Exam 1

Cheat Sleet

2/22

$x = r\cos\theta$ $\cos\theta = \frac{x}{r} = \frac{x}{\sqrt{x^2+y^2}}$

$y = r\sin\theta$ $\sin\theta = \frac{y}{r}$ $\frac{y}{\sqrt{x^2+y^2}}$

$r = \sqrt{x^2+y^2}$

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$\vec{v} = \frac{d\vec{r}}{dt}$ $v = |\vec{v}| = \frac{ds}{dt}$

$\vec{T} = \frac{d\vec{r}}{ds} = \frac{dr/dt}{ds/dt} = \frac{\vec{v}}{|\vec{v}|}$

$\frac{d\vec{T}}{ds} = \frac{dT/dt}{ds/dt} \leftarrow |\vec{v}| = \kappa \vec{N}$

$|\vec{v}| = \sqrt{\hat{u}_r + \hat{u}_\theta}$

$\vec{v} = \dot{r}\,\hat{u}_r + r\dot{\theta}\,\hat{u}_\theta$

determinate = volume 3D = $\vec{u} \cdot (\vec{v} \times \vec{w})$
area 2D

$\vec{a} = \frac{d^2 s}{dt^2}\vec{T} + v^2 \kappa N$ ← not useful I think

$\tan^{-1}\left(\frac{r}{dr/d\theta}\right)$ $\tan^{-1}\left(\frac{\text{coord in } \hat{u}_\theta}{\text{coords in } \hat{u}_r}\right)$

not inversable = det = 0

homogenious novel solutions determinant = 0

18.02 Practice Exam 1 (50 mins.)

1. (20) Consider the points in xyz-space $P:(1,2,-2)$, $Q:(-2,1,2)$, and the origin $O:(0,0,0)$.

a) (6) Find the cosine of angle POQ.

b) (6) Find a vector perpendicular to both OP and OQ.

c) (5) Find the xyz-equation of a plane parallel to the one through O, P and Q, but intersecting the z-axis at $z=2$.

d) (3) Where does the plane you found in (c) intersect the x-axis?

2. (20) Let $A = \begin{pmatrix} 1 & 3 & 2 \\ 0 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$. Its matrix of cofactors is (in part) $C = \begin{pmatrix} 3 & 1 & -2 \\ -4 & 0 & 2 \\ & & \end{pmatrix}$.

a) (15) Confirm (mentally) the entry -4 in C, then fill in the bottom row of C and from this find A^{-1}.

b) (5) Use the result of part (a) to solve the system

$$x+3y+2z=1, \qquad 2y+z=2, \qquad x+y+2z=-1 \ .$$

3. (8) Find all values of the constant c for which the system of homogeneous equations

$$cx+y+4z=0, \qquad -x+y+z=0, \qquad y+cz=0$$

has a non-trivial solution (i.e., a solution other than $x=y=z=0$)

4. (12: 10,2) Scotch tape is being peeled off a stationary roll, modeled as a circle of radius a, and center at the origin. The end $P:(x,y)$ of the tape is initially at the point $A:(a,0)$ on the x-axis. During the process, the pulled-off length of tape is always tangent to the rest of the roll – call the point of tangency Q on the circle, and of the two possible directions for the pulled-off tape, it's the one where the sticky side faces away from the roll (not towards it).

a) Use vector methods to derive parametric equations for x and y in terms of the central angle $AOQ = \theta$, for $0 \le \theta \le 2\pi$. Show work, indicating reasoning.

b) Show on a separate sketch where P is when $\theta = \pi$, and verify that your equations give the correct position of P when $\theta = \pi$.

5. (20) A point P moves in space so that its position vector is given by

$$OP = \mathbf{r} = (\cos t)\,\mathbf{i} + (\sqrt{2}\sin t)\,\mathbf{j} + (\cos t)\,\mathbf{k} \ .$$

a) (10: 5,3,2) Find its velocity vector $\mathbf{v}$, its speed $\dfrac{ds}{dt}$, and its unit tangent vector $\mathbf{T}$.

b) (5) Find its curvature $\kappa = \left|\dfrac{d\mathbf{T}}{ds}\right|$.

c) (5) At what point(s) in the xz-plane does P pass through this plane? skipped ???

6. (10) At what point P on the line given by the position vector $\mathbf{r}(t) = \langle 1+t,\ 3-t,\ 1+2t \rangle$ will the origin vector OP be perpendicular to the line?

7. (10) A point P moves in the polar coordinate (r,θ)-plane so that its velocity vector at time t is given in the $\mathbf{u}_r, \mathbf{u}_\theta$ system by $\mathbf{v} = 2\,\mathbf{u}_r + 2\,\mathbf{u}_\theta$.

At time $t=0$, the point P has coodinates $r=1$ and $\theta=0$.

Answer the following, showing work or brief indication of reason.

a) How long is the path that P travels from $t=0$ to $t=3$?

b) How far is P from the origin when $t=3$?

c) What angle does the path of P make with its position vector, when $t=3$?

d) Where is the point P at time t?

18.02 Practice Exam 1 Solutions – S-2010

$\boxed{1}$ $\vec{OP} = \langle 1,2,-2\rangle$ $\vec{OQ} = \langle -2,1,2\rangle$

a) $\cos POQ = \dfrac{OP\cdot OQ}{|OP||OQ|} = \dfrac{-4}{3\cdot 3} = \dfrac{-4}{9}$

b) $OP\times OQ = \begin{vmatrix} i & j & k \\ 1 & 2 & -2 \\ -2 & 1 & 2\end{vmatrix} = \langle 6,2,5\rangle$

c) $6x + 2y + 5(z-2) = 0$
or $6x + 2y + 5z = 10$ ..

d) Since $y=z=0$, $6x = 10$
$\therefore$ at $x = \boxed{5/3}$

$\boxed{2}$ $A = \begin{pmatrix} 1 & 3 & 2 \\ 0 & 2 & 1 \\ 1 & 1 & 2\end{pmatrix}$ $C = \begin{pmatrix} 3 & 1 & -2 \\ -4 & 0 & 2 \\ -1 & -1 & 2\end{pmatrix}$

a)
$|A| = 4+3-(4+1)$
$= 2$

$A^{-1} = \frac{1}{2}C^T = \frac{1}{2}\begin{pmatrix} 3 & -4 & -1 \\ 1 & 0 & -1 \\ -2 & 2 & 2\end{pmatrix}$

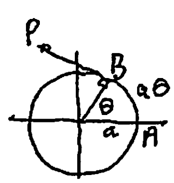
b) The system is $A\vec{x} = \begin{pmatrix}1\\2\\-1\end{pmatrix}$
so $\begin{pmatrix}x\\y\\z\end{pmatrix} = \frac{1}{2}\begin{pmatrix} 3 & -4 & -1 \\ 1 & 0 & -1 \\ -2 & 2 & 2\end{pmatrix}\begin{pmatrix}1\\2\\-1\end{pmatrix} = \begin{pmatrix}-2\\1\\0\end{pmatrix}$

$\boxed{3}$ Requires coeff. det. be 0: $\begin{vmatrix} c & 1 & 4 \\ -1 & 1 & 1 \\ 0 & 1 & c\end{vmatrix} = (c^2-4)-(c-c)$
$= c^2-4$

$c^2-4=0 \Leftrightarrow c = \pm 2$

$\boxed{4}$

a)

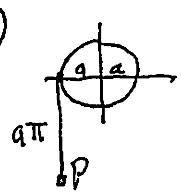


$\vec{OP} = \vec{OB} + \vec{BP}$
$OB = a\langle\cos\theta, \sin\theta\rangle$

$BP = \underset{\substack{\uparrow \\ \text{length}}}{a\theta}\,\underset{\substack{\uparrow \\ \text{OB rotated 90° to the left}}}{\langle -\sin\theta, \cos\theta\rangle}$

Adding: $\vec{OP} = a\langle\cos\theta - \theta\sin\theta,\ \sin\theta + \theta\cos\theta\rangle$

b)



P: $(-a, -a\pi)$ from picture
$\vec{OP} = a\langle -1-\pi\cdot 0,\ 0+\pi\cdot -1\rangle$
$= a\langle -1, -\pi\rangle$ ✓

$\boxed{5}$ $\vec{r} = \langle\cos t, \sqrt{2}\sin t, \cos t\rangle$

a) $\vec{v} = \langle -\sin t, \sqrt{2}\cos t, -\sin t\rangle$

$|\vec{v}| = \frac{ds}{dt} = (\sin^2 t + 2\cos^2 t + \sin^2 t)^{1/2}$
$= \sqrt{2(\sin^2 t + \cos^2 t)}$
$= \sqrt{2}$

$\vec{T} = \dfrac{\vec{v}}{|\vec{v}|} = \left\langle -\dfrac{\sin t}{\sqrt{2}}, \cos t, -\dfrac{\sin t}{\sqrt{2}}\right\rangle$

b) $K = \left|\dfrac{d\vec{T}}{ds}\right| = \dfrac{|d\vec{T}/dt|}{|ds/dt|} = \dfrac{|d\vec{T}/dt|}{\sqrt{2}}$

$= \dfrac{1}{\sqrt{2}}\left|\left\langle -\dfrac{\cos t}{\sqrt{2}}, -\sin t, -\dfrac{\cos t}{\sqrt{2}}\right\rangle\right|$

$= \dfrac{1}{\sqrt{2}}\sqrt{\cos^2 t + \sin^2 t} = \dfrac{1}{\sqrt{2}}$.

c) Crosses xz-plane when $y=0$
$\therefore$ when $\sqrt{2}\sin t = 0$: $t = 0, \pi$
The corresponding points are resp'y:
$(1,0,1)$ and $(-1,0,-1)$.
$t=0$

$\boxed{6}$

The line has dir'n vector $\vec{A} = \langle 1,-1,2\rangle$
$\vec{OP} = \langle 1+t, 3-t, 1+2t\rangle$
These are $\perp$ if $\vec{OP}\cdot\vec{A} = 0$:
$(1+t) - (3-t) + 2(1+2t) = 0$
or $6t = 0$
$\therefore t = 0$, corresponding to
$P = (1,3,1)$

$\boxed{7}$ $\vec{V} = 2\hat{u}_r + 2\hat{u}_\theta = \dot{r}\hat{u}_r + r\dot{\theta}\hat{u}_\theta$

a) $|\vec{V}| = \sqrt{4+4} = 2\sqrt{2} = v = \frac{ds}{dt}$
so $s = 2\sqrt{2}\cdot 3 = \boxed{6\sqrt{2}}$

b) $\dot{r} = 2 \Rightarrow r = 2t+1$ $\therefore r(3) = 7$
$r(0) = 1$

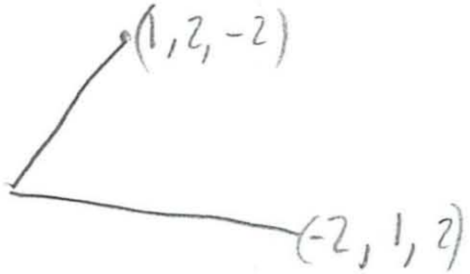
c) $\tan\psi = \frac{2}{2} = 1$, $\therefore \psi = \pi/4$ (45°)

d) Find $\theta(t)$: $r\dot{\theta} = 2$, $r = 2t+1$
$\therefore \dot{\theta} = \dfrac{2}{2t+1}$, $\begin{cases}\theta = \ln(2t+1) + \cancel{c} \\ r = 2t+1\end{cases}$
($c = 0$ since $\theta(0) = 0$)

18.02 Exam 1 Review 2/22/10

1. $P(1,2,-2)$ $O(0,0,0)$
$Q(-2,1,2)$

Find cosign POQ



? How do
? find equations of line $\langle 1,2,-2\rangle \langle -2,1,2\rangle$

$\vec{P}\cdot\vec{Q} = |\vec{P}|\,|Q|\cos\theta$ ← ?? could it be?

$(1\cdot -2)+(2\cdot 1)+(-2\cdot 2)$

$-2+2+-4$

$-4 = \sqrt{1^2+2^2+(-2)^2}\cdot\sqrt{(-2)^2+1^2+2^2}\cos\theta$

$\sqrt{1+4+4}\cdot\sqrt{4+1+4}$

$3\cdot 3\cos\theta$

$\boxed{\frac{-4}{9} = \cos\theta}$ ✓ Yes!

$\theta = \cos^{-1}\left(\frac{-4}{9}\right)$ ← does not $\neq$ nice

(2)

b) Find vector perpendicular

$\vec{OP} \cdot \vec{OQ}$

$\cancel{\langle 1,2,-2\rangle \cdot \langle -2,1,2\rangle}$

$\cancel{9}$

to find perpendicular → cross product

to ✓ dot prod = 0

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -2 \\ -2 & 1 & 2 \end{vmatrix}$$

$$\begin{vmatrix} 2 & -2 \\ 1 & 2 \end{vmatrix} \hat{i} + \begin{vmatrix} 1 & -2 \\ -2 & 2 \end{vmatrix} \hat{j} + \begin{vmatrix} 1 & 2 \\ -2 & 1 \end{vmatrix} \hat{k}$$

$$(4 - -2)\hat{i} \cancel{+} - (2-4)\hat{j} + (1 - -4)\hat{k}$$

$$6\hat{i} + 2\hat{j} + 5\hat{k}$$

forgot

c) Find xyz = of plane parallel to O, P, Q
but intersecting z = 2

So I kinda rembber - but not exactly

$6x + 2y + 5(z-2)$ ← easy

$6x + 2y + 5z - 10 = 0$

$6x + 2y + 5z = 10$

③ d Where does this intersect x- axis

$$6x + 2y + 5z = 10$$

where is $y = 0$ $z = 0$? as well
don't know 3D
yeah

$$6x + 2(0) + 5(0) = 10$$

$$6x = 10$$

$$x = \frac{10}{6} = \frac{5}{3}$$ ✓

2. $A = \begin{pmatrix} 1 & 3 & 2 \\ 0 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$ $C = \begin{pmatrix} 3 & 1 & -2 \\ -4 & 0 & 2 \\ \_ & \_ & \_ \end{pmatrix}$
cofactors

~~Con~~ Fill in bottom of C

- what is cofactors again?
  - signed minor of a matrix

for example 1

$$\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 4 - 1 = 3$$

$\underline{-1}$ $\underline{-1}$ $\underline{2}$

why wrong? ?

(4)

A^{-1} = This long process I forget

1. Get determinant
2. Minors
3. Signs of minor
4. Transpose signed minor
5. $\frac{1}{\det}$ transposed signed minor

don't forget

1. $\det = 1(4-1) - 3(0-1) + 2(0-2)$

$4 - 1 + -3 + -4$

$\cancel{-4}$ $+2$

2. $\begin{pmatrix} 3 & 1 & -2 \\ -4 & 0 & 2 \\ -1 & -1 & 2 \end{pmatrix}$

3. $\begin{pmatrix} 3 & +1 & -2 \\ 4 & 0 & +2 \\ -1 & -1 & 2 \end{pmatrix}$

$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$? what went wrong ???

4. $\begin{pmatrix} 3 & 4 & -1 \\ +1 & 0 & -1 \\ -2 & +2 & 2 \end{pmatrix}$

5. $-\frac{1}{2}\begin{pmatrix} 3 & 4 & -1 \\ +1 & 0 & -1 \\ -2 & +2 & 2 \end{pmatrix}$

$\begin{pmatrix} -3/4 & -1 & 1/4 \\ 1/4 & 0 & -1/4 \\ 1/2 & 1/2 & -1/2 \end{pmatrix}$

There is no way you are getting this right

5.) Use the result of inverse to solve system

$x + 3y + 2z = 1$

$2y + z = 2$

$x + y + 2z = -1$

? what is rule again $A^{-1}x = B$

Start w/

$A \quad x = d$

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Multiply both sides by A^{-1}

$AA^{-1}x = A^{-1}d$

$(A \cdot A^{-1} = I)$ $\quad x = A^{-1}d$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{2}\begin{pmatrix} 3 & -4 & -1 \\ 1 & 0 & -1 \\ -2 & 2 & 2 \end{pmatrix}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Then something

I think

Copy error

$$\begin{bmatrix} 3 \cdot 1 + 2 \cdot -4 + -1 \cdot -1 \\ 1 \cdot \cancel{1} + 0 \cdot 2 + -1 \cdot -\cancel{1} \\ -2 \cdot 1 + 2 \cdot 2 + 2 \cdot -1 \end{bmatrix} = \begin{bmatrix} 3 - 8 + 1 \\ \cancel{1} + 0 + \cancel{1} \\ -2 + 4 - 2 \end{bmatrix} \begin{matrix} \leftarrow -4 \\ \leftarrow 2 \\ \\ \end{matrix}$$

$$\frac{1}{2}\begin{bmatrix} -4 \\ \cancel{0} \\ 0 \end{bmatrix} \quad x = -\cancel{4}2 \quad \cancel{x} = \cancel{0} \quad z = 0$$

? don't forget

(6) 3. Find all values of constant c for which systems of homogenious equations

$$cx + y + 4z = 0$$
$$-x + y + z = 0$$
$$y + cz = 0$$

non trivial means something = or $\neq 0$

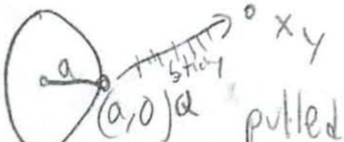
non trivial if $|A| = 0$

$$\begin{vmatrix} c & 1 & 4 \\ -1 & 1 & 1 \\ 0 & 1 & c \end{vmatrix} = 0$$

$$c(1c-1) - 1(-c-0) + 4(-1-0) = 0$$
$$c^2 - c + c - 4 = 0$$
$$c^2 = 4$$
$$c = \pm 2$$ right - just know when $|A| = 0$

So ok on vectors + matricies - just memorize

4. Scotch tape peeled off a roll



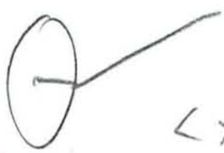
pulled tangent

can't visualize the which side is sticky
- not that helpful of a hint

This seems very familiar to problems we've done - but I have no clue
what is it even asking?

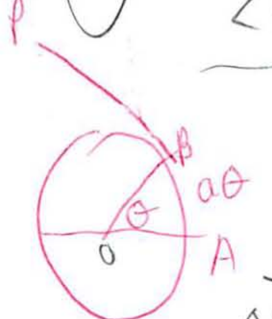
7

a) Use vector methods to derive parametric equations for x y in terms of $AOQ = \theta$
Show work, indicate reasoning



$OP = OQ + QP$

$\langle x, y\rangle = \langle a, 0\rangle + \langle x-a, y-0\rangle$



↙ picture like this

so does $a\theta = \overrightarrow{BP}$

$OP = \vec{OB} + \vec{PB}$ ✓

r, in polar coords
no I think rect

$\cancel{\langle x}$

$\vec{OB} = a \langle \cos\theta, \sin\theta\rangle$

↑ duh - know that circ is $\langle \cos, \sin\rangle$ and then put r in front

$\vec{PB} = a\theta \langle -\sin\theta, \cos\theta\rangle$

↑ know that is lenght

↑ so got lenght

↑ $\vec{OB}$ rotated 90° to left

↑ just direction

flip and neg on 1st

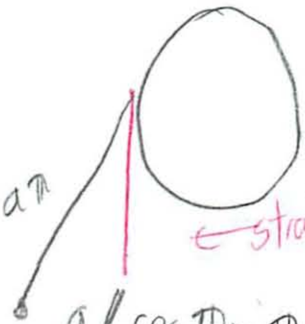
Add $\vec{OP} = a \langle \cos\theta - \theta\sin\theta, \sin\theta + \theta\cos\theta\rangle$

hopefully I got these type of problems now but I drew wrong initial pic

~ could still do it

8) well the assumption I missed was that point
Where touches roll changes as it turns

b. Show on a sketch when $\Phi = P$ and verify



← straight down duh
tangent at exactly y here

$\sin\pi = 0$
$\cos\pi = \cancel{a}$ ← not a, but $-1 = \frac{-a}{a}$
↑ come on, know this

$a\langle \cos\pi - \pi\sin\theta,\ \sin\pi,\ + \pi\cos\pi \rangle$
↑ copy error

$\cancel{a\langle a,\ 0 - \pi a \rangle}$ $\quad a\langle -1,\ \pi \cdot -1 \rangle$

$\cancel{\langle a^2, -\pi a^2 \rangle}$ $\quad a\langle -1, -\pi \rangle$

5. A point P moves so position vector

$OP = \vec{r} = \langle \cos t,\ \sqrt{2}\sin t,\ \cos t \rangle$

$\vec{V} =$ need to remember derivs + integral rules

$\int \frac{1}{x} = \ln x$

$\int x^2 = \frac{x^3}{3}$ $\quad \int x^n = \frac{x^{n+1}}{n+1} + C$

$\int x^3 = \frac{x^4}{4}$

$\int \frac{1}{x^2} = \frac{x^{-1}}{-2} = -\frac{1}{\cancel{2}x}$

$\int \frac{1}{x^3} = \quad = \frac{1}{2x^2}$

Integral Review-

(1)

Integral Review

$\int k = kx + C$

$\int \sin x = -\cos x + C$ $\quad \int \cos x \, dx = \sin x + C$

$\int \sec^2 x = \tan x$ $\quad \int \sec x \tan x \, dx = \sec x + C$

$\int \csc^2 x = -\cot x$ $\quad \int \csc x \cot x \, dx = -\csc x + C$

$\int e^x = e^x + C$

$\int a^x = \frac{a^x}{\ln a} + C$

$\int \frac{1}{x} = \int x^{-1} = \ln|x| + C$

$\int \frac{1}{x^8} = \int x^{-8} = -\frac{1}{7} x^{-7}$

so still +1

$\int \frac{1}{x^2} = \int x^{-2} = \frac{x^{-2+1}}{-2+1} = -\frac{1}{x}$

duh $-2+1=-1$

$\int \frac{1}{x^3} = \int x^{-3} = \frac{x^{-3+1}}{-3+1} = \frac{x^{-2}}{-2} = \frac{1}{2x}$

So I did know it I made a simple math error

(10) negitive

$\vec{v} = \langle -\sin t,\ \sqrt{2} - \cos t,\ \sin t \rangle$

$d\cos = -\sin$
$d\sin = \cos$) I got it backwards

$\frac{ds}{dt} = |\vec{v}| = \sqrt{\sin^2 t + \sqrt{2}^2 \cos^2 t + \sin^2 t}$

did I do this right

yes $\sqrt{2}^2 = 2$

$(-\cos t)^2 = \cos t^2$

and screwed up here

$\vec{T} = \frac{|v|}{|\vec{v}|} = \frac{\langle \sin t,\ \sqrt{2} - \cos t,\ \sin t \rangle}{\sqrt{\sin^2 t + 2\cos^2 t + \sin^2 t}}$ $\frac{1}{\sqrt{2}}$

b) Find curvature $\left|\frac{d\vec{T}}{ds}\right|$ - don't just divide by $|v|$ again learned today
- and need to simplify
- which is not easy

$\frac{d\vec{T}}{ds} = \frac{d\vec{T}/dt}{ds/dt}$ ← take deriv w/ chain rule

$\leftarrow |\vec{v}|$

but the real challenge is simplifing

$\frac{1}{\sqrt{2\sin^2 t + 2\cos^2 t}}$ $\frac{1}{\sqrt{2(\sin^2 t \cos^2 t)}} = \frac{1}{\sqrt{2}}$ or not :)

$\frac{1}{\sqrt{2}} \langle \sin t,\ \sqrt{2} - \cos t,\ \sin t \rangle$

$2^{-1/2}$ deriv of constant = 0 duh

$\cancel{-\frac{1}{2} \cdot 2 \cdot^{-3/2}} \langle \sin t,\ \sqrt{2} - \cos t,\ \sin t \rangle \cdot \frac{1}{\sqrt{2}} \langle \cos t,\ \sqrt{2} + \sin t,\ \cos$

$\cancel{-\frac{1}{2 \cdot \sqrt{2}^3}}$ ~~$\langle \sin\ \sqrt{2} - \cos t,\ \sin t \rangle$~~ $\cdot$ $\frac{\langle \cos t,\ \sqrt{2} + \sin t,\ \cos t \rangle}{\sqrt{2}}$

(11)

$$\frac{\text{all of that}}{\sqrt{2}}$$

now more simplifing

$$\frac{\frac{\langle \cos t, \sqrt{2} + \sin t, \cos t \rangle}{\sqrt{2}}}{\sqrt{2}}$$

note $\frac{\frac{x}{\sqrt{2}}}{\sqrt{2}} = \cancel{\frac{x}{\sqrt{2}} \cdot \frac{\sqrt{2}}{1}} \quad \frac{x}{\sqrt{2}\sqrt{2}} = \frac{x}{2}$

? is right ?? — see Clarification — depends on ()

$$\frac{\langle \cos t, \sqrt{2} + \sin t, \cos t \rangle}{2}$$

so $\kappa = \pm \frac{1}{2}$ $\quad N = \langle \cos t, \sqrt{2} + \sin t, \cos t \rangle$

(need to visualize for which is which of original function plot pts

made a lot of small mistakes so not checking

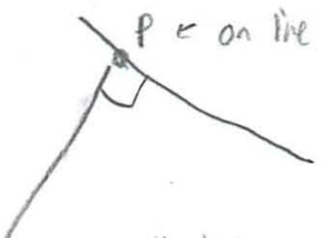
- basically $d \sin \to \cos$
  $d \cos \to -\sin$
- and ~~t~~ treating a neg as minus got messed up

but I think I got concepts
try again tonight

⑫ 6. At what pt on line $r(t) = \langle 1+t, 3-t, 1+2t \rangle$
will $\vec{OP}$ be perpendicular
- when dot product $= 0$

So far seems fairly easy
Just need to know steps
and not make mistakes
- Not the hard p-set qu

but how do we know $\vec{OP}$
↑ well we have a line



$\langle OP \rangle = \langle x-0, y-0, z-0 \rangle$
So ~~where is~~ need (x, y, z)
But where does t come into this
just where line is at that pt in time

$(x \cdot 1+t) + (y(3-t)) + (z(1+2t)) = 0$
$x + xt + 3y - yt + z + 2zt = 0$

↙ coefficents of t
line has direction $\langle 1, -1, 2 \rangle$
$\vec{OP} \cdot \vec{A} = 0$

13 $\langle 1, -1, 2 \rangle \cdot \langle 1+t, 3-t, 1+2t \rangle = 0$

? how does this help - ask in OH

$(1(1+t)) + (-1(3-t)) + (2(1+2t)) = 0$

$1 + t - 3 + t + 2 + 4t = 0$

$-1 \quad +3 \quad -2$

$t + t + 4t = 0$

so need to find time when line is like that →

$6t = 0$

$t = 0$

Corresponds to $P(1, 3, 1)$

? plug in

then plug in to see lines likes that →

$\langle 1+0, 3-0, 1+2\cdot 0 \rangle$

$\langle 1, 3, 1 \rangle$

I remember qus like this - go back + look

(14) A point P moves in (r,θ) plane

So $\vec{v} = 2\vec{u}_r + 2\vec{u}_\theta$

did in recitation - lets see what I remember - also don't understand this type of qu

$t = 0$ $r = 1$ $\theta = 0$

? $r(0) = 1$? $\theta(0) = \theta$

a) How long path b/w $t=0$ and $t=3$?

$S = \int \frac{ds}{dt}\, dt$

recognize $\frac{ds}{dt} = |\vec{v}|$

$|\vec{v}| = \sqrt{2^2 + 2^2}$ ← same as before

$\sqrt{8}$

$2\sqrt{2}$

its getting started to work

$S = \int_0^3 2\sqrt{2}\, dt$

$2\sqrt{2}\, t\, dt \Big|_0^3$

$(2\sqrt{2})\cdot 3 - (2\sqrt{2})\, 0$

$6\sqrt{2}$ ✓

(15)

b) How far is it from origin at $t=3$

we want $\vec{r}(3)$ ← I'm guessing we just want $|\vec{r}(3)|$ but he shows us it they ask for the vector

so $\int |\vec{v}|$

$v = \dot{r}\,\hat{u}_r + r\dot{\theta}\,\hat{u}_\theta$

↳ know = 2 ↳ know = 2

coefficents what we had

So if $\dot{r} = 2$

$\int$

↓

$r = 2t + c$

↑ know $r(0) = 1$

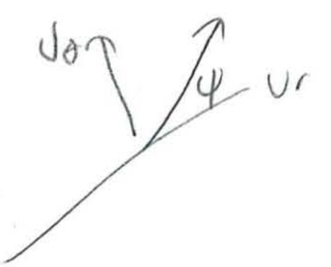
$\vec{r}(t) = 2t + 1$

So plug in $r(3) = 2(3) + 1$

7

(no need $\sqrt{a^2+b^2}$ just r!)

Now need to find angle



$\tan\left(\frac{-\hat{u}_\theta}{-u_r}\right) = \tan\left(\frac{2}{2}\right) = \frac{\pi}{4}$

need to know or beable to see

when its $\frac{\pi}{4}$, 

(16) c What is the point at time t?

We know $r(t) = 2t + 1$

We need $\theta(t)$

wasn't that the last one

$\dot{\theta} = \frac{2}{r}$ ← did not know could do that

$\dot{\theta}$ $\frac{2}{2t+1}$

$\theta = \int \frac{2}{2t+1}$

$\theta = 2(2t+1)^{-1}$

$2t \cdot \frac{(2t+1)^{0}}{0 \cdot 1}$

no $\underline{\ln}$

$\cancel{2} \ln(2t+1) + C$

← given $\theta(0) = 0$

Seperate

$X = r \cos \theta$

$= (2t+1) \cos(\ln(2t+1))$

$Y = r \sin \theta$

$= (2t+1) \sin(\ln(2t+1))$

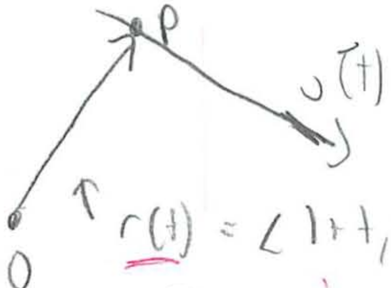
← unintuitive, but obvious

1

Oliver

~~Recitati.~~ Office Hrs

5 from quiz



$\underline{r(t)} = \langle 1+t, 3-t, 1+2t \rangle$

the r is that one

$r(t) = \vec{OP}$ "always"

* Don't want to find a cross product
- because there are many
- but know $\vec{v}$ as velocity
- want to know where $\vec{r} \perp \vec{v}$

* $\vec{v}(t)$ is the velocity of $\vec{r}$

$r(t) \cdot \langle 1, -1, 2 \rangle = 0$

find t where this is case

So take $\boxed{\text{derivitive}}$

So when we have t plug it in

to get $\vec{OP} = \vec{r}$ at that point in time

~~(or plug the~~ get 1 single pt back

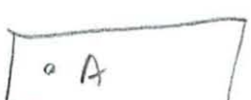
t_1 ↗
t_2 →
t_3 →

when is it $\perp$ to?

$\vec{N} = \vec{OP} \times \vec{OQ}$

$\vec{ON} \cdot \vec{N} = 0$ ← plane equation

need vector perpendicular to it



- Review this one more 1c pratie-exam

② $\vec{u} \cdot (\vec{v} \times \vec{w}) = $ Volume

? height $\cdot$ area

dot product of 1 vector × oter

$\cancel{(u \cdot v)} \times w$

$\det \begin{bmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{bmatrix}$ = volume as well = determinant

-area in 2D

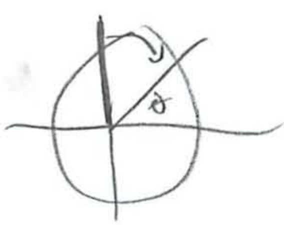
---

more examples of problems similar to scotch tape

- what is u_r u_θ
- how to express $\vec{v}$ in terms of that

$\vec{v} = \dot{r}\,\hat{u}_r + r\dot{\theta}\,\hat{u}_\theta$

? need a lot more help w



$x = a\cos\theta$
$y = a\sin\theta$

$\theta(t) = at + b$

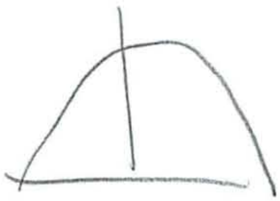
$\theta(0) = \frac{\pi}{2} = b$

$\theta(1) = \frac{\pi}{2} - 2\pi = a + b$

compute

$\theta(t) = -2\pi t + \frac{\pi}{2}$

③



$$\text{speed} = 10 = \frac{r\theta}{t}$$

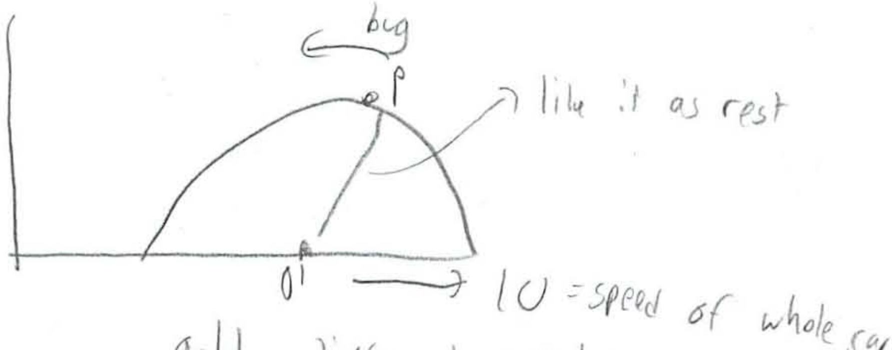
parametric eq

$$\theta(t) = 10t$$

$$x(t) = r\cos\theta = 1\cos 10t$$

$$y = r\sin\theta = \underset{\substack{\uparrow \\ \text{radius}}}{1}\sin \underset{\substack{\uparrow \\ \text{speed}}}{10}t$$

Now moving



add different vectors

$$\vec{r}(t) = \vec{OP}(t) = OO' + O'P$$

$$= \langle 10t, 0\rangle + \underset{\substack{\uparrow \\ \text{from above}}}{\langle\cos\theta, \sin\theta\rangle}$$

$1\cos 1t$
$1\sin 1t$

know this

$$= \langle 10t + \cos t, \sin t\rangle$$

and then to find max and min speed

Differentiate and $\sqrt{x^2+y^2}$

and see where largest / smalles

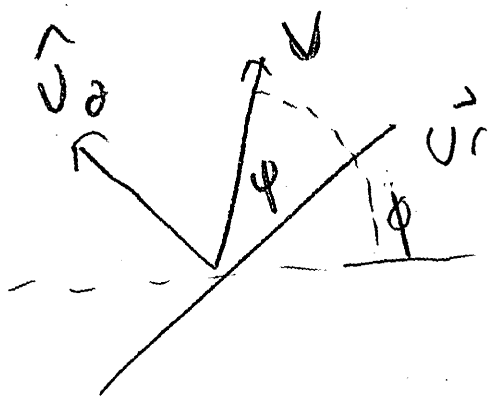
$$\vec{a} = \frac{d^2 s}{dt^2}\vec{T} + v^2 \kappa N$$

$$V = \dot{r}\,\hat{u}_r + r\dot{\theta}\,\hat{u}_\theta$$

$$\tan^{-1}\left(\frac{r}{dr/d\theta}\right) = \tan^{-1}\left(\frac{\hat{u}_\theta}{\hat{u}_r}\right)$$

↑ coords in that dir

know what direction



φ between u_r and v

$\tan \phi \; \frac{v_\theta}{v_r}$

κ thing in notes

Math Textbook

2/22

Polar Coords

much of the same
at basic level
↳ I kinda miss that basic level
and no guarentee this is on test

Parametric Curves

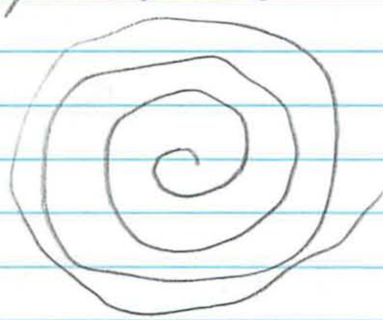
Lines tangent to parametric curve

$$\frac{dy}{dt} = \frac{dy}{dt} \cdot \frac{dx}{dt}$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{f'(t)}{g'(t)}$$

Spiral of Archimedes

$$x = a\theta \cos\theta$$
$$y = a\theta \sin\theta$$



$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$$

I feel like this is filling my head w/
stuff I don't need to know

Skip for now

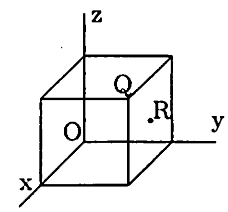
and in future prob
- how I was led astray in Sadoway 3.091

2008

18.02 Practice Exam 1 A

Problem 1. (15 points)

A unit cube lies in the first octant, with a vertex at the origin (see figure).



a) Express the vectors $\overrightarrow{OQ}$ (a diagonal of the cube) and $\overrightarrow{OR}$ (joining O to the center of a face) in terms of $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$.

b) Find the cosine of the angle between OQ and OR.

Problem 2. (10 points)

The motion of a point P is given by the position vector $\vec{R} = 3\cos t\,\hat{\imath} + 3\sin t\,\hat{\jmath} + t\,\hat{k}$. Compute the velocity and the speed of P.

Problem 3. (15 points: 10, 5)

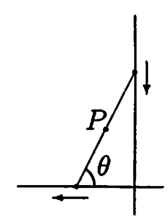
a) Let $A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix}$; then $\det(A) = 2$ and $A^{-1} = \frac{1}{2}\begin{bmatrix} 1 & a & b \\ -1 & -2 & 5 \\ 2 & 2 & -6 \end{bmatrix}$; find a and b.

b) Solve the system $AX = B$, where $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

c) In the matrix A, replace the entry 2 in the upper-right corner by c. Find a value of c for which the resulting matrix M is not invertible.
For this value of c the system $MX = 0$ has other solutions than the obvious one $X = 0$: find such a solution by using vector operations. (*Hint:* call U, V and W the three rows of M, and observe that $MX = 0$ if and only if X is orthogonal to the vectors U, V and W.)

Problem 4. (15 points)

The top extremity of a ladder of length L rests against a vertical wall, while the bottom is being pulled away. Find parametric equations for the midpoint P of the ladder, using as parameter the angle θ between the ladder and the horizontal ground.



Problem 5. (25 points: 10, 5, 10)

a) Find the area of the space triangle with vertices $P_0 : (2,1,0)$, $P_1 : (1,0,1)$, $P_2 : (2,-1,1)$.

b) Find the equation of the plane containing the three points P_0, P_1, P_2.

c) Find the intersection of this plane with the line parallel to the vector $\vec{V} = \langle 1,1,1 \rangle$ and passing through the point $S : (-1,0,0)$.

Problem 6. (20 points: 5, 5, 10)

a) Let $\vec{R} = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$ be the position vector of a path. Give a simple intrinsic formula for $\frac{d}{dt}(\vec{R}\cdot\vec{R})$ in vector notation (not using coordinates).

b) Show that if $\vec{R}$ has constant length, then $\vec{R}$ and $\vec{V}$ are perpendicular.

c) let $\vec{A}$ be the acceleration: still assuming that $\vec{R}$ has constant length, and using vector differentiation, express the quantity $\vec{R}\cdot\vec{A}$ in terms of the velocity vector only.

18.02 Practice Exam 1 A – Solutions

Problem 1.

a) $\overrightarrow{OQ} = \hat{\imath} + \hat{\jmath} + \hat{k}$; $\overrightarrow{OR} = \frac{1}{2}\hat{\imath} + \hat{\jmath} + \frac{1}{2}\hat{k}$.

b) $\cos\theta = \dfrac{\overrightarrow{OQ} \cdot \overrightarrow{OR}}{|\overrightarrow{OQ}|\,|\overrightarrow{OR}|} = \dfrac{\langle 1,1,1\rangle \cdot \langle \frac{1}{2}, 1, \frac{1}{2}\rangle}{\sqrt{3}\sqrt{\frac{3}{2}}} = \dfrac{2\sqrt{2}}{3}$.

Problem 2.

Velocity: $\vec{V} = \dfrac{d\vec{R}}{dt} = \langle -3\sin t, 3\cos t, 1\rangle$. Speed: $|\vec{V}| = \sqrt{9\sin^2 t + 9\cos^2 t + 1} = \sqrt{10}$.

Problem 3.

a) Minors: $\begin{bmatrix} 1 & 1 & 2 \\ -2 & -2 & -2 \\ -3 & -5 & -6 \end{bmatrix}$. Cofactors: $\begin{bmatrix} 1 & -1 & 2 \\ 2 & -2 & 2 \\ -3 & 5 & -6 \end{bmatrix}$. Inverse: $\dfrac{1}{2}\begin{bmatrix} 1 & \boxed{2} & \boxed{-3} \\ -1 & -2 & 5 \\ 2 & 2 & -6 \end{bmatrix}$.

b) $X = A^{-1}B = \begin{bmatrix} -3 \\ 4 \\ -4 \end{bmatrix}$.

Problem 4.

Q = top of the ladder: $\overrightarrow{OQ} = \langle 0, L\sin\theta\rangle$; R = bottom of the ladder: $\overrightarrow{OR} = \langle -L\cos\theta, 0\rangle$.
Midpoint: $\overrightarrow{OP} = \frac{1}{2}(\overrightarrow{OQ} + \overrightarrow{OR}) = \langle -\frac{L}{2}\cos\theta, \frac{L}{2}\sin\theta\rangle$.
Parametric equations: $x = -\frac{L}{2}\cos\theta$, $y = \frac{L}{2}\sin\theta$.

Problem 5.

a) $\overrightarrow{P_0P_1} \times \overrightarrow{P_0P_2} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -1 & -1 & 1 \\ 0 & -2 & 1 \end{vmatrix} = \hat{\imath} + \hat{\jmath} + 2\hat{k}$. Area $= \frac{1}{2}|\overrightarrow{P_0P_1} \times \overrightarrow{P_0P_2}| = \frac{1}{2}\sqrt{6}$.

b) Normal vector: $\overrightarrow{P_0P_1} \times \overrightarrow{P_0P_2} = \hat{\imath} + \hat{\jmath} + 2\hat{k}$. Equation: $x + y + 2z = 3$.

c) Parametric equations for the line: $x = -1 + t$, $y = t$, $z = t$.
Substituting: $-1 + 4t = 3$, $t = 1$, intersection point $(0, 1, 1)$.

Problem 6.

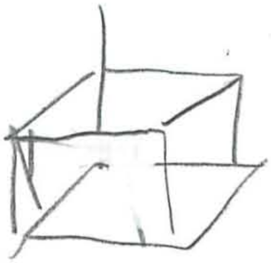
a) $\dfrac{d}{dt}(\vec{R} \cdot \vec{R}) = \vec{V} \cdot \vec{R} + \vec{R} \cdot \vec{V} = 2\vec{R} \cdot \vec{V}$.

b) Assume $|\vec{R}|$ is constant: then $\dfrac{d}{dt}(\vec{R} \cdot \vec{R}) = 2\vec{R} \cdot \vec{V} = 0$, i.e. $\vec{R} \perp \vec{V}$.

c) $\vec{R} \cdot \vec{V} = 0$, so $\dfrac{d}{dt}(\vec{R} \cdot \vec{V}) = \vec{V} \cdot \vec{V} + \vec{R} \cdot \vec{A} = 0$. Therefore $\vec{R} \cdot \vec{A} = -|\vec{V}|^2$.

18.02 2008 Pratice Exam 1A 2/22

1. Small cube



$\vec{OQ}$ diagional

$\langle 1, 1, 1 \rangle$ $\hat{i} + \hat{j} + \hat{k}$ ← same qu as 3.091

$\vec{Or}$ $\langle \frac{1}{2}, 1, \frac{1}{2} \rangle$

$\frac{1}{2}\hat{i} + \hat{j} + \frac{1}{2}\hat{k}$

b. Cosine of angle

- involves $A \cdot B = |A||B| \cos\theta =$

$(1 \cdot \frac{1}{2}) + (1 \cdot 1) + (\frac{1}{2} \cdot 1)$

$\frac{1}{2} + 1 + \frac{1}{2}$

2

$\boxed{\sqrt{n} \cdot \sqrt{n} = n}$

$\sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$

$\sqrt{\frac{1}{2}^2 + 1 + \frac{1}{2}^2} = \sqrt{1.5} = \sqrt{\frac{3}{2}}$

$2 = \sqrt{3}\sqrt{1.5} \cos\theta$

$\cos\theta = \frac{2}{\sqrt{3} \cdot \sqrt{1.5}} = \frac{2\sqrt{2}}{3}$ ← so I missed simplifing furter

- and no roots in denom

Can't simplify furter

$\frac{2\sqrt{3}}{3\sqrt{1.5}} = \frac{2\sqrt{3}\sqrt{\frac{3}{2}}}{3 \cdot \frac{3}{2}} =$ idk

② The motion of p is $\vec{r} = 3\cos t\,\hat{i} + 3\sin t\,\hat{j} + t\hat{k}$

- helix

d sin = cos
d cos = -sin

$\vec{v} = \langle -3\sin t, 3\cos t, 1 \rangle$ ✓

$|\vec{v}| = \sqrt{(-3)^2 \sin^2 t + 3^2 \cos t^2 + 1^2}$

$\sqrt{9\sin^2 t + 9\cos^2 t + 1}$

$\sqrt{9 + 1}$

$\sqrt{10}$ ← speed is just that remember

$\cancel{\dfrac{\langle -3\sin t, 3\cos t, 1 \rangle}{\sqrt{10}}}$

this is going pretty well
Jay said real is just like pratice

3. $A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix}$ $\det(A) = 2$

$A^{-1} = \frac{1}{2} \begin{bmatrix} 1 & a & b \\ -1 & -2 & 5 \\ 2 & 2 & 6 \end{bmatrix}$ Find b + c

- oh still have to ~~the~~ do the work
  - ~~better~~ you work backwards
  - the better you know it, the faster it goes.

(3)

$$\begin{bmatrix} 1 & -1 & 2 \\ a & -2 & 2 \\ b & 5 & -6 \end{bmatrix}$$

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 2 \\ -a & -2 & -2 \\ b & -5 & -6 \end{bmatrix}$$

$$-a = \begin{vmatrix} 3 & 2 \\ 1 & 0 \end{vmatrix} = -(3 \cdot 0 - 2 \cdot 1)$$

$$2$$

$$b = \begin{vmatrix} 3 & 2 \\ 0 & -1 \end{vmatrix} = (3 \cdot -1 - 2 \cdot 0)$$

$$-3 - 0$$

$$\cancel{-6} \quad -3$$

← duh just a little 2·0 error here stupid

b. $AX = B$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$A X A^{-1} = B A^{-1}$$

$$X = B A^{-1}$$

in the zone now that's b/c "winning" - knowing them

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 2 & -3 \\ -1 & -2 & 5 \\ 2 & 2 & -6 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 2 & -3 \\ -1 & -2 & 5 \\ 2 & 2 & -6 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1\cdot 1 + 2\cdot -2 + -3\cdot 1 \\ -1\cdot 1 + -2\cdot -2 + 5\cdot 1 \\ 2\cdot 1 + 2\cdot -2 + -6\cdot 1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1-4-3 \\ -1+4+5 \\ 2-4-6 \end{bmatrix}$$

copy error was thinking ⊖ too stupid

$$= \frac{1}{2}\begin{bmatrix} -6 \\ 8 \\ -8 \end{bmatrix}$$

$$= \begin{bmatrix} -3 \\ 4 \\ -4 \end{bmatrix}$$

c) In matrix A $\begin{bmatrix} 1 & 3 & c \\ 2 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix}$

Find c where matrix not invertable

- opps did not study this

Hint $MX = 0$ has other solutions than $X = 0$

$\begin{bmatrix} U \\ V \\ W \end{bmatrix}$ $MX = 0$ if X is orthogonal to U
↑ right angle

- So if they are providing the hint did that mean we did not learn it ?

(5) Invertable Matrix WP

$$AB = BA = I$$

identiy

- So usually matrix is invertiable
- I (I think)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ (1) only on main diagonal

No solution on sheet ....
from notes Idenity

$AB = I$ $\quad$ $Ix = x$

$I / A = B$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & c \\ 2 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix} = [$$

← thats not how they say

(7?) Inverse Notes

$AM = I$

$MA = I$

$M = A^{-1}$

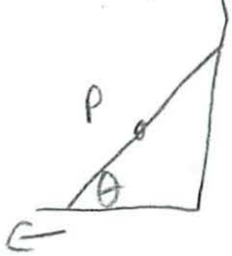
and then it talks about solving linear systems

I'm guessing this is the novel av.

See Sahar help notes

Not inversable = when det = 0

6 4. I was never good at these - lets see how it goes



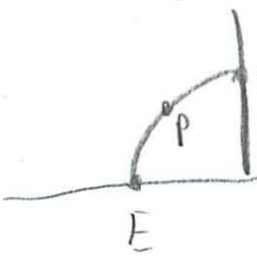
Ladder lenght L on wall

Parametric eq for pt P w/ θ

So $t = 0$

$\vec{R}(0) = (0, L/2)$ $\theta = \frac{y}{x} = \frac{L/2}{0} =$ invalid

$R(\text{end}) = (\frac{L}{2}, 0)$ $\theta = \frac{0}{L/2} = 0$



? vector method ? ? when is this ??

$\vec{OS} + \vec{SP}$

$(0, \frac{L}{2}) + \langle -\cos\theta, \sin\theta \rangle$

? so at θ 90

$-0, 1$

multiply $L/2$ → multiply

$\langle -\frac{L}{2}\cos\theta, \frac{L}{2}\sin\theta \rangle$

? at $\theta = 90$

$\langle 0, \frac{L}{2} - 0 \rangle$

So didn't really get - hopefully started right

$Q = \langle 0, L\sin\theta \rangle$

R $\langle -L\cos\theta, 0 \rangle$

Midpoint $= \frac{1}{2}(OQ + OR) =$

$= \langle -\frac{L}{2}\cos\theta, \frac{L}{2}\sin\theta \rangle$

↑ x ↑ y

⑦ So what did I do wrong

- they used top + bottom
- then found mid in between
- but I could have done that too

* no need to write $\frac{L}{2}$ → write $\frac{1}{2}$ *

* ~~should not use L~~

- so was closer than I thought
- have it multiply not subtract from L
  - why did I not realize that ??

$$\left\langle -\frac{L}{2}\cos\theta, \frac{L}{2}\sin\theta \right\rangle$$

* don't try to add vectors
since O is the pivot pt *

And write it as parametric equation

$$x = -\frac{L}{2}\cos\theta$$
$$y = \frac{L}{2}\cos\theta$$

5. Find the area of the space triangle

- what I learned in Office hrs today

$$\det\begin{vmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ 2 & -1 & 1 \end{vmatrix} = 2(0--1) - 1(1-2) + 0$$

$2+1$

$\cancel{3}$ nope thats not it

?: cube $\frac{1}{2}\sqrt{6} \approx 1.22$

(8) So why did that not work
- since triangle

their strategy
· cross multiply 2 lines
and then $\cdot \frac{1}{2}$

$\overrightarrow{P_0P_1} = \langle -1, -1, 1 \rangle$

$\overrightarrow{P_0P_2} = \langle 0, -2, 1 \rangle$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & -1 & 1 \\ 0 & -2 & 1 \end{vmatrix}$$ ✓

matrix errors
grr

$(-1-\cancel{2})\hat{i} - (-1-0)\hat{j} + (2-\cancel{0})\hat{k}$

$\cancel{1}\hat{i} + 1\hat{j} + \cancel{3}\hat{k}$ $\quad \hat{i} + \hat{j} + 2\hat{k}$

$\frac{1}{2} \left| +\cancel{1}\hat{i} + \hat{j} + \cancel{3}\hat{k} \right|$

$\frac{1}{2}\sqrt{\cancel{1}^2 + 1^2 + \cancel{3}^2}$

$\frac{1}{2}\sqrt{\cancel{11}\ 6}$

$\cancel{\frac{1}{2}\sqrt{\ }}$

b. Find the equation of a plane containing the vectors
- so the eq of a plane is just a line ⊥ to it
* I know this

→

1)

$\cancel{\langle -1,-1,1 \rangle \cdot \langle 0,-2,1 \rangle}$

$\cancel{(-1\cdot 0)+(-1\cdot -2)+(1\cdot 1)}$

that is to v
we want to find

$\hat{N} = \hat{i} + \hat{j} + 2\hat{k}$ we did

v) $\langle -1,-1,1 \rangle \cdot \langle 1,1,2 \rangle = 0$

$(-1\cdot 1)+(-1\cdot 1)+(1\cdot 2) = 0$

$-1 \quad -1 \quad +2 \quad = 0$

v

need eq of plane

oh- plug in a pt

$1(x+1)+1\cdot(y+1)+2(z-1) = 0$

$x + 1 + y + 1 + 2z - 2 = 0$

$\quad\; -1 \qquad\; -1 \qquad\quad +2$

done wright $(x-1)$

$x + y + 2z = 0$

nope not that

Should be $x + y + 2z = 3$

where does this come from?

$x +$

(10) c Find the intersection of this plane $x+y+2z=3$

w/ line parralle l $\vec{v}\ \langle 1,1,1\rangle$ passing through $(-1,0,0)$

$1(x+1)+y+2z=3$

$x+1+y+2z=3$

$x+y+2z=2$ So how is parallel to that?

? Solutions don't make sense

parametric $x=-1+t$

$y=t$

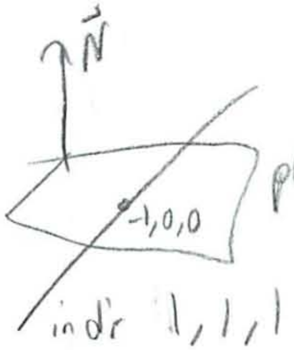
$z=t$

Sub in $-1+4t=3$

$t=3$

$(0,1,1)$ intersection

try and picture



in

So $-x+y+2$

(11) Ok so take

line in that dir

$\langle -1+t, t, t \rangle$ ← so is it

plane $\vec{N}$ ↙pt

$\langle a, b, c \rangle \; (d, e, f)$

$(-1+t) + (t) + 2(t) = 3$

$x = d + at,$

$y = e + bt$

$z = f + ct$

Plug into

$-1 + t + t + 2t = 3$

$4t = 4$

$t = 1$

plug back in

~~$-1 + 1 + 1 + 2(1) = 3$~~

$\langle -1+1, 1, 1 \rangle$

$(0, 1, 1)$ point

I don't think I will be able to navigate the subtities of this type of problem

6. $\vec{R} = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$

simple intrinic formula

↑ did we do?

for $\frac{d}{dt}(R \cdot R)$

↑ so $R \cdot R$ is something special I think.

No do deriv w/ cross product

$R \cdot V + V \cdot R = 2R \cdot V$

(12) But how were you supposed to know that?

b) Show that $\vec{R} \perp \vec{V}$

- just find deriv
- dot product $= 0$
- but no #, so how supposed to show?

$2 R \cdot V = 0$

but how do you know that

and how does $\frac{d}{dt}(R \cdot R)$ factor into this?

c) Let $\vec{A}$ be acceleration

express $\vec{R} \cdot \vec{A}$ in terms of V only

$\int \vec{R} \cdot \frac{d}{dt} \vec{A}$

$\vec{R} \cdot \vec{V} = 0$ (proved above)

$\frac{d}{dt} R \cdot V = V \cdot V + R \cdot A = 0$

chain rule?

$\vec{R} \cdot \vec{A} = -|\vec{V}|^2$

I don't get why you would write that

18.02 Practice Exam 1 B

Problem 1.

Let P, Q and R be the points at 1 on the x-axis, 2 on the y-axis and 3 on the z-axis, respectively.

a) (6) Express $\overrightarrow{QP}$ and $\overrightarrow{QR}$ in terms of $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$.

b) (9) Find the cosine of the angle PQR.

Problem 2. Let $P = (1,1,1)$, $Q = (0,3,1)$ and $R = (0,1,4)$.

a) (10) Find the area of the triangle PQR.

b) (5) Find the plane through P, Q and R, expressed in the form $ax + by + cz = d$.

c) (5) Is the line through $(1,2,3)$ and $(2,2,0)$ parallel to the plane in part (b)? Explain why or why not.

Problem 3. A ladybug is climbing on a Volkswagen Bug (= VW). In its starting position, the the surface of the VW is represented by the unit semicircle $x^2 + y^2 = 1$, $y \geq 0$ in the xy-plane. The road is represented as the x-axis. At time $t = 0$ the ladybug starts at the front bumper, $(1,0)$, and walks counterclockwise around the VW at unit speed relative to the VW. At the same time the VW moves to the right at speed 10.

a) (15) Find the parametric formula for the trajectory of the ladybug, and find its position when it reaches the rear bumper. (At $t = 0$, the rear bumper is at $(-1,0)$.)

b) (10) Compute the speed of the bug, and find where it is largest and smallest. Hint: It is easier to work with the square of the speed.

Problem 4.

$$M = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & -1 & -1 \end{pmatrix} \qquad M^{-1} = \frac{1}{12}\begin{pmatrix} 1 & 1 & 4 \\ a & 7 & -8 \\ b & -5 & 4 \end{pmatrix}$$

(a) (5) Compute the determinant of M.

b) (10) Find the numbers a and b in the formula for the matrix M^{-1}.

c) (10) Find the solution $\vec{r} = \langle x, y, z \rangle$ to $\begin{aligned} x + 2y + 3z &= 0 \\ 3x + 2y + z &= t \\ 2x - y - z &= 3 \end{aligned}$ as a function of t.

d) (5) Compute $\dfrac{d\vec{r}}{dt}$.

Problem 5.

(a) (5) Let $P(t)$ be a point with position vector $\vec{r}(t)$. Express the property that $P(t)$ lies on the plane $4x - 3y - 2z = 6$ in vector notation as an equation involving $\vec{r}$ and the normal vector to the plane.

(b) (5) By differentiating your answer to (a), show that $\dfrac{d\vec{r}}{dt}$ is perpendicular to the normal vector to the plane.

18.02 Practice Exam 1 B Solutions

Problem 1.

a) $P = (1,0,0)$, $Q = (0,2,0)$ and $R = (0,0,3)$. Therefore $\overrightarrow{QP} = \hat{\imath} - 2\hat{\jmath}$ and $\overrightarrow{QR} = -2\hat{\jmath} + 3\hat{k}$.

b) $\cos\theta = \dfrac{\overrightarrow{QP}\cdot\overrightarrow{QR}}{\left|\overrightarrow{QP}\right|\left|\overrightarrow{QR}\right|} = \dfrac{\langle 1,-2,0\rangle\cdot\langle 0,-2,3\rangle}{\sqrt{1^2+2^2}\sqrt{2^2+3^2}} = \dfrac{4}{\sqrt{65}}$

Problem 2.

a) $\overrightarrow{PQ} = \langle -1,2,0\rangle$, $\overrightarrow{PR} = \langle -1,0,3\rangle$.

$$\overrightarrow{PQ}\times\overrightarrow{PR} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -1 & 2 & 0 \\ -1 & 0 & 3 \end{vmatrix} = 6\hat{\imath} + 3\hat{\jmath} + 2\hat{k}.$$

Then $\quad area(\Delta) = \dfrac{1}{2}\left|\overrightarrow{PQ}\times\overrightarrow{PR}\right| = \dfrac{1}{2}\sqrt{6^2+3^2+2^2} = \dfrac{1}{2}\sqrt{49} = \dfrac{7}{2}.$

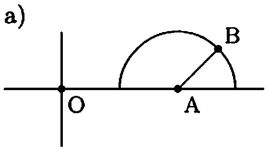
b) A normal to the plane is given by $\overrightarrow{N} = \overrightarrow{PQ}\times\overrightarrow{PR} = \langle 6,3,2\rangle$. Hence the equation has the form $6x + 3y + 2z = d$. Since P is on the plane $d = 6\cdot 1 + 3\cdot 1 + 2\cdot 1 = 11$. In conclusion the equation of the plane is

$$6x + 3y + 2z = 11.$$

c) The line is parallel to $\langle 2-1, 2-2, 0-3\rangle = \langle 1,0,-3\rangle$. Since $\overrightarrow{N}\cdot\langle 1,0,-3\rangle = 6-6 = 0$, the line is parallel to the plane.

Problem 3.

a)



$\overrightarrow{OA} = \langle 10t, 0\rangle$ and $\overrightarrow{AB} = \langle \cos t, \sin t\rangle$, hence

$$\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB} = \langle 10t + \cos t, \sin t\rangle.$$

The rear bumper is reached at time $t = \pi$ and the position of B is $(10\pi - 1, 0)$.

b) $\overrightarrow{V} = \langle 10 - \sin t, \cos t\rangle$, thus

$$|\overrightarrow{V}|^2 = (10-\sin t)^2 + \cos^2 t = 100 - 20\sin t + \sin^2 t + \cos^2 t = 101 - 20\sin t.$$

The speed is then given by $\sqrt{101 - 20\sin t}$. The speed is smallest when $\sin t$ is largest i.e. $\sin t = 1$. It occurs when $t = \pi/2$. At this time, the position of the bug is $(5\pi, 1)$. The speed is largest when $\sin t$ is smallest; that happens at the times $t = 0$ or π for which the position is then $(0,0)$ and $(10\pi - 1, 0)$.

Problem 4.

a) $|M| = -12$.

b) $a = -5$, $b = 7$.

c) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \dfrac{1}{12}\begin{bmatrix} 1 & 1 & 4 \\ -5 & 7 & -8 \\ 7 & -5 & 4 \end{bmatrix}\begin{bmatrix} 0 \\ t \\ 3 \end{bmatrix} = \begin{bmatrix} t/12 + 1 \\ 7t/12 - 2 \\ -5t/12 + 1 \end{bmatrix}$

d) $\dfrac{d\vec{r}}{dt} = \left\langle \dfrac{1}{12}, \dfrac{7}{12}, -\dfrac{5}{12}\right\rangle.$

Problem 5.

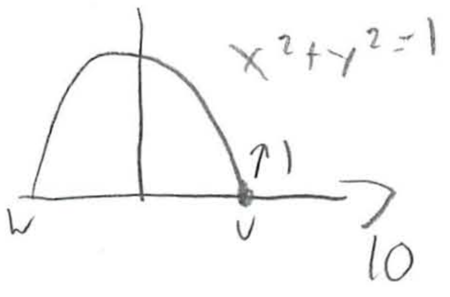
a) $\overrightarrow{N}\cdot\vec{r}(t) = 6$, where $\overrightarrow{N} = \langle 4,-3,-2\rangle$.

b) We differentiate $\overrightarrow{N}\cdot\vec{r}(t) = 6$:

$0 = \dfrac{d}{dt}\left(\overrightarrow{N}\cdot\vec{r}(t)\right) = \dfrac{d}{dt}\overrightarrow{N}\cdot\vec{r}(t) + \overrightarrow{N}\cdot\dfrac{d}{dt}\vec{r}(t) = \vec{0}\cdot\vec{r}(t) + \overrightarrow{N}\cdot\dfrac{d}{dt}\vec{r}(t)$ and hence $\overrightarrow{N}\perp\frac{d}{dt}\vec{r}(t)$.

will do 3 + 5

3. Started in Office Hrs



$OP = OV + VP$

$\langle 10t, 0 \rangle + \langle 1\cos\theta, 1\sin\theta \rangle$

look at each individualy

$\vec{OP} = \langle 10t + \cos\theta, \sin\theta \rangle$

?: how to reconcile t and θ

~~$\theta = \frac{y}{x} = \frac{\sqrt{1-x^2}}{\sqrt{1-y^2}}$~~

just write $\theta = t$ as the parameter

don't get tripped up

When is bug at rear bumper

So when is $\langle \cos\theta, \sin\theta \rangle = \langle -1, 0 \rangle$

at $\theta = \pi$

So where is total → calculate

$\langle 10\pi + \cos\pi, \sin\pi \rangle$ $\langle 10\pi - 1, 0 \rangle$

$\frac{d}{} \sin = \cos$
$\frac{d}{} \cos = -\sin$

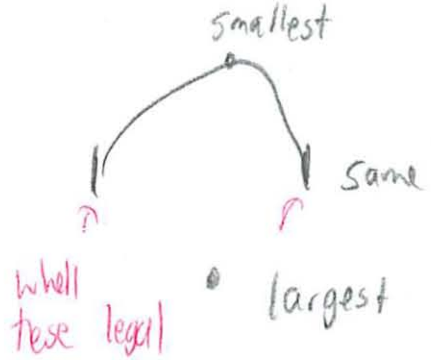
b. Speed of bug

- integrate

$\langle 10t \quad t\cos\theta, \sin\theta \rangle$

$\langle 10 + -\sin\theta, \cos\theta \rangle$

$\langle 10 - \sin\theta, \cos\theta \rangle$



Smallest $\frac{\pi}{2}$

$\langle 10 - 1, 0 \rangle$

largest $\frac{3\pi}{2}$

$\langle 10 + 1, 0 \rangle$

Speed is $|\vec{v}|$ → so $\sqrt{x^2 + y^2}$

(not fixing now)

5. $p(t)$ is point on $\vec{OP} = r(t)$

p lies on $4x - 3y - 2z = 6$

w/ eq w/ $\vec{r}$ and $\vec{N}$

so $r(t)\hat{i} + 4x - (r(t)\hat{j} + 3y + r(t)\hat{k} - 2z) = 6$

$\vec{N} \cdot \vec{r}(t) = 6$

$\vec{N} = \langle 4, -3, 2 \rangle$

oh from OH - just differentiate to find
Velocity of that - again screwed up this type of problem

By differentiating ans to a show $\frac{dr}{dt}$ is $\perp$ to plane

$\langle 4, -3, 2 \rangle$

?? how can we $\frac{dr}{dt}$?

$\frac{d}{dt}(N \cdot r(t))$

$N' \cdot r(t) + r(t)' \cdot N$

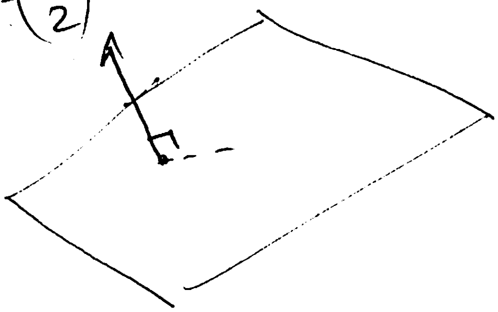
$\cancel{0 \cdot r(t)} + N \cdot \frac{dr(t)}{dt}$

$\vec{N} \perp \frac{d}{dt}\vec{r(t)}$

??? don't get

$x + y + 2z = 3$ Sahor Help Coefficents plane = $\hat{N}$

$\vec{n} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$



$\vec{r}(x,y,z) = \langle -1, 0, 0 \rangle + t \cancel{2} \langle 1, 1, 1 \rangle$

offset

$x = -1 + \cancel{2} t$
$y = t$
$z = t$

get in terms of 1 variable

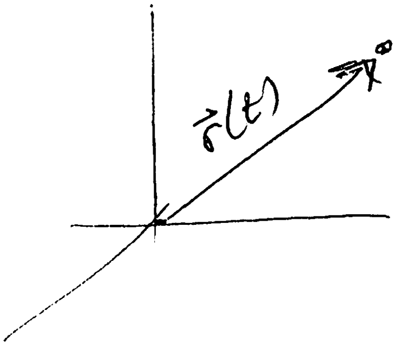
$\cancel{z}$

$(-1 + t) + (t) + (2t) = 3$ ← general solution

$4t = 4 \longrightarrow \boxed{t = 1}$ solve for t

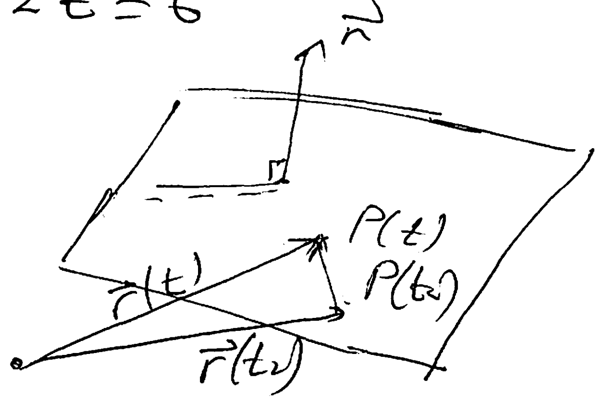
find intersection

Point $\begin{pmatrix} \overset{-1+t}{0}, & \overset{t}{1}, & \overset{t}{1} \end{pmatrix}$ ← specific solution



$4x - 3y - 2z = 6$

$\vec{n} = \begin{pmatrix}4\\-3\\-2\end{pmatrix}$



~~P(t)~~

$\left(\vec{r}(t_2) - \vec{r}(t_1)\right)\cdot\vec{n} = 0$

$\cancel{\vec{r}(t_2)}$

$\boxed{\vec{n}\cdot\vec{r}(t_2) = \vec{n}\cdot\vec{r}(t_1)}$

$\cancel{\vec{n}}\cdot$

$\vec{r}(t) = \begin{pmatrix}x\\y\\z\end{pmatrix}\cdot\begin{pmatrix}4\\-3\\-2\end{pmatrix} = 6$

$P(t)$ $\qquad$ $\uparrow\vec{N}$

(6) $\vec{N}\cdot\vec{r}(t) = 6$

$$\frac{d\vec{N}}{dt}\cdot\vec{r}(t) + \frac{d\vec{r}(t)}{dt}\cdot\vec{N} = 0$$

$$0 + \boxed{\frac{d\vec{r}(t)}{dt}\cdot\vec{N} = 0}$$

↑Normal is constant $d = 0$

perpendicular

$\vec{v} = \frac{dr}{dt}$

$$\begin{bmatrix} 1 & 3 & c \\ 2 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix}$$

not invertible = singular

not invertiable = singular

= det(A) = 0

$\cancel{\text{det}}$ $\frac{1}{\det(A)} = \frac{1}{0}$ = opps = AA

Do det

Set = 0

Solve for C

$$\frac{2}{\sqrt{3}\sqrt{\frac{3}{2}}}$$

$\cancel{\frac{2}{\sqrt{3}}}$

the algebra

$$\frac{2}{\frac{3}{\sqrt{2}}} = \frac{2\sqrt{2}}{3}$$

$$\frac{2}{\bigcirc}$$

$$\frac{2}{3} \cdot \frac{\sqrt{2}}{1} = \frac{2\sqrt{2}}{3}$$

└see clarification depends on where ( )

$$\boxed{\frac{2}{1/2} \quad \frac{2}{1} \cdot \frac{2}{1} = \frac{4}{1} = 4}$$

$\cancel{*}$

$$\sqrt{n} \cdot \sqrt{n} = n$$

$$\sqrt{3} \cdot \sqrt{3} \cdot \sqrt{\tfrac{1}{2}}$$

$$3\sqrt{\tfrac{1}{2}}$$

$$\frac{3}{\sqrt{2}}$$

all legal

(a) $\frac{d\vec{R}}{dt}\cdot\vec{R} + \frac{d\vec{R}}{dt}\cdot\vec{R} = 2\,\frac{d\vec{R}}{dt}\cdot\vec{R}$

$2\vec{V}\cdot\vec{R} = 2\vec{V}\vec{R}.$

(b) $\cancel{2\vec{V}\vec{R} = 0}$

$2\underline{V}R = 0$ - becase if R constant

$\frac{dR}{dt} = 0$

$\boxed{\vec{V}\cdot\vec{R} = 0}$

$0\cdot R + 0\cdot R$

0

$\vec{V} \perp \vec{R}$

(c) $\vec{V}\cdot\vec{R} = 0$

$\frac{d(\vec{V}\cdot\vec{R})}{dt} = \frac{d\vec{V}}{dt}\cdot\vec{R} + \cancel{\frac{d\vec{R}}{dt}\cdot\vec{V}}$ (constant) $\vec{V}\cdot\vec{V} = 0$

$\cancel{\vec{A}\cdot\vec{R} = 0}$

$\vec{R}\cdot\vec{A} = -|\vec{V}|^2$

$\vec{R}\cdot\vec{A} + \vec{V}\cdot\vec{V} = 0$

$a + b = 0$

$\boxed{a = -b}$

$\vec{R}\cdot\vec{A} = -\vec{V}\cdot\vec{V}$

$R\cdot\frac{dv}{dt} = -V\cdot V$

$-|V|^2$

Clarification (Jay)

$$\frac{\left(\frac{x}{\sqrt{2}}\right)}{\sqrt{2}} \qquad \frac{x}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}$$

$$\left(\frac{\frac{x}{\sqrt{2}}}{\sqrt{2}}\right) \qquad \frac{x}{\sqrt{2}} \cdot \frac{\sqrt{2}}{1}$$