval original integral

b) sin

$\sqrt{\sin}$ same

$\int$ increasing

concave ~~down~~ up

take 2nd deriv to look at concavity

$F'(x) = \sqrt{3+\sin x}$

$F''(x) = \dfrac{\cos x}{2\sqrt{3+\sin x}}$ ⇒ ∪ ← did not know should calc this for

$F(x) = \int F'(x)$

that 1st line really threw me off!

C) Terms of values

$F(2) - F(1)$

aka actually do it

again - was that not forbidden?

integrate

$u = 2t$ ← constant

$du = 2dt$

where bounds change

$\int_1^2 \sqrt{3+\sin 2x}\, dt = \int_2^4 \sqrt{3+\sin x} \cdot \frac{1}{2}\, du$ ← it integrates

$= \frac{1}{2} F(u) \Big]_2^4 = \frac{1}{2} \left[F(4) - F(2)\right]$ ← well gives it in terms of f

(5) $\int_0^x f(t)\,dt = e^{2x} \cos x + C$ find C and $f(t)$

original function

← yeah deriv of

$f(t) \overset{?}{=} f(x) - f(0)$ find c 1st

know $\int_0^0 f(t)\, dt = 0$

set = 0 find c

$e^{2 \cdot 0} \cos 0 + C$

$C = -1$

then FTC #2

$f(x) = D\left(e^{2x} \cos x - 1\right)$

take deriv

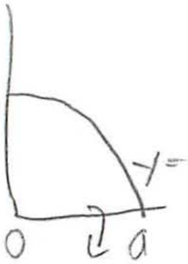
deriv

take deriv

$2 e^{2x} \cos x + -\sin x\, e^{2x}$

find it

(6)



Find ~~surface area~~ ? or solid so find volume ? going with

could papus if find com

$\int_0^a \sqrt{x}\, dx$? a what for rotation ? 2π ?

they use shell method

$$\int_0^{\sqrt{a}} (\pi a^2 - \pi x^2)\, dy$$

- look over

$$\boxed{\begin{aligned} dV &= \text{height} \cdot \text{circ} \;\; dx \\ V &= \int^{\sqrt{a}} \text{height} \cdot \text{sirc} \cdot dx \end{aligned}}$$

$$\pi \int_0^{\sqrt{a}} a^2 - \cancel{x^2} \; y^4 \quad x^2 = y$$

← how ?!!

$$\pi\left(\frac{a^3 y}{3} - \frac{y^5}{5}\right)\Big|_0^{\sqrt{a}}$$

↑ don't forget

$$\cancel{\pi\left(\frac{\sqrt{a}^3}{3} - \frac{\pi\sqrt{x}^3}{3}\right)}$$

eval

$$\pi\left(a^{5/2} - \frac{1}{5} a^{5/2}\right)$$

↑ fill in $\sqrt{a}$ for y

$$\pi \frac{4}{5} a^{5/2}$$

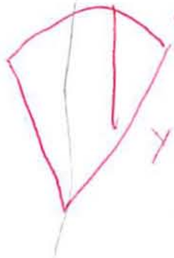
could not have gotten

(7)

$y = 6 - x^2$ ~~the~~ the overfilling

? why don't read close enough or badly worded

$V =$ not good at shells since skipped that day



? again - know

$$\int (6-x^2) - (5x)\,dx$$

$$\int_0^1 (6-x^2) - (5x)\,dx \qquad \int_{-1}^{0} (6-x^2) + 5x\,dx$$

* now rotate around y axis $= 2\pi$

why? ? widths?

$$2\pi \int_0^1 \cancel{x}(6 - x^2 - (5x))\,dx$$

so forgot x here

$$\cancel{2\pi\, x\left(6x - \frac{x^3}{3} - 5\frac{x^2}{2}\right)}\Big|_0^1$$

just rotate

think

$$2\pi x\left(6\cdot 1 - \frac{1^3}{3} - 5\frac{1^2}{2}\right)$$

$$2\pi x\left(6 - \frac{1}{3} - \frac{5}{2}\right)$$

$$\cancel{2\pi\left(5\tfrac{2}{3} - \frac{5}{2}\right)}$$

? figure later

$$2\pi\left(\frac{6x^2}{2} - \frac{x^4}{4} - \frac{5x^3}{3}\right)$$

$$2\pi \frac{\sqrt{3}}{\sqrt{2}}$$

Calc Review

1/9/27

FTC 2 $F(x) = \int_0^x f(t)\,dt$

$= \cancel{f(k(x)) \cdot k'(x)}$

Concepts

improper integral

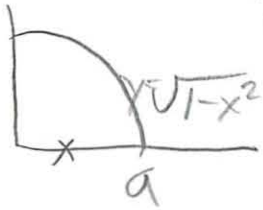
- use n ($n \to \infty$)
- might work out $F(n) - F(0)$
- or go to ∞

trap $(b-a)\dfrac{f(a)+f(b)}{2}$

$\cancel{(b-a)}$ base $(b-a)$ Δx $\left(\dfrac{1+4+2+4+2+4+1}{6}\right)$

sections

8. ↑ think did already



↑ so what does x mean

Seems like standard area problem

avg value ↙ area

$$\frac{1}{b-a}\int_0^a x\sqrt{a^2-x^2}\,dx$$

↑ not 1 but a^2

opps — assumed $a=1$

$$\uparrow \frac{1}{a}$$

~~① concept~~

$$\frac{1}{a}\int_0^a x\sqrt{1-x^2}\,dx$$

↑ $u = 1-x^2$

$du = -2x\,dx$

$x\,dx = -\frac{1}{2}\,du$

$$\frac{1}{a}\int_0^a -\frac{1}{2}\sqrt{u}$$

$$-\frac{1}{2}\,\frac{1}{a}\,\frac{u^{3/2}}{\frac{3}{2}} \qquad -\frac{1}{2}\cdot\frac{1}{a}\cdot\frac{2}{3}\,u^{3/2}\Big|_0^a$$

$$-\frac{1}{2}\cdot\frac{1}{\cancel{a}}\cdot\frac{2}{3}\sqrt{a}^{\cancel{3}\,2}$$

$$-\frac{a^2}{3}$$

close

but calculus right

9. $\int_0^{\pi/2} \sin^6 x \, dx$

$\approx \frac{\pi}{4}$

$\frac{\Delta x}{3} f\left(\frac{\pi}{4}\right)$ ← just that

$0 + 4 \cdot \frac{1}{8} + 1$

$\frac{\pi/4}{3} f\left(\frac{\pi}{4}\right)$ ← how figure this out

always a always b

$\frac{1}{\sqrt{2}} \cdot \boxed{\frac{\sqrt{2}}{2}^6}$ $\frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2}$

$\frac{8}{64} \to \frac{4}{32} \to \frac{2}{16} \to \frac{1}{8}$

that is just $f\left(\frac{\pi}{4}\right)$

$\frac{\pi/4}{3} \cdot 0 + 4 \cdot \frac{1}{8} + 1 = \frac{\pi}{8}$

$\frac{\pi}{12} \cdot \frac{3}{2}$

finish table problem

but forget Δ

$\cancel{(b-a)\frac{f(a)+f(b)}{2}}$

$\left(\frac{\pi}{2} - 0\right) \frac{0+1}{2}$ $\frac{\pi}{2} \cdot \frac{1}{2} = \frac{\pi}{4}$

+ mid pt

$\frac{\pi}{4}\left(\frac{1}{2} \cdot 0 + \frac{1}{8} + \frac{1}{2} \cdot 1\right)$ $\frac{5\pi}{32}$

$\frac{1}{2}$ start end

trapizotal $\frac{\Delta x}{2}\left[f(x) + \underbrace{2f(x) + 2f(x)}_{\text{middle}} + f(x)\right]$

$D(x\ln x) = 1\ln x + \frac{1}{x}x$

$\ln x + 1$

$\int_1^e \ln x\, dx$

$\int_1^e (x\ln x \quad -1)$

I don't really get ~~it~~ weird

$e\ln e -$

$x\ln x\Big|_1^e = \int_1^e \ln x + \int 1$

$(e-1) - 0 = \ln x\, dx + (e-1)$

$\int_1^e \ln x\, dx = 1$

$\int x\, dy = y\, dy$?

$\int_0^x t^2 e^{-t^2}\, dt$

- was a hard one to integrate

old exam

18.01 Exam 3

Problem 1. (20 pts) Evaluate the following integrals

a) $\int_0^2 \frac{xdx}{(1+x^2)^2}$

b) $\int_{-\pi/2}^{\pi/2} \sin^6 x \cos x dx$

Problem 2. (20 pts.) Find the following approximations to

$$\int_0^{\pi/2} \cos x dx$$

(Do not give a numerical approximation to square roots; leave them alone.)

a) Using the upper Riemann sum with two intervals

b) Using the trapezoidal rule with two intervals

c) Using Simpson's rule with two intervals

Problem 3. (20 points) Find the volume of the solid of revolution formed by revolving the y-axis the region enclosed by

$$y=\cos(x^2)$$

and the x-axis (central hump, only).

Problem 4. (20 points) Students studying for an exam get x hours of sleep in the two days leading up to the exam, where x is the range $0 \leq x \leq a$. The numbers of students who got between x_1 and x_2 hours of sleep in given by

$$\int_{x_1}^{x_2} cxdx; 0 \leq x_1 \leq x_2 \leq a$$

a) What fraction of the student got less than a/2 hours of sleep?

b) Their scores are proportional to the amount of sleep they got:
$S(x) = 100\ (x/a)$. Find the (correctly weighted) average score in the class.

Problem 5. (20 points) Let

$$F(x) = \int_0^x \sqrt{t}\sin t\,dt$$

a) Find $F'(x)$ for $x > 0$ identify the points $a > 0$ $F'(a) = 0$

b) Decide whether F has a local maximum at the smallest critical point $a > 0$ that you found in part (a) by evaluating F''.

c) Say whether $F(x)$ is positive, negative or zero at each of the following points, and give a reason in each case.

   i) $x=0$

   ii) $x=\pi$

   iii) $x=2\pi$

d) Use a change of the variable to express $G(x) = \int_0^x u^2 \sin(u^2)\,du$ in terms of F.

Problem 1. (20 points) Evaluate the following integrals

10 pts a) $\int_0^2 \frac{x\,dx}{(1+x^2)^2} = \frac{1}{2}\int_1^5 \frac{du}{u^2} = -\frac{1}{2u}\Big|_1^5$

$u = 1+x^2$

$du = 2x\,dx$

$= -\frac{1}{10} - \left(-\frac{1}{2}\right) =$

$= \frac{1}{2} - \frac{1}{10} = \boxed{2/5}$

10 pts b) $\int_{-\pi/2}^{\pi/2} \sin^6 x \cos x\,dx = \int_{-1}^{1} u^6\,du = \frac{1}{7}u^7\Big|_{-1}^{1}$

$u = \sin x$

$du = \cos x\,dx$

$= \boxed{\frac{2}{7}}$

Problem 2. (20 points) Find the following approximations to

$$\int_0^{\pi/2} \cos x\,dx$$

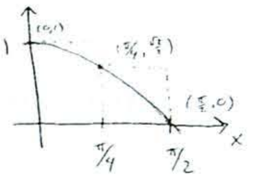
(Do not give a numerical approximation to square roots; leave them alone.)

a) using the upper Riemann sum with two intervals

$\int_0^{\pi/2} \cos x\,dx \approx \frac{\pi}{4}\cdot 1 + \frac{\pi}{4}\cos\frac{\pi}{4}$

$\boxed{= \frac{\pi}{4}\left(1+\frac{\sqrt{2}}{2}\right)}$

(≈ 1.341)

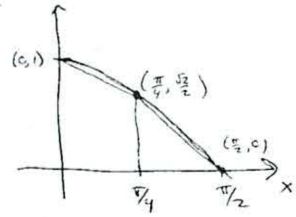


b) using the trapezoidal rule with two intervals

$\int_0^{\pi/2} \cos x\,dx \approx \frac{1}{2}\frac{\pi}{4}\left(1+\frac{\sqrt{2}}{2}\right) + \frac{1}{2}\frac{\pi}{4}\cdot\frac{\sqrt{2}}{2}$

$\boxed{= \frac{\pi}{8}\left(1+\sqrt{2}\right)}$

(≈ 0.948)



c) using Simpson's rule with two intervals

$\int_0^{\pi/2} \cos x\,dx \approx \frac{1}{3}\frac{\pi}{4}\left(1+4\cdot\frac{\sqrt{2}}{2}+0\right)$

$= \frac{\pi}{12}\left(1+2\sqrt{2}\right)$

(≈ 1.002)

Problem 3. (20 points) Find the volume of the solid of revolution formed by revolving around the y-axis the region enclosed by

$$y = \cos(x^2)$$

and the x-axis (central hump, only).

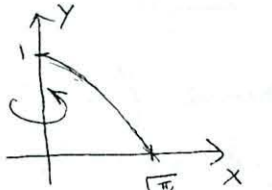
$V = \int_0^{\sqrt{\pi/2}} 2\pi x\cdot\cos(x^2)\,dx$

$u = x^2$

$du = 2x\,dx$

$= \pi\int_0^{\pi/2} \cos u\,du$

$= \pi\sin u\Big|_0^{\pi/2} = \boxed{\pi}$



Problem 4. (20 points) Students studying for an exam get x hours of sleep in the two days leading up to the exam, where x is in the range $0 \le x \le a$. The number of students who got between x_1 and x_2 hours of sleep is given by

$$\int_{x_1}^{x_2} cx\,dx; \quad 0 \le x_1 \le x_2 \le a$$

10 pts a) What fraction of the students got less than $a/2$ hours of sleep?

Total number of students $= \int_0^a cx\,dx = \frac{ca^2}{2}$ (all students got between 0 and a hours of sleep).

number of students who got between 0 and $\frac{a}{2}$ hours of sleep $= \int_0^{a/2} cx\,dx = c\frac{a^2}{8}$

ratio $= \frac{ca^2/8}{ca^2/2} = \boxed{\frac{1}{4}}$

10 pts b) Their scores are proportional to the amount of sleep they got: $S(x) = 100(x/a)$. Find the (correctly weighted) average score in the class.

$N =$ Total number of students $= \int_0^a cx\,dx = \frac{ca^2}{2}$

Average score $= \frac{1}{N}\int_0^a cx\,S(x)\,dx = \frac{1}{N}\int_0^a \frac{100c}{a}x^2\,dx$

$= \frac{1}{ca^2/2}\,\frac{100c}{a}\,\frac{a^3}{3} = \boxed{\frac{200}{3}}$ $(\approx 66.\overline{6})$

Problem 5. (20 points) Let

$$F(x) = \int_0^x \sqrt{t}\sin t\,dt$$

5 pts a) Find $F'(x)$ for $x > 0$ and identify the points $a > 0$ where $F'(a) = 0$.

$$F'(x) = \boxed{\sqrt{x}\sin x}$$

$F'(a) = 0$, $a > 0$; for $\boxed{a = K\pi}$ K a positive integer. $(1, 2, 3, 4, \dots)$

5 pts b) Decide whether F has a local maximum or minimum at the smallest critical point $a > 0$ that you found in part (a) by evaluating F''.

The smallest positive critical point is at $a = \pi$

$$F''(x) = \frac{1}{2\sqrt{x}}\sin x + \sqrt{x}\cos x$$

$$F''(\pi) = \frac{1}{2\sqrt{\pi}}\underbrace{\sin\pi}_{0} + \sqrt{\pi}\underbrace{\cos\pi}_{-1} = \boxed{-\sqrt{\pi} < 0}$$

So π is a $\boxed{\text{local maximum.}}$

5 pts c) Say whether $F(x)$ is positive, negative or zero at each of the following points, and give a reason in each case.

1 pt. i) $x = 0$ $F(0) = \int_0^0 \sqrt{t}\sin t\,dt \boxed{= 0}$ since the interval of integration has length 0.

2 pts ii) $x = \pi$ $F(\pi) = \int_0^\pi \sqrt{t}\sin t\,dt \boxed{> 0}$ since for t between 0 and π the integrand, $\sqrt{t}\sin t$, is positive.

2 pts iii) $x = 2\pi$ $F(2\pi) = \int_0^{2\pi} \sqrt{t}\sin t\,dt = \int_0^{\pi} \sqrt{t}\sin t\,dt + \int_\pi^{2\pi} \sqrt{t}\sin t\,dt$

$= \int_0^\pi \sqrt{t}|\sin t|\,dt - \int_\pi^{2\pi} \sqrt{t}|\sin t|\,dt \boxed{< 0}$ since $\sqrt{t}\,|\sin t| < \sqrt{t+\pi}\,|\sin(t+\pi)|$.

In other words, "there is more negative area between π and 2π than positive area from 0 to π."

5 pts d) Use a change of variable to express $G(x) = \int_0^x u^2\sin(u^2)\,du$ in terms of F.

Let $t = u^2$, $dt = 2u\,du$

$$G(x) = \int_0^x u^2\sin(u^2)\,du = \int_0^{x^2} t\sin t\,\frac{dt}{2\sqrt{t}}$$

$$= \frac{1}{2}\int_0^{x^2} \sqrt{t}\sin t\,dt = \boxed{\frac{1}{2}F(x^2)}$$

1a

~~old~~ Old exam for makeup

Evaluate $\int_0^2 \frac{x\,dx}{(1+x^2)^2} = \frac{x}{(1+x^2)^2}$ ¿is that what we are doing now?

$\int x \cdot \frac{1}{1+2x^2+x^4}\,dx$

got it after 3 mistakes

① did not see U-sub + got stuck on long way

② don't forget −0 part

$x^2 \cdot \frac{1}{x^2+\frac{2x^3}{3}+\frac{x^5}{5}}$

$\frac{x^2}{x^2+\frac{2}{3}x^3+\frac{x^5}{5}}\Big|_0^2 \to \frac{2^2}{2^2+\frac{2}{3}2^3+\frac{2^5}{5}} = \frac{4}{4+\frac{2}{3}\cdot 8+\frac{32}{5}}$

$\cancel{\frac{4}{4+\frac{2}{3}\cdot\frac{8}{1}+\frac{32}{5}}}$

$\cancel{\frac{20}{5}+\frac{16}{3}+\frac{32}{5}}$

peeked at ans do U-sub

- why did I not see that in 1st place!

~~at then~~ first I thought trig

$U = 1+x^2$

$dU = 2x\,dx$

$x\,dx = \frac{dU}{2}$

$\frac{\frac{dU}{2}}{(U)^2}$ $\int \frac{dU}{2U^2} \to \frac{1}{2}\int U^{-2}\,dU \to \frac{1}{2}\frac{U^{-1}}{-1}\Big|_0^2$

$-\frac{1}{2U}$ $-\frac{1}{2(1+x^2)}$

$-\frac{1}{2(1+2^2)}$ ★ don't forget 0 $+\frac{1}{2}$

$-\frac{1}{10}+\frac{1}{2} = \frac{5}{10} = \frac{4}{10} \to \frac{2}{5}$

b) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^6 x \cos x \, dx$

- ~~know the trig identies~~

$u = \sin x$ ← did not identity - looked up

$du = \cos x \, dx$ -guess I would have just goe w/ it

~~$dx =$~~

$\int u^6 \, du$

$\frac{u^7}{7} \to \frac{\sin^7 u}{7}$

$\frac{\sin^7 \frac{\pi}{2}}{7} - \frac{\sin^7\left(-\frac{\pi}{2}\right)}{7}$ ← unsure how to eval $\sin \cdot 7$

$\frac{1^7}{7} - \frac{(-1)^7}{7}$

$\frac{1}{7} - -\frac{1}{7} = \frac{2}{7}$ ✓ bingo

---

2 Find approx to -So this is av on approxes

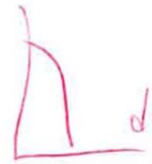
$\int_0^{\pi/2} \cos x \, dx$

$\frac{\pi}{4} \cdot \cos \frac{\pi}{4} + \frac{\pi}{4} \cdot \cos \frac{\pi}{2}$ think rectangles

then generize

$\frac{\pi}{4}\left(1 + \frac{\sqrt{2}}{2}\right)$

Bn 2b Trapizodal Rule

Also  decreasing

$\frac{b-a}{\cancel{3}2} \frac{\pi}{\cancel{2}4} \left(\frac{1}{2}\cdot 0 + \cos\frac{\pi}{4} + \frac{1}{2}\cos\frac{\pi}{2}\right)$

n interval - duh

$\frac{\frac{\pi}{2}-0}{2}$ $\frac{\pi}{4}$

$\overbrace{\left(\frac{1}{2}\cdot\left(1+\frac{\sqrt{2}}{2}\right)\right)}^{\text{section}} + \frac{\pi}{4}\frac{1}{2}\overbrace{\left(\frac{\sqrt{2}}{2}+0\right)}^{\text{section}}$

half for avg

then just add / do math

$\frac{\pi}{8}\left(1+\sqrt{2}\right)$

I think the pain w/ these formulas is that there are so many ways to write them

Simp $\frac{b-a}{n}$ $\left(\frac{1}{\cancel{}}\cdot 1 + \frac{4}{\cancel{}}\cdot\frac{\sqrt{2}}{2}\cdot \cancel{}\cdot 0\right)$ $\cos 0$ $\cos\frac{\pi}{4}$

$\frac{\pi}{4}\cdot\frac{1}{3}$

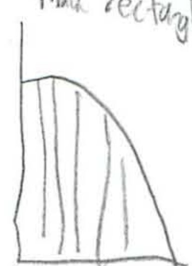
constant $\frac{\pi}{12}\left(1+\frac{\sqrt{2}}{2}\right)$

- just mem formulas + variations

3. $y=\cos(x^2)$ Find volume through x axis

- 2 ways to solve
  - find COM (lot of math) · ~~area~~ dist it rotates · area
  - Find area · $2\pi x$

think rectangles



$\int_0^{\pi/4} 2\pi x\cdot\cos x^2\, dx$ ← the around ?? "circumference"

$-\sin x^2\, 2x$ ~~← chain rule~~

$\frac{\pi}{4}$ ← when is $\cos x^2 = 0$

$\frac{\pi}{4}$

$u = x^2$

$du = 2x\,dx$ ← pay attension to what is remaind

$\pi\int \cos u\, du$

$\pi + \sin u \Big|_0^{\frac{\pi}{2}}$ ← convert bounds!

~~$\pi \sin x^2$~~

~~$+\pi \sin\left(\frac{\pi}{2}\right)^2$~~

$\pi \sin\frac{\pi}{2}$

π

4, $\int_{x_1}^{x_2} cx \, dx \qquad 0 \le x_1 \le x_2 \le a$

What fraction of students $< \frac{a}{2}$ hrs of sleep

- seems like too few #

$c\left(x^2 \Big|_{x_1}^{x_2}\right.$

$X = $ # hrs sleep
$x_1 = $ # students
$x_2 = $ # students

$c\left(x_2^2 - x_1^2\right)$

$c\left(\frac{a}{2}\right)^2$

total # students $= \int_0^a cx\,dx = c\frac{a^2}{2}$ ← why? oh in 2 days leading up to exam tricky

Oh! - probability proplem

$\int_0^a cx\,dx = 1$ ↗ rat'l-ize it $\frac{1}{4}$

$c\frac{a^2}{\cancel{4}8}$ ← since whole/2 right

- got it in concept, missed trick and not fare fond way

b. $S(x) = 100\frac{x}{a}$ — avg value $\frac{1}{b-a}\int_0^b f(x)$ ← # students

$\frac{1}{100}\int_a^b 100\frac{x}{a}\,dx \qquad \frac{1}{N}\int_0^a cx\,S(x)\,dx$ ← sleep · score $\boxed{y=ha}$ $y\,dx$

score = # student · sleep

total # students $\frac{1}{N}\int_0^a \frac{100c}{a}x^2\,dx$

$\frac{1}{c^{a^2}/_2} \cdot \frac{100c}{a} \cdot \frac{a^3}{3} = \frac{200}{3} = 66.\overline{6}$

always bad at these problems
- hard to set up

5, $F(x) = \int_0^x \sqrt{t}\ \sin t\, dt$

a) Find $F'(x)$ for $x > 0$ and identify $a > 0$ $F(a) = 0$

ftc ~~FTC~~

$F'(x) = \sqrt{x}\ \sin x$ ← just plug in ~~deriv of $\int$~~

~~I think it's better and always~~ like that?

$F'(a) = 0$ $a > 0$ for $a = k\pi$

When is $\sqrt{x}\sin x = 0$ ↑ integers

↑ (+) values   ↑ multiples of π is when $\sin = 0$

b) Local max

~~F~~ $\sqrt{\pi}\ \sin\pi$

2nd deriv

$\cancel{\sqrt{\pi}\ \sin\pi}$ (+) yes

actually ~~integrate~~ differentiate

$\frac{1}{2\sqrt{x}} \sin x + \sqrt{x}\ \cos x$

plug in π

$-\sqrt{\pi}$ ← so is local max

↑ (−) → concave down

c) Say whether $F(x)$ is $+, -, 0$?

$x = 0$

$\int \sqrt{x}\sin x$

– have not solved yet

but deriv will tell you if slope there ↑, ↓ or →

– and concavity

– eval all 3

a) $\sqrt{0}\ \sin 0$ $\frac{1}{2\sqrt{0}}\sin 0 + \sqrt{0}$

0   0

$= 0$ ⇒ not moving

↑ can you say that

b) π

0 local max

c) 0 − (+) (−)

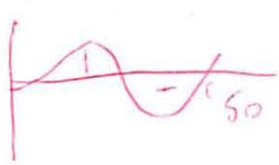
1 -

2 $\int_0^{\pi} \sqrt{t}\sin t\,dt$ since for t between 0 and π the integral $\sqrt{t}\sin t$ is + ∴ thus integral must be + ∴

$$\int_0^{2\pi} \sqrt{t}\sin t\,dt = \int_0^{\pi} - \int_{\pi}^{2\pi}$$

$$? \sqrt{t}\,|\sin t| < \sqrt{1+\pi}\,|\sin(t+\pi)|$$

- lot to think about



- here ?
or amt area ∴
would not cancle = 0 ?∴
don't really get

d Use the change in variable to express $G(x) = \int_0^t u^2 \sin(u^2)\,du$
in terms of F
- at 1st glance → no clue

let $t = u^2$

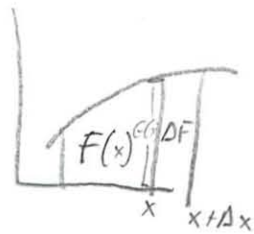
$dt = 2u\,du$

$$G(x) = \int_0^x u^2 \sin(u^2)\,du = \int_0^{x^2} t \sin t \frac{dt}{2\sqrt{t}}$$

$$= \frac{1}{2}\int_0^{x^2} \sqrt{t}\sin t\,dt = \frac{1}{2}F(x^2)$$

ftC #1 review (again)

$$F(x) = \int_a^x f(t)\,dt \qquad \text{f continuous} \quad F'(x) = f(x)$$



$$\Delta F = F(x+\Delta x) - F(x)$$

$$\Delta F = \text{base} \cdot \text{height} \approx \Delta x\, f(x)$$

$$\frac{\Delta F}{\Delta x} = f(x)$$

$$\lim \frac{\Delta F}{\Delta x} = f(x)$$

$$\frac{d}{dx} = F'(x)$$

$$\frac{ds}{dt} = -10\tfrac{\#}{m} \cdot \frac{s(t)}{1000} \leftarrow \text{change}$$

~~integration~~ get s on 1 side + # on other

solve

$$ds \frac{1000}{s(t)} = \frac{-10t}{100\cancel{0}} = -\frac{1}{100}t$$

← bad division :-)

would be a lot better if did not make stupid mistakes

$$\frac{ds}{s(t)} = -\frac{1}{100}t$$

$$\ln \qquad \ln$$

$$s(t) = e^{-\frac{1}{100}t} \leftarrow \text{exponential change}$$

18.01 EXAM III

Tuesday, Nov. 10, 2009

Name: Michael Plasmeier

E-mail: theplaz@mit.edu

Recitation Instructor (Circle One): C. Breiner / I. Losev / X. Ma / S. Ramakrishnan / B. Rhoades

Recitation Hour: ______________

Instructions: You may not use calculators, notes, textbooks, or personal electronic devices. As a courtesy to other students, please turn off all cell phones. Read each question carefully. Whenever possible, include justification for your reasoning and show your work. Answers without any work or explanation will receive little or no credit. There are 6 questions and a 50 minute time limit on this exam. Good luck.

Question	Score	Maximum
1	4	4
2	1	5
3	0	6
4	0.5	5
5	2	5
6	1	6
Total	9	31

← thought I got

≥ 11 = passing

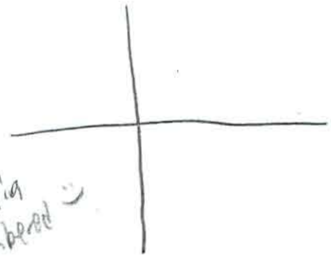
no usch on whole test?

 Name: ______________________

Like $y = x^2 - 4x$ $y = 2x - x^2$ $x^2 - 4y = 2x - x^2$? what is x where = ?

1. Compute the area between the curves $x = y^2 - 4y$ and $x = 2y - y^2$.

Find where intersect



intersection remembered I think

$$y^2 - 4y = 2y - y^2$$

$$2y^2 = 6y$$

What is y where =

$$y^2 = 3y \qquad x = 3^2 - 4(3)$$

$$9 - 12$$

$$x = -3$$

$$y = 3, 0 \qquad (-3, 3)$$

$$\int_0^3 (y^2 - 4y) - (2y - y^2)\, dy$$

$$\int_0^3 y^2 - 4y - 2y + y^2$$

$$\int_0^3 2y^2 - 6y\, dy$$

$$\frac{2y^3}{3} - \frac{6y^2}{2} \Big|_0^3$$

$$\frac{2}{3}y^3 - 3y^2 \Big|_0^3$$

$$\frac{2}{3}(3)^3 - 3(3)^2 - \left[\frac{2}{3}(0)^3 - 3(0)^2\right]$$

$$18 - 27$$

$$-9 \to \text{flip } \# \to 9$$

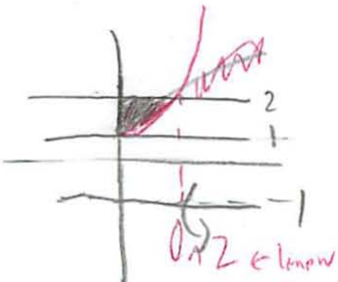
4. ✓

 Name: ______________________

?ash

2. Find the volume of the solid obtained by revolving the region bounded by the curves $y = e^x$, $y = 2$, and $x = 0$ about the line $y = -1$.

Setup integral only



$\ln 2$ ← know $e^{\ln 2} = 2$

$$\sum (\text{area}) \cdot 2\pi$$

$$\sum_{n\to\infty}^{n} x(2-e^x)2\pi$$ omissing n — but don't need

$$2\pi \int_{0}^{e^2} x(2-e^x)\,dx$$

did not remember steps since studies ineffectivly

↑ washers

go slow + re... it

↑ but must take into account distance to the axis

forget

$$2\pi \int_{0}^{e^2\ \ln 2} x\left((2+1) - (e^x+1)\right)\,dx$$

↑ if do w/ shells

Wrong bounds

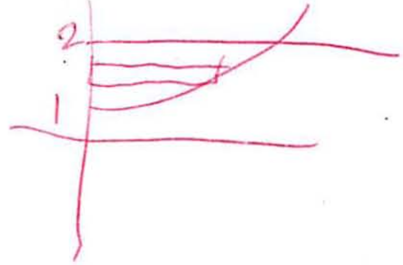
why 2 - since x_i

$$\int_0^{\ln 2} \pi\left(3^2 - (1+e^x)^2\right)dx$$

?2 why 3 - oh yeah

$\boxed{\text{subtraction error}}$

$$\int_1^2 2\pi(y+1)\ln y\,dy$$



 Name: ____________

3. Evaluate each of the following expressions

I did not get this one at all

(a)

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left(1+i\cdot\frac{3}{n}\right)^2\frac{3}{n}$$

→ see back

$(1+1\cdot\infty)^2\cdot\frac{3}{\infty}$

$\infty\cdot\frac{3}{\infty}$

$\infty\cdot 0$

So still will $= 0$

$\left(1+\frac{3n}{n}\right)^2\frac{3}{n}$

$4^2\cdot\frac{3}{n}$

$16\cdot 0$

0

but we are summing the values - will eventually not add more

0/3

(b) The value $f(4)$ for the continuous function f satisfying

is that the value or have to find ? ?

$$x\sin\pi x=\int_0^{x^2}f(t)\,dt$$

$4\sin(4\pi)=\int_0^{4^2}f(t)dt$

$4\cdot 0=f(16)-f(0)$

$0=f(16)$

$\sin(0)=0$

differentiate both sides to find $f(x)$

0/3

?? what is it asking wtf

don't even get know

$\frac{d}{dx}x\sin(\pi x)=\frac{d}{dx}\int_0^{x^2}f(t)\,dt$

constant $\sin\pi x+\pi x(\cos\pi x)=f(t)\,dt\leftarrow f(x^2)\,2x$

did not know u could integrate def. integral

? integrate → differentiate etc

$f(x^2)=\frac{1}{2x}\sin\pi x+\frac{\pi}{2}\cos\pi x$

So $f(4)=f(2^2)=\frac{1}{4}\sin 2\pi+\frac{\pi}{2}\cos 2\pi=\frac{\pi}{2}$

? given

*key was to recognize that it was Right Riemann sum

since $\frac{3}{n}$

Consider $[0, 3]$, cut into n parts

Consider the function $f(x) = (1+x)^2$ ← how do I know that ????

Right Rieman sum

$$\sum_{i=1}^{n} \left(1 + i\,\frac{3}{n}\right)^2 \frac{3}{n}$$

← generic Riemann sum formula

$\frac{b-a}{n} \sum f(x_i)$

$x_i = a + i\,\frac{b-a}{n}$

← get rid of n

convert integral

$$\int_0^3 (1+x)^2\,dx$$

Solve integral

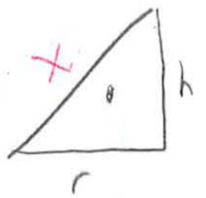
$$\int_0^3 x^2 + 2x + 1\,dx$$

$$\frac{x^3}{3} + \frac{2x^2}{2} + x \Big|_0^3$$

$$9 + 9 + 3 = 21$$

same as $\int_1^4 x^2\,dx$

4. (a) Find the centroid (i.e. center of mass) of a right triangle with height h and base r (assuming the triangle has uniform density).



~~Symetrical shape~~ $\bar{x} = \bar{y}$ ← can be - does not have to be

$\int_0^r x \cdot h \, dx$ decide ||| or ≡ $\int_0^h r y \, dy$

$\int_0^r \underline{\frac{h}{r} x \, dx}$ wrong integral $r \frac{y^2}{2} \Big|_0^h$

$\frac{h}{r} \frac{x^2}{2} \Big|_0^h$ $f = \frac{h}{r}$ $r \frac{(h^2)}{2}$

~~$\frac{h}{2}(r^2) - (\frac{h}{2} \cdot 0)^2$~~ $\frac{h}{r} \cdot \frac{h^2}{2}$

$\bar{x} = \frac{r^2 h}{2}$

$\bar{y} = \frac{r h^2}{2}$

$\left(\frac{r^2 h}{2}, \frac{r h^2}{2} \right)$

0

Thanks I think! from p-set which got from Brendon

~~back~~

See ans sheet

(b) Use Pappus' Theorem and your answer in the previous part to find the volume of a cone with height h and base radius r.

Papus → Volume = ~~com~~ area · 2π dist travell by com ← which is $\frac{r}{3}$

~~travel around~~

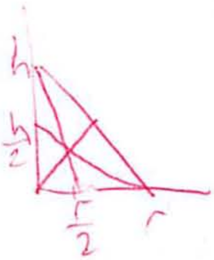
$\frac{1}{2} h r \cancel{\frac{r^2 h}{2}} \cdot \cancel{2\pi}$ $2\pi \frac{r}{3}$ where is area

$V = \boxed{\pi r^2 h}$ $\frac{}{3}$ 0.5

?✓ have heard before

can't ever get this right → put w/ x+1 my fav

$area = \frac{1}{2} hr$



$$\frac{h/2}{r} = \frac{h/3}{2r/3} \quad (h/3, r/3)$$

I don't get

S S from something to something

Brier not helping me

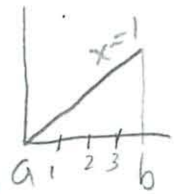
5. Given a definite integral

$$\int_a^b f(x)\,dx,$$

let T_n be the *trapezoid* approximation with n intervals, M_n the *midpoint* approximation using n intervals, and S_{2n} the *Simpson's rule* approximation using $2n$ intervals. Prove that

$$\frac{1}{3}T_n + \frac{2}{3}M_n = S_{2n}.$$

Defining a scenario



$n=4$? generic $y=$

complex way to ask

weird way of writing that they all =

$$T_n = \frac{\Delta x}{2}\left(y_0 + 2y_1 + 2y_2 + 2y_3 + y_4\right) \checkmark$$

$$M_n = \Delta x \left(\cancel{\frac{y_1 - y_0}{2} + \frac{y_2 - y_1}{2} + \frac{y_3 - y_2}{2} + \frac{y_4 - y_3}{2}}\right) \times$$

$\hookrightarrow \sum_{i=1}^{n} f(x_{2x_i - 1})$ ← know the rules!

$$S_n = \frac{\Delta x}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + y_4\right)_{2n} \checkmark$$

diff. Δx than above

This was closer I though

T_{2n} should have fixed

→ back

$$\cancel{T_n = \frac{1}{4} \cdot \frac{1}{2}\left(0 + 2\cdot .25 + 2\cdot .5 + 2\cdot .75 + 1\right)}$$

$$\frac{1}{8}\cdot 4 = \frac{1}{2}$$

$$M_n = \frac{1}{4}\left(\frac{.25 - 0}{2} + \frac{.5 - .25}{2} + \frac{.75 - .5}{2} + \frac{1 - .75}{2}\right)$$

$$\frac{1}{4}\left(.125 + .375 + .625 + .875\right)$$

$$= \frac{1}{2}$$

$$S_n = \frac{1/4}{3}\left(0 + 4\cdot .25 + 2\cdot .5 + 4\cdot .75 + 1\right)$$

$$\frac{1}{12}\cdot 6 = \frac{1}{2}$$

Can't prove w/ #

$$S_{2n} = \frac{1/8}{3}\left(0 + 4\cdot .125 + 2\cdot .25 + 4\cdot .375 + 2\cdot .5 + 4\cdot .625 + 2\cdot .75 + 4\cdot .875 + \right)) = \frac{1}{2}$$

$$\frac{1}{3}\cdot\frac{1}{2} + \frac{2}{3}\cdot\frac{1}{2} = \frac{1}{2}$$

$$\frac{1}{6} + \frac{3}{6} = \frac{1}{2} \quad \checkmark$$

← proved w/ # but basically you are saying $\frac{1}{3}x + \frac{2}{3}x = x$ – all should be the same

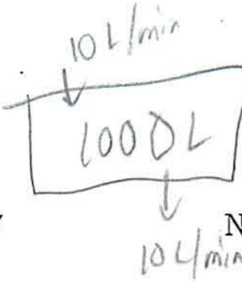
$$\frac{1}{3}T_n + \frac{2}{3}M_n = \frac{1}{3}T_n + \frac{4}{6}M_n$$

$$= \frac{b-a}{6n}\left(f(x_0) + f(x_{2n}) + 2\sum_{i=1}^{n-1} f(x_{2i}) + 4\sum_{i=1}^{n} f(x_{2i-1})\right)$$

what is this

$$= S_n$$

?? how does this prove anything



* did not understand how to set up *

 Name: ____________________

6. A tank contains 1000 L of brine (that is, salt water) with 15 kg of dissolved salt. Pure water enters the top of the tank at a constant rate of 10 L / min. The solution is thoroughly mixed and drains from the bottom of the tank at the same rate so that the volume of liquid in the tank is constant.

$\left(1 - \frac{1}{n}\right)$

$S(0) = 15 = \frac{15}{1000}$

$S(t) = ?$ ← part b

S = concentration of salt

$S(t) = \frac{15}{1000} - \frac{ds}{dt}$

(a) Find a differential equation expressing the rate at which salt leaves the tank.

hard to visualize & write equation

$\cancel{\frac{dS}{dt} = \frac{15}{(1000-10t)} dt}$ ← changes

t in min

but it continuously mixes

concentration if water filtered when leaving tank but leaving pure amt salt changes

$\frac{ds}{dt} = 15 - \frac{1000-10t}{15} dt$

variable

$\frac{ds}{dt} = -10 \text{ L/m} \cdot \frac{s(t)}{1000}$

$= -\frac{s(t)}{100} \frac{\text{kg}}{\text{min}}$

hint that uses ln

prob wrong since no lag

1

not right d.e. but some good obs.

(b) Solve this differential equation to find an expression for the amount of salt (in kg) in the mixture at time t.

can't visualize like fb problem need to be for better at this

$\cancel{S(t) = \int 15 - \frac{1000-10t}{15} dt}$

$\cancel{15t - \frac{1000t - 10t^2}{30}}$

$S(0) = 15$ so $k = 15$

$S(t) = 15e^{-1/100t}$

Integrate

$\frac{ds}{S(t)} = -\frac{1}{100} dt$

$\ln(s(t)) = -\frac{1}{100}t + c$

exponentiate

$s(t) = k \cdot e^{-1/100 t}$

$+ e^c = k$

(c) How long does it take for the total amount of salt in the brine to be reduced by half its original amount? (Recall $\ln 2 \approx .693$.) ← hint

$S(t) = $ [illegible]

t?

prob wrong since no ln 2

$.5 \cdot 15 = 15 - \frac{1000-10t^2}{30}$

$7.5 = 15 - \frac{1000-10t^2}{30}$ ✓

$-7.5 = -\left(\frac{1000-10t^2}{30}\right)$

$7.5 = \frac{1000-10t^2}{30}$ → back

$225 = 1000 - 10t^2$

$-775 = -10t^2$

$775 = 10t^2$

$7.75 = t^2$

$t = \sqrt{7.75}$

30
7.5
150
2100
225

focus a lot on working backwards - just learn to solve the problem

$e^{-1/100t} = \frac{1}{2}$

$\ln \qquad \ln$

$1/100 t = \ln 2$

$t = 100 \ln 2$

69.2 min

18.01 EXAM III

Tuesday, Nov. 10, 2009

Name: Solutions.

E-mail:

Recitation Instructor (Circle One): C. Breiner / I. Losev / X. Ma / S. Ramakrishnan / B. Rhoades

Recitation Hour:

Instructions: You may not use calculators, notes, textbooks, or personal electronic devices. As a courtesy to other students, please turn off all cell phones. Read each question carefully. Whenever possible, include justification for your reasoning and show your work. Answers without any work or explanation will receive little or no credit. There are 6 questions and a 50 minute time limit on this exam. Good luck.

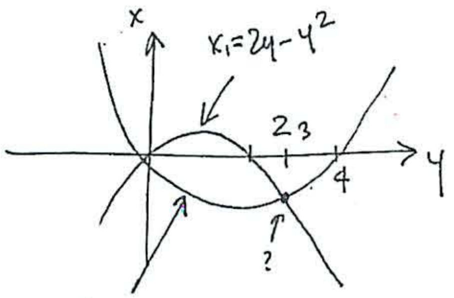
Question	Score	Maximum
1		4
2		5
3		6
4		5
5		5
6		6
Total		31

 Name: ____________________

1. Compute the area between the curves $x = y^2 - 4y$ and $x = 2y - y^2$.

$y(y-4)$ $\qquad$ $y(2-y)$

$0, 4$ $\qquad$ $0, 2$



$y^2 - 4y = 2y - y^2$

$2y^2 - 6y = 0$

$y^2 - 3y = 0$

$y = 0$ or $y = 3$

$x_2 = y^2 - 4y$ $\qquad$ height $= x_1 - x_2$

$$\int_0^3 2y - y^2 - (y^2 - 4y)\,dy$$

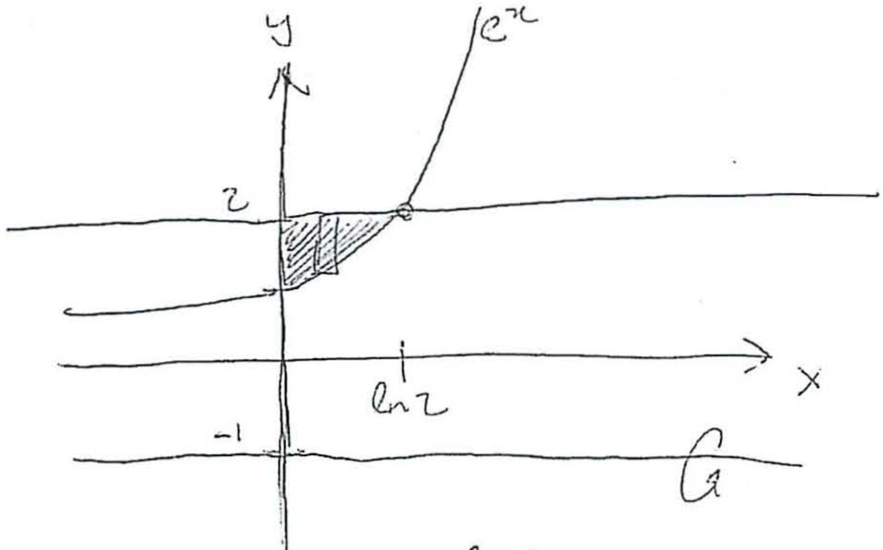
$$\int_0^3 6y - 2y^2\,dy = 3y^2 - \frac{2}{3}y^3\Big|_0^3 = \cancel{27 - 2\cdot}$$

$$3\cdot 3^2 - 2\cdot 3^2 = $$

$$= 1\cdot 3^2 = \boxed{9}$$

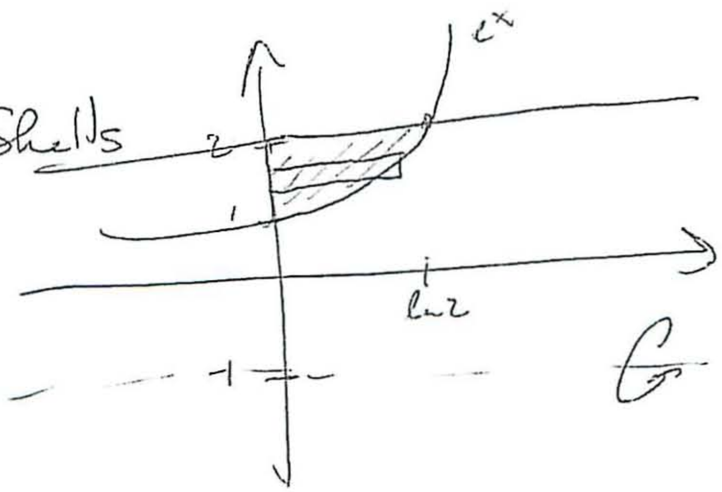
 Name: ______________________

2. Find the volume of the solid obtained by revolving the region bounded by the curves $y = e^x$, $y = 2$, and $x = 0$ about the line $y = -1$. You only need to give a definite integral expressing the volume. Do not solve the integral.



Washers: $\int_0^{\ln 2} \pi \left(3^2 - (1+e^x)^2\right) dx$

Shells



$\int_1^2 2\pi(y+1) \ln y \, dy$

3. Evaluate each of the following expressions

(a)

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(1 + i\cdot\frac{3}{n}\right)^2 \frac{3}{n}$$

Consider the interval $[0, 3]$, cut into n parts.

Consider the function $f(x) = (1+x)^2$.

Then the right RS is.

$$\sum_{i=1}^{n} \left(1 + i\frac{3}{n}\right)^2 \cdot \frac{3}{n}$$

So $\displaystyle\lim_{n\to 0} \sum_{i=1}^{n} (1+i\tfrac{3}{n})^2 \frac{3}{n} = \int_0^3 (1+x)^2 dx$

$\displaystyle = \int_0^3 1\cdot dx + \int_0^3 2x\,dx + \int_0^3 x^2 dx = 3 + 9 + \tfrac{1}{3}27 = 21$

(b) The value $f(4)$ for the continuous function f satisfying

$$x \sin \pi x = \int_0^{x^2} f(t)\, dt$$

$\frac{d}{dx}(x \sin \pi x) = \frac{d}{dx} \int_0^{x^2} f(t) dt$

$\Rightarrow \sin \pi x + \pi x \cos \pi x = f(x^2)\cdot 2x$

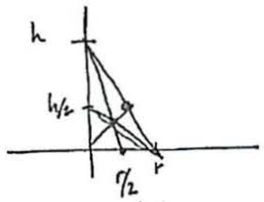
$\Rightarrow f(x^2) = \frac{1}{2x} \sin \pi x + \frac{\pi}{2} \cos \pi x$

So $f(4) = f(2^2) = \frac{1}{4} \sin 2\pi + \frac{\pi}{2} \cos(2\pi)$

$= \frac{\pi}{2}.$

 Name: ____________________

4. (a) Find the centroid (i.e. center of mass) of a right triangle with height h and base r (assuming the triangle has uniform density).



something like $(h/3, r/3)$

$\frac{h/2}{r} = \frac{h/3}{2r/3}$ ✓

<See solutions to HW> 1a2

Com $\frac{\sum_{i=1}^{n} m_i \cdot x_i}{\sum_{i=1}^{n} m_i} \rightarrow \frac{\lim_{n\to\infty} \sum x_i f(x_i)\Delta x}{f(x_i)\Delta x}$ mass

$\frac{\int x f(x)\,dx}{\text{Mass} = \text{Area}}$

integral way · Briner office hrs

← to find $\frac{\Delta y}{\Delta x} = \frac{h-0}{0-r} = \frac{-h}{r}$

$y = mx + b$

$h = -\frac{h}{r}\cdot 0 + b \quad b = h$

$0 = -\frac{h}{r}\cdot r + b \quad b = h$

$y = -\frac{h}{r}x + h$

x) $\frac{\int_0^r x\left(-\frac{h}{r}x + h\right)dx}{\frac{1}{2}hr}$

y) $\sum \frac{m_i y_i}{m_i} = f(x_i)\Delta x \cdot \frac{f(x_i)}{2} \rightarrow \frac{\int_0^r \frac{f(x)}{2}\cdot f(x)\,dx}{\frac{1}{2}hr}$

now integrate

(b) Use Pappus' Theorem and your answer in the previous part to find the volume of a cone with height h and base radius r.

Area · dist travelled.

$\frac{1}{2}hr \cdot \frac{2\pi r}{3} = \frac{\pi r^2 h}{3}$.

$\frac{\int_0^r -\frac{h}{r}x^2 + hx\,dx}{\frac{1}{2}hr}$

$\frac{-\frac{h}{r}\frac{x^3}{3} + h\frac{x^2}{2}\Big|_0^r}{\frac{1}{2}hr}$

$\frac{-\frac{h}{r}\cdot\frac{r^3}{3} + \frac{hr^2}{2}}{\frac{1}{2}hr}$

$\frac{-\frac{hr^2}{3} + \frac{hr^2}{2}}{\frac{1}{2}hr}$

$\frac{\frac{-2hr^2 + 3hr^2}{6}}{\frac{hr}{2}}$

$\frac{hr^2}{3\cdot 2}\cdot\frac{2}{hr}$ $\frac{r}{3}$

$$\frac{\int_0^r \frac{f(x)^2}{2}\,dx}{\frac{1}{2}hr} \rightarrow \frac{\int_0^r \frac{\left(-\frac{h}{r}x+h\right)^2}{2}\,dx}{\frac{1}{2}hr}$$

$$\frac{\int_0^r \frac{\left(\left(\frac{h}{r}\right)^2 x^2 - 2\cdot\frac{h^2}{r}x + h^2\right)}{2}\,dx}{\frac{1}{2}hr}$$ watch signs

✓

$$\frac{\frac{\frac{h^2}{r^2}x^3}{3} - \frac{2h^2}{r}\frac{x^2}{2} + h^2x \Big|_0^r}{\frac{2}{\frac{1}{2}hr}}$$

$$\frac{\frac{\frac{h^2}{\cancel{r^2}}\frac{r^{\cancel{3}}}{3} - \frac{2h^2}{\cancel{r}}\cdot\frac{r^{\cancel{2}}}{\cancel{2}} + h^2r}{2}}{\frac{hr}{2}}$$

Wow a lot of algebra
- made a lot of mistakes
- I am knowing the rules now - but still a bit unsure

$$\frac{\frac{h^{\cancel{2}}\cancel{r}}{3} - h^{\cancel{2}}\cancel{r} + h^{\cancel{2}}\cancel{r}}{\cancel{2}} \cdot \frac{\cancel{2}}{\cancel{hr}}$$

$$\frac{h}{3} - h + h$$

$$\frac{h}{3}$$

should $= \frac{h}{3}$

 Name: ____________________

5. Given a definite integral
$$\int_a^b f(x)\,dx,$$
let T_n be the *trapezoid* approximation with n intervals, M_n the *midpoint* approximation using n intervals, and S_{2n} the *Simpson's rule* approximation using $2n$ intervals. Prove that
$$\frac{1}{3}T_n + \frac{2}{3}M_n = S_{2n}.$$

Easiest to divide $[a,b]$ into $2n$ intervals:

Let $x_0 = a$, $x_{2n} = b$, $x_i = a + \frac{(b-a)i}{2n}$.

Then $T_n = \frac{b-a}{2n}\left(f(x_0) + f(x_{2n}) + 2\sum_{i=1}^{n-1} f(x_{2i})\right)$

$$M_n = \frac{b-a}{n}\left(\sum_{i=1}^{n} f(x_{2i-1})\right)$$

$$S_n = \frac{b-a}{6n}\left(f(x_0) + 4f(x_1) + 2f(x_2) + \dots + f(x_{2n})\right)$$
$$= \frac{b-a}{6n}\left(f(x_0) + f(x_{2n}) + 4\sum_{i=1}^{n} f(x_{2i-1}) + 2\sum_{i=1}^{n-1} f(x_{2i})\right)$$

$$\frac{1}{3}T_n + \frac{2}{3}M_N = \frac{1}{3}T_n + \frac{4}{6}M_n$$
$$= \frac{b-a}{6n}\left(f(x_0) + f(x_{2n}) + 2\sum_{i=1}^{n-1} f(x_{2i}) + 4\sum_{i=1}^{n} f(x_{2i-1})\right)$$

$$= S_n$$

6. A tank contains 1000 L of brine (that is, salt water) with 15 kg of dissolved salt. Pure water enters the top of the tank at a constant rate of 10 L / min. The solution is thoroughly mixed and drains from the bottom of the tank at the same rate so that the volume of liquid in the tank is constant.

(a) Find a differential equation expressing the rate at which salt leaves the tank.

Let $s(t)$: amount of salt in kg. at time t.

$$\frac{ds}{dt} = -10 \text{ L/min} \cdot \frac{s(t) \text{ kg}}{1000 \text{ L}} = -\frac{s(t)}{100} \text{ kg/min}$$

(b) Solve this differential equation to find an expression for the amount of salt (in kg) in the mixture at time t.

Use separation of variables.

$$\frac{ds}{s(t)} = \frac{-1}{100} dt \longrightarrow \text{integrate: } \ln(s(t)) = -\frac{1}{100}t + C$$

Exponentiate both sides:

$$s(t) = k \cdot e^{-1/100 t}$$ where we've written $e^{C} = k$ for some constant k.

$s(0) = 15$, so $k = 15$

$$\boxed{s(t) = 15 e^{-1/100 t}}$$

(c) How long does it take for the total amount of salt in the brine to be reduced by half its original amount? (Recall $\ln 2 \approx .693$.)

We need $e^{-1/100 t} = 1/2$

$$\ln(e^{-1/100 t}) = \ln(1/2)$$

$$1/100\, t = \ln 2$$

$$t = 100 \cdot \ln 2 \approx \boxed{69.3 \text{ minutes}}$$