Math + Physics Review

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Maxwell's equations

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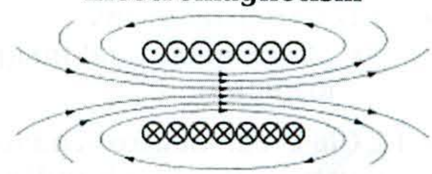
Maxwell's equations are a set of four partial differential equations that relate the electric and magnetic fields to their sources, charge density and current density. These equations can be combined to show that light is an electromagnetic wave. Individually, the equations are known as Gauss's law, Gauss's law for magnetism, Faraday's law of induction, and Ampère's law with Maxwell's correction. The set of equations is named after James Clerk Maxwell.

These four equations, together with the Lorentz force law are the complete set of laws of classical electromagnetism. The Lorentz force law itself was actually derived by Maxwell under the name of *Equation for Electromotive Force* and was one of an earlier set of eight equations by Maxwell.

Electromagnetism



Electricity · Magnetism

Electrostatics

Electric charge · Coulomb's law · Electric field · Electric flux · Gauss's law · Electric potential · Electrostatic induction · Electric dipole moment · Polarization density

Magnetostatics

Ampère's law · Electric current · Magnetic field · Magnetization · Magnetic flux · Biot–Savart law · Magnetic dipole moment · Gauss's law for magnetism

Electrodynamics

Free space · Lorentz force law · emf · Electromagnetic induction · Faraday's law · Lenz's law · Displacement current · **Maxwell's equations** · EM field · Electromagnetic radiation · Liénard-Wiechert Potential · Maxwell tensor · Eddy current

Electrical Network

Electrical conduction · Electrical resistance · Capacitance · Inductance · Impedance · Resonant cavities · Waveguides

Covariant formulation

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Electromagnetic tensor · EM Stress-energy tensor · Four-current · Electromagnetic four-potential

Scientists

Ampère · Coulomb · Faraday · Gauss · Heaviside · Henry · Hertz · Lorentz · Maxwell · Tesla · Volta · Weber · Ørsted

Conceptual description

This section will conceptually describe each of the four Maxwell's equations, and also how they link together to explain the origin of electromagnetic radiation such as light. The exact equations are set out in later sections of this article.

- Gauss' law describes how an electric field is generated by electric charges: The electric field tends to point away from positive charges and towards negative charges. More technically, it relates the electric flux through any hypothetical closed "Gaussian surface" to the electric charge within the surface.

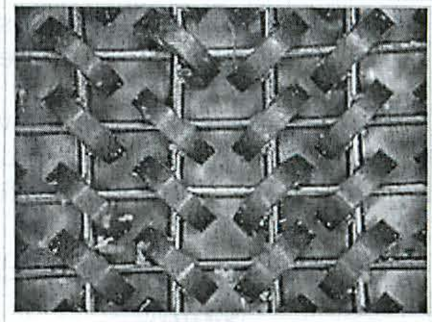
- Gauss' law for magnetism states that there are no "magnetic charges" (also called magnetic monopoles), analogous to electric charges.[1] Instead the magnetic field is generated by a configuration called a dipole, which has no magnetic charge but resembles a positive and negative charge inseparably bound together. Equivalent technical statements are that the total magnetic flux through any Gaussian surface is zero, or that the magnetic field is a solenoidal vector field.

allwork?

key difference I've missed in past is difference b/w work + flux
—Complete opposite!

Come to realitization math is more about patterns — not like HS

? other way around



An Wang's magnetic core memory (1954) is an application of Ampere's law. Each core stores one bit of data.

- Faraday's law describes how a changing magnetic field can create ("induce") an electric field.[1] This aspect of electromagnetic induction is the operating principle behind many electric generators: A bar magnet is rotated to create a changing magnetic field, which in turn generates an electric field in a nearby wire. (Note: The "Faraday's law" that occurs in Maxwell's equations is a bit different than the version originally written by Michael Faraday. Both versions are equally true laws of physics, but they have different scope, for example whether "motional EMF" is included. See Faraday's law of induction for details.)
- Ampère's law with Maxwell's correction states that magnetic fields can be generated in two ways: by electrical current (this was the original "Ampère's law") and by changing electric fields (this was "Maxwell's correction").

Solenoid problem

Maxwell's correction to Ampère's law is particularly important: It means that a changing magnetic field creates an electric field, *and* a changing electric field creates a magnetic field.[1][2] Therefore, these equations allow self-sustaining "electromagnetic waves" to travel through empty space (see electromagnetic wave equation).

The speed calculated for electromagnetic waves, which could be predicted from experiments on charges and currents,[note 1] exactly matches the speed of light; indeed, light *is* one form of electromagnetic radiation (as are X-rays, radio waves, and others). Maxwell understood the connection between electromagnetic waves and light in 1864, thereby unifying the previously-separate fields of electromagnetism and optics.

General formulation

The equations in this section are given in SI units. Unlike the equations of mechanics (for example), Maxwell's equations are *not* unchanged in other unit systems. Though the general form remains the same, various definitions get changed and different constants appear at different places. Other than SI (used in engineering), the units commonly used are Gaussian units (based on the cgs system and considered to have some theoretical advantages over SI[3]), Lorentz-Heaviside units (used mainly in particle physics) and Planck units (used in theoretical physics). See below for CGS-Gaussian units.

Two equivalent, general formulations of Maxwell's equations follow. The first separates bound charge and bound current (which arise in the context of dielectric and/or magnetized materials) from free charge and free current (the more conventional type of charge and current). This separation is useful for calculations involving dielectric or magnetized materials. The second formulation treats all charge equally, combining free and bound charge into *total* charge (and likewise with current). This is the more fundamental or microscopic point of view, and is particularly useful when no dielectric

think I know exactly what to study → good or magnetic material is present. More details, and a proof that these two formulations are mathematically equivalent, are given in section 4.

Symbols in **bold** represent vector quantities, and symbols in *italics* represent scalar quantities. The definitions of terms used in the two tables of equations are given in another table immediately following.

Formulation in terms of *free* charge and current

Name	Differential form	Integral form
Gauss's law	$\nabla \cdot \mathbf{D} = \rho_f$	$\oiint_{\partial V} \mathbf{D} \cdot \mathrm{d}\mathbf{A} = Q_f(V)$
Gauss's law for magnetism	$\nabla \cdot \mathbf{B} = 0$	$\oiint_{\partial V} \mathbf{B} \cdot \mathrm{d}\mathbf{A} = 0$
Maxwell–Faraday equation (Faraday's law of induction)	$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	$\oint_{\partial S} \mathbf{E} \cdot \mathrm{d}\mathbf{l} = -\frac{\partial \Phi_{B,S}}{\partial t}$
Ampère's circuital law (with Maxwell's correction)	$\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}$	$\oint_{\partial S} \mathbf{H} \cdot \mathrm{d}\mathbf{l} = I_{f,S} + \frac{\partial \Phi_{D,S}}{\partial t}$

Formulation in terms of *total* charge and current[note 2]

Name	Differential form	Integral form
Gauss's law	$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$	$\oiint_{\partial V} \mathbf{E} \cdot \mathrm{d}\mathbf{A} = \frac{Q(V)}{\varepsilon_0}$
Gauss's law for magnetism	$\nabla \boxed{\cdot} \mathbf{B} = 0$ large difference!	$\oiint_{\partial V} \mathbf{B} \cdot \mathrm{d}\mathbf{A} = 0$
Maxwell–Faraday equation (Faraday's law of induction)	$\nabla \boxed{\times} \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$ - realize	$\oint_{\partial S} \mathbf{E} \cdot \mathrm{d}\mathbf{l} = -\frac{\partial \Phi_{B,S}}{\partial t}$
Ampère's circuital law (with Maxwell's correction)	$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$	$\oint_{\partial S} \mathbf{B} \cdot \mathrm{d}\mathbf{l} = \mu_0 I_S + \mu_0 \varepsilon_0 \frac{\partial \Phi_{E,S}}{\partial t}$

our study

The following table provides the meaning of each symbol and the SI unit of measure:

Of course written differently

Definitions and units

Symbol	Meaning (first term is the most common)	SI Unit of Measure
$\mathbf{E}$	electric field	volt per meter or, equivalently, newton per coulomb
$\mathbf{B}$	magnetic field also called the magnetic induction also called the magnetic field density also called the magnetic flux density	tesla, or equivalently, weber per square meter, volt-second per square meter
$\mathbf{D}$	electric displacement field also called the electric induction also called the electric flux density	coulombs per square meter or equivalently, newton per volt-meter
$\mathbf{H}$	magnetizing field also called auxiliary magnetic field also called magnetic field intensity also called magnetic field	ampere per meter
$\nabla\cdot$	the divergence operator Study difference!	per meter (factor contributed by applying either operator)
$\nabla\times$	the curl operator	
$\frac{\partial}{\partial t}$	partial derivative with respect to time	per second (factor contributed by applying the operator)
$d\mathbf{A}$ really understand diff + S	differential vector element of surface area A, with infinitesimally small magnitude and direction normal to surface S	square meters
$d\mathbf{l}$	differential vector element of *path length* tangential to the path/curve	meters
ε_0 - think I kinda do know	permittivity of free space, also called the electric constant, a universal constant	farads per meter
μ_0	permeability of free space, also called the magnetic constant, a universal constant	henries per meter, or newtons per ampere squared
ρ_f	free charge density (not including bound charge)	coulombs per cubic meter
ρ	total charge density (including both free and bound charge)	coulombs per cubic meter
$\mathbf{J}_f$	free current density (not including bound current)	amperes per square meter
$\mathbf{J}$	total current density (including both free and bound current)	amperes per square meter
$Q_f(V)$	net free electric charge within the three-dimensional volume V (not including bound charge)	coulombs

$Q(V)$	net electric charge within the three-dimensional volume V (including both free and bound charge)	coulombs
$\oint_{\partial S} \mathbf{E} \cdot \mathrm{d}\mathbf{l}$	line integral of the electric field along the boundary ∂S of a surface S (∂S is always a closed curve).	joules per coulomb
$\oint_{\partial S} \mathbf{B} \cdot \mathrm{d}\mathbf{l}$	line integral of the magnetic field over the closed boundary ∂S of the surface S	tesla-meters
$\oiint_{\partial V} \mathbf{E} \cdot \mathrm{d}\mathbf{A}$	the electric flux (surface integral of the electric field) through the (closed) surface ∂V (the boundary of the volume V)	joule-meter per coulomb
$\oiint_{\partial V} \mathbf{B} \cdot \mathrm{d}\mathbf{A}$	the magnetic flux (surface integral of the magnetic B-field) through the (closed) surface ∂V (the boundary of the volume V)	tesla meters-squared or webers
$\iint_S \mathbf{B} \cdot \mathrm{d}\mathbf{A} = \Phi_{B,S}$	magnetic flux through any surface S, not necessarily closed	webers or equivalently, volt-seconds
$\iint_S \mathbf{E} \cdot \mathrm{d}\mathbf{A} = \Phi_{E,S}$	electric flux through any surface S, not necessarily closed	joule-meters per coulomb
$\iint_S \mathbf{D} \cdot \mathrm{d}\mathbf{A} = \Phi_{D,S}$	flux of electric displacement field through any surface S, not necessarily closed	coulombs
$\iint_S \mathbf{J}_f \cdot \mathrm{d}\mathbf{A} = I_{f,S}$	net free electrical current passing through the surface S (not including bound current)	amperes
$\iint_S \mathbf{J} \cdot \mathrm{d}\mathbf{A} = I_S$	net electrical current passing through the surface S (including both free and bound current)	amperes

really recognize $\int$ vs $\iint$ vs $\oint$ vs $\iiint$ And how to do each and when Shortcuts work

Maxwell's equations are generally applied to *macroscopic averages* of the fields, which vary wildly on a microscopic scale in the vicinity of individual atoms (where they undergo quantum mechanical effects as well). It is only in this averaged sense that one can define quantities such as the permittivity and permeability of a material. At microscopic level, Maxwell's equations, ignoring quantum effects, describe fields, charges and currents in free space—but at this level of detail one must include all charges, even those at an atomic level, generally an intractable problem.

History

Although James Clerk Maxwell is said by some not to be the originator of these equations, he nevertheless derived them independently in conjunction with his molecular vortex model of Faraday's "lines of force". In doing so, he made an important addition to Ampère's circuital law.

All four of what are now described as Maxwell's equations can be found in recognizable form (albeit without any trace of a vector notation, let alone ∇) in his 1861 paper *On Physical Lines of Force*, in his 1865 paper A Dynamical Theory of the Electromagnetic Field, and also in vol. 2 of Maxwell's "A Treatise on Electricity & Magnetism", published in 1873, in Chapter IX, entitled "General Equations of the Electromagnetic Field". This book by Maxwell pre-dates publications by Heaviside, Hertz and others.

The term *Maxwell's equations*

The term *Maxwell's equations* originally applied to a set of eight equations published by Maxwell in 1865, but nowadays applies to modified versions of four of these equations that were grouped together in 1884 by Oliver Heaviside,[5] concurrently with similar work by Willard Gibbs and Heinrich Hertz.[6] These equations were also known variously as the Hertz-Heaviside equations and the Maxwell-Hertz equations,[5] and are sometimes still known as the Maxwell–Heaviside equations.[7] Coverup method

Maxwell's contribution to science in producing these equations lies in the correction he made to Ampère's circuital law in his 1861 paper *On Physical Lines of Force*. He added the displacement current term to Ampère's circuital law and this enabled him to derive the electromagnetic wave equation in his later 1865 paper *A Dynamical Theory of the Electromagnetic Field* and demonstrate the fact that light is an electromagnetic wave. This fact was then later confirmed experimentally by Heinrich Hertz in 1887.

The concept of fields was introduced by, among others, Faraday. Albert Einstein wrote:

> The precise formulation of the time-space laws was the work of Maxwell. Imagine his feelings when the differential equations he had formulated proved to him that electromagnetic fields spread in the form of polarised waves, and at the speed of light! To few men in the world has such an experience been vouchsafed . . it took physicists some decades to grasp the full significance of Maxwell's discovery, so bold was the leap that his genius forced upon the conceptions of his fellow-workers
> —(*Science*, May 24, 1940)

The equations were called by some the Hertz-Heaviside equations, but later Einstein referred to them as the Maxwell-Hertz equations.[5] However, in 1940 Einstein referred to the equations as *Maxwell's equations* in "The Fundamentals of Theoretical Physics" published in the Washington periodical *Science*, May 24, 1940.

Heaviside worked to eliminate the potentials (electrostatic potential and vector potential) that Maxwell had used as the central concepts in his equations;[5] this effort was somewhat controversial,[8] though it was understood by 1884 that the potentials must propagate at the speed of light like the fields, unlike the concept of instantaneous action-at-a-distance like the then conception of gravitational potential.[6] Modern analysis of, for example, radio antennas, makes full use of Maxwell's vector and scalar potentials to separate the variables, a common technique used in formulating the solutions of differential equations. However the potentials can be introduced by algebraic manipulation of the four fundamental equations.

The net result of Heaviside's work was the symmetrical duplex set of four equations,[5] all of which originated in Maxwell's previous publications, in particular Maxwell's 1861 paper *On Physical Lines of Force*, the 1865 paper *A Dynamical Theory of the Electromagnetic Field* and the Treatise. The fourth was a partial time derivative version of Faraday's law of induction that doesn't include motionally induced EMF; this version is often termed the *Maxwell-Faraday equation* or *Faraday's law in differential form* to keep clear the distinction from Faraday's law of induction, though it expresses the same law.[9][10]

Maxwell's *On Physical Lines of Force* (1861)

The four modern day Maxwell's equations appeared throughout Maxwell's 1861 paper *On Physical Lines of Force*:

i. Equation (56) in Maxwell's 1861 paper is $\nabla \cdot \mathbf{B} = 0$.
ii. Equation (112) is Ampère's circuital law with Maxwell's displacement current added. It is the addition of displacement current that is the most significant aspect of Maxwell's work in electromagnetism, as it enabled him to later derive the electromagnetic wave equation in his 1865 paper A Dynamical Theory of the Electromagnetic Field, and hence show that light is an electromagnetic wave. It is therefore this aspect of Maxwell's work which gives the equations their full significance. (Interestingly, Kirchhoff derived the telegrapher's equations in 1857 without using displacement current. But he did use Poisson's equation and the equation of continuity which are the mathematical ingredients of the displacement current. Nevertheless, Kirchhoff believed his equations to be applicable only inside an electric wire and so he is not credited with having discovered that light is an electromagnetic wave).
iii. Equation (115) is Gauss's law.
iv. Equation (54) is an equation that Oliver Heaviside referred to as 'Faraday's law'. This equation caters for the time varying aspect of electromagnetic induction, but not for the motionally induced aspect, whereas Faraday's original flux law caters for both aspects. Maxwell deals with the motionally dependent aspect of electromagnetic induction, $\mathbf{v} \times \mathbf{B}$, at equation (77). Equation (77) which is the same as equation (D) in the original eight Maxwell's equations listed below, corresponds to all intents and purposes to the modern day force law $\mathbf{F} = q\,(\mathbf{E} + \mathbf{v} \times \mathbf{B})$ which sits adjacent to Maxwell's equations and bears the name Lorentz force, even though Maxwell derived it when Lorentz was still a young boy.

The difference between the B and the H vectors can be traced back to Maxwell's 1855 paper entitled *On Faraday's Lines of Force* which was read to the Cambridge Philosophical Society. The paper presented a simplified model of Faraday's work, and how the two phenomena were related. He reduced all of the current knowledge into a linked set of differential equations.

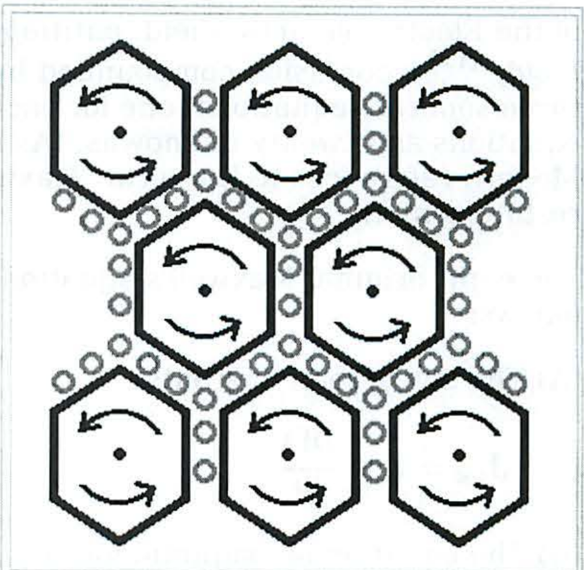


Figure of Maxwell's molecular vortex model. For a uniform magnetic field, the field lines point outward from the display screen, as can be observed from the black dots in the middle of the hexagons. The vortex of each hexagonal molecule rotates counter-clockwise. The small green circles are clockwise rotating particles sandwiching between the molecular vortices.

It is later clarified in his concept of a sea of molecular vortices that appears in his 1861 paper On Physical Lines of Force - 1861 (http://upload.wikimedia.org/wikipedia/commons/b/b8/On_Physical_Lines_of_Force.pdf) . Within that context, $\mathbf{H}$ represented pure vorticity (spin), whereas $\mathbf{B}$ was a weighted vorticity that was weighted for the density of the vortex sea. Maxwell considered magnetic permeability μ to be a measure of the density of the vortex sea. Hence the relationship,

(1) **Magnetic induction current** causes a magnetic current density

$$\mathrm{B} = \mu \mathrm{H}$$

was essentially a rotational analogy to the linear electric current relationship,

(2) **Electric convection current**

$$\mathbf{J} = \rho \mathbf{v}$$

where ρ is electric charge density. $\mathbf{B}$ was seen as a kind of magnetic current of vortices aligned in their axial planes, with $\mathbf{H}$ being the circumferential velocity of the vortices. With μ representing vortex density, it follows that the product of μ with vorticity $\mathbf{H}$ leads to the magnetic field denoted as $\mathbf{B}$.

The electric current equation can be viewed as a convective current of electric charge that involves linear motion. By analogy, the magnetic equation is an inductive current involving spin. There is no linear motion in the inductive current along the direction of the $\mathbf{B}$ vector. The magnetic inductive current represents lines of force. In particular, it represents lines of inverse square law force.

The extension of the above considerations confirms that where $\mathbf{B}$ is to $\mathbf{H}$, and where $\mathbf{J}$ is to ρ, then it necessarily follows from Gauss's law and from the equation of continuity of charge that $\mathbf{E}$ is to $\mathbf{D}$. i.e. $\mathbf{B}$ parallels with $\mathbf{E}$, whereas $\mathbf{H}$ parallels with $\mathbf{D}$.

Maxwell's *A Dynamical Theory of the Electromagnetic Field* (1864)

Main article: A Dynamical Theory of the Electromagnetic Field

In 1864 Maxwell published **A Dynamical Theory of the Electromagnetic Field** in which he showed that light was an electromagnetic phenomenon. Confusion over the term "Maxwell's equations" is exacerbated because it is also sometimes used for a set of eight equations that appeared in Part III of Maxwell's 1864 paper A Dynamical Theory of the Electromagnetic Field, entitled "General Equations of the Electromagnetic Field,"[11] a confusion compounded by the writing of six of those eight equations as three separate equations (one for each of the Cartesian axes), resulting in twenty equations and twenty unknowns. (As noted above, this terminology is not common: Modern references to the term "Maxwell's equations" refer to the Heaviside restatements.)

The eight original Maxwell's equations can be written in modern vector notation as follows:

(A) The law of total currents

$$\mathbf{J}_{tot} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

(B) The equation of magnetic force

$$\mu \mathbf{H} = \nabla \times \mathbf{A}$$

(C) Ampère's circuital law

$$\nabla \times \mathbf{H} = \mathbf{J}_{tot}$$

(D) Electromotive force created by convection, induction, and by static electricity. (This is in effect the Lorentz force)

$$\mathbf{E} = \mu \mathbf{v} \times \mathbf{H} - \frac{\partial \mathbf{A}}{\partial t} - \nabla \phi$$

(E) The electric elasticity equation

$$\mathbf{E} = \frac{1}{\epsilon} \mathbf{D}$$

(F) Ohm's law

$$\mathbf{E} = \frac{1}{\sigma} \mathbf{J}$$

(G) Gauss's law

$$\nabla \cdot \mathbf{D} = \rho$$

(H) Equation of continuity

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}$$

or

$$\nabla \cdot \mathbf{J}_{tot} = 0$$

CALCULUS III

Line Integrals

Paul Dawkins

Table of Contents

Preface

Here are my online notes for my Calculus III course that I teach here at Lamar University. Despite the fact that these are my "class notes", they should be accessible to anyone wanting to learn Calculus III or needing a refresher in some of the topics from the class.

These notes do assume that the reader has a good working knowledge of Calculus I topics including limits, derivatives and integration. It also assumes that the reader has a good knowledge of several Calculus II topics including some integration techniques, parametric equations, vectors, and knowledge of three dimensional space.

Here are a couple of warnings to my students who may be here to get a copy of what happened on a day that you missed.

1. Because I wanted to make this a fairly complete set of notes for anyone wanting to learn calculus I have included some material that I do not usually have time to cover in class and because this changes from semester to semester it is not noted here. You will need to find one of your fellow class mates to see if there is something in these notes that wasn't covered in class.

2. In general I try to work problems in class that are different from my notes. However, with Calculus III many of the problems are difficult to make up on the spur of the moment and so in this class my class work will follow these notes fairly close as far as worked problems go. With that being said I will, on occasion, work problems off the top of my head when I can to provide more examples than just those in my notes. Also, I often don't have time in class to work all of the problems in the notes and so you will find that some sections contain problems that weren't worked in class due to time restrictions.

3. Sometimes questions in class will lead down paths that are not covered here. I try to anticipate as many of the questions as possible in writing these up, but the reality is that I can't anticipate all the questions. Sometimes a very good question gets asked in class that leads to insights that I've not included here. You should always talk to someone who was in class on the day you missed and compare these notes to their notes and see what the differences are.

4. This is somewhat related to the previous three items, but is important enough to merit its own item. THESE NOTES ARE NOT A SUBSTITUTE FOR ATTENDING CLASS!! Using these notes as a substitute for class is liable to get you in trouble. As already noted not everything in these notes is covered in class and often material or insights not in these notes is covered in class.

This should be a very good review

Line Integrals

Introduction

In this section we are going to start looking at Calculus with vector fields (which we'll define in the first section). In particular we will be looking at a new type of integral, the line integral and some of the interpretations of the line integral. We will also take a look at one of the more important theorems involving line integrals, Green's Theorem.

Here is a listing of the topics covered in this chapter.

Vector Fields

We need to start this chapter off with the definition of a vector field as they will be a major component of both this chapter and the next. Let's start off with the formal definition of a vector field.

Definition

A vector field on two (or three) dimensional space is a function $\vec{F}$ that assigns to each point (x,y) (or (x,y,z)) a two (or three dimensional) vector given by $\vec{F}(x,y)$ (or $\vec{F}(x,y,z)$).

That may not make a lot of sense, but most people do know what a vector field is, or at least they've seen a sketch of a vector field. If you've seen a current sketch giving the direction and magnitude of a flow of a fluid or the direction and magnitude of the winds then you've seen a sketch of a vector field.

The standard notation for the function $\vec{F}$ is,

$$\vec{F}(x,y) = P(x,y)\vec{i} + Q(x,y)\vec{j}$$
$$\vec{F}(x,y,z) = P(x,y,z)\vec{i} + Q(x,y,z)\vec{j} + R(x,y,z)\vec{k}$$

depending on whether or not we're in two or three dimensions. The function P, Q, R (if it is present) are sometimes called **scalar functions**.

just modify

Let's take a quick look at a couple of examples.

Example 1 Sketch each of the following direction fields.

(a) $\vec{F}(x,y) = -y\vec{i} + x\vec{j}$ [Solution]

(b) $\vec{F}(x,y,z) = 2x\vec{i} - 2y\vec{j} - 2x\vec{k}$ [Solution]

Solution

(a) $\vec{F}(x,y) = -y\vec{i} + x\vec{j}$

Okay, to graph the vector field we need to get some "values" of the function. This means plugging in some points into the function. Here are a couple of evaluations.

$$\vec{F}\left(\frac{1}{2},\frac{1}{2}\right) = -\frac{1}{2}\vec{i} + \frac{1}{2}\vec{j}$$

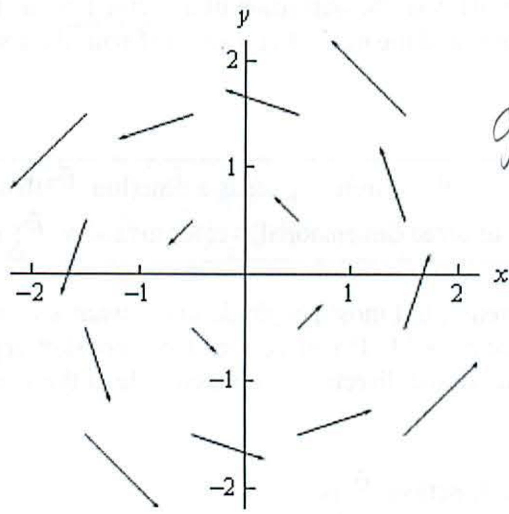
different at each ~~starting~~ point

$$\vec{F}\left(\frac{1}{2},-\frac{1}{2}\right) = -\left(-\frac{1}{2}\right)\vec{i} + \frac{1}{2}\vec{j} = \frac{1}{2}\vec{i} + \frac{1}{2}\vec{j}$$
$$\vec{F}\left(\frac{3}{2},\frac{1}{4}\right) = -\frac{1}{4}\vec{i} + \frac{3}{2}\vec{j}$$

then plug in

So, just what do these evaluations tell us? Well the first one tells us that at the point $\left(\frac{1}{2},\frac{1}{2}\right)$ we will plot the vector $-\frac{1}{2}\vec{i} + \frac{1}{2}\vec{j}$. Likewise, the third evaluation tells us that at the point $\left(\frac{3}{2},\frac{1}{4}\right)$ we will plot the vector $-\frac{1}{4}\vec{i} + \frac{3}{2}\vec{j}$.

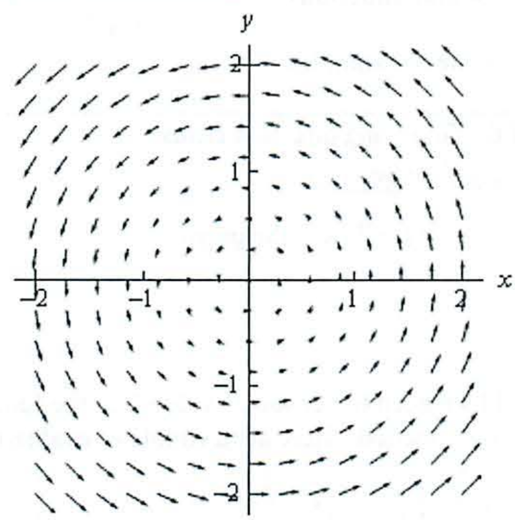
We can continue in this fashion plotting vectors for several points and we'll get the following sketch of the vector field.



get this

remember different at each point is a key fact - but overlooked often I think

If we want significantly more points plotted then it is usually best to use a computer aided graphing system such as Maple or Mathematica. Here is a sketch with many more vectors included that was generated with Mathematica.



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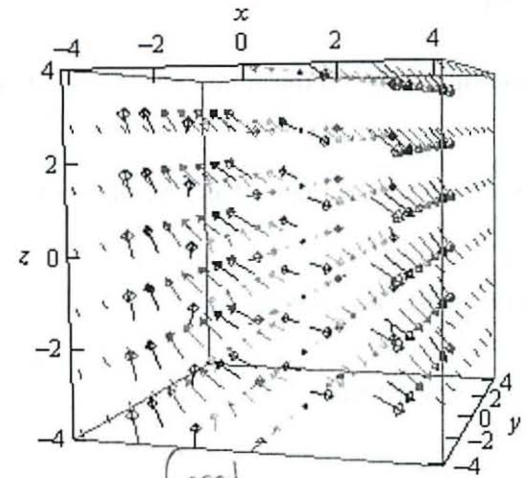
(b) $\vec{F}(x,y,z) = 2x\vec{i} - 2y\vec{j} - 2x\vec{k}$

In the case of three dimensional vector fields it is almost always better to use Maple, Mathematica, or some other such tool. Despite that let's go ahead and do a couple of evaluations anyway.

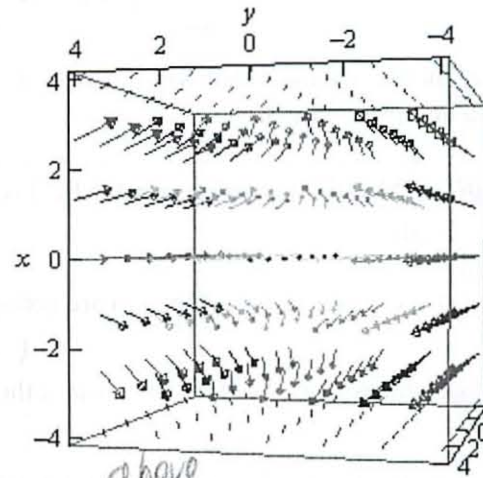
$$\vec{F}(1,-3,2) = 2\vec{i} + 6\vec{j} - 2\vec{k}$$
$$\vec{F}(0,5,3) = -10\vec{j}$$

Notice that z only affect the placement of the vector in this case and does not affect the direction or the magnitude of the vector. Sometimes this will happen so don't get excited about it when it does.

Here is a couple of sketches generated by Mathematica. The sketch on the left is from the "front" and the sketch on the right is from "above".



front



above

[Return to Problems]

Now that we've seen a couple of vector fields let's notice that we've already seen a vector field function. In the second chapter we looked at the gradient vector. Recall that given a function $f(x,y,z)$ the gradient vector is defined by,

$$\nabla f = \langle f_x, f_y, f_z \rangle$$

This is a vector field and is often called a **gradient vector field**.

In these cases the function $f(x,y,z)$ is often called a scalar function to differentiate it from the vector field.

Example 2 Find the gradient vector field of the following functions.

(a) $f(x,y) = x^2 \sin(5y)$

(b) $f(x,y,z) = z\mathbf{e}^{-xy}$

take deriv of each part
but how does it look like ???

Solution

(a) $f(x,y) = x^2 \sin(5y)$

Note that we only gave the gradient vector definition for a three dimensional function, but don't forget that there is also a two dimension definition. All that we need to drop off the third component of the vector.

Here is the gradient vector field for this function.

$$\nabla f = \langle 2x\sin(5y), 5x^2\cos(5y) \rangle$$

(b) $f(x,y,z) = z\mathbf{e}^{-xy}$

There isn't much to do here other than take the gradient.

$$\nabla f = \left\langle -yz\mathbf{e}^{-xy}, -xz\mathbf{e}^{-xy}, \mathbf{e}^{-xy} \right\rangle$$

Let's do another example that will illustrate the relationship between the gradient vector field of a function and its contours.

Example 3 Sketch the gradient vector field for $f(x,y) = x^2 + y^2$ as well as several contours for this function.

Solution

Recall that the contours for a function are nothing more than curves defined by,

$$f(x,y) = k$$

for various values of k. So, for our function the contours are defined by the equation,

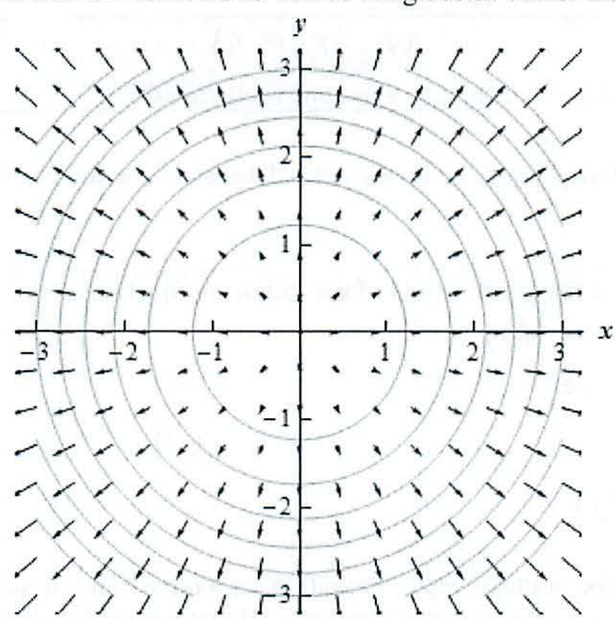
$$x^2 + y^2 = k$$

and so they are circles centered at the origin with radius $\sqrt{k}$.

Here is the gradient vector field for this function.

$$\nabla f(x,y) = 2x\vec{i} + 2y\vec{j}$$

Here is a sketch of several of the contours as well as the gradient vector field.



what is difference b/w Field + gradiant graphically -derivitive (shows change)

Notice that the vectors of the vector field are all perpendicular (or orthogonal) to the contours. This will always be the case when we are dealing with the contours of a function as well as its gradient vector field.

The k's we used for the graph above were 1.5, 3, 4.5, 6, 7.5, 9, 10.5, 12, and 13.5. Now notice that as we increased k by 1.5 the contour curves get closer together and that as the contour curves get closer together the larger vectors become. In other words, the closer the contour curves are (as k is increased by a fixed amount) the faster the function is changing at that point. Also recall that the direction of fastest change for a function is given by the gradient vector at that point. Therefore, it should make sense that the two ideas should match up as they do here.

The final topic of this section is that of conservative vector fields. A vector field $\vec{F}$ is called a **conservative vector field** if there exists a function f such that $\boxed{\vec{F} = \nabla f}$. If $\vec{F}$ is a conservative vector field then the function, f, is called a **potential function** for $\vec{F}$.

if it is always a gradiant

All this definition is saying is that a vector field is conservative if it is also a gradient vector field for some function.

For instance the vector field $\vec{F} = y\vec{i} + x\vec{j}$ is a conservative vector field with a potential function of $f(x, y) = xy$ because $\nabla f = \langle y, x \rangle$.

On the other hand, $\vec{F} = -y\vec{i} + x\vec{j}$ is not a conservative vector field since there is no function f such that $\vec{F} = \nabla f$. If you're not sure that you believe this at this point be patient, we will be able to prove this in a couple of sections. In that section we will also show how to find the potential function for a conservative vector field.

- Conservation
- does not change w/ what pt you start at

remember $\vec{F}$ is the gradient of
$\vec{f}$ is original
Seems backward in lettering -
but oh well

Seperate mentally the 3 types of S

Line Integrals – Part I

In this section we are now going to introduce a new kind of integral. However, before we do that it is important to note that you will need to remember how to parameterize equations, or put another way, you will need to be able to write down a set of parametric equations for a given curve. You should have seen some of this in your Calculus II course. If you need some review you should go back and review some of the basics of parametric equations and curves.

Here are some of the more basic curves that we'll need to know how to do as well as limits on the parameter if they are required.

know how to paramatrize!

Curve	**Parametric Equations** Counter-Clockwise	Clockwise
$\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ (Ellipse)	$x=a\cos(t)$ $y=b\sin(t)$ $0\le t\le 2\pi$	$x=a\cos(t)$ $y=-b\sin(t)$ $0\le t\le 2\pi$
$x^2+y^2=r^2$ (Circle)	Counter-Clockwise $x=r\cos(t)$ $y=r\sin(t)$ $0\le t\le 2\pi$	Clockwise $x=r\cos(t)$ $y=-r\sin(t)$ $0\le t\le 2\pi$
$y=f(x)$	$x=t$ $y=f(t)$	
$x=g(y)$	$x=g(t)$ $y=t$	
Line Segment From (x_0,y_0,z_0) to (x_1,y_1,z_1)	$\vec{r}(t)=(1-t)\langle x_0,y_0,z_0\rangle+t\langle x_1,y_1,z_1\rangle$, $0\le t\le 1$ or $x=(1-t)x_0+t\,x_1$ $y=(1-t)y_0+t\,y_1$, $0\le t\le 1$ $z=(1-t)z_0+t\,z_1$	

With the final one we gave both the vector form of the equation as well as the parametric form and if we need the two-dimensional version then we just drop the z components. In fact, we will be using the two-dimensional version of this in this section.

For the ellipse and the circle we've given two parameterizations, one tracing out the curve clockwise and the other counter-clockwise. As we'll eventually see the direction that the curve is traced out can, on occasion, change the answer. Also, both of these "start" on the positive x-axis at $t=0$.

Now let's move on to line integrals. In Calculus I we integrated $f(x)$, a function of a single variable, over an interval $[a,b]$. In this case we were thinking of *x* as taking all the values in this interval starting at *a* and ending at *b*. With line integrals we will start with integrating the function $f(x,y)$, a function of two variables, and the values of *x* and *y* that we're going to use will be the points, (x,y), that lie on a curve *C*. Note that this is different from the double integrals that we were working with in the previous chapter where the points came out of some two-dimensional region.

Let's start with the curve *C* that the points come from. We will assume that the curve is *smooth* (defined shortly) and is given by the parametric equations,

$$x = h(t) \qquad y = g(t) \qquad a \le t \le b$$

We will often want to write the parameterization of the curve as a vector function. In this case the curve is given by,

$$\vec{r}(t) = h(t)\vec{i} + g(t)\vec{j} \qquad a \le t \le b$$

rewrite

will do practice tests later for actual practice

The curve is called **smooth** if $\vec{r}'(t)$ is continuous and $\vec{r}'(t) \ne 0$ for all *t*.

The **line integral** of $f(x,y)$ along *C* is denoted by,

$$\int_C f(x,y)\,ds$$

We use a *ds* here to acknowledge the fact that we are moving along the curve, *C*, instead of the *x*-axis (denoted by *dx*) or the *y*-axis (denoted by *dy*). Because of the *ds* this is sometimes called the **line integral of *f* with respect to arc length**.

We've seen the notation *ds* before. If you recall from Calculus II when we looked at the arc length of a curve given by parametric equations we found it to be,

✓ never saw that before

$$L = \int_a^b ds\,, \qquad \text{where} \quad ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

It is no coincidence that we use *ds* for both of these problems. The *ds* is the same for both the arc length integral and the notation for the line integral.

So, to compute a line integral we will convert everything over to the parametric equations. The line integral is then,

$$\int_C f(x,y)\,ds = \int_a^b f\left(h(t),g(t)\right)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

Don't forget to plug the parametric equations into the function as well.

If we use the vector form of the parameterization we can simplify the notation up somewhat by noticing that,

$$\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}=\|\vec{r}'(t)\|$$

the simplification trick
– but it always works?

where $\|\vec{r}'(t)\|$ is the <u>magnitude</u> or norm of $\vec{r}'(t)$. Using this notation the line integral becomes,

$$\boxed{\int_C f(x,y)\,ds=\int_a^b f\left(h(t),g(t)\right)\,\|\vec{r}'(t)\|\,dt}$$

Note that as long as the parameterization of the curve C is traced out exactly once as t increases from a to b the value of the line integral will be independent of the parameterization of the curve.

Let's take a look at an example of a line integral.

Example 1 Evaluate $\int_C xy^4\,ds$ where C is the right half of the circle, $x^2+y^2=16$ rotated in the counter clockwise direction.

Solution

We first need a parameterization of the circle. This is given by,

$$x=4\cos t \qquad y=4\sin t$$

We now need a range of t's that will give the right half of the circle. The following range of t's will do this.

$$-\frac{\pi}{2}\le t\le\frac{\pi}{2}$$

Now, we need the derivatives of the parametric equations and let's compute ds.

$$\frac{dx}{dt}=-4\sin t \qquad \frac{dy}{dt}=4\cos t$$

$$ds=\sqrt{16\sin^2 t+16\cos^2 t}\,dt=4\,dt$$

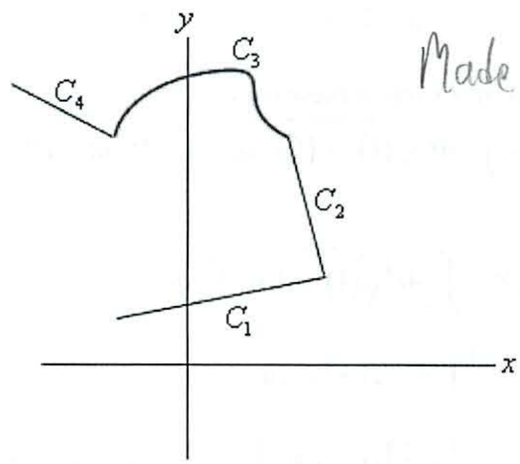
don't remember doing that
– did each part separately
and multiplied

The line integral is then,

$$\begin{aligned}\int_C xy^4\,ds&=\int_{-\pi/2}^{\pi/2}4\cos t\left(4\sin t\right)^4(4)\,dt\\&=4096\int_{-\pi/2}^{\pi/2}\cos t\,\sin^4 t\,dt\\&=\frac{4096}{5}\sin^5 t\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\\&=\frac{8192}{5}\end{aligned}$$

Next we need to talk about line integrals over **piecewise smooth curves**. A piecewise smooth curve is any curve that can be written as the union of a finite number of smooth curves, $C_1,\ldots,C_n$

where the end point of C_i is the starting point of C_{i+1}. Below is an illustration of a piecewise smooth curve.



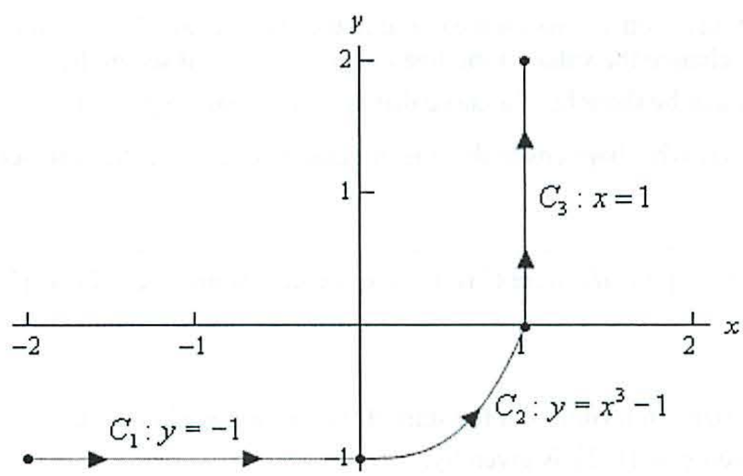
Made of pieces

Evaluation of line integrals over piecewise smooth curves is a relatively simple thing to do. All we do is evaluate the line integral over each of the pieces and then add them up. The line integral for some function over the above piecewise curve would be,

$$\int_C f(x,y)\,ds = \int_{C_1} f(x,y)\,ds + \int_{C_2} f(x,y)\,ds + \int_{C_3} f(x,y)\,ds + \int_{C_4} f(x,y)\,ds$$

Let's see an example of this.

Example 2 Evaluate $\int_C 4x^3\,ds$ where C is the curve shown below.



Solution

So, first we need to parameterize each of the curves.

$$C_1 : x = t, \; y = -1, \qquad -2 \le t \le 0$$
$$C_2 : x = t, \; y = t^3 - 1, \qquad 0 \le t \le 1$$
$$C_3 : x = 1, \; y = t, \qquad 0 \le t \le 2$$

Now let's do the line integral over each of these curves.

$$\int_{C_1} 4x^3 \, ds = \int_{-2}^{0} 4t^3 \sqrt{(1)^2 + (0)^2} \, dt = \int_{-2}^{0} 4t^3 \, dt = t^4 \Big|_{-2}^{0} = -16$$

$$\begin{aligned}\int_{C_2} 4x^3 \, ds &= \int_0^1 4t^3 \sqrt{(1)^2 + (3t^2)^2} \, dt \\ &= \int_0^1 4t^3 \sqrt{1 + 9t^4} \, dt \\ &= \frac{1}{9}\left(\frac{2}{3}\right)\left(1 + 9t^4\right)^{\frac{3}{2}}\Big|_0^1 = \frac{2}{27}\left(10^{\frac{3}{2}} - 1\right) = 2.268\end{aligned}$$

$$\int_{C_3} 4x^3 \, ds = \int_0^2 4(1)^3 \sqrt{(0)^2 + (1)^2} \, dt = \int_0^2 4 \, dt = 8$$

Finally, the line integral that we were asked to compute is,

$$\begin{aligned}\int_C 4x^3 \, ds &= \int_{C_1} 4x^3 \, ds + \int_{C_2} 4x^3 \, ds + \int_{C_3} 4x^3 \, ds \\ &= -16 + 2.268 + 8 \\ &= -5.732\end{aligned}$$

Notice that we put direction arrows on the curve in the above example. The direction of motion along a curve *may* change the value of the line integral as we will see in the next section. Also note that the curve can be thought of a curve that takes us from the point $(-2,-1)$ to the point $(1,2)$. Let's first see what happens to the line integral if we change the path between these two points.

Example 3 Evaluate $\int_C 4x^3 \, ds$ were C is the line segment from $(-2,-1)$ to $(1,2)$.

Solution

From the parameterization formulas at the start of this section we know that the line segment start at $(-2,-1)$ and ending at $(1,2)$ is given by,

$$\begin{aligned}\vec{r}(t) &= (1-t)\langle -2,-1\rangle + t\langle 1,2\rangle \\ &= \langle -2+3t, -1+3t\rangle\end{aligned}$$

for $0 \le t \le 1$. This means that the individual parametric equations are,

$$x = -2 + 3t \qquad\qquad y = -1 + 3t$$

Using this path the line integral is,

$$\int_C 4x^3\,ds = \int_0^1 4\left(-2+3t\right)^3 \sqrt{9+9}\,dt$$

$$= 12\sqrt{2}\left(\tfrac{1}{12}\right)\left(-2+3t\right)^4\Big|_0^1$$

$$= 12\sqrt{2}\left(-\frac{5}{4}\right)$$

$$= -15\sqrt{2} = -21.213$$

When doing these integrals don't forget simple Calc I substitutions to avoid having to do things like cubing out a term. Cubing it out is not that difficult, but it is more work than a simple substitution. never remember that stuff

So, the previous two examples seem to suggest that if we change the path between two points then the value of the line integral (with respect to arc length) will change. While this will happen fairly regularly we can't assume that it will always happen. In a later section we will investigate this idea in more detail

make it simple in your mind

Next, let's see what happens if we change the direction of a path.

Example 4 Evaluate $\int_C 4x^3\,ds$ were C is the line segment from $(1,2)$ to $(-2,-1)$.

Solution

This one isn't much different, work wise, from the previous example. Here is the parameterization of the curve.

$$\vec{r}(t) = (1-t)\langle 1,2\rangle + t\langle -2,-1\rangle$$

$$= \langle 1-3t, 2-3t\rangle$$

key

for $0 \le t \le 1$. Remember that we are switch the direction of the curve and this will also change the parameterization so we can make sure that we start/end at the proper point.

Here is the line integral.

$$\int_C 4x^3\,ds = \int_0^1 4\left(1-3t\right)^3 \sqrt{9+9}\,dt$$

$$= 12\sqrt{2}\left(-\tfrac{1}{12}\right)\left(1-3t\right)^4\Big|_0^1$$

$$= 12\sqrt{2}\left(-\frac{5}{4}\right)$$

$$= -15\sqrt{2} = -21.213$$

So, it looks like when we switch the direction of the curve the line integral (with respect to arc length) will not change. This will always be true for these kinds of line integrals. However, there are other kinds of line integrals in which this won't be the case. We will see more examples of this in the next couple of sections so don't get it into your head that changing the direction will never change the value of the line integral.

Before working another example let's formalize this idea up somewhat. Let's suppose that the curve *C* has the parameterization $x = h(t)$, $y = g(t)$. Let's also suppose that the initial point on the curve is *A* and the final point on the curve is *B*. The parameterization $x = h(t)$, $y = g(t)$ will then determine an **orientation** for the curve where the positive direction is the direction that is traced out as *t* increases. Finally, let $-C$ be the curve with the same points as *C*, however in this case the curve has *B* as the initial point and *A* as the final point, again *t* is increasing as we traverse this curve. In other words, given a curve *C*, the curve $-C$ is the same curve as *C* except the direction has been reversed.

We then have the following fact about line integrals with respect to arc length.

same

Fact

$$\int_C f(x,y)\,ds = \int_{-C} f(x,y)\,ds$$

So, for a line integral with respect to arc length we can change the direction of the curve and not change the value of the integral. This is a useful fact to remember as some line integrals will be easier in one direction than the other.

force field

Now, let's work another example

Example 5 Evaluate $\int_C x\,ds$ for each of the following curves.

(a) $C_1 : y = x^2,\ -1 \le x \le 1$ [Solution]

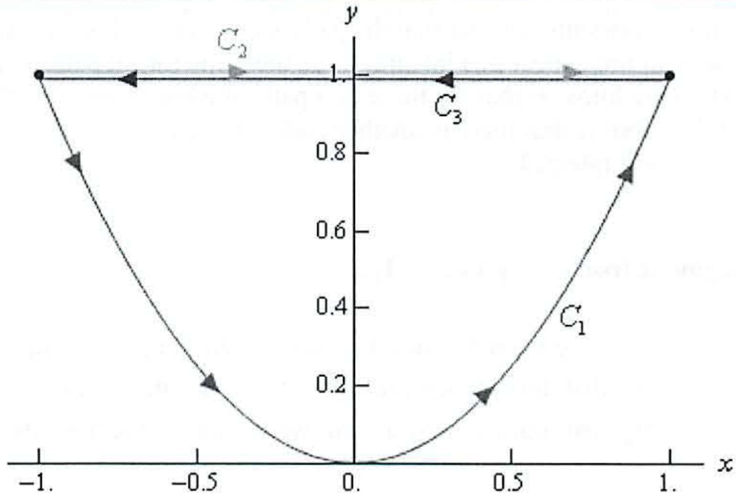
(b) C_2 : The line segment from $(-1,1)$ to $(1,1)$. [Solution]

(c) C_3 : The line segment from $(1,1)$ to $(-1,1)$. [Solution]

Solution

Before working any of these line integrals let's notice that all of these curves are paths that connect the points $(-1,1)$ and $(1,1)$. Also notice that $C_3 = -C_2$ and so by the fact above these two should give the same answer.

Here is a sketch of the three curves and note that the curves illustrating C_2 and C_3 have been separated a little to show that they are separate curves in some way even thought they are the same line.



(a) $C_1 : y = x^2,\ -1 \le x \le 1$

Here is a parameterization for this curve.

$$C_1 : x = t,\ y = t^2,\ -1 \le t \le 1$$

Here is the line integral.

$$\int_{C_1} x\, ds = \int_{-1}^{1} t\sqrt{1+4t^2}\, dt = \frac{1}{12}\left(1+4t^2\right)^{\frac{3}{2}}\Bigg|_{-1}^{1} = 0$$

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(b) C_2 **: The line segment from** $(-1,1)$ **to** $(1,1)$**.**

There are two parameterizations that we could use here for this curve. The first is to use the formula we used in the previous couple of examples. That parameterization is,

$$\begin{aligned} C_2 : \vec{r}(t) &= (1-t)\langle -1,1\rangle + t\langle 1,1\rangle \\ &= \langle 2t-1, 1\rangle \end{aligned}$$

for $0 \le t \le 1$.

Sometimes we have no choice but to use this parameterization. However, in this case there is a second (probably) easier parameterization. The second one uses the fact that we are really just graphing a portion of the line $y = 1$. Using this the parameterization is,

$$C_2 : x = t,\ y = 1,\ -1 \le t \le 1$$

This will be a much easier parameterization to use so we will use this. Here is the line integral for this curve.

$$\int_{C_2} x\, ds = \int_{-1}^{1} t\sqrt{1+0}\, dt = \frac{1}{2}t^2\Bigg|_{-1}^{1} = 0$$

Note that this time, unlike the line integral we worked with in Examples 2, 3, and 4 we got the same value for the integral despite the fact that the path is different. This will happen on occasion. We should also not expect this integral to be the same for all paths between these two points. At this point all we know is that for these two paths the line integral will have the same value. It is completely possible that there is another path between these two points that will give a different value for the line integral.

Conservative Field

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(c) C_3 : The line segment from $(1,1)$ to $(-1,1)$.

Now, according to our fact above we really don't need to do anything here since we know that $C_3 = -C_2$. The fact tells us that this line integral should be the same as the second part (*i.e.* zero). However, let's verify that, plus there is a point we need to make here about the parameterization.

Here is the parameterization for this curve.

$$\begin{aligned} C_3 : \vec{r}(t) &= (1-t)\langle 1,1\rangle + t\langle -1,1\rangle \\ &= \langle 1-2t, 1\rangle \end{aligned}$$

for $0 \le t \le 1$.

Note that this time we can't use the second parameterization that we used in part (b) since we need to move from right to left as the parameter increases and the second parameterization used in the previous part will move in the opposite direction.

Here is the line integral for this curve.

$$\int_{C_3} x\,ds = \int_0^1 (1-2t)\sqrt{4+0}\,dt = 2\left(t-t^2\right)\Big|_0^1 = 0$$

Sure enough we got the same answer as the second part.

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To this point in this section we've only looked at line integrals over a two-dimensional curve. However, there is no reason to restrict ourselves like that. We can do line integrals over three-dimensional curves as well.

Let's suppose that the three-dimensional curve C is given by the parameterization,

$$x = x(t), \qquad y = y(t) \qquad z = z(t) \qquad a \le t \le b$$

then the line integral is given by,

$$\boxed{\int_C f(x,y,z)\,ds = \int_a^b f\left(x(t), y(t), z(t)\right)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt}$$

Note that often when dealing with three-dimensional space the parameterization will be given as a vector function.

$$\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$$

Notice that we changed up the notation for the parameterization a little. Since we rarely use the function names we simply kept the *x*, *y*, and *z* and added on the (t) part to denote that they may be functions of the parameter.

Also notice that, as with two-dimensional curves, we have,

$$\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}=\left\|\vec{r}'(t)\right\|$$

and the line integral can again be written as,

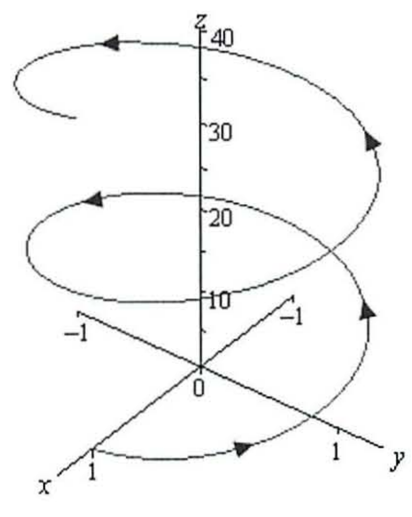
$$\int_C f(x,y,z)\,ds=\int_a^b f\left(x(t),y(t),z(t)\right)\left\|\vec{r}'(t)\right\|dt$$

So, outside of the addition of a third parametric equation line integrals in three-dimensional space work the same as those in two-dimensional space. Let's work a quick example.

Example 6 Evaluate $\int_C xyz\,ds$ where *C* is the helix given by, $\vec{r}(t)=\langle\cos(t),\sin(t),3t\rangle$, $0\le t\le 4\pi$.

Solution

Note that we first saw the vector equation for a helix back in the Vector Functions section. Here is a quick sketch of the helix.



Here is the line integral.

$$
\begin{aligned}
\int_C xyz\,ds &= \int_0^{4\pi} 3t\cos(t)\sin(t)\sqrt{\sin^2 t+\cos^2 t+9}\,dt \\
&= \int_0^{4\pi} 3t\left(\frac{1}{2}\sin(2t)\right)\sqrt{1+9}\,dt \\
&= \frac{3\sqrt{10}}{2}\int_0^{4\pi} t\sin(2t)\,dt \\
&= \frac{3\sqrt{10}}{2}\left(\frac{1}{4}\sin(2t)-\frac{t}{2}\cos(2t)\right)\Bigg|_0^{4\pi} \\
&= -3\sqrt{10}\,\pi
\end{aligned}
$$

You were able to do that integral right? It required integration by parts.

So, as we can see there really isn't too much difference between two- and three-dimensional line integrals.

This might end up being my biggest problem

Line Integrals – Part II

In the previous section we looked at line integrals with respect to arc length. In this section we want to look at line integrals with respect to x and/or y. why – just an axis ?

As with the last section we will start with a two-dimensional curve C with parameterization,

$$x = x(t) \qquad y = y(t) \qquad a \le t \le b$$

The **line integral of f with respect to x** is,

$$\int_C f(x,y)\,dx = \int_a^b f(x(t),y(t))\,x'(t)\,dt$$

The **line integral of f with respect to y** is,

$$\int_C f(x,y)\,dy = \int_a^b f(x(t),y(t))\,y'(t)\,dt$$

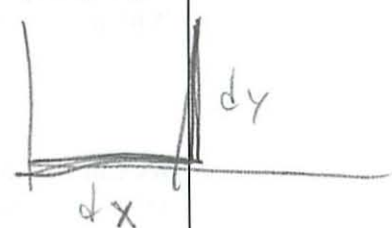
Note that the only notational difference between these two and the line integral with respect to arc length (from the previous section) is the differential. These have a dx or dy while the line integral with respect to arc length has a ds. So when evaluating line integrals be careful to first note which differential you've got so you don't work the wrong kind of line integral.

These two integral often appear together and so we have the following shorthand notation for these cases.

$$\int_C P\,dx + Q\,dy = \int_C P(x,y)\,dx + \int_C Q(x,y)\,dy$$

Let's take a quick look at an example of this kind of line integral.

Example 1 Evaluate $\int_C \sin(\pi y)\,dy + yx^2\,dx$ where C is the line segment from $(0,2)$ to $(1,4)$.



Solution
Here is the parameterization of the curve.

$$\vec{r}(t) = (1-t)\langle 0,2\rangle + t\langle 1,4\rangle = \langle t, 2+2t\rangle \qquad 0 \le t \le 1$$

The line integral is,

$$\begin{aligned}\int_C \sin(\pi y)\,dy + yx^2\,dx &= \int_C \sin(\pi y)\,dy + \int_C yx^2\,dx \\ &= \int_0^1 \sin(\pi(2+2t))(2)\,dt + \int_0^1 (2+2t)(t)^2(1)\,dt \\ &= -\frac{1}{\pi}\cos(2\pi + 2\pi t)\Big|_0^1 + \left(\frac{2}{3}t^3 + \frac{1}{2}t^4\right)\Big|_0^1 \\ &= \frac{7}{6}\end{aligned}$$

In the previous section we saw that changing the direction of the curve for a line integral with respect to arc length doesn't change the value of the integral. Let's see what happens with line integrals with respect to x and/or y.

Example 2 Evaluate $\int_C \sin(\pi y)\,dy + yx^2\,dx$ where C is the line segment from $(1,4)$ to $(0,2)$.

Solution
So, we simply changed the direction of the curve. Here is the new parameterization.

$$\vec r(t) = (1-t)\langle 1,4\rangle + t\langle 0,2\rangle = \langle 1-t, 4-2t\rangle \qquad 0 \le t \le 1$$

The line integral in this case is,

$$\begin{aligned}\int_C \sin(\pi y)\,dy + yx^2\,dx &= \int_C \sin(\pi y)\,dy + \int_C yx^2\,dx\\ &= \int_0^1 \sin(\pi(4-2t))(-2)\,dt + \int_0^1 (4-2t)(1-t)^2(-1)\,dt\\ &= \frac{1}{\pi}\cos(4\pi - 2\pi t)\Big|_0^1 - \left(-\frac{1}{2}t^4 + \frac{8}{3}t^3 - 5t^2 + 4t\right)\Big|_0^1\\ &= -\frac{7}{6}\end{aligned}$$

So, switching the direction of the curve got us a different value or at least the opposite sign of the value from the first example. In fact this will always happen with these kinds of line integrals.

Fact

If C is any curve then,

$$\int_{-C} f(x,y)\,dx = -\int_C f(x,y)\,dx \qquad \text{and} \qquad \int_{-C} f(x,y)\,dy = -\int_C f(x,y)\,dy$$

With the combined form of these two integrals we get,

$$\int_{-C} P\,dx + Q\,dy = -\int_C P\,dx + Q\,dy$$

We can also do these integrals over three-dimensional curves as well. In this case we will pick up a third integral (with respect to z) and the three integrals will be.

$$\int_C f(x,y,z)\,dx = \int_a^b f(x(t),y(t),z(t))\,x'(t)\,dt$$

$$\int_C f(x,y,z)\,dy = \int_a^b f(x(t),y(t),z(t))\,y'(t)\,dt$$

$$\int_C f(x,y,z)\,dz = \int_a^b f(x(t),y(t),z(t))\,z'(t)\,dt$$

where the curve C is parameterized by

$$x = x(t) \qquad y = y(t) \qquad z = z(t) \qquad a \le t \le b$$

?, is it that function same everywhere

What is difference?

where field can change

Line Integrals of Vector Fields

In the previous two sections we looked at line integrals of functions. In this section we are going to evaluate line integrals of vector fields. We'll start with the vector field,

$$\vec{F}(x,y,z) = P(x,y,z)\,\vec{i} + Q(x,y,z)\,\vec{j} + R(x,y,z)\,\vec{k}$$

depending on starting pt

and the three-dimensional, smooth curve given by

$$\vec{r}(t) = x(t)\,\vec{i} + y(t)\,\vec{j} + z(t)\,\vec{k} \qquad a \le t \le b$$

-yeah think that is it

The **line integral of $\vec{F}$ along C** is

$$\int_C \vec{F}\cdot d\,\vec{r} = \int_a^b \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt$$

Note the notation in the left side. That really is a dot product of the vector field and the differential and the differential really is a vector. Also, $\vec{F}(\vec{r}(t))$ is a shorthand for,

$$\vec{F}(\vec{r}(t)) = \vec{F}(x(t), y(t), z(t))$$

We can also write line integrals of vector fields as a line integral with respect to arc length as follows,

$$\int_C \vec{F}\cdot d\,\vec{r} = \int_C \vec{F}\cdot\vec{T}\,ds$$

where $\vec{T}(t)$ is the unit tangent vector and is given by,

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{\left\|\vec{r}\,'(t)\right\|}$$

If we use our knowledge on how to compute line integrals with respect to arc length we can see that this second form is equivalent to the first form given above.

$$\begin{aligned}
\int_C \vec{F}\cdot d\,\vec{r} &= \int_C \vec{F}\cdot\vec{T}\,ds \\
&= \int_a^b \vec{F}(\vec{r}(t))\cdot\frac{\vec{r}\,'(t)}{\left\|\vec{r}\,'(t)\right\|}\left\|\vec{r}\,'(t)\right\|dt \\
&= \int_a^b \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt
\end{aligned}$$

In general we use the first form to compute these line integral as it is usually much easier to use. Let's take a look at a couple of examples.

As with the two-dimensional version these three will often occur together so the shorthand we'll be using here is,

$$\int_C P\,dx + Q\,dy + R\,dz = \int_C P(x,y,z)\,dx + \int_C Q(x,y,z)\,dy + \int_C R(x,y,z)\,dz$$

Let's work an example.

Example 3 Evaluate $\int_C y\,dx + x\,dy + z\,dz$ where C is given by $x = \cos t$, $y = \sin t$, $z = t^2$, $0 \le t \le 2\pi$.

Solution

So, we already have the curve parameterized so there really isn't much to do other than evaluate the integral.

$$\begin{aligned}
\int_C y\,dx + x\,dy + z\,dz &= \int_C y\,dx + \int_C x\,dy + \int_C z\,dz \\
&= \int_0^{2\pi} \sin t(-\sin t)\,dt + \int_0^{2\pi} \cos t(\cos t)\,dt + \int_0^{2\pi} t^2(2t)\,dt \\
&= -\int_0^{2\pi} \sin^2 t\,dt + \int_0^{2\pi} \cos^2 t\,dt + \int_0^{2\pi} 2t^3\,dt \\
&= -\frac{1}{2}\int_0^{2\pi} (1-\cos(2t))\,dt + \frac{1}{2}\int_0^{2\pi} (1+\cos(2t))\,dt + \int_0^{2\pi} 2t^3\,dt \\
&= \left(-\frac{1}{2}\left(t - \frac{1}{2}\sin(2t) \right) + \frac{1}{2}\left(t + \frac{1}{2}\sin(2t) \right) + \frac{1}{2}t^4 \right)\Bigg|_0^{2\pi} \\
&= 8\pi^4
\end{aligned}$$

Example 1 Evaluate $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F}(x,y,z) = 8x^2yz\,\vec{i} + 5z\,\vec{j} - 4xy\,\vec{k}$ and C is the curve given by $\vec{r}(t) = t\,\vec{i} + t^2\,\vec{j} + t^3\,\vec{k}$, $0 \le t \le 1$.

Solution
Okay, we first need the vector field evaluated along the curve.

$$\vec{F}(\vec{r}(t)) = 8t^2(t^2)(t^3)\vec{i} + 5t^3\,\vec{j} - 4t(t^2)\vec{k} = 8t^7\,\vec{i} + 5t^3\,\vec{j} - 4t^3\,\vec{k}$$

Next we need the derivative of the parameterization.

$$\vec{r}'(t) = \vec{i} + 2t\,\vec{j} + 3t^2\,\vec{k}$$

Finally, let's get the dot product taken care of.

$$\vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) = 8t^7 + 10t^4 - 12t^5$$

dot product result = #
not vector

The line integral is then,

$$\begin{aligned}\int_C \vec{F} \cdot d\vec{r} &= \int_0^1 8t^7 + 10t^4 - 12t^5\,dt \\ &= \left(t^8 + 2t^5 - 2t^6\right)\Big|_0^1 \\ &= 1\end{aligned}$$

Example 2 Evaluate $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F}(x,y,z) = xz\,\vec{i} - yz\,\vec{k}$ and C is the line segment from $(-1,2,0)$ and $(3,0,1)$.

Solution
We'll first need the parameterization of the line segment. We saw how to get the parameterization of line segments in the first section on line integrals. We've been using the two dimensional version of this over the last couple of sections. Here is the parameterization for the line.

$$\begin{aligned}\vec{r}(t) &= (1-t)\langle -1,2,0\rangle + t\langle 3,0,1\rangle \\ &= \langle 4t-1, 2-2t, t\rangle, \qquad\qquad 0 \le t \le 1\end{aligned}$$

So, let's get the vector field evaluated along the curve.

$$\begin{aligned}\vec{F}(\vec{r}(t)) &= (4t-1)(t)\vec{i} - (2-2t)(t)\vec{k} \\ &= \left(4t^2 - t\right)\vec{i} - \left(2t - 2t^2\right)\vec{k}\end{aligned}$$

Now we need the derivative of the parameterization.

$$\vec{r}'(t) = \langle 4, -2, 1\rangle$$

wait, why?

The dot product is then,

$$\vec{F}\left(\vec{r}\left(t\right)\right)\bullet\vec{r}'\left(t\right)=4\left(4t^2-t\right)-\left(2t-2t^2\right)=18t^2-6t$$

The line integral becomes,

$$\begin{aligned}\int_C \vec{F}\bullet d\,\vec{r} &= \int_0^1 18t^2-6t\,dt\\ &=\left(6t^3-3t^2\right)\Big|_0^1\\ &=3\end{aligned}$$

Let's close this section out by doing one of these in general to get a nice relationship between line integrals of vector fields and line integrals with respect to x, y, and z.

Given the vector field $\vec{F}\left(x,y,z\right)=P\,\vec{i}+Q\,\vec{j}+R\,\vec{k}$ and the curve C parameterized by $\vec{r}\left(t\right)=x\left(t\right)\vec{i}+y\left(t\right)\vec{j}+z\left(t\right)\vec{k}$, $a\le t\le b$ the line integral is,

$$\begin{aligned}\int_C \vec{F}\bullet d\,\vec{r} &= \int_a^b \left(P\,\vec{i}+Q\,\vec{j}+R\,\vec{k}\right)\bullet\left(x'\,\vec{i}+y'\,\vec{j}+z'\,\vec{k}\right)dt\\ &=\int_a^b Px'+Qy'+Rz'\,dt\\ &=\int_a^b Px'\,dt+\int_a^b Qy'\,dt+\int_a^b Rz'\,dt\\ &=\int_C P\,dx+\int_C Q\,dy+\int_C R\,dz\\ &=\int_C P\,dx+Q\,dy+R\,dz\end{aligned}$$

So, we see that,

$$\int_C \vec{F}\bullet d\,\vec{r}=\int_C P\,dx+Q\,dy+R\,dz$$

Note that this gives us another method for evaluating line integrals of vector fields.

This also allows us to say the following about reversing the direction of the path with line integrals of vector fields.

Fact

$$\int_{-C} \vec{F}\bullet d\,\vec{r}=-\int_C \vec{F}\bullet d\,\vec{r}$$

This should make some sense given that we know that this is true for line integrals with respect to x, y, and/or z and that line integrals of vector fields can be defined in terms of line integrals with respect to x, y, and z.

Fundamental Theorem for Line Integrals

In Calculus I we had the Fundamental Theorem of Calculus that told us how to evaluate definite integrals. This told us,

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

was never good at that

It turns out that there is a version of this for line integrals over certain kinds of vector fields. Here it is.

Theorem

Suppose that C is a smooth curve given by $\vec{r}(t)$, $a \le t \le b$. Also suppose that f is a function whose gradient vector, $\boxed{\nabla f}$, is continuous on C. Then,

$$\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

Note that $\vec{r}(a)$ represents the initial point on C while $\vec{r}(b)$ represents the final point on C. Also, we did not specify the number of variables for the function since it is really immaterial to the theorem. The theorem will hold regardless of the number of variables in the function.

Only for gradient fields

Proof

This is a fairly straight forward proof.

For the purposes of the proof we'll assume that we're working in three dimensions, but it can be done in any dimension.

Let's start by just computing the line integral.

$$\begin{aligned}\int_C \nabla f \cdot d\vec{r} &= \int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}'(t)\,dt \\ &= \int_a^b \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}\right)dt\end{aligned}$$

Now, at this point we can use the Chain Rule to simplify the integrand as follows,

$$\begin{aligned}\int_C \nabla f \cdot d\vec{r} &= \int_a^b \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}\right)dt \\ &= \int_a^b \frac{d}{dt}\left[f(\vec{r}(t))\right]dt\end{aligned}$$

To finish this off we just need to use the Fundamental Theorem of Calculus for single integrals.

$$\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

■

Let's take a quick look at an example of using this theorem.

Example 1 Evaluate $\int\limits_C \nabla f \cdot d\vec{r}$ where $f(x,y,z) = \cos(\pi x) + \sin(\pi y) - xyz$ and C is *any* path that starts at $\left(1,\frac{1}{2},2\right)$ and ends at $(2,1,-1)$.

Solution
First let's notice that we didn't specify the path for getting from the first point to the second point. The reason for this is simple. The theorem above tells us that all we need are the initial and final points on the curve in order to evaluate this kind of line integral.

So, let $\vec{r}(t)$, $a \le t \le b$ be any path that starts at $\left(1,\frac{1}{2},2\right)$ and ends at $(2,1,-1)$. Then,

$$\vec{r}(a) = \left\langle 1,\frac{1}{2},2 \right\rangle \qquad\qquad \vec{r}(b) = \langle 2,1,-1 \rangle$$

The integral is then,

$$\begin{aligned} \int\limits_C \nabla f \cdot d\vec{r} &= f(2,1,-1) - f\left(1,\frac{1}{2},2\right) \\ &= \cos(2\pi) + \sin\pi - 2(1)(-1) - \left(\cos\pi + \sin\left(\frac{\pi}{2}\right) - 1\left(\frac{1}{2}\right)(2)\right) \\ &= 4 \end{aligned}$$

Notice that we also didn't need the gradient vector to actually do this line integral. However, for the practice of finding gradient vectors here it is,

$$\nabla f = \langle -\pi\sin(\pi x) - yz, \pi\cos(\pi y) - xz, -xy \rangle$$

The most important idea to get from this example is not how to do the integral as that's pretty simple, all we do is plug the final point and initial point into the function and subtract the two results. The important idea from this example (and hence about the Fundamental Theorem of Calculus) is that, for these kinds of line integrals, we didn't really need to know the path to get the answer. In other words, we could use any path we want and we'll always get the same results.

In the first section on line integrals (even though we weren't looking at vector fields) we saw that often when we change the path we will change the value of the line integral. We now have a type of line integral for which we know that changing the path will NOT change the value of the line integral.

Let's formalize this idea up a little. Here are some definitions. The first one we've already seen before, but it's been a while and it's important in this section so we'll give it again. The remaining definitions are new.

Definitions

First suppose that $\vec{F}$ is a continuous vector field in some domain D.

1. $\vec{F}$ is a **conservative** vector field if there is a function *f* such that $\vec{F} = \nabla f$. The function *f* is called a **potential function** for the vector field. We first saw this definition in the first section of this chapter.

2. $\int_C \vec{F}\cdot d\vec{r}$ is **independent of path** if $\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_2} \vec{F}\cdot d\vec{r}$ for any two paths C_1 and C_2 in D with the same initial and final points. for conservative fields i.e.

always but for non conservative fields if same start + end pts i.e.

3. A path C is called **closed** if its initial and final points are the same point. For example a circle is a closed path.

4. A path C is **simple** if it doesn't cross itself. A circle is a simple curve while a figure 8 type curve is not simple.

5. A region D is **open** if it doesn't contain any of its boundary points.

6. A region D is **connected** if we can connect any two points in the region with a path that lies completely in D.

7. A region D is **simply-connected** if it is connected and it contains no holes. We won't need this one until the next section, but it fits in with all the other definitions given here so this was a natural place to put the definition.

With these definitions we can now give some nice facts.

Facts — as gradient field —

1. $\int_C \boxed{\nabla f}\cdot d\vec{r}$ is independent of path.

   This is easy enough to prove since all we need to do is look at the theorem above. The theorem tells us that in order to evaluate this integral all we need are the initial and final points of the curve. This in turn tells us that the line integral must be independent of path.

2. If $\vec{F}$ is a conservative vector field then $\int_C \vec{F}\cdot d\vec{r}$ is independent of path.

   This fact is also easy enough to prove. If $\vec{F}$ is conservative then it has a potential function, f, and so the line integral becomes $\int_C \vec{F}\cdot d\vec{r} = \int_C \nabla f\cdot d\vec{r}$. Then using the first fact we know that this line integral must be independent of path.

3. If $\vec{F}$ is a continuous vector field on an open connected region D and if $\int_C \vec{F}\cdot d\vec{r}$ is independent of path (for any path in D) then $\vec{F}$ is a conservative vector field on D.

4. If $\int_C \vec{F}\cdot d\vec{r}$ is independent of path then $\int_C \vec{F}\cdot d\vec{r} = 0$ for every closed path C.

5. If $\int\limits_C \vec{F} \cdot d\,\vec{r} = 0$ for every closed path C then $\int\limits_C \vec{F} \cdot d\,\vec{r}$ is independent of path.

These are some nice facts to remember as we work with line integrals over vector fields. Also notice that 2 & 3 and 4 & 5 are converses of each other.

Conservative Vector Fields

In the previous section we saw that if we knew that the vector field $\vec{F}$ was conservative then $\int_C \vec{F} \bullet d\,\vec{r}$ was independent of path. This in turn means that we can easily evaluate this line integral provided we can find a potential function for $\vec{F}$. gradiant field?

In this section we want to look at two questions. First, given a vector field $\vec{F}$ is there any way of determining if it is a conservative vector field? Secondly, if we know that $\vec{F}$ is a conservative vector field how do we go about finding a potential function for the vector field?

The first question is easy to answer at this point if we have a two-dimensional vector field. For higher dimensional vector fields we'll need to wait until the final section in this chapter to answer this question. With that being said let's see how we do it for two-dimensional vector fields.

Theorem

Let $\vec{F} = P\,\vec{i} + Q\,\vec{j}$ be a vector field on an <u>open</u> and <u>simply-connected</u> region *D*. Then if *P* and *Q* have continuous first order partial derivatives in *D* and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

the vector field $\vec{F}$ is conservative.

wait why is it conservative if same

Let's take a look at a couple of examples.

Example 1 Determine if the following vector fields are conservative or not.

(a) $\vec{F}(x,y) = (x^2 - yx)\vec{i} + (y^2 - xy)\vec{j}$ [Solution]

(b) $\vec{F}(x,y) = (2x\mathbf{e}^{xy} + x^2 y\mathbf{e}^{xy})\vec{i} + (x^3\mathbf{e}^{xy} + 2y)\vec{j}$ [Solution]

Solution

Okay, there really isn't too much to these. All we do is identify *P* and *Q* then take a couple of derivatives and compare the results.

(a) $\vec{F}(x,y) = (x^2 - yx)\vec{i} + (y^2 - xy)\vec{j}$

In this case here is *P* and *Q* and the appropriate partial derivatives.

$$P = x^2 - yx \qquad \frac{\partial P}{\partial y} = -x$$

$$Q = y^2 - xy \qquad \frac{\partial Q}{\partial x} = -y$$

So, since the two partial derivatives are not the same this vector field is NOT conservative.

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(b) $\vec{F}(x,y) = \left(2x\mathbf{e}^{xy} + x^2 y\mathbf{e}^{xy}\right)\vec{i} + \left(x^3\mathbf{e}^{xy} + 2y\right)\vec{j}$

Here is P and Q as well as the appropriate derivatives.

$$P = 2x\mathbf{e}^{xy} + x^2 y\mathbf{e}^{xy} \qquad \frac{\partial P}{\partial y} = 2x^2\mathbf{e}^{xy} + x^2\mathbf{e}^{xy} + x^3 y\mathbf{e}^{xy} = 3x^2\mathbf{e}^{xy} + x^3 y\mathbf{e}^{xy}$$

$$Q = x^3\mathbf{e}^{xy} + 2y \qquad \frac{\partial Q}{\partial x} = 3x^2\mathbf{e}^{xy} + x^3 y\mathbf{e}^{xy}$$

The two partial derivatives are equal and so this is a conservative vector field.

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Now that we know how to identify if a two-dimensional vector field is conservative we need to address how to find a potential function for the vector field. This is actually a fairly simple process. First, let's assume that the vector field is conservative and so we know that a potential function, $f(x,y)$ exists. We can then say that,

$$\vec{F} = \nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} = P\,\vec{i} + Q\,\vec{j} = \vec{F}$$

didn't we learn the 3 things

opps

Or by setting components equal we have,

$$\frac{\partial f}{\partial x} = P \qquad \text{and} \qquad \frac{\partial f}{\partial y} = Q$$

By integrating each of these with respect to the appropriate variable we can arrive at the following two equations.

$$f(x,y) = \int P(x,y)\,dx \qquad \text{or} \qquad f(x,y) = \int Q(x,y)\,dy$$

We saw this kind of integral briefly at the end of the section on iterated integrals in the previous chapter.

It is usually best to see how we use these two facts to find a potential function in an example or two.

did we learn this? -seems unfamiliar

Example 2 Determine if the following vector fields are conservative and find a potential function for the vector field if it is conservative.

(a) $\vec{F} = \left(2x^3y^4 + x\right)\vec{i} + \left(2x^4y^3 + y\right)\vec{j}$ [Solution]

(b) $\vec{F}(x,y) = \left(2x\mathbf{e}^{xy} + x^2y\mathbf{e}^{xy}\right)\vec{i} + \left(x^3\mathbf{e}^{xy} + 2y\right)\vec{j}$ [Solution]

Solution

(a) $\vec{F} = \left(2x^3y^4 + x\right)\vec{i} + \left(2x^4y^3 + y\right)\vec{j}$

Let's first identify P and Q and then check that the vector field is conservative..

$$P = 2x^3y^4 + x \qquad \frac{\partial P}{\partial y} = 8x^3y^3$$

$$Q = 2x^4y^3 + y \qquad \frac{\partial Q}{\partial x} = 8x^3y^3$$

So, the vector field is conservative. Now let's find the potential function. From the first fact above we know that,

$$\frac{\partial f}{\partial x} = 2x^3y^4 + x \qquad \frac{\partial f}{\partial y} = 2x^4y^3 + y$$

From these we can see that

$$f(x,y) = \int 2x^3y^4 + x\,dx \qquad \text{or} \qquad f(x,y) = \int 2x^4y^3 + y\,dy$$

We can use either of these to get the process started. Recall that we are going to have to be careful with the "constant of integration" which ever integral we choose to use. For this example let's work with the first integral and so that means that we are asking what function did we differentiate with respect to x to get the integrand. This means that the "constant of integration" is going to have to be a function of y since any function consisting only of y and/or constants will differentiate to zero when taking the partial derivative with respect to x.

Here is the first integral.

$$\begin{aligned} f(x,y) &= \int 2x^3y^4 + x\,dx \\ &= \frac{1}{2}x^4y^4 + \frac{1}{2}x^2 + h(y) \end{aligned}$$

oh the reversed thing
need to practice this

where $h(y)$ is the "constant of integration".

We now need to determine $h(y)$. This is easier that it might at first appear to be. To get to this point we've used the fact that we knew P, but we will also need to use the fact that we know Q to complete the problem. Recall that Q is really the derivative of f with respect to y. So, if we differentiate our function with respect to y we know what it should be.

So, let's differentiate f (including the $h(y)$) with respect to y and set it equal to Q since that is what the derivative is supposed to be.

$$\frac{\partial f}{\partial y} = 2x^4y^3 + h'(y) = 2x^4y^3 + y = Q$$

From this we can see that,

$$h'(y) = y$$

Notice that since $h'(y)$ is a function only of y so if there are any x's in the equation at this point we will know that we've made a mistake. At this point finding $h(y)$ is simple.

$$h(y) = \int h'(y)\,dy = \int y\,dy = \frac{1}{2}y^2 + c$$

So, putting this all together we can see that a potential function for the vector field is,

$$f(x,y) = \frac{1}{2}x^4y^4 + \frac{1}{2}x^2 + \frac{1}{2}y^2 + c$$

Note that we can always check our work by verifying that $\nabla f = \vec{F}$. Also note that because the c can be anything there are an infinite number of possible potential functions, although they will only vary by an additive constant.

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(b) $\vec{F}(x,y) = \left(2x\mathbf{e}^{xy} + x^2y\mathbf{e}^{xy}\right)\vec{i} + \left(x^3\mathbf{e}^{xy} + 2y\right)\vec{j}$

Okay, this one will go a lot faster since we don't need to go through as much explanation. We've already verified that this vector field is conservative in the first set of examples so we won't bother redoing that.

Let's start with the following,

$$\frac{\partial f}{\partial x} = 2x\mathbf{e}^{xy} + x^2y\mathbf{e}^{xy} \qquad\qquad \frac{\partial f}{\partial y} = x^3\mathbf{e}^{xy} + 2y$$

This means that we can do either of the following integrals,

$$f(x,y) = \int 2x\mathbf{e}^{xy} + x^2y\mathbf{e}^{xy}\,dx \qquad \text{or} \qquad f(x,y) = \int x^3\mathbf{e}^{xy} + 2y\,dy$$

While we can do either of these the first integral would be somewhat unpleasant as we would need to do integration by parts on each portion. On the other hand the second integral is fairly simple since the second term only involves y's and the first term can be done with the substitution $u = xy$. So, from the second integral we get,

$$f(x,y) = x^2\mathbf{e}^{xy} + y^2 + h(x)$$

Notice that this time the "constant of integration" will be a function of x. If we differentiate this with respect to x and set equal to P we get,

$$\frac{\partial f}{\partial x} = 2x\mathbf{e}^{xy} + x^2y\mathbf{e}^{xy} + h'(x) = 2x\mathbf{e}^{xy} + x^2y\mathbf{e}^{xy} = P$$

So, in this case it looks like,

$$h'(x)=0 \qquad \Rightarrow \qquad h(x)=c$$

So, in this case the "constant of integration" really was a constant. Sometimes this will happen and sometimes it won't.

Here is the potential function for this vector field.

$$f(x,y)=x^2\mathbf{e}^{xy}+y^2+c$$

[Return to Problems]

Now, as noted above we don't have a way (yet) of determining if a three-dimensional vector field is conservative or not. However, if we are given that a three-dimensional vector field is conservative finding a potential function is similar to the above process, although the work will be a little more involved.

In this case we will use the fact that,

$$\nabla f=\frac{\partial f}{\partial x}\vec{i}+\frac{\partial f}{\partial y}\vec{j}+\frac{\partial f}{\partial z}\vec{k}=P\,\vec{i}+Q\,\vec{j}+R\,\vec{k}=\vec{F}$$

Let's take a quick look at an example.

Example 3 Find a potential function for the vector field,

$$\vec{F}=2xy^3z^4\,\vec{i}+3x^2y^2z^4\,\vec{j}+4x^2y^3z^3\,\vec{k}$$

Solution

Okay, we'll start off with the following equalities.

$$\frac{\partial f}{\partial x}=2xy^3z^4 \qquad \frac{\partial f}{\partial y}=3x^2y^2z^4 \qquad \frac{\partial f}{\partial z}=4x^2y^3z^3$$

To get started we can integrate the first one with respect to x, the second one with respect to y, or the third one with respect to z. Let's integrate the first one with respect to x.

$$f(x,y,z)=\int 2xy^3z^4\,dx=x^2y^3z^4+g(y,z)$$

Note that this time the "constant of integration" will be a function of both y and z since differentiating anything of that form with respect to x will differentiate to zero.

Now, we can differentiate this with respect to y and set it equal to Q. Doing this gives,

$$\frac{\partial f}{\partial y}=3x^2y^2z^4+g_y(y,z)=3x^2y^2z^4=Q$$

Of course we'll need to take the partial derivative of the constant of integration since it is a function of two variables. It looks like we've now got the following,

$$g_y(y,z)=0 \qquad \Rightarrow \qquad g(y,z)=h(z)$$

Since differentiating $g(y,z)$ with respect to y gives zero then $g(y,z)$ could at most be a function of z. This means that we now know the potential function must be in the following form.

$$f(x,y,z) = x^2y^3z^4 + h(z)$$

To finish this out all we need to do is differentiate with respect to z and set the result equal to R.

$$\frac{\partial f}{\partial z} = 4x^2y^3z^3 + h'(z) = 4x^2y^3z^3 = R$$

So,

$$h'(z) = 0 \qquad \Rightarrow \qquad h(z) = c$$

The potential function for this vector field is then,

$$f(x,y,z) = x^2y^3z^4 + c$$

Note that to keep the work to a minimum we used a fairly simple potential function for this example. It might have been possible to guess what the potential function was based simply on the vector field. However, we should be careful to remember that this usually won't be the case and often this process is required.

Also, there were several other paths that we could have taken to find the potential function. Each would have gotten us the same result.

Let's work one more slightly (and only slightly) more complicated example.

f = main function
– why is it called
potential function

Example 4 Find a potential function for the vector field,

$$\vec{F} = \left(2x\cos(y) - 2z^3\right)\vec{i} + \left(3 + 2y\mathbf{e}^z - x^2\sin(y)\right)\vec{j} + \left(y^2\mathbf{e}^z - 6xz^2\right)\vec{k}$$

Solution

Here are the equalities for this vector field.

$$\frac{\partial f}{\partial x} = 2x\cos(y) - 2z^3 \qquad \frac{\partial f}{\partial y} = 3 + 2y\mathbf{e}^z - x^2\sin(y) \qquad \frac{\partial f}{\partial z} = y^2\mathbf{e}^z - 6xz^2$$

For this example let's integrate the third one with respect to z.

$$f(x,y,z) = \int y^2\mathbf{e}^z - 6xz^2\,dz = y^2\mathbf{e}^z - 2xz^3 + g(x,y)$$

The "constant of integration" for this integration will be a function of both x and y.

Now, we can differentiate this with respect to x and set it equal to P. Doing this gives,

$$\frac{\partial f}{\partial x} = -2z^3 + g_x(x,y) = 2x\cos(y) - 2z^3 = P$$

So, it looks like we've now got the following,

$$g_x(x,y) = 2x\cos(y) \qquad \Rightarrow \qquad g(x,y) = x^2\cos(y) + h(y)$$

The potential function for this problem is then,

$$f(x,y,z) = y^2\mathbf{e}^z - 2xz^3 + x^2\cos(y) + h(y)$$

To finish this out all we need to do is differentiate with respect to y and set the result equal to Q.

$$\frac{\partial f}{\partial y} = 2y\mathbf{e}^{z} - x^2\sin(y) + h'(y) = 3 + 2y\mathbf{e}^{z} - x^2\sin(y) = Q$$

So,

$$h'(y) = 3 \qquad \Rightarrow \qquad h(y) = 3y + c$$

The potential function for this vector field is then,

$$f(x,y,z) = y^2\mathbf{e}^{z} - 2xz^3 + x^2\cos(y) + 3y + c$$

So, a little more complicated than the others and there are again many different paths that we could have taken to get the answer.

We need to work one final example in this section.

Example 5 Evaluate $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F} = \left(2x^3y^4 + x\right)\vec{i} + \left(2x^4y^3 + y\right)\vec{j}$ and C is given by $\vec{r}(t) = \left(t\cos(\pi t) - 1\right)\vec{i} + \sin\left(\frac{\pi t}{2}\right)\vec{j}$, $0 \le t \le 1$.

Solution

Now, we could use the techniques we discussed when we first looked at line integrals of vector fields however that would be particularly unpleasant solution.

Instead, let's take advantage of the fact that we know from Example 2a above this vector field is conservative and that a potential function for the vector field is,

$$f(x,y) = \frac{1}{2}x^4y^4 + \frac{1}{2}x^2 + \frac{1}{2}y^2 + c$$

Using this we know that integral must be independent of path and so all we need to do is use the theorem from the previous section to do the evaluation.

$$\int_C \vec{F}\cdot d\vec{r} = \int_C \nabla f\cdot d\vec{r} = f\left(\vec{r}(1)\right) - f\left(\vec{r}(0)\right)$$

where,

$$\vec{r}(1) = \langle -2, 1\rangle \qquad \vec{r}(0) = \langle -1, 0\rangle$$

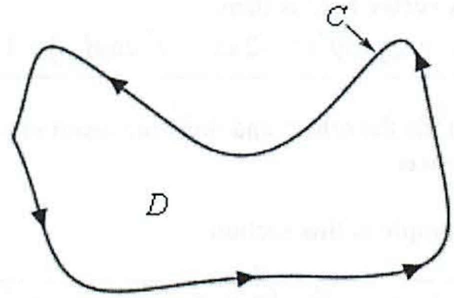
So, the integral is,

$$\begin{aligned}\int_C \vec{F}\cdot d\vec{r} &= f(-2,1) - f(-1,0)\\ &= \left(\frac{21}{2} + c\right) - \left(\frac{1}{2} + c\right)\\ &= 10\end{aligned}$$

Green's Theorem

In this section we are going to investigate the relationship between certain kinds of line integrals (on closed paths) and double integrals.

Let's start off with a simple (recall that this means that it doesn't cross itself) closed curve C and let D be the region enclosed by the curve. Here is a sketch of such a curve and region.



First, notice that because the curve is simple and closed there are no holes in the region D. Also notice that a direction has been put on the curve. We will use the convention here that the curve C has a **positive orientation** if it is traced out in a counter-clockwise direction. Another way to think of a positive orientation (that will cover much more general curves as well see later) is that as we traverse the path following the positive orientation the region D must always be on the left.

Given curves/regions such as this we have the following theorem.

Green's Theorem

Let C be a positively oriented, piecewise smooth, simple, closed curve and let D be the region enclosed by the curve. If P and Q have continuous first order partial derivatives on D then,

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

Before working some examples there are some alternate notations that we need to acknowledge. When working with a line integral in which the path satisfies the condition of Green's Theorem we will often denote the line integral as,

$$\oint_C P\,dx + Q\,dy \qquad \text{or} \qquad \circlearrowleft\!\!\!\!\!\!\int_C P\,dx + Q\,dy$$

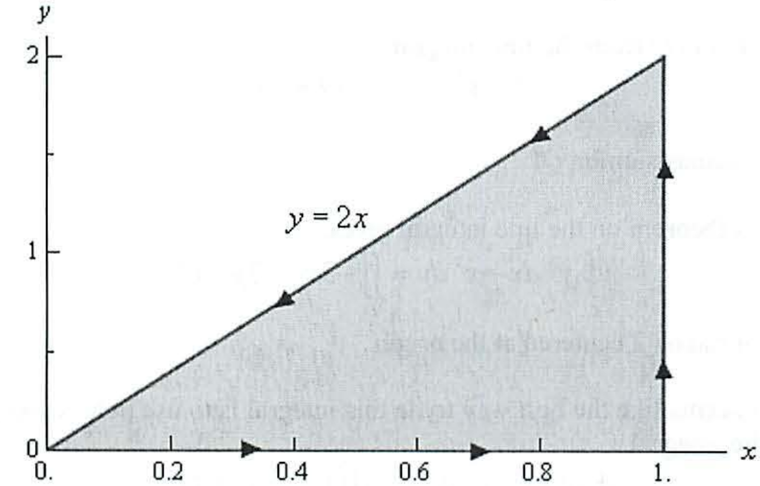
Both of these notations do assume that C satisfies the conditions of Green's Theorem so be careful in using them.

Also, sometimes the curve C is not thought of as a separate curve but instead as the boundary of some region D and in these cases you may see C denoted as ∂D.

Let's work a couple of examples.

Example 1 Use Green's Theorem to evaluate $\oint_C xy\,dx + x^2y^3\,dy$ where C is the triangle with vertices $(0,0)$, $(1,0)$, $(1,2)$ with positive orientation.

Solution
Let's first sketch C and D for this case to make sure that the conditions of Green's Theorem are met for C and will need the sketch of D to evaluate the double integral.



So, the curve does satisfy the conditions of Green's Theorem and we can see that the following inequalities will define the region enclosed.

$$0 \le x \le 1 \qquad\qquad 0 \le y \le 2x$$

We can identify P and Q from the line integral. Here they are.

$$P = xy \qquad\qquad Q = x^2y^3$$

So, using Green's Theorem the line integral becomes,

$$\oint_C xy\,dx + x^2y^3\,dy = \iint_D 2xy^3 - x\,dA$$

rather than calculating this, do this → right?

$$= \int_0^1\int_0^{2x} 2xy^3 - x\,dy\,dx$$

$$= \int_0^1 \left(\frac{1}{2}xy^4 - xy\right)\Big|_0^{2x} dx$$

$$= \int_0^1 8x^5 - 2x^2\,dx$$

$$= \left(\frac{4}{3}x^6 - \frac{2}{3}x^3\right)\Big|_0^1$$

$$= \frac{2}{3}$$

isn't the pt of GT is that you could use either one

Example 2 Evaluate $\oint_C y^3\,dx - x^3\,dy$ where C is the positively oriented circle of radius 2 centered at the origin.

Solution
Okay, a circle will satisfy the conditions of Green's Theorem since it is closed and simple and so there really isn't a reason to sketch it.

Let's first identify P and Q from the line integral.

$$P = y^3 \qquad\qquad Q = -x^3$$

Be careful with the minus sign on Q!

Now, using Green's theorem on the line integral gives,

$$\oint_C y^3\,dx - x^3\,dy = \iint_D -3x^2 - 3y^2\,dA$$

where D is a disk of radius 2 centered at the origin.

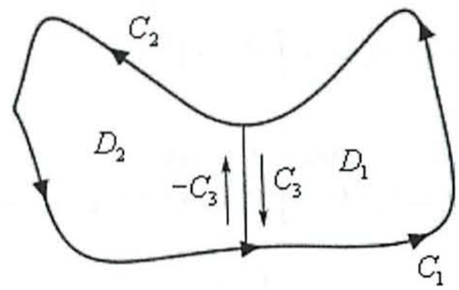
Since D is a disk it seems like the best way to do this integral is to use polar coordinates. Here is the evaluation of the integral.

$$\begin{aligned}\oint_C y^3\,dx - x^3\,dy &= -3\iint_D \left(x^2 + y^2\right) dA \\ &= -3\int_0^{2\pi}\int_0^2 \underline{r^3\,dr\,d\theta} \quad \text{Polor coords} \\ &= -3\int_0^{2\pi} \left.\frac{1}{4}r^4\right|_0^2 d\theta \\ &= -3\int_0^{2\pi} 4\,d\theta \\ &= -24\pi\end{aligned}$$

So, Green's theorem, as stated, will not work on regions that have holes in them. However, many regions do have holes in them. So, let's see how we can deal with those kinds of regions.

Let's start with the following region. Even though this region doesn't have any holes in it the arguments that we're going to go through will be similar to those that we'd need for regions with holes in them, except it will be a little easier to deal with and write down.

learned more in math than I thought



The region D will be $D_1 \cup D_2$ and recall that the symbol $\cup$ is called the union and means that we'll D consists of both D_1 and D_2. The boundary of D_1 is $C_1 \cup C_3$ while the boundary of D_2 is $C_2 \cup (-C_3)$ and notice that both of these boundaries are positively oriented. As we traverse each boundary the corresponding region is always on the left. Finally, also note that we can think of the whole boundary, C, as,

$$C = (C_1 \cup C_3) \cup (C_2 \cup (-C_3)) = C_1 \cup C_2$$

since both C_3 and $-C_3$ will "cancel" each other out.

Now, let's start with the following double integral and use a basic property of double integrals to break it up.

$$\iint_D (Q_x - P_y)\, dA = \iint_{D_1 \cup D_2} (Q_x - P_y)\, dA = \iint_{D_1} (Q_x - P_y)\, dA + \iint_{D_2} (Q_x - P_y)\, dA$$ "holes"

Next, use Green's theorem on each of these and again use the fact that we can break up line integrals into separate line integrals for each portion of the boundary.

$$\begin{aligned}
\iint_D (Q_x - P_y)\, dA &= \iint_{D_1} (Q_x - P_y)\, dA + \iint_{D_2} (Q_x - P_y)\, dA \\
&= \oint_{C_1 \cup C_3} P\,dx + Q\,dy + \oint_{C_2 \cup (-C_3)} P\,dx + Q\,dy \\
&= \oint_{C_1} P\,dx + Q\,dy + \oint_{C_3} P\,dx + Q\,dy + \oint_{C_2} P\,dx + Q\,dy + \oint_{-C_3} P\,dx + Q\,dy
\end{aligned}$$

Next, we'll use the fact that,

$$\oint_{-C_3} P\,dx + Q\,dy = -\oint_{C_3} P\,dx + Q\,dy$$

Recall that changing the orientation of a curve with line integrals with respect to x and/or y will simply change the sign on the integral. Using this fact we get,

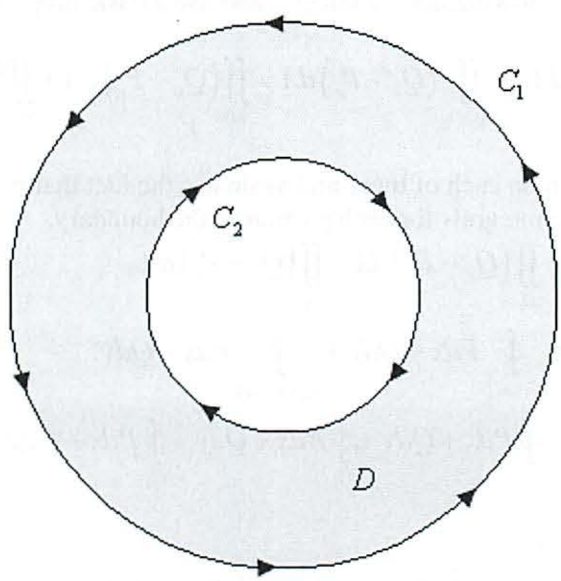
$$\begin{aligned}\iint_D \left(Q_x - P_y\right) dA &= \oint_{C_1} P\,dx + Q\,dy + \oint_{C_3} P\,dx + Q\,dy + \oint_{C_2} P\,dx + Q\,dy - \oint_{C_3} P\,dx + Q\,dy \\ &= \oint_{C_1} P\,dx + Q\,dy + \oint_{C_2} P\,dx + Q\,dy\end{aligned}$$

Finally, put the line integrals back together and we get,

$$\begin{aligned}\iint_D \left(Q_x - P_y\right) dA &= \oint_{C_1} P\,dx + Q\,dy + \oint_{C_2} P\,dx + Q\,dy \\ &= \oint_{C_1 \cup C_2} P\,dx + Q\,dy \\ &= \oint_C P\,dx + Q\,dy\end{aligned}$$

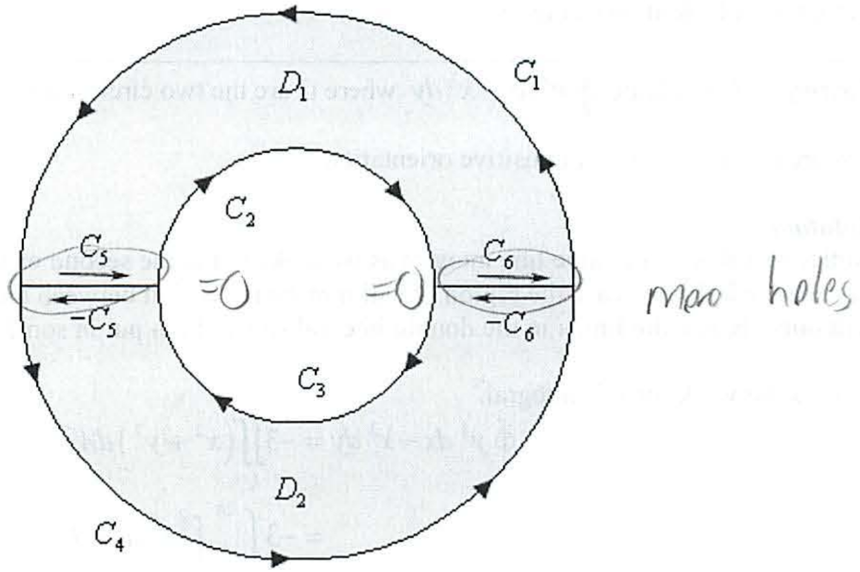
So, what did we learn from this? If you think about it this was just a lot of work and all we got out of it was the result from Green's Theorem which we already knew to be true. What this exercise has shown us is that if we break a region up as we did above then the portion of the line integral on the pieces of the curve that are in the middle of the region (each of which are in the opposite direction) will cancel out. This idea will help us in dealing with regions that have holes in them.

To see this let's look at a ring.



Notice that both of the curves are oriented positively since the region D is on the left side as we traverse the curve in the indicated direction. Note as well that the curve C_2 seems to violate the original definition of positive orientation. We originally said that a curve had a positive orientation if it was traversed in a counter-clockwise direction. However, this was only for regions that do not have holes. For the boundary of the hole this definition won't work and we need to resort to the second definition that we gave above.

Now, since this region has a hole in it we will apparently not be able to use Green's Theorem on any line integral with the curve $C = C_1 \cup C_2$. However, if we cut the disk in half and rename all the various portions of the curves we get the following sketch.



The boundary of the upper portion (D_1)of the disk is $C_1 \cup C_2 \cup C_5 \cup C_6$ and the boundary on the lower portion (D_2)of the disk is $C_3 \cup C_4 \cup (-C_5) \cup (-C_6)$. Also notice that we can use Green's Theorem on each of these new regions since they don't have any holes in them. This means that we can do the following,

$$\iint_D (Q_x - P_y)\,dA = \iint_{D_1} (Q_x - P_y)\,dA + \iint_{D_2} (Q_x - P_y)\,dA$$
$$= \oint_{C_1 \cup C_2 \cup C_5 \cup C_6} P\,dx + Q\,dy + \oint_{C_3 \cup C_4 \cup (-C_5) \cup (-C_6)} P\,dx + Q\,dy$$

Now, we can break up the line integrals into line integrals on each piece of the boundary. Also recall from the work above that boundaries that have the same curve, but opposite direction will cancel. Doing this gives,

$$\iint_D (Q_x - P_y)\,dA = \iint_{D_1} (Q_x - P_y)\,dA + \iint_{D_2} (Q_x - P_y)\,dA$$
$$= \oint_{C_1} P\,dx + Q\,dy + \oint_{C_2} P\,dx + Q\,dy + \oint_{C_3} P\,dx + Q\,dy + \oint_{C_4} P\,dx + Q\,dy$$

But at this point we can add the line integrals back up as follows,

$$\iint_D (Q_x - P_y)\,dA = \oint_{C_1 \cup C_2 \cup C_3 \cup C_4} P\,dx + Q\,dy$$
$$= \oint_C P\,dx + Q\,dy$$

The end result of all of this is that we could have just used Green's Theorem on the disk from the start even though there is a hole in it. This will be true in general for regions that have holes in them.

Let's take a look at an example.

Example 3 Evaluate $\oint_C y^3\,dx - x^3\,dy$ where C are the two circles of radius 2 and radius 1 centered at the origin with positive orientation.

Solution

Notice that this is the same line integral as we looked at in the second example and only the curve has changed. In this case the region D will now be the region between these two circles and that will only change the limits in the double integral so we'll not put in some of the details here.

Here is the work for this integral.

$$\begin{aligned}\oint_C y^3\,dx - x^3\,dy &= -3\iint_D \left(x^2+y^2\right)dA \\ &= -3\int_0^{2\pi}\int_1^2 r^3\,dr\,d\theta \\ &= -3\int_0^{2\pi} \left.\frac{1}{4}r^4\right|_1^2 d\theta \\ &= -3\int_0^{2\pi}\frac{15}{4}\,d\theta \\ &= -\frac{45\pi}{2}\end{aligned}$$

We will close out this section with an interesting application of Green's Theorem. Recall that we can determine the area of a region D with the following double integral.

$$A = \iint_D dA$$

Let's think of this double integral as the result of using Green's Theorem. In other words, let's assume that

$$Q_x - P_y = 1$$

and see if we can get some functions P and Q that will satisfy this.

There are many functions that will satisfy this. Here are some of the more common functions.

$$\begin{array}{c|c|c} \begin{aligned} P&=0\\ Q&=x\end{aligned} & \begin{aligned} P&=-y\\ Q&=0\end{aligned} & \begin{aligned} P&=-\frac{y}{2}\\ Q&=\frac{x}{2}\end{aligned}\end{array}$$

Then, if we use Green's Theorem in reverse we see that the area of the region D can also be computed by evaluating any of the following line integrals.

$$A = \oint_C x\,dy = -\oint_C y\,dx = \frac{1}{2}\oint_C x\,dy - y\,dx$$

where C is the boundary of the region D.

Let's take a quick look at an example of this.

Example 4 Use Green's Theorem to find the area of a disk of radius a.

Solution
We can use either of the integrals above, but the third one is probably the easiest. So,

$$A = \frac{1}{2}\oint_C x\,dy - y\,dx$$

where C is the circle of radius a. So, to do this we'll need a parameterization of C. This is,

$$x = a\cos t \qquad y = a\sin t \qquad 0 \le t \le 2\pi$$

The area is then,

$$\begin{aligned} A &= \frac{1}{2}\oint_C x\,dy - y\,dx \\ &= \frac{1}{2}\left(\int_0^{2\pi} a\cos t\left(a\cos t\right)dt - \int_0^{2\pi} a\sin t\left(-a\sin t\right)dt\right) \\ &= \frac{1}{2}\int_0^{2\pi} a^2\cos^2 t + a^2\sin^2 t\,dt \\ &= \frac{1}{2}\int_0^{2\pi} a^2\,dt \\ &= \pi a^2 \end{aligned}$$

Curl and Divergence

In this section we are going to introduce a couple of new concepts, the curl and the divergence of a vector.

Let's start with the curl. Given the vector field $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ the curl is defined to be,

$$\text{curl}\,\vec{F} = \left(R_y - Q_z\right)\vec{i} + \left(P_z - R_x\right)\vec{j} + \left(Q_x - P_y\right)\vec{k}$$

There is another (potentially) easier definition of the curl of a vector field. We use it we will first need to define the ∇ *operator*. This is defined to be, Work

del
$$\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$$

We use this as if it's a function in the following manner.

$$\nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$$

So, whatever function is listed after the ∇ is substituted into the partial derivatives. Note as well that when we look at it in this light we simply get the gradient vector.

Using the ∇ we can define the curl as the following cross product,

$$\text{curl}\,\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$ ← Short cut

We have a couple of nice facts that use the curl of a vector field.

Facts

1. If $f(x, y, z)$ has continuous second order partial derivatives then $\text{curl}(\nabla f) = \vec{0}$. This is easy enough to check by plugging into the definition of the derivative so we'll leave it to you to check.

2. If $\vec{F}$ is a conservative vector field then $\text{curl}\,\vec{F} = \vec{0}$. This is a direct result of what it means to be a conservative vector field and the previous fact.

3. If $\vec{F}$ is defined on all of $\mathbb{R}^3$ whose components have continuous first order partial derivative and $\text{curl}\,\vec{F} = \vec{0}$ then $\vec{F}$ is a conservative vector field. This is not so easy to verify and so we won't try.

what check for, right?

Example 1 Determine if $\vec{F} = x^2 y\,\vec{i} + xyz\,\vec{j} - x^2y^2\,\vec{k}$ is a conservative vector field.

Solution
So all that we need to do is compute the curl and see if we get the zero vector or not.

$$\begin{aligned}\operatorname{curl}\vec{F} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 y & xyz & -x^2y^2 \end{vmatrix} \\ &= -2x^2y\,\vec{i} + yz\,\vec{k} - \left(-2xy^2\,\vec{j}\right) - xy\,\vec{i} - x^2\vec{k} \\ &= -\left(2x^2y + xy\right)\vec{i} + 2xy^2\,\vec{j} + \left(yz - x^2\right)\vec{k} \\ &\neq \vec{0}\end{aligned}$$

So, the curl isn't the zero vector and so this vector field is not conservative.

Next we should talk about a physical interpretation of the curl. Suppose that $\vec{F}$ is the velocity field of a flowing fluid. Then $\operatorname{curl}\vec{F}$ represents the tendency of particles at the point (x, y, z) to rotate about the axis that points in the direction of $\operatorname{curl}\vec{F}$. If $\operatorname{curl}\vec{F} = \vec{0}$ then the fluid is called irrotational.

? work – or clarify intepretation

Let's now talk about the second new concept in this section. Given the vector field $\vec{F} = P\,\vec{i} + Q\,\vec{j} + R\,\vec{k}$ the divergence is defined to be,

$$\operatorname{div}\vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

flux

There is also a definition of the divergence in terms of the ∇ operator. The divergence can be defined in terms of the following dot product.

$$\operatorname{div}\vec{F} = \nabla\bullet\vec{F}$$

Example 2 Compute $\operatorname{div}\vec{F}$ for $\vec{F} = x^2 y\,\vec{i} + xyz\,\vec{j} - x^2y^2\,\vec{k}$

Solution
There really isn't much to do here other than compute the divergence.

$$\operatorname{div}\vec{F} = \frac{\partial}{\partial x}\left(x^2y\right) + \frac{\partial}{\partial y}\left(xyz\right) + \frac{\partial}{\partial z}\left(-x^2y^2\right) = 2xy + xz$$

We also have the following fact about the relationship between the curl and the divergence.

$$\operatorname{div}\left(\operatorname{curl}\vec{F}\right) = 0$$

Example 3 Verify the above fact for the vector field $\vec{F} = yz^2\,\vec{i} + xy\,\vec{j} + yz\,\vec{k}$.

Solution
Let's first compute the curl.

$$\begin{aligned}\operatorname{curl}\vec{F} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ yz^2 & xy & yz \end{vmatrix} \\ &= z\,\vec{i} + 2yz\,\vec{j} + y\,\vec{k} - z^2\vec{k} \\ &= z\vec{i} + 2yz\,\vec{j} + \left(y - z^2\right)\vec{k}\end{aligned}$$

Now compute the divergence of this.

$$\operatorname{div}\left(\operatorname{curl}\vec{F}\right) = \frac{\partial}{\partial x}(z) + \frac{\partial}{\partial y}(2yz) + \frac{\partial}{\partial z}\left(y - z^2\right) = 2z - 2z = 0$$

We also have a physical interpretation of the divergence. If we again think of $\vec{F}$ as the velocity field of a flowing fluid then $\operatorname{div}\vec{F}$ represents the net rate of change of the mass of the fluid flowing from the point (x, y, z) per unit volume. This can also be thought of as the tendency of a fluid to diverge from a point. If $\operatorname{div}\vec{F} = 0$ then the $\vec{F}$ is called incompressible.

not flux — reread carefully

The next topic that we want to briefly mention is the **Laplace** operator. Let's first take a look at,

$$\operatorname{div}(\nabla f) = \nabla\bullet\nabla f = f_{xx} + f_{yy} + f_{zz}$$

The Laplace operator is then defined as,

$$\nabla^2 = \nabla\bullet\nabla$$

The Laplace operator arises naturally in many fields including heat transfer and fluid flow.

The final topic in this section is to give two vector forms of Green's Theorem. The first form uses the curl of the vector field and is,

$$\oint_C \vec{F}\bullet d\,\vec{r} = \iint_D \left(\operatorname{curl}\vec{F}\right)\bullet\vec{k}\,dA$$

where $\vec{k}$ is the standard unit vector in the positive z direction.

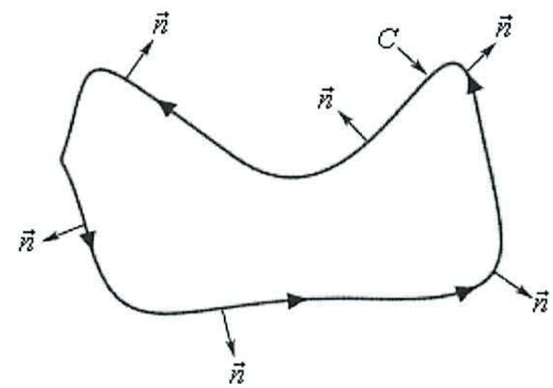
The second form uses the divergence. In this case we also need the outward unit normal to the curve C. If the curve is parameterized by

$$\vec{r}(t) = x(t)\,\vec{i} + y(t)\,\vec{j}$$

then the outward unit normal is given by,

$$\vec{n} = \frac{y'(t)}{\left\|\vec{r}\,'(t)\right\|}\vec{i} - \frac{x'(t)}{\left\|\vec{r}\,'(t)\right\|}\vec{j}$$

Here is a sketch illustrating the outward unit normal for some curve C at various points.



The vector form of Green's Theorem that uses the divergence is given by,

$$\oint_C \vec{F} \cdot \vec{n}\, ds = \iint_D \operatorname{div} \vec{F}\, dA$$

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The idea of divergence and curl

Vector fields

We can think of a vector-valued function $\mathbf{F} : \mathbf{R}^2 \to \mathbf{R}^2$ as representing fluid flow in two dimensions, so that $\mathbf{F}(x,y)$ gives the velocity of a fluid at the point (x,y). In this case, we may call $\mathbf{F}(x,y)$ the velocity field of the fluid. More generally, we refer to a function like $\mathbf{F}(x,y)$ as a two-dimensional *vector field*. You can read more about how we can visualize the fluid flow by plotting the velocity $\mathbf{F}(x,y)$ as vector positioned at the point (x,y).

We can do the same thing for a three-dimensional fluid flow with velocity represented by a function $\mathbf{F} : \mathbf{R}^3 \to \mathbf{R}^3$. In this case, $\mathbf{F}(x,y,z)$ is the velocity of the fluid at the point (x,y,z), and we can visualize it as the vector $\mathbf{F}(x,y,z)$ positioned a the point (x,y,z). We refer to $\mathbf{F}(x,y,z)$ as a three-dimensional vector field.

Divergence

The **divergence** of a vector field is relatively easy to understand intuitively. Imagine that the vector field $\mathbf{F}$ below gives the velocity of some fluid flow. It appears that the fluid is exploding outward from the origin.

So it is like flux?

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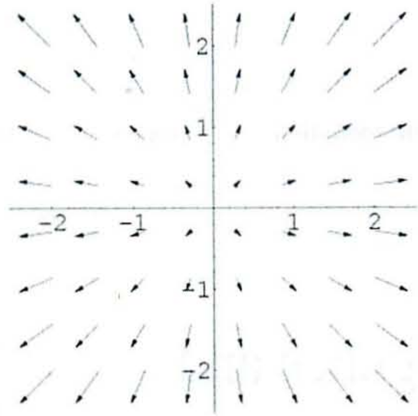
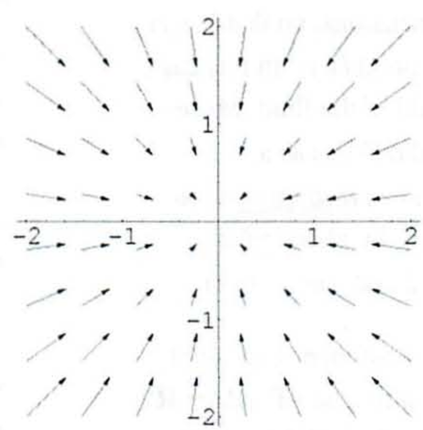


This expansion of fluid flowing with velocity field **F** is captured by the divergence of **F**, which we denote div **F**. The divergence of the above vector field is positive since the flow is expanding.

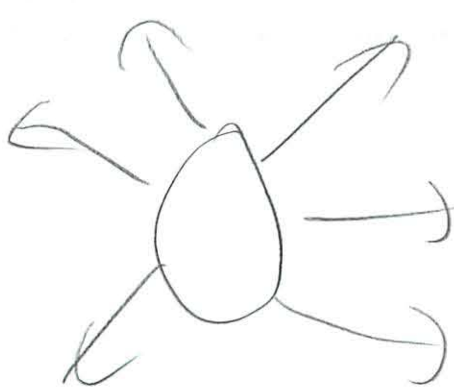
In contrast, the below vector field represents fluid flowing so that it compresses as it moves toward the origin. Since this compression of fluid is the opposite of expansion, the divergence of this vector field is negative.



The divergence is defined for both two-dimensional vector fields $\mathbf{F}(x,y)$ and three-dimensional vector fields $\mathbf{F}(x,y,z)$. A three-dimensional vector field **F** showing expansion of fluid flow is shown in the below CVT. Again, because of the expansion, we can conclude that div $\mathbf{F} > 0$.

net rate of change
of the mass of the fluid
- like people entering a room

Now, imagine that one placed a sphere S centered at the origin. It is clear that the fluid is flowing out of the sphere.



fluid added from imaginary pt, right?

Later, when we introduce the divergence theorem, we will show that the divergence of a vector field and the flow out of spheres are closely related. For now, it's enough to see that if a fluid is expanding (i.e., the flow has positive divergence everywhere inside the sphere), the net flow out of a sphere will be positive.

Since the above vector field has positive divergence everywhere, the flow out of the sphere will be positive even if we move the sphere away from the origin. Can you see why flow out is still positive even when you move the sphere around using the sliders?

i same

$$F(x,y,z) = \frac{(x,y,z)}{(x^2+y^2+z^2)^{3/2}}$$ not at origin

Still coming out of box

arrows did not seem to change

(Notice that the arrows continue to get longer as one moves away from the origin. Moreover, since the arrows are radiating outward, the fluid is always entering the sphere over less than half its surface and is exiting the sphere over greater than half its surface. Hence, the flow out of the sphere is always greater than the flow into the sphere.)

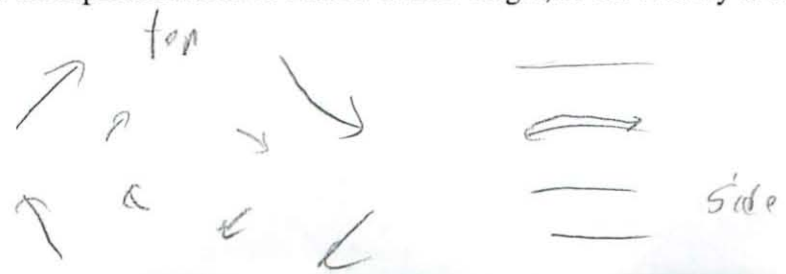
One last observation about the divergence: the divergence is a scalar. At a given point, the divergence of a vector field is just a single number that represents how much the flow is expanding at that point.

Care to read about some subtleties about the divergence or an example of calculating the divergence?

The curl

field not expanding or compressing = 0

The **curl** of a vector field is slightly more complicated than the divergence. It captures the idea of how a fluid may rotate. Imagine that the below vector field **F** represents fluid flow. It appears that fluid is circulating around a bit. From the figure's original perspective (i.e., before you rotate the graph with your mouse), the fluid appears to circulate in a counter clockwise fashion. (If you rotate the graph, you might see dots floating along the axis of rotation. These dots are representations of vectors of zero length, as the velocity is zero there.)



This macroscopic circulation of fluid around circles (i.e., the rotation you can easily view in the above graph) isn't exactly what curl measures. But, it turns out that this vector field also has curl, which we might think of as "microscopic circulation." To test for curl, imagine that you immerse a small sphere into the fluid flow, and you fix the center of the sphere at some point so that the sphere cannot follow the fluid around. Although you fix the center of the sphere, you allow the sphere to rotate in any direction around its center point. The rotation of such a sphere is illustrated below. To see the rotation of the sphere, you must hold your mouse cursor over the figure. (If you double-click, the animation will stop; double-click again to restart the animation.) The rotation of the sphere measures the curl of the vector field **F** at the point in the center of the sphere. (The sphere should actually be really really small, because, remember, the curl is *microscopic* circulation.)

It is at center

not in center !!

What if sphere at other point?

The vector field **F** determines both *in what direction* the sphere rotates, and *the speed* at which it rotates. We define the curl of **F**, denoted curl **F**, by a vector that points along the axis of the rotation and whose length corresponds to the speed of the rotation. As the curl is a vector, it is very different from the divergence, which is a scalar.

We can draw the vector corresponding to curl **F** as follows. As mentioned above, the length of the vector curl **F** is determined by how fast the sphere is rotating. The direction of curl **F** points along the axis of rotation, but we need to specify in which direction along this axis the vector should point. We will (arbitrarily?) set the direction of the curl vector by using the right hand rule, as follows. To see where curl **F** should point, curl the fingers of your right hand in the direction the sphere is rotating; your thumb will point in the direction of curl **F**. For our example, curl **F** is shown by the green arrow. (You can rotate the graph to see the green arrow better.)

I never knew curl meant this!
thanks good explination and 3D pictures!

ok

For this particular vector field, it turns out that curl **F** doesn't change with position (this, of course, is not true in general). For example, if we move the sphere to another location, it will still spin in the same direction with the same speed. Can you see why the sphere spins the same way when the sphere is in the location shown below?

what does this mean

why? field is conservative

(Notice that the arrows continue to get longer as one moves away from the axis around which the fluid is rotating. For this reason, the fluid flow pushes the sphere more strongly on the side away from this axis, causing the sphere to spin in the same direction and speed as before. The general rotation of the flow also contributes to the sphere's spinning, as it causes the fluid to push against the sphere for a greater distance on the side away from the fluid's axis of rotation.)

You can read more about how one can determine the components of the vector curl **F**. You can also see an example of calculating the divergence and curl of a vector field. As usual, pictures can be deceiving; so if you want to make sure you really understand curl, check out some subtleties about the curl.

burnt out of this
do another night.

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Divergence and curl example

For $\mathbf{F} : \mathbf{R}^3 \to \mathbf{R}^3$, the formulas for the divergence and curl are

$$\text{div}\,\mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$\text{curl}\,\mathbf{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right).$$

(The formula for curl was somewhat motivated in an earlier reading.)

Given these formulas, there isn't a whole lot to computing the divergence and curl. Just "plug and chug," as they say.

Example

Calculate the divergence and curl of $\mathbf{F} = (-y, xy, z)$.

Solution: Since

$$\frac{\partial F_1}{\partial x} = 0, \frac{\partial F_2}{\partial y} = x, \frac{\partial F_3}{\partial z} = 1$$

we calculate that

$$\text{div}(\mathbf{F}) = 0 + x + 1 = x + 1.$$

Since

$$\frac{\partial F_1}{\partial y} = -1, \frac{\partial F_2}{\partial x} = y,$$

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$$\frac{\partial F_1}{\partial z} = \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial x} = \frac{\partial F_3}{\partial y} = 0,$$

we calculate that

$$\text{curl}(\mathbf{F}) = (0 - 0, 0 - 0, y + 1) = (0, 0, y + 1).$$

Good things we can do this with math. If you can figure out the divergence or curl from the picture of the vector field (below), you doing better than I can.

Open MathML version of reading. You can read more about the beta MathML version of this site.

analysis

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Divergence and curl notation

For $\mathbf{F} : \mathbf{R}^3 \to \mathbf{R}^3$, the formulas for the divergence and curl are

$$\text{div } \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$\text{curl } \mathbf{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right).$$

These formulas are easy to memorize using a tool called the "del" operator, denoted by ∇. Think of ∇ as a "fake" vector composed of all the partial derivatives that we use just to help us remember the formulas:

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right).$$

Although it may not seem to make sense to just have the partial derivatives without them acting on a function, we won't worry about that. This is just notation.

Now, let's take the dot product of the ∇ vector with $\mathbf{F} = (F_1, F_2, F_3)$:

$$\begin{aligned} \nabla \cdot \mathbf{F} &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot (F_1, F_2, F_3) \\ &= \frac{\partial}{\partial x} F_1 + \frac{\partial}{\partial y} F_2 + \frac{\partial}{\partial z} F_3 \end{aligned}$$

If we think of each "multiplication" in the dot product as instead being the derivative of the corresponding F, then we have the formula for the divergence. So, if you can remember the del operator ∇ and how to take a dot

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product, you can easily remember the formula for the divergence

$$\operatorname{div} \mathbf{F} = \nabla\cdot \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

This notation is also helpful because you will always know that $\nabla\cdot \mathbf{F}$ is a scalar (since, of course, you know that the dot product is a scalar product).

The curl, on the other hand, is a vector. We know one product that gives a vector: the cross product. And, yes, it turns out that curl $\mathbf{F}$ is equal to $\nabla\times \mathbf{F}$. To see this, let's take the cross product of the ∇ vector with $\mathbf{F}$:

$$\begin{aligned}
\nabla\times \mathbf{F} &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\times (F_1, F_2, F_3)\\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ F_1 & F_2 & F_3 \end{vmatrix}\\
&= \mathbf{i}\left(\frac{\partial}{\partial y}F_3 - \frac{\partial}{\partial z}F_2\right) - \mathbf{j}\left(\frac{\partial}{\partial x}F_3 - \frac{\partial}{\partial z}F_1\right) + \mathbf{k}\left(\frac{\partial}{\partial x}F_2 - \frac{\partial}{\partial y}F_1\right)\\
&= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k}
\end{aligned}$$

This is exactly the formula we gave above. So if you can use the rule that "multiplication" by $\frac{\partial}{\partial x}$ is the same as taking the partial derivative with respect to x (and similar for the other derivatives), then you can remember the curl formula by

$$\operatorname{curl} \mathbf{F} = \nabla\times \mathbf{F}.$$

Open MathML version of reading. You can read more about the beta MathML version of this site.

More details about the components of the curl

Once you've learned about line integrals, you may be able make sense of the following description about the origin of the formula for the curl.

In the previous reading, we denoted the components of the curl by

$$\text{curl}\,\mathbf{F} = \mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}.$$

We visualized the component of the curl in the x direction as the rotation of a ball on a rod parallel to the x-axis.

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The component of the curl in the x direction is $v_1 = \mathbf{v} \cdot \mathbf{i} = \operatorname{curl} \mathbf{F} \cdot \mathbf{i}$. We could derive an expression for this component of the curl just like we derived an expression for the "microscopic circulation" used in Green's theorem. To see this, rotate the above animation so that the x-axis is coming straight out of the screen and the yz-plane is parallel to the screen. You can see that the rotation of the sphere is affected only by the components of $\mathbf{F}$ that are parallel to the yz-plane (and perpendicular to the x-axis), i.e., F_2 and F_3. We have reduced the situation to a two-dimensional case of rotation parallel to the yz-plane. We simply need to find the "microscopic circulation" of (F_2, F_3).

To estimate this "microscopic circulation," we can construct a curve C (shown in red below) centered at the sphere's location, and parallel to the yz-plane. The circulation of $\mathbf{F}$ around C is just the line integral $\int_C \mathbf{F} \cdot d\,\mathbf{s}$.

The "microscopic circulation" or "circulation per unit area," is just the circulation around C, divided by the the area of the region inside C, in the limit where C shrinks down to a point (drag the red point on the slider to the left). If we repeat the calculation used for Green's theorem, we could conclude that this microscopic circulation is

$$v_1 = \text{curl } \mathbf{F} \cdot \mathbf{i} = \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}.$$

One can perform similar calculations to determine the formulas for the other components of the curl, as given in the previous reading.

Subtleties about divergence

Picture of divergence as expansion

We have shown in a previous reading about the divergence that the divergence measures expansion or compression of a vector field. We ended that section with the example where we immersed a sphere into a vector field that had positive divergence everyone. No matter where one moves the sphere (with the sliders), more fluid flows out of the sphere than into the sphere, indicating the fluid is expanding.

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Open non-MathML version of reading. You can read more about the beta MathML version of this site.

The vector field pictured was

$$\mathbf{F}(x, y, z) = (x, y, z). \qquad (1)$$

Its divergence is

$$\operatorname{div} \mathbf{F}(x, y, z) = \frac{\partial}{\partial x}x + \frac{\partial}{\partial y}y + \frac{\partial}{\partial z}z = 1 + 1 + 1 = 3,$$

which is a positive constant independent of the point (x, y, z). The picture of the vector field looks like fluid exploding outward, so it makes sense that the fluid is expanding.

Can a picture be misleading?

As one becomes more sophisticated in mathematical thinking, one discovers that pictures can sometimes be misleading. (One reason mathematicians demand mathematical proof is to ensure they aren't fooling themselves into jumping to conclusions based on incomplete information, such as the information gained solely by exploring pictures.) With regard to divergence, one might wonder if an outward flow, such as pictured above, always means that the divergence of the vector field is positive?

Here's a picture of a different vector field showing fluid flowing outward from the origin. However, it differs from the above vector field in that the arrows get shorter the further they are from the origin. Is the divergence of this vector field positive? In other words, is the fluid expanding as it may look like from the picture?

To answer this question, we have to compute the divergence. This vector field is

$$\mathbf{F}(x, y, z) = \frac{(x, y, z)}{(x^2 + y^2 + z^2)^{3/2}}, \qquad (2)$$

for $(x, y, z) \neq (0, 0, 0)$. (It is not defined at the origin.) This new vector field is the same as the vector field in equation (1) except that we have divided it by its magnitude raised to the third power. (We could write this vector field as $\mathbf{F}(\mathbf{x}) = \frac{\mathbf{x}}{\|\mathbf{x}\|^3}$, where $\mathbf{x} = (x, y, z)$.) In this way, the vector field gets smaller as one moves away from the origin.

We calculate the divergence of $\mathbf{F}$:

$$\begin{aligned}
\text{div}\,\mathbf{F}(x, y, z) &= \frac{\partial}{\partial x}\frac{x}{(x^2+y^2+z^2)^{3/2}} + \frac{\partial}{\partial y}\frac{y}{(x^2+y^2+z^2)^{3/2}} + \frac{\partial}{\partial z}\frac{z}{(x^2+y^2+z^2)^{3/2}} \\
&= \frac{\left(x^2+y^2+z^2\right)-3x^2}{(x^2+y^2+z^2)^{5/2}} + \frac{\left(x^2+y^2+z^2\right)-3y^2}{(x^2+y^2+z^2)^{5/2}} + \frac{\left(x^2+y^2+z^2\right)-3z^2}{(x^2+y^2+z^2)^{5/2}} \\
&= \frac{3\left(x^2+y^2+z^2\right)-3\left(x^2+y^2+z^2\right)}{(x^2+y^2+z^2)^{5/2}} = 0
\end{aligned}$$

Hence, as long as we are not at the origin, the divergence is zero and the flow is neither expanding nor contracting.

How can we reconcile this with the picture? If the sphere is at the origin, clearly the flow is out of the sphere. But the divergence is not defined at the origin, so we have to ignore that point. If you move the sphere away from the origin, it is not clear if there is more fluid flowing into the sphere or more fluid flowing out. On one hand, the flow out of the sphere is slower than the flow into the sphere, as the arrows are getting shorter. On the other hand, because the flow is radiating outward, the fluid is flowing out of the sphere across more than half of its surface. For this particular vector field, I balanced those two effects (by carefully choosing how quickly the vector field shrinks as one moves away from the origin) so that the net flow into the sphere is exactly equal to the net flow out of the sphere. Hence, if we stay away from the origin, the fluid is neither expanding nor compressing and the divergence is zero.

Dependence on dimension

Here's one more subtlety just for fun. To make the divergence zero in the above example, I balanced the outward flow of the vector field by shrinking the vector field as one moves away from the origin. Hence, the flow out of the sphere was equal to the flow into the sphere and there was no expansion or compression.

What happens if I take the two-dimensional version of the vector field from equation (2)? The 2D vector field is

$$\mathbf{F}(x, y) = \frac{(x, y)}{(x^2+y^2)^{3/2}},$$

for $(x, y) \neq (0, 0)$. (It is not defined at the origin.) This vector field is shown below along with a circle that you can move by dragging its top red point with your mouse. Move the circle so that it is away from the origin. In this case, is the divergence positive, negative, or zero?

We calculate the divergence of $\mathbf{F}$:

$$\begin{aligned}
\operatorname{div} \mathbf{F}(x, y) &= \frac{\partial}{\partial x} \frac{x}{(x^2+y^2)^{3/2}} + \frac{\partial}{\partial y} \frac{y}{(x^2+y^2)^{3/2}} \\
&= \frac{\left(x^2+y^2\right) - 3x^2}{(x^2+y^2)^{5/2}} + \frac{\left(x^2+y^2\right) - 3y^2}{(x^2+y^2)^{5/2}} \\
&= \frac{2\left(x^2+y^2\right) - 3\left(x^2+y^2\right)}{(x^2+y^2)^{5/2}} = \frac{-1}{(x^2+y^2)^{3/2}} < 0
\end{aligned}$$

In this case, away from the origin, the divergence is negative. The fluid is compressing even though it is flowing outward.

Why did the dimension make a difference? One can see the difference from the calculations, but what is the difference in the geometric picture? As in the three-dimensional case, the fluid flows into the circle faster than it flows out of the circle, as the arrows are getting shorter. And, as in the three-dimensional case, because the flow is radiating outward, the fluid is flowing out of the circle over more than half the boundary of the circle. But, because we are only in two dimensions, the effect from the boundary is smaller. I chose the vector field to balance the two effects and make the divergence zero in three dimensions. But, this makes the divergence of the two-dimensional analogue be negative.

You can check that the divergence of the vector field

$$\mathbf{F}(x, y) = \frac{(x, y)}{x^2 + y^2}$$

is zero but that the divergence of the three-dimensional analogue

$$\mathbf{F}(x, y, z) = \frac{(x, y, z)}{x^2 + y^2 + z^2}$$

is positive. In general, for a number p, the divergence of the vector field

$$\mathbf{F}(\mathbf{x}) = \frac{\mathbf{x}}{\|\mathbf{x}\|^p}$$

is $\operatorname{div} \mathbf{F}(\mathbf{x}) = (3-p)/\|\mathbf{x}\|^p$ in three dimensions and is $\operatorname{div} \mathbf{F}(\mathbf{x}) = (2-p)/\|\mathbf{x}\|^p$ in two dimensions. So you need $p = 3$ to have zero divergence in three dimensions and $p = 2$ to have zero divergence in two dimensions.

More on Curl

5/13

X - Component of curl

- put rod parallel to x axis
- can ~~th~~ only rotate around the rod
- dir vector points along rod

* if curl $\vec{F}$ not parallel to X axis rotates more slowly

Y - componant

in this case axis $\perp$ to fluid

- not flowing anywhere!

$$\frac{\partial F}{\partial y} - \frac{\partial F_z}{\partial z} \ldots$$

what we learned for curl

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ - & - & \ldots \end{vmatrix}$$

Stokes Theorm

do later

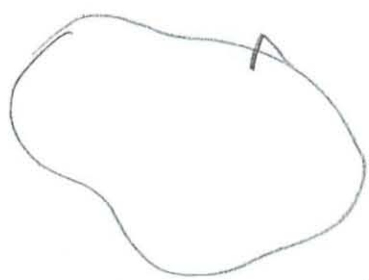
more curl

line integral at rod

- need to get good at knowing all the differences

" microscopic circulation "

Green's Theorem



circulation around path

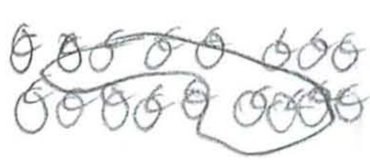
$\int F \cdot ds$ macroscopic circulation

Can compute line integral directly

but what if 2d field

Green's theorem alternate way to calculate

$\iint$ over the region



← of microscopic circulation

$\int F \cdot ds = \iint (\text{microscopic circulations}) dA$

microscopic circulations = curl $\vec{F}$ $\hat{k}$
(Greens only in a plane)

curl $\vec{F}$ only in $\hat{k}$

$N_x - M_y$ ← oh that is notation never realized where that came from

- clockwise $\oplus$
  CCW $\ominus$

$\int M\,dx + N\,dy$

When it applies
$\boxed{\text{Green's Theorm}}$

- only for closed curves
- they are the only ones w/ D inside
- think about idea 
- only vector fied in 2D
- field not conservative (otherwise 0)

(4)

Div + curl notation

↙ putting in my notation

$\text{div} = M_x + N_y + P_z$ ← # ↓vector

$\text{curl} = (P_y - N_z, M_z - P_x, N_x - M_y)$

↳ or that ∇ matrix

$\text{div} = \nabla \cdot F$

↑ dot product

$\text{curl} = \nabla \times F$

↑ cross

(need to put all out on page to make it clear)

Div + Curl examples

- just take the formulas + plug + chug

(But what is it if field conservative again)

- Ah! - If all of the pieces are 0

$P_y - N_z = 0 \quad M_z - P_x = 0 \quad N_x - M_y = 0$

then $\vec{0}$

Makes so much more sense now!

Need to study today's (Physics) notes

(5.)

~~(Actually this is ~~

Subtleties about divergence

- remember compression + expansion
- is the picture misleading

often
- div $\neq 0$ only at center
  which is not included

* Not the in one side out the other from physics *
just calculate
- he had a certain field for it to $= 0$
- not always true
- so calculate

↳ mysterious sources / sinks
- still kinad weird

Subtleies about curl

its not a sphere  circling

its will it spin when ⊙ axis fixed

"microscopic measurement"

(6)

Again, depends on field

Circulation may not be obvious

Stokes Theorm

- combines Green's w/ curl into 3D
- instead of just $\hat{k}$ dir
- now normal/perp to surface $= \hat{n}$

$$\int F \cdot ds = \underbrace{\iint}_{\text{double } \int \text{ over surface floating in space}} \text{curl } F \cdot \hat{n} \, ds$$

double $\int$ over

~~Region~~

surface floating in space

(recall surface $\iint$ is component $\perp$ to $\hat{n}$)

~~$\iint$ curl D~~ curl F over surface

= circulation of F around bound'y

- special $\iint$
  - if we move surface further away → same

(7)

(Is that why they eval as flat surface?)

as long as $\int$ boundry is C

Orienting surface

- $\hat{n}$ vector
- could be off by - sign

If green microscopic + red large curve C match right hand rule

$\circlearrowleft$ is $\uparrow$up
ccw

Surface integral

- similar to path integrals
- 2 types of $\int$ over paths
  - path $\int$ of scalar functions
  - line $\int$ of vector fields

$\uparrow$ mostly

8 Diversion

Path $\int$ of scalar-valued function

- remembered calced lenght of slinky =
  lenght of path
- what if δ of slinky varried
- add up all the density sections

(writing these notes = trick for me to pay attention)

- as decrease lenght of each section (dt)
  reach actual mass
- ~~p~~ paramatrize by the lenght

but then next heading is path $\int$ do not depend on paramitization

- or do they mean it does not depend on which poramitization since it all works out of course

Line $\int$ of vector field

What ~~do~~ work does magnetic field do?
but work calc in direction of movement

$\int F \cdot T \, ds$

(9)

Back to Surface S

2 types of surface S

- S of scalor-valued functions

S vetor fields

↑ in fundemental theorms

$S = \Phi(u,v)$ paramatrized

$f(x)$ = density

↑ only depends on x

to obtain mass from density

multiply ~~xxx~~ density · surface area?

↑ will do math from my book — ? correct?? Volume??

Surface S of vector fields

line integral is like work done by field as move along path

Surface S → ~~amt~~ fluid flowing through surfece = flux

(so 2D work; 3D flux???)

(10)

If water $\perp$ to surface flows through

$\hat{n}$ = normal

remember $\vec{F} \circ \hat{n} = 0$ when $\perp$

$F \times \hat{n} = F_n$

(use this to remember which is which)

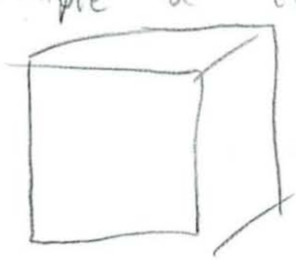
Triple $\int\int\int$

(when are they used in vector field)

~~(surface as well)~~ (is it just flat ??)

(good w/ 2x $\int$)

- same thing as 2D in 3D
- again most of the trick is determining limits
- for example a cube



Chop up into small boxes

$\Delta V = \Delta x \Delta y \Delta z$

(11)

$$\int_0^z \int_0^y \int_0^x \sigma \, dx \, dy \, dz$$

(What about non sq example?)

- again do integrals in order

- set up ⟶ (with the limits)
- solve ⟵

(it is vector calculus I am really struggling with
- pehaps read overview)

Changing variables
- what is this again?
- Oh is it like paramitrization / changing variables

Compose function g w/ change of variables function T

(I am not getting this explination

- never forget to compensate for change in ~~variables~~ volume when chaning variables

$\int_t^{t+1}$ ← oh here

yeah - I did that a few times

(12)

Vector Calculus

- wp article
- Vector operators

- Gradiant $\text{grad}(f) = \nabla f$ Rate + dir of change in scalar fied
- Curl $\text{curl}(F) = \nabla \times F$ tendency to rotate
- Divergence $\text{div}(F) = \nabla \cdot F$ magnitude source or sink
- Laplacian $\nabla^2 f = \nabla \cdot \nabla f$ div + gradiant

Theorms

- gradiant → line S through vector field = diff in its scalar field at endpoint

greens
stokes
divergence

Div Theorm (back to notes)

rigid container w/ gas
gas expanding → compressing/decompressing

$$\iiint \text{div } F = \iint F \, ds$$

13

= to flux/surface S of F over surface

~~OK think~~

gas expanding → must be leaking through wall

ok think done w/ concept review

- do pratice + past tests
  ↑2nd ↑1st

Well first topic review

The integrals

To help you organize the integral calculus portion of the course, I'm outlining the integrals you've learned, methods you can use to solve them, and their relationship to the fundamental theorems.

Path integral of scalar-valued function

The path integral over path C of a scalar-valued function f(x) is written as

$\int_C f ds$

If, for example, f were the density of a wire, the integral would be the mass.

The only way we've encountered to evaluate this integral is the direct method. We must parametrize C by some function c(t), for $a \leq t \leq b$. Then,

$\int_C f ds = \int_a^b f(c(t))||c'(t)||dt$

Note that ds became $||c'(t)||dt$. This measures how c(t) stretches or shrinks the interval [a,b] as it maps it onto C.

Line integral of a vector field

The line integral over path C of a vector field F(x) is written as

$\int_C F \cdot ds$

If, for example, F were a force acting on a particle moving along C, then the integral would be the total work performed by the force on the particle.

This integral is one of the most important for this course. We have four alternatives to evaluate the integral, although most of the alternatives work only in special cases.

1. We can compute the integral directly. We parametrize C by some function c(t), for $a \leq t \leq b$. Then

   $\int_C F \cdot ds = \int_a^b F(c(t)) \cdot c'(t)dt$

2. This method always applies. Sometimes, though, the integral will be difficult or we won't even be able to evaluate it. Our lives can be made easier by using one of the fundamental theorems to convert the line integral into something else.
3. Since this integral is really a path integral of the scalar-valued function $f = F \cdot T$ where T is the unit tangent vector

$$T = \frac{c'(t)}{||c'(t)||},$$

4. the formula for the direct method is the same as the formula for the scalar-valued path integral.
5. If the vector field F happens to be path-independent, then we could use the gradient theorem for line integrals. We reduce the problem from an integral over the curve C to something just depending on the "boundary" of C, i.e., its endpoints. We need to find a potential function f so that $\nabla f = F$. Then,

$$\int_C F \cdot ds = f(q) - f(p),$$

6. where p and q are the endpoints of C.
7. Note, if C also happens to be a closed curve, then the integral of F will be zero. Note also, that if you know F is path-independent, another thing you can do is just change the curve C to another curve that has the same endpoints as C. In this case, the line integral of F over C is the same as the line integral of F over any other curve with the same endpoints.
8. If the vector field F and the curve C happen to be in two dimensions and if C happens to be a closed curve, then we can use Green's theorem. Green's theorem converts the line integral over C to a double integral over the interior of C, which we call D,

$$\int_C F \cdot ds = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA.$$

9. Note that F must be defined everywhere in D for this to work. Sometimes we write $C = \partial D$ to denote that C is the boundary of D. C must be oriented in a counterclockwise fashion, otherwise, we'll be off by a minus sign.
10. If the vector field F and the curve C happen to be in three dimensions and if C happens to be a closed curve, then we can use Stokes' theorem. Stokes' theorem converts the line integral over C to a surface integral over any surface S for which C is a boundary,

$$\int_C F \cdot ds = \iint_S \text{curl } F \cdot dS$$

11. Sometimes we write $C = \partial S$ to denote that C is the boundary of S. C must be a positively (consistently) oriented boundary of S, otherwise, we'll be off by a minus sign.

Surface integral of a scalar-valued function

The surface integral over surface S of a scalar-valued function f(x) is written as

$$\iint_S f dS$$

If, for example, f were the density of a sheet, the integral would be the mass.

The only way we've encountered to evalute this integral is the direct method. We must parametrize S by some function $\Phi(u,v)$, for $(u,v) \in D$. Then,

$$\iint_S f dS = \iint_D f(\Phi(u,v)) \left\| \frac{\partial \Phi}{\partial u}(u,v) \times \frac{\partial \Phi}{\partial v}(u,v) \right\| dudv$$

Note that dS became $\left\| \frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v} \right\|$dudv. This measures how $\Phi(u,v)$ stretches or shrinks the region D as it maps it onto S.

Surface integral of a vector field

The surface integral over surface S of a vector field F(x) is written as

$$\iint_S F \cdot dS.$$

If, for example, F were the flow of fluid, then the integral would be the flux of the fluid through S. For this reason, we often refer to the integral as a "flux integral."

Like the line integral of a vector field, this integral plays a big role in this course. We have three alternatives to evaluate the integral, although most of the alternatives work only in special cases.

1. We can compute the integral directly. We parametrize S by some function $\Phi(u,v)$, for $(u,v) \in D$. Then,

$$\iint_S F \cdot dS = \iint_D F(\Phi(u,v)) \cdot \left(\frac{\partial \Phi}{\partial u}(u,v) \times \frac{\partial \Phi}{\partial v}(u,v) \right) dudv$$

2. This method always applies. Sometimes, though, the integral will be difficult or we won't even be able to evaluate it. Our lives can be made easier by using one of the fundamental theorems to convert the surface integral into something else.
3. Since this integral is really a surface integral of the scalar-valued function $f = F \cdot n$ where n is the unit normal vector

$$n = \frac{\frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v}}{\left\| \frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v} \right\|},$$

4. the formula for the direct method is the same as the formula for the scalar-valued surface integral.
5. If the vector field F happens to be the curl of another vector field G, i.e., F = curl G, then we can apply Stokes' theorem to convert the surface integral of curl G into the line integral of G around the positively (consistently) oriented boundary of S, which we denote ∂S,

$$\iint_S F \cdot dS = \iint_S \text{curl } G \cdot dS = \int_C G \cdot ds$$

6. We don't have any methods to find G from F. We can use Stokes' theorem to convert a surface integral into a line integral only if we are told outright that F = curl G and are given what G is. But, if given the surface integral that looks like $\iint_S \text{curl } G \cdot dS$, we can immediately recognize that Stokes' theorem is an option.
7. Note that Stokes' theorem allows us to do one more thing to the integral $\iint_S \text{curl } G \cdot dS$. We can switch the surface S to any other surface S' as long as the boundaries of S and S' are the same, i.e., $\partial S = \partial S'$ (assuming both boundaries are positively (consistently) oriented). If S is a complicated surface, we could feasibly save ourselves some work by integrating over another surface S' if that surface is simpler than S.
8. If the surface S happens to be a closed surface so that it is the boundary of some solid W, i.e., $S = \partial W$, then we can use the divergence theorem to convert the surface integral into the triple integral of div F over W,

$$\iint_S F \cdot dS = \iiint_W \text{div } F dV,$$

9. where we orient S so that it has an outward pointing normal vector. This works, of course, only if F is defined everywhere in the solid W.

Double integrals

The double integral of a (scalar-valued) function f(x) over a two-dimensional region D is written as

$$\iint_D f dA.$$

If, for example, f were the density of the region, the integral would be its mass.

We have encountered three alternatives to evaluate the integral.

1. We can compute the integral directly in terms of the original variables x and y. In this case, dA = dxdy.

2. We can compute the integral by changing to the variables u and v by finding a function (x,y) = T(u,v). Then the integral is

$$\iint_D f\,dA = \iint_{D^*} f(T(u,v))\left|\det DT(u,v)\right| du\,dv,$$

3. where D is parametrized by (x,y) = T(u,v) for (u,v) in D^*. We often write the determinant of the matrix of partial derivatives of T(u,v) as det DT(u,v) = $\frac{\partial(x,y)}{\partial(u,v)}$.
4. If f happens to be equal to $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$ for some vector field F, then we could use Green's theorem to convert the double integral into the integral of F around the boundary of D, which we denote ∂D,

$$\iint_D f\,dA = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA = \int_{\partial D} F \cdot ds.$$

5. To orient the boundary properly, outside boundaries must be counterclockwise and inner boundaries must be clockwise.
6. We usually think of Green's theorem going the other way, i.e., converting a line integral into a double integral. One reason for this is that we don't have any methods to find F from f. We can use Green's theorem to convert a double integral into a line integral only if we are told outright that f = $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$ and are given what F is. But, if given the double integral that looks like $\iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA$, we can immediately recognize that Green's theorem is an option. As a special case, if we are given an integral $\iint_D dA$ (i.e., finding the area), we can let F(x,y) = (-y,x)/2 so that $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 1$ and $\iint_D dA = \int_{\partial D} F \cdot ds$.

Triple integrals

The triple integral of a (scalar-valued) function f(x) over a three-dimensional solid W is written as

$$\iiint_W f\,dV.$$

If, for example, f were the density of the solid, the integral would be its mass.

We have encountered three alternatives to evaluate the integral.

1. We can compute the integral directly in terms of the original variables x, y, and z. In this case, $dV = dxdydz$.
2. We can compute the integral by changing to the variables u, v, and w by finding a function $(x,y,z) = T(u,v,w)$. Then the integral is

$$\iiint_W f dV = \iiint_{W^*} f(T(u,v,w)) |\det DT(u,v,w)| dudvdw,$$

3. where W is parametrized by $(x,y,z) = T(u,v,w)$ for (u,v,w) in W^*. We often write the determinant of the matrix of partial derivatives of $T(u,v,w)$ as $\det DT(u,v,w) = \frac{\partial(x,y,z)}{\partial(u,v,w)}$.
4. If f happens to be equal to div F for some vector field F, then we could use the divergence theorem to convert the triple integral into the surface integral of F around the boundary of W, which we denote ∂W,

$$\iiint_W f dV = \iiint_W \operatorname{div} F dV = \iint_{\partial W} F \cdot dS.$$

5. We usually think of the divergence theorem going the other way, i.e., converting a surface integral into a triple integral. One reason for this is that we don't have any methods to find F from f. We can use the divergence theorem to convert a triple integral into a surface integral only if we are told outright that f = div F and are given what F is. But, if given the triple integral that looks like $\iiint_W \operatorname{div} F dV$, we can immediately recognize that the divergence theorem is an option.

The fundamental theorems

To help you organize the integral calculus portion of the course, I'm outlining the fundamental theorems you've learned and their relationship to the various integrals.

The gradient theorem for line integrals

The gradient theorem for line integrals relates a line integral to the values of a function at the "boundary" of the path i.e., its endpoints. It says that

$$\int_C \nabla f \cdot ds = f(q) - f(p),$$

where p and q are the endpoints of C. In words, this means the line integral of the gradient of some function is just the function evaluated at the endpoints of the curve. In particular, this means that the integral of ∇f does not depend on the curve itself; the integral is path-independent.

We usually use this theorem when trying to integrate $\int_C F \cdot ds$. We can use it only when F is path-independent, i.e., only when there exists a potential function f so that $\nabla f = F$. Then,

$$\int_C F \cdot ds = f(q) - f(p),$$

where p and q are the endpoints of C.

Even if you can't find f, but still know that F is path-independent, you could use the gradient theorem for line integrals to change the line integral of F over C to the line integral of F over any other curve with the same endpoints. Moreover, the integral of any path-independent F over a closed curve is zero.

Green's theorem

Green's theorem relates a double integral over a region to a line integral over the boundary of the region. If a path C is the boundary of some region D, i.e., $C = \partial D$, then Green's theorem says that

$$\int_C F \cdot ds = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA.$$

The integrand of the double integral can be thought of as the "microscopic circulation" of F. Green's theorem then says that the total "microscopic circulation" in D is equal to the circulation $\int_C F \cdot ds$ around the boundary $C = \partial D$. Thinking of Green's theorem in terms of circulation will help prevent you from erroneously attempting to use it when C is an open curve.

In order for Green's theorem to work, the curve C has to be oriented properly. Outer boundaries must be counterclockwise and inner boundaries must be clockwise.

Stokes' theorem

Stokes' theorem relates a line integral over a closed curve to a surface integral. If a path C is the boundary of some surface S, i.e., $C = \partial S$, then Stokes' theorem says that

$$\int_C F \cdot ds = \iint_S \text{curl } F \cdot dS.$$

The integrand of the surface integral can be thought of as the "microscopic circulation" of F. Stokes' theorem then says that the total "microscopic circulation" in S is equal to the circulation $\int_C F \cdot ds$ around the boundary $C = \partial S$. Thinking of Stokes' theorem in terms of circulation will help prevent you from erroneously attempting to use it when C is an open curve.

In order for Stokes' theorem to work, the curve C has to be oriented properly compared to the surface S. To check for proper orientation, use the right hand rule.

Since the line integral $\int_C \mathbf{F} \cdot d\mathbf{s}$ depends only on the boundary of S (remember $C = \partial S$), the surface integral on the right hand side of Stokes' theorem must also depend only on the boundary of S. Therefore, Stokes' theorem says you can change the surface to another surface S', as long as $\partial S' = \partial S$. This works, of course, only when integrating curl F.

The divergence theorem

The divergence theorem relates a surface integral to a triple integral. If a surface S is the boundary of some solid W, i.e., $S = \partial W$, then the divergence theorem says that

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_W \operatorname{div} \mathbf{F} \, dV,$$

where we orient S so that it has an outward pointing normal vector.

The integrand of the triple integral can be thought of as the expansion of some fluid. The divergence theorem then says that the total expansion of the fluid in W is equal to the total flux of the fluid out of the boundary $S = \partial W$.

Length, area, and volume factors

Along with the multitude of integrals came a bunch of factors for length, area, and volume. In many cases, these factors adjusted for the expansion or compression by functions that transform between different integrals. I hope you will see the similarity among these factors.

Length in the ordinary one-variable integral

If we integrate a function f(x) from x = a to x = b, the length measurement is the familiar dx:

$$\int_a^b f(x)dx.$$

Length when change variables in one-variable integrals

The following is attempt to tie one-variable change of variables to multivariable change of variables. If it is too confusing, just skip it and move on.

When you perform a "u-substitution" in one-variable calculus, you are changing variables. To help you link one-variable u-substitution to multivariable change of variables, we can write a u-substitution in the same language as multivariable calculus.

You are given some integral $\int_a^b f(x)dx$. Let $x = T(u)$ be our invertible "change of variables" function. Then the u-substitution is $u = T^{-1}(x)$, where $T^{-1}(x)$ is the inverse of T(u). To perform the u-substitution, you replace x with T(u), integrate from $T^{-1}(a)$ to $T^{-1}(b)$, and replace dx with T'(u)du:

$$\int_a^b f(x)dx = \int_{T^{-1}(a)}^{T^{-1}(b)} f(T(u))T'(u)du.$$

We could go a little further and make this formula even closer to what we write in multivariable calculus. We could write the interval [a,b] as I. The integral is over the interval I = [a,b], so we could write the integral as

$$\int_I f(x)dx.$$

If $x = T(u)$ is our change of variables, then T maps an interval I^* in "u-space" to the interval I in "x-space." If $T^{-1}(b)$ is greater than $T^{-1}(a)$, then I^* is the interval $[T^{-1}(a),T^{-1}(b)]$. Otherwise, I^* is the interval $[T^{-1}(b),T^{-1}(a)]$. Our change of variables formula is then

$$\int_I f(x)dx = \int_{I^*} f(f(u))|T'(u)|du.$$

Note that in this case, the change of variables "length expansion factor" is $|T'(u)|$. We need the absolute value because of how we defined I^* in the case where $T^{-1}(b) > T^{-1}(a)$. (Technical detail: if $T'(u) < 0$ then $T^{-1}(b) < T^{-1}(a)$ and we would have flipped the order in our definition of $I^* = [T^{-1}(b),T^{-1}(a)]$. This flipping changes the sign of the integral. Adding the absolute value $|T'(u)|$ changes the sign back to the correct sign.)

The factor $|T'(u)|$ indicates how much T expands or contracts I^* when it maps I^* onto I.

Length in path integrals

In path integrals, a path C is parametrized by a function c(t). In this case, the length measure on the path is $ds = ||c'(t)||dt$. The factor $||c'(t)||$ accounts for expansion or contraction by c when it maps some interval I = [a,b] onto C. Hence, the integral of a scalar-valued function f(x) is

$$\int_C f ds = \int_a^b f(c(t))||c'(t)||dt.$$

For line integral of vector fields, we integrate $f = F \cdot T$, where T is the unit tangent vector of the curve:

$$T = \frac{c'(t)}{||c'(t)||}.$$

In this case, the denominator cancels the $||c'(t)||$ factor,

$\int_C F \cdot ds = \int_C F \cdot T ds = \int_a^b F(c(t)) \cdot c'(t)dt,$

but the expansion or contraction of c(t) is still included in the c'(t) factor.

Area in double integrals

If we integrate a function f(x,y) over a region D, the area measurement dA in the double integral is simply dxdy

$$\iint_D f dA = \iint_D f(x,y) dxdy.$$

Area when change variables in double integrals

To change variables in a double integral, we find a function (x,y) = T(u,v) that maps some new region D^* in (u,v)-space to the original region D in (x,y)-space. We then need a factor that accounts for the expansion or contraction of T as it maps D^* onto D. That factor is the absolute value of the determinant of the matrix of parital derivatives of T:

$$|\det D\mathbf{T}(u,v)|$$

We often write this is

$$|\det D\mathbf{T}(u,v)| = \left|\frac{\partial(x,y)}{\partial(u,v)}\right|$$

In the end, the formula for changing variables in double integrals is

$$\iint_D f dA = \iint_{D^*} f(T(u,v)) |\det D\mathbf{T}(u,v)| dudv.$$

Area in surface integrals

In surface integrals, a surface S is parametrized by a function $\Phi(u,v)$. In this case, the area measure on the surface is

$$dS = \left\| \frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v} \right\| dudv.$$

The factor $\left\|\frac{\partial\Phi}{\partial u}\times\frac{\partial\Phi}{\partial v}\right\|$ accounts for expansion or contraction by Φ when it maps some region D onto S. Hence, the integral of a scalar-valued function f(x) is

$$\iint_S f\,dS = \iint_D f(\Phi(u,v))\left\|\frac{\partial\Phi}{\partial u}(u,v)\times\frac{\partial\Phi}{\partial v}(u,v)\right\|dudv$$

For surface integrals of vector fields, we integrate f = F · n, where n is the unit normal vector of the surface:

$$n = \frac{\frac{\partial\Phi}{\partial u}\times\frac{\partial\Phi}{\partial v}}{\left\|\frac{\partial\Phi}{\partial u}\times\frac{\partial\Phi}{\partial v}\right\|},$$

In this case, the denominator cancels the $\left\|\frac{\partial\Phi}{\partial u}\times\frac{\partial\Phi}{\partial v}\right\|$ factor,

$$\iint_S F\cdot dS = \iint_D F(\Phi(u,v))\cdot\left(\frac{\partial\Phi}{\partial u}(u,v)\times\frac{\partial\Phi}{\partial v}(u,v)\right)dudv$$

but the expansion or contraction of Φ(u,v) is still included in the $\left(\frac{\partial\Phi}{\partial u}\times\frac{\partial\Phi}{\partial v}\right)$ factor.

Volume in triple integrals

If we integrate a function f(x,y,z) over a solid W, the volume measurement dV in the triple integral is simply dxdydz

$$\iiint_W f\,dV = \iiint_W f(x,y,z)dxdydz.$$

Volume when change variables in triple integrals

To change variables in a triple integral, we find a function (x,y,z) = T(u,v,w) that maps some new solid W^* in (u,v,w)-space to the original solid W in (x,y,z)-space. We then need a factor that accounts for the expansion or contraction of T as it maps W^* onto W. That factor is the absolute value of the determinant of the matrix of partial derivatives of T:

$|\det D\mathbf{T}(u,v,w)|$

We often write this is

$$|\det D\mathbf{T}(u,v,w)| = \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|$$

In the end, the formula for changing variables in triple integrals is

$$\iiint_W f\,dV = \iiint_{W^*} f(\mathbf{T}(u,v,w))\,|\det D\mathbf{T}(u,v,w)|\,du\,dv\,dw.$$

5/14

18.02 - Practice Final A - Spring 2006

Problem 1. Let $P = (0,1,0)$, $Q = (2,1,3)$, $R = (1,-1,2)$. Compute $\overrightarrow{PQ} \times \overrightarrow{PR}$ and find the equation of the plane through P, Q, and R, in the form $ax + by + cz = d$.

Problem 2. Find the point of intersection of the line through $P_1 = (-1,2,-1)$ and $P_2 = (1,4,0)$ with the plane $3x - 2y + z = 1$.
Is P_2 on the same side of the plane as the origin $(0,0,0)$ or not?

Problem 3. Let $A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 4 & c \\ 3 & c & 2 \end{bmatrix}$.

a) Find all values of c for which A is not invertible.

b) Let $c = 1$, and find the two entries marked $*$ in $A^{-1} = \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & * \\ \cdot & \cdot & * \end{bmatrix}$.

Problem 4. Consider the plane curve given by $x(t) = e^t \cos t$, $y(t) = e^t \sin t$.

a) Find the velocity vector, and show that the speed is equal to $\sqrt{2}\, e^t$.

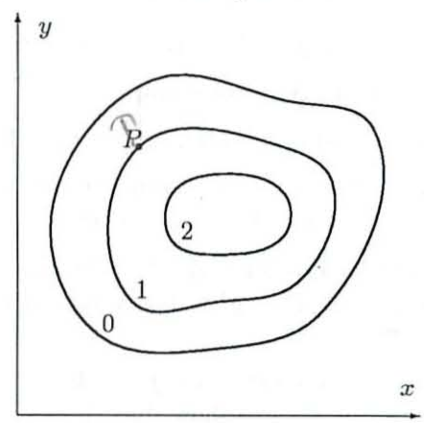
b) Find the angle between the velocity vector and the position vector, and show that it is the same for every t.

Problem 5. Let $f(x,y) = x^3 + xy^2 - 2y$.

a) Find the gradient of f at $(1,2)$ and use an approximation formula to estimate the value of $f(1.1, 1.9)$.

b) Use the chain rule to find the rate of change of f, df/dt, along the parametric curve $x(t) = t^3$, $y(t) = 2t^2$, at the time $t = 1$.

Problem 6. In the contour plot below: mark a point where $f = 1$, $f_x < 0$ and $f_y = 0$, and draw the direction of the gradient vector at the point P.



Problem 7. Let $f(x,y) = x^3 - xy + \frac{1}{2}y^2$.

a) Find all the critical points of f.

b) Determine the type of the critical point at the origin.

c) What are the maximum and the minimum of f in the region $x \geq 0$? (Justify your answer.)

Problem 8. a) Find the equation of the tangent plane to the surface $x^3 + yz = 1$ at $(-1, 2, 1)$.

b) Assume that x, y, z are constrained by the relation $x^3 + yz = 1$, and let f be a function of x, y, z whose gradient at $(-1, 2, 1)$ is $\langle a, b, c \rangle$. Find the value of $\left(\frac{\partial f}{\partial y}\right)_z$ at $(-1, 2, 1)$. Express your answer in terms of a, b, c.

Problem 9. Evaluate the integral $\int_0^1 \int_0^{\sqrt{x}} \frac{2xy}{1-y^4}\, dy\, dx$ by changing the order of integration.

Problem 10. Evaluate the work done by the vector field $\mathbf{F} = -y^3\, i + x^3\, j$ around the circle of radius a centered at the origin, oriented counterclockwise in two ways: directly, or by using Green's theorem.

Problem 11. Find the flux of $x\hat{\imath}$ out of each side of the square of sidelength 2, $-1 \leq x \leq 1$, $-1 \leq y \leq 1$. Explain why the total flux out of any square of sidelength 2 is the same regardless of its center and orientation.

Problem 12. Let $\mathbf{F} = (x^2 - xy)\hat{\imath} + 2y\hat{\jmath}$, and let C be the ellipse $(2x - y)^2 + (5x + y)^2 = 3$, oriented counterclockwise.
Use the normal form of Green's theorem to express the flux of $\mathbf{F}$ through C as a double integral.
(Give the integrand and region of integration, but do **not** provide limits for an iterated integral.)
Use a change of variables to evaluate the double integral you found.

Problem 13. Express the volume of the cylinder $0 \leq z \leq a$, $x^2 + y^2 \leq 1$ first as a triple integral in cylindrical coordinates and then as the sum of two triple integrals in spherical coordinates.

Problem 14. Let $\mathbf{F} = z^2\hat{\imath} + (z \sin y)\hat{\jmath} + (2z + axz + b\cos y)\hat{k}$.

a) Find values of a and b such that $\mathbf{F}$ is conservative.

b) For these values of a and b, find a potential function for $\mathbf{F}$ using a systematic method.

c) Still using the same values of a and b you found in part (a), calculate $\int_C \mathbf{F} \cdot dr$ where C is the portion of the curve $x = t^3$, $y = 1 - t^2$, $z = t$ for $-1 \leq t \leq 1$.

Problem 15. Calculate the flux of $\mathbf{F} = x\hat{\imath} + y\hat{\jmath} + (1 - 2z)\hat{k}$ out of the solid bounded by the xy-plane and the paraboloid $z = 4 - x^2 - y^2$ in two ways: directly, or using the divergence theorem.

do again

Problem 16. Let $\mathbf{F} = (-6y^2 + 6y)\hat{\imath} + (x^2 - 3z^2)\hat{\jmath} - x^2\hat{k}$.
Calculate curl $\mathbf{F}$ and use Stokes' theorem to show that the work done by $\mathbf{F}$ along any simple closed curve contained in the plane $x + 2y + z = 1$ is equal to zero.

Oliver's Math Review

5/14

Spring 06 Practice Final
- google for answers

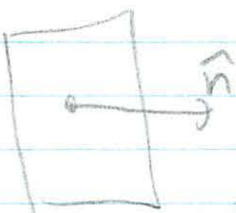
1. Compute cross product

$\langle 6, -1, -4 \rangle$
- remembered after a min

$\vec{n} \cdot \langle x, y, z \rangle = \vec{n} \cdot \overrightarrow{OP}$

(0, 1, 0) could pk any pt

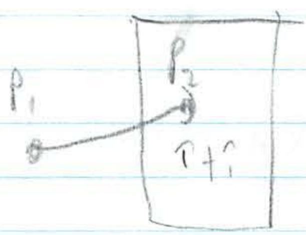
$= 6x - y - 4z \cdot (0, 1, 0)$
$= -1$



$6x - y - 4z = -1$

2.

So all of this forget - but when see familiar



What is intersection of plane

$P(t) = P_1 + \overrightarrow{P_1P_2}\, t$

Solve for t

Paramtize line

$= \langle -1 + 2t, 2 + 2t, -1 + t \rangle$

Plug into plane eq solve for t

$3(-1+2t) - 2(2+2t) + (-1+t) = 1$

$t = 3$

$3x - 2y + z > 1$ one side
$3x - 2y + z < 1$ other side

Plug in pts to see what is true

3.

$$A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 4 & c \\ 3 & c & 2 \end{bmatrix}$$ Values for c that are <u>not</u> invertable

forget all this!

not invartable → det(A) = 0
Compute
Solve for c

b) Compute certain areas of inverse matrix

$$\frac{-1}{\det(A)} \begin{bmatrix} 1 & 2 & 1 \\ -1 & 4 & c \\ 3 & c & 2 \end{bmatrix}$$

parity checkbox rule

Compute the proper lines

4. $x(t) =$ $\qquad$ $y(t) =$

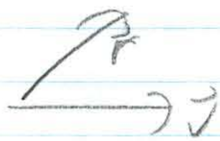
a) $\vec{v} = < x'(t), y'(t), z'(t)$

$|\vec{v}| = \sqrt{x'(t)^2 + y'(t)^2}$

b) $\vec{r} = < x(t), y(t) >$

want angles b/w 2 vectors

Angle → dot product



$$\cos\theta = \frac{\vec{r}\cdot\vec{v}}{|\vec{r}||\vec{v}|}$$

5. $f(x,y) = x^3 + xy^2 - 2y$

a) Find gradient at $(1,2)$

$\nabla f = < 3x^2 + y^2, 2xy - 2 >$

$\nabla f(1,2) = < 7, 2 >$

If I think about it makes sense

error $f(1.1, 1.9) \approx f(1,2) + \overset{7}{f_x} \underset{.1}{\Delta x} + \overset{2}{f_y} \underset{-.1}{\Delta y}$

$= 1.5$

b) note $\frac{d}{dt} f(x(t), y(t))$

$= x'(t) f_x + y'(t) f_y$

6. Level curves

On paper

how long?

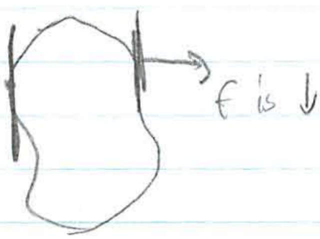
* $f_x \to$ when go in x dir, f should $\downarrow$

$f_y = 0 \to$ " " " y " f should not change

$\nabla f = \langle f_x, f_y \rangle$ — gradiant ~~horizontal~~ vertical

curve

— always $\perp$ to level curve



Oh 2 seperate things to draw!

at p draw tangent line

toward ↑ values

$\boxed{\left(\frac{\partial f}{\partial s}\right)_{p, \vec{u}} = \nabla f \cdot u}$ rate of change if you go in dir $\vec{u}$ at speed 1

$\approx \frac{\Delta f}{\Delta s}$ ← distance to next line in dir $\vec{u}$

$|\nabla f| = \dfrac{\Delta f}{\Delta s}$ ← smallest distance to next curve

7. Critical point

- not interesting
- $\left| \begin{array}{l} f_x = 0 \\ f_y = 0 \end{array} \right|$ both

- min, max
- saddle pts

b) type 2nd deriv test

c) Did not discuss in lecture

$y = 0$
$x \to \infty$

$f \to \infty$ no max

canidates for min max
- ∞
- critical pts in interior
- lagrange pts on boundry

(time consuming)

8. Tangent plane

- take the gradient of

$\vec{n} = \vec{\nabla} f$

going through fast - recognize less now

$\vec{n} \cdot \langle x, y, z \rangle = \vec{n} \cdot \vec{OP}$

b) Non independent variables

$$\left(\frac{\partial f}{\partial y}\right)_z$$

Choosing y, z as independent

$x = x(y,z)$

$$\left[\frac{\partial f((x,y), y, z)}{\partial y}\right]_z$$

Chain rule

$$f_x \left(\frac{\partial x}{\partial y}\right)_z + f_y \cdot b$$

a

need to determine this part

use knowledge of how things are dependent

$x^3 + yz = 1$

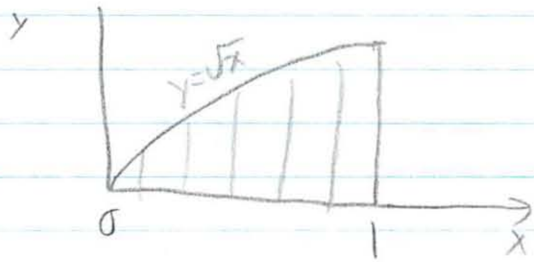
differenitate eq

$$3x^2\left(\frac{\partial x}{\partial y}\right)_z + z = 0$$

$$\left(\frac{\partial x}{\partial y}\right)_z = \frac{-z}{3x^2}$$

9. $\int_0^1 \int_0^{\sqrt{x}} \frac{2xy}{1-y^4}\, dy\, dx$

draw region



now reverse variables

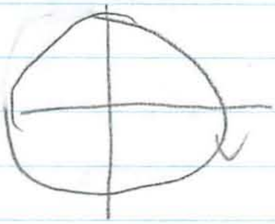
$$\int_0^1 \int_{y^2}^1 \frac{2xy}{1-y^4}\, dx\, dy$$

same!

calc (can do)

10. Work done by a field

$\vec{F} = -y^3\hat{\imath} + x^3\hat{\jmath}$



directly

$\oint$ paramitrize circle

$x = r\cos\theta$

$y = r\sin\theta$

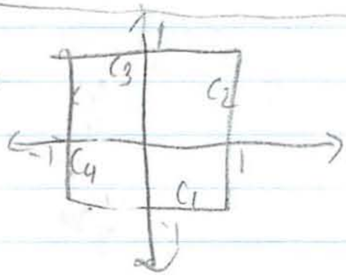
<u>Greens</u>

memorize → $\oint \vec{F}\cdot d\vec{r} = \iint \text{curl}\,\vec{F}$

pass to polar coords

This is familar again - spent time studying

11.



$\vec{F} = x\hat{\imath}$

$\text{flux} = \int_C \vec{F}\cdot\vec{n}\,dr$

flux → $C_1 + C_3 = 0$ since parallel

C_2

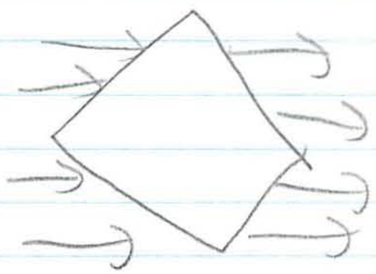
$\int_{C_2} \vec{F}\cdot\vec{n}\,ds = \int 1\,ds = 2$

$\overset{\shortparallel}{|\vec{F}|\,ds}$

(4) Same side → same sign

b) Why the same?

flux = 4
does not depend on position



(I thing fairly elementry)

Green's $\int \vec{F} \cdot \vec{n}\, ds = \iint \operatorname{div}(F)\, dA$
$\qquad\qquad\qquad\qquad\qquad\qquad\quad \| $
$\qquad\qquad\qquad\qquad\qquad\qquad\quad 1$

12. Given field + curve

flux f through curve
Green's theorm

$$\int \vec{F} \cdot \vec{n}\, ds = \iint_R \operatorname{div}(\vec{F})\, dA$$

$$(2x-y)^2 + (5x+y)^2 < 3$$

Change variables to compute $\int$

$$\operatorname{div}(F) = 2x - y + 2$$

$u = 2x - y$
$v = 5x + y$

$$dA = dx\,dy = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

need help w/ this one

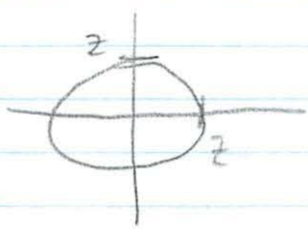
$$\frac{1}{\left|\frac{\partial(u,v)}{\partial(x,y)}\right|}$$

<u>Limits</u>

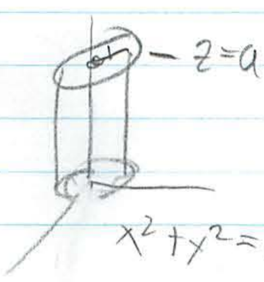
$u^2 + v^2 \le 3$

$$\int_{u\,-\sqrt{3}}^{\sqrt{3}} \int_{v\,-\sqrt{3-u^2}}^{\sqrt{3-u^2}}$$

could also do polar



13. Triple $\iiint$ cylinder



cyl $\int_0^{2\pi} \int_0^1 \int_0^0 dz\, r\, dr\, d\theta$

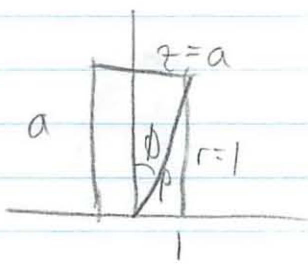
$\theta \quad r \quad z$

does not matter order

sph $\int_0^{2\pi} \int \int$

$\theta \quad \phi \quad \rho$

does matter



$\phi = \tan^{-1}\left(\frac{1}{a}\right)$

cut into
2 blocks

$\int_0^{2\pi} \int_0^{\phi_0} \int_0$ (depends on ϕ! ← need to translate $z=a$ into coords

$\theta \quad \phi \quad \rho$

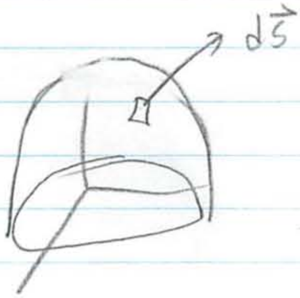
$z = a$

$\rho \cos\phi = a$

$\rho = \frac{a}{\cos\phi}$

$\int_0^{2\pi} \int_0^{\tan^{-1}\left(\frac{1}{a}\right)} \int_0^{\frac{a}{\cos\phi}} \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$

$+ \int_0^{2\pi} \int_{\tan^{-1}\left(\frac{1}{a}\right)}^{\pi/2} \int_0^{\frac{1}{\sin\phi}} \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$

15. 

flux across surface

$\int \vec{F} \cdot d\vec{s}$ directly

$d\vec{s} = \langle -z_x, -z_y, 1 \rangle \, dx\,dy$

$= \langle +2x, +2y, 1 \rangle$

↳ for parabola

also the bottom

$ds = \langle 0, 0, -1 \rangle$

where $\hat{n}$ pts

and $\iint$ over disk

- might want to change to polar

b. div theorem

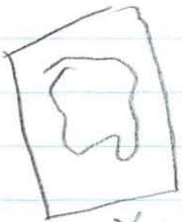
$$\iint \vec{F} \cdot d\vec{s} = \iiint \operatorname{div} \vec{F} \, dV$$

$= 0$ ← realize, easy

no source or sink

(do have to calculation in head)

16. Compute curl



$\oint_C \vec{F} \cdot d\hat{r} \stackrel{?}{=} 0$

want to prove

$x + 2y + z = 1$

$\text{curl}(F) = \langle 6z,\ 2x,\ 2x + 12y - 6 \rangle$

write out + memorize all of the theorms

Stokes Theorm

$\oint F \cdot dr = \iint_S \text{curl}(F) \cdot d\vec{s}$

↑ portion of plane

$d\vec{s} = \hat{n}\, ds$

$\hat{n} =$ plane eq! $\perp$ to plane

$= \dfrac{\langle 1, 2, 1 \rangle}{\sqrt{6}}$ ← "normalize" unit vector no magnitude

$\text{curl}(F) \cdot \hat{n} = \dfrac{6x + 4x + 2x + 12y - 6}{\sqrt{6}}$

→ should $= 0$

$= \dfrac{6x + 12y + 6z - 6}{\sqrt{6}}$

$= 0$ because will reduce to plane eq

18.02

5/14

Ok did pratice final - go through on problems I did not understand

Vectors + matrices

- was coming back to me quickly

$\vec{PQ} = Q - P$

$(2-0, 1-1, 3-0)$

Cross product multipy each part → vector

$2 \cdot 3\, \hat{i} + 3 \cdot 4\, \hat{j}$

Dot product just add → scalar

$(2 \cdot 3) + (3 \cdot 4) = 6 + 12 = 18$

Plane eq is the normal to the plane

Then the = is $\hat{n} \cdot \vec{OP}$

? what is $\hat{OP}$?

just a pt I think

or vector origin → pt

then not = 0

so origin it would = 0

②

wh had $\langle 6, -1, -4 \rangle$ was normal

~~need~~ put at a pt $(0, 1, 0)$

So ~~???~~ $\langle 6, -1, -4 \rangle \cdot \langle 0, 1, 0 \rangle$

$= -1$

So $6x - 1y - 4z = -1$

Ok think got that

Pt of intersection

$P_2 \ (1, 4, 0)$

$P_1 \ (-1, 2, -1)$

~~(vect~~ $\langle 1 - -1, 4 - 2, 0 - 1 \rangle$

$\langle 2, 2, -1 \rangle$

intersection w/ plane

~~3)~~ on wrong track

paramatrize line + solve for t

$P(t) = P_1 + \overrightarrow{P_1 P_2}\, t$

I think $-t$

$\langle -1 + 2t, 2 + 2t, -1 - t \rangle = 1$

$3(-1 + 2t) - 2(2 + 2t) + (-1 - t) = 1$

Solve for t

③ Ok I can handle that

Plane (one pt + (pt − pt) t) + ... + ... = plane eq
x z

But what is broader concept?

- ~~se~~ online writings seem more complex

just follow pattern

Right side of origin?

$$3(1) - 2(4) + (0) =$$

$$3 - 4 = -1$$

~~0~~ same ✓

$$3(0) - 2(0) + 0 = 0$$

also after get t, don't forget to plug back in

3. ~~I~~ forgot a lot of the matrix rules.

invertable

(at least no algebra problems this semester)

$$A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 4 & 0 \\ 3 & 0 & 2 \end{bmatrix}$$

(4) Invertable → <u>inverse</u>

- cover up row and colum

$-\ {}^{+}_{-}\ {}^{-}_{+}$

for the 1 → $4 \cdot 2 - c^2$

<u>inverse</u> $\cancel{\begin{bmatrix} 8-c^2 & -2-3c & -c-12 \\ & & \\ & & \end{bmatrix}}$

but how does this figure in

invertable = if det $\neq 0$

So how find determinate

$-$

$1 \begin{vmatrix} 4 & c \\ c & 2 \end{vmatrix}$

$1\,(8-c^2)$ So was like what I was doing with main # first and add

remember

just 1st row $1(8-c^2) \cancel{+} 2(-2-3c) + 1(-c-12) \neq 0$

$8 - c^2 + 4 + 6c + -c - 12$

$c^2 + 6c - c + 0 \cancel{\neq 0}$ $\cancel{5c}$

know what factor? $c^2 - 5c \rightarrow c(5-c)$ ✓ it will $= 0$ $c = 0$ and $c = 5$ thus ... be inv...

5

So I was close
not an equation
Only 1st row

b) $c = 1$

$$\begin{bmatrix} 1 & 2 & 1 \\ -1 & 4 & 1 \\ 3 & 1 & 2 \end{bmatrix}$$ find those 2 inverses

$\cancel{A} = 1 \cdot 1 - 6 = -5$ +5

$\cancel{A} = 4 - -2 = 6$ −6

remember ~~check~~

multiple steps

$\frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ Checkerboard $\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$

det

but remember that is after rotate

$\begin{bmatrix} \circ & \circ \\ & \end{bmatrix}$ ← before rotate

$\begin{bmatrix} \cdot & \end{bmatrix} = -2-3 = -5 \to 5$

$\begin{bmatrix} & \cdot \end{bmatrix} = -1-12 = -13$

$\frac{1}{|A|} \begin{bmatrix} 5 \\ -13 \end{bmatrix}$

↳ can use ~~n~~ first part)

↳ $-c^2+5c$

$-(1)^2+5(1)$

$-1+5=4$

$\frac{1}{4} \begin{bmatrix} 5 \\ -13 \end{bmatrix}$ $\cancel{\begin{bmatrix} \frac{5}{4} \\ \frac{-13}{4} \end{bmatrix}}$

not even close

from checkerboard $\to \boxed{-}\frac{1}{4} \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix}$

↳ that is from $\begin{bmatrix} & \\ \cdot & \cdot \end{bmatrix}$

did I rotate wrong?

⑦

(endless pratice problems
If I am getting it eaiser)

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \quad \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}$$

opps rotated wrong

$- so \dfrac{1}{det}$

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

rotate correcty

just do whole thing on exam -
would be eaiser

~~Uh Ca~~

4. Curve $x(t) = e^t \cos t$
$y(t) = e^t \sin t$) parametrization

a) Find velocity vector and show that speed
$= \sqrt{2}\, e^t$

⑧ ~~No clue~~ No clue even how to start

Velocity vector def

$$v = \frac{ds}{dt}$$

points in the dir of the velocity

remember velocity is deriv of position

$$\frac{\partial x}{\partial t}, \frac{\partial y}{\partial t}$$

grrr deriv of sin + cos

$\vec{v} = \langle -\cancel{*}e^t \sin t, \cancel{*} e^t \cos t \rangle$

$e^t \langle \cos t - \sin t, \sin t + \cos t \rangle$

* $\boxed{\text{deriv } (e^t) = e^t}$

* $\boxed{\text{deriv } e^t \sin t = e^t \sin t + e^t \cos t}$

So why am I screwing up elementry deriv

chain rule

$\rightarrow e^t \cos t + \sin t\, e^t$

So I have moved from screwing up algebra to screwing up derivs

(9)

$\vec{v} = e^t \langle \cos t - \sin t, \sin t + \cos t \rangle$

Now speed $= \sqrt{x^2+y^2}$

$e^t \left(-\cos^2 t - \sin^2 t + 1 \right)$

Can keep on side

$\cos^2 + \sin^2 = 1$

$-\cos 2 - \sin 2 = -1$

$-(\text{that}) = -1$

$+ \text{that} = 1$

$\sqrt{1+1}$

$\sqrt{2}\, e^t$

b) Find angle b/w vectors
(this is kinda coming back now)

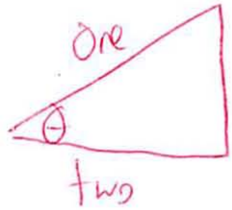
$\langle e^t \cos t - e^t \sin t, e^t \sin t + e^t \cos t \rangle$

$\langle e^t \cos t, e^t \cancel{\cos} \sin t \rangle$

So how find angle b/w again

(10)

$$\cos\theta = \frac{\vec{r} \cdot \vec{v}}{|\vec{r}|\,|\vec{v}|}$$



$\leftarrow \cos^{-1}\left(\frac{two}{one}\right)$

just stop + think about trig
is possible!

and want just dir, so that is why
the denominator?

Ok lets try it

$(e^t \cos t \cdot e^t \cos t - e^t \sin t) + (\ldots$

$e^{2t} \boxed{\cos^2 t} - e^t \sin t + e^{2t} \boxed{\sin^2 t} - e^t \cos t$

2 squared

$$\frac{e^{2t} - e^t \sin t - e^t \cos t}{|r| \sqrt{2} e^t}$$

$|r| = \sqrt{e^t \cos t^2 + e^t \sin t^2}$

$e^{2t}(\cos^2 t + \sin^2 t)$

$\sqrt{e^{2t}} \to e^t$

(11)

$$\frac{e^{2t} - e^{t}\sin t - e^{t}\cos t}{\sqrt{2}\, e^{2t}}$$

(pealed at ans - so not exactly fair)

$$\frac{\cancel{e^{2t}} \langle \cos t, \sin t \rangle \cdot \overset{1}{\langle} \cos t - \sin t, \sin t + \cos t \rangle}{\sqrt{2}\, \cancel{e^{2t}}}$$

$= \dfrac{\sqrt{2}}{2}$ (got top wrong

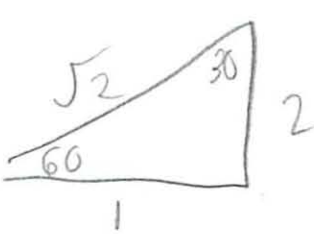
~~prob~~ and this is factored

$$\frac{1}{\sqrt{2}}$$

yeah top should be 1

now $\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$

60° $\pm \frac{\pi}{4}$



t droped out

So does not depend on t

(12)

5. $f(x,y) = x^3 + xy^2 - 2y$

find gradiants

- (reminded of this today)

$\langle 3x^2 + y^2, 2xy - 2 \rangle$

$(1,2)$

$\langle 3(1)^2 + (2)^2, 2(1)(2) - 2 \rangle$

$3+4, \quad 4-2$

$\langle 7, 2 \rangle$

Now what was estimating again

$f(1.1, 1.9)$ - never really got this

$\uparrow +.1 \quad \uparrow -.1$

$\approx \underbrace{f(1,2)}_{\text{original}} + \underbrace{\langle .1, -.1 \rangle}_{\text{diff}} \cdot \underbrace{\nabla f(1,2)}_{\text{gradiant}}$

- just memorize I guess

or $f_x \Delta x + f_y \Delta y$ which that is - different forms confuse

adding original optiod depending on what you want

(13)

$(1)^3 + (1)(2)^2 - 2(2) + .1 \cdot 7 + -.1 \cdot 2$

$1 + 4 - 4 + .7 - .2$

1.5 ✓

totally forget diagrams behind

~~f2 ... of dist~~ just do it

b) Use the chain rule to find rate of chain

of f

↑; ∇f

$\frac{df}{dt}$ along parametric $x(t) = t^3$ $t=1$

$y(t) = 2t^2$

Seems familiar

note $\frac{d}{dt} f(x(t), y(t))$

↓ backwards so confusing

$= x'(t) f_x + y'(t) f_y$

$3t^2 \cdot 3x^2 + y^2 + 4t \cdot 2xy - 2$

↑ paramatrize

and isn't this backwords

no idea of whats going on - do it

$\cancel{3t^2}$ $[3(t^3)^2 + (2t^2)^2] \cdot 3t^2 +$ →

(14) $\left[2(t^3)(2t^2)-2\right]4t$

and $t=1$

$[3+4]\,3 + [2\cdot 2-2]\,4$

$21 + 8$

29 ⓪ wow worked at

weird - just remember rules

$f_x\ x'(t) + f_y\ y'(t)$

then paramatize ? the paramotors

6. Remerber this from class

-2 points (thought ~~the~~ it was only asking for one)

$f_y=0$ mens horizontal (no change in y)

$f_x < 1$ means down hill

$f=1$ means on 1 line

gradiant dir = up hill tangent to line

(5)

7. $f(x,y) = x^3 - xy + \frac{1}{2}y^2$

find critical pts

the did not cover well

when both $\left(\begin{array}{l} f_x = 0 \\ f_y = 0 \end{array}\right.$

(This was 18.01 - strange that is what I messed up the most - whatever I did in hs)

Critical pt is first dir or both 1st/2nd

both I think

well take gradient

$\langle 3x^2 - y, -x + \frac{1}{2} \cdot 2y \rangle$

$\langle 3x^2 - y, -x + y \rangle$

that $= 0$ that $= 0$

$3x^2 - y = 0$ 2eq, 2 unknown

$-x + y = 0$

$y = 0 + x$

$y = x$

$3x^2 - x = 0$

$\frac{3x^2}{x} = \frac{x}{x}$

$3x = 1$

$\boxed{x = \frac{1}{3}} = y$

(16)

Oh so did get it
0 as well of course
↳ ~~did~~ realize $(0,0)$ $(\frac{1}{3}, \frac{1}{3})$] critical pts
I ~~a~~ failed ↑ to put it in this step

b) (woot done half)
What type of crit pt at origin
- forgot again
2nd deriv test

max

↓ (I do this first

max $f_{xx} < 0$ and $f_{xx} f_{yy} - f_{xy}^2 > 0$

min $f_{xx} > 0$ and $f_{xx} f_{yy} - f_{xy}^2 > 0$

saddle $f_{xx} f_{yy} - f_{xy}^2 < 0$

↑ do this first

then this

$f_{xx} = 6x$
$f_{xy} = -1$
$f_{yy} = 1$

$6x \cdot 1 - (-1)^2$
$6x - 1 =$
↑ plug pt in

7

$6\left(\frac{1}{3}\right)-1$ (+)

so min or max

$f_{xx} = (+)$ so $\boxed{\text{min}}$ $\left(\frac{1}{3}, \frac{1}{3}\right)$ ✓

$6(0) \cdot 1 - (-1)^2$

(−) $\boxed{\text{saddle}}$ $(0,0)$ ✓

So know rules!

(thanks Youtube video)

- works w/ Taylor approx

max is ∞

Can say if find no point I guess

c) What are min + max of f in $x \geq 0$

what is f?

- how is that diff from what I found?

So similar

max ∞ was from above

local min in a region $(0,0)$ or $\left(\frac{1}{3}, \frac{1}{3}\right)$ so plug in values to see which

8. Find the equation of the tangent plane

$\cancel{x^3 + xy}$ $x^3 + yz = 1$ at $(-1, 2, 1)$

So is this like before?

take gradiant of $\vec{n} = \nabla f$

$\vec{n}\langle x, y, z\rangle = \vec{n} \cdot \vec{OP}$ So $\perp$

yeah like begining

$g(x,y,z) = x^3 + yz - 1$ ← why change variable IDK

$\nabla g = \langle 3x^2, z, y \rangle$

plug values in

$\langle 3(-1)^2, 1, 2 \rangle$

$\langle 3, 1, 2 \rangle$

$3x + y + 2z = d$

From start

passes through

$3(-1) + (2) + 2(1) = ?$

$-3 + 2 + 2 = 1$

$\boxed{3x + y + 2z = 1}$

(19)

So review

end part was familar from begining

So left → take gradiant

oh yeah $n = \nabla f$

$\nabla f \langle x, y, z \rangle$

1. Take function
2. take gradiant
3. Plug values for point in
4. multiply by $\langle x, y, z \rangle$ to make

$$\_x + \_y + \_z = d$$

5. Then plug values in again for d
6. Rewrite with function and d

b) Constrained by relation $x^3 + yz = 1$

at $(-1, 2, 1)$ $\nabla f = \langle a, b, c \rangle$

find $\left(\frac{\partial f}{\partial y}\right)_z$ at $(-1, 2, 1)$

what is a, b, c?

20

Absoutly no clue what asking
Remeber it is something about partial derivs
(w/ respect to)
(really need to study these topics)

$$\left(\frac{\partial f}{\partial y}\right)_z$$

$\uparrow$ $x = x(y, z)$
x is a function of y and z

$$\left[\frac{\partial}{\partial y} f(x(y), y, z\right]_z$$

chain rule

$$f_x \left(\frac{\partial x}{\partial y}\right)_z + f_y$$

$\uparrow$ a $\qquad$ $\uparrow$ need to determine $\qquad$ $\uparrow$ b

$$x^3 + y z = 1$$

$\hookrightarrow$ diff eq $\qquad$ $\cancel{3x^2} + \left(\frac{\partial x}{\partial y}\right)_z$ $\qquad$ diff w/ respect to y
treat z as constant

(21)

z

$\boxed{-\dfrac{z}{3x^2}}$ How in all world did they get that ?

Completly forget non independent variables + la grange multipliers

also did later in Oliver's OH

23

N. Non-independent Variables

ie Dependent

1. Partial differentiation with non-independent variables.

Up to now in calculating partial derivatives of functions like $w = f(x,y)$ or $w = f(x,y,z)$, we have assumed the variables x, y (or x, y, z) were independent. However in real-world applications this is frequently not so. Computing partial derivatives then becomes confusing, but it is better to face these complications now while you are still in a calculus course, than wait to be hit with them at the same time that you are struggling to cope with the thermodynamics or economics or whatever else is involved.

For example, in thermodynamics, three variables that are associated with a contained gas are its

$$p = \text{pressure}, \qquad v = \text{volume}, \qquad T = \text{temperature},$$

and you can express other thermodynamic variables like the internal energy U and entropy S in terms of p, v, and T.

However, p, v, and T are not independent variables. If the gas is a so-called "ideal gas", they are related by the equation

constraint

(1) $$pv = nRT \qquad (n, R \text{ constants}).$$ $T(P,V)$

To see what complications this produces, let's consider first a purely mathematical example.

Example 1. Let $w = x^2 + y^2 + z^2$, where $z = x^2 + y^2$. Calculate $\dfrac{\partial w}{\partial x}$.

Discussion.

(a) If we think of x and y as the independent variables, then we can calculate $\dfrac{\partial w}{\partial x}$ by two different methods:

(i) using $z = x^2 + y^2$ to get rid of z, we get ↓ replace function

$$\begin{aligned} w &= x^2 + y^2 + (x^2 + y^2)^2 \\ &= x^2 + y^2 + x^4 + 2x^2y^2 + y^4; \\ \frac{\partial w}{\partial x} &= 2x + 4x^3 + 4xy^2 \end{aligned}$$

don't think suppose to do this

(ii) or by using the chain rule, remembering z is a function of x and y,

$$\begin{aligned} w &= x^2 + y^2 + z^2 \quad \text{not slip} \\ \frac{\partial w}{\partial x} &= 2x + 2z\boxed{\frac{\partial z}{\partial x}} = 2x + 2z \cdot 2x \\ &= 2x + 2(x^2 + y^2) \cdot 2x, \end{aligned}$$

← here

so the two methods agree. replace + expand

(b) On the other hand, if we think of x and z as the independent variables, using say method (i) above, we get rid of y by using the relation $y^2 = z - x^2$, and get

$$\begin{aligned} w &= x^2 + y^2 + z^2 = x^2 + (z - x^2) + z^2 \\ &= z + z^2; \\ \frac{\partial w}{\partial x} &= 0. \end{aligned}$$

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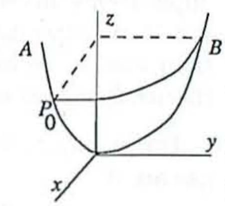
These answers are genuinely different — we cannot convert one into the other by using the relation $z = x^2 + y^2$. Will the right $\partial w/\partial x$ please stand up?

The answer is, that there *is* no one right answer, because the problem was not well-stated. *When the variables are not independent, an expression like $\partial w/\partial x$ does not have a definite meaning.*

stupid in textbooks always use valid examples or have a highly visible X over it!

To see why this is so, we interpret the above example geometrically. Saying that x, y, z satisfy the relation $z = x^2 + y^2$ means that the point (x, y, z) lies on the paraboloid surface formed by rotating $z = y^2$ about the z-axis. The function

$$w = x^2 + y^2 + z^2$$



measures the square of the distance from the origin. To be definite, let's suppose we are at the starting point $P = P_0 : (1, 0, 1)$ indicated, and we want to calculate $\partial w/\partial x$ at this point.

Case (a) If we take x and y to be the independent variables, then to find $\partial w/\partial x$, we *hold y fixed and let x vary.* So P moves in the xz-plane towards A, along the path shown.

As P moves along this path, evidently w, the square of its distance from the origin, is steadily increasing: $\dfrac{\partial w}{\partial x} > 0$ and in fact the calculations for (a) on the previous page show that $\dfrac{\partial w}{\partial x} = 6$.

Case (b) If we take x and z to be the independent variables, then to find $\partial w/\partial x$, we *hold z fixed and let x vary.* Now P moves in the plane $z = 1$, along the circular path towards B.

As P moves on this path, the square of its distance from the origin is *not changing*, and therefore $\dfrac{\partial w}{\partial x} = 0$, as we calculated in (b) before.

To sum up, the value of $\partial w/\partial x$ depends on which variables we take to be independent, because we are actually measuring different rates of change, as P moves along different paths.

There is only one way out of our difficulty. When we ask for $\partial w/\partial x$, we must at the same time specify which variables are to be taken as the independent ones. This is done by using the following notation:

Case (a): x, y are the independent variables: $\left(\dfrac{\partial w}{\partial x}\right)_y$ independent

Case (b): x, z are the independent variables: $\left(\dfrac{\partial w}{\partial x}\right)_z$

These are read, "the partial of w with respect to x, with y (resp. z) held constant".

Note how in each case the two lower letters give you the two independent variables. If we had more variables, we would use a similar notation. For instance if

(2) $$w = f(x, y, z, t), \qquad \text{where } xy = zt,$$

then only three of the variables x, y, z, t can be independent; the fourth is then determined

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by the equation on the right of (2). Thus we would write expressions like

$$\left(\frac{\partial w}{\partial x}\right)_{y,t} \qquad \text{"partial of } w \text{ with respect to } x;\ y \text{ and } t \text{ held constant";}$$

$$\left(\frac{\partial w}{\partial y}\right)_{x,z} \qquad \text{"partial of } w \text{ with respect to } y;\ x \text{ and } z \text{ held constant";}$$

in the first, x, y, t are the independent variables; in the second, x, y, z are independent.

example

2. Differentials vs. Chain Rule

An alternative way of calculating partial derivatives uses total differentials. We illustrate with an example, doing it first with the chain rule, then repeating it using differentials. By definition, the differential of a function of several variables, such as $w = f(x, y, z)$ is

$$dw = f_x dx + f_y dy + f_z dz, \tag{3}$$

where the three partial derivatives $f_x,\ f_y,\ f_z$ are the *formal* partial derivatives, i.e., the derivatives calculated as if x, y, z were independent.

Example 2. Find $\left(\frac{\partial w}{\partial y}\right)_{x,t}$, where $w = x^3y - z^2t$ and $xy = zt$.

Solution 1. Using the chain rule and the two equations in the problem, we have

$$\left(\frac{\partial w}{\partial y}\right)_{x,t} = x^3 - 2zt\left(\frac{\partial z}{\partial y}\right)_{x,t} = x^3 - 2zt\,\frac{x}{t} = x^3 - 2zx.$$

Solution 2. We take the differentials of both sides of the two equations in the problem:

$$dw = 3x^2y\,dx + x^3dy - 2zt\,dz - z^2dt, \qquad y\,dx + x\,dy = z\,dt + t\,dz. \tag{4}$$

Since the problem indicates that x, y, t are the independent variables, we eliminate dz from the equations in (4) by multiplying the second equation by $2z$, adding it to the first, then grouping the terms, which gives

$$dw = (3x^2y - 2zy)\,dx + (x^3 - 2zx)\,dy + z^2dt$$

Comparing this with (3) — after replacing z by t in (3) — we see that

$$\left(\frac{\partial w}{\partial x}\right)_{y,t} = 3x^2y - 2zy, \qquad \left(\frac{\partial w}{\partial y}\right)_{x,t} = x^3 - 2zx, \qquad \left(\frac{\partial w}{\partial t}\right)_{x,y} = z^2\,.$$

(The actual partial derivatives are the same as the formal partial derivatives w_x, w_y, w_t because x, y, t are independent variables.)

Notice that the differential method here takes a bit more calculation, but gives us three derivatives, not just one; this is fine if you want all three, but a little wasteful if you don't. The main thing to keep in mind for the method is that differentials are treated like vectors, with the $dx,\ dy,\ dz, \ldots$ playing the role of $\mathbf{i}, \mathbf{j}, \mathbf{k}, \ldots$. That is:

(26)

D1. *Differentials can be added, subtracted, and multiplied by scalar functions;*

D2. *If the variables $x, y, \ldots$ are independent, two differentials are equal if and only if their corresponding coefficients are equal:*

$$(5) \qquad A\,dx + B\,dy + \ldots = A_1\,dx + B_1\,dy + \ldots \quad \Leftrightarrow \quad A = A_1,\ B = B_1, \ldots\,;$$

D3. *One differential can be substituted into another.*

Remarks.

1. In Example 2, Solution 2, we used the operations in **D1** to do the calculations. We used **D2** in the last step, taking advantage of the fact that the x, y, t were independent.

We could have done the calculations using **D3** instead, by solving the second equation in (4) for dz and substituting it into the first equation. **D3** is a consequence of the chain rule. Illustrations of its use will be given in the next section.

2. The main advantage of calculating with differentials is that one need not take into account whether the variables are dependent or not, or which variables depend on which others; the method does this automatically for you. Examples will illustrate.

3. If the variables are not independent, **D2** is emphatically *not* true; the second equation in (4) gives a counterexample.

Note also that in **D1**, there is no attempt to include a "multiplication" or "division" of differentials to the list of operations. If u and v are functions of several variables, then their "product" $du\,dv$ makes no sense as a differential, nor does their "quotient" du/dv, which despite appearances is not in general related to any derivative, or function, or even defined. (There is no elementary analogue of the dot and cross product of vectors, though in advanced differential geometry courses a certain type of product for differentials is defined and used for multiple integration.)

Example 3. Let $w = x^2 - yz + t^2$, where x, y, z, t satisfy the two equations
$$z^2 = x + y^2 \qquad \text{and} \qquad xy = zt.$$
Using these equations, we can express first z and then t in terms of x and y; this means that w can also be expressed in terms of x and y. Without actually calculating $w(x, y)$ explicitly, find its gradient vector $\nabla w(x, y)$.

Solution. Since we need both partial derivatives $(\partial w/\partial x)_y$ and $(\partial w/\partial y)_x$, it makes sense to use the differential method. Taking the differential of w and of the two equations connecting the variables gives us

$$(6) \quad dw = 2xdx - zdy - ydz + 2tdt, \qquad xdy + ydx = zdt + tdz, \qquad 2zdz = dx + 2ydy.$$

We want x and y to be the independent variables; using the operations in **D1**, first eliminate dt by solving for it in the second equation, and substituting for it into the first equation; then eliminate dz by solving for it in the last equation and substituting into the first equation; the result is

$$(7) \qquad dw = \left(2x - \frac{y}{2z} + \frac{2ty}{z} - \frac{t^2}{z^2}\right) dx + \left(-z - \frac{y^2}{z} + \frac{2xt}{z} - \frac{2t^2y}{z^2}\right) dy.$$

Since x and y are independent, comparing the two expressions for dw in (7) and (3) (using x and y), and then using **D2**, shows that the two coefficients in (7) are respectively the two partial derivatives w_x and w_y, i.e., the two components of the gradient ∇w.

(27)

Example 4. Suppose the variables x, y, z satisfy an equation $g(x, y, z) = 0$. Assume the point $P : (1, 1, 1)$ lies on the surface $g = 0$ and that $(\nabla g)_P = \langle -1, 1, 2 \rangle$.

Let $f(x, y, z)$ be another function, and assume that $(\nabla f)_P = \langle 1, 2, 1 \rangle$.

Find the gradient of the function $w = f(x, y, z(x, y))$ of the two independent variables x and y, at the point $x = 1,\ y = 1$.

Solution. Using differentials, we have, by (3) and our hypotheses,

$$(dw)_P = dx + 2dy + dz; \qquad (dg)_P = -dx + dy + 2dz = 0, \quad \text{since } dg = 0 \text{ for all } x, y, z;$$

eliminating dz by solving the second equation for it and substituting into the first, or by dividing the second equation by 2 and substracting it from the first, we get

$$(dw)_P = \tfrac{3}{2}dx + \tfrac{3}{2}dy; \qquad (\nabla w)_P = \tfrac{3}{2}\,\mathbf{i} + \tfrac{3}{2}\,\mathbf{j}\,.$$

3. Abstract partial differentiation; rules relating partial derivatives

Often in applications, the function w is not given explicitly, nor are the equations connecting the variables. Thus you need to be able to work with functions and equations just given abstractly. The previous ideas work perfectly well, as we will illustrate. However, we will need (as in section 2) to distinguish between

formal partial derivatives, written here $f_x,\ f_y, \ldots$ (calculated as if all the variables were independent), and

actual partial derivatives, written $\partial f/\partial x, \ldots$, which take account of any relations between the variables.

Example 5. If $f(x, y, z) = xy^2z^4$, where $z = 2x + 3y$, then the three formal derivatives are

$$f_x = y^2z^4, \qquad f_y = 2xyz^4, \qquad f_z = 4xy^2z^3,$$

while three of the many possible actual partial derivatives are (we use the chain rule)

$$\begin{aligned}
\left(\frac{\partial f}{\partial x}\right)_y &= f_x + f_z\left(\frac{\partial z}{\partial x}\right)_y = y^2z^4 + 8xy^2z^3;\\
\left(\frac{\partial f}{\partial y}\right)_x &= f_y + f_z\left(\frac{\partial z}{\partial y}\right)_x = 2xyz^4 + 12xy^2z^3;\\
\left(\frac{\partial f}{\partial z}\right)_x &= f_y\left(\frac{\partial y}{\partial z}\right)_x + f_z = \tfrac{2}{3}xyz^4 + 4xy^2z^3.
\end{aligned}$$

Rules connecting partial derivatives. These rules are widely used in the applications, especially in thermodynamics. Here we will use them as an excuse for further practice with the chain rule and differentials.

With an eye to thermodynamics, we assume a set of variables $t, u, v, w, x, y, z, \ldots$ connected by several equations in such a way that

- any *two* are independent;
- any *three* are connected by an equation.

Thus, one can choose any two of them to be the independent variables, and then each of the other variables can be expressed in terms of these two.

(28)

We give each rule in two forms—the second form is the one ordinarily used, while the first is easier to remember. (The first two rules are fairly simple in either form.)

(8a,b) $$\left(\frac{\partial x}{\partial y}\right)_z\left(\frac{\partial y}{\partial x}\right)_z = 1 \qquad \left(\frac{\partial x}{\partial y}\right)_z = \frac{1}{(\partial y/\partial x)_z} \quad \textbf{reciprocal rule}$$

(9a,b) $$\left(\frac{\partial x}{\partial y}\right)_z\left(\frac{\partial y}{\partial t}\right)_z = \left(\frac{\partial x}{\partial t}\right)_z \qquad \left(\frac{\partial x}{\partial y}\right)_z = \frac{(\partial x/\partial t)_z}{(\partial y/\partial t)_z}, \quad \textbf{chain rule}$$

(10a,b) $$\left(\frac{\partial x}{\partial y}\right)_z\left(\frac{\partial y}{\partial z}\right)_x\left(\frac{\partial z}{\partial x}\right)_y = -1 \qquad \left(\frac{\partial x}{\partial y}\right)_z = -\frac{(\partial x/\partial z)_y}{(\partial y/\partial z)_x}, \quad \textbf{cyclic rule}$$

Note how the successive factors in the cyclic rule are formed: the variables are used in the successive orders x, y, z; y, z, x; z, x, y; one says they are permuted cyclically, and this explains the name.

Proof of the rules. The first two rules are simple: since z is being held fixed throughout, each variable becomes a function of just one other variable, and (9) is just the one-variable chain rule. Then (8) is just the special case of (9) where $x = t$.

The cyclic rule is less obvious — on the right side it looks almost like the chain rule, but different variables are being held constant in each of the differentiations, and this changes it entirely. To prove it, we suppose $f(x, y, z) = 0$ is the equation satisfied by x, y, z; taking y and z as the independent variables and differentiating $f(x, y, z) = 0$ with respect to y gives:

(11) $$f_x\left(\frac{\partial x}{\partial y}\right)_z + f_y = 0; \qquad \text{therefore} \quad \left(\frac{\partial x}{\partial y}\right)_z = -\frac{f_y}{f_x}.$$

Permuting the variables in (11) and multiplying the resulting three equations gives (10a):

$$\left(\frac{\partial x}{\partial y}\right)_z\left(\frac{\partial y}{\partial z}\right)_x\left(\frac{\partial z}{\partial x}\right)_y = -\frac{f_x}{f_y}\cdot -\frac{f_y}{f_z}\cdot -\frac{f_z}{f_x} = -1.$$

Example 6. Suppose $w = w(x, r)$, with $r = r(x, \theta)$. Give an expression for $\left(\frac{\partial w}{\partial r}\right)_\theta$ in terms of formal partial derivatives of w and r.

Solution. Evidently the independent variables are to be r and θ, since these are the ones that occur in the lower part of the partial derivative, with x dependent on them. Since θ is viewed as a constant, the chain rule gives

$$\left(\frac{\partial w}{\partial r}\right)_\theta = w_x\left(\frac{\partial x}{\partial r}\right)_\theta + w_r\,;$$
$$\left(\frac{\partial x}{\partial r}\right)_\theta = \frac{1}{(\partial r/\partial x)_\theta},$$

by the reciprocal rule (8). and therefore finally,

$$\left(\frac{\partial w}{\partial r}\right)_\theta = \frac{w_x}{r_x} + w_r\,.$$

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4. Changing the independent variables.* For those of you who will study thermodynamics, a major use of the rules of the preceding section is to change physical laws expressed in terms of one pair of independent variables to another pair which is better adapted to the particular problem at hand.

In thermodynamics, some of the variables associated with a confined gas are p (pressure), V (volume), T (temperature), U (internal energy), S (entropy), and H (enthalpy). Any two are independent, and their values then determine all the others.

To avoid confusion, it is better to state our general problem in terms of a neutral list of variables — we will use u, v, w, x, y. Then we can state the problem this way: a partial derivative $\left(\dfrac{\partial A}{\partial B}\right)_C$ is given, where the A, B, C are three of these variables, and we want to use x and y as the new independent variables; i.e., we want to express $\left(\dfrac{\partial A}{\partial B}\right)_C$ in terms of partial derivatives that look like $\left(\dfrac{\partial *}{\partial x}\right)_y$ and $\left(\dfrac{\partial *}{\partial y}\right)_x$, where $*$ stands for any of the variables.

It looks like there will be many cases, but outside of the trivial ones, the most commonly occurring ones are all handled by the rules of the previous section.

The trivial cases are when two of A, B, C are equal:

$$\left(\frac{\partial A}{\partial B}\right)_C = \begin{cases} 1, & A = B; \\ 0, & A = C; \\ \text{undefined}, & B = C. \end{cases} \tag{12}$$

Two more "trivial" cases are when B and C are x and y, in either order, since then the partial derivative is already in the desired form.

The rest of the cases are non-trivial, but are covered by the rules. Remembering that x and y are to be the new variables, the commonly occurring cases are these two:

$$\left(\frac{\partial A}{\partial B}\right)_x = \frac{(\partial A/\partial y)_x}{(\partial B/\partial y)_x}, \qquad \text{(chain rule (9))} \tag{13}$$

$$\left(\frac{\partial x}{\partial y}\right)_u = \frac{(\partial x/\partial u)_y}{(\partial y/\partial u)_x} = \frac{(\partial u/\partial y)_x}{(\partial u/\partial x)_y}, \qquad \text{by (10b) and (9b)} \tag{14}$$

In the above, x and y can be interchanged; A, B, C stand for any variables; u, v, w are any variables other than x or y. The reciprocal rule can be used as a preliminary step to put a given partial derivative into one of the above forms.

Example 7. One of the laws of thermodynamics is expressed by the equation

$$\left(\frac{\partial U}{\partial p}\right)_T + T\left(\frac{\partial V}{\partial T}\right)_p + p\left(\frac{\partial V}{\partial p}\right)_T = 0.$$

What is the equation for this law when V and T are the independent variables?

30

Solution. Looking at each derivative in turn, the first has the form (13) and needs the chain rule; the second has the form (14) and needs the cyclic rule; the last needs only the reciprocal rule. Using these, the equation is transformed into

$$\frac{\partial U/\partial V}{\partial p/\partial V} - T\frac{\partial p/\partial T}{\partial p/\partial V} + \frac{p}{\partial p/\partial V} = 0.$$

The subscripts are unnecessary, if it is known that T and V are the independent variables; however there is no harm in including them and removing the common denominator, which gives finally

$$\left(\frac{\partial U}{\partial V}\right)_T - T\left(\frac{\partial p}{\partial T}\right)_V + p = 0$$

as the form the law takes when referred to the variables V and T.

5. Additional rules.* For the sake of completeness, we add two more rules which will enable you to make even uncommon selections of independent variables.

To state these last two rules, we need a determinant called the **Jacobian**. We give the notation and definition for two functions $u(x,y)$ and $v(x,y)$:

(15) $$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} \qquad \textit{(the Jacobian);}$$

for three functions of three variables, etc. the definition would be analogous.

(16a,b) $$\left(\frac{\partial u}{\partial x}\right)_v\left(\frac{\partial v}{\partial y}\right)_x = \frac{\partial(u,v)}{\partial(x,y)}; \qquad \left(\frac{\partial u}{\partial x}\right)_v = \left(\frac{\partial u}{\partial x}\right)_y - \left(\frac{\partial u}{\partial v}\right)_x\left(\frac{\partial v}{\partial x}\right)_y$$

Jacobian rule

(17) $$\left(\frac{\partial u}{\partial v}\right)_w = \frac{\partial(u,w)/\partial(x,y)}{\partial(v,w)/\partial(x,y)} \qquad \textbf{two-Jacobian rule}$$

We leave the proof of the Jacobian rule (16b) as a good exercise in the use of differentials; the form (16a) follows from it by applying the chain rule (9b) and the definition (15).

The two-Jacobian rule can be proved directly either with differentials or the standard chain rule for functions of several variables. It is the mother of all rules: the other four can be derived from it by making some of the variables equal to each other.

As in section 4, these new rules allow the remaining choices of independent variable:

(18) $$\left(\frac{\partial u}{\partial x}\right)_v = \left(\frac{\partial u}{\partial x}\right)_y - \frac{(\partial u/\partial y)_x}{(\partial v/\partial y)_x}\left(\frac{\partial v}{\partial x}\right)_y, \qquad \text{by (16b)}$$

(19) $$\left(\frac{\partial u}{\partial v}\right)_w = \frac{\partial(u,w)/\partial(x,y)}{\partial(v,w)/\partial(x,y)}, \qquad \text{by (17)}$$

Exercises: Section 2J

(31)

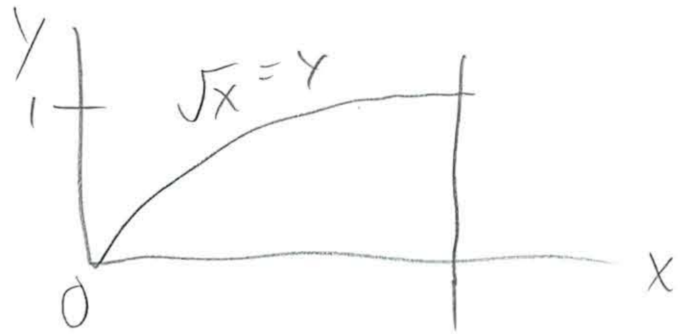
Going to skip this one for now
- will go back if have time
- or give up on this "sacrifice"
- or if presented elsewhere

9. Now into unit 3

$$\int_0^1 \int_0^{\sqrt{x}} \frac{2xy}{1-y^4}\, dy\, dx$$

looks scary

but rember class - just draw



$$\int_0^1 \int_{y^2}^1 \frac{2xy}{1-y^4}\, dx\, dy$$

y x

? same

(32)

Now actually solve

$$\int_{y^2}^{1} \frac{2xy}{1-y^4}\, dx$$

$$\frac{2y}{1-y^4} \int_{y^2}^{1} \frac{x}{1-y^4}$$

here is the problem 18.01

$$\frac{\cancel{2}y\, x^2}{\cancel{2}(1-y^4)}\Bigg|_{y^2}^{1} dx$$

cancles

$$\downarrow (y^2)^2 \cdot y$$

$$\frac{\cancel{2}y - \cancel{2}y^{5}}{\cancel{2}(1-y^4)}$$

apperently oh yeah

$$\frac{y-y^5}{1-y^4} = \frac{y(\cancel{1-y^4})}{\cancel{1-y^4}}$$

$$\downarrow$$

$$\cancel{\int_0^1 \; dy \quad \frac{2y^2}{4(1-y^4)} - \frac{2y^4}{8(1-y^4)}\Bigg|_0^1}$$

$$\int_0^1 y\, dy$$

$$\frac{y^2}{2} = \frac{1}{2}$$

$$\cancel{\frac{2(1)^2}{4(1-1)}}$$ – So had made 2 math errors which made me fail to recognize a reduction that made problem simpler

div by 0

~~10.~~ ~~the~~

10. there we go

work $F = -y^3\,\hat{i} + x^3\,\hat{j}$

circle radius a

directly Green's

first know work

$$\oint F \cdot dr = \iint N_x - M_y \, dA$$

how to do the $\oint$

- paramitrize

$x = r\cos\theta \qquad x' = -r\sin\theta$

$y = r\sin\theta \qquad y' = +r\cos\theta$

$-(r\cos\theta)^3\, r\sin\theta \;+ +(r\sin\theta)^3\, r\cos\theta$

$-r^4\sin\theta\cos^3\theta + r^4\sin^3\cos\theta$

(34)

Glad I remembered that

What next though?

not sure - yeah did it

sign error

$(-\sin\theta\cos^3\theta + \sin^3\cos\theta)$

$\int_0^{2\pi}$ then $\int$ $d\theta$

$\int_0^{2\pi} -\sin\theta\cos^3\theta + \sin^3\theta\cos\theta$

how do you $\int$ that?

$U = -\sin\theta$

$du = -\cos\theta$

$-\int u^3$

$-\frac{u^4}{4}$

$+\frac{1}{4}\sin^4(\theta)$

$U = \cos x$

$du = -\sin(x)$

$-\int u^3\, du$

$-\frac{u^4}{4}$

$-\frac{1}{4}\cos^4(\theta) + C$

$\frac{1}{4}(\sin^4\theta + \cos^4\theta)$

5 "factor answer in a different way"

$\frac{1}{16}\left(\cos(4x)+3\right)$ ← double angle ???

$\frac{1}{16} \cos 4x + C$

- would never have realized

they have in answer sheet

$8a^4 \int_0^{\pi/2} \sin^4\theta \, d\theta$

(using table)

$\frac{3\pi}{2} a^4$

I guess that is why you use Green's theorm!

$\iint N_x - M_y \, dA$ ✓

$M = -y^3 \qquad N_x = 3x^2$
$N = x^3 \qquad M_y = -3y^2$

$\iint 3x^2 + 3y^2 \, dA$

(36) Convert to polar

$x = r\cos\theta$

$y = r\sin\theta$

$$\int_0^{2\pi}\int_0^1 3(r\cos\theta)^2 + 3(r\sin\theta)^2 \, r\,dr\,d\theta$$

$$3r^2\cos^2\theta + 3r^2\sin^2\theta \quad r\,dr\,d\theta$$

did something wrong here - felt wrong too

factor out

$$\int_0^{2\pi}\int_0^1 3r^3 \, dr\,d\theta$$

$$\frac{3r^4}{4}\Big|_0^1$$

$$\frac{3}{4}\cdot 2\pi$$

$$\frac{3\pi}{2}$$

or r may not have been defined

$$\int_0^{2\pi}\int_0^a$$

$$\frac{3r^4}{4}\Big|_0^a$$

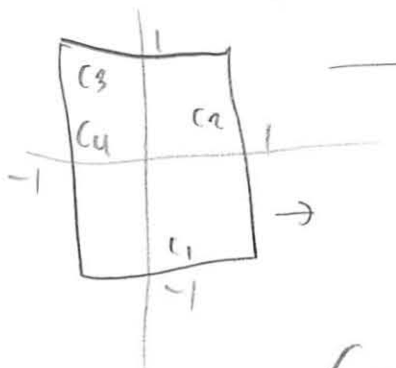
$$\frac{3a^4}{4}\cdot 2\pi$$

$$\frac{3\pi a^4}{2}$$

woot - it seems I learned this just be confident + execute + know formulas

(37)

11. Find flux ~~⊘~~ out of square (remember from class)



$\rightarrow x\hat{\imath}$

$\hat{n} = \underline{1\,\hat{\imath}}$ ← just know

flux $= \oint F \cdot \hat{n}\, ds = \iint M_x + N_y \, dA$

$C_1 + C_3 = 0$ since $=$

$\int_{-1}^{1} x\hat{\imath} \cdot \cancel{x}\,\hat{\imath}\; dx$

$x^{\cancel{2}}$ ~~x~~

~~$x^2 + x^2 = 2x^2$???~~ don't bother why?

~~perhaps use other way?~~

other one 2 as well

Here is other way for ✓

$\iint_S 1 \cdot dA = \text{Area}(S) = 2^2 = 4$

static → so trick

So that would have worked

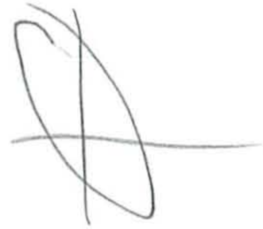
38

12. $\vec{F} = (x^2 - xy)\,\hat{\imath} + 2y\,\hat{\jmath}$

$C = (2x - y)^2 + (5x + y)^2 = 3$?

Flux through C as 2x integral

so in class he just drew up ellipse weird



$\text{flux} = \iint M_x + N_y \, dA$

$M = x^2 - xy \qquad M_x = 2x - y$

$N = 2y \qquad N_y = 2$

$\iint 2x - y + 2 \; dA$

? what here for ellipse

- no - direct is what would need paramitrizing

why is div in answer?

Green's theorm normal form is

Something else remember

$\oint -N\,dx + M\,dy = \iint M_x + N_y = \iint \operatorname{div} F$

Here they are doing that weird convert variables
I was never really good at

$\text{div}(F) = 2x - y + 2$

$U = 2x - y$
$V = 5x + y$

) where in all world
from

$$\iint_{\underbrace{(2x-y)}_{U}^2 + \underbrace{(5x-y)}_{V}^2 < 3} (2x-y+2)\, dx\, dy$$

to be simpler

$\left|\frac{\partial(U,V)}{\partial(x,y)}\right|$

$$dx\, dy = \left|\frac{\partial(U,V)}{\partial(x,y)}\right|^{-1} du\, dv$$

Jacobean $\det\begin{vmatrix} U_x & U_y \\ V_x & V_y \end{vmatrix}$

Did in Oliver's OH later

$$= \left|\det\begin{pmatrix} 2 & -1 \\ 5 & -1 \end{pmatrix}\right|^{-1} du\, dv$$

$$= \frac{1}{3}\, du\, dv$$

(40)

Limits

$U^2 + V^2 \leq 3$

$$\int_{-\sqrt{3}}^{\sqrt{3}} \int_{-\sqrt{3-U^2}}^{\sqrt{3-U^2}} \frac{U+2}{3}\, du\, dv$$

$U \qquad V$

could also use polar

Using symmetry $(U, V) \mapsto (-U, V)$

$$\iint_{U^2+V^2<3} \frac{U}{3}\, du\, dv = 0$$

flux given by

$$\iint_{U^2+V^2<3} \frac{2}{3}\, du\, dv$$

$$= \frac{2}{3} \pi (\sqrt{3})^2$$

$$= 2\pi$$

another problem I don't get at all!

(41.)

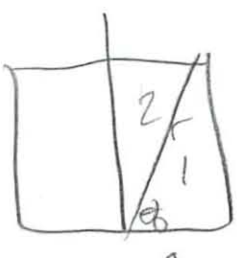
13. Cylinder as $\iiint$

← remember this
  - Fairly simple

$$\int_0^{2\pi}\int_0^{a}\int_0^{1} \sigma \; dz\, r\,dr\, d\theta \quad \checkmark$$

$\theta \quad r \quad z$

This is the complicated one
  - don't remember too much



$$\int_0^{2\pi}\int_0^{\phi_0}\int_a^{\frac{?}{}}$$

are the hard ones
Should be able to engineer geometry

$\theta \quad \phi \quad \rho$

$\rho^2 \sin\phi$

$$\cos\phi_0 = \frac{a}{x^2+y^2}$$

$$\phi_0 = \cos^{-1}\left(\frac{a}{x^2+y^2}\right) \quad \text{arc tan } \frac{1}{a}$$

↑ could also I guess

$\underline{?}$ = the hyp as ϕ increases

$$\cos\phi = \frac{a}{?}$$

$$?\cos\phi = a$$

$$? = \frac{a}{\cos\phi} \quad \checkmark$$

(42)

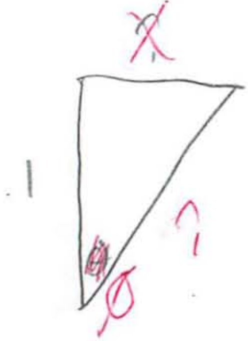
+ Region 2

$$\int_{0}^{2\pi}\int_{\phi_0}^{\frac{\pi}{2}}\int_{?}^{1} \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

$\theta \quad \phi \quad \rho$

not same value
must write different expression

? → no $\frac{1}{\sin\phi}$



~~sin θ~~

~~$\tan\theta = \frac{?}{1}$~~

~~$\tan\phi = ?$~~

$\cos\phi = \frac{1}{?}$

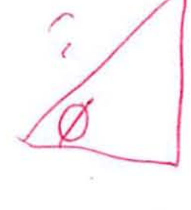
$?\cos\phi = 1$

~~Cos~~ ~~ϕ~~

$? = \frac{1}{\cos\phi}$ but why did they get sin ?? seems wrong

Oliver had $\frac{1}{\sin}$

Oh wrong angle



but for spherical I was right

(43)

14. $\vec{F} = z^2\,\hat{i} + z\sin y\,\hat{j} + (2z + axz$
$+ b\cos y)\,\hat{k}$

Find values so that conservative

$$\begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ M & N & P \end{vmatrix}$$

$$P_y - N_z\ \hat{i} \cancel{+} M_z - P_x\ \hat{j} + N_x - M_y\ \hat{k}$$

$-b\sin y = \sin y$

$\boxed{b = -1}$

$2z = az$ got this

$\boxed{a = 2}$

$0 = 0$

b) Find a potential function
- that means $\int$
- lets see if I remember complex steps

(14) $(z^2)\hat{\imath} + (z\sin y)\hat{\jmath} + (2z + 2xz - b\cos y)\hat{k}$

~~z^2~~ No I Forget

$f_z = 2z + 2xz - \cos y$

start w/ one and integrate

$f = \int z^2 + xz^2 - z\cos y + g(x,y)$

↓ differeintate

$f_y = -z\sin y + g(x,y)$

↓

$\cancel{f = z\cos}$

compare w/ real

$f_y = -z\sin y + 0$

$f = \cancel{+ y z\cos y + g(x)}$

$f_x = z^2 + xz^2 - yz\cos y + h(x)$

$\cancel{xz^2} +$ differenitatez

$z^2 + h(x)$

compare $h(x) = 0$

$f = z^2 + xz^2 - yz\cos y$

45

Eventually figured that one out, but struggled + used notes

Steps

Write f_z

↓ integrate

$f = \quad + g(x, y)$

↓ differentiate y

$f_y = \quad + g'(x, y)$

compare w/ real f_y

fill in $g(x, y)$ add back to real f

differentiate x $\qquad + h(x)$

$f_x = \quad + h'(x)$

compare w/ real

fill in $h(x)$

done

try again on back w/o looking

b)

No ~~Chat~~ peaking

$f_z = 2z + 2xz - \cos y$

$f = z^2 + xz^2 - z\cos y + g(x,y)$

$f_y = -z\sin y + g'(x,y)$

check

$g'(x,y) = 0$

$f = z^2 + xz^2 - z\cos y + 0 + h(x)$

$f_x = z^2 + h'(x)$

check

$h'(x) = 0$

$f = z^2 + xz^2 - z\cos y + 0 + 0$ 0

wow! That was easy!

c) Calculate $\int_C F \cdot dr$ where C is still the portion of curve

$x = t^3$

$y = 1 - t^2$ $\quad -1 \le t \le 1$

$z = 1$

17) So what are these types of problems

$$\int_{-1}^{1} z^2 + z\sin y + 2z + 2xz - \cos y$$

paramitrize

$$t^2 + t\sin(1-t^2) + 2t + 2(t^3)(t) - \cos(1-t^2)\,ds$$

$$\int t^2 + 2t^4 + 2t + t\sin(1-t^2) - (\cos(1-t^2)$$

$$\frac{t^3}{3} + \frac{2t^5}{5} + \frac{2t^2}{2} + \frac{t^2\sin(1-t^2)}{2}$$

$$- t\cos(1-t^2)\cdot \underbrace{\cancel{2t}}_{\text{integrate}} - \sin(1-t^2)\cdot \cancel{2t}$$

$$\frac{t^3}{3} \qquad \frac{t^3}{3}$$

$$\frac{t^3}{3} + \frac{2}{5}t^5 + t^2 + \frac{t^2\sin(1-t^2)}{2} - \frac{t^4}{3}\cos(1-t^2)$$

$$- \frac{t^3\sin(1-t^2)}{3}\Bigg|_{-1}^{1}$$

Plug in values

Did I do this wrong?

(48)

Fudge FTC

- did same thing on make up test
- no wonder impossibly difficult

$$f(1,0,1) - (-1,0,-1) = 1 - 1 = 0$$

Why does it apply?

$$\int_A^B F \cdot dr = f(B) - f(A)$$

~~the deriv~~

the original function

* Continuous since gradient function

* really need to recognize

① when gradiant function

② after integrating

not too sure on rules

beware of sneaky FTC

(49)

15. Calculate : Flux of $\vec{F}$

$F = x\hat{i} + y\hat{j} + (1-2z)\hat{k}$

$z = 4 - x^2 - y^2$

2^2

div theorm $\iint_S F \cdot d\vec{S} = \iiint_D (\nabla \cdot F)\,dV$

~~$\oint_C \vec{F} \cdot \hat{n}\,ds = \iint_D \operatorname{div} F\,dA$~~

directly

$\hat{n} =$?   remember $\hat{n} = \nabla f$

gradient

$ds = \langle -z_x, -z_y, 1 \rangle\,dxdy$

where $\hat{n}$ points

know $z =$

top $= \langle 2x, 2y, 1 \rangle$

botton $\langle 0, 0, 1 \rangle$

how do you know this?

$\oint$ over the shadow

(30)

$r=2$

top $\cancel{\int_0^{2\pi}}$ $x \cdot 2x + y \cdot 2y + (1-2z) \cdot 1 \; dS$

$2x^2 + 2y^2 + 1 - 2z$

$x = r\cos\theta$

$y = r\sin\theta$

$z = 4 - x^2 - y^2$

$4 - r^2\cos^2\theta - r^2\sin^2\theta$

$2r^2\cos^2\theta + 2r^2\sin^2\theta + 1 - 2(4 - r^2\cos^2\theta - r^2\sin^2\theta)$

$2r^2 + 1 - 8 + 2r^2\cos^2\theta + 2r^2\sin^2\theta$

$2r^2 - 7 + 2r^2 \; r\,dr \; d\theta$

r was right

$\int_0^{2\pi}\int_0^2 (4r^2 - 7) \; r\,dr\,d\theta$

$\frac{4r^3}{3} - 7r \Big|_0^2$

$\frac{4 \cdot 8}{3} - 14$

$10\frac{2}{3} - 14$

$-3\frac{1}{3}$

(51)

bottom $\iint (1-2z)\cdot -1$

$\iint -1 + 2z$

$-\pi(2)^2 = -4\pi$

? how did I get this?

$-1\cdot$ area

b) div theorm

$\iiint \nabla\cdot F \, ds$

$\cancel{\partial x}\cdot F_x + F_y + F_z \, ds$

oh easy could have done → $1 + 1 - 2 = 0$

0

no sources or sinks

52

16. Last problem!

$$\vec{F} = (-6y^2 + 6y)\,\hat{i} + (x^2 - 3z^2)\,\hat{j} - x^2\,\hat{k}$$

curl $F = \nabla \times F$

$$\begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ M & N & P \end{vmatrix}$$

$$P_y - N_z\,\hat{i} + P_x - M_z\,\hat{j} + N_x - M_y\,\hat{k}$$

$$0 - 6z\,\hat{i} + -2x - 0\,\hat{j} + 2x - 12y + 6\,\hat{k}$$

Stokes what is theorm again

$$\oint F \cdot dr = \iint \text{curl } F \cdot ds$$

✓ $\iint -6z\,\hat{i} - 2x\,\hat{j} + 2x - 12y + 6\,\hat{k}\;dA$

any $\hat{n}$ should show conservative
Can't since its <u>not</u>

(53)

$$\hat{n} = \frac{\langle 1,2,1 \rangle}{\sqrt{1^2+2^2+1^2}} = \frac{\langle 1,2,1 \rangle}{\sqrt{6}}$$

$$\nabla \times \underbrace{F \circ \hat{n}}$$

this part I forget

$$\frac{6z + 2(2x) + (2x + 12y - 6)}{\sqrt{6}}$$

= can simplify

$$\frac{6z + 4x + 2x + 12y - 6}{\sqrt{6}}$$

$$\frac{6z + 6x + 12y - 6}{\sqrt{6}}$$

Oh yeah this weirdness

$$\frac{6(x + 2y + z) - 6}{\sqrt{6}}$$

$$\frac{6(1) - 6}{\sqrt{6}} = 0$$

(54) Would never have recognized that
Ok done that pratice test

3 things to work on

know vector calculus theorems
- make sheet

memorize all steps
- like the S thing
- do more pratice

learn
- changing variables u, v
non independant - variables
don't understand at all

Integral
Vector Calculus Theorems
(unverified)

5/14

Work (Green)

$$\int_D F \cdot ds = \int_C M dx + N dy = \iint \underset{\nabla \times F}{\text{curl}\, \vec{F}} = \iint N_x - M_y \, dA$$

Flux (Green Normal)

$$\oint_C F \cdot \hat{n} ds = \iint \text{div}\, \vec{F} \, dA = \iint M_x + N_y \, dA$$

Stokes

$$\int \vec{F} \cdot d\vec{r} = \iint (\underset{\text{curl}}{\vec{\nabla} \times} \vec{F} \cdot \hat{n}) \, ds$$

Div

$$\oiint_C \vec{F} \cdot d\vec{S} = \iiint_V \underset{\text{div}}{\nabla \cdot} F \, dV$$

don't forget

FTC $\int_A^B F \cdot da = f(B) - f(A)$ if conservative

18.02 - Solutions of Practice Final A - Spring 2006

Problem 1. $\overrightarrow{PQ} = \langle 2,0,3\rangle$; $\overrightarrow{PR} = \langle 1,-2,2\rangle$; $\overrightarrow{PQ}\times\overrightarrow{PR} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 0 & 3 \\ 1 & -2 & 2 \end{vmatrix} = 6\hat{\imath} - \hat{\jmath} - 4\hat{k}$

Equation of the plane: $6x - y - 4z = d$. Plane passing through P: $6\cdot 0 - 1 - 4\cdot 0 = d$.
Equation of the plane: $6x - y - 4z = -1$.

Problem 2. Parametric equation for the line: $P_1 + t\overrightarrow{P_1P_2} = (-1,2,-1) + t\langle 2,2,1\rangle =$
$= (-1+2t, 2+2t, -1+t)$, that is $x(t) = -1+2t$, $y(t) = 2+2t$, $z(t) = -1+t$.
Intersection: $3x(t) - 2y(t) + z(t) = 1 \implies -3+6t-4-4t-1+t = 1 \implies -8+3t = 1$, that is $t = 3$, which corresponds to the point $(5,8,2)$.
The function $3x - 2y + z - 1$ takes value -1 at the origin and -6 at P_2, which are both negative. So P_2 and the origin are in the same half-space.

Problem 3. a) A is not invertible if and only if $\det(A) = 0$.
$\det(A) = 1\begin{vmatrix} 4 & c \\ c & 2 \end{vmatrix} - 2\begin{vmatrix} -1 & c \\ 3 & 2 \end{vmatrix} + \begin{vmatrix} -1 & 4 \\ 3 & c \end{vmatrix} = (8-c^2) - 2(-2-3c) + (-c-12) = -c^2+5c =$
$= c(5-c)$, hence A is not invertible if and only if $c = 0$ or $c = 5$.
b) For $c = 1$, $\det(A) = 4$.

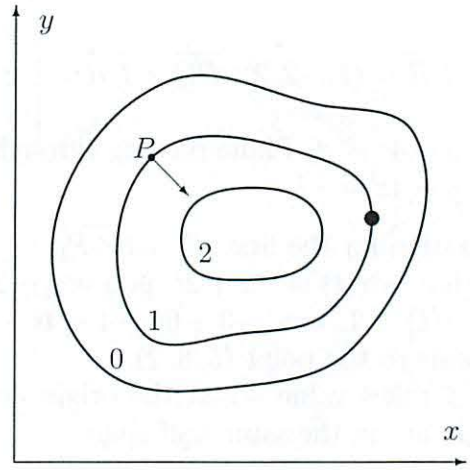
If $A^{-1} = \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & a \\ \cdot & \cdot & b \end{pmatrix}$, then $a = -\frac{1}{4}\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = -\frac{1}{2}$ and $b = \frac{1}{4}\begin{vmatrix} 1 & 2 \\ -1 & 4 \end{vmatrix} = \frac{3}{2}$.

Problem 4. a) $\vec{v}(t) = e^t\langle \cos t - \sin t, \sin t + \cos t\rangle$ and $|\vec{v}(t)|^2 = e^{2t}(\cos^2 t + \sin^2 t - 2\sin t\cos t +$
$+ \sin^2 t + \cos^2 t + 2\sin t\cos t) = 2e^{2t}$, so the speed is $|\vec{v}(t)| = \sqrt{2}e^t$.

b) $\cos\theta = \dfrac{\vec{r}\cdot\vec{v}}{|\vec{r}|\,|\vec{v}|} = \dfrac{e^{2t}\langle \cos t, \sin t\rangle \cdot \langle \cos t - \sin t, \sin t + \cos t\rangle}{\sqrt{2}e^{2t}} = \dfrac{\sqrt{2}}{2}$, so $\theta = \pm\pi/4$.

Problem 5. a) $\nabla f = \langle 3x^2 + y^2, 2xy - 2\rangle$ and $\nabla f(1,2) = \langle 7, 2\rangle$.
$f(1.1, 1.9) \approx f(1,2) + \langle 0.1, -0.1\rangle \cdot \nabla f(1,2) = 1 + 0.7 - 0.2 = 1.5$.
b) The velocity is $\vec{v}(t) = \langle 3t^2, 4t\rangle$ and $\vec{v}(1) = \langle 3, 4\rangle$.
$t = 1$ corresponds to the point $(1,2)$, so
$\frac{df}{dt}(1) = \frac{\partial f}{\partial x}(1,2)\frac{dx}{dt}(1) + \frac{\partial f}{\partial y}(1,2)\frac{dy}{dt}(1) = 7\cdot 3 + 2\cdot 4 = 29$.

Problem 6.



Problem 7. a) $\nabla f = \langle 3x^2 - y, -x + y\rangle$.

Critical points: $\nabla f = 0 \iff \begin{cases} y = 3x^2 \\ x = y \end{cases}$

The critical points are $(0,0)$ and $(1/3, 1/3)$.

b) $f_{xx} = 6x$, $f_{xy} = -1$, $f_{yy} = 1$, so $\Delta = 6x - 1$. At the origin $\Delta(0,0) = -1 < 0$, so it is a saddle point.

c) On the boundary $x = 0$ and $f(0,y) = y^2/2$, so the minimum at the boundary is 0 attained at $(0,0)$. The maximum value is $+\infty$.

$f(x,y) = x^3 - \dfrac{x^2}{2} + \dfrac{1}{2}(y-x)^2$, so $f(x,y) \to +\infty$ for $x \to +\infty$ and/or $y \to \pm\infty$. Hence the minimum can be either at $(0,0)$ or at $(1/3, 1/3)$. Because $f(1/3, 1/3) = -1/54$, this is the minimum value.

Problem 8. a) Let $g(x,y,z) = x^3 + yz - 1$. Then $\nabla g = \langle 3x^2, z, y\rangle$ and $\nabla g(-1,2,1) = \langle 3,1,2\rangle$, hence the equation of the tangent plane is $3x + y + 2z = d$.
It must pass through $(-1,2,1)$, so $3(-1) + 2 + 2(1) = d \implies d = 1$.
Equation of the tangent plane: $3x + y + 2z = 1$.

b) Constraint $\implies 3dx + dy + 2dz = 0$ at $(-1,2,1)$. Keeping z fixed, we get $dx = -dy/3$. Because $df = a\,dx + b\,dy + c\,dz$ at $(-1,2,1)$, we obtain $df = (-a/3 + b)dy$, that is $\left(\dfrac{\partial f}{\partial y}\right)_z (-1,2,1) = b - \dfrac{a}{3}$.

Problem 9. $\displaystyle\int_0^1 \int_0^{\sqrt{x}} \frac{2xy}{1-y^4}\,dy\,dx = \int_0^1 \int_{y^2}^1 \frac{2xy}{1-y^4}\,dx\,dy = \int_0^1 \frac{y}{1-y^4}\Big[x^2\Big]_{x=y^2}^{x=1} dy =$

$\displaystyle = \int_0^1 y\,dy = 1/2.$

Problem 10. *Direct method.* The circle is parametrized by $x(\theta) = a\cos\theta$, $y(\theta) = a\sin\theta$, for $0 \le \theta \le 2\pi$. The work is $\displaystyle\int_C \vec{\mathbf{F}} \cdot d\vec{r} = \int_C -y^3 dx + x^3 dy =$

$$= \int_0^{2\pi} -a^3\sin^3\theta(-a\sin\theta\,d\theta) + a^3\cos^3\theta(a\cos\theta\,d\theta) = a^4 \int_0^{2\pi} (\sin^4\theta + \cos^4\theta)d\theta =$$

$$= 8a^4 \int_0^{\pi/2} \sin^4\theta \, d\theta = \text{(using the table)} = \frac{3\pi}{2}a^4.$$

Using Green's theorem. $\int_C \vec{\mathbf{F}} \cdot d\vec{r} = \iint_R (N_x - M_y)dA$, where R is the disc of radius a, $M = -y^3$ and $N = x^3$, so that $N_x - M_y = 3x^2 + 3y^2 = 3r^2$.

Hence the work is $\int_0^{2\pi}\int_0^a 3r^2 \cdot r \, dr \, d\theta = \int_0^{2\pi} d\theta \left[\frac{3r^4}{4}\right]_0^a = \frac{3\pi}{2}a^4.$

Problem 11. Call $\vec{\mathbf{F}} = x\hat{\imath}$ and recall that $(\text{Flux}) = \int_C \vec{\mathbf{F}} \cdot \hat{\mathbf{n}} \, ds$.

Side $x = -1$: $\hat{\mathbf{n}} = -\hat{\imath}$, $\vec{\mathbf{F}} \cdot \hat{\mathbf{n}} = 1$, so the flux is 2.
Side $x = 1$: $\hat{\mathbf{n}} = \hat{\imath}$, $\vec{\mathbf{F}} \cdot \hat{\mathbf{n}} = 1$, so the flux is 2.
Side $y = -1$: $\hat{\mathbf{n}} = -\hat{\jmath}$, $\vec{\mathbf{F}} \cdot \hat{\mathbf{n}} = 0$, so the flux is 0.
Side $y = 1$: $\hat{\mathbf{n}} = \hat{\jmath}$, $\vec{\mathbf{F}} \cdot \hat{\mathbf{n}} = 0$, so the flux is 0.
The total flux out of any square S of sidelength 2 is always 4, because Green's theorem in normal form says it is equal to $\iint_S (M_x + N_y)dA = \iint_S 1 \cdot dA = \text{Area}(S) = 2^2 = 4$.

Problem 12. Green's theorem in normal form: $\int_C \vec{\mathbf{F}} \cdot \hat{\mathbf{n}} \, ds = \iint_R \text{div}(\vec{\mathbf{F}})dA$, where R is the region enclosed by C.

why ?

$\text{div}(\vec{\mathbf{F}}) = 2x - y + 2$, so the flux is given by $\iint_{(2x-y)^2+(5x-y)^2<3} (2x - y + 2) \, dx \, dy$.

Change of variables: $u = 2x - y$, $v = 5x \cancel{-} + y$, so

mistake in exersize

$$dx \, dy = \left|\frac{\partial(u,v)}{\partial(x,y)}\right|^{-1} du \, dv = \left|\det\begin{pmatrix} 2 & -1 \\ 5 & -1 \end{pmatrix}\right|^{-1} du \, dv = \frac{1}{3} du \, dv.$$

See done in Oliver's OH

The integral becomes $\iint_{u^2+v^2<3} \frac{u+2}{3} du \, dv$. Using the symmetry $(u,v) \mapsto (-u,v)$, we have that the integral $\iint_{u^2+v^2<3} \frac{u}{3} du \, dv = 0$, so that the flux is given by $\iint_{u^2+v^2<3} \frac{2}{3} du \, dv = \frac{2}{3}\pi(\sqrt{3})^2 = 2\pi$.

So this wrong

Problem 13. In cylindrical coordinates the volume is $\int_0^a \int_0^{2\pi} \int_0^1 r \, dr \, d\theta \, dz$.

In spherical coordinates $\int_0^{2\pi}\int_0^{\arctan(1/a)}\int_0^{a/\cos\varphi} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta +$

$$+ \int_0^{2\pi}\int_{\arctan(1/a)}^{\pi/2}\int_0^{1/\sin\varphi} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta.$$

Problem 14. a) $\vec{\mathbf{F}}$ is conservative if and only if $\vec{\nabla} \times \vec{\mathbf{F}} = 0$ (because $\vec{\mathbf{F}}$ is continuous and differentiable everywhere).

$\overrightarrow{\nabla} \times \overrightarrow{\mathbf{F}} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ z^2 & z\sin y & 2z + axz + b\cos y \end{vmatrix} = (-b\sin y - \sin y)\hat{\imath} - (az - 2z)\hat{\jmath}$, so we must have $a = 2$ and $b = -1$.

b) Let $\overrightarrow{\mathbf{F}} = \nabla f$. We must have $f_z = 2z + 2xz - \cos y$, so $f(x, y, z) = z^2 + xz^2 - z\cos y + g(x, y)$. Moreover, $z\sin y + g_y(x, y) = f_y = z\sin y \implies g(x, y) = h(x)$. Finally, $z^2 + h'(x) = z^2$ $\implies h(x) = \text{constant}$. Hence, $f(x, y, z) = z^2 + xz^2 - z\cos y$ is a potential for $\overrightarrow{\mathbf{F}}$.

c) The curve goes from $(-1, 0, -1)$ to $(1, 0, 1)$. Fundamental theorem of calculus for line integrals: $\int_C \overrightarrow{\mathbf{F}} \cdot d\vec{r} = f(1, 0, 1) - f(-1, 0, -1) = 1 - 1 = 0$.

Problem 15. *Direct method.* On the xy-plane, $\hat{\mathbf{n}} = -\hat{k}$, $\overrightarrow{\mathbf{F}} \cdot \hat{\mathbf{n}} = -1$, so the flux is $\pi(2)^2 = -4\pi$. On the portion S of paraboloid, we compute $\iint_S \overrightarrow{\mathbf{F}} \cdot d\overrightarrow{\mathbf{S}}$ by integrating over the shadow of S in the xy-plane.

$d\overrightarrow{\mathbf{S}} = \langle 2x, 2y, 1\rangle\, dx\, dy$, so $\overrightarrow{\mathbf{F}} \cdot d\overrightarrow{\mathbf{S}} = (2x^2 + 2y^2 + 1 - 2z)\, dx\, dy =$
$= [2x^2 + 2y^2 + 1 - 2(4 - x^2 - y^2)]\, dx\, dy = (4r^2 - 7)r\, dr\, d\theta$.

The flux is $\int_0^{2\pi}\int_0^2 (4r^3 - 7r)\, dr\, d\theta = 2\pi\left[r^4 - \frac{7r^2}{2}\right]_0^2 = 2\pi(16 - 14) = 4\pi$.

The total flux is $4\pi - 4\pi = 0$.

Using divergence theorem. The flux is given by $\iiint_D (\overrightarrow{\nabla} \cdot \overrightarrow{\mathbf{F}})dV$, where D is the solid region enclosed. In our case $\overrightarrow{\nabla} \cdot \overrightarrow{\mathbf{F}} = 1 + 1 - 2 = 0$, hence the total flux is 0.

Problem 16. $\overrightarrow{\nabla} \times \overrightarrow{\mathbf{F}} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ -6y^2 + 6y & x^2 - 3z^2 & -x^2 \end{vmatrix} = 6z\hat{\imath} + 2x\hat{\jmath} + (2x + 12y - 6)\hat{k}$.

Call R the region of the plane $x + 2y + z = 1$ enclosed by a simple closed curve C lying entirely on that plane. Stokes' theorem: $\int_C \overrightarrow{\mathbf{F}} \cdot d\vec{r} = \iint_R \left(\overrightarrow{\nabla} \times \overrightarrow{\mathbf{F}} \cdot \hat{\mathbf{n}}\right) dS$.

On R we have $\hat{\mathbf{n}} = \dfrac{\langle 1, 2, 1\rangle}{\sqrt{6}}$ and $\overrightarrow{\nabla} \times \overrightarrow{\mathbf{F}} \cdot \hat{\mathbf{n}} = \dfrac{6z + 2(2x) + (2x + 12y - 6)}{\sqrt{6}} =$
$= \sqrt{6}(x + 2y + z - 1) = 0$, because R belongs to the plane $x + 2y + z = 1$.

We conclude that $\int_C \overrightarrow{\mathbf{F}} \cdot d\vec{r} = \iint_R \left(\overrightarrow{\nabla} \times \overrightarrow{\mathbf{F}} \cdot \hat{\mathbf{n}}\right) dS = 0$ because $\overrightarrow{\nabla} \times \overrightarrow{\mathbf{F}} \cdot \hat{\mathbf{n}} = 0$.

18.02 Exam 1 Tues. Feb. 23, 2010 11:05-11:55

Directions:

1. There are 3 sheets, printed on both sides: seven problems in all.
2. Do all the work on these sheets; use the blank part below if truly necessary. Write down enough to show you are not guessing.
3. No books, notes, calculators, use of cell-phones, etc.
4. Please don't start until the signal is given; stop at the end when asked to; don't talk until your paper is handed in.
5. When the exam starts, read through the exam and start with what you are surest of.
6. Fill out the information below now.

Name Michael Plasmeier e-mail@mit.edu theplaz

Recitation teacher Oliver Rec. hour 12

pg.1 16

pg.2 15

pg.3 15

pg.4 12 ✓

pg.5 8

Total. 66/90

$\frac{d\vec{r}}{dt} = \vec{v}$

$\frac{ds}{dt} = |\vec{v}|$

$\frac{dT}{ds} = kN$

$\vec{v} = \dot{r}\,\hat{u}_r + \dot{\theta} r\,\hat{u}_\theta$

(16)

Problem 1. (20). Three points in xyz-space are $P: (-1,1,2)$, $Q: (1,2,1)$, and $O: (0,0,0)$.

a) (5) Find angle POQ.

$\vec{PO} = \langle -1,1,2 \rangle$ $\quad PQ \cdot QO = |PQ|\,|QO| \cos\theta$

$\vec{QO} = \langle 1,2,1 \rangle$ $\quad \sqrt{(-1)^2+1^2+2^2} = \sqrt{6}$

$(-1\cdot 1)+(1\cdot 2)+(2\cdot 1)$ $\quad \sqrt{1^2+2^2+1^2} = \sqrt{6}$

$-1+2+2$ $\quad 3 = \sqrt{6}\sqrt{6}\cos\theta$

3 $\quad 3 = 6\cos\theta$

$\frac{1}{2} = \cos\theta \quad \theta = 60^\circ$ (?)

5

b) (5) Find the scalar component of $\mathbf{i}+\mathbf{j}+\mathbf{k}$ in the direction of the vector PQ.

why not remember this

? scalar component

$\vec{PQ} = \langle 2,1,-1 \rangle$ ← direction

magnitude

$2\hat{\imath} + \hat{\jmath} - \hat{k}$

2

$|\vec{PQ}| = \sqrt{2^2+1^2+(-1)^2} = \sqrt{\hat{\imath}^2+\hat{\jmath}^2+k^2}$

$\sqrt{6}$ $\quad$ " component of $\vec{A}$ in dir $\vec{B}$

Oliver: they never told us this $\quad \vec{A} \cdot \frac{B}{|B|} = |\vec{A}|\cos\theta$ ← don't know θ

c) Find the equation of the plane through O, P, and Q.

5

eq of a plane $\quad$ → plug in pt

= Normal vector to it

– cross multiply

$-3(x--1) + 3(y-1) - 3(z-2)$

$-3x - 3 + 3y - 3 - 3z + 6 = 0$

$+3 \quad +3 \quad -6 \quad =6$

$\vec{PO} = \langle -1,1,2 \rangle$

$\vec{QO} = \langle 1,2,1 \rangle$ $\quad$ $\boxed{-3x + 3y - 3z = 0}$ ✓

$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -1 & 1 & 2 \\ 1 & 2 & 1 \end{vmatrix}$ $\quad (1-4)\hat{\imath} - (-1-2)\hat{\jmath} + (-2-1)\hat{k}$

$-3\hat{\imath} + 3\hat{\jmath} - 3\hat{k}$

d) Find the area of the space triangle OPQ.

4

$\frac{1}{2}\begin{vmatrix} i & j & k \\ -1 & 1 & 2 \\ 1 & 2 & 1 \end{vmatrix} = \frac{1}{2}(-3\hat{\imath} + 3\hat{\jmath} - 3\hat{k})$

$\frac{1}{2}\sqrt{3^2+3^2+3^2}$

$\frac{1}{2}\sqrt{27}$

$\frac{1}{2}\,\cancel{\frac{1}{3}}\,3\sqrt{3} = \sqrt{3}$

– opps, why did I do that

Problem 2. (20) 15

Let $A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 2 \\ 1 & 0 & 2 \end{pmatrix}$. Its matrix of cofactors is (in part) $C = \begin{pmatrix} 2 & -2 & -1 \\ -4 & 2 & a \\ 4 & -2 & b \end{pmatrix}$.

10 a) (15) Confirm (mentally) the entry -4 in the first column of C, then fill in the last column of C and from this find A^{-1}.

$a = \begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} = (0-2) = +2$

$b = \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = (1-4) = -3$

det A

$1(2-0) - 2(4-2) + 0$

$2 - 8 + 4$

-2

$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix} \to \begin{bmatrix} 2 & 2 & -1 \\ 4 & 2 & 2 \\ 4 & 2 & -3 \end{bmatrix}$

Oh they already did it w/ signs (when you say cofactor)

1) done twice! flip

$\begin{bmatrix} 2 & -4 & 4 \\ -2 & 2 & -2 \\ -1 & 2 & -3 \end{bmatrix} \to -\frac{1}{2}\begin{bmatrix} 2 & -4 & 4 \\ -2 & 2 & -2 \\ -1 & 2 & -3 \end{bmatrix}$

5 b) (5) Use the matrices of part (a) to solve the following system (no credit for solving the system by elimination):

$$x + 2y = 1, \qquad 2x + y + 2z = 0, \qquad x + 2z = 0\ .$$

$Ax = d$

$AA^{-1}x = dA^{-1}$

$x = dA^{-1}$

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} 2 & 4 & 4 \\ 2 & 2 & 2 \\ -1 & 2 & -3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$-\frac{1}{2}\begin{bmatrix} 2\cdot 1 + 2\cdot 0 + 0\cdot 0 \\ 2\cdot 1 + 1\cdot 0 + 2\cdot 0 \\ -1\cdot 1 + 0\cdot 0 + 2\cdot 0 \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ \frac{1}{2} \end{bmatrix}$

fixed at last min!

$x = -1 \quad y = -1 \quad z = \frac{1}{2}$

Problem 3. (5) Find the value(s) of c for which the system of homogeneous equations

$$cx + 2y + z = 0, \qquad 2x - y + z = 0, \qquad x + 3y - 2z = 0$$

has a solution other than $x = y = z = 0$. (No credit for solving by elimination.)

where det $= 0$

$$\begin{bmatrix} c & 2 & 1 \\ 2 & -1 & 1 \\ 1 & 3 & -2 \end{bmatrix}$$

$$c(2-3) - 2(-4-1) + 1(6--1) = 0$$

$$2c - 3c + 8 + 2 + 6 + 1 = 0$$

$$-c = -17$$

$$c = 17$$

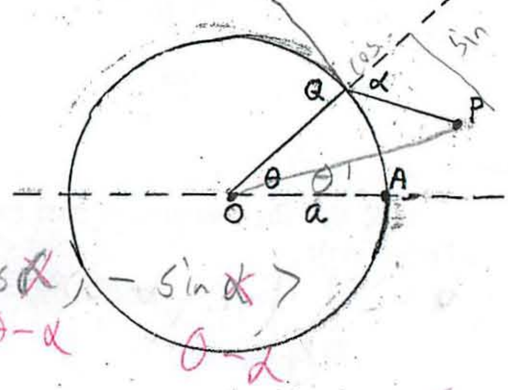
✓

Problem 4 (15) Scotch® tape is being unwound from a stationary circular spool having radius a. The end $P : (x, y)$ of the tape is initially at the point $A : (a, 0)$ on the x-axis; Q is the point on the circumference where the tape is leaving the spool. During the process, the unwound length of tape QP is held taut, and held so that it makes a constant negative angle $-\alpha$, $0 < \alpha < \pi/2$ with the radial vector OQ (as measured clockwise from OQ to QP).

Use vector methods to derive parametric equations for x and y in terms of the central angle θ and the constants a and α, for $0 \le \theta \le 2\pi$. Show work, indicating reasoning.

(If stuck, for 5 points less, you can take $\alpha = \pi/2$, so that the unwound tape is always held tangent at Q, in the direction where its sticky side faces the spool.)

$\vec{OP} = OQ + \vec{QP}$ + something w/ α when $\alpha \neq \frac{\pi}{2}$



$a \langle \cos\theta, \sin\theta \rangle + a\theta \langle + \cos\alpha, -\sin\alpha \rangle$

$\uparrow$ arclenght $\qquad \theta - \alpha \qquad \theta - \alpha$

$\vec{r} = a \langle \cos\theta + \theta\cos\alpha, \ \sin\theta - \theta\sin\alpha \rangle$

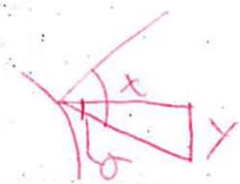
$\theta' = \tan^{-1}\left(\frac{y}{x}\right) = \left(\frac{a\sin\theta - a\theta\sin\alpha}{a\cos\theta + a\theta\cos\alpha}\right)$

$x = r\cos\theta' = a\cos\theta + a\theta\cos\alpha$

$y = r\sin\theta' = a\sin\theta - a\theta\sin\alpha$

note $\sin(-\alpha) = -\sin\alpha$

$\cos(-\alpha) = \cos\alpha$



create a new angle (was thinking about that)

$\sigma = \theta - \alpha$

10/15

Problem 5. (15) The path of a point P is a circular helix in space having position vector

$$OP = \mathbf{r}(t) = \langle 2\cos t,\ 2\sin t,\ t\rangle .$$

Find in order the following, in terms of t, giving enough calculation or reasoning to show you are not guessing or writing down answers from memory:

(3) a) the velocity vector $\mathbf{v}$

$d \sin = \cos$

$d \cos = -\sin$

derivative

$\vec{v}(t) = \langle -2\sin t,\ 2\cos t,\ 1\rangle$

3/3

(4) b) the speed $|\mathbf{v}|$ and the length of one complete turn of the helix, i.e., the length between two successive points lying over the same point in the xy-plane.

$|\vec{v}| = \sqrt{(-2)^2 \sin^2 t + 2^2 \cos t + 1^2}$

$\sqrt{4\sin^2 t + 4\cos^2 t + 1}$

$\sqrt{4(\sin^2 t + \cos^2 t) + 1}$

$\sqrt{4+1}$

$\sqrt{5}$

$S = \int_0^{2\pi} \frac{ds}{dt}\, dt$

$S = \int_0^{2\pi} \sqrt{5}\, dt$

$S = \sqrt{5}t \Big|_0^{2\pi}$

4/4 $S = 2\pi\sqrt{5} - 0$

$\cancel{(5)^{1/2}}$

$\cancel{5^{3/2}}$

$\cancel{3/2 \text{ constant}}$

(8) c) the unit tangent vector $\mathbf{T}$, the unit normal vector $\mathbf{N}$, and the curvature κ (k in the book), at time t.

$T = \frac{V}{|\vec{V}|} = \frac{\langle -2\sin t,\ 2\cos t,\ 1\rangle}{\sqrt{5}}$

5

$dT = \frac{dT/dt}{ds/dt} = \frac{?\ \text{differentiate (product rule)}}{|\vec{v}|}$

$d\ \frac{1}{\sqrt{5}} \cdot \langle -2\sin t,\ 2\cos t,\ 1\rangle + \langle -2\cos t, -2\sin t, 0\rangle \cdot \frac{1}{\sqrt{5}}$

$0 + \left(\frac{\langle -2\cos t,\ -2\sin t,\ 0\rangle}{\sqrt{5}}\right) \Big/ \sqrt{5}$

$\frac{\langle -2\cos t,\ -2\sin t,\ 0\rangle}{\sqrt{5}\sqrt{5}}$

$\langle -2\cos t, -2\sin t, 0\rangle$

$N = \langle -2\cos t, -2\sin t, 0\rangle$

$k = \pm 1/5$

"right"

so $\ominus$

$k = -1/5$

factor $\frac{\sqrt{2}}{5}$ out of normal vector

$\frac{\sqrt{2}}{5} / \sqrt{5} = \boxed{\frac{2}{5} = k}$

Problem 6. (5)

Find the length of the exponential spiral curve $r = e^{2\theta}$ in the plane, between the point on the curve where $r = 1$, $\theta = 0$, and the next point on the curve where it crosses the x axis as θ increases.

$S = \int_0^{2\pi} \frac{ds}{2t}\, dt$ $\qquad r = e^{2\theta}$

$V = 2\theta e^{2\theta}$ ①

$V^2 = 2\theta e^{2\theta^2} + ?$

$\frac{ds}{dt} = |\vec{v}| = \sqrt{\hat{v}_r^2 + \hat{v}_\theta^2}$

$S = \int_0^{2\pi}$

$\vec{V} = \dot{r}\hat{v}_r + r\dot{\theta}\hat{v}_\theta$

out of the

$S = \int_0^{2\pi} \sqrt{\hat{v}_r^2 + \hat{v}_\theta^2}\, dt$

$r = e^{2\theta}$

$S = \sqrt{\hat{v}_r^2 + \hat{v}_\theta^2}\, t \Big|_0^{2\pi}$

$\theta =$

See other sheet

$S = \sqrt{\hat{v}_r^2 + \hat{v}_\theta^2}\; 2\pi$

$S = \sqrt{1^2 + 0^2}\; 2\pi = 2\pi$

Problem 7. (10) The velocity vector of a moving point in the polar-coordinate $\mathbf{u}_r - \mathbf{u}_\theta$ system is given in general by $\mathbf{v} = r'\mathbf{u}_r + r\theta'\mathbf{u}_\theta$.

A point P moves with velocity vector $\mathbf{v} = -\sin t\, \mathbf{u}_r + \sin 2t\, \mathbf{u}_\theta$.

If it is at $r = 1$, $\theta = 0$ at time $t = 0$, what are the parametric equations $r = r(t)$, $\theta = \theta(t)$ that describe its motion?

Should have remembered better

$\vec{V} = \dot{r}\hat{U}_r + r\dot{\theta}\hat{U}_\theta$

Don't forget constant of integration

Know: $\sin 2t = 2\cos t \sin t$

$\cos(a+b) = \cos a \cos b - \sin a \sin b$

$\dot{r} = -\sin t$

$\int$

$r(t) = \cos t + 1$

$r = \cos t + c$

$\theta(t) = \frac{-\cos 2t}{\sin t + t} + \frac{\pi}{4}$

$r(1) = \cos(1) + ? = 1$

$r = \cos t + 1$ $+4$??? cos at 1?

$+4$

$\sin(a+b) = \cos a \sin b + \sin a \cos b$

$r\dot{\theta} = \sin 2t$

$\dot{\theta} = \frac{\sin 2t}{\cos t + 1}$

$\int$

I did integral wrong prob

$\theta = \frac{-\cos 2t}{\sin t + t} + C$

$0 = \frac{-\cos 2t}{\sin t + t} = \frac{-\cos 0}{\sin 0 + 0} \rightarrow \theta = \frac{-\cos 2t}{\sin t + t} + \pi/4$

6.

lenght $= \int_t |\vec{v}|\, dt$ need access to that

no t given

take $\theta = t$

$r(t) = r(\theta) = e^{2t}$

$\vec{v} = \dot{r}\hat{v}_r + r\dot{\theta}\hat{v}_\theta$

$|v| = \sqrt{\dot{r}^2 + (r\dot{\theta})^2}$ ← oh duh

⚠ $|\vec{v}| = \left|\frac{dr}{dt}\right|$ $\boxed{|\vec{v}| \neq \dot{r}}$ ← be careful

See answer key as well

18.02 Exam I Sol'ns   Spring 2010

[1] a) $\cos(POQ) = \dfrac{\vec{OP}\cdot\vec{OQ}}{|OP|\,|OQ|} = \dfrac{3}{\sqrt6\sqrt6} = \dfrac12$

$\therefore\ POQ = \pi/3$ or $60°$

b) $\vec{PQ} = \langle 2,1,-1\rangle$

$\langle 1,1,1\rangle \cdot \dfrac{\langle 2,1,-1\rangle}{\sqrt6} = \dfrac{2}{\sqrt6} = \dfrac{\sqrt6}{3}$

c) $\vec{OP}\times\vec{OQ} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & 2 \\ 1 & 2 & 1 \end{vmatrix} = \langle -3,3,-3\rangle$ $= \vec{N}$ to plane

Plane through $(0,0,0)$:

$-x+y-z=0$ or ($x - y + z = 0$ or a multiple)

d) $\frac12|\vec{OP}\times\vec{OQ}| = \frac32|\langle -1,1,-1\rangle| = \frac32\sqrt3$

---

[2] a) $\begin{pmatrix} 2 & -2 & -1 \\ -4 & 2 & \boxed{2} \\ 4 & -2 & \boxed{-3} \end{pmatrix} = C$ $\quad \begin{vmatrix} 1 & 2 & 0 \\ 2 & 1 & 2 \\ 1 & 0 & 2 \end{vmatrix} = -2$

$C^T = \begin{pmatrix} 2 & -4 & 4 \\ -2 & 2 & -2 \\ -1 & 2 & -3 \end{pmatrix}$ $\quad A^{-1} = \begin{pmatrix} -1 & 2 & -2 \\ 1 & -1 & 1 \\ 1/2 & -1 & 3/2 \end{pmatrix}$

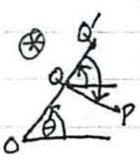
(or $\frac12 C^T$)

b) $A\vec{x} = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $\vec{x} = A^{-1}\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}-1\\1\\1/2\end{pmatrix} = \begin{pmatrix}x\\y\\z\end{pmatrix}$

[3] $\begin{vmatrix} c & 2 & 1 \\ 2 & -1 & 1 \\ 1 & 3 & -2 \end{vmatrix} = \begin{matrix} 2c+2+6 \\ -(-1+3c-8) \end{matrix} = -c+17$ $=0$ if $\boxed{c=17}$

[4] $\vec{OQ} = \langle a\cos\theta, a\sin\theta\rangle$

$\vec{QP} = a\theta\langle \cos(\theta-\alpha), \sin(\theta-\alpha)\rangle$



$\vec{OP} = a\langle \cos\theta + \theta\cos(\theta-\alpha),$
$\sin\theta + \theta\sin(\theta-\alpha)\rangle$

---

if $\alpha = \pi/2$, $\vec{QP} = a\theta\langle \sin\theta, -\cos\theta\rangle$

$\vec{OP} = a\langle \cos\theta + \theta\sin\theta, \sin\theta - \theta\cos\theta\rangle$

⊛ $\angle PQQ' = \theta - \alpha$

[5] $\vec{r}(t) = \langle 2\cos t, 2\sin t, t\rangle$

a) $\vec{v} = \langle -2\sin t, 2\cos t, 1\rangle$

b) $|\vec{v}| = \dfrac{ds}{dt} = \sqrt{4(\sin^2 t+\cos^2 t)+1}$

$= \sqrt5$

$s = \int_0^{2\pi} \dfrac{ds}{dt}\cdot dt = 2\sqrt5\pi$

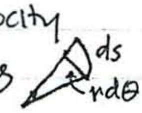
c) $\vec{T} = \dfrac{\vec{v}}{|\vec{v}|} = \dfrac{1}{\sqrt5}\langle -2\sin t, 2\cos t, 1\rangle$

$\vec{N} = \mathrm{dir}\left(\dfrac{d\vec{T}}{dt}\right) = \langle -\cos t, -\sin t, 0\rangle$

$K = \left|\dfrac{d\vec{T}}{ds}\right| = \left|\dfrac{d\vec{T}/dt}{ds/dt}\right|$

$= \dfrac{\frac{2}{\sqrt5}|\vec{N}|}{\sqrt5} = \dfrac25$

[6] Taking $t=\theta$ in the velocity formula (see prob. 7), or using



$\dfrac{ds}{d\theta} = \sqrt{r^2+r'^2}$

$= \sqrt{(e^{2\theta})^2+(2e^{2\theta})^2} = e^{2\theta}\sqrt5$

$\therefore\ s = \int_0^{2\pi} e^{2\theta}\sqrt5\,d\theta = \dfrac{e^{2\theta}}{2}\sqrt5\Big]_0^{2\pi} = \dfrac{\sqrt5}{2}(e^{4\pi}-1)$

[or $\int_0^{\pi} e^{2\theta}\sqrt5\,d\theta = \frac{\sqrt5}{2}(e^{2\pi}-1)$]

[7] Comparing two formulas for $\vec{v}$,

$r' = -\sin t \qquad r\theta' = \sin 2t$

$\therefore\ r = \cos t + c_1;\ r(0)=1 \Rightarrow c_1 = 0$

$r\theta' = \sin 2t \Rightarrow \cos t\cdot\theta' = 2\sin t\cos t$

$\therefore\ \theta' = 2\sin t$

$\theta = -2\cos t + c_2 \qquad \theta(0) = 0 = -2+c_2$

$\therefore\ c_2 = 2$

$\begin{cases} r(t) = \cos t \\ \theta(t) = 2-2\cos t \end{cases}$

$\sin 2t = 2\cos t\sin t$

18.02 Exam 1 Redo

5/14

1. 3 points in space

P (-1, 1, 2)

Q (1, 2, 1)

O (0, 0, 0)

Find angle POQ

? how do you do this

Cross product ??

? from a plane

$PQ \cdot QO = |PQ||QO| \cos\theta$

$\vec{PO} = \langle -1, 1, 2 \rangle$

$\vec{QO} = \langle 1, 2, 1 \rangle$

So dot product = mag · mag $\cos\theta$

~~So~~ $3 = \sqrt{6}\sqrt{6} \cos\theta$

$\cos\theta = \frac{1}{2}$

$\theta = 60°$

②

Not a good sign forgetting that

~~Will pratice test~~

Will main test match pratice test ???

b) Find scalar component of $\hat{i} + \hat{j} + \hat{k}$ in $\vec{PQ}$

- guess tey mean $1\hat{i} + 1\hat{j} + 1\hat{k}$

$$\vec{PQ} = \langle 1--1,\ 2-1,\ 1-2 \rangle$$

$$\text{d'r} \quad \frac{\langle 2,\ 1,\ -1 \rangle}{\sqrt{2^2+1^2+1^2}} \quad \frac{\langle 2, 1, -1 \rangle}{\sqrt{6}}$$

$$\frac{1 \cdot 2 + 1 \cdot 1 + -1 \cdot 1}{\sqrt{6}}$$

$$\frac{2}{\sqrt{6}} \rightarrow \frac{2\sqrt{6}}{\cancel{6}} \quad \boxed{\frac{\sqrt{6}}{3}}$$

reduce one more

Missed on real test

Got here → cool!

(using what I learned to figure out)

(3)

c) Find the eq of the plane through O, P, Q

$PQ \times QO$

some normal to

no eq

$P_1 \cdot \overline{P_1P_2}\, t$

$$\begin{array}{c|ccc|}
 & \hat{i} & \hat{j} & \hat{k} \\
PO & -1 & 1 & 2 \\
QO & 1 & 2 & 1
\end{array}$$

$1 - 4\,\hat{i} + -1 + 2\,\hat{j} + 1 + 2\,\hat{k}$

$-3\hat{i} + 1\hat{j} + 3k$

$-3 + 1 + 3 = ?$

~~?plug in pt~~?

~~or~~ 0

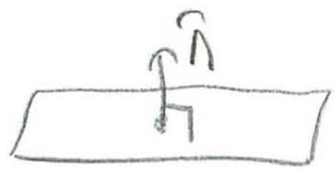
$-3(x - -1) + 3(y - 1) - 3(z - 2) = 0$

wtf → what is the formula

(4)

Review of plane (wolfram math)

Normal vector



$\hat{n} = (a, b, c)$

$$ax + by + cz + d = 0$$

$$ax + by + cz = -d$$

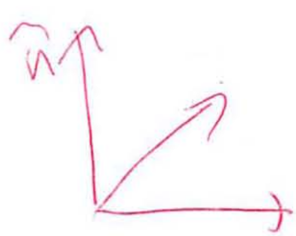
$$d = -ax_0 - by_0 - cz_0$$

Another problem example

(getting more confused hurts test performance)

$AB \times AC$

↑ first part



Then use points as coord

So <u>before</u> was find intersection point

(5)

try again

PO × QO

$\langle -1\cdot 1,\ 1\cdot 2,\ 2\cdot 1 \rangle$

~~$\langle -1,\ 2,\ 2 \rangle$~~

not how you do cross product

plug in pt 0

~~$-1(x-0)+2(y-0)+2(z-0)$~~

~~$-1x+2y+z=0$~~

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & 2 \\ 1 & 2 & 1 \end{vmatrix}$$

$1-4\,\hat{i} \; - \; -1+2\,\hat{j} + -2\,\cancel{+}1\,\hat{k}$

matrix rules error

$-3\hat{i}+3\hat{j}-3\hat{k}=0$

$-3(x-0)+3(y-0)-3(z-0)=0$ can I pick any pt?

yeah

$-3x+3y-3z=0$

on test picked harder one

(6)

2. $A = \begin{vmatrix} 1 & 2 & 0 \\ 2 & 1 & 2 \\ 1 & 0 & 2 \end{vmatrix}$ Cofactors $\begin{vmatrix} 2 & -2 & -1 \\ -4 & 2 & a \\ 4 & -2 & b \end{vmatrix}$

$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$

~~Confirm~~

fill in a

$1 \cdot 0 - 2 \cdot 1$

$a = +2$ ~~don't change sign~~

b) $1 - 4$

$b = -3$

~~do~~ sign should change ✓

~~[illegible]~~

?what are cofactors again

Think kinda like inverse

~~before~~ after $+-$ and rotate before

and before $\frac{1}{\det}$

Cofactor $\xrightarrow{\text{rotate}}$ adjoint

$\frac{1}{\det}$ adjoint = inverse

another one didn't knew on test

now need to rotate and $\frac{1}{\det(A)}$

$\det(A) = 1(2-0) - 2(4-2) + 0$

$2 - 4 = -2$

(7)

$$-\frac{1}{2}\begin{vmatrix} 2 & -4 & 4 \\ -2 & 2 & -2 \\ -1 & 2 & -3 \end{vmatrix}$$

↓ actually do

$$\begin{vmatrix} -1 & 2 & -2 \\ 1 & -1 & 1 \\ \frac{1}{2} & -1 & \frac{3}{2} \end{vmatrix}$$ ✓

b) Use matrix of part a to solve

Forget what the rules

$x + 2y = 1$

$2x + y + 2z = 0$

$x + 2z = 0$

What are rules again

$X = dA^{-1}$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 & 2 & -2 \\ 1 & -1 & 1 \\ \frac{1}{2} & -1 & \frac{3}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

But how does actual fit in?

(8)

Now that part at end

Oh yeah the ~~ot~~ equations are same as matrix

$$\begin{bmatrix} -1 & 2 & -2 \\ 1 & -1 & 1 \\ \frac{1}{2} & -1 & \frac{3}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$-1 \cdot 1 + 2 \cdot 0 - 2 \cdot 0 \rightarrow$ add

$1 \cdot 1$

$1 \cdot \frac{1}{2}$

$$\begin{bmatrix} -1 \\ 1 \\ \frac{1}{2} \end{bmatrix}$$

$x = -1 \quad y = 1 \quad z = \frac{1}{2}$

hopefully will remeber this

(9)

3. Find the values of c for which system of homogenous equations has solution other than $x = y = z = 0$

$cx + 2y + z = 0$

$2x - y + z = 0$

$x + 3y - 2z = 0$

What does "homogeneas" mean again?

When det $= 0$

$$\begin{bmatrix} c & 2 & 1 \\ 2 & -1 & 1 \\ 1 & 3 & -2 \end{bmatrix}$$

$c(2-3) - 2(-4-1) + 1(6+1) = 0$

$-c + 10 + 7 = 0$

$-c = -17$

$\boxed{c = 17}$ (1)

(10)

Remember

Homogeneous = det = 0

~~I got a~~ at least relearned what det was

4. Oh know this problem

I remember could never do it

Let me come back to this one

5. ~~P~~ Position vector

$OP = r(t) = \langle 2\cos t, 2\sin t, t \rangle$

~~a~~) Find $\vec{v}$

- did this earlier

$\langle -2\sin t, 2\cos t, 1 \rangle$ ✓

b) Speed

$\sqrt{(2\sin t)^2 + (2\cos t)^2 + 1^2}$

$\sqrt{4\sin^2 t + 4\cos^2 t + 1}$

$\sqrt{4(1) + 1} = \sqrt{5}$ ✓

(11)

2 part question

s → arc lenght

$$\int_0^{2\pi} \frac{ds}{dt}\, dt$$

$$|\vec{v}|$$

$$\int_0^{2\pi} \sqrt{5}\, dt$$

$$\sqrt{5}\, t$$

$$s = 2\pi\sqrt{5}$$

memorize this

c) Unit tangent vector T, N, k

normal curvature

forget all et tese

— no $n = \langle -2\sin t, 2\cos t \rangle$

$\vec{T} = \langle 2\cos t, 2\sin t \rangle$

$k = \frac{\text{something}}{\text{something}}$

12

$$T = \frac{\vec{v}}{|v|} = \frac{\langle -2\sin t, 2\cos t, 1 \rangle}{\sqrt{5}}$$

$N =$ dir $\frac{dT}{dt}$

oh yeah - another deriv

$$\frac{dT}{dt} = \frac{\langle -2\cos t, -2\sin t, 0 \rangle}{\cancel{\sqrt{5}}}$$

↑ constant factors out

$\langle -\cancel{2}\cos t, -\cancel{2}\sin t, 0 \rangle$

↑ direction means no #

$$\kappa = \left|\frac{d\vec{T}}{ds}\right| = \left|\frac{d\vec{T}/dt}{ds/dt}\right|$$

← found above

← ~~length~~ ~~velocity~~ speed

$$\left|\frac{\frac{\langle -2\cos t, -2\sin t, 0 \rangle}{\sqrt{5}}}{\sqrt{5}}\right|$$

notice reduces

$$\frac{\frac{2}{\sqrt{5}}\,|N|}{\sqrt{5}} = \boxed{\frac{2}{5}}$$

(all 3 of these = pratice test)

6. So I completly blow this on test

Find the lenghs of curve $r = e^{2\theta}$

from $r=1$ $\theta = 0$ to next place it crosses

X axis as $\theta \uparrow$

lenght $= \int_t |\vec{v}| \, dt$

need access to that

no t given

take $\theta = t$

$r(t) = r(\theta) = e^{2t}$

$\vec{v} = \dot{r} \, \hat{v}_r + r\dot{\theta} \, \hat{v}_\theta$

what is this again? r derivative?

and what is this?

$|v| = \sqrt{\dot{r}^2 + (r\dot{\theta})^2}$

⚠ $|\vec{v}| = \left|\frac{dr}{dt}\right|$ $\vec{v} \neq \hat{r}$

(14)

Oh answer key has something else

$t = \theta$

$r\,d\theta$

$$\frac{ds}{d\theta} = \sqrt{r^2 + r'^2}$$

$$= \sqrt{(e^{2\theta})^2 + (2e^{2\theta})^2}$$

$$= e^{2\theta}\sqrt{5}$$

$$s = \int_0^{2\pi} e^{2\theta}\sqrt{5}\,d\theta$$

$$= \frac{e^{2\theta}}{2}\sqrt{5}\,\Big|_0^{2\pi} = \frac{\sqrt{5}}{2}\left(e^{4\pi}-1\right)$$

$\boxed{\text{Or}}$ $\int_0^{\pi} e^{2\theta}\sqrt{5}\,d\theta$

$$= \frac{\sqrt{5}}{2}\left(e^{2\pi}-1\right)$$

(15)

Yeah I don't get that at all.
Oh well skip + hope it does not come up

7. u_r u_θ System

$\vec{v} = r' u_r + r\theta' u_\theta$

Point P $v = -\sin t\, u_r + \sin 2t\, u_\theta$

If $r=1$ $\theta=0$ at $t=0$

What are parametric eq that describe its motion

$r = r(t)$

$\theta = \theta(t)$

Wrote on test that I forgot
- forgot again now I think

Comparing 2 formulas for $\vec{v}$

$r' = -\sin t$

$r\theta' = \sin 2t$

$r = \cos t + c_1 \qquad r(0) = 1 \rightarrow c_1 = 0$

$r\theta' = \sin 2t \rightarrow \cos t \cdot \theta' = 2\sin t \cos t$

$\theta' = 2\sin t$

$\theta = -2\cos t + c_2$

$c_2 = 2$

Oh is this one of those reverse things?

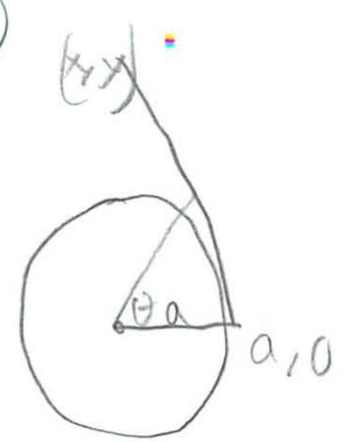
$\theta(0) = 0$

$= -2 + c_2$

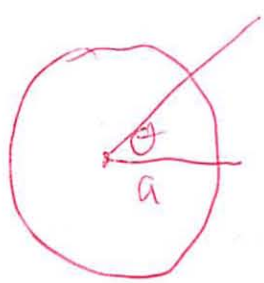
$$\begin{cases} r(t) = \cos t \\ \theta(t) = 2 - 2\cos t \end{cases}$$

$\sin 2t = 2\cos t \sin t$

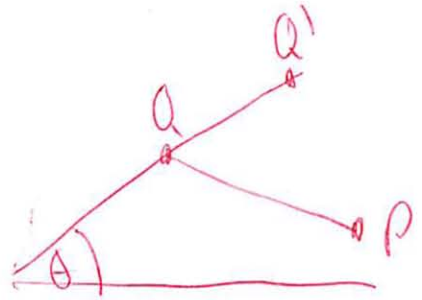
(17)



forget these pictures
- how do I draw it



Still don't get from the answer!



$\vec{OQ} = \langle a\cos\theta, a\sin\theta \rangle$

$\vec{QP} = a\theta \langle \cos(\theta-\alpha), \sin(\theta-\alpha) \rangle$

$\vec{OP} = a \langle \cos\theta + \theta\sin(\theta-\alpha),$
$\sin\theta + \theta\sin(\theta-\alpha) \rangle$

if $\alpha = \frac{\pi}{2}$ $\vec{QP} = a\theta \langle \sin\theta, -\cos\theta \rangle$

$\vec{OP} = a \langle \cos\theta + \theta\sin\theta,$
$\sin\theta - \theta\cos\theta \rangle$

$PQQ' = \theta - \alpha$

Just coping answer will not help me understand
must understand picture

⑧ No still don't get picture
How tape being pulled off spool.

So for this test better on matrix stuff
but 2nd half not as good at
- what is that even called?
- parametric equations
lecture 5

So paramatrize curve into functions of t

$\vec{A}$ is direction moving in

$\overrightarrow{P_0 P} + \vec{A}$

$= \langle a_1 t, a_2 t, a_3 t \rangle$

like the find pt on plane

$P_0 + \underset{\uparrow \text{vector}}{A} t$

Plug into plane

(19)

Steps when given a parametric eq

1. Find curve
   - maybe eliminate t

2. Given geometry, find parametric eq

3. Given parametric eq, get info

Introduction of vectors

Cycloid example

$$\vec{r} = \vec{OP}$$

$$\vec{OP} = \vec{OA} + \vec{AB} + \vec{BP}$$

~~thake~~ take derivs using vectors

velocity $\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta r}{\Delta t} = \frac{d\tilde{r}}{dt}$ (know this)

~~I~~ Using chain rule $\frac{dr}{ds} \cdot \underbrace{\frac{ds}{dt}}_{\text{speed}} = \vec{v}$

$$T \cdot \frac{ds}{dt}$$

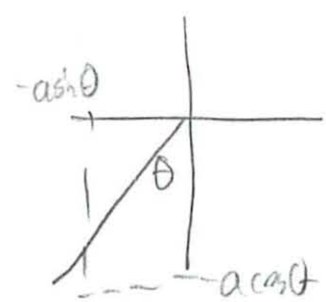
20

Lecture 6 was polar coords
- good on those

Lecture 5 Arox

OP = OA + AB + PP

$\langle a(\theta), 0 \rangle + \langle 0, a \rangle + \langle -a\sin\theta, a\cos\theta \rangle$



$= \langle \underbrace{a\theta - a\sin\theta}_{x(\theta)}, \underbrace{a - a\cos\theta}_{y(\theta)} \rangle$

guess scotch tape is like that
just more complex

(21)

$$V = \frac{dr}{dt} = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle$$

arc lengnt = s = distance traveled

$\hat{T}$ = unit tangent vector $\dfrac{\vec{V}}{|V|}$ ← same dir as v ← make unit lengnth

Arav covered this so much better!

$$p(t) = A + t\vec{AB} = B$$

$$(0,-1) + t\langle 1, 3\rangle)$$

$$(t, -1+3t)$$

$r_{x(t)}$ $r_{y(t)}$

Intersection line on a plane

1. paramitrize line
2. Find the t that corresponds to intersection

1. $P(t) = A + \vec{N}$
2. Plug into plane eq

get t

3. Plug t in to get final eq

18.02 Exam 2 Thurs. Apr.1, 2010 11:05-11:55

Directions:

1. There are 3 sheets, printed on both sides: nine problems in all.
2. Do all the work on these sheets; use the blank part below if truly necessary. Write down enough to show you are not guessing.
3. No books, notes, calculators, use of cell-phones, etc.
4. Please don't start until the signal is given; stop at the end when asked to; don't talk until your paper is handed in.
5. When the exam starts, read through the exam and start with what you are surest of.
6. Fill out the information below now.

Name Michael Plasmeier e-mail@mit.edu theplaz

Recitation teacher Oliver Rec. hour 12

pg.1 19

pg.2 12

pg.3 8

pg.4 14

pg.5 19

Total. 72

Recitation

Mean 69

Median 74

Problem 1. (10) For the function $w = x^2y - xy^3$, find its directional derivative $\left.\frac{dw}{ds}\right|_{P,\hat{\mathbf{u}}}$ at the point $P:(1,1)$ in the direction $\hat{\mathbf{u}}$ of the vector $\mathbf{i}+\mathbf{j}$.

$= \nabla W \cdot \vec{u}$

$W_x = 2xy - y^3$

$W_y = x^2 - x3y^2$

Plug pts in

$W_x = 2(1)(1) - (1)^3 = 1$

$W_y = 1^2 - (1)\cdot 3(1)^2 = -2$

$\nabla = \langle 1, -2 \rangle$

$\sqrt{1^2+1^2} = \sqrt{2}$

$\left.\frac{dw}{ds}\right|_{P\cdot\vec{u}} = \langle 1, -2 \rangle \cdot \frac{\langle 1, 1 \rangle}{\sqrt{2}}$

$\left(1\cdot\frac{1}{\sqrt{2}}\right) + \left(-2\cdot\frac{1}{\sqrt{2}}\right)$

$\frac{1}{\sqrt{2}} - \frac{2}{\sqrt{2}}$

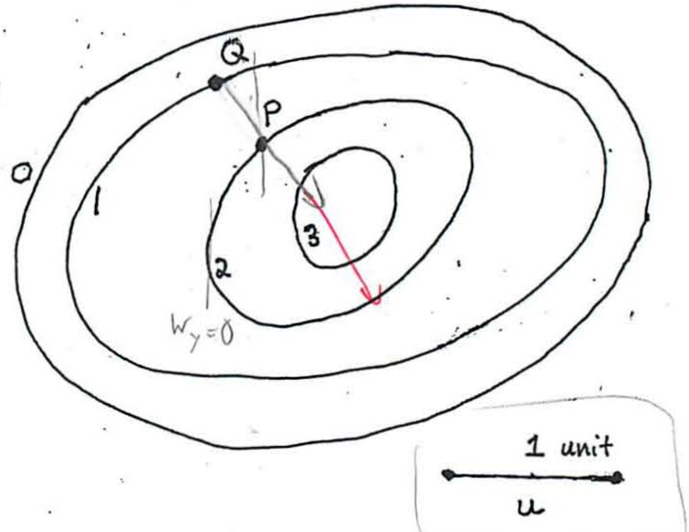
$\frac{-1}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}} = \frac{-\sqrt{2}}{2}$

✓

Problem 2 (10: 4,6) Some level curves for $w = f(x,y)$ are shown; $\mathbf{u}$ is a unit distance.

a) At P, estimate the value of w_y.

b) At Q, draw the vector $(\nabla f)_Q$.



$W_y\big|_P = \frac{\partial w}{\partial y} = \frac{1}{-1/2}$ change of curve / distance in y dir

$= -2$ ✓

$Q = (\nabla f)_Q$ draw gradient

towards increasing ground

$= \left\langle \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y} \right\rangle$

$= \left\langle \frac{1}{-\frac{1}{2}}, \frac{-1}{\frac{1}{2}} \right\rangle$

$\langle 2, -2 \rangle$

5|6

Problem 3. (20: 3, 12, 5) Find the point P on the surface $x^2 + yz + 3z - 8 = 0$ which is closest to the origin, by following the steps below.

optimization

(a) It suffices to find the point P which minimizes the square of the distance to the origin. Show this leads to finding the point which minimizes $w(y, z) = y^2 + z^2 - yz - 3z + 8$.

rewrite constraint

$d = \sqrt{x^2 + y^2 + z^2}$

0

(b) Find the point (y_0, z_0) which minimizes $w(y, z)$, and use it to find P. (You don't have to prove it is a minimum point.)

$W_y = 2y + 0 - z - 0 + 0$

$W_z = 0 + 2z - y - 3 + 0$

Set = to 0

$2y - z = 0 \qquad 0 = 2z - y - 3$

find pts $\qquad 3 = 2z - y$

$(y, z) \quad (1, 2) \qquad (1, 2)$

$(-1, -2) \qquad (-1, -2)$

P = ?

10

Try each

$1^2 + (2)^2 - 1 \cdot 2 - 3 \cdot 2 + 8 = 5$ ← minimum point $(1, 2)$

$(-1)^2 + (-2)^2 - (-1)(-2) - 3(-2) + 8 = 15$

what is difference (y_0, x_0) and P

(c) If this problem is solved by Lagrange multipliers instead, give one of the equations involving the multiplier λ, and use it to determine the value of λ corresponding to the point P.

$2x + 0 + 0 - 0 = \lambda$

$0 + z + 0 - 0 = \lambda$

$0 + y + 3 - 0 = \lambda$

$$\boxed{\begin{gathered} 2x = \lambda \\ z = \lambda \\ y + 3 = \lambda \end{gathered}}$$

2

$2x = z = y + 3$

Out of the

5

Problem 4 (10) Let $w = f(x,y)$, where in turn $x = 2u - v^2$ and $y = uv$.

If in xy-coordinates $\nabla f = \langle 2,3 \rangle$ at the point $P : (4,0)$, find the value of $\frac{\partial w}{\partial v}$ at the point in uv-coordinates corresponding to P.

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial v} + ?$$

$$= 2 \cdot 0 - 2v$$

$$= 2 \cdot -2v$$

$$= 2 \cdot -2(0)$$

$$= 0$$

point

$4 = 2u - v^2$ 2eq, 2 unknowns

$0 = uv$

$\cancel{u = \frac{v}{0}}$?

$4 = 2(2) - (0)^2$

$u = 2$

$v = 0$

3

4

can't picture graphically at all

Problem 5 (10: 5,5))

they are not independent - have some relation

a) Suppose $f(x,y,z) = 0$. Derive a formula for $\left(\frac{\partial z}{\partial x}\right)_y$ in terms of the formal partial derivatives f_x, f_y, f_z, i.e., the derivatives taken as if x, y, z were independent; use the chain rule or differentials.

find formula for

$$\left(\frac{\partial f(x,y,z)}{\partial x}\right)_y = f_x \underbrace{\left(\frac{\partial x}{\partial x}\right)_y}_{=1} + f_y \underbrace{\left(\frac{\partial y}{\partial x}\right)_y}_{=0} + f_z \left(\frac{\partial z}{\partial x}\right)_y$$

z is function of x and y

recall $= 0$ ← by definition

$$0 = \left(\frac{\partial f}{\partial x}\right)_y = f_x + f_z \left(\frac{\partial z}{\partial x}\right)_y$$

we want

$$\left(\frac{\partial z}{\partial x}\right)_y = \frac{\left(\frac{\partial f}{\partial x}\right)_y - f_x}{f_z} = \frac{-f_x}{f_z}$$

← don't think what they are asking for

almost

b) Letting x, y, r, θ be the usual rectangular and polar coordinates, calculate $\left(\frac{\partial r}{\partial \theta}\right)_x$ in terms of r and θ.

has relation to part a

$x = r\cos\theta$ $\qquad r = \sqrt{x^2+y^2}$

$y = r\sin\theta$ $\qquad \theta = \tan^{-1}\left(\frac{y}{x}\right)$

→ so $f(x,r,\theta) = 0$ for $f(x,r,\theta) = x - r\cos\theta$

of this

$$\cancel{\frac{\partial r}{\partial x} \cdot \frac{\partial x}{\partial \theta}} \quad ?$$

$$\cancel{\sqrt{2x+y^2} \cdot \cos\theta}$$

$$\left(\frac{\partial r}{\partial \theta}\right)_x = \frac{-f_\theta}{f_r} = \frac{-r\sin\theta}{-\cos\theta} = r\tan\theta$$

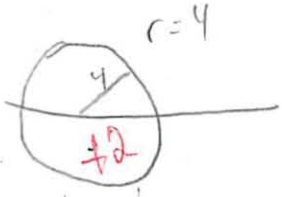
rr

Problem 6 (10: 3,7) Set up a double iterated integral in polar coordinates which gives the volume of the solid lying under the graph of $z = 16 - x^2 - y^2$ and above the xy-plane, as follows.

a) Show the region of integration is the interior of the circle $x^2 + y^2 = 16$.

b) Then set up the integral. *Do not evaluate the integral.*

the region



top view

$$\int_0^{2\pi}\int_0^{4} \underbrace{(16 - x^2 - y^2)}_{r^2}\, r\, dr\, d\theta$$

θ r

+6


Sid view

Problem 7 (10) By changing the order of integration, evaluate $\int_0^1 \int_{\sqrt{x}}^1 \cos(y^3)\, dy\, dx$.

don't Change inside.

$$\int_0^1 \int_0^{y^2} \cos(y^3)\, dx\, dy$$

y x

+3

y = √x

y² = x

$$\int_0^{y^2} \cos(y^3)\, dx$$

$$\cos(y^3)\, x \Big|_0^{y^2}$$

$$\int_0^1 y^2 \cos(y^3)\, dy$$

$u = \cos y^3$

$du = \sin y^3 \cdot 3y^2\, dy$

$$\frac{y^3 \cos(y^4)}{3 \cdot 4} \Big|_0^1$$

integration wrong

+3

$$\frac{1^3 \cos(1^4)}{12} = \frac{\cos(1)}{12}$$

Problem 8 (10) A uniform metal plate has the form of an isosceles right triangle having its two legs both of length 1; find its moment of inertia about one of its legs L, taking the density $\delta = 1$.
(Place the triangle in the first quadrant so the right angle is at the origin, and L lies along the y-axis.)

$\delta = 1$

Moment of inertia$_y$ = $\iint_R x^2 \, \delta \; \text{Area } dA$

Mass = area $\cdot$ density

$\frac{1}{2} \cdot 1 \cdot 1 \cdot 1$

$= \frac{1}{2}$

$\int_0^1 \int_0^{1-x} x^2 \cdot \cancel{\frac{1}{2}} \; dy \, dx$

$\frac{1}{2} \cdot \int_0^{1-x} x^2 \, dy$

$\frac{1}{2} \cdot x^2 y \Big|_0^{1-x}$

$\frac{1}{2} \int_0^1 x^2 (1-x) \, dx$

$\frac{1}{2} \int_0^1 x^2 - x^3 \, dx$

$\frac{1}{2} \left(\frac{x^3}{3} - \frac{x^4}{4} \right) \Big|_0^1$

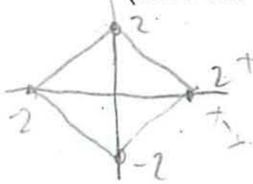
$\frac{1}{2} \left(\frac{1^3}{3} - \frac{1^4}{4} \right)$

$\frac{1}{2} \left(\frac{4}{12} - \frac{3}{12} \right)$

$\frac{1}{24}$

Problem 9. (10) Consider the double integral $\iint_R \sin(x-y)\cos(x+y)\, dy\, dx$, where R is the square xy-region having its vertices at the four points ± 2 on the x- and y- axes.

Change it to a double iterated integral in uv-coordinates, where $u = x - y$ and $v = x + y$. (Give the new limits, integrand, and area element dA, but *do not evaluate*.)



$U = x - y$ $\qquad x = U + y \qquad y = x - U$

$V = x + y$ $\qquad x = V - y \qquad y = V - x$

want x in terms of u and v

~~$x = U + y = V - y$~~

~~$x = x - y + y$~~

~~$x = x$~~

$x = U + x - U$

look graphically

$\int_{-2}^{2} \int_{-2}^{2} \sin(U) \cos(V) \left| \begin{matrix} x_U & y_U \\ x_V & y_V \end{matrix} \right| \, du \, dv$

$\frac{1}{\left| \begin{matrix} U_x & U_y \\ V_x & V_y \end{matrix} \right|} = \frac{1}{\left| \begin{matrix} 1 & -1 \\ 1 & 1 \end{matrix} \right|} = \frac{1}{2}$ ✓ +4

18.02 Exam 2 Solns – Spring 2010

$\boxed{1}$ $w = x^2y - xy^3$

$\vec{\nabla} w = \langle 2xy - y^3,\ x^2 - 3xy^2 \rangle$

$(\vec{\nabla} w)_{(1,1)} = \langle 1, -2 \rangle$

$\left.\frac{dw}{ds}\right|_{P,\hat{u}} = \langle 1,-2 \rangle \cdot \frac{\langle 1, 1 \rangle}{\sqrt{2}} = \frac{-1}{\sqrt{2}}$

$\boxed{2}$ $(w_y)_P \approx \frac{\Delta w}{\Delta y} = \frac{-1}{1/2} = -2$

$(\vec{\nabla} f)_Q$: direction $\hat{u}$

$|\vec{\nabla} f|_Q \approx \left.\frac{\Delta w}{\Delta s}\right|_{\hat{u}} = \frac{1}{1/2} = 2$

so it should have length 2

$\boxed{3}$ $w = x^2 + y^2 + z^2$ (**)

a) $x^2 = -yz - 3z + 8$ (*)

$\therefore \boxed{w = y^2 + z^2 - yz - 3z + 8}$

b) $\frac{\partial w}{\partial y} = 2y - z = 0 : z = 2y$

$\frac{\partial w}{\partial z} = -y + 2z - 3 = 0 : 3y = 3$

solving: $y = 1,\ z = 2$

using (*) $x^2 = -2 - 6 + 8 = 0$

$\therefore P : (0, 1, 2)$

c) $\vec{\nabla} w = \langle 2x, 2y, 2z \rangle = \lambda \vec{\nabla} g$

(from **) $= \lambda \langle 2x, z, y+3 \rangle$

$g(x,y,z) = x^2 + yz + 3z - 8$

$2x = \lambda \cdot 2x$ useless (since $x=0$)

$\begin{cases} 2y = \lambda z & \text{— OK: } \lambda = 1 \\ 2z = \lambda(y+3) & \text{— OK: } \lambda = 1 \end{cases}$

$\boxed{4}$ $\frac{\partial w}{\partial v} = \underbrace{\frac{\partial w}{\partial x}}_{2} \cdot \frac{\partial x}{\partial v} + \underbrace{\frac{\partial w}{\partial y}}_{3} \cdot \frac{\partial y}{\partial v}$ $\quad \begin{cases} x = 4 \\ y = 0 \end{cases}$

$x = 2u - v^2 \quad \frac{\partial x}{\partial v} = -2v$ corresponds to

$y = uv \quad \frac{\partial y}{\partial v} = u \quad \begin{cases} v = 0 \\ u = 2 \end{cases}$

At $u = 2,\ v = 0$, ($u = 0$ says $4 = -v^2$) — no sol'n

$\frac{\partial w}{\partial v} = 2 \cdot 0 + 3 \cdot 2 = \boxed{6}$

$\boxed{5}$ a) $f(x,y,z) = 0$

$f_x \underbrace{\left(\frac{\partial x}{\partial x}\right)_y}_{1} + f_y \underbrace{\left(\frac{\partial y}{\partial x}\right)_y}_{0} + f_z \left(\frac{\partial z}{\partial x}\right)_y = 0$

$\therefore \left(\frac{\partial z}{\partial x}\right)_y = \frac{-f_x}{f_z}$

b) $x - r\cos\theta = 0$

By formula: $\left(\frac{\partial r}{\partial \theta}\right)_x = \frac{-f_\theta}{f_r} = \frac{-r\sin\theta}{-\cos\theta}$

$\boxed{\text{OR}}$ $= r\tan\theta$

Directly:

$r = \frac{x}{\cos\theta} = x\sec\theta \quad \left(\frac{\partial r}{\partial \theta}\right)_x = x\sec\theta\tan\theta = r\tan\theta$

[can give ans. in sin + cos also]

$\boxed{6}$ a) Graph intersects xy-plane where $z = 0$: $16 - x^2 - y^2 = 0$

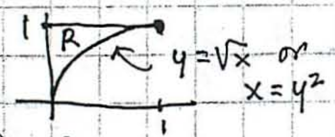
$x^2 + y^2 = 16$ (R) radius 4

inside R, $16 - (x^2 + y^2) > 0$

(outside: < 0)

b) $\int_0^{2\pi} \int_0^4 (16 - r^2) \cdot r\,dr\,d\theta$

$\boxed{7}$ $\int_0^1 \int_{\sqrt{x}}^1 \cos(y^3)\,dy\,dx$

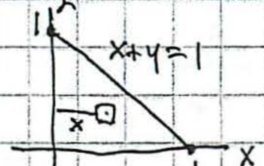
Region 

Changed order:

$$\int_0^1 \int_0^{y^2} \cos(y^3)\,dx\,dy$$

Inner: $\cos(y^3)\,x \Big]_0^{y^2} = y^2\cos(y^3)$

Outer: $\frac{1}{3}\sin(y^3)\Big]_0^1 = \frac{1}{3}\sin(1)$

$\boxed{8}$ 

Moment of inertia about y-axis

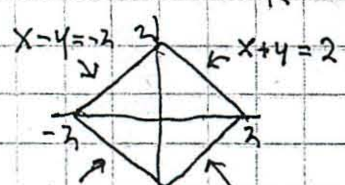
$$= \int_0^1 \int_0^{1-x} x^2\,dy\,dx$$

Inner $= x^2 y\Big]_0^{1-x} = x^2(1-x)$

Outer $= \frac{1}{3}x^3 - \frac{x^4}{4}\Big]_0^1 =$

$$= \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

$\boxed{9}$ $\iint_R \sin(x-y)\cos(x+y)\,dy\,dx$



$u = x - y$

$v = x + y$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2$$

$$\therefore \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{2}$$

$$\int_{-2}^{2}\int_{-2}^{2} \sin u \cos v \cdot \frac{1}{2}\,dv\,du$$

1)

Test 2 Redo

5/15

1. For the function $w = x^2 y - x y^3$
find directional derivative $\left.\frac{dw}{ds}\right|_{P,\hat{u}}$
at pt $(1,1)$ in dir $\hat{u}$ of $\hat{i}+\hat{j}$

So what is it asking?
Deriv in that dir $1\hat{i} + 1\hat{j}$

$\frac{dw}{dt}$ $\frac{ds}{dt}$ ← dir $\langle 1, 1 \rangle$
velocity

$\langle 2xy - y^3, x^2 - 3xy^2 \rangle$

$\langle 2xy - y^3, x^2 - 3xy^2 \rangle$

Started out right

$\nabla w \cdot \hat{u}$ ← did that

Plug points in

$w_x = 2(1)(1) - (1)^3 = 1$

$w_y = 1^2 - 1 \cdot 3(1)^2 = -2$

$(1, -2)$ ← so I just forgot to plug pts in

(2)

$$\frac{dw}{ds} = \langle 1, -2 \rangle \cdot \frac{\langle 1, 1 \rangle}{\sqrt{2}}$$ ← forgot to make unit vector

Now actually solve

$$\left(1 \cdot \frac{1}{\sqrt{2}}\right) + \left(-2 \cdot \frac{1}{\sqrt{2}}\right)$$

$$\frac{1}{\sqrt{2}} - \frac{2}{\sqrt{2}}$$

$$\frac{-1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{-\sqrt{2}}{2}$$

↑ remember from HS!

So forgot to plug pt in and make $\hat{v}$ an unit vector (Note it had ^ unit hat)

2. Estimate w_y

So changes in y dir at given point

$$\frac{\Delta \text{ slope}}{\Delta \text{ length (look at scale}} \quad \cancel{\frac{-1/2}{1}} \quad \frac{1}{\frac{1}{2}} = -\frac{1}{2}$$

* opps thought error moved 1 line in $\frac{1}{2}$ d distance — really stupid mistake

3)

At Q draw $(\nabla f)_q$

So draw gradiant
1 unit long
toward top of "mantah"

I calculated on test

$\langle \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y} \rangle$

$\langle -\frac{1}{\frac{1}{2}}, -\frac{1}{\frac{1}{2}} \rangle$

$\langle 2, -2 \rangle$

3. Find the pt P. on surface $x^2 + yz + 3z - 8 = 0$
$x^2 + yz + 3z = 8$

optimize

minimize $w(y, z) - y^2 + z^2 - yz - 3z + 8$

What do the 2 things mean?

Oh did not solve on test eaiter
did not start

(4)

$W = x^2 + y^2 + z^2$

a) $x^2 = -yz - 3z + 8$

$W = y^2 + z^2 - yz - 3z + 8$

still don't get what they did

They just wrote the line that minimizes

Oh I see distance b/w pts

$x^2 + y^2 + z^2$

(plug in from solving eq for plane

~~idk~~ idk → That just seems weird

b) Find the pt which minimizes $w(y,z)$
and use it to find P

So find min of $x^2 + yz + 3z - 8$

take deriv set = to 0
w/ respect to? both w_y w_z

of the w eq

$W_y = \cancel{z} = 2y - z$

$W_z = \cancel{y + 3} = 2z - y - 3$

⑤

Set = to 0

$\cancel{z = 0}$ $\cancel{y + 3 = 0}$

$\cancel{y = -3}$

$2y - z = 0$ $2z - y - 3 = 0$

now find (y, z) pts

$2y = z$ $\cancel{2z - y} = 3$

$3y = 3$

Solve systems

$\cancel{2(2y - y) = 3}$ $y = 1$

$\cancel{2y = 3}$ $z = 2$

$\cancel{y = 1.5}$

$(0, 1, 2)$

Can check to make sure $= 0$

c) If problem used Lagrange multipliers instead.

α forget Lagrange

Oh it's that complex thing

(6)

$$\vec{\nabla w} = \langle 2x, 2y, 2z \rangle = \lambda \nabla \vec{g}$$

$$= \lambda \langle 2x, z, y+3 \rangle$$

$$g(x,y,z) = x^2 + y^2 + 3z - 8$$

$2x = \lambda 2x$ usless

$2y = \lambda z$ ✓

$2z = \lambda (y+3)$ ✓ $\} \lambda = 1$

Online Lagrange multiplior (Paul's Notes)

- in previas section optimized (found absolute extrema) a function on a region that contaied a boundry
- was fairly long + messy

Want to minimize $f(x,y,z)$

Constraint $g(x,y,z) = k$

1. Solve

$$\boxed{\begin{gathered} \vec{\nabla} f(x,y,z) = \lambda \nabla g(x,y,z) \\ g(x,y,z) = c \end{gathered}}$$

(7)

2. Plug in all solutions (x, y, z) from the 1st step into $f(x,y,z)$ and identify the min/max vaues provided they exist

λ = Lagrange multiplior

$$\langle f_x, f_y, f_z \rangle = \lambda \langle g_x, g_y, g_z \rangle$$
$$= \langle \lambda g_x, \lambda g_y, \lambda g_z \rangle$$

$$\left.\begin{aligned} f_x &= \lambda g_x \\ f_y &= \lambda g_y \\ f_z &= \lambda g_z \end{aligned}\right\}$$ this is prob best thing to remember

$$g(x,y,z) = c \}$$ don't forget constraint!

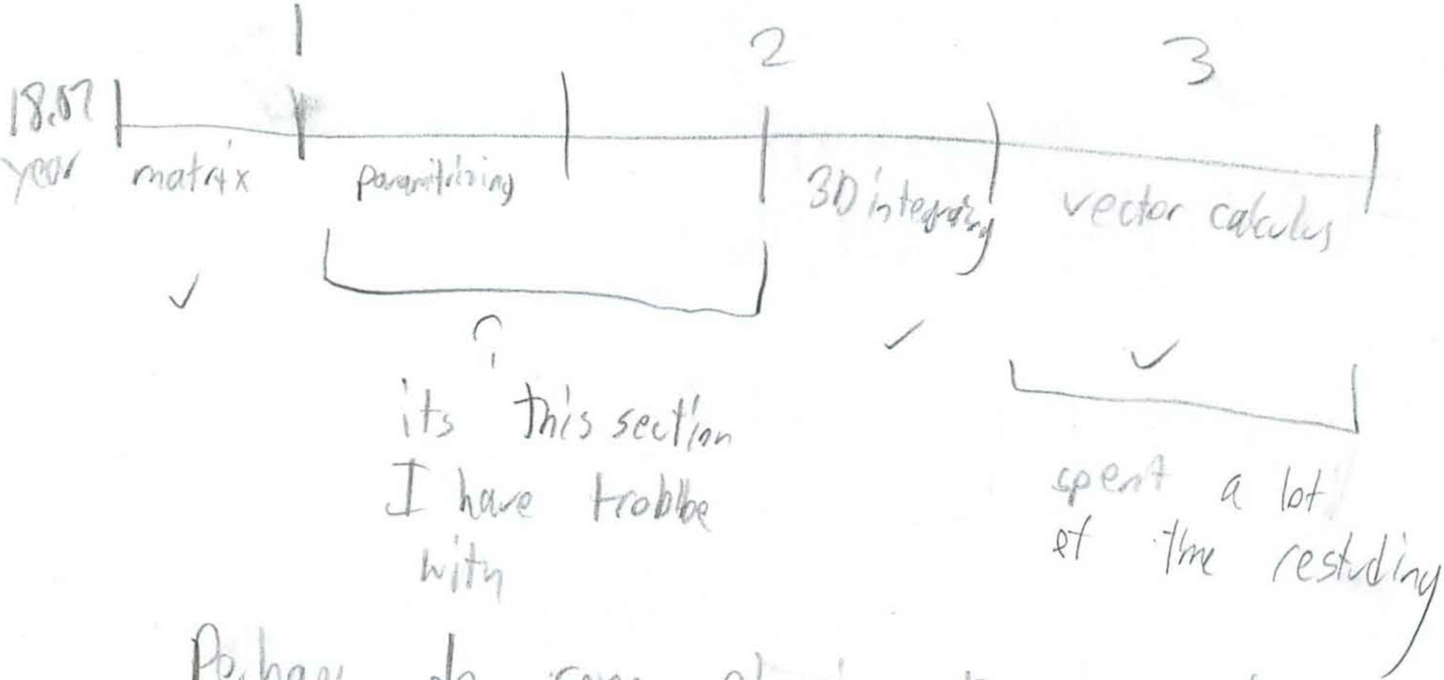
4. Let $w = f(x,y)$ $\quad x = 2u - v^2$

$y = uv$

$\nabla f = \langle 2, 3 \rangle$ at $(4,0)$ point

$\dfrac{\partial w}{\partial v}$ these next two problems lots of trouble w/

(8) Then the ∫ I think I fairly got
integrating
So its just these 2 next problems



Perhaps do some physics then come back
Or work through + study
Since this is current part
~~Then work on physics~~
no can't concentrate on this

study this before Mon
Then ~~some~~ after 8.02 do another whole
practice final - the one from this year

Oliver Office Hrs   5/17

Non indepdent variables

#86 again

not related to a

b) a) $x^3 + yz = 1$

$f(x, y, z)$

$\left(\frac{\partial f}{\partial y}\right)_z = \langle a, b, c \rangle$

free variables (independent)

x = dependent = $f(y, z)$

When differenitate remember x is function of y, z

$$\frac{\partial f}{\partial y} = f_x \left(\frac{\partial x}{\partial y}\right)_z + f_y \underbrace{\left(\frac{\partial y}{\partial y}\right)_z}_{=1} + f_z \left(\frac{\partial z}{\partial y}\right)_z$$

does not depend on y so 0

$$= f_x \left(\frac{\partial x}{\partial y}\right)_z + f_y$$

$$= f_x \left(\frac{\partial x}{\partial y}\right)_z + f_y$$

$$= a \left(\frac{\partial x}{\partial y}\right)_z + b$$ express in terms of a, b

↑↑↑ use equation to get something don't know

$$\left(\frac{\partial(x^3 + yz = 1)}{\partial y}\right)_z$$

z is like a constant

$$3x^2\left(\frac{\partial x}{\partial y}\right)_z + z = 0$$

↑ solve for

$$\left(\frac{\partial x}{\partial y}\right)_z = \frac{-z}{3x}$$

$$= a \frac{-z}{3x} + b$$ ✓

12.

$R = (2x-y)^2 + (5x+y)^2 \leq 3$

$\iint_R 2x - y + 2 \; dx\,dy$

↑ Green's Theorm (stipping to integral)

$= \int_x \int_y$ hard to set limits

So change to u, v

Variables so region simple

$u = 2x - y$

$v = 5x + y$

abs value of the deremant aka (Jacobian)

$\iint_R u + 2 \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$

↑

$u^2 + v^2 \leq 3$

$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$

eaiser way trick

$= \left(\frac{\partial(u,v)}{\partial(x,y)} \right)^{-1}$ ← eaiser to solve

$$= \begin{vmatrix} U_x & U_y \\ V_x & V_y \end{vmatrix}^{-1}$$

$$= \begin{vmatrix} 2 & -1 \\ 5 & 1 \end{vmatrix}^{-1}$$

Compute determinant

$$(2 - -5)^{-1} = (7)^{-1} = \frac{1}{7}$$

back to original function

$$= \iint (U+2)\, \frac{1}{7}\, dU\, dV$$

$$\int_U \int_V$$

Could go to polar (another change in variables)
but min + max of $U^2 + V^2 \leq 3$
so $U \to -\sqrt{3} \to \sqrt{3}$
and $V = -\sqrt{3-U^2} \to \sqrt{3-U^2}$

$$\int_{-\sqrt{3}}^{\sqrt{3}} \int_{-\sqrt{3-U^2}}^{\sqrt{3-U^3}} \frac{(U+2)}{7}\, dV\, dU$$

Not easy to integrate

Use polar coords instead

$$\int_{\theta=0}^{2\pi}\int_{r=0}^{\sqrt{3}}$$

what is v in terms of r, θ

define $\begin{bmatrix} U = r\cos\theta \\ V = r\sin\theta \end{bmatrix}$ 2nd change in variables

$$\int_0^{2\pi}\int_0^{\sqrt{3}} \frac{(r\cos\theta + 2)}{7}\, r\, dr\, d\theta$$

$$\int_0^{\sqrt{3}} \frac{r^2\cos\theta + 2r}{7}\, dr$$

$$\frac{r^3}{3\cdot 7}\cos\theta + \frac{2r^2}{2\cdot 7}\Big|_0^{\sqrt{3}}$$

$$\frac{r^3}{21}\cos\theta + \frac{r^2}{7}\Big|_0^{\sqrt{3}}$$

$$\frac{\sqrt{3}^3}{21}\cos\theta + \frac{(\sqrt{3})^2}{7}$$

? can leave it like that

$(\sqrt{3})^2 \cdot \sqrt{3}$

$$\frac{3\sqrt{3}}{21}\cos\theta + \frac{3}{7} \rightarrow \frac{\sqrt{3}}{7}\cos\theta + \frac{3}{7}$$

$$\int_{-\sqrt{3-v^2}}^{\sqrt{3-v^2}} \frac{v+2}{7}\, dv$$

$$\left.\frac{uv+2v}{7}\right|_{-\sqrt{3-v^2}}^{\sqrt{3-v^2}}$$

$$\frac{v\sqrt{3-v^2}+2\sqrt{3-v^2}}{7} - \frac{v(-\sqrt{3-v^2})+2(-\sqrt{3-v^2})}{7}$$

$$\frac{2v\sqrt{3-v^2}+4\sqrt{3-v^2}}{7}$$

$$\frac{(2v+4)\sqrt{3-v^2}}{7}$$

$$\int_{-\sqrt{3}}^{\sqrt{3}} \frac{2(v+2)\sqrt{3-v^2}}{7}\, dv$$

$$\int_0^{2\pi} \frac{\sqrt{3}}{7}\cos\theta + \frac{3}{7}\,d\theta$$

$$\frac{\sqrt{3}}{7}\sin\theta + \frac{3}{7}\theta \Big|_0^{2\pi}$$

$$\frac{\sqrt{3}}{7}\sin(2\pi) + \frac{3\cdot 2\pi}{7}$$

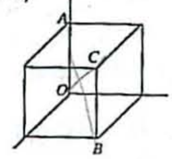
$$0 + \boxed{\frac{6\pi}{7}}$$

18.02 PRACTICE QUESTIONS FOR FINAL – Part A (2 hours) Spring 2010

Definite integral formulas:

$$\int_0^{\frac{\pi}{2}} \sin^n x\,dx = \int_0^{\frac{\pi}{2}} \cos^n x\,dx = \frac{(n-1)!!}{n!!}A_n\,; \quad A_n = \begin{cases} 1, & n \text{ odd integer } \geq 3; \\ \pi/2, & n \text{ even integer} \geq 2; \end{cases} \qquad n!! = n(n-2)(n-4)\cdots.$$

Problem 1. (30: 5 each) The cube shown has edges of unit length.



a) Find the $\mathbf{i}\,\mathbf{j}\,\mathbf{k}$-components of the vectors AB and OC, and use them to find $\cos(\theta)$, where θ = the acute angle between AB and OC.

b) If $O = (0,0,0)$, $A = (1,2,-1)$, $B = (-1,1,1)$ are the vertices of a space triangle, find $OA \times OB$ and the area of the triangle.

c) If $\mathbf{A}$, $\mathbf{B}$, and C are vectors in 3-space, circle those expressions which make sense, put a diagonal line through those which do not (for each: +1 if right, -1 if wrong, 0 if unmarked).

$(\mathbf{A}\cdot\mathbf{B})\mathbf{C}$ $\qquad$ $\mathbf{A}\cdot(\mathbf{B}\cdot\mathbf{C})$ $\qquad$ $(\mathbf{A}\times\mathbf{B})\cdot\mathbf{C}$ $\qquad$ $(\mathbf{A}\times\mathbf{B})\times\mathbf{C}$ $\qquad$ $\mathbf{A}\times(\mathbf{B}\cdot\mathbf{C})$

d) Let $A = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 1 \\ -1 & 1 & 2 \end{pmatrix}$. In the matrix A^{-1}, what is the entry in the lower left corner?

e) For which value of the constant a is the line given parametrically by $x = 1+t, \quad y = 1-t \quad z = 2+at$ parallel to the plane $2x+3y+z=2$?

f) For which value of c is there a non-zero vector $\langle x,y,z\rangle$ perpendicular to each of the vectors $\langle 1,3,-1\rangle$, $\langle 2,c,1\rangle$, $\langle 1,1,2\rangle$?

Problem 2. (20) $OP = \mathbf{r} = \langle 4\cos t, -3\cos t, 5\sin t\rangle$ is the position vector for a point P moving in 3-space. (In each of the questions, show work or indicate reasoning.)

a) (10: 4,3,3) Find its velocity vector $\mathbf{v}$, its speed $\dfrac{ds}{dt}$, and its unit tangent vector $\mathbf{T}$.

b) (5) Find its curvature $\kappa = \left|\dfrac{d\mathbf{T}}{ds}\right|$.

c) (5) Show that P moves in a vertical plane containing the origin.

Problem 3. (20: 8,2,5,5) For the function $w = y(1+x) + \sin(xy)$,

a) Write an approximate formula showing how Δw depends on Δx and Δy, at the point $(0,1)$.

b) At the point $(0,1)$, is w more sensitive to x or y? (give reason)

c) Find the directional derivative $\left.\dfrac{dw}{ds}\right|_{\mathbf{u}}$ at the point $(0,1)$ in the direction of the vector $3\mathbf{i}-4\mathbf{j}$.

d) Starting at the point $(0,1)$, what is the minimal distance you could travel to increase the value of w by .2? (show work or indicate reasoning)

Problem 4. (15) Some level curves for a function $w = f(x,y)$ are shown, with a unit distance $\mathbf{u}$ in the xy-plane.

a) At the point P, draw in the gradient vector $(\text{grad} f)_P$ (Use $\mathbf{u}$ to estimate its length.)

b) Estimate the value of $\left(\dfrac{\partial w}{\partial x}\right)$ at Q.

c) Mark a point R where $f(R) = 3$ and $\dfrac{\partial w}{\partial y} = 0$.

Problem 5. (25: 5, 10, 5, 5) A wooden rectangular drawer with a capacity of one cubic foot is to be constructed. The wood costs \$1/sq.ft. for the bottom and the back, \$2/sq.ft. for the two sides, and \$3/sq.ft. for the front; there is no top. Let x be the end width, y the side width, and z the height, and C the total cost. What values for x, y, z minimize the total cost?

y

a) Show this leads to minimizing $C = xy + \dfrac{2}{x} + \dfrac{4}{y}$.

b) Find the minimizing values for x, y, z.

c) Use the second derivative test to show it is actually a minimum.

d) Give one of the equations for the Lagrange multiplier method, and use it to determine the value of the multiplier λ corresponding to the minimum.

Problem 6. (10) Where does the tangent plane to the surface $x^2+2y^2+3z^2 = 12$ at the point $(1, 2, -1)$ intersect the y-axis?

Problem 7. (15: 7,8) Let $w = w(x, y)$, and let r, θ be the usual polar coordinates.

a) Express $\dfrac{\partial w}{\partial r}$ and $\dfrac{\partial w}{\partial \theta}$ in terms of w_x, w_y, r and θ.

b) If the gradient ∇w at the point $(x, y) = (1, 1)$ has the value $2\mathbf{i} + 3\mathbf{j}$, find the value of $\dfrac{\partial w}{\partial r}$ and $\dfrac{\partial w}{\partial \theta}$ at this point.

Problem 8. (15) Let $w = xy + xz + yz$, where the variables x, y, z are not independent, but constrained by a relation $y = f(x, z)$.

Express $\left(\dfrac{\partial w}{\partial y}\right)_z$ in terms of x, y, z and the formal partial derivatives f_x and f_z. You can use either method: the chain rule or differentials.

Problem 9. (10) Find the volume of the region in space lying under the graph of $z = x^2 + y^2$ and over the triangle in the xy-plane having vertices at $(0, 0)$, $(1, 0)$, $(0, 1)$.

Problem 10. (15: 5,5,5) Let R be the upper half of the circular disc of radius a centered at the origin. Express the average distance of a point in R from the x-axis by an iterated integral in

(a) rectangular coordinates; and (b) polar coordinates;

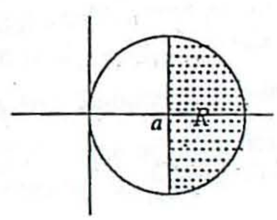
(c) evaluate the integral in either (a) or (b).

Problem 11. (10) Evaluate $\displaystyle\int_0^1 \int_x^1 \frac{dy\,dx}{\sqrt{1+y^2}}$ by changing the order of integration.

Problem 12. (15: 7,8) *Using polar coordinates for both parts,*

a) set up an iterated integral giving the moment of inertia about the y-axis of the pictured shaded semicircular region R of radius a. Assume the density $\delta = 1$. *Do not evaluate.*

b) Calculate the moment of inertia about the y-axis of the entire circular disc ($\delta = 1$).



5/17

Pratice Test

Let's see how much I remember

a) $\hat{i}\,\hat{j}\,\hat{k}$ vector componants

Find $\cos(\theta)$

↑ angle b/w them

$A = (0,0,1)$ $O = (0,0,0)$

$B = (1,1,0)$

$C = (1,1,1)$

$AB = \langle 1, 1, -1 \rangle$ ✓

$OC = \langle 1, 1, 1 \rangle$ ✓

how find angle b/w vector again?

$$\cos\theta = \frac{\vec{AB}}{|AB|} \cdot \frac{\vec{OC}}{|OC|}$$

$$\frac{\langle 1, 1, -1 \rangle}{\sqrt{1^2+1^2+1^2}} \cdot \frac{\langle 1, 1, 1 \rangle}{\sqrt{3}}$$

$$\frac{(1\cdot1)+(1\cdot1)+(1\cdot-1)}{\sqrt{3}} = \frac{1+1-1}{\sqrt{3}} = \frac{1}{\sqrt{3}}$$ ✓

②

b) If $O = (0,0,0)$

$A = (1, 2, -1)$

$B = (-1, 1, 1)$ space triangle

$\vec{OA} \times \vec{OB} = ?$ and area

$\vec{OA} = \langle 1, 2, -1 \rangle$

$\vec{OB} = \langle -1, 1, 1 \rangle$

~~cross~~ $\cancel{\langle -1, 2, -1 \rangle}$ $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -1 \\ -1 & 1 & 1 \end{vmatrix} = \langle 3, 0, 3 \rangle$

area $= \cancel{|base| \cdot |height| \cdot \frac{1}{2}}$

$\frac{1}{2} |\vec{A} \times \vec{B}|$

$\cancel{\sqrt{1^2 + 2^2 + 1^2} \cdot \sqrt{-1^2 + 1^2 + 1^2} \cdot \frac{1}{2}}$

$\cancel{\sqrt{6} \cdot \sqrt{3} \cdot \frac{1}{2}}$

$\frac{1}{2} \sqrt{3} \sqrt{6}$

$\frac{3\sqrt{2}}{2}$

③

Cross product defined wrong

$(1\cdot 1 - -2)\,\hat{\imath} + (1-1)\,\hat{\jmath} + (1 - -2)\,\hat{k}$

$3\hat{\imath} + 0\hat{\jmath} + 3\hat{k}$

$\langle 3, 0, 3 \rangle$

just remember the rules!

area of triangle I forgot too

$\frac{1}{2}|\vec{A} \times \vec{B}|$

$\frac{1}{2}\sqrt{3^2 + 0^2 + 3^2}$

$\frac{1}{2}\sqrt{18}$

$\frac{1}{2}\sqrt{2}\cdot\sqrt{9} = \frac{3}{2}\sqrt{2}$ ✓

c) what make sense
- idk in what manner?

$(A \cdot B) \cdot C$

$(A \times B) \cdot C$

$(A \times B) \times C$

Not

$A \cdot (B \cdot C)$ ← go in order?

$A \times (B \cdot C)$

I don't get why it is wrong though?

(4)

d.) $A = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 1 \\ -1 & 1 & 2 \end{pmatrix}$ $A^{-1} = ?$

So lets see if I remember

$\frac{1}{\det} \left[\text{cofactor flipped signed} \right]$

$\det = 1(2-1) - 0(\ ) + (-1)(2+1)$

$1 - 3 = -2$

So for that corner

$\begin{bmatrix} & & \\ & & \\ & & x \end{bmatrix}$ need $\begin{bmatrix} & & x \\ & & \\ & & \end{bmatrix}$

$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$ → So no sign flip

Cofactor $(2 - -1)$ 3

$\frac{3}{-2}$ ✓

⑤

e) For what value ~~of~~ of the constant a is line given parametrically

$x = 1+t$

$y = 1-t$ parallel $2x+3y+z=2$

$z = 2+at$

$2(1+t)+3(1-t)+(2+at)=2$

$2+2t+3-3t+2+at=2$

-5 -5

$-1t+at=-3$

$(a-1)t=-3$

$t = \frac{-3}{a-1}$ 2 variables

i need to find t, or on wrong path?

* eq of plane is normal to it

$\perp \hat{n}$ they want parallel to plane

⑥

~~[illegible]~~ When something is normal to something it $\overset{\text{dot product}}{=} 0$

$\langle 2, 3, 1 \rangle \cdot \cancel{\langle 1+t, 1-t, 2+at \rangle = 0}$

$\langle 1, -1, a \rangle$

$\cancel{2(1+t) + 3(1-t) + 2 + at = 0}$

division of line

$(P_1 + \overrightarrow{P_1P_2})$

$2 \cdot 1 + 3 \cdot -1 + 1 \cdot a = 0$

$2 - 3 + a = 0$

$+1 \qquad\qquad +1$

$\boxed{a = 1}$ ✓

f) For what value of c is there a non 0 vector $\perp$ to each of vectors

$\langle 1, 3, -1 \rangle \quad \langle 2, c, 1 \rangle \quad \langle 1, 1, 2 \rangle$

perp means dot product = 0

$\langle 1, 3, -1 \rangle \cdot \langle 2, c, 1 \rangle$

$2 + 3c - 1 = 0$

$3c = -1$

$c = -\frac{1}{3}$

⑦

$\langle 2, c, 1 \rangle \cdot \langle 1, 1, 2 \rangle$

$2 + c + 2 = 0$

$c = 4$

but not $\perp$ to both; whats wrong?

matrix idea was right

$$\begin{vmatrix} 1 & 3 & -1 \\ 2 & c & 1 \\ 1 & 1 & 2 \end{vmatrix} \quad \det = 0$$

$1(2c-1) - 3(4-1) - 1(2-c) = 0$

$2c - 1 - 9 \cancel{-} 2 + c = 0$ ← copy error

$3c - \cancel{8}12 = 0$

$3c = \cancel{8}12$

$\boxed{c = \cancel{\frac{8}{3}}}$ 4

matrix determinate = 0 means what again?

- not invertable
- "singular"
- has unique solutions

I guess if $\perp$ to each that is not true but not visualizing this

(8)

2. $\vec{OP} = \vec{r} = \langle 4\cos t, -3\cos t, 5\sin t \rangle$ is a position vector

a) $\vec{v} = \langle -4\sin t, +3\sin t, 5\cos t \rangle$ ✓ remember this :)

$|v| = \frac{ds}{dt} = \sqrt{(4\sin t)^2 + (3\sin t)^2 + (5\cos t)^2}$

$\sqrt{16\sin^2 t + 9\sin^2 t + 25\cos^2 t}$ ✓

$\sqrt{25(1)}$

5

$\hat{T} = \frac{\vec{v}}{|v|} = \left\langle \frac{-4\sin t}{5}, \frac{3\sin t}{5}, \cos t \right\rangle$ ✓

they write as $\frac{1}{5}\vec{v}$

I think that is cheap

b) find $k = \frac{d\vec{T}}{ds}$

What was the trick for this again?

$\frac{dT/dt}{ds/dt}$ ← had a feeling

but what is that again?

(9)

$$\frac{dT}{dt} = \langle -\frac{4}{5}\cos t, \frac{3}{5}\cos t, -\sin t \rangle$$

$$\frac{ds}{dt} = 5$$

$$\langle -\frac{4}{25}\cos t, \frac{3}{25}\cos t, -\frac{1}{5}\sin t \rangle$$

$$= \frac{\frac{1}{5} dv/dt}{5}$$

$$= \frac{1}{25} |\langle -4\cos t, 3\cos t, -5\sin t \rangle|$$

which is what I had

but $\vec{T} = \left|\frac{dT}{dt}\right|$

magnitude of

$$= \frac{1}{25} = \sqrt{(-4\cos t)^2 + (3\cos t)^2 + (5\sin t)^2}$$

$$16\cos^2 t + 9\cos^2 t + 25\sin^2 t$$

$$\sqrt{25(1)}$$

$$5$$

$$\frac{5}{25} = \frac{1}{5}$$

(10)

c) Show that P moves in a vertical plane containing the origin

Using a plane q4

$P + \overrightarrow{P_2P_1}$

$(0,0,0) + \langle 4\cos t, -3\cos t, 5\sin t \rangle$

$4\cos t\, x - 3\cos t\, y + 5\sin t\, z = 0$

vertical plane

$3x + 4y = 0$

since $3(4\cos t) + 4(-3\cos t) + 0(5\sin t) = 0$ for all t

Did I have it?

- I just needed to solve for t?

But where did they get $3x + 4y = 0$

(should have asked about)

- should review lecture notes

(11)

Course Notes

~~dir~~ $\text{dir} = \dfrac{\langle a_1, a_2 \rangle}{\sqrt{a_1^2 + a_2^2}}$ "unit dir"

dot product

$$\cos\theta = \frac{\vec{A}\ \vec{B}}{|A|\,|B|}$$

$$\vec{A} \cdot \vec{B} = 0$$

either $A \perp B$ or $\vec{A} = 0$, $\vec{B} = 0$

$$\vec{A} \cdot \vec{A} = |\vec{A}|^2$$

Cross product

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \qquad \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Item in plane $\overrightarrow{P_1 P_2} \cdot N = 0$

$$\overrightarrow{P_1 P} \cdot (P_1 P_2 \times P_1 P_3) = 0$$

(Don't forget how to solve from last pratice test

$$\begin{matrix} x \\ y \\ z \end{matrix} = \begin{bmatrix} A^{-1} & & \\ & & \\ & & \end{bmatrix} \begin{bmatrix} B \\ \\ \\ \end{bmatrix}$$

known vector

(12)

It ~~...~~ starts $A \cdot X = D$

$a_1 x + a_2 y + a_3 z = d_1$

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ & & \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = d_1$$

So it ends $X = A' \underset{\text{or } d}{B}$

3)

$w = y(1+x) + \sin(xy)$

a) Write an approximat formula showing how Δw depends on Δx Δy at $(0,1)$

Need to revise this topic, but

what was it

~~Δw~~ $w = w(0,1) + \Delta x \, \Delta y$

$\left.\begin{array}{l} w_x = \\ w_y = \end{array}\right]$ need to find first

~~$w=$~~ $w = y + xy + \sin(xy)$

$w_x = y + \cos(xy) \cdot y$

$w_y = 1 + x + \cos(xy) \cdot x$

at each pt

$w_x(0,1) = 1 + \cos(0 \cdot 1) \cdot 1$

2

$w_y(0,1) = 1 + 0 + 0$

1

(14)

Look up what is approx formula

Lecture 12

$$\boxed{W - W_0 = \left(\frac{\partial f}{\partial x}\right)_0 (x - x_0) + \left(\frac{\partial f}{\partial y}\right)_0 (y - y_0)}$$

$$\boxed{\Delta w = f_x \, \Delta x + f_y \, \Delta y}$$

when that $= 0$

~~both~~ x-horizontal

So $\Delta w \approx 2 \Delta x + \Delta y$ ✓

just want you to show relation

b) At pt $(0,1)$ is w more sensitive to Δx or Δy

Δx because has larger coefficent ✓ bingo

c) Find the directional derivitive

$\boxed{\left(\frac{dw}{ds}\right)_u}$ at $(0,1)$ $3\hat{i} - 4\hat{j}$

So what is this again?

$\boxed{= \nabla \vec{w} \times \hat{u}} = \langle 2,1 \rangle \cdot \frac{\langle 3,-4 \rangle}{5} = \frac{2}{5}$

don't forget unit vector

(15)

d) Start out at (0, 1) what is min distance you could travel to increas value of w by .2 ?

Could possibly do graphically, but not math

$$\frac{\Delta w}{\Delta s} = \left(\frac{dw}{ds}\right)_{\hat{u}} = |\langle 2, 1 \rangle| = \sqrt{5}$$

go in dir $\hat{u}$ = dir $\vec{\nabla w}$

to get most rapid change

$$\Delta s = \frac{\Delta w}{\sqrt{5}}$$

$$= \frac{.2}{3.2} \approx .1 \text{ or } \frac{.2}{\sqrt{5}}$$

Don't get this at all

Review class notes

$$\frac{dw}{dt} = \vec{\nabla} w \cdot \frac{d\vec{r}}{dt} \qquad r = \langle x, y \rangle$$

besides this did not find much on this

(6)

Is this one of those keep 1 variable constant things?
Oh well as I said ~~a~~ ~~day~~ Friday its the $\frac{2}{5}$
section of the class I am particular bad at.

4. level curve

$\frac{\Delta w}{\Delta s} = \frac{1}{1/2} = 2$ must calculate magnitude

a) grad vector uphill one unit length

b) $\left(\frac{\partial w}{\partial x}\right)$ at q so $\frac{\text{\# lines crossed}}{\text{1 unit vector}} = \frac{2}{1}$

since down (including)

c) $f(R) = 3$ means on line

$\frac{\partial w}{\partial y} = 0$ means no change in y dir

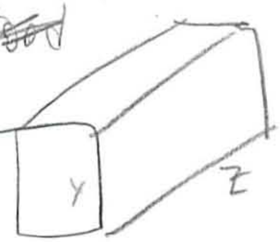
more like only change in y oh well

(17)

5. Optimization - can I remember?

$V = 1 ft^3$

~~Wood~~



$\$ = xz + xy + 2 \cdot 2yz + 3xy$

$\$ = xz + 4xy + 4yz$

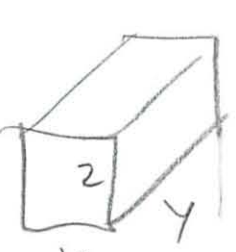
what values minimize cost?

$1 = xyz$

a) Show this leads to minimizing $C = xy + \frac{2}{x} + \frac{4}{y}$

what is this?
How did they find

* They used different variables



$\$ = xy + xz + 4yz + 3xz$

$\$ = xy + 4yz + 4xz$

Guess they pull constraint in

~~AA~~ $1 = xyz$

$z = \frac{1}{xy}$

$C = xy + 4yz + 4xz$

$= xy + \frac{4y}{xy} + \frac{4x}{xy}$

$= xy + \frac{4}{x} + \frac{4}{y}$

close except of 2

but ~~$~~ 2 for both sides

~~$~~ $2 \cdot 2 \cdot yz$ is correct

They get 4 too

must be mistake in problem

mistakes in problem very dangerous since break my confidence + knowledge of problem

Find minimizing values
- just skipped over in notes

$C_x = 0$] for critical pts
$C_y = 0$

$C = xy + 4x^{-1} + 4y^{-1}$

$C_x = y + 4 \cdot -1 x^{-2} = y - \frac{4}{x^2} = 0$

$C_y = x + 4 \cdot -1 y^{-2} = x - \frac{4}{y^2} = 0$

Solve for $x = y$

$x^3 = 4$

$x = \sqrt[3]{4}$

$y - \sqrt[3]{4}$

$z = \frac{1}{4^{3/2}}$

c) Check to see if min

$f_{xx} = -4 \cdot -2x^{-3} = \frac{8}{x^3}$

$f_{xy} = 1$

$f_{yy} = -4 \cdot -2y^{-3} = \frac{8}{y^3}$

$f_{xx} f_{yy} - (f_{xy})^2$

$\frac{8}{x^3} \frac{8}{y^3} - 1$

)

Is $\oplus$, so what
-memorize rule)

look at f_{xx} which is $\oplus$ so minimum

really need to memorize rules
but did ok w/ this question

d) Lagrange multiplier
- Let me review notes first

$$\vec{\nabla} w = \lambda \vec{\nabla} g$$

$$w_x = \lambda g_x$$

$$w_y = \lambda g_y$$

So for box

take deriv of cost ea every way

C_x
C_y
C_z

$$C_x = y + 4z$$

$$C_y = x + 4y$$

$$C_z = 4y + 4x$$

$g\circlearrowright$

Since $xyz-1=0$

Set each equal to right side

$y+4z=\lambda yz$ $\qquad g=0$

$x+4z=\lambda xz$

$4y+4x=\lambda xy$

Now get λ on one side

$\frac{y}{yz}+\frac{4z}{yz}=\lambda$ $\qquad g=0$

$\frac{x}{xz}+\frac{4z}{xz}=\lambda$

$\frac{4y}{xy}+\frac{4x}{xy}=\lambda$

Reduce

$\frac{1}{z}+\frac{4}{y}=\lambda$ $\qquad g=0$

$\frac{1}{z}+\frac{4}{z}=\lambda$

$\frac{4}{x}+\frac{4}{y}=\lambda$

(22) Set them all = to each other

$$\frac{1}{z}+\frac{4}{y}=\frac{1}{z}+\frac{4}{z}=\frac{4}{x}+\frac{4}{y}$$

Now it seems they take first part I don't know what they are doing ??

? or one of each variable

$$\frac{4}{y}=\frac{4}{z}=\frac{4}{x}$$

Smush together

$$\frac{64}{xyz}=?$$ ← something

explination awful

look at recitation

here they do a much ~~[illegible]~~ different job

they solved for $x = \_\lambda$

$y = \_\lambda$

$z = \_\lambda$

but how to do this on this problem?

look at answer sheet

~~$C_x = \lambda t$~~

$C_x = \lambda g_x \rightarrow y + 4z = \lambda yz$

$C_y = \lambda g_y \rightarrow x + 4z = \lambda xz$

$C_z = \lambda g_z \rightarrow 4(x+y) = \lambda xy$

had that

Ok so now what?

$x = y = \sqrt{4}$

$8 \cdot 4^{1/3} = \lambda 4^{2/3}$

$\lambda = 2 \cdot 4^{2/3}$

How in all world did they get this is a complete mess

Watching ~~a~~ video from youtube

form $f(x, y, z)$

$\underbrace{g(x, y, z)}$ $= k$ constraint

form $F(x, y, z, \lambda)$

$= f(x, y, z) - \lambda (g(x, y, z) - k)$

and then solve for

$F_x = 0$

$F_y = 0$

$F_z = 0$

$F_\lambda = 0$

sub solutions into original

(24)

$C(x,y) = 6x^2 + 12y^2$

$x + y = 90$

minimize cost

$6x^2 + 12y^2 - \lambda(x + y - 90) = F(x,y,\lambda)$

$6x^2 + 12y^2 - \lambda x - \lambda y + 90\lambda$

$F_x = 12x - \lambda$

$F_y = 24y - \lambda$

$F_\lambda = -x - y + 90$

Set all $= 0$ and solve for x, y

$x = \frac{\lambda}{12}$

$y = \frac{\lambda}{24}$

Plug into F_λ

$-\left[\frac{\lambda}{12}\right] - \left[\frac{\lambda}{24}\right] + 90 = 0$

now solve For λ

~~the~~ Then solve for x, y

These videos always make things much clearer✓

25

Ok back to my problem

$C_x - \lambda(g_x) = 0$ is what he had

We did $C_x = \lambda(g_x)$

Same thing

but what then

- same as answer key
- solve for the letters

$$\frac{1}{z} + \frac{4}{y} = \lambda$$

$$\frac{1}{z} + \frac{4}{z} = \lambda$$

$$\frac{4}{x} + \frac{4}{y} = \lambda$$

$$\frac{1}{z} = \lambda - \frac{4}{y}$$

$$z = \frac{1}{\lambda} - \frac{y}{4}$$

$$\cancel{z = \frac{1}{\lambda} - \frac{z}{4}}$$

$$\frac{5}{z} = \lambda$$

$$\boxed{z = \frac{5}{\lambda}}$$

(26)

$\frac{5}{\lambda} = \frac{5}{\lambda} = \frac{1}{\lambda} - \frac{y}{4}$ or $\frac{y}{4} = \frac{1}{\lambda} - \frac{5}{\lambda}$

$\boxed{5 = 1 - \frac{y\lambda}{4}}$

$\frac{y}{4} = \frac{-4}{\lambda}$

$y = \frac{-16}{\lambda}$

$\lambda = \frac{4}{x} + \frac{4}{y}$

$\frac{4}{x} = \lambda - \frac{4}{y}$

$\frac{x}{4} = \frac{1}{\lambda} - \frac{y}{4}$

$\boxed{x = \frac{4}{\lambda} - y}$

~~Shi~~ i have 4 eq

$z = \frac{5}{\lambda}$ $y = \frac{-16}{\lambda}$ $x = \frac{4}{\lambda} - y$

$x = \frac{4}{\lambda} - \frac{16}{\lambda}$

$x = \frac{-12}{\lambda}$

$\lambda = \frac{5}{z}$ $\lambda = \frac{-16}{y}$ $\lambda = \frac{-12}{x}$

$\frac{5}{z} = \frac{-16}{y} = \frac{-12}{x}$

(27)

Then constraint again? (from lecture)

$$\frac{5}{z} = \frac{-16}{y} = \frac{-12}{x}$$

Completly confused

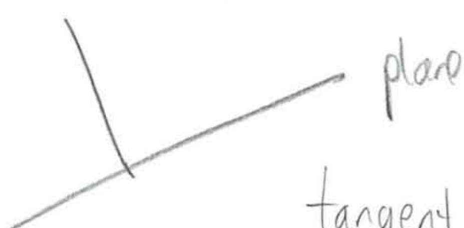
Hope this problem does not come on exam

- spent 2 pages trying to figure out

6. Where does tangent plane to surface $x^2+2y^2+3z^2 = 12$ ~~m~~ at pt $(1,2,-1)$ intersect y axis

(~~I don't feel like doing this~~)

Visualize



tangent $\langle 1, 2, 3 \rangle$

So confused on plane eq

$$\vec{\nabla w}_{(1,2,-1)} = \langle 2x, 4y, 6z \rangle_{(1,2,-1)}$$

why?

$$= \langle 2, 8, -6 \rangle$$

reduce

$$\langle 1, 4, -3 \rangle$$

(28)

~~a~~ $x + 4y - 3z = 12$

intersects y-axis where $x = z = 0$

$\boxed{y = 3}$

So confused on plane stuff - can't visualize at all

7. $w = w(x, y)$ $\quad r, \theta$

a) Express $\frac{\partial w}{\partial r}$ $\frac{\partial w}{\partial \theta}$

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial r}$$

$$w_x \, r \sin\theta + w_y \, r \cos\theta$$

~~b)~~ Can't even get that right!

$$w_r = w_x \cos\theta + w_y \sin\theta$$

what they were asking for

~~w_θ~~

$$w_\theta = -w_x \, r \sin\theta + w_y \, r \cos\theta$$

tangent

(29)

b) If the gradient ∇w at $(x,y) = (1,1)$ has value $\underset{w_x}{2\hat{\imath}} + \underset{w_y}{3\hat{\jmath}}$ find $\frac{\partial w}{\partial r}$ $\frac{\partial w}{\partial \theta}$

$$\frac{\partial w}{\partial r} = w_x \cos\theta + w_y \sin\theta$$

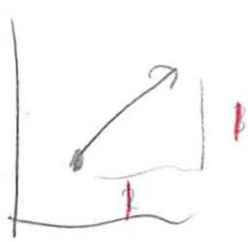
$$= 2\cos\theta + 3\sin\theta$$

$$w_\theta = -2r\sin\theta + 3r\cos\theta$$

must I do something about r, θ?

Yes know that $r = \sqrt{2}$ $\theta = \frac{\pi}{4}$

how



$\sqrt{2^2+3^2} = \sqrt{4+9} = \cancel{4}\sqrt{3}$

$2\sqrt{3}$

$\tan\theta = \frac{3}{2} =$

$\sqrt{1^2+1^2} = \sqrt{2}$

$\tan\theta^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{2}$ ~~that is not right but what is?~~

do in original variables

$$w_r = 2 \cdot \frac{\sqrt{3}}{2} + 3\frac{\sqrt{2}}{2} = \frac{5\sqrt{2}}{2}$$

$$w_\theta = -2 \cdot 1 + 3 \cdot 1 = 1$$

0)

Let $w = xy + xz + yz$

riables Constrained

$y = f(x, z)$

(Lets see how much I remember)

Express $\left(\frac{\partial w}{\partial y}\right)_z$ in terms of $f_x\ f_z$

$f_x \left(\frac{\partial x}{\partial y}\right)_z + f_y \left(\frac{\partial y}{\partial y}\right)_z + f_z \left(\frac{\partial z}{\partial y}\right)_z$

$\uparrow 0\ ?$ $\quad \uparrow 1$ $\quad \uparrow 0$

Constant

$f_y + f_z \left(\frac{\partial z}{\partial y}\right)_z$

$f_y = x + z$

$f_z = y' + x + z$

$\left(\frac{\partial z}{\partial y}\right)_z = -x - z$

$x + z + x + z - x - z$

$x + z$

(31)

And they used both w + z

- previous one left a + b and did not work w/ them

use right letters tho

$$\left(\frac{\partial w}{\partial y}\right)_z = w_x \left(\frac{\partial x}{\partial y}\right)_z + w_y$$

$$\boxed{1 = f_x \left(\frac{\partial x}{\partial y}\right)_z + f_z(0)}$$

this does not involve w whatsoever

$$= w_x \frac{1}{f_x} + w_y$$

$$= \frac{(y+z)}{f_x} + (x+z)$$

---

9. Find volume of region in space

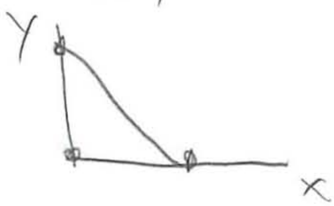
$z = x^2 + y^2$ and over triangle (0,0) (1,0) (0,1)

Finally something I understand better!

(32)

$$\int_0^1\int_0^{1-x}\int_0^{x^2+y^2} dz\ dy\ dx$$

x y z



$y = 1-x$

they do $\int_0^1\int_0^{1-x}\left(x^2+y^2\right)\ dy\,dx$

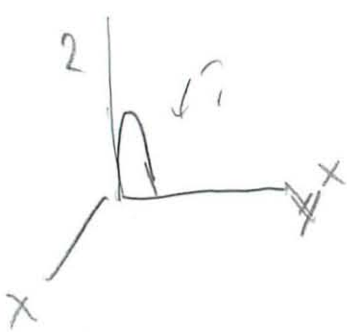
and then solve (which is easy)

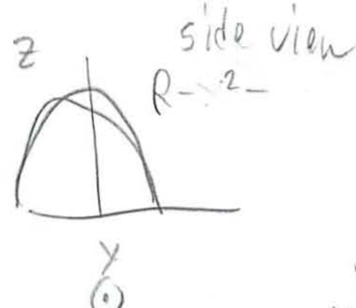
but why the part in middle

clue i written differently

10. Let R be upper half of circular disk

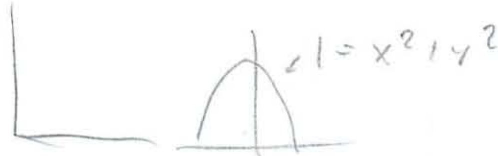
radius a centered at origin





$$\int_{-R}^{R}\int_0^{1-x^2+z^2} dz\ dx$$

x



(33)

I think I should just study this stuff really well and forget that quarter of the test I don't know how to do.

$$\int_{-a}^{a}\int_{0}^{\sqrt{a^2-x^2}} \frac{y\, dA}{\frac{\pi a^2}{2}}$$

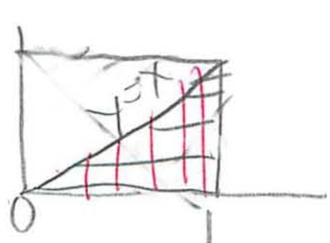
r wtf

$$\int_{-\pi/2}^{\pi/2}\int_{0}^{R} r\,dr\,d\theta$$

$$\frac{1}{\frac{\pi a^2}{2}} \int_{0}^{\pi}\int_{0}^{a} r\sin\theta\; r\,dr\,d\theta$$

11. Eval by changing order of $\int$

$$\int_0^1\int_x^1 \frac{dy\,dx}{\sqrt{1+y^2}}$$



$$\int_0^1\int_{\cancel{0}}^{y} \frac{dx\,dy}{\sqrt{1+y^2}}$$

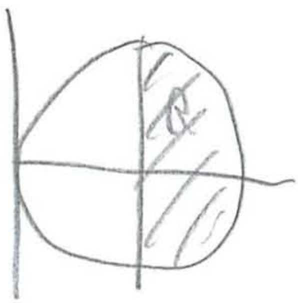
duh go slow and think it over

34

12. Use polar coords

a) Set up iterated $\int$ of moment of inertia

density $= \sigma = 1$



I remember this is hard

$$\int_{-\pi/4}^{\pi/4} \int_{a\sec\theta}^{2a\cos\theta} r^2\cos^2\theta \, r\,dr\,d\theta$$

b) Moment of inertia about y axis of whole disc

$$2\int_0^{\pi/2}\int_0^{2a\cos\theta} r^3\cos^2\theta \, dr\,d\theta$$

calculate

SOLUTIONS TO PART A

1 a) $\vec{AB} = \langle 1,1,-1\rangle \quad \vec{OC} = \langle 1,1,1\rangle$

$\cos(\theta) = \dfrac{\langle 1,1,-1\rangle}{\sqrt3}\cdot\dfrac{\langle 1,1,1\rangle}{\sqrt3} = \frac13$

b) $\begin{vmatrix} \hat\imath & \hat\jmath & \hat k \\ 1 & 2 & -1 \\ -1 & 1 & 1\end{vmatrix} = \langle 3,0,3\rangle$

area $= \frac12 |\vec A\times\vec B| = \dfrac{3\sqrt2}{2}$

c) ○ / ○ ○ /

d) cofactor of $(\cdot\cdot\cdot)$ $= \begin{vmatrix} 2 & 1 \\ -1 & 1\end{vmatrix} = 3$

$\det A = -2 \qquad$ Ans: $-\frac32$

e) $\langle 1,-1,a\rangle \cdot \langle 2,3,1\rangle = 0$

dir. of line $\quad$ normal to plane

$\therefore 2-3+a=0, \quad a=1$

f) $\begin{vmatrix} 1 & 3 & -1 \\ 2 & c & 1 \\ 1 & 1 & 2\end{vmatrix} = 3c-12=0$

$\therefore c=4$

2. $\vec V = \langle -4\sin t, 3\sin t, 5\cos t\rangle$

a) $\frac{ds}{dt} = |\vec v| = \sqrt{25\sin^2 t + 25\cos^2 t} = 5$

$\vec T = \frac15 \vec v$

b) $\kappa = \left|\frac{d\vec T}{ds}\right| = \dfrac{d\vec T/dt}{ds/dt} = \dfrac{\frac15 d\vec v/dt}{5}$

$= \frac{1}{25}\left|\langle -4\cos t, 3\cos t, -5\sin t\rangle\right|$

$= \frac{1}{25}\cdot 5 = \frac15$

c) $3x+4y=0$ (vertical plane),

since $3(4\cos t)+4(-3\cos t)+0\cdot(5\sin t) = 0$ for all t

3. $w = y(1+x)+\sin(xy)$

(a) $w_x = y + y\cos(xy)$; $= 2$ at $(0,1)$

$w_y = 1+x+x\cos(xy)$, $=1$ at $(0,1)$

$\therefore \Delta w \approx 2\Delta x + \Delta y$

b) to x; since coeff. of Δx is bigger.

c) $\left.\frac{dw}{ds}\right|_{\hat u} = \vec\nabla w\cdot\hat u = \langle 2,1\rangle\cdot\dfrac{\langle 3,-4\rangle}{5} = \frac25$

+ how?

d) $\frac{\Delta w}{\Delta s} = \left.\frac{dw}{ds}\right|_{\hat u} = |\langle 2,1\rangle| = \sqrt5$

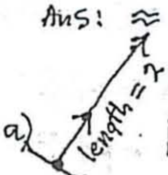
(go in dir $\hat u$ = dir $\vec\nabla w$ to get most rapid increase)

$\therefore \Delta s = \frac{\Delta w}{\sqrt5}$

$= \frac{.2}{2.2}$

Ans: $\approx .1$ or $\frac{.2}{\sqrt5}$

4. a)

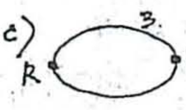


$|\vec\nabla w|_P = \frac{\Delta w}{\Delta s} = \frac{1}{1/2} = 2$

⊥ to contour line $\quad$ (twice length of $\hat u$)

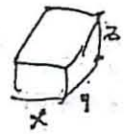
b) $\left.\frac{\partial w}{\partial x}\right|_Q \approx \frac{\Delta w}{\Delta x} = \frac{-1}{1/2} = -2$

c)



either point (where tan. line is vertical)

5.



$C = xy + xz + 2\cdot 2yz + 3\cdot xz \qquad xyz=1$

a) $\therefore C = xy+4yz+4xz$

$= xy + \frac4x + \frac4y$

or using: $xy + \frac2x + \frac4y$

b) $C_x = y - \frac{4}{x^2} = 0$

$C_y = x - \frac{4}{y^2} = 0$

soln: $x=y$ (by symmetry) $\therefore x^3=4$

$x = \sqrt[3]4$

$y = \sqrt[3]4$

$z = 1/4^{2/3}$

$C_x = y - \frac{2}{x^2} = 0$

$C_y = x - \frac{4}{y^2} = 0$

$\therefore x=1$, $y=2$, $z=\frac12$

c) $C_{xx} = \frac{8}{x^3} = 2$

$C_{yy} = \frac{8}{y^3} = 2$

$C_{xy} = 1$

2nd der. test: In both cases, $AC-B^2=3$, $A>0$ $\therefore$ minimum

$A = C_{xx} = \frac{4}{x^3} = 4$

$C = C_{yy} = 8/y^3 = 1$

$B = C_{xy} = 1$

←

d) Lagrange:

$C_x = \lambda g_x$: $y+4z = \lambda yz$

$C_y = \lambda g_y$: $x+4z = \lambda xz$

$C_z = \lambda g_z$: $4(x+y) = \lambda xy$

$x=y=\sqrt[3]4 \quad 8\cdot 4^{1/3} = \lambda 4^{2/3}$

$\lambda = 2\cdot 4^{2/3}$

or:

$y+4z = \lambda yz$

$x+2z = \lambda xz$

$4x+2y = \lambda xy$

$x=1, y=2 \quad 8 = 2\lambda$

$\lambda = 4$

6. $x^2+2y^2+3z^2=12$, $(1,2,-1)$

tan. plane has normal

$$(\vec{\nabla} w)_{(1,2,-1)} = \langle 2x, 4y, 6z\rangle_{(1,2,-1)}$$

$$= \langle 2, 8, -6\rangle$$

or $\langle 1, 4, -3\rangle$

$\boxed{x+4y-3z=12}$. (since it goes through $(1,2,-1)$)

intersects y-axis where $x=z=0$, $\therefore \boxed{y=3}$.

7. a) $w_r = w_x\cos\theta + w_y\sin\theta$ $\begin{pmatrix} x=r\cos\theta \\ y=r\sin\theta\end{pmatrix}$

$w_\theta = -w_x r\sin\theta + w_y r\cos\theta$

b) $(x,y)=(1,1) \Rightarrow r=\sqrt{2},\ \theta=\pi/4$

$\therefore w_r = 2\cdot\frac{\sqrt{2}}{2} + 3\cdot\frac{\sqrt{2}}{2} = \frac{5\sqrt{2}}{2}$

$w_\theta = -2\cdot 1 + 3\cdot 1 = 1$

8. $\left(\frac{\partial w}{\partial y}\right)_z = w_x\left(\frac{\partial x}{\partial y}\right)_z + w_y\underbrace{\left(\frac{\partial y}{\partial y}\right)_z}_{=1} + w_z\underbrace{\left(\frac{\partial z}{\partial y}\right)_z}_{=0}$

$1 = f_x\left(\frac{\partial x}{\partial y}\right)_z + f_z\underbrace{\left(\frac{\partial z}{\partial y}\right)_z}_{=0}$

$\therefore \left(\frac{\partial w}{\partial y}\right)_z = w_x\cdot\frac{1}{f_x} + w_y = \frac{(y+z)}{f_x} + (x+z)$

Differentials:

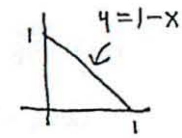
$dw = w_x dx + w_y dy + w_z dz$

$dy = f_x dx + f_z dz$ eliminate dx

$dw = w_x\left(\frac{dy}{f_x} - \frac{f_z}{f_x}dz\right) + w_y dy + w_z dz$

$= \underbrace{\left(\frac{w_x}{f_x} + w_y\right)}_{=\left(\frac{\partial w}{\partial y}\right)_z} dy + \left(-\frac{f_z}{f_x} + w_z\right)dz$

9. $\int_0^1\int_0^{1-x}(x^2+y^2)\,dy\,dx$

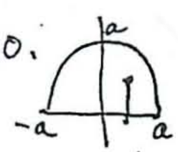


Inner: $x^2y + \frac{1}{3}y^3\Big]_0^{1-x}$

$= x^2(1-x) + \frac{1}{3}(1-x)^3$

Outer: $\frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{-1}{12}(1-x)^4\Big]_0^1 = \frac{1}{12} - \left(\frac{-1}{12}\right)$

$= \frac{1}{6}$

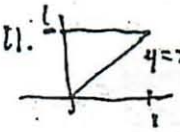
10. 

$\int_{-a}^{a}\int_0^{\sqrt{a^2-x^2}} y\,dA \Big/ \frac{\pi a^2}{2}$

$\frac{1}{\pi a^2/2}\int_0^{\pi}\int_0^a r\sin\theta\cdot r\,dr\,d\theta$

Evaluating (2nd form): $= \frac{2}{\pi a^2}\cdot\int_0^{\pi}\sin\theta\,d\theta\cdot\int_0^a r^2\,dr$

$= \frac{2}{\pi a^2}\cdot 2\cdot\frac{1}{3}a^3 = \frac{4a}{3\pi}$

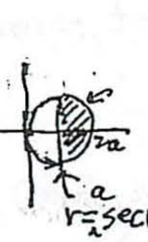
11. 

$\int_0^1\int_x^1 \frac{dy\,dx}{\sqrt{1+y^2}}$

$= \int_0^1\int_0^y \frac{dx\,dy}{\sqrt{1+y^2}}$

Inner: $\frac{y}{\sqrt{1+y^2}}$, Outer: $\sqrt{1+y^2}\Big]_0^1$

$= \sqrt{2}-1$

12. a) 

$\int_{-\pi/4}^{\pi/4}\int_{a\sec\theta}^{2a\cos\theta} r^2\cos^2\theta\cdot r\,dr\,d\theta$

b) $2\int_0^{\pi/2}\int_0^{2a\cos\theta} r^3\cos^2\theta\,dr\,d\theta$

Inner: $2\cdot\frac{1}{4}r^4\cos^2\theta\Big]_0^{2a\cos\theta} = 2\cdot\frac{1}{4}\cdot 16a^4\cos^6\theta$

Outer: $8a^4\cdot\int_0^{\pi/2}\cos^6\theta\,d\theta = 8a^4\cdot\frac{5\cdot 3}{6\cdot 4\cdot 2}\cdot\frac{\pi}{2}$

$= \frac{5\pi}{4}a^4$

PRACTICE PROBLEMS FOR 18.02 FINAL (Part B – 2 hours) – Spring 2010

Problem 1.

a) In the xy-plane, let $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$. Give in terms of P and Q the line integral representing the flux of $\mathbf{F}$ across a simple closed curve C, with outward-pointing normal.

b) Let $\mathbf{F} = ax\,\mathbf{i} + by\,\mathbf{j}$. How should the constants a and b be related if the flux of $\mathbf{F}$ over any simple closed curve C is equal to the area inside C?

Problem 2.

A solid hemisphere of radius 1 has its lower flat base on the xy-plane and center at the origin. Its density function is $\delta = z$. Find the force of gravitational attraction it exerts on a unit point mass at the origin.

Problem 3.

Evaluate $\int_C (y-x)dx + (y-z)dz$ over the line segment C from $P : (1,1,1)$ to $Q : (2,4,8)$.

Problem 4.

Consider a solid sphere of radius a with center at the origin; let H be its solid upper hemisphere (i.e., the part above the xy-plane). Set up a triple integral in spherical coordinates which gives the average distance of a point in H from the xy-plane.

(Give integrand, limits, and the constant factor in front, but *do not evaluate.*)

Problem 5.

Let C be a solid right circular cone having base radius 1 and vertex angle $60°$. Set up an integral in cylindrical coordinates which represents the moment of inertia of C about its central axis; assume the density $\delta = 1$.

(Place the cone so its axis is the z-axis and its vertex is at the origin; supply integrand and limits, but *do not evaluate.*)

Problem 6.

a) Let $\mathbf{F} = ay^2\,\mathbf{i} + 2y(x+z)\,\mathbf{j} + (by^2 + z^2)\,\mathbf{k}$. For what values of the constants a and b will $\mathbf{F}$ be conservative? Show work.

b) Using these values, find a function $f(x,y,z)$ such that $\mathbf{F} = \nabla f$.

c) Using these values, give the equation of a surface S having the property: $\int_P^Q \mathbf{F}\cdot d\mathbf{r} = 0$ for any two points P and Q on the surface S.

Problem 7.

Let S be the surface formed by the part of the graph of the paraboloid $z = x^2 + y^2$ lying below the plane $z = 1$, and let $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + (1-2z)\,\mathbf{k}$.

Calculate the flux of $\mathbf{F}$ across S, taking the outward direction (i.e., the one pointing away from the z-axis) as the one for which the flux is positive. Do this two ways:

a) by a method which calculates $\iint_S \mathbf{F}\cdot d\mathbf{S}$ directly;

b) by using the divergence theorem.

Problem 8.

Let S be the infinite circular cylindrical surface given by the equation $x^2 + y^2 = 1$ having the whole z-axis as its central axis, and let $\mathbf{F} = (zx - y)\,\mathbf{i} + zy\,\mathbf{j} + z\,\mathbf{k}$.

a) Calculate $\nabla \times \mathbf{F}$ (i.e., curl $\mathbf{F}$).

b) Deduce that $\iint_R \nabla\times\mathbf{F}\cdot\mathbf{n}\,dS = 0$ for any finite portion R of the surface S.

c) Let C be any closed curve on S going once around S (and oriented as in the picture). Show by using the result of part (b) and Stokes' theorem that $\oint_C \mathbf{F}\cdot d\mathbf{r}$ always has a constant value independent of C, and determine this value.

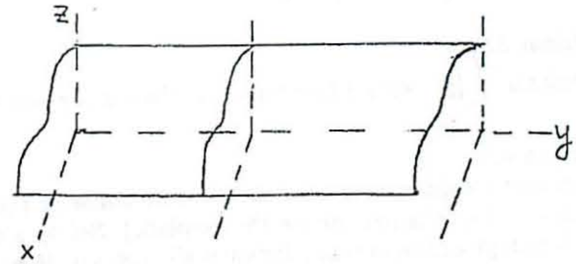
Problem 9.

Let $\phi(x, y, z)$ be a function with continuous second partial derivatives.
Prove that $\nabla \times \nabla \phi = 0$

Problem 10.

An xz-cylinder in 3-space is a surface given by an equation $f(x, z) = 0$ in x and z alone; its section by any plane $y = c$ perpendicular to the y-axis is always the same xz-curve.

Show that if $\mathbf{F} = x^2\,\mathbf{i} + y^2\,\mathbf{j} + xz\,\mathbf{k}$, then $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ for any simple closed curve C lying on an xz-cylinder. (Use Stokes' theorem.)



Part B Practice final 5/17

Let's hope part B is easier

Someone ~~at~~ at dinner ~~said~~ said he found part A hard

1. In the xy plane $\vec{F} = P\hat{i} + Q\hat{j}$

~~flux~~ in outward normal

now just need to remember it all

- find cheatsheet
- looks like never made

$\int M N_x - M_y$

$\oint Q_x \cancel{- M_y}$ ← is that line $\int$ form?

$\oint P dy - Q dx = \cancel{\iint Q_x - P_y \, dx \, dy}$

$= \iint Q_x - P_y \, dx \, dy$

remember flux is green's theorm in $\boxed{\text{normal}}$ form

2nd one on sheet

paramitrize dt as well $\quad 0 \le t < 1$

$= \int_0^1 2t\,dt - 4\cdot 7t\,dt$

$= \int_0^1 -26t\,dt$

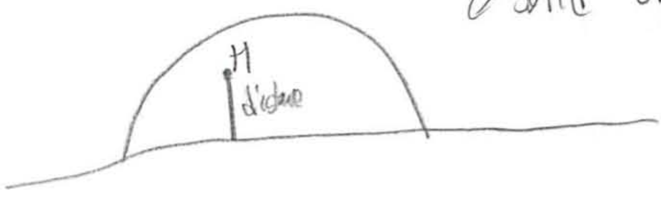
↑actually I got that
but I set it = to 0
instead of $\int$ - stupid move
think about what doing!

$= -13t^2 \Big|_0^1$

$= \boxed{-13}$

4. Solid sphere w/ radius a at center origin

triple $\iiint$
spherical coordinates
average distance of point H



← solid so any point right?

Part B Pratice Final

5/17

Let's hope part B is easier
Someone ~~at~~ at dinner ~~said~~ said he found part A hard

1. In the xy plane $\vec{F} = P\hat{\imath} + Q\hat{\jmath}$

~~flux~~ in outward normal
now just need to remember it all
- find cheatsheet
- looks like never made

$\int \cancel{M} N_x - M_y$

$\oint Q_x - M_y$ ← is that line $\int$ form?

$\oint P dy - Q dx = \cancel{\iint Q_x - P_y \, dx \, dy}$

$= \iint Q_x - P_y \, dx \, dy$

remember flux is green's theorm in $\boxed{\text{normal}}$ form
2nd one on sheet

② b) $\vec{F} = ax\hat{\imath} + by\hat{\jmath}$ How should the

Constants $a + b$ be related

$$\int P\,dx - Q\,dy = \iint \operatorname{div} F$$

$$= \iint M_x + N_y$$

$$= \iint a + b \, dV$$

~~solid~~ constant $(a+b)$

So $(a \cdot b) = \text{Volume } R$

$=$

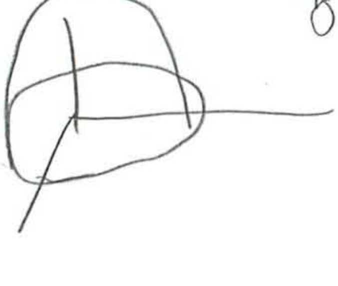
was this ever defined

2. Solid hemisphere

radius 1

Find force of gravitational attraction

$\delta = z$



Need to study that day before spring break again (3/27) make up

3/19 original day

(3) Physical applications of double $\iint$

$\iint f(x,y)\, dA$ - volume

σ = density

<u>Mass</u> $= \iint \sigma \, dA$

<u>Moment</u>
- mass w/ respect to axis
- multiplied by lever arm

$M_x = \iint y \sigma \, dA$

<u>Center of Mass</u>

$$\bar{x} = \frac{M_y}{M} = \frac{\iint x \sigma \, dA}{\iint \sigma \, dA}$$

$$\bar{y} = \frac{M_x}{M} = \frac{\iint y \sigma \, dA}{\iint \sigma \, dA}$$

<u>Moment of Inertia</u>

$$I_x = \iint_R y^2 \sigma \, dA$$

(4)

$\dfrac{GmM}{r^2}$ ← but at origin, so how does that work?

Notes From Book

$$F = \frac{GM}{|\vec{R}|^2} r \qquad \frac{r}{|R|} = \text{direction}$$

$$\Delta m = \sigma(x,y,z)\Delta V$$

$$F_z = \frac{G \Delta m}{R^2} \hat{r} \text{ ok}$$

- guess assuming origin has no mass

$$\vec{F} = G \iiint \frac{\cos\varphi}{\rho^2} \sigma \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

↑ what is this?
prob the paramitrization of σ — $\sigma = z = \rho\cos\varphi$
— over $|R|^2$

$$= G \int_0^{2\pi}\int_0^{\pi/2}\int_0^1 \rho \cos^2\varphi \sin\varphi \, d\rho \, d\varphi \, d\theta$$

$$= \text{ ⋮ solve } \frac{\pi G}{3}$$

(5)

3. Evaluate $\int_C (y-x)\,dx + (y-z)\,dz$

over line segment C $\quad P=(1,1,1)$

$Q=(2,4,8)$

$\vec{QP} = \langle 1, 3, 7 \rangle$

$x = 1+t$

$y = 1+3t$

$z = 1+7t$ that plane stuff again

$\dot{P} + \vec{PP}$

$P + \vec{QP}$

$dx = 1$

$dy = 3$

$dz = 7$

$\int (1+3t)-(1+t)\cdot 1 + (1+3t)-(1+7t)\cdot 7$

$1+3t-1-t+7+21t-7-49t=0$

$49-23t$ $\quad 0 = 26t$

26 $\quad t=0$

(6)

paramitrize dt as well $\quad 0 \le t \le 1$

$$= \int_0^1 2t\,dt - 4\cdot 7t\,dt$$

$$= \int_0^1 -26t\,dt$$

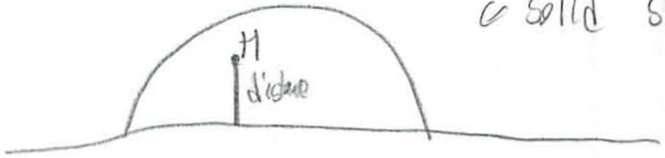
↑actually I got that
but I set it = to 0
instead of $\int$ - stupid move
think about what doing!

$$= -13t^2 \Big|_0^1$$

$$= \boxed{-13}$$

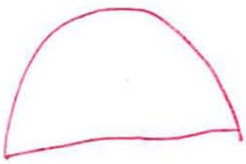
4. Solid sphere w/ radius a at center origin

triple $\int\int\int$
spherical coordinates
average distance of point H



← solid so any point right?

7



Volume = $\frac{2}{3}\pi a^3$

$z = \rho \cos\varphi$

? paramitrize

(I'm not really trying ....

- just writing down ans

not helping to memorize)

$$\frac{3}{2\pi a^3}\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^{a} \rho\cos\varphi\,\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$$

Ok so what in all world did they do

they just considered on the shell

not inside shell

(well why is shell solid then)

and why did they multiply by $\frac{1}{M}$

One of those ~~moment things~~

— ~~* Center of mass)~~

---

or perhaps they did all pts

- yeah 'cause they $\int$

- and then divide by mass to make it aveage

- need to prevent this

⑧ Skipped # 5 opps

6. $\vec{F} = ay^2 + 2y(x+z)\,\hat{j} + (by^2+z^2)\,\hat{k}$

$2xy + 2yz$

what values a, b is $\vec{F}$ conservative
(finally!)

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ M & N & P \end{vmatrix}$$

$P_y = N_z$
$P_x = M_z$
$N_x = M_y$

$2by = 2y$

$\boxed{b=1}$ ✓

$0 = 2z$

$2y = 2ay$ ✓

$\boxed{a=1}$

b) Find $\vec{f}$
- lets see if I remember the ~~a~~ long steps

~~$f_z = y^2 + z^2$~~
~~$f = 2y^2 +$~~

$f_x = y^2$

$f = x\underline{y}^2 + h(y, z)$

must I go in a certain order?

$f_y = \underline{\frac{xy^3}{3}} + h'(y,z)$

why in all world is this not working?

Final answer $xy^2 + y^2 z + \frac{z^3}{3}$

(1)

Try starting w/ z

$f_z = b \; y^2 + z^2$

$= y^2 + z^2$

$f = y^2 z + \frac{z^3}{3} + h(x,y)$

↓ good ↓ differentiate!

$f_y = 2yz + \frac{yz^3}{3} + h'(x,y)$

— up to here good

I think working from wrong thing

$h'(x,y) = 6yz^{1/3}$

~~$h(x,y)$~~ = ↑ then what w/ this

$h(x,y) = 3y^2 z^{1/3}$

$f = y^2 z + \frac{z^3}{3} + 3y^2 z^{1/3} + g(x)$

$f_x = xy^2 z + \frac{xz^3}{3} + 3x^2 z^{1/3} + g(x)$

Why such a disaster?

I don't know when we covered this and don't want to get into

(10)

from last pratice test

$f_z = y^2 + z^2$

$f = zy^2 + \frac{z^3}{3} + g(x,y)$

$f_y = 2yz +$ ~~differentiate~~ $+ g'(x,y)$

really stupid!

$g'(x,y) = 2xy$

$g(x,y) = xy^2$

$f = zy^2 + \frac{z^3}{3} + xy^2 + h(x)$

$f_x = 0 + 0 + y^2 + h'(x)$

$h'(x) = 0$

$h(x) = C$

$zy^2 + \frac{z^3}{3} + xy^2 + C$ (1)

woot! figured it out!

(took 45 min - but glad figured it out!)

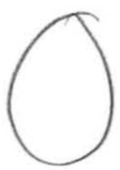
(1)

) Using these values give eq of surface

S having property $\int_P^Q F \cdot dr = 0$

forget what surface even is by now!

$Q = P$ same point



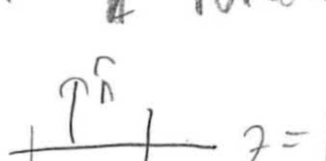
Circle
Conservative

any surface

$xy^2 + y^2 z + \frac{z^3}{3} = C$

I like my answer better + I am pretty sure
it is correct

7. Parabola $z = x^2 + y^2$



$\vec{F} = x\vec{i} + y\vec{j} + (1-2z)\vec{k}$

calculate flux

a) directly

div $\iint_S F \cdot ds = \iiint \operatorname{div} F \, dV$

$\iiint \nabla \cdot F \, dV$

(12)

Still must do #5

$\iint_S F \cdot ds$ around what the surface

So many variations in this section, So hard to choose what to use

$\vec{F} = \langle x, y, 1-2z \rangle$ I did not even try to do the problem!

$\iint F \cdot ds =$

What is ds → lenght of circle

But must paramitize

$d\vec{s} = \langle 2x, 2y, -1 \rangle$

∂x, ∂y, ~~R~~ direction So this is over surface

~~$\langle x \cdot 2x, y \cdot 2y$~~ scalar result

$\iint 2x^2 + 2y^2 - 1 + 2z \, \cancel{dS} \, dx \, dy$

get rid of $z = x^2 + y^2$

$2x^2 + 2y^2 - 1 + 2(x^2 + y^2)$

$4x^2 + 4y^2 - 1$

Now must $\int$ - lets actually do this time

 ← circle base

$$\cancel{\int_{-1}^{+1}\int_{-1}^{1} 4x^2+4y^2-1}$$

duh convert to polar

$x = r\cos\theta$

$y = r\sin\theta$

$$\int_0^{2\pi}\int_0^{r} 4r^2\cos^2\theta + 4r^2\sin^2\theta - 1$$

← r dr dθ grrrrr tried to take a shortcut

$$\int_0^{r} (4r^2-1)\, r \, dr$$

$4r^3 - r$

$$\frac{4r^4}{4} - \frac{r^2}{2}\Big|_0^{r}$$

$$\int_0^{2\pi} \cancel{\frac{4r^3\cdot 2\pi}{3} - 2\pi r}$$

$$\cancel{\frac{8\pi r^3}{3} - 2\pi r}$$

$$\int_0^{2\pi} r^4 - \frac{r^2}{2} \, d\theta$$

they define $r=1$

$$2\pi\left(r^4 - \frac{r^2}{2}\right) = \boxed{\pi}$$

(14)

Why am I messing up problem after problem
- not focusing on the differences
- esp if problem out of order

b) Div theorm

$\iiint$ div

$\nabla \cdot F$

$\iiint F_x + F_y + F_z \; dV$

$\cancel{1} + 1 + -2$

0 (ㆆ_ㆆ) 0

$\iint_{0} + \iint_{\text{top disc} \uparrow} \vec{F} \cdot d\vec{s} = 0$

$\iint_{\text{disc}} \tilde{F} \cdot d\vec{s} = \iint_{\text{circle/disc}} (1-2z)\, dx\, dy$

$= - \cdot$ area of circle $= \boxed{-\pi}$

$\iint F \cdot ds = \boxed{\pi}$

I don't get what they did
- is only divergance though the "lid" ???

(15)

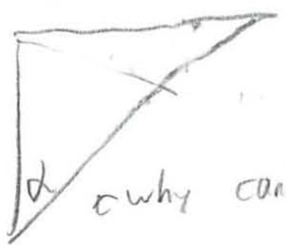
← go back

5\. C = solid right circular cone

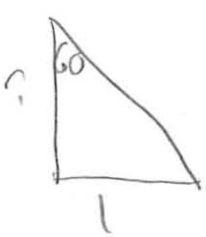
Set up SSS cyc coordinates

$$\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_1^{\frac{\sqrt{3}}{3}} \quad \text{← need a function in there } \frac{1}{\cos\alpha}$$

$\theta \quad \varphi \quad \rho$

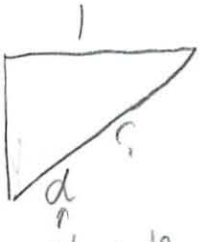


← why can't angle be from top?



$\tan 60 = \frac{1}{?}$

$? = \frac{1}{\tan 60} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$



this angle

$\cos\alpha = \frac{1}{?}$

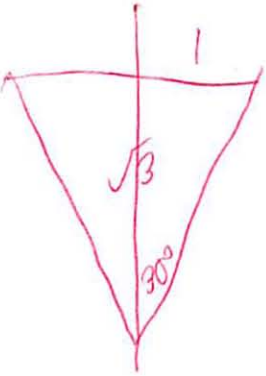
$? = \frac{1}{\cos\alpha}$

try $\alpha = \cancel{0}\ \frac{\pi}{2}$

$\frac{1}{\cos\frac{\pi}{2}} = 0$ wrong

why?

(16)



↓ oh - remember only goes to 30°

$$z = r\sqrt{3}$$

What angle is φ

and cylindrical coords is z, r, θ

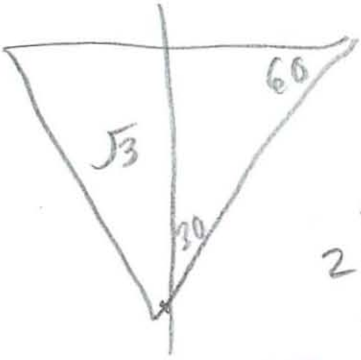
↳ but is

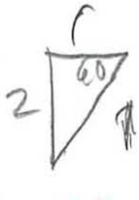
was right 1st time

Ok try again

$$\int_0^{2\pi} \int_0^{z\tan 30} \int_0^{\sqrt{3}} dz\, r\, dr\, d\theta$$

$\theta \quad r \quad z$





$$\tan 60 = \frac{z}{r}$$

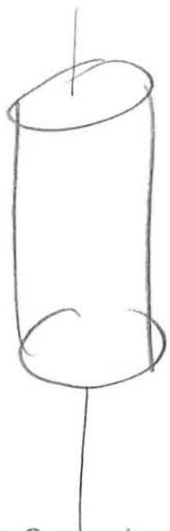
$$r = \frac{z}{\tan 60} = z \tan 30 = r\sqrt{3}$$

$$\int_0^{2\pi} \int_0^{1} \int_{r\sqrt{3}}^{\sqrt{3}} \underbrace{r^2}_{\text{why ???}} dz\, r\, dr\, d\theta$$

(17)

8. S = infinite circular cylinder

$x^2+y^2=1$



$\vec{F} = (zx-y)\hat{i} + zy\hat{j} + z\hat{k}$

a) Calculate $\nabla \times F$ = curl F

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ M & N & P \end{vmatrix}$$ ← I remembered this

$$(P_y - N_z)\hat{i} - (P_x - M_z)\hat{j} + (N_x - M_y)\hat{k}$$

$$0 - y\hat{i} - (0 - x)\hat{j} + (0 - -1)\hat{k}$$

$$-y\hat{i} + x\hat{j} + \hat{k}$$ ✓ bingo

b) Deduce that $\iint_R \nabla \times F \cdot \hat{n}\, dS = 0$ for any finite portion R on surface S

(18)

$-y\hat{\imath} + x\hat{\jmath} + \hat{k} \cdot \hat{n} = 0$

$\langle x, y, 0 \rangle$ was right before

$\hat{n} = \langle \cancel{-y}, \cancel{x}, 0 \rangle$

$\cancel{-y^2 + x^2} = 0$

which is tangent to cylinder, so it $= 0$

$-xy \; + xy = 0$ oh I see

no vector!

c) Let C be a closed curve on S going once around S

- Show Stokes's Theorm that $\oint F \cdot dr$ always has a constant value independent of C + show value

Add a horizontal disk at top

$\oint_C F \cdot dr = \iint_S \nabla \times F \cdot ds + \iint_T \vec{\nabla} \times \vec{F} \cdot ds$

$= 0 + \iint_T 1 ds = \pi$

area of T

(9)

Let $\phi(x, y, z)$ be a function w/ contineous 2nd deriv

Prove $\nabla \times \nabla \phi = 0$

is this like ∇^2 ~~.~~ Laplace

or something like that

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ \phi_x & \phi_y & \phi_z \end{vmatrix}$$

$$(\partial_y \phi_z - \partial_z \phi_y)\hat{i} - (\partial_x \phi_z - \partial_z \phi_x)\hat{j} + (\partial_x \phi_y - \partial_y \phi_x)\hat{k}$$

and then it evens out

think I would have gotten that

10. xz-cylinder is given by eq $f(x, z) = 0$

in x and z alone ~~th~~

its section by any plane $y = c$ $\perp$ to y axis is sam xz curve

Show that if $F = x^2\hat{i} + y^2\hat{j} + xz\hat{k}$

then $\oint F \cdot dr = 0$ for any simple closed curve C lying on xz cylinder

- what is that weird pic they drew?

(20) Stokes $\int_C F \cdot ds = \iint \text{curl } F \, ds$

$$= (\nabla \times F \cdot n)\, ds$$

$$\nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ M & N & P \end{vmatrix}$$

$$(P_y - N_z)\,\hat{i} - (P_x - M_z)\,\hat{j} + (N_x - M_y)\,\hat{k}$$

$$0 - 0\,\hat{i} - (z - 0)\,\hat{j} + (0 - 0)$$

$$-z\,\hat{j}$$

normal is

$$\hat{n} = \frac{\nabla f}{|\nabla f|} = \frac{f_x\,\hat{i} + f_z\,\hat{k}}{|\nabla f|}$$

fairly standard

$\nabla \times F \cdot \hat{n} = 0$ so $\oint F \cdot dr = 0$

SOLUTIONS TO PART B

1. $\oint P\,dy - Q\,dx$ $\left[\text{or: } \oint -Q\,dx + P\,dy\right]$

b) By Green's Thm: above

$$= \iint_R (P_x + Q_y)\,dx\,dy = \iint_R (a+b)\,dx\,dy.$$

$$= \text{area of } R \iff \boxed{a+b=1}$$

2.

$$F = G\iiint \frac{\cos\varphi}{\rho^2}\cdot\delta\cdot\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$$

$$\delta = z = \rho\cos\varphi$$

$$\therefore F = G\int_0^{2\pi}\int_0^{\pi/2}\int_0^1 \rho\cos^2\varphi\sin\varphi\,d\rho\,d\varphi\,d\theta$$

$$= G\cdot 2\pi\cdot\int_0^{\pi/2}\cos^2\varphi\sin\varphi\,d\varphi\cdot\int_0^1\rho\,d\rho$$

$$= G\cdot 2\pi\cdot \frac{-\cos^3\varphi}{3}\Big]_0^{\pi/2}\cdot\frac{1}{2}\rho^2\Big]_0^1$$

$$= 2\pi G\cdot\frac{1}{3}\cdot\frac{1}{2} = \boxed{\frac{\pi G}{3}}$$

3. Line from $P:(1,1,1)$ to $Q:(2,4,8)$ is: $x = 1+t$, $y = 1+3t$, $z = 1+7t$ (since $\vec{PQ} = \langle 1,3,7\rangle$). $0 \le t \le 1$

$$\therefore \int_C (y-x)\,dx + (y-z)\,dz = \int_0^1 2t\,dt - 4\cdot 7t\,dt$$

$$= \int_0^1 -26t\,dt = -13t^2\Big]_0^1 = \boxed{-13}$$

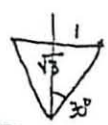
4.

Volume $= \frac{2}{3}\pi a^3$

$z = \rho\cos\varphi$

$$\frac{3}{2\pi a^3}\int_0^{2\pi}\int_0^{\pi/2}\int_0^a \rho\cos\varphi\cdot\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$$

5.



Equation of cone: $z = r\sqrt{3}$

$$\int_0^{2\pi}\int_0^1\int_{r\sqrt3}^{\sqrt3} r^2\cdot dz\cdot r\,dr\,d\theta$$

---

6. a) $\vec F = \langle ay^2,\ 2yx+2yz,\ by^2+z^2\rangle$

Test: $2ay = 2y \quad \therefore a = 1$

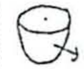
$2y = 2by \quad \therefore b = 1$

$0 = 0$ ✓

b) By any method, $f(x,y,z) = \boxed{xy^2 + y^2z + \frac{z^3}{3}}$

c) Any surface S: (contour surface of f) $\boxed{xy^2+y^2z+\frac{z^3}{3} = C}$

7.



a) $z = x^2+y^2$

$d\vec S = \langle z_x, z_y, -1\rangle\,dx\,dy$

$= \langle 2x, 2y, -1\rangle\,dx\,dy$

$\vec F = \langle x, y, 1-2z\rangle$

$$\therefore \iint_S \vec F\cdot d\vec S = \iint_{\text{radius }1} (2x^2+2y^2-1+2z)\,dx\,dy$$

$$= \iint (4x^2+4y^2-1)\,dx\,dy$$

$$= \int_0^{2\pi}\int_0^1 (4r^2-1)\,r\,dr\,d\theta$$

$$= 2\pi\left(r^4 - \frac{1}{2}r^2\right)\Big]_0^1 = \pi$$

b) $\operatorname{div}\vec F = 1+1-2 = 0$

$$\therefore \iint + \iint_{\text{top disc }\uparrow} \vec F\cdot d\vec S = 0 \quad (*) \text{ by divergence theorem}$$

$$\iint_{\text{disc}} \vec F\cdot d\vec S = \iint (1-2\underset{(z=1)}{z})\,dx\,dy$$

$$= -(\text{area of circle}) = -\pi$$

$$\therefore \iint \vec F\cdot d\vec S = \pi, \text{ by } (*)$$

8. a) $\begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ 2x-y & zy & z\end{vmatrix} = -y\hat\imath + x\hat\jmath + \hat k$

b) $\hat n = x\hat\imath + y\hat\jmath$ (on cylinder) $\quad\therefore \nabla\times\vec F\cdot\hat n = 0$

c)

Add a horizontal disc T as shown ($\hat n = \hat k$)

$$\oint_C F\cdot dr = \iint_S \nabla\times F\cdot dS + \iint_T \nabla\times\vec F\cdot\hat k\,dS$$

$$= \underset{\text{(by part b)}}{0} + \iint_T 1\,dS = \pi \quad (\text{area of } T)$$

9. $\begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ \varphi_x & \varphi_y & \varphi_z\end{vmatrix} = \hat\imath(\varphi_{zy} - \varphi_{yz}) - \hat\jmath(\varphi_{zx} - \varphi_{xz}) + \hat k(\varphi_{yx} - \varphi_{xy}) = \vec 0$

since mixed 2nd partials are equal: $\varphi_{xy} = \varphi_{yx}$, $\varphi_{yz} = \varphi_{zy}$, $\varphi_{zx} = \varphi_{xz}$

10. $$\oint_C \vec F\cdot d\vec r = \iint_S \vec\nabla\times\vec F\cdot d\vec S$$

$$\vec\nabla\times\vec F = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ x^2 & y^2 & xz\end{vmatrix} = -z\hat\jmath$$

The normal vector to $f(x,z) = 0$ is

$$\hat n = \frac{\vec\nabla f}{|\nabla f|} = \frac{f_x\hat\imath + f_z\hat k}{|\nabla f|}$$

$$\therefore \vec\nabla\times\vec F\cdot\hat n = 0, \text{ so } \oint_C \vec F\cdot d\vec r = 0$$

Cheat Sheet

5/17

Green's Work

$$\oint F \cdot \underset{\text{tangent}}{ds} = \iint_R \underset{\text{rotation}}{\text{curl } \vec{F}}\, dA \qquad dr = \hat{T}\, ds$$

$$\oint M\,dx + N\,dy = \iint_R (N_x - M_y)\, dA$$

Greens Normal Flux

$$\oint \vec{F} \cdot \underset{\text{normal}}{\hat{n}}\, ds = \iint \text{div } \vec{F}$$

$$\oint M\,dy - N\,dx = \iint (M_x + N_y)\, dA$$

Stokes

$$\int_C F \cdot ds = \iint \text{curl } \vec{F}\, ds = \iint (\vec{\nabla} \times \vec{F} \cdot \hat{n})\, ds$$

main circulation = microscopic circulation

Div

$$\iint_S F \cdot ds = \iiint_W \text{div } \vec{F}\, dV = \iiint \nabla \cdot F\, dV$$

expansion / shrinkage of fluid

Review Notes

5/17

$$\vec{T} = \frac{\vec{r}'(t)}{|r'(t)|}$$

remember gradiant = $\langle f_x, f_y, f_z \rangle$

↑ simple things like that I miss

Paramitrize r

div = how much fluid flowing out

Curl = circle "how fluid may rotate"

direction + speed

Stokes → curl in 3D

double $\iint$ over surface in space

remember circle radius doesn't matter

Going through first pratice test → it makes a lot more sense now

now just be able to do the math

is it just me or is it easier

- or is it looking at ans sheet
- this one is prob realistic where as the other was harder to train
  - but ended up confusing

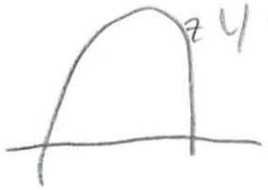
---

never realized $\hat{n} = \nabla f$

1 5/17

#15 Redo from scratch

Calculate flux out of solid



$z=4-x^2-y^2$ $\quad F = x\hat{i} + y\hat{j} + (1-2z)\hat{k}$

$x=0$

Div theorm

$$\iint_S F\cdot ds = \iiint \text{div } F \, dV$$

$$= \iint \nabla\cdot F \, dV$$

$\vec{n} = \langle 2x, 2y, 1 \rangle$

$\hat{n} = \langle 0, 0, -1 \rangle$

actually thought error → wrong flux is ↑ up

$F \cdot n$

$$\iint 2x^2 + 2y^2 + (1-2z)$$

$*z = 4-x^2-y^2$

$$\iint 2x^2 + 2y^2 + 1 - 2(4-x^2-y^2)$$

$$\iint \cancel{2}\, 4x^2 + 4y^2 - \cancel{9}$$

what w/ 2x SS ?

$dx\,dy$

or convert $r\,dr\,d\theta$

r

$x = r\cos\theta$

$y = r\sin\theta$

$4(r\cos\theta)^2 + 4(r\sin\theta)^2 - \cancel{9}\,7 \cdot r\,dr\,d\theta$

$$\int_0^{2\pi}\int_0^2 (4r^2 - \cancel{9}\,7)\,r\,dr\,d\theta$$

close

$$\int_0^2 4r^3 - \cancel{9}\,7r\,dr$$

$$\frac{4r^4}{4} - \frac{\cancel{9}\,7r^2}{2}\Big|_0^2$$

seems strangely familiar

$$r^4 - \frac{\cancel{9}\,7}{2}r^2$$

$$2^4 - \frac{\cancel{9}\,7}{2}2^2$$

$\cancel{32}\ 16 - 14$

Got 2^4 wrong

$\cancel{14}$ $2 \cdot 2\pi$

forgot 2π

$= 4\pi$

(3)

b) Now w/ div theorm

$$\iiint \text{div } F \, dV$$

$$\iiint \nabla \cdot F \, dV$$

$$\iiint F_x + F_y + F_z \, dV$$

$$1 + 1 - 2 = 0$$

also forgot to do w/ disk in part a

$n = \langle 0, 0, -1 \rangle$

$$\iint -(1-2z)$$

Wrong z here → yeah if = 0 then −1 · area that checks out

$$-1 + 2(\underbrace{4 - x^2 - y^2})$$

$$-1 + 8 - 2x^2 - 2y^2$$

$$-2x^2 - 2y^2 + 7$$

$x = r\cos\theta$

$y = r\sin\theta$ (I know could do this eaisor)

9)

$-2(r\cos\theta)^2 - 2(r\sin\theta)^2 - 7$

$-2r^2\cos^2\theta - 2r^2\sin\theta - 7$

$-2r^2 - 7$

$\int_0^{2\pi}\int_0^2 (-2r^2 - 7)\, r\, dr\, d\theta$

$\int_0^2 -2r^3 - 7r$

$\frac{-2r^4}{4} - \frac{7r^2}{2}\Big|_0^2$

$(-8 - 14)\,2\pi$

\# ? wrong

$n = -k$ ✓

$F \cdot \hat{n} = -1$

$4\pi + -4\pi = 0$ of course

So flux is $-\pi(2)^2 = -4\pi$

? makes sense but why did I miss shortcut

8

16. $\vec{F} = (-6y^2 + 6y)\hat{i} + (x^2 - 3z^2)\hat{j} - x^2\hat{k}$

Calc curl $\vec{F}$ and use Stoke's theorm to

Show work done along closed path $= 0$

$$\int F \cdot ds = \iint (\nabla \times F) \cdot \hat{n} \, dA$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ M & N & P \end{vmatrix} \quad (P_y - N_z)\hat{i} - (P_x - M_z)\hat{j} + (N_x - M_y)\hat{k}$$

$$(0 - 6z)\hat{i} - (-2x - 0)\hat{j} + (2x - (-12y + 6))\hat{k}$$

$$-6z\,\hat{i} + 2x\,\hat{j} + (2x + 12y - 6)\hat{k}$$

unit vector

$$\hat{n} = \frac{\langle 1, 2, 1 \rangle}{\sqrt{6}} \leftarrow$$

$$-6z + 4x + 2x + 12y - 6$$

$$6x + 12y - 6z - 6$$

kinda remember from class now

$$\frac{6(x + 2y - z - 1)}{\sqrt{6}} = 0$$

$\sqrt{6}$ ~~6~~ ← ~~why here though~~ ← I knew something here

$$x + 2y - z = 1$$

$$6(1 - 1) = 0$$

⑥ Now how is Stoke's theorm involved in this

Just conclude that related

What is ds?

5/18

$\hat{n} = \frac{\nabla f}{|\nabla f|}$

Or paramitrize

or dx dy

$\int M dx + N dy$ ← guess paramitrize here

$\vec{A} \cdot \frac{B}{|B|} = |\vec{A}| \cos\theta$

Area of space triangle $\frac{1}{2} |\vec{AB} \times \vec{BC}|$

$Ax = d$

$x = d A^{-1}$

$T = \frac{\vec{v}}{|\vec{v}|}$

$N = \text{dir } \frac{dT}{dt}$

$\kappa = \left|\frac{dT}{ds}\right| = \left|\frac{dT/dt}{ds/dt}\right|$ ← N ← v and don't forget magnitude

directional deriv $= \frac{dw}{ds} = \nabla w \cdot \hat{u}$

∇ at pt plug in

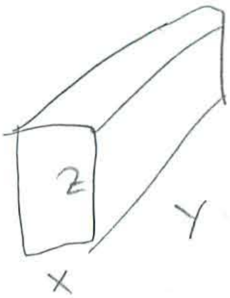
$\hat{u}$ the vector unit fized! $\sqrt{1^2+1^2}$

Minimize

$f_x = 0$

$f_y = 0$

Redo Problems 5/18



$C = xy + xz + 2 \cdot 2zy + 3xz$

$C = xy + 4xz + 4zy$

$x \cdot y \cdot z = 1$

$z = \dfrac{1}{xy}$

$C = xy + \dfrac{4x\cancel{z}}{xy} + \dfrac{4\cancel{z}x}{xy}$ match

b) Minimize $xy + 4\cancel{z}y^{-1} + 4\cancel{z}x^{-1}$ forgot to remove

$C_x = 0$

$C_y = 0$

$\cancel{C_x = y + 4 \cdot -1}$

$\cancel{y - 4 = 0}$ $\cancel{x = 4}$

$\cancel{C_y = x + 4 \cdot -1}$

$\cancel{x - 4 = 0}$ $\cancel{x = 4}$

did wrong got confused w/ forgotten z

$C_x = y + 4x^{-2} \cdot -1$

$$y - \frac{4}{x^2} = 0$$

$$C_y = x - \frac{4}{y^2} = 0$$ ✓

now solve for x + y

$$x = \frac{4}{y^2}$$

$$y - \frac{4}{\left(\frac{4}{y^2}\right)^2} = 0$$

← must have screwed up somewhere

$$y = \cancel{16}\ 4 \cdot \frac{y^4}{16}$$

$$y = \frac{y^4}{4} \qquad x = \frac{x^4}{4}$$

now what find values that fit? Yeah page not clear

$$x = \sqrt[3]{4} \qquad y = \sqrt[3]{4}$$

figure out how that was done

set = to each other ?

d) Lagrange

Have the

$f_x(x,y,z) = \lambda g_x$

$''_y \qquad '' _y$

$''_z \qquad '' _z$

$g = xyz$

$f = xy + \frac{4}{x} + \frac{4}{y}$

$\cancel{y - \frac{4}{x^2} = \lambda yz}$

$\cancel{x - \frac{4}{y^2} = \lambda xz}$

Now what

$\downarrow (x$

$y + 4z = \lambda yz$

$x + 4z = \lambda xz$

$4(x+y) = \lambda xy$

So what did I do wrong

a step too far ahead

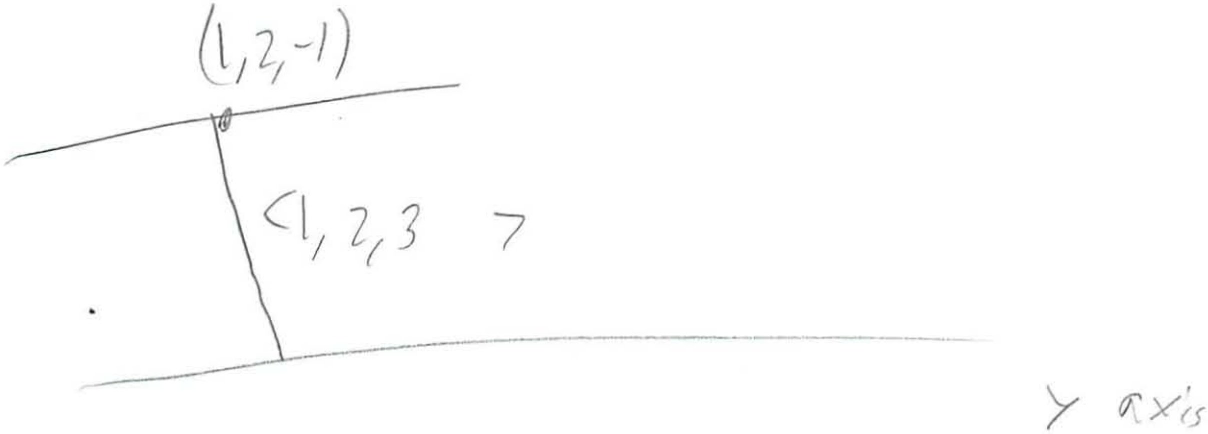
did not have a problem last time

$\frac{1}{z} + \frac{4}{y} = \lambda$

?

Solve for

set = to



at y axis $x=0$
$z=0$

✱ $\hat{n} = \nabla f$ (!) dont forget

$\nabla w = \langle 2x, -4y, 6z \rangle$

$\langle 2, 8, -6 \rangle$

$\langle 1, 4, 3 \rangle$ $\in$ line

Then $P + \overrightarrow{PP} = 0$

plug in for d

$\cancel{\nabla w \cdot \hat{u}} =$

$\vec{\nabla} w \cdot \hat{u} = \frac{dw}{ds}$

$= \nabla w \cdot \frac{\overset{\hat{n}}{\nabla w}}{|\nabla w|}$

$\nabla w =$ level curve

$\hat{u} = \text{dir}(\nabla w)$

$\int w_x \, dx'$

x'

$w_x \frac{dx}{ds}$

Seems so obvious, easy
But I had a hard time this semester
and stuff ~~em~~ mentioned in passing in lecture is big

$$\frac{\Delta w}{\Delta s} = \left|\frac{dw}{ds}\right| = |\langle 2,1 \rangle| = \sqrt{5} = \text{number}$$

gradiant $\frac{\Delta w}{\Delta s}$

$\frac{dw}{ds}$ = numer as well

dir $\hat{u}$ = dir ∇w?

I don't get how $a \neq b$