Limits

Definitions

Precise Definition : We say $\lim_{x \to a} f(x) = L$ if for every $\varepsilon > 0$ there is a $\delta > 0$ such that whenever $0 < |x - a| < \delta$ then $|f(x) - L| < \varepsilon$.

"Working" Definition : We say $\lim_{x \to a} f(x) = L$ if we can make $f(x)$ as close to L as we want by taking x sufficiently close to a (on either side of a) without letting $x = a$.

Right hand limit : $\lim_{x \to a^+} f(x) = L$. This has the same definition as the limit except it requires $x > a$.

Left hand limit : $\lim_{x \to a^-} f(x) = L$. This has the same definition as the limit except it requires $x < a$.

Limit at Infinity : We say $\lim_{x \to \infty} f(x) = L$ if we can make $f(x)$ as close to L as we want by taking x large enough and positive.

There is a similar definition for $\lim_{x \to -\infty} f(x) = L$ except we require x large and negative.

Infinite Limit : We say $\lim_{x \to a} f(x) = \infty$ if we can make $f(x)$ arbitrarily large (and positive) by taking x sufficiently close to a (on either side of a) without letting $x = a$.

There is a similar definition for $\lim_{x \to a} f(x) = -\infty$ except we make $f(x)$ arbitrarily large and negative.

Relationship between the limit and one-sided limits

$\lim_{x \to a} f(x) = L \Rightarrow \lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L$ $\qquad$ $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L \Rightarrow \lim_{x \to a} f(x) = L$

$\lim_{x \to a^+} f(x) \neq \lim_{x \to a^-} f(x) \Rightarrow \lim_{x \to a} f(x)$ Does Not Exist

Properties

Assume $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ both exist and c is any number then,

1. $\lim_{x \to a} \left[c f(x) \right] = c \lim_{x \to a} f(x)$
2. $\lim_{x \to a} \left[f(x) \pm g(x) \right] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$
3. $\lim_{x \to a} \left[f(x) g(x) \right] = \lim_{x \to a} f(x) \; \lim_{x \to a} g(x)$
4. $\lim_{x \to a} \left[\frac{f(x)}{g(x)} \right] = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$ provided $\lim_{x \to a} g(x) \neq 0$
5. $\lim_{x \to a} \left[f(x) \right]^n = \left[\lim_{x \to a} f(x) \right]^n$
6. $\lim_{x \to a} \left[\sqrt[n]{f(x)} \right] = \sqrt[n]{\lim_{x \to a} f(x)}$

Basic Limit Evaluations at $\pm \infty$

Note : $\operatorname{sgn}(a) = 1$ if $a > 0$ and $\operatorname{sgn}(a) = -1$ if $a < 0$.

1. $\lim_{x \to \infty} \mathbf{e}^x = \infty$ & $\lim_{x \to -\infty} \mathbf{e}^x = 0$
2. $\lim_{x \to \infty} \ln(x) = \infty$ & $\lim_{x \to 0^-} \ln(x) = -\infty$
3. If $r > 0$ then $\lim_{x \to \infty} \frac{b}{x^r} = 0$
4. If $r > 0$ and x^r is real for negative x then $\lim_{x \to -\infty} \frac{b}{x^r} = 0$
5. n even : $\lim_{x \to \pm\infty} x^n = \infty$
6. n odd : $\lim_{x \to \infty} x^n = \infty$ & $\lim_{x \to -\infty} x^n = -\infty$
7. n even : $\lim_{x \to \pm\infty} a x^n + \cdots + b x + c = \operatorname{sgn}(a) \infty$
8. n odd : $\lim_{x \to \infty} a x^n + \cdots + b x + c = \operatorname{sgn}(a) \infty$
9. n odd : $\lim_{x \to -\infty} a x^n + \cdots + c x + d = -\operatorname{sgn}(a) \infty$

Evaluation Techniques

Continuous Functions

If $f(x)$ is continuous at a then $\lim_{x \to a} f(x) = f(a)$

Continuous Functions and Composition

$f(x)$ is continuous at b and $\lim_{x \to a} g(x) = b$ then

$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x) \right) = f(b)$

Factor and Cancel

$$\lim_{x \to 2} \frac{x^2 + 4x - 12}{x^2 - 2x} = \lim_{x \to 2} \frac{(x-2)(x+6)}{x(x-2)} = \lim_{x \to 2} \frac{x+6}{x} = \frac{8}{2} = 4$$

Rationalize Numerator/Denominator

$$\lim_{x \to 9} \frac{3 - \sqrt{x}}{x^2 - 81} = \lim_{x \to 9} \frac{3 - \sqrt{x}}{x^2 - 81} \frac{3 + \sqrt{x}}{3 + \sqrt{x}} = \lim_{x \to 9} \frac{9 - x}{(x^2 - 81)(3 + \sqrt{x})} = \lim_{x \to 9} \frac{-1}{(x+9)(3 + \sqrt{x})} = \frac{-1}{(18)(6)} = -\frac{1}{108}$$

Combine Rational Expressions

$$\lim_{h \to 0} \frac{1}{h}\left(\frac{1}{x+h} - \frac{1}{x} \right) = \lim_{h \to 0} \frac{1}{h}\left(\frac{x - (x+h)}{x(x+h)} \right) = \lim_{h \to 0} \frac{1}{h}\left(\frac{-h}{x(x+h)} \right) = \lim_{h \to 0} \frac{-1}{x(x+h)} = -\frac{1}{x^2}$$

L'Hospital's Rule

If $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{0}{0}$ or $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\pm\infty}{\pm\infty}$ then,

$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$ a is a number, ∞ or $-\infty$

Polynomials at Infinity

$p(x)$ and $q(x)$ are polynomials. To compute $\lim_{x \to \pm\infty} \frac{p(x)}{q(x)}$ factor largest power of x out of both $p(x)$ and $q(x)$ and then compute limit.

$$\lim_{x \to -\infty} \frac{3x^2 - 4}{5x - 2x^2} = \lim_{x \to -\infty} \frac{x^2\left(3 - \frac{4}{x^2}\right)}{x^2\left(\frac{5}{x} - 2\right)} = \lim_{x \to -\infty} \frac{3 - \frac{4}{x^2}}{\frac{5}{x} - 2} = -\frac{3}{2}$$

Piecewise Function

$\lim_{x \to -2} g(x)$ where $g(x) = \begin{cases} x^2 + 5 & \text{if } x < -2 \\ 1 - 3x & \text{if } x \geq -2 \end{cases}$

Compute two one sided limits,

$\lim_{x \to -2^-} g(x) = \lim_{x \to -2^-} x^2 + 5 = 9$

$\lim_{x \to -2^+} g(x) = \lim_{x \to -2^+} 1 - 3x = 7$

One sided limits are different so $\lim_{x \to -2} g(x)$ doesn't exist. If the two one sided limits had been equal then $\lim_{x \to -2} g(x)$ would have existed and had the same value.

Some Continuous Functions

Partial list of continuous functions and the values of x for which they are continuous.

1. Polynomials for all x.
2. Rational function, except for x's that give division by zero.
3. $\sqrt[n]{x}$ (n odd) for all x.
4. $\sqrt[n]{x}$ (n even) for all $x \geq 0$.
5. $\mathbf{e}^x$ for all x.
6. $\ln x$ for $x > 0$.
7. $\cos(x)$ and $\sin(x)$ for all x.
8. $\tan(x)$ and $\sec(x)$ provided $x \neq \cdots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \cdots$
9. $\cot(x)$ and $\csc(x)$ provided $x \neq \cdots, -2\pi, -\pi, 0, \pi, 2\pi, \cdots$

Intermediate Value Theorem

Suppose that $f(x)$ is continuous on $[a, b]$ and let M be any number between $f(a)$ and $f(b)$. Then there exists a number c such that $a < c < b$ and $f(c) = M$.

Derivatives

Definition and Notation

If $y=f(x)$ then the derivative is defined to be $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$.

If $y=f(x)$ then all of the following are equivalent notations for the derivative.

$$f'(x)=y'=\frac{df}{dx}=\frac{dy}{dx}=\frac{d}{dx}(f(x))=Df(x)$$

If $y=f(x)$ all of the following are equivalent notations for derivative evaluated at $x=a$.

$$f'(a)=y'\big|_{x=a}=\frac{df}{dx}\bigg|_{x=a}=\frac{dy}{dx}\bigg|_{x=a}=Df(a)$$

Interpretation of the Derivative

If $y=f(x)$ then,

1. $m=f'(a)$ is the slope of the tangent line to $y=f(x)$ at $x=a$ and the equation of the tangent line at $x=a$ is given by $y=f(a)+f'(a)(x-a)$.
2. $f'(a)$ is the instantaneous rate of change of $f(x)$ at $x=a$.
3. If $f(x)$ is the position of an object at time x then $f'(a)$ is the velocity of the object at $x=a$.

Basic Properties and Formulas

If $f(x)$ and $g(x)$ are differentiable functions (the derivative exists), c and n are any real numbers,

1. $(cf)'=cf'(x)$
2. $(f\pm g)'=f'(x)\pm g'(x)$
3. $(fg)'=f'g+fg'$ – **Product Rule**
4. $\left(\frac{f}{g}\right)'=\frac{f'g-fg'}{g^2}$ – **Quotient Rule**
5. $\frac{d}{dx}(c)=0$
6. $\frac{d}{dx}(x^n)=nx^{n-1}$ – **Power Rule**
7. $\frac{d}{dx}(f(g(x)))=f'(g(x))g'(x)$

   This is the **Chain Rule**

Common Derivatives

$\frac{d}{dx}(x)=1$

$\frac{d}{dx}(\sin x)=\cos x$

$\frac{d}{dx}(\cos x)=-\sin x$

$\frac{d}{dx}(\tan x)=\sec^2 x$

$\frac{d}{dx}(\sec x)=\sec x\tan x$

$\frac{d}{dx}(\csc x)=-\csc x\cot x$

$\frac{d}{dx}(\cot x)=-\csc^2 x$

$\frac{d}{dx}(\sin^{-1}x)=\frac{1}{\sqrt{1-x^2}}$

$\frac{d}{dx}(\cos^{-1}x)=-\frac{1}{\sqrt{1-x^2}}$

$\frac{d}{dx}(\tan^{-1}x)=\frac{1}{1+x^2}$

$\frac{d}{dx}(a^x)=a^x\ln(a)$

$\frac{d}{dx}(e^x)=e^x$

$\frac{d}{dx}(\ln(x))=\frac{1}{x},\ x>0$

$\frac{d}{dx}(\ln|x|)=\frac{1}{x},\ x\neq 0$

$\frac{d}{dx}(\log_a(x))=\frac{1}{x\ln a},\ x>0$

Chain Rule Variants

The chain rule applied to some specific functions.

1. $\frac{d}{dx}\left([f(x)]^n\right)=n[f(x)]^{n-1}f'(x)$
2. $\frac{d}{dx}\left(e^{f(x)}\right)=f'(x)e^{f(x)}$
3. $\frac{d}{dx}\left(\ln[f(x)]\right)=\frac{f'(x)}{f(x)}$
4. $\frac{d}{dx}\left(\sin[f(x)]\right)=f'(x)\cos[f(x)]$
5. $\frac{d}{dx}\left(\cos[f(x)]\right)=-f'(x)\sin[f(x)]$
6. $\frac{d}{dx}\left(\tan[f(x)]\right)=f'(x)\sec^2[f(x)]$
7. $\frac{d}{dx}\left(\sec[f(x)]\right)=f'(x)\sec[f(x)]\tan[f(x)]$
8. $\frac{d}{dx}\left(\tan^{-1}[f(x)]\right)=\frac{f'(x)}{1+[f(x)]^2}$

Higher Order Derivatives

The Second Derivative is denoted as

$f''(x)=f^{(2)}(x)=\frac{d^2f}{dx^2}$ and is defined as

$f''(x)=(f'(x))'$, *i.e.* the derivative of the first derivative, $f'(x)$.

The n^{th} Derivative is denoted as

$f^{(n)}(x)=\frac{d^nf}{dx^n}$ and is defined as

$f^{(n)}(x)=\left(f^{(n-1)}(x)\right)'$, *i.e.* the derivative of the $(n-1)^{st}$ derivative, $f^{(n-1)}(x)$.

Implicit Differentiation

Find y' if $e^{2x-9y}+x^3y^2=\sin(y)+11x$. Remember $y=y(x)$ here, so products/quotients of x and y will use the product/quotient rule and derivatives of y will use the chain rule. The "trick" is to differentiate as normal and every time you differentiate a y you tack on a y' (from the chain rule). After differentiating solve for y'.

$$e^{2x-9y}(2-9y')+3x^2y^2+2x^3y\,y'=\cos(y)\,y'+11$$

$$2e^{2x-9y}-9y'e^{2x-9y}+3x^2y^2+2x^3y\,y'=\cos(y)\,y'+11 \quad\Rightarrow\quad y'=\frac{11-2e^{2x-9y}-3x^2y^2}{2x^3y-9e^{2x-9y}-\cos(y)}$$

$$\left(2x^3y-9e^{2x-9y}-\cos(y)\right)y'=11-2e^{2x-9y}-3x^2y^2$$

Increasing/Decreasing – Concave Up/Concave Down

Critical Points

$x=c$ is a critical point of $f(x)$ provided either

1. $f'(c)=0$ or **2.** $f'(c)$ doesn't exist.

Increasing/Decreasing

1. If $f'(x)>0$ for all x in an interval I then $f(x)$ is increasing on the interval I.
2. If $f'(x)<0$ for all x in an interval I then $f(x)$ is decreasing on the interval I.
3. If $f'(x)=0$ for all x in an interval I then $f(x)$ is constant on the interval I.

Concave Up/Concave Down

1. If $f''(x)>0$ for all x in an interval I then $f(x)$ is concave up on the interval I.
2. If $f''(x)<0$ for all x in an interval I then $f(x)$ is concave down on the interval I.

Inflection Points

$x=c$ is a inflection point of $f(x)$ if the concavity changes at $x=c$.

Extrema

Absolute Extrema

1. $x = c$ is an absolute maximum of $f(x)$ if $f(c) \ge f(x)$ for all x in the domain.
2. $x = c$ is an absolute minimum of $f(x)$ if $f(c) \le f(x)$ for all x in the domain.

Fermat's Theorem

If $f(x)$ has a relative (or local) extrema at $x = c$, then $x = c$ is a critical point of $f(x)$.

Extreme Value Theorem

If $f(x)$ is continuous on the closed interval $[a,b]$ then there exist numbers c and d so that, **1.** $a \le c, d \le b$, **2.** $f(c)$ is the abs. max. in $[a,b]$, **3.** $f(d)$ is the abs. min. in $[a,b]$.

Finding Absolute Extrema

To find the absolute extrema of the continuous function $f(x)$ on the interval $[a,b]$ use the following process.

1. Find all critical points of $f(x)$ in $[a,b]$.
2. Evaluate $f(x)$ at all points found in Step 1.
3. Evaluate $f(a)$ and $f(b)$.
4. Identify the abs. max. (largest function value) and the abs. min.(smallest function value) from the evaluations in Steps 2 & 3.

Relative (local) Extrema

1. $x = c$ is a relative (or local) maximum of $f(x)$ if $f(c) \ge f(x)$ for all x near c.
2. $x = c$ is a relative (or local) minimum of $f(x)$ if $f(c) \le f(x)$ for all x near c.

1st Derivative Test

If $x = c$ is a critical point of $f(x)$ then $x = c$ is

1. a rel. max. of $f(x)$ if $f'(x) > 0$ to the left of $x = c$ and $f'(x) < 0$ to the right of $x = c$.
2. a rel. min. of $f(x)$ if $f'(x) < 0$ to the left of $x = c$ and $f'(x) > 0$ to the right of $x = c$.
3. not a relative extrema of $f(x)$ if $f'(x)$ is the same sign on both sides of $x = c$.

2nd Derivative Test

If $x = c$ is a critical point of $f(x)$ such that $f'(c) = 0$ then $x = c$

1. is a relative maximum of $f(x)$ if $f''(c) < 0$.
2. is a relative minimum of $f(x)$ if $f''(c) > 0$.
3. may be a relative maximum, relative minimum, or neither if $f''(c) = 0$.

Finding Relative Extrema and/or Classify Critical Points

1. Find all critical points of $f(x)$.
2. Use the 1st derivative test or the 2nd derivative test on each critical point.

Mean Value Theorem

If $f(x)$ is continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b) then there is a number $a < c < b$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

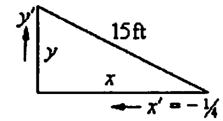
Newton's Method

If x_n is the n^{th} guess for the root/solution of $f(x) = 0$ then $(n+1)^{st}$ guess is $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$ provided $f'(x_n)$ exists.

Related Rates

Sketch picture and identify known/unknown quantities. Write down equation relating quantities and differentiate with respect to t using implicit differentiation (*i.e.* add on a derivative every time you differentiate a function of t). Plug in known quantities and solve for the unknown quantity.

Ex. A 15 foot ladder is resting against a wall. The bottom is initially 10 ft away and is being pushed towards the wall at $\frac{1}{4}$ ft/sec. How fast is the top moving after 12 sec?



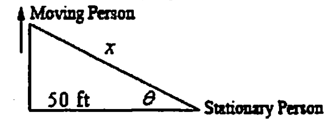
x' is negative because x is decreasing. Using Pythagorean Theorem and differentiating,

$$x^2 + y^2 = 15^2 \quad \Rightarrow \quad 2x\,x' + 2y\,y' = 0$$

After 12 sec we have $x = 10 - 12\left(\frac{1}{4}\right) = 7$ and so $y = \sqrt{15^2 - 7^2} = \sqrt{176}$. Plug in and solve for y'.

$$7\left(-\tfrac{1}{4}\right) + \sqrt{176}\; y' = 0 \Rightarrow y' = \frac{7}{4\sqrt{176}} \text{ ft/sec}$$

Ex. Two people are 50 ft apart when one starts walking north. The angle θ changes at 0.01 rad/min. At what rate is the distance between them changing when $\theta = 0.5$ rad?



We have $\theta' = 0.01$ rad/min. and want to find x'. We can use various trig fcns but easiest is,

$$\sec\theta = \frac{x}{50} \quad \Rightarrow \quad \sec\theta\tan\theta\;\theta' = \frac{x'}{50}$$

We know $\theta = 0.05$ so plug in θ' and solve.

$$\sec(0.5)\tan(0.5)(0.01) = \frac{x'}{50}$$

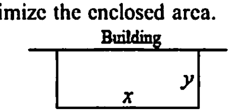
$$x' = 0.3112 \text{ ft/sec}$$

Remember to have calculator in radians!

Optimization

Sketch picture if needed, write down equation to be optimized and constraint. Solve constraint for one of the two variables and plug into first equation. Find critical points of equation in range of variables and verify that they are min/max as needed.

Ex. We're enclosing a rectangular field with 500 ft of fence material and one side of the field is a building. Determine dimensions that will maximize the enclosed area.



Maximize $A = xy$ subject to constraint of $x + 2y = 500$. Solve constraint for x and plug into area.

$$x = 500 - 2y \quad \Rightarrow \quad A = y(500 - 2y) = 500y - 2y^2$$

Differentiate and find critical point(s).

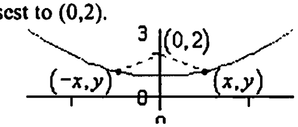
$$A' = 500 - 4y \quad \Rightarrow \quad y = 125$$

By 2nd deriv. test this is a rel. max. and so is the answer we're after. Finally, find x.

$$x = 500 - 2(125) = 250$$

The dimensions are then 250 x 125.

Ex. Determine point(s) on $y = x^2 + 1$ that are closest to (0,2).



Minimize $f = d^2 = (x - 0)^2 + (y - 2)^2$ and the constraint is $y = x^2 + 1$. Solve constraint for x^2 and plug into the function.

$$x^2 = y - 1 \Rightarrow f = x^2 + (y-2)^2 = y - 1 + (y-2)^2 = y^2 - 3y + 3$$

Differentiate and find critical point(s).

$$f' = 2y - 3 \quad \Rightarrow \quad y = \tfrac{3}{2}$$

By the 2nd derivative test this is a rel. min. and so all we need to do is find x value(s).

$$x^2 = \tfrac{3}{2} - 1 = \tfrac{1}{2} \quad \Rightarrow \quad x = \pm\tfrac{1}{\sqrt{2}}$$

The 2 points are then $\left(\frac{1}{\sqrt{2}}, \frac{3}{2}\right)$ and $\left(-\frac{1}{\sqrt{2}}, \frac{3}{2}\right)$.

Integrals

Definitions

Definite Integral: Suppose $f(x)$ is continuous on $[a,b]$. Divide $[a,b]$ into n subintervals of width Δx and choose x_i^* from each interval. Then $\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{\infty} f(x_i^*)\Delta x$.

Anti-Derivative : An anti-derivative of $f(x)$ is a function, $F(x)$, such that $F'(x)=f(x)$.

Indefinite Integral : $\int f(x)\,dx = F(x)+c$ where $F(x)$ is an anti-derivative of $f(x)$.

Fundamental Theorem of Calculus

Part I : If $f(x)$ is continuous on $[a,b]$ then $g(x)=\int_a^x f(t)\,dt$ is also continuous on $[a,b]$ and $g'(x)=\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$.

Part II : $f(x)$ is continuous on $[a,b]$, $F(x)$ is an anti-derivative of $f(x)$ (*i.e.* $F(x)=\int f(x)\,dx$) then $\int_a^b f(x)\,dx = F(b)-F(a)$.

Variants of Part I :

$$\frac{d}{dx}\int_a^{u(x)} f(t)\,dt = u'(x)f[u(x)]$$

$$\frac{d}{dx}\int_{v(x)}^{b} f(t)\,dt = -v'(x)f[v(x)]$$

$$\frac{d}{dx}\int_{v(x)}^{u(x)} f(t)\,dt = u'(x)f[u(x)]-v'(x)f[v(x)]$$

Properties

$\int f(x)\pm g(x)\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

$\int_a^b f(x)\pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

$\int_a^a f(x)\,dx = 0$

$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

$\int cf(x)\,dx = c\int f(x)\,dx$, c is a constant

$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$, c is a constant

$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$

$\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$

If $f(x)\ge g(x)$ on $a\le x\le b$ then $\int_a^b f(x)\,dx \ge \int_b^a g(x)\,dx$

If $f(x)\ge 0$ on $a\le x\le b$ then $\int_a^b f(x)\,dx \ge 0$

If $m\le f(x)\le M$ on $a\le x\le b$ then $m(b-a)\le \int_a^b f(x)\,dx \le M(b-a)$

Common Integrals

$\int k\,dx = kx+c$

$\int x^n\,dx = \frac{1}{n+1}x^{n+1}+c, n\ne -1$

$\int x^{-1}\,dx = \int \frac{1}{x}\,dx = \ln|x|+c$

$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b|+c$

$\int \ln u\,du = u\ln(u)-u+c$

$\int e^u\,du = e^u+c$

$\int \cos u\,du = \sin u+c$

$\int \sin u\,du = -\cos u+c$

$\int \sec^2 u\,du = \tan u+c$

$\int \sec u\tan u\,du = \sec u+c$

$\int \csc u\cot u\,du = -\csc u+c$

$\int \csc^2 u\,du = -\cot u+c$

$\int \tan u\,du = \ln|\sec u|+c$

$\int \sec u\,du = \ln|\sec u+\tan u|+c$

$\int \frac{1}{a^2+u^2}\,du = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right)+c$

$\int \frac{1}{\sqrt{a^2-u^2}}\,du = \sin^{-1}\left(\frac{u}{a}\right)+c$

Standard Integration Techniques

Note that at many schools all but the Substitution Rule tend to be taught in a Calculus II class.

***u* Substitution :** The substitution $u=g(x)$ will convert $\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$ using $du=g'(x)\,dx$. For indefinite integrals drop the limits of integration.

Ex. $\int_1^2 5x^2\cos(x^3)\,dx$

$u=x^3 \Rightarrow du=3x^2dx \Rightarrow x^2dx=\frac{1}{3}du$

$x=1 \Rightarrow u=1^3=1 :: x=2 \Rightarrow u=2^3=8$

$\int_1^2 5x^2\cos(x^3)\,dx = \int_1^8 \frac{5}{3}\cos(u)\,du$

$=\frac{5}{3}\sin(u)\Big|_1^8 = \frac{5}{3}(\sin(8)-\sin(1))$

Integration by Parts : $\int u\,dv = uv-\int v\,du$ and $\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$. Choose u and dv from integral and compute du by differentiating u and compute v using $v=\int dv$.

Ex. $\int xe^{-x}\,dx$

$u=x \quad dv=e^{-x} \Rightarrow du=dx \quad v=-e^{-x}$

$\int xe^{-x}\,dx = -xe^{-x}+\int e^{-x}\,dx = -xe^{-x}-e^{-x}+c$

Ex. $\int_3^5 \ln x\,dx$

$u=\ln x \quad dv=dx \Rightarrow du=\frac{1}{x}dx \quad v=x$

$\int_3^5 \ln x\,dx = x\ln x\Big|_3^5 - \int_3^5 dx = (x\ln(x)-x)\Big|_3^5$

$=5\ln(5)-3\ln(3)-2$

Products and (some) Quotients of Trig Functions

For $\int \sin^n x\cos^m x\,dx$ we have the following :

1. ***n* odd.** Strip 1 sine out and convert rest to cosines using $\sin^2 x = 1-\cos^2 x$, then use the substitution $u=\cos x$.
2. ***m* odd.** Strip 1 cosine out and convert rest to sines using $\cos^2 x = 1-\sin^2 x$, then use the substitution $u=\sin x$.
3. ***n* and *m* both odd.** Use either 1. or 2.
4. ***n* and *m* both even.** Use double angle and/or half angle formulas to reduce the integral into a form that can be integrated.

For $\int \tan^n x\sec^m x\,dx$ we have the following :

1. ***n* odd.** Strip 1 tangent and 1 secant out and convert the rest to secants using $\tan^2 x=\sec^2 x-1$, then use the substitution $u=\sec x$.
2. ***m* even.** Strip 2 secants out and convert rest to tangents using $\sec^2 x = 1+\tan^2 x$, then use the substitution $u=\tan x$.
3. ***n* odd and *m* even.** Use either 1. or 2.
4. ***n* even and *m* odd.** Each integral will be dealt with differently.

Trig Formulas : $\sin(2x)=2\sin(x)\cos(x)$, $\cos^2(x)=\frac{1}{2}(1+\cos(2x))$, $\sin^2(x)=\frac{1}{2}(1-\cos(2x))$

Ex. $\int \tan^3 x\sec^5 x\,dx$

$\int \tan^3 x\sec^5 x\,dx = \int \tan^2 x\sec^4 x\tan x\sec x\,dx$

$=\int(\sec^2 x-1)\sec^4 x\tan x\sec x\,dx$

$=\int(u^2-1)u^4\,du \qquad (u=\sec x)$

$=\frac{1}{7}\sec^7 x-\frac{1}{5}\sec^5 x+c$

Ex. $\int \frac{\sin^5 x}{\cos^3 x}\,dx$

$\int \frac{\sin^5 x}{\cos^3 x}\,dx = \int \frac{\sin^4 x\sin x}{\cos^3 x}\,dx = \int \frac{(\sin^2 x)^2\sin x}{\cos^3 x}\,dx$

$=\int \frac{(1-\cos^2 x)^2\sin x}{\cos^3 x}\,dx \qquad (u=\cos x)$

$=-\int \frac{(1-u^2)^2}{u^3}\,du = -\int \frac{1-2u^2+u^4}{u^3}\,du$

$=\frac{1}{2}\sec^2 x+2\ln|\cos x|-\frac{1}{2}\cos^2 x+c$

Trig Substitutions : If the integral contains the following root use the given substitution and formula to convert into an integral involving trig functions.

$\sqrt{a^2-b^2x^2} \Rightarrow x=\frac{a}{b}\sin\theta$	$\sqrt{b^2x^2-a^2} \Rightarrow x=\frac{a}{b}\sec\theta$	$\sqrt{a^2+b^2x^2} \Rightarrow x=\frac{a}{b}\tan\theta$
$\cos^2\theta=1-\sin^2\theta$	$\tan^2\theta=\sec^2\theta-1$	$\sec^2\theta=1+\tan^2\theta$

Ex. $\int \frac{16}{x^2\sqrt{4-9x^2}}dx$

$x=\frac{2}{3}\sin\theta \Rightarrow dx=\frac{2}{3}\cos\theta\, d\theta$

$\sqrt{4-9x^2}=\sqrt{4-4\sin^2\theta}=\sqrt{4\cos^2\theta}=2\left|\cos\theta\right|$

Recall $\sqrt{x^2}=\left|x\right|$. Because we have an indefinite integral we'll assume positive and drop absolute value bars. If we had a definite integral we'd need to compute θ's and remove absolute value bars based on that and,

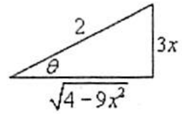
$$\left|x\right|=\begin{cases} x & \text{if } x\ge 0 \\ -x & \text{if } x<0 \end{cases}$$

In this case we have $\sqrt{4-9x^2}=2\cos\theta$.

$$\int \frac{16}{\frac{4}{9}\sin^2\theta(2\cos\theta)}\left(\tfrac{2}{3}\cos\theta\right)d\theta=\int\frac{12}{\sin^2\theta}d\theta$$

$$=\int 12\csc^2 d\theta=-12\cot\theta+c$$

Use Right Triangle Trig to go back to x's. From substitution we have $\sin\theta=\frac{3x}{2}$ so,



From this we see that $\cot\theta=\frac{\sqrt{4-9x^2}}{3x}$. So,

$$\int\frac{16}{x^2\sqrt{4-9x^2}}dx=-\frac{4\sqrt{4-9x^2}}{x}+c$$

Partial Fractions : If integrating $\int\frac{P(x)}{Q(x)}dx$ where the degree of $P(x)$ is smaller than the degree of $Q(x)$. Factor denominator as completely as possible and find the partial fraction decomposition of the rational expression. Integrate the partial fraction decomposition (P.F.D.). For each factor in the denominator we get term(s) in the decomposition according to the following table.

Factor in $Q(x)$	Term in P.F.D	Factor in $Q(x)$	Term in P.F.D
$ax+b$	$\frac{A}{ax+b}$	$(ax+b)^k$	$\frac{A_1}{ax+b}+\frac{A_2}{(ax+b)^2}+\cdots+\frac{A_k}{(ax+b)^k}$
ax^2+bx+c	$\frac{Ax+B}{ax^2+bx+c}$	$(ax^2+bx+c)^k$	$\frac{A_1x+B_1}{ax^2+bx+c}+\cdots+\frac{A_kx+B_k}{(ax^2+bx+c)^k}$

Ex. $\int\frac{7x^2+13x}{(x-1)(x^2+4)}dx$

$$\int\frac{7x^2+13x}{(x-1)(x^2+4)}dx=\int\frac{4}{x-1}+\frac{3x+16}{x^2+4}dx$$

$$=\int\frac{4}{x-1}+\frac{3x}{x^2+4}+\frac{16}{x^2+4}dx$$

$$=4\ln\left|x-1\right|+\tfrac{3}{2}\ln\left(x^2+4\right)+8\tan^{-1}\left(\tfrac{x}{2}\right)$$

Here is partial fraction form and recombined.

$$\frac{7x^2+13x}{(x-1)(x^2+4)}=\frac{A}{x-1}+\frac{Bx+C}{x^2+4}=\frac{A(x^2+4)+(Bx+C)(x-1)}{(x-1)(x^2+4)}$$

Set numerators equal and collect like terms.

$$7x^2+13x=(A+B)x^2+(C-B)x+4A-C$$

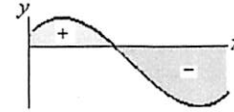
Set coefficients equal to get a system and solve to get constants.

$A+B=7$	$C-B=13$	$4A-C=0$
$A=4$	$B=3$	$C=16$

An alternate method that *sometimes* works to find constants. Start with setting numerators equal in previous example : $7x^2+13x=A\left(x^2+4\right)+\left(Bx+C\right)\left(x-1\right)$. Chose *nice* values of x and plug in. For example if $x=1$ we get $20=5A$ which gives $A=4$. This won't always work easily.

Applications of Integrals

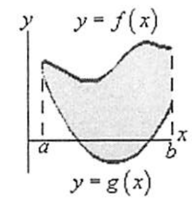
Net Area : $\int_a^b f(x)dx$ represents the net area between $f(x)$ and the x-axis with area above x-axis positive and area below x-axis negative.



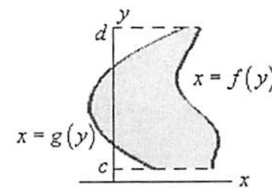
Area Between Curves : The general formulas for the two main cases for each are,

$y=f(x) \Rightarrow A=\int_a^b\left[\text{upper function}\right]-\left[\text{lower function}\right]dx$ & $x=f(y) \Rightarrow A=\int_c^d\left[\text{right function}\right]-\left[\text{left function}\right]dy$

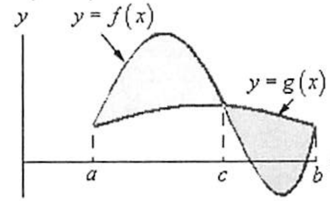
If the curves intersect then the area of each portion must be found individually. Here are some sketches of a couple possible situations and formulas for a couple of possible cases.



$A=\int_a^b f(x)-g(x)dx$



$A=\int_c^d f(y)-g(y)dy$

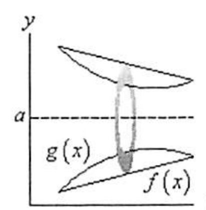


$A=\int_a^c f(x)-g(x)dx+\int_c^b g(x)-f(x)dx$

Volumes of Revolution : The two main formulas are $V=\int A(x)dx$ and $V=\int A(y)dy$. Here is some general information about each method of computing and some examples.

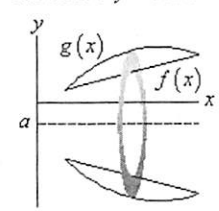
Rings		Cylinders	
$A=\pi\left((\text{outer radius})^2-(\text{inner radius})^2\right)$		$A=2\pi\,(\text{radius})(\text{width / height})$	
Limits: x/y of right/bot ring to x/y of left/top ring		Limits : x/y of inner cyl. to x/y of outer cyl.	
Horz. Axis use $f(x)$, $g(x)$, $A(x)$ and dx.	Vert. Axis use $f(y)$, $g(y)$, $A(y)$ and dy.	Horz. Axis use $f(y)$, $g(y)$, $A(y)$ and dy.	Vert. Axis use $f(x)$, $g(x)$, $A(x)$ and dx.

Ex. Axis : $y=a>0$



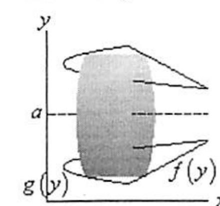
outer radius : $a-f(x)$
inner radius : $a-g(x)$

Ex. Axis : $y=a\le 0$



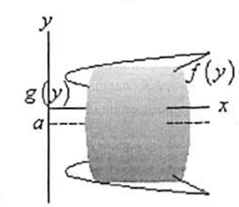
outer radius: $\left|a\right|+g(x)$
inner radius: $\left|a\right|+f(x)$

Ex. Axis : $y=a>0$



radius : $a-y$
width : $f(y)-g(y)$

Ex. Axis : $y=a\le 0$



radius : $\left|a\right|+y$
width : $f(y)-g(y)$

These are only a few cases for horizontal axis of rotation. If axis of rotation is the x-axis use the $y=a\le 0$ case with $a=0$. For vertical axis of rotation ($x=a>0$ and $x=a\le 0$) interchange x and y to get appropriate formulas.

Work : If a force of $F(x)$ moves an object in $a \le x \le b$, the work done is $W = \int_a^b F(x)\,dx$

Average Function Value : The average value of $f(x)$ on $a \le x \le b$ is $f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx$

Arc Length Surface Area : Note that this is often a Calc II topic. The three basic formulas are,

$L = \int_a^b ds \qquad SA = \int_a^b 2\pi y\,ds$ (rotate about x-axis) $\qquad SA = \int_a^b 2\pi x\,ds$ (rotate about y-axis)

where ds is dependent upon the form of the function being worked with as follows.

$ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$ if $y = f(x)$, $a \le x \le b$ $\qquad ds = \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$ if $x = f(t), y = g(t)$, $a \le t \le b$

$ds = \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$ if $x = f(y)$, $a \le y \le b$ $\qquad ds = \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$ if $r = f(\theta)$, $a \le \theta \le b$

With surface area you *may* have to substitute in for the x or y depending on your choice of ds to match the differential in the ds. With parametric and polar you will always need to substitute.

Improper Integral

An improper integral is an integral with one or more infinite limits and/or discontinuous integrands. Integral is called convergent if the limit exists and has a finite value and divergent if the limit doesn't exist or has infinite value. This is typically a Calc II topic.

Infinite Limit

1. $\int_a^{\infty} f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx$
2. $\int_{-\infty}^{b} f(x)\,dx = \lim_{t\to-\infty}\int_t^b f(x)\,dx$
3. $\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_c^{\infty} f(x)\,dx$ provided BOTH integrals are convergent.

Discontinuous Integrand

1. Discont. at a: $\int_a^b f(x)\,dx = \lim_{t\to a^+}\int_t^b f(x)\,dx$
2. Discont. at b: $\int_a^b f(x)\,dx = \lim_{t\to b^-}\int_a^t f(x)\,dx$
3. Discontinuity at $a < c < b$: $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ provided both are convergent.

Comparison Test for Improper Integrals : If $f(x) \ge g(x) \ge 0$ on $[a,\infty)$ then,

1. If $\int_a^{\infty} f(x)\,dx$ conv. then $\int_a^{\infty} g(x)\,dx$ conv.
2. If $\int_a^{\infty} g(x)\,dx$ divg. then $\int_a^{\infty} f(x)\,dx$ divg.

Useful fact : If $a > 0$ then $\int_a^{\infty} \frac{1}{x^p}\,dx$ converges if $p > 1$ and diverges for $p \le 1$.

Approximating Definite Integrals

For given integral $\int_a^b f(x)\,dx$ and a n (must be even for Simpson's Rule) define $\Delta x = \frac{b-a}{n}$ and divide $[a,b]$ into n subintervals $[x_0,x_1]$, $[x_1,x_2]$, … , $[x_{n-1},x_n]$ with $x_0 = a$ and $x_n = b$ then,

Midpoint Rule : $\int_a^b f(x)\,dx \approx \Delta x\left[f(x_1^*)+f(x_2^*)+\cdots+f(x_n^*)\right]$, x_i^* is midpoint $[x_{i-1},x_i]$

Trapezoid Rule : $\int_a^b f(x)\,dx \approx \frac{\Delta x}{2}\left[f(x_0)+2f(x_1)++2f(x_2)+\cdots+2f(x_{n-1})+f(x_n)\right]$

Simpson's Rule : $\int_a^b f(x)\,dx \approx \frac{\Delta x}{3}\left[f(x_0)+4f(x_1)+2f(x_2)+\cdots+2f(x_{n-2})+4f(x_{n-1})+f(x_n)\right]$

Inverse Trig Functions

$\int \frac{1}{\sqrt{a^2-u^2}}\,du = \sin^{-1}\left(\frac{u}{a}\right)+c$ $\qquad \int \sin^{-1}u\,du = u\sin^{-1}u+\sqrt{1-u^2}+c$

$\int \frac{1}{a^2+u^2}\,du = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right)+c$ $\qquad \int \tan^{-1}u\,du = u\tan^{-1}u-\frac{1}{2}\ln\left(1+u^2\right)+c$

$\int \frac{1}{u\sqrt{u^2-a^2}}\,du = \frac{1}{a}\sec^{-1}\left(\frac{u}{a}\right)+c$ $\qquad \int \cos^{-1}u\,du = u\cos^{-1}u-\sqrt{1-u^2}+c$

Hyperbolic Trig Functions

$\int \sinh u\,du = \cosh u+c$ $\qquad \int \cosh u\,du = \sinh u+c$ $\qquad \int \operatorname{sech}^2 u\,du = \tanh u+c$

$\int \operatorname{sech} u\tanh u\,du = -\operatorname{sech} u+c$ $\qquad \int \operatorname{csch} u\coth u\,du = -\operatorname{csch} u+c$ $\qquad \int \operatorname{csch}^2 u\,du = -\coth u+c$

$\int \tanh u\,du = \ln(\cosh u)+c$ $\qquad \int \operatorname{sech} u\,du = \tan^{-1}\left|\sinh u\right|+c$

Miscellaneous

$\int \frac{1}{a^2-u^2}\,du = \frac{1}{2a}\ln\left|\frac{u+a}{u-a}\right|+c$ $\qquad \int \frac{1}{u^2-a^2}\,du = \frac{1}{2a}\ln\left|\frac{u-a}{u+a}\right|+c$

$\int \sqrt{a^2+u^2}\,du = \frac{u}{2}\sqrt{a^2+u^2}+\frac{a^2}{2}\ln\left|u+\sqrt{a^2+u^2}\right|+c$

$\int \sqrt{u^2-a^2}\,du = \frac{u}{2}\sqrt{u^2-a^2}-\frac{a^2}{2}\ln\left|u+\sqrt{u^2-a^2}\right|+c$

$\int \sqrt{a^2-u^2}\,du = \frac{u}{2}\sqrt{a^2-u^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{u}{a}\right)+c$

$\int \sqrt{2au-u^2}\,du = \frac{u-a}{2}\sqrt{2au-u^2}+\frac{a^2}{2}\cos^{-1}\left(\frac{a-u}{a}\right)+c$

Standard Integration Techniques

Note that all but the first one of these tend to be taught in a Calculus II class.

u Substitution

Given $\int_a^b f(g(x))g'(x)\,dx$ then the substitution $u=g(x)$ will convert this into the integral, $\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$.

Integration by Parts

The standard formulas for integration by parts are,

$$\int u\,dv = uv-\int v\,du \qquad \int_a^b u\,dv = uv\Big|_a^b-\int_a^b v\,du$$

Choose u and dv and then compute du by differentiating u and compute v by using the fact that $v=\int dv$.

Trig Substitutions

If the integral contains the following root use the given substitution and formula.

$\sqrt{a^2-b^2x^2} \Rightarrow x=\frac{a}{b}\sin\theta$ and $\cos^2\theta = 1-\sin^2\theta$

$\sqrt{b^2x^2-a^2} \Rightarrow x=\frac{a}{b}\sec\theta$ and $\tan^2\theta = \sec^2\theta-1$

$\sqrt{a^2+b^2x^2} \Rightarrow x=\frac{a}{b}\tan\theta$ and $\sec^2\theta = 1+\tan^2\theta$

Partial Fractions

If integrating $\int \frac{P(x)}{Q(x)}\,dx$ where the degree (largest exponent) of $P(x)$ is smaller than the degree of $Q(x)$ then factor the denominator as completely as possible and find the partial fraction decomposition of the rational expression. Integrate the partial fraction decomposition (P.F.D.). For each factor in the denominator we get term(s) in the decomposition according to the following table.

Factor in $Q(x)$	Term in P.F.D	Factor in $Q(x)$	Term in P.F.D
$ax+b$	$\frac{A}{ax+b}$	$(ax+b)^k$	$\frac{A_1}{ax+b}+\frac{A_2}{(ax+b)^2}+\cdots+\frac{A_k}{(ax+b)^k}$
ax^2+bx+c	$\frac{Ax+B}{ax^2+bx+c}$	$(ax^2+bx+c)^k$	$\frac{A_1x+B_1}{ax^2+bx+c}+\cdots+\frac{A_kx+B_k}{(ax^2+bx+c)^k}$

Products and (some) Quotients of Trig Functions

$\int \sin^n x\cos^m x\,dx$

1. **If n is odd.** Strip one sine out and convert the remaining sines to cosines using $\sin^2 x = 1-\cos^2 x$, then use the substitution $u=\cos x$
2. **If m is odd.** Strip one cosine out and convert the remaining cosines to sines using $\cos^2 x = 1-\sin^2 x$, then use the substitution $u=\sin x$
3. **If n and m are both odd.** Use either 1. or 2.
4. **If n and m are both even.** Use double angle formula for sine and/or half angle formulas to reduce the integral into a form that can be integrated.

$\int \tan^n x\sec^m x\,dx$

1. **If n is odd.** Strip one tangent and one secant out and convert the remaining tangents to secants using $\tan^2 x = \sec^2 x-1$, then use the substitution $u=\sec x$
2. **If m is even.** Strip two secants out and convert the remaining secants to tangents using $\sec^2 x = 1+\tan^2 x$, then use the substitution $u=\tan x$
3. **If n is odd and m is even.** Use either 1. or 2.
4. **If n is even and m is odd.** Each integral will be dealt with differently.

Convert Example : $\cos^6 x = \left(\cos^2 x\right)^3 = \left(1-\sin^2 x\right)^3$

Doble Angle

$\sin(2\theta) =$

Formulas and Identities

Tangent and Cotangent Identities

$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ $\qquad \cot\theta = \dfrac{\cos\theta}{\sin\theta}$

Reciprocal Identities

$\csc\theta = \dfrac{1}{\sin\theta}$ $\qquad \sin\theta = \dfrac{1}{\csc\theta}$

$\sec\theta = \dfrac{1}{\cos\theta}$ $\qquad \cos\theta = \dfrac{1}{\sec\theta}$

$\cot\theta = \dfrac{1}{\tan\theta}$ $\qquad \tan\theta = \dfrac{1}{\cot\theta}$

Pythagorean Identities

$\sin^2\theta + \cos^2\theta = 1$

$\tan^2\theta + 1 = \sec^2\theta$

$1 + \cot^2\theta = \csc^2\theta$

Even/Odd Formulas

$\sin(-\theta) = -\sin\theta$ $\qquad \csc(-\theta) = -\csc\theta$

$\cos(-\theta) = \cos\theta$ $\qquad \sec(-\theta) = \sec\theta$

$\tan(-\theta) = -\tan\theta$ $\qquad \cot(-\theta) = -\cot\theta$

Periodic Formulas

If n is an integer.

$\sin(\theta + 2\pi n) = \sin\theta$ $\qquad \csc(\theta + 2\pi n) = \csc\theta$

$\cos(\theta + 2\pi n) = \cos\theta$ $\qquad \sec(\theta + 2\pi n) = \sec\theta$

$\tan(\theta + \pi n) = \tan\theta$ $\qquad \cot(\theta + \pi n) = \cot\theta$

Double Angle Formulas

$\sin(2\theta) = 2\sin\theta\cos\theta$

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$$

$\tan(2\theta) = \dfrac{2\tan\theta}{1 - \tan^2\theta}$

Degrees to Radians Formulas

If x is an angle in degrees and t is an angle in radians then

$$\frac{\pi}{180} = \frac{t}{x} \quad \Rightarrow \quad t = \frac{\pi x}{180} \quad \text{and} \quad x = \frac{180t}{\pi}$$

Half Angle Formulas

Can find

$\sin^2\theta = \frac{1}{2}\left(1 - \cos(2\theta)\right)$

$\cos^2\theta = \frac{1}{2}\left(1 + \cos(2\theta)\right)$

$\tan^2\theta = \dfrac{1 - \cos(2\theta)}{1 + \cos(2\theta)}$

Sum and Difference Formulas

$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$

$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$

$\tan(\alpha \pm \beta) = \dfrac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$

Product to Sum Formulas

$\sin\alpha\sin\beta = \frac{1}{2}\left[\cos(\alpha - \beta) - \cos(\alpha + \beta)\right]$

$\cos\alpha\cos\beta = \frac{1}{2}\left[\cos(\alpha - \beta) + \cos(\alpha + \beta)\right]$

$\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha + \beta) + \sin(\alpha - \beta)\right]$

$\cos\alpha\sin\beta = \frac{1}{2}\left[\sin(\alpha + \beta) - \sin(\alpha - \beta)\right]$

Sum to Product Formulas

$\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha + \beta}{2}\right)\cos\left(\frac{\alpha - \beta}{2}\right)$

$\sin\alpha - \sin\beta = 2\cos\left(\frac{\alpha + \beta}{2}\right)\sin\left(\frac{\alpha - \beta}{2}\right)$

$\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha + \beta}{2}\right)\cos\left(\frac{\alpha - \beta}{2}\right)$

$\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha + \beta}{2}\right)\sin\left(\frac{\alpha - \beta}{2}\right)$

Cofunction Formulas

$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$ $\qquad \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$

$\csc\left(\frac{\pi}{2} - \theta\right) = \sec\theta$ $\qquad \sec\left(\frac{\pi}{2} - \theta\right) = \csc\theta$

$\tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta$ $\qquad \cot\left(\frac{\pi}{2} - \theta\right) = \tan\theta$

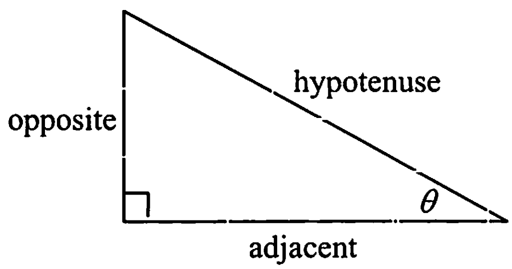
Trig Cheat Sheet

Definition of the Trig Functions

Right triangle definition

For this definition we assume that

$0<\theta<\frac{\pi}{2}$ or $0^\circ<\theta<90^\circ$.



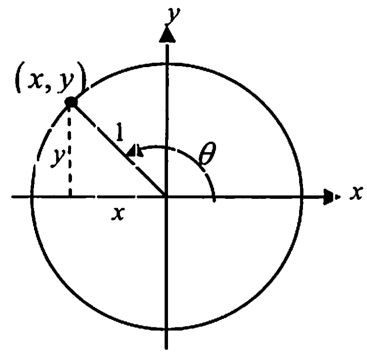
$$\sin\theta=\frac{\text{opposite}}{\text{hypotenuse}} \qquad \csc\theta=\frac{\text{hypotenuse}}{\text{opposite}}$$

$$\cos\theta=\frac{\text{adjacent}}{\text{hypotenuse}} \qquad \sec\theta=\frac{\text{hypotenuse}}{\text{adjacent}}$$

$$\tan\theta=\frac{\text{opposite}}{\text{adjacent}} \qquad \cot\theta=\frac{\text{adjacent}}{\text{opposite}}$$

Unit circle definition

For this definition θ is any angle.



$$\sin\theta=\frac{y}{1}=y \qquad \csc\theta=\frac{1}{y}$$

$$\cos\theta=\frac{x}{1}=x \qquad \sec\theta=\frac{1}{x}$$

$$\tan\theta=\frac{y}{x} \qquad \cot\theta=\frac{x}{y}$$

Facts and Properties

Domain

The domain is all the values of θ that can be plugged into the function.

$\sin\theta$, θ can be any angle

$\cos\theta$, θ can be any angle

$\tan\theta$, $\theta\neq\left(n+\frac{1}{2}\right)\pi,\ \ n=0,\pm1,\pm2,\ldots$

$\csc\theta$, $\theta\neq n\pi,\ \ n=0,\pm1,\pm2,\ldots$

$\sec\theta$, $\theta\neq\left(n+\frac{1}{2}\right)\pi,\ \ n=0,\pm1,\pm2,\ldots$

$\cot\theta$, $\theta\neq n\pi,\ \ n=0,\pm1,\pm2,\ldots$

Range

The range is all possible values to get out of the function.

$-1\le\sin\theta\le1$ $\qquad$ $\csc\theta\ge1$ and $\csc\theta\le-1$

$-1\le\cos\theta\le1$ $\qquad$ $\sec\theta\ge1$ and $\sec\theta\le-1$

$-\infty\le\tan\theta\le\infty$ $\qquad$ $-\infty\le\cot\theta\le\infty$

Period

The period of a function is the number, T, such that $f(\theta+T)=f(\theta)$. So, if ω is a fixed number and θ is any angle we have the following periods.

$$\sin(\omega\theta)\ \rightarrow\ T=\frac{2\pi}{\omega}$$

$$\cos(\omega\theta)\ \rightarrow\ T=\frac{2\pi}{\omega}$$

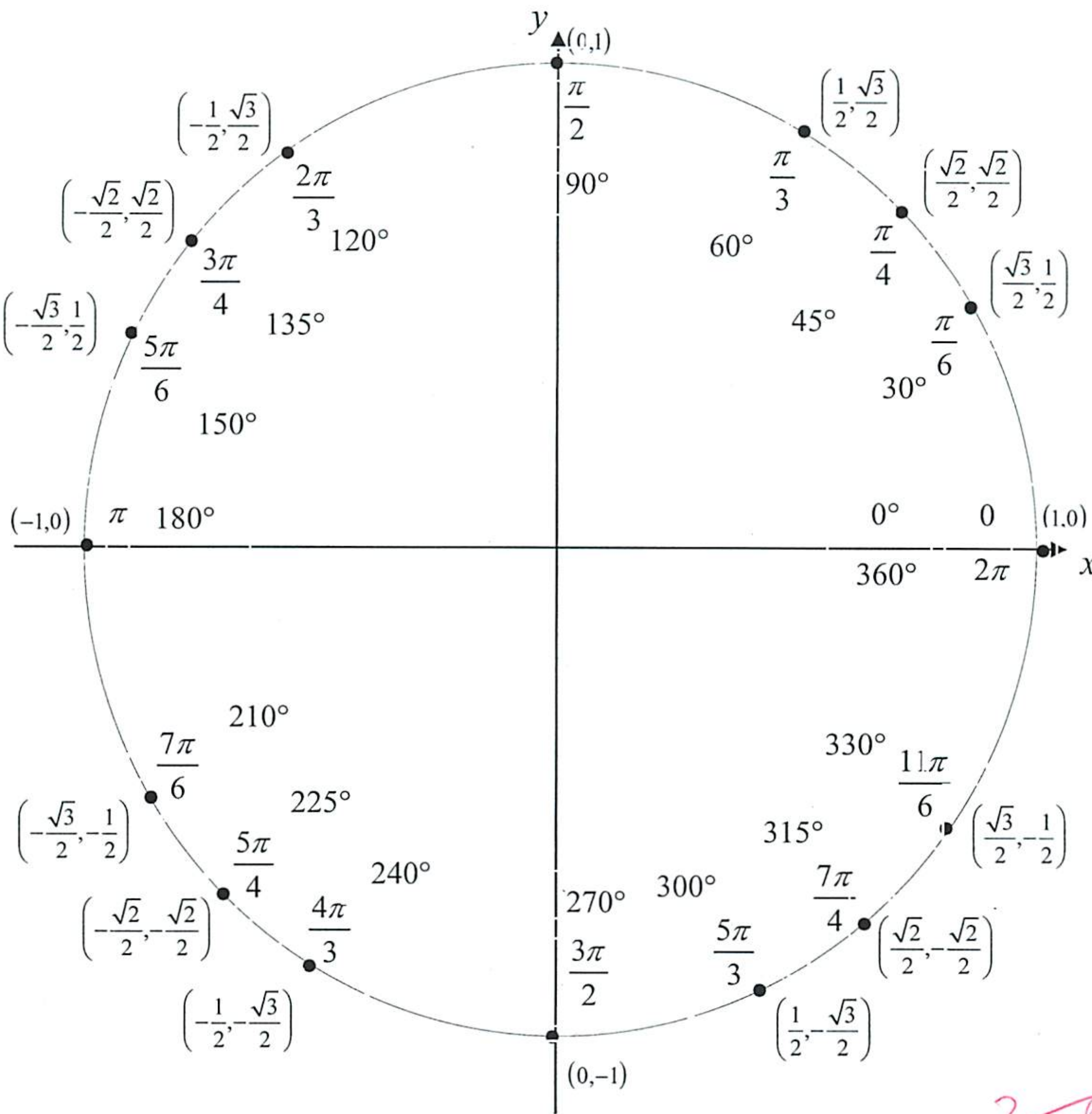
$$\tan(\omega\theta)\ \rightarrow\ T=\frac{\pi}{\omega}$$

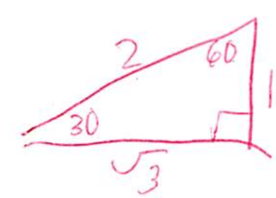
$$\csc(\omega\theta)\ \rightarrow\ T=\frac{2\pi}{\omega}$$

$$\sec(\omega\theta)\ \rightarrow\ T=\frac{2\pi}{\omega}$$

$$\cot(\omega\theta)\ \rightarrow\ T=\frac{\pi}{\omega}$$

Unit Circle

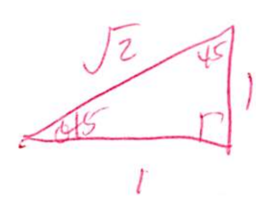




For any ordered pair on the unit circle (x, y) : $\cos\theta = x$ and $\sin\theta = y$

Example

$$\cos\left(\frac{5\pi}{3}\right) = \frac{1}{2} \qquad \sin\left(\frac{5\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$



Inverse Trig Functions

Definition

$y=\sin^{-1}x$ is equivalent to $x=\sin y$

$y=\cos^{-1}x$ is equivalent to $x=\cos y$

$y=\tan^{-1}x$ is equivalent to $x=\tan y$

Domain and Range

Function	Domain	Range
$y=\sin^{-1}x$	$-1\le x\le 1$	$-\frac{\pi}{2}\le y\le\frac{\pi}{2}$
$y=\cos^{-1}x$	$-1\le x\le 1$	$0\le y\le\pi$
$y=\tan^{-1}x$	$-\infty<x<\infty$	$-\frac{\pi}{2}<y<\frac{\pi}{2}$

Inverse Properties

$\cos\left(\cos^{-1}(x)\right)=x \qquad \cos^{-1}\left(\cos(\theta)\right)=\theta$

$\sin\left(\sin^{-1}(x)\right)=x \qquad \sin^{-1}\left(\sin(\theta)\right)=\theta$

$\tan\left(\tan^{-1}(x)\right)=x \qquad \tan^{-1}\left(\tan(\theta)\right)=\theta$

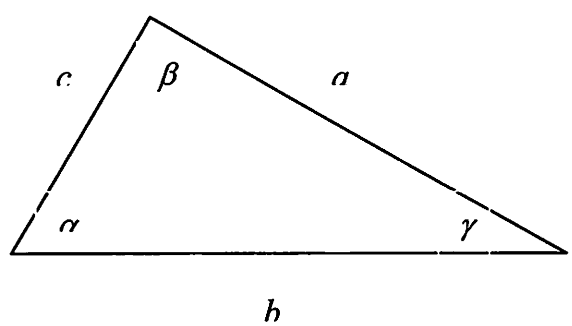
Alternate Notation

$\sin^{-1}x=\arcsin x$

$\cos^{-1}x=\arccos x$

$\tan^{-1}x=\arctan x$

Law of Sines, Cosines and Tangents



Law of Sines

$$\frac{\sin\alpha}{a}=\frac{\sin\beta}{b}=\frac{\sin\gamma}{c}$$

Law of Cosines

$$a^2=b^2+c^2-2bc\cos\alpha$$

$$b^2=a^2+c^2-2ac\cos\beta$$

$$c^2=a^2+b^2-2ab\cos\gamma$$

Mollweide's Formula

$$\frac{a+b}{c}=\frac{\cos\frac{1}{2}(\alpha-\beta)}{\sin\frac{1}{2}\gamma}$$

Law of Tangents

$$\frac{a-b}{a+b}=\frac{\tan\frac{1}{2}(\alpha-\beta)}{\tan\frac{1}{2}(\alpha+\beta)}$$

$$\frac{b-c}{b+c}=\frac{\tan\frac{1}{2}(\beta-\gamma)}{\tan\frac{1}{2}(\beta+\gamma)}$$

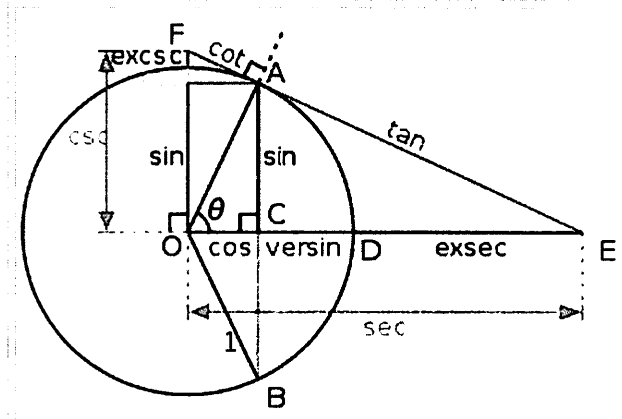
$$\frac{a-c}{a+c}=\frac{\tan\frac{1}{2}(\alpha-\gamma)}{\tan\frac{1}{2}(\alpha+\gamma)}$$

List of trigonometric identities

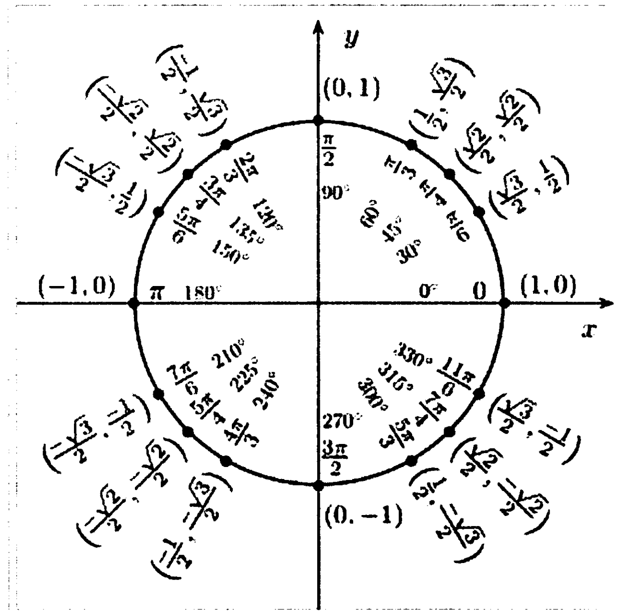
A list-class article from Wikipedia, the free encyclopedia

In mathematics, **trigonometric identities** are equalities that involve trigonometric functions and are true for every single value of the occurring variables (see Identity). Geometrically, these are identities involving certain functions of one or more angles. These are distinct from triangle identities, which are identities involving both angles and side lengths of a triangle. Only the former are covered in this article.

These identities are useful whenever expressions involving trigonometric functions need to be simplified. An important application is the integration of non-trigonometric functions: a common technique involves first using the substitution rule with a trigonometric function, and then simplifying the resulting integral with a trigonometric identity.



All of the trigonometric functions of an angle θ can be constructed geometrically in terms of a unit circle centered at *O*.



Cosines & Sines around the unit circle

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Angles

This article uses Greek letters such as alpha (α), beta (β), gamma (γ), and theta (θ) to represent angles. Several different units of angle measure are widely used, including degrees, radians, and grads:

$$1 \text{ full circle} = 360 \text{ degrees} = 2\pi \text{ radians} = 400 \text{ grads}.$$

The following table shows the conversions for some common angles:

Degrees	30°	60°	120°	150°	210°	240°	300°	330°
Radians	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$
Grads	33⅓ grad	66⅔ grad	133⅓ grad	166⅔ grad	233⅓ grad	266⅔ grad	333⅓ grad	366⅔ grad

Degrees	45°	90°	135°	180°	225°	270°	315°	360°
Radians	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
Grads	50 grad	100 grad	150 grad	200 grad	250 grad	300 grad	350 grad	400 grad

Unless otherwise specified, all angles in this article are assumed to be in radians, though angles ending in a degree symbol (°) are in degrees.

Trigonometric functions

The primary trigonometric functions are the sine and cosine of an angle. These are usually abbreviated sin(θ) and cos(θ), respectively, where θ is the angle. In addition, the parentheses around the angle are sometimes omitted, e.g. sin θ and cos θ.

The tangent (tan) of an angle is the ratio of the sine to the cosine:

$$\tan\theta = \frac{\sin\theta}{\cos\theta}.$$

Finally, the reciprocal functions secant (sec), cosecant (csc), and cotangent (cot) are the reciprocals of the cosine, sine, and tangent:

$$\sec\theta = \frac{1}{\cos\theta}, \quad \csc\theta = \frac{1}{\sin\theta}, \quad \cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}.$$

These definitions are sometimes referred to as ratio identities.

Inverse functions

Main article: Inverse trigonometric functions

The inverse trigonometric functions are partial inverse functions for the trigonometric functions. For example, the inverse function for the sine, known as the **inverse sine** ($\sin^{-1}$) or **arcsine** (arcsin or asin), satisfies

$$\sin(\arcsin x) = x$$

and

$$\arcsin(\sin\theta) = \theta \quad \text{for} \quad -\pi/2 \le \theta \le \pi/2.$$

This article uses the following notation for inverse trigonometric functions:

Function	sin	cos	tan	sec	csc	cot
Inverse	arcsin	arccos	arctan	arcsec	arccsc	arccot

The Pythagorean identity

The basic relationship between the sine and the cosine is the Pythagorean trigonometric identity:

$$\cos^2\theta + \sin^2\theta = 1.$$

This can be viewed as a version of the Pythagorean theorem, and follows from the equation $x^2 + y^2 = 1$ for the unit circle. This equation can be solved for either the sine or the cosine:

$$\sin\theta = \pm\sqrt{1-\cos^2\theta} \quad \text{and} \quad \cos\theta = \pm\sqrt{1-\sin^2\theta}.$$

Related identities

Dividing the Pythagorean identity through by either $\cos^2\theta$ or $\sin^2\theta$ yields two other identities:

$$1 + \tan^2\theta = \sec^2\theta \quad \text{and} \quad 1 + \cot^2\theta = \csc^2\theta.$$

Using these identities together with the ratio identities, it is possible to express any trigonometric function in terms of any other (up to a plus or minus sign):

Each trigonometric function in terms of the other five. [1]

sinθ =	$\sin\theta$	$\pm\sqrt{1-\cos^2\theta}$	$\pm\frac{\tan\theta}{\sqrt{1+\tan^2\theta}}$	$\frac{1}{\csc\theta}$	$\pm\frac{\sqrt{\sec^2\theta-1}}{\sec\theta}$	$\pm$
cosθ =	$\pm\sqrt{1-\sin^2\theta}$	$\cos\theta$	$\pm\frac{1}{\sqrt{1+\tan^2\theta}}$	$\pm\frac{\sqrt{\csc^2\theta-1}}{\csc\theta}$	$\frac{1}{\sec\theta}$	$\pm$
tanθ =	$\pm\frac{\sin\theta}{\sqrt{1-\sin^2\theta}}$	$\pm\frac{\sqrt{1-\cos^2\theta}}{\cos\theta}$	$\tan\theta$	$\pm\frac{1}{\sqrt{\csc^2\theta-1}}$	$\pm\sqrt{\sec^2\theta-1}$	
cscθ =	$\frac{1}{\sin\theta}$	$\pm\frac{1}{\sqrt{1-\cos^2\theta}}$	$\pm\frac{\sqrt{1+\tan^2\theta}}{\tan\theta}$	$\csc\theta$	$\pm\frac{\sec\theta}{\sqrt{\sec^2\theta-1}}$	$\pm$
secθ =	$\pm\frac{1}{\sqrt{1-\sin^2\theta}}$	$\frac{1}{\cos\theta}$	$\pm\sqrt{1+\tan^2\theta}$	$\pm\frac{\csc\theta}{\sqrt{\csc^2\theta-1}}$	$\sec\theta$	$\pm$
cotθ =	$\pm\frac{\sqrt{1-\sin^2\theta}}{\sin\theta}$	$\pm\frac{\cos\theta}{\sqrt{1-\cos^2\theta}}$	$\frac{1}{\tan\theta}$	$\pm\sqrt{\csc^2\theta-1}$	$\pm\frac{1}{\sqrt{\sec^2\theta-1}}$	

Historic shorthands

The versine, coversine, haversine, and exsecant were used in navigation. For example the haversine formula was used to calculate the distance between two points on a sphere. They are rarely used today.

Name(s)	Abbreviation(s)	Value [2]
versed sine, versine	$\text{versin}(\theta)$ $\text{vers}(\theta)$ $\text{ver}(\theta)$	$1 - \cos(\theta)$
versed cosine, vercosine	$\text{vercosin}(\theta)$	$1 + \cos(\theta)$
coversed sine, coversine	$\text{coversin}(\theta)$ $\text{cvs}(\theta)$	$1 - \sin(\theta)$
coversed cosine, covercosine	$\text{covercosin}(\theta)$	$1 + \sin(\theta)$
haversed sine, haversine	$\text{haversin}(\theta)$	$\frac{1-\cos(\theta)}{2}$
haversed cosine, havercosine	$\text{havercosin}(\theta)$	$\frac{1+\cos(\theta)}{2}$
hacoversed sine, hacoversine cohaversine	$\text{hacoversin}(\theta)$	$\frac{1-\sin(\theta)}{2}$

hacoversed cosine, hacovercosine cohavercosine	$\text{hacovercosin}(\theta)$	$\dfrac{1+\sin(\theta)}{2}$
exterior secant, exsecant	$\text{exsec}(\theta)$	$\sec(\theta) - 1$
exterior cosecant, excosecant	$\text{excsc}(\theta)$	$\csc(\theta) - 1$

Symmetry, shifts, and periodicity

By examining the unit circle, the following properties of the trigonometric functions can be established.

Symmetry

When the trigonometric functions are reflected from certain angles, the result is often one of the other trigonometric functions. This leads to the following identities:

Reflected in θ = 0[3]	**Reflected in θ = π / 2 (co-function identities)**[4]	**Reflected in θ = π**
$\sin(-\theta) = -\sin\theta$	$\sin(\frac{\pi}{2}-\theta) = +\cos\theta$	$\sin(\pi-\theta) = +\sin\theta$
$\cos(-\theta) = +\cos\theta$	$\cos(\frac{\pi}{2}-\theta) = +\sin\theta$	$\cos(\pi-\theta) = -\cos\theta$
$\tan(-\theta) = -\tan\theta$	$\tan(\frac{\pi}{2}-\theta) = +\cot\theta$	$\tan(\pi-\theta) = -\tan\theta$
$\csc(-\theta) = -\csc\theta$	$\csc(\frac{\pi}{2}-\theta) = +\sec\theta$	$\csc(\pi-\theta) = +\csc\theta$
$\sec(-\theta) = +\sec\theta$	$\sec(\frac{\pi}{2}-\theta) = +\csc\theta$	$\sec(\pi-\theta) = -\sec\theta$
$\cot(-\theta) = -\cot\theta$	$\cot(\frac{\pi}{2}-\theta) = +\tan\theta$	$\cot(\pi-\theta) = -\cot\theta$

Shifts and periodicity

By shifting the function round by certain angles, it is often possible to find different trigonometric functions that express the result more simply. Some examples of this are shown by shifting functions round by π/2, π and 2π radians. Because the periods of these functions are either π or 2π, there are cases where the new function is exactly the same as the old function without the shift.

Shift by π/2	**Shift by π Period for tan and cot**[5]	**Shift by 2π Period for sin, cos, csc and sec**[6]

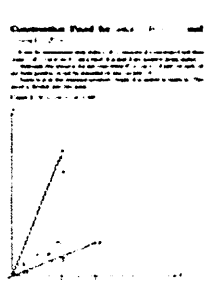
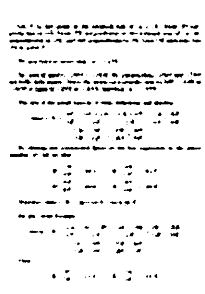
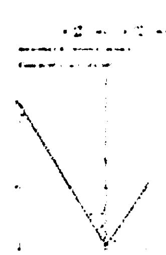
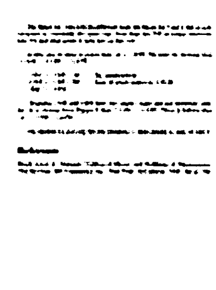
$\sin(\theta + \frac{\pi}{2}) = +\cos\theta$	$\sin(\theta + \pi) = -\sin\theta$	$\sin(\theta + 2\pi) = +\sin\theta$
$\cos(\theta + \frac{\pi}{2}) = -\sin\theta$	$\cos(\theta + \pi) = -\cos\theta$	$\cos(\theta + 2\pi) = +\cos\theta$
$\tan(\theta + \frac{\pi}{2}) = -\cot\theta$	$\tan(\theta + \pi) = +\tan\theta$	$\tan(\theta + 2\pi) = +\tan\theta$
$\csc(\theta + \frac{\pi}{2}) = +\sec\theta$	$\csc(\theta + \pi) = -\csc\theta$	$\csc(\theta + 2\pi) = +\csc\theta$
$\sec(\theta + \frac{\pi}{2}) = -\csc\theta$	$\sec(\theta + \pi) = -\sec\theta$	$\sec(\theta + 2\pi) = +\sec\theta$
$\cot(\theta + \frac{\pi}{2}) = -\tan\theta$	$\cot(\theta + \pi) = +\cot\theta$	$\cot(\theta + 2\pi) = +\cot\theta$

Angle sum and difference identities

These are also known as the *addition and subtraction theorems* or *formulæ*. They were originally established by the 10th century Persian mathematician Abū al-Wafā' Būzjānī. The quickest way to prove these is Euler's formula. The use of the symbols $\pm$ and $\mp$ is described at Plus-minus sign.

Sine	$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$ [7][8]
Cosine	$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$ [9][8]
Tangent	$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$ [10][8]
Arcsine	$\arcsin\alpha \pm \arcsin\beta = \arcsin(\alpha\sqrt{1-\beta^2} \pm \beta\sqrt{1-\alpha^2})$ [11]
Arccosine	$\arccos\alpha \pm \arccos\beta = \arccos(\alpha\beta \mp \sqrt{(1-\alpha^2)(1-\beta^2)})$ [12]
Arctangent	$\arctan\alpha \pm \arctan\beta = \arctan\left(\frac{\alpha \pm \beta}{1 \mp \alpha\beta}\right)$ [13]

Detailed, diagrammed construction proofs, by geometric construction, of formulas for the sine and cosine of the sum of two angles are available for download as a four-page PDF document at

File:Sine Cos Proofs.pdf			
			

Matrix form

See also: matrix multiplication

The sum and difference formulæ for sine and cosine can be written in matrix form, thus:

$$\begin{bmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{bmatrix}\begin{bmatrix}\cos\beta & -\sin\beta\\ \sin\beta & \cos\beta\end{bmatrix} = \begin{bmatrix}\cos(\alpha+\beta) & -\sin(\alpha+\beta)\\ \sin(\alpha+\beta) & \cos(\alpha+\beta)\end{bmatrix}.$$

Sines and cosines of sums of infinitely many terms

$$\sin\left(\sum_{i=1}^{\infty}\theta_i\right) = \sum_{\text{odd } k\geq 1}(-1)^{(k-1)/2}\sum_{\substack{A\subseteq\{1,2,3,\dots\}\\ |A|=k}}\left(\prod_{i\in A}\sin\theta_i\prod_{i\notin A}\cos\theta_i\right)$$

$$\cos\left(\sum_{i=1}^{\infty}\theta_i\right) = \sum_{\text{even } k\geq 0}(-1)^{k/2}\sum_{\substack{A\subseteq\{1,2,3,\dots\}\\ |A|=k}}\left(\prod_{i\in A}\sin\theta_i\prod_{i\notin A}\cos\theta_i\right)$$

In these two identities an asymmetry appears that is not seen in the case of sums of finitely many terms: in each product, there are only finitely many sine factors and cofinitely many cosine factors.

If only finitely many of the terms θ_i are nonzero, then only finitely many of the terms on the right side will be nonzero because sine factors will vanish, and in each term, all but finitely many of the cosine factors will be unity.

Tangents of sums of finitely many terms

Let e_k (for $k \in \{0, \dots, n\}$) be the kth-degree elementary symmetric polynomial in the variables:

$$x_i = \tan\theta_i$$

for $i \in \{0, \dots, n\}$, i.e.

$$e_0 = 1$$

$$e_1 = \sum_{1\le i\le n} x_i = \sum_{1\le i\le n} \tan\theta_i$$

$$e_2 = \sum_{1\le i<j\le n} x_i x_j = \sum_{1\le i<j\le n} \tan\theta_i \tan\theta_j$$

$$e_3 = \sum_{1\le i<j<k\le n} x_i x_j x_k = \sum_{1\le i<j<k\le n} \tan\theta_i \tan\theta_j \tan\theta_k$$

$$\vdots \qquad \vdots$$

Then

$$\tan(\theta_1 + \cdots + \theta_n) = \frac{e_1 - e_3 + e_5 - \cdots}{e_0 - e_2 + e_4 - \cdots},$$

the number of terms depending on n.

For example:

$$\tan(\theta_1 + \theta_2) = \frac{e_1}{e_0 - e_2} = \frac{x_1 + x_2}{1 \ - \ x_1 x_2} = \frac{\tan\theta_1 + \tan\theta_2}{1 \ - \ \tan\theta_1 \tan\theta_2},$$

$$\tan(\theta_1 + \theta_2 + \theta_3) = \frac{e_1 - e_3}{e_0 - e_2} = \frac{(x_1 + x_2 + x_3) \ - \ (x_1 x_2 x_3)}{1 \ - \ (x_1 x_2 + x_1 x_3 + x_2 x_3)},$$

$$\tan(\theta_1 + \theta_2 + \theta_3 + \theta_4) = \frac{e_1 - e_3}{e_0 - e_2 + e_4}$$

$$= \frac{(x_1 + x_2 + x_3 + x_4) \ - \ (x_1 x_2 x_3 + x_1 x_2 x_4 + x_1 x_3 x}{1 \ - \ (x_1 x_2 + x_1 x_3 + x_1 x_4 + x_2 x_3 + x_2 x_4 + x_3 x_4) \ +}$$

and so on. The general case can be proved by mathematical induction.

Secants of sums of finitely many terms

$$\sec(\theta_1 + \cdots + \theta_n) = \frac{\sec\theta_1 \cdots \sec\theta_n}{e_0 - e_2 + e_4 - \cdots}$$

where e_k is the kth-degree elementary symmetric polynomial in the n variables $x_i = \tan\theta_i$, $i = 1, \ldots, n$, and the number of terms in the denominator depends on n.

For example,

$$\sec(\alpha+\beta+\gamma) = \frac{\sec\alpha\sec\beta\sec\gamma}{1-\tan\alpha\tan\beta-\tan\alpha\tan\gamma-\tan\beta\tan\gamma}.$$

Multiple-angle formulae

T_n is the nth Chebyshev polynomial	$\cos n\theta = T_n(\cos\theta)$ [14]
S_n is the nth spread polynomial	$\sin^2 n\theta = S_n(\sin^2\theta)$
de Moivre's formula, i is the Imaginary unit	$\cos n\theta + i\sin n\theta = (\cos(\theta)+i\sin(\theta))^n$ [15]

$$1+2\cos(x)+2\cos(2x)+2\cos(3x)+\cdots+2\cos(nx) = \frac{\sin\left(\left(n+\frac{1}{2}\right)x\right)}{\sin(x/2)}.$$

(This function of x is the Dirichlet kernel.)

Double-, triple-, and half-angle formulae

See also: Tangent half-angle formula

These can be shown by using either the sum and difference identities or the multiple-angle formulae.

	Double-angle formulae[16][17]		
$\sin 2\theta = 2\sin\theta\cos\theta = \frac{2\tan\theta}{1+\tan^2\theta}$	$\cos 2\theta = \cos^2\theta-\sin^2\theta = 2\cos^2\theta-1 = 1-2\sin^2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}$	$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$	cot
	Triple-angle formulae[18][14]		
$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$	$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$	$\tan 3\theta = \frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}$	cot
	Half-angle formulae[19][20]		

$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$	$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$	$\tan\frac{\theta}{2} = \csc\theta - \cot\theta = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \frac{\sin\theta}{1+\cos\theta} = \frac{1-\cos\theta}{\sin\theta}$	$\cot$

The fact that the triple-angle formula for sine and cosine only involves powers of a single function allows one to relate the geometric problem of a compass and straightedge construction of angle trisection to the algebraic problem of solving a cubic equation, which allows one to prove that this is in general impossible, by field theory.

A formula for computing the trigonometric identities for the third-angle exists, but it requires finding the zeroes of the cubic equation $x^3 - \frac{3x+d}{4} = 0$, where *x* is the value of the sine function at some angle and *d* is the known value of the sine function at the triple angle. However, the discriminant of this equation is negative, so this equation has three real roots (of which only one is the solution within the correct third-circle) but none of these solutions is reducible to a real algebraic expression, as they use intermediate complex numbers under the cube roots, (which may be expressed in terms of real-only functions only if using hyperbolic functions). As a consequence, it is not possible to express the trigonometric values of angles that are not multiples of 3 degrees divided by any power of two, if using a real-only algebraic expression (for example sin(1°)).

See also Casus irreducibilis.

Sine, cosine, and tangent of multiple angles

For specific multiples, these follow from the angle addition formulas, while the general formula was given by 16th century French mathematician Vieta.

$$\sin n\theta = \sum_{k=0}^{n} \binom{n}{k} \cos^k\theta \, \sin^{n-k}\theta \, \sin\left(\frac{1}{2}(n-k)\pi\right)$$

$$\cos n\theta = \sum_{k=0}^{n} \binom{n}{k} \cos^k\theta \, \sin^{n-k}\theta \, \cos\left(\frac{1}{2}(n-k)\pi\right)$$

tan $n\theta$ can be written in terms of tan θ using the recurrence relation:

$$\tan(n+1)\theta = \frac{\tan n\theta + \tan\theta}{1 - \tan n\theta \, \tan\theta}.$$

cot $n\theta$ can be written in terms of cot θ using the recurrence relation:

$$\cot(n+1)\theta = \frac{\cot n\theta \cot\theta - 1}{\cot n\theta + \cot\theta}.$$

Chebyshev method

The Chebyshev method is a recursive algorithm for finding the n^{th} multiple angle formula knowing the $(n-1)^{\text{th}}$ and $(n-2)^{\text{th}}$ formulae.[21]

The cosine for nx can be computed from the cosine of $(n - 1)$ and $(n - 2)$ as follows:

$$\cos nx = 2 \cdot \cos x \cdot \cos(n-1)x - \cos(n-2)x$$

Similarly sin(nx) can be computed from the sines of $(n - 1)x$ and $(n - 2)x$

$$\sin nx = 2 \cdot \cos x \cdot \sin(n-1)x - \sin(n-2)x$$

For the tangent, we have:

$$\tan nx = \frac{H + K\tan x}{K - H\tan x}$$

where $H/K = \tan(n - 1)x$.

Tangent of an average

$$\tan\left(\frac{\alpha+\beta}{2}\right) = \frac{\sin\alpha + \sin\beta}{\cos\alpha + \cos\beta} = -\frac{\cos\alpha - \cos\beta}{\sin\alpha - \sin\beta}$$

Setting either α or β to 0 gives the usual tangent half-angle formulæ.

Euler's infinite product

$$\cos\left(\frac{\theta}{2}\right) \cdot \cos\left(\frac{\theta}{4}\right) \cdot \cos\left(\frac{\theta}{8}\right) \cdots = \prod_{n=1}^{\infty} \cos\left(\frac{\theta}{2^n}\right) = \frac{\sin(\theta)}{\theta} = \operatorname{sinc}\theta.$$

Power-reduction formulas

Obtained by solving the second and third versions of the cosine double-angle formula.

Sine	Cosine	
$\sin^2\theta = \frac{1-\cos 2\theta}{2}$	$\cos^2\theta = \frac{1+\cos 2\theta}{2}$	$\sin^2\theta\cos$

$\sin^3\theta = \frac{3\sin\theta - \sin 3\theta}{4}$	$\cos^3\theta = \frac{3\cos\theta + \cos 3\theta}{4}$	$\sin^3\theta\cos^3$
$\sin^4\theta = \frac{3 - 4\cos 2\theta + \cos 4\theta}{8}$	$\cos^4\theta = \frac{3 + 4\cos 2\theta + \cos 4\theta}{8}$	$\sin^4\theta\cos^4$
$\sin^5\theta = \frac{10\sin\theta - 5\sin 3\theta + \sin 5\theta}{16}$	$\cos^5\theta = \frac{10\cos\theta + 5\cos 3\theta + \cos 5\theta}{16}$	$\sin^5\theta\cos^5$

and in general terms of powers of sin θ or cos θ the following is true, and can be deduced using De Moivre's formula, Euler's formula and binomial theorem.

	Cosine	
if n is odd	$\cos^n\theta = \frac{2}{2^n}\sum_{k=0}^{\frac{n-1}{2}}\binom{n}{k}\cos((n-2k)\theta)$	$\sin^n\theta = \frac{2}{2^n}\sum_{k=0}^{\frac{n-1}{2}}(-1)^{(\frac{n-1}{2}}$
if n is even	$\cos^n\theta = \frac{1}{2^n}\binom{n}{\frac{n}{2}} + \frac{2}{2^n}\sum_{k=0}^{\frac{n}{2}-1}\binom{n}{k}\cos((n-2k)\theta)$	$\sin^n\theta = \frac{1}{2^n}\binom{n}{\frac{n}{2}} + \frac{2}{2^n}\sum_{k}^{\frac{n}{2}}$

Product-to-sum and sum-to-product identities

The product-to-sum identities can be proven by expanding their right-hand sides using the angle addition theorems. See beat (acoustics) for an application of the sum-to-product formulæ.

Product-to-sum[22]
$\cos\theta\cos\varphi = \frac{\cos(\theta-\varphi) + \cos(\theta+\varphi)}{2}$
$\sin\theta\sin\varphi = \frac{\cos(\theta-\varphi) - \cos(\theta+\varphi)}{2}$
$\sin\theta\cos\varphi = \frac{\sin(\theta+\varphi) + \sin(\theta-\varphi)}{2}$
$\cos\theta\sin\varphi = \frac{\sin(\theta+\varphi) - \sin(\theta-\varphi)}{2}$

Sum-to-product[23]
$\sin\theta \pm \sin\varphi = 2\sin\left(\frac{\theta\pm\varphi}{2}\right)\cos\left(\frac{\theta\mp\varphi}{2}\right.$
$\cos\theta + \cos\varphi = 2\cos\left(\frac{\theta+\varphi}{2}\right)\cos\left(\frac{\theta-}{2}\right.$
$\cos\theta - \cos\varphi = -2\sin\left(\frac{\theta+\varphi}{2}\right)\sin\left(\frac{\theta-}{2}\right.$

Other related identities

If x, y, and z are the three angles of any triangle, or in other words

$$\begin{aligned}&\text{if } x+y+z=\pi=\text{half circle,}\\&\qquad\text{then }\ \tan(x)+\tan(y)+\tan(z)=\tan(x)\tan(y)\tan(z).\end{aligned}$$

(If any of x, y, z is a right angle, one should take both sides to be ∞. This is neither +∞ nor −∞; for present purposes it makes sense to add just one point at infinity to the real line, that is approached by tan(θ) as tan(θ) either increases through positive values or decreases through negative values. This is a one-point compactification of the real line.)

$$\begin{aligned}&\text{If } x+y+z=\pi=\text{half circle,}\\&\qquad\text{then }\ \sin(2x)+\sin(2y)+\sin(2z)=4\sin(x)\sin(y)\sin(z).\end{aligned}$$

Ptolemy's theorem

$$\begin{aligned}&\text{If } w+x+y+z=\pi=\text{half circle,}\\&\qquad\text{then }\ \sin(w+x)\sin(x+y)\\&\qquad\qquad=\sin(x+y)\sin(y+z)\\&\qquad\qquad=\sin(y+z)\sin(z+w)\\&\qquad\qquad=\sin(z+w)\sin(w+x)=\sin(w)\sin(y)+\sin(x)\sin(z).\end{aligned}$$

(The first three equalities are trivial; the fourth is the substance of this identity.) Essentially this is Ptolemy's theorem adapted to the language of trigonometry.

Linear combinations

For some purposes it is important to know that any linear combination of sine waves of the same period or frequency but different phase shifts is also a sine wave with the same period or frequency, but a different phase shift. In the case of a linear combination of a sine and cosine wave[24] (which is just a sine wave with a phase shift of $\pi/2$), we have

$$a\sin x+b\cos x=\sqrt{a^2+b^2}\cdot\sin(x+\varphi)$$

where

$$\varphi=\begin{cases}\arcsin\left(\frac{b}{\sqrt{a^2+b^2}}\right) & \text{if } a\geq 0,\\ \pi-\arcsin\left(\frac{b}{\sqrt{a^2+b^2}}\right) & \text{if } a<0,\end{cases}$$

or equivalently

$$\varphi=\arctan\left(\frac{b}{a}\right)+\begin{cases}0 & \text{if } a\geq 0,\\ \pi & \text{if } a<0.\end{cases}$$

More generally, for an arbitrary phase shift, we have

$$a \sin x + b \sin(x + \alpha) = c \sin(x + \beta)$$

where

$$c = \sqrt{a^2 + b^2 + 2ab \cos \alpha},$$

and

$$\beta = \arctan\left(\frac{b \sin \alpha}{a + b \cos \alpha}\right) + \begin{cases} 0 & \text{if } a + b \cos \alpha \geq 0, \\ \pi & \text{if } a + b \cos \alpha < 0. \end{cases}$$

Other sums of trigonometric functions

Sum of sines and cosines with arguments in arithmetic progression:

$$\sin \varphi + \sin(\varphi + \alpha) + \sin(\varphi + 2\alpha) + \cdots + \sin(\varphi + n\alpha) = \frac{\sin\left(\frac{(n+1)\alpha}{2}\right) \cdot \sin\left(\varphi + \frac{n}{\cdot}\right.}{\sin \frac{\alpha}{2}}$$

$$\cos \varphi + \cos(\varphi + \alpha) + \cos(\varphi + 2\alpha) + \cdots + \cos(\varphi + n\alpha) = \frac{\sin\left(\frac{(n+1)\alpha}{2}\right) \cdot \cos\left(\varphi +\right.}{\sin \frac{\alpha}{2}}$$

For any *a* and *b*:

$$a \cos(x) + b \sin(x) = \sqrt{a^2 + b^2} \cos(x - \operatorname{atan2}(b, a))$$

where atan2(*y*, *x*) is the generalization of arctan(*y*/*x*) which covers the entire circular range.

$$\tan(x) + \sec(x) = \tan\left(\frac{x}{2} + \frac{\pi}{4}\right).$$

The above identity is sometimes convenient to know when thinking about the Gudermannian function, which relates the circular and hyperbolic trigonometric functions without resorting to complex numbers.

If *x*, *y*, and *z* are the three angles of any triangle, i.e. if $x + y + z = \pi$, then

$$\cot(x)\cot(y) + \cot(y)\cot(z) + \cot(z)\cot(x) = 1.$$

Certain linear fractional transformations

If $f(x)$ is given by the linear fractional transformation

$$f(x) = \frac{(\cos \alpha)x - \sin \alpha}{(\sin \alpha)x + \cos \alpha},$$

and similarly

$$g(x) = \frac{(\cos\beta)x - \sin\beta}{(\sin\beta)x + \cos\beta},$$

then

$$f(g(x)) = g(f(x)) = \frac{(\cos(\alpha+\beta))x - \sin(\alpha+\beta)}{(\sin(\alpha+\beta))x + \cos(\alpha+\beta)}.$$

More tersely stated, if for all α we let f_α be what we called f above, then

$$f_\alpha \circ f_\beta = f_{\alpha+\beta}.$$

If x is the slope of a line, then $f(x)$ is the slope of its rotation through an angle of $-\alpha$.

Inverse trigonometric functions

$$\arcsin(x) + \arccos(x) = \pi/2$$

$$\arctan(x) + \operatorname{arccot}(x) = \pi/2.$$

$$\arctan(x) + \arctan(1/x) = \begin{cases} \pi/2, & \text{if } x > 0 \\ -\pi/2, & \text{if } x < 0 \end{cases}$$

Compositions of trig and inverse trig functions

$\sin[\arccos(x)] = \sqrt{1-x^2}$	$\tan[\arcsin(x)] = \frac{x}{\sqrt{1-x^2}}$
$\sin[\arctan(x)] = \frac{x}{\sqrt{1+x^2}}$	$\tan[\arccos(x)] = \frac{\sqrt{1-x^2}}{x}$
$\cos[\arctan(x)] = \frac{1}{\sqrt{1+x^2}}$	$\cot[\arcsin(x)] = \frac{\sqrt{1-x^2}}{x}$
$\cos[\arcsin(x)] = \sqrt{1-x^2}$	$\cot[\arccos(x)] = \frac{x}{\sqrt{1-x^2}}$

Relation to the complex exponential function

$e^{ix} = \cos(x) + i\sin(x)$[25] (Euler's formula),

$$e^{-ix} = \cos(-x) + i\sin(-x) = \cos(x) - i\sin(x)$$

$$e^{i\pi} = -1$$

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2}$$ [26]

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}$$ [27]

and hence the corollary:

$$\tan(x) = \frac{e^{ix} - e^{-ix}}{i(e^{ix} + e^{-ix})} = \frac{\sin(x)}{\cos(x)}$$

where $i^2 = -1$.

Infinite product formula

For applications to special functions, the following infinite product formulae for trigonometric functions are useful:[28][29]

$$\sin x = x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{\pi^2 n^2}\right) \qquad \cos x = \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{\pi^2 (n - \frac{1}{2})^2}\right)$$

$$\sinh x = x \prod_{n=1}^{\infty} \left(1 + \frac{x^2}{\pi^2 n^2}\right) \qquad \cosh x = \prod_{n=1}^{\infty} \left(1 + \frac{x^2}{\pi^2 (n - \frac{1}{2})^2}\right)$$

$$\frac{\sin x}{x} = \prod_{n=1}^{\infty} \cos\left(\frac{x}{2^n}\right)$$

Identities without variables

The curious identity

$$\cos 20^\circ \cdot \cos 40^\circ \cdot \cos 80^\circ = \frac{1}{8}$$

is a special case of an identity that contains one variable:

$$\prod_{j=0}^{k-1} \cos(2^j x) = \frac{\sin(2^k x)}{2^k \sin(x)}.$$

A similar-looking identity is

$$\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{3\pi}{7}=\frac{1}{8},$$

and in addition

$$\sin 20^\circ\cdot\sin 40^\circ\cdot\sin 80^\circ=\frac{\sqrt{3}}{8}.$$

The following is perhaps not as readily generalized to an identity containing variables (but see explanation below):

$$\cos 24^\circ+\cos 48^\circ+\cos 96^\circ+\cos 168^\circ=\frac{1}{2}.$$

Degree measure ceases to be more felicitous than radian measure when we consider this identity with 21 in the denominators:

$$\cos\left(\frac{2\pi}{21}\right)+\cos\left(2\cdot\frac{2\pi}{21}\right)+\cos\left(4\cdot\frac{2\pi}{21}\right)+\cos\left(5\cdot\frac{2\pi}{21}\right)+\cos\left(8\cdot\frac{2\pi}{21}\right)+\cos\left(10\cdot\frac{2\pi}{21}\right)=\frac{1}{2}.$$

The factors 1, 2, 4, 5, 8, 10 may start to make the pattern clear: they are those integers less than 21/2 that are relatively prime to (or have no prime factors in common with) 21. The last several examples are corollaries of a basic fact about the irreducible cyclotomic polynomials: the cosines are the real parts of the zeroes of those polynomials; the sum of the zeroes is the Möbius function evaluated at (in the very last case above) 21; only half of the zeroes are present above. The two identities preceding this last one arise in the same fashion with 21 replaced by 10 and 15, respectively.

Computing π

An efficient way to compute π is based on the following identity without variables, due to Machin:

$$\frac{\pi}{4}=4\arctan\frac{1}{5}-\arctan\frac{1}{239}$$

or, alternatively, by using an identity of Leonhard Euler:

$$\frac{\pi}{4}=5\arctan\frac{1}{7}+2\arctan\frac{3}{79}.$$

A useful mnemonic for certain values of sines and cosines

For certain simple angles, the sines and cosines take the form $\sqrt{n}/2$ for $0\leq n\leq 4$, which makes them easy to remember.

$$\begin{array}{lllllllll}
\sin 0 & = & \sin 0^\circ & = & \sqrt{0}/2 & = & \cos 90^\circ & = & \cos\left(\frac{\pi}{2}\right) \\
\sin\left(\frac{\pi}{6}\right) & = & \sin 30^\circ & = & \sqrt{1}/2 & = & \cos 60^\circ & = & \cos\left(\frac{\pi}{3}\right) \\
\sin\left(\frac{\pi}{4}\right) & = & \sin 45^\circ & = & \sqrt{2}/2 & = & \cos 45^\circ & = & \cos\left(\frac{\pi}{4}\right) \\
\sin\left(\frac{\pi}{3}\right) & = & \sin 60^\circ & = & \sqrt{3}/2 & = & \cos 30^\circ & = & \cos\left(\frac{\pi}{6}\right) \\
\sin\left(\frac{\pi}{2}\right) & = & \sin 90^\circ & = & \sqrt{4}/2 & = & \cos 0^\circ & = & \cos 0
\end{array}$$

Other interesting values

$$\sin\frac{\pi}{7} = \frac{\sqrt{7}}{6} - \frac{\sqrt{7}}{189}\sum_{j=0}^{\infty}\frac{(3j+1)!}{189^j j!\,(2j+2)!}$$

$$\sin\frac{\pi}{18} = \frac{1}{6}\sum_{j=0}^{\infty}\frac{(3j)!}{27^j j!\,(2j+1)!}$$

With the golden ratio φ:

$$\cos\left(\frac{\pi}{5}\right) = \cos 36^\circ = \frac{\sqrt{5}+1}{4} = \varphi/2$$

$$\sin\left(\frac{\pi}{10}\right) = \sin 18^\circ = \frac{\sqrt{5}-1}{4} = \frac{\varphi-1}{2} = \frac{1}{2\varphi}$$

Also see exact trigonometric constants.

Calculus

In calculus the relations stated below require angles to be measured in radians; the relations would become more complicated if angles were measured in another unit such as degrees. If the trigonometric functions are defined in terms of geometry, their derivatives can be found by verifying two limits. The first is:

$$\lim_{x\to 0}\frac{\sin x}{x} = 1,$$

verified using the unit circle and squeeze theorem. It may be tempting to propose to use L'Hôpital's rule to establish this limit. However, if one uses this limit in order to prove that the derivative of the sine is the cosine, and then uses the fact that the derivative of the sine is the cosine in applying L'Hôpital's rule, one is reasoning circularly—a logical fallacy. The second limit is:

$$\lim_{x\to 0}\frac{1-\cos x}{x}=0,$$

verified using the identity $\tan(x/2) = (1 - \cos x)/\sin x$. Having established these two limits, one can use the limit definition of the derivative and the addition theorems to show that $(\sin x)' = \cos x$ and $(\cos x)' = -\sin x$. If the sine and cosine functions are defined by their Taylor series, then the derivatives can be found by differentiating the power series term-by-term.

$$\frac{d}{dx}\sin x=\cos x$$

The rest of the trigonometric functions can be differentiated using the above identities and the rules of differentiation:[30][31][32]

$$\frac{d}{dx}\sin x=\cos x, \qquad \frac{d}{dx}\arcsin x=\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\cos x=-\sin x, \qquad \frac{d}{dx}\arccos x=\frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\tan x=\sec^2 x, \qquad \frac{d}{dx}\arctan x=\frac{1}{1+x^2}$$

$$\frac{d}{dx}\cot x=-\csc^2 x, \qquad \frac{d}{dx}\operatorname{arccot} x=\frac{-1}{1+x^2}$$

$$\frac{d}{dx}\sec x=\tan x\sec x, \qquad \frac{d}{dx}\operatorname{arcsec} x=\frac{1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx}\csc x=-\csc x\cot x, \qquad \frac{d}{dx}\operatorname{arccsc} x=\frac{-1}{|x|\sqrt{x^2-1}}$$

The integral identities can be found in "list of integrals of trigonometric functions". Some generic forms are listed below.

$$\int\frac{du}{\sqrt{a^2-u^2}}=\sin^{-1}\left(\frac{u}{a}\right)+C$$

$$\int\frac{du}{a^2+u^2}=\frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right)+C$$

$$\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C$$

Implications

The fact that the differentiation of trigonometric functions (sine and cosine) results in linear combinations of the same two functions is of fundamental importance to many fields of mathematics, including differential equations and Fourier transforms.

Exponential definitions

Function	Inverse function[33]
$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$	$\arcsin x = -i \ln\left(ix + \sqrt{1 - x^2}\right)$
$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$	$\arccos x = -i \ln\left(x + \sqrt{x^2 - 1}\right)$
$\tan\theta = \frac{e^{i\theta} - e^{-i\theta}}{i(e^{i\theta} + e^{-i\theta})}$	$\arctan x = \frac{i}{2} \ln\left(\frac{i + x}{i - x}\right)$
$\csc\theta = \frac{2i}{e^{i\theta} - e^{-i\theta}}$	$\operatorname{arccsc} x = -i \ln\left(\frac{i}{x} + \sqrt{1 - \frac{1}{x^2}}\right)$
$\sec\theta = \frac{2}{e^{i\theta} + e^{-i\theta}}$	$\operatorname{arcsec} x = -i \ln\left(\frac{1}{x} + \sqrt{1 - \frac{i}{x^2}}\right)$
$\cot\theta = \frac{i(e^{i\theta} + e^{-i\theta})}{e^{i\theta} - e^{-i\theta}}$	$\operatorname{arccot} x = \frac{i}{2} \ln\left(\frac{x - i}{x + i}\right)$
$\operatorname{cis}\theta = e^{i\theta}$	$\operatorname{arccis} x = \frac{\ln x}{i}$

Miscellaneous

Dirichlet kernel

The **Dirichlet kernel** $D_n(x)$ is the function occurring on both sides of the next identity:

$$1 + 2\cos(x) + 2\cos(2x) + 2\cos(3x) + \cdots + 2\cos(nx) = \frac{\sin\left[\left(n + \frac{1}{2}\right)x\right]}{\sin\left(\frac{x}{2}\right)}.$$

The convolution of any integrable function of period 2π with the Dirichlet kernel coincides with the function's *n*th-degree Fourier approximation. The same holds for any measure or generalized function.

Extension of half-angle formula

If we set

$$t = \tan\left(\frac{x}{2}\right),$$

then[34]

$$\sin(x) = \frac{2t}{1+t^2} \text{ and } \cos(x) = \frac{1-t^2}{1+t^2} \text{ and } e^{ix} = \frac{1+it}{1-it}$$

where $e^{ix} = \cos(x) + i\sin(x)$, sometimes abbreviated to cis(x).

This substitution of t for tan(x/2), with the consequent replacement of sin(x) by $2t/(1+t^2)$ and cos(x) by $(1-t^2)/(1+t^2)$ is useful in calculus for converting rational functions in sin(x) and cos(x) to functions of t in order to find their antiderivatives. For more information see tangent half-angle formula.

See also

- Trigonometry
- Proofs of trigonometric identities
- Uses of trigonometry
- Tangent half-angle formula
- Law of cosines
- Law of sines
- Law of tangents
- Mollweide's formula
- Pythagorean theorem
- Exact trigonometric constants (values of sine and cosine expressed in surds)
- Derivatives of trigonometric functions
- List of integrals of trigonometric functions
- Hyperbolic function
- Prosthaphaeresis
- Versine and haversine
- Exsecant

Notes

1. ^ Abramowitz and Stegun, p. 73, 4.3.45
2. ^ Abramowitz and Stegun, p. 78, 4.3.147
3. ^ Abramowitz and Stegun, p. 72, 4.3.13–15
4. ^ The Elementary Identities (http://jwbales.home.mindspring.com/precal/part5/part5.1.html)
5. ^ Abramowitz and Stegun, p. 72, 4.3.9
6. ^ Abramowitz and Stegun, p. 72, 4.3.7–8
7. ^ Abramowitz and Stegun, p. 72, 4.3.16
8. ^ *a b c* Weisstein, Eric W., "Trigonometric Addition Formulas (http://mathworld.wolfram.com/TrigonometricAdditionFormulas.html) " from MathWorld.
9. ^ Abramowitz and Stegun, p. 72, 4.3.17
10. ^ Abramowitz and Stegun, p. 72, 4.3.18
11. ^ Abramowitz and Stegun, p. 80, 4.4.42
12. ^ Abramowitz and Stegun, p. 80, 4.4.43
13. ^ Abramowitz and Stegun, p. 80, 4.4.36
14. ^ *a b* Weisstein, Eric W., "Multiple-Angle Formulas (http://mathworld.wolfram.com/Multiple-AngleFormulas.html) " from MathWorld.
15. ^ Abramowitz and Stegun, p. 74, 4.3.48
16. ^ Abramowitz and Stegun, p. 72, 4.3.24–26
17. ^ Weisstein, Eric W., "Double-Angle Formulas (http://mathworld.wolfram.com/Double-AngleFormulas.html) " from MathWorld.

18. ^ Abramowitz and Stegun, p. 72, 4.3.27–28
19. ^ Abramowitz and Stegun, p. 72, 4.3.20–22
20. ^ Weisstein, Eric W., "Half-Angle Formulas (http://mathworld.wolfram.com/Half-AngleFormulas.html) " from MathWorld.
21. ^ Ken Ward's Mathematics Pages, http://www.trans4mind.com/personal_development/mathematics/trigonometry/multipleAnglesRecursiveFormula.htm
22. ^ Abramowitz and Stegun, p. 72, 4.3.31–33
23. ^ Abramowitz and Stegun, p. 72, 4.3.34–39
24. ^ Proof at http://pages.pacificcoast.net/~cazelais/252/lc-trig.pdf
25. ^ Abramowitz and Stegun, p. 74, 4.3.47
26. ^ Abramowitz and Stegun, p. 71, 4.3.2
27. ^ Abramowitz and Stegun, p. 71, 4.3.1
28. ^ Abramowitz and Stegun, p. 75, 4.3.89–90
29. ^ Abramowitz and Stegun, p. 85, 4.5.68–69
30. ^ Abramowitz and Stegun, p. 77, 4.3.105–110
31. ^ Abramowitz and Stegun, p. 82, 4.4.52–57
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External links

- Values of Sin and Cos, expressed in surds, for integer multiples of 3° and of 5⅝° (http://www.jdawiseman.com/papers/easymath/surds_sin_cos.html) , and for the same angles Csc and Sec (http://www.jdawiseman.com/papers/easymath/surds_csc_sec.html) and Tan (http://www.jdawiseman.com/papers/easymath/surds_tan.html) .

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