18.02 Practice Exam 1 (50 mins.)

1. (20) Consider the points in xyz-space $P:(1,2,-2)$, $Q:(-2,1,2)$, and the origin $O:(0,0,0)$.

a) (6) Find the cosine of angle POQ.

b) (6) Find a vector perpendicular to both OP and OQ.

c) (5) Find the xyz-equation of a plane parallel to the one through O, P and Q, but intersecting the z-axis at $z=2$.

d) (3) Where does the plane you found in (c) intersect the x-axis?

2. (20) Let $A = \begin{pmatrix} 1 & 3 & 2 \\ 0 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$. Its matrix of cofactors is (in part) $C = \begin{pmatrix} 3 & 1 & -2 \\ -4 & 0 & 2 \\ & & \end{pmatrix}$.

a) (15) Confirm (mentally) the entry -4 in C, then fill in the bottom row of C and from this find A^{-1}.

b) (5) Use the result of part (a) to solve the system

$$x+3y+2z=1, \qquad 2y+z=2, \qquad x+y+2z=-1\ .$$

3. (8) Find all values of the constant c for which the system of homogeneous equations

$$cx+y+4z=0, \qquad -x+y+z=0, \qquad y+cz=0$$

has a non-trivial solution (i.e., a solution other than $x=y=z=0$)

4. (12: 10,2) Scotch tape is being peeled off a stationary roll, modeled as a circle of radius a, and center at the origin. The end $P:(x,y)$ of the tape is initially at the point $A:(a,0)$ on the x-axis. During the process, the pulled-off length of tape is always tangent to the rest of the roll – call the point of tangency Q on the circle, and of the two possible directions for the pulled-off tape, it's the one where the sticky side faces away from the roll (not towards it).

a) Use vector methods to derive parametric equations for x and y in terms of the central angle $AOQ=\theta$, for $0 \le \theta \le 2\pi$. Show work, indicating reasoning.

b) Show on a separate sketch where P is when $\theta=\pi$, and verify that your equations give the correct position of P when $\theta=\pi$.

5. (20) A point P moves in space so that its position vector is given by

$$OP = \mathbf{r} = (\cos t)\,\mathbf{i} + (\sqrt{2}\sin t)\,\mathbf{j} + (\cos t)\,\mathbf{k}\ .$$

a) (10: 5,3,2) Find its velocity vector $\mathbf{v}$, its speed $\dfrac{ds}{dt}$, and its unit tangent vector $\mathbf{T}$.

b) (5) Find its curvature $\kappa = \left|\dfrac{d\mathbf{T}}{ds}\right|$.

c) (5) At what point(s) in the xz-plane does P pass through this plane? skipped???

6. (10) At what point P on the line given by the position vector $\mathbf{r}(t) = \langle 1+t,\ 3-t,\ 1+2t\rangle$ will the origin vector OP be perpendicular to the line?

7. (10) A point P moves in the polar coordinate (r,θ)-plane so that its velocity vector at time t is given in the $\mathbf{u}_r, \mathbf{u}_\theta$ system by $\mathbf{v} = 2\,\mathbf{u}_r + 2\,\mathbf{u}_\theta$.

At time $t=0$, the point P has coodinates $r=1$ and $\theta=0$.

Answer the following, showing work or brief indication of reason.

a) How long is the path that P travels from $t=0$ to $t=3$?

b) How far is P from the origin when $t=3$?

c) What angle does the path of P make with its position vector, when $t=3$?

d) Where is the point P at time t?

18.02 Practice Exam 1 Solutions – S-2010

$\boxed{1}$ $\vec{OP} = \langle 1, 2, -2 \rangle$ $\vec{OQ} = \langle -2, 1, 2 \rangle$

a) $\cos POQ = \dfrac{OP \cdot OQ}{|OP||OQ|} = \dfrac{-4}{3 \cdot 3} = \dfrac{-4}{9}$

b) $OP \times OQ = \begin{vmatrix} i & j & k \\ 1 & 2 & -2 \\ -2 & 1 & 2 \end{vmatrix} = \langle 6, 2, 5 \rangle$

c) $6x + 2y + 5(z-2) = 0$

or $6x + 2y + 5z = 10$ . .

d) Since $y = z = 0$, $6x = 10$

$\therefore$ at $x = \boxed{5/3}$

$\boxed{2}$ $A = \begin{pmatrix} 1 & 3 & 2 \\ 0 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$ $C = \begin{pmatrix} 3 & 1 & -2 \\ -4 & 0 & 2 \\ -1 & -1 & 2 \end{pmatrix}$

a)

$|A| = 4 + 3 - (4+1)$

$= 2$

$A^{-1} = \frac{1}{2} C^T = \frac{1}{2} \begin{pmatrix} 3 & -4 & -1 \\ 1 & 0 & -1 \\ -2 & 2 & 2 \end{pmatrix}$

b) The system is $A\vec{x} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$

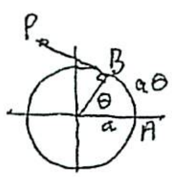
so $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 3 & -4 & -1 \\ 1 & 0 & -1 \\ -2 & 2 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$

$\boxed{3}$ Requires coeff. det.: be 0 $\begin{vmatrix} c & 1 & 4 \\ -1 & 1 & 1 \\ 0 & 1 & c \end{vmatrix} = (c^2 - 4) - (c - c)$

$= c^2 - 4$

$c^2 - 4 = 0 \Leftrightarrow c = \pm 2$

$\boxed{4}$



$\vec{OP} = \vec{OB} + \vec{BP}$

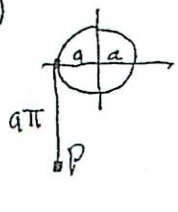
$OB = a\langle \cos\theta, \sin\theta \rangle$

a)

$BP = \underset{\text{length}}{a\theta} \underset{\vec{OB} \text{ rotated } 90° \text{ to the left}}{\langle -\sin\theta, \cos\theta \rangle}$

Adding: $\vec{OP} = a\langle \cos\theta - \theta\sin\theta, \sin\theta + \theta\cos\theta \rangle$

b)



P: $(-a, -a\pi)$ from picture

$\vec{OP} = a\langle -1 - \pi \cdot 0, 0 + \pi \cdot -1 \rangle$

$= a\langle -1, -\pi \rangle$ ✓

$\boxed{5}$ $\vec{r} = \langle \cos t, \sqrt{2}\sin t, \cos t \rangle$

a) $\vec{v} = \langle -\sin t, \sqrt{2}\cos t, -\sin t \rangle$

$|\vec{v}| = \dfrac{ds}{dt} = (\sin^2 t + 2\cos^2 t + \sin^2 t)^{1/2}$

$= \sqrt{2(\sin^2 t + \cos^2 t)}$

$= \sqrt{2}$

$\vec{T} = \dfrac{\vec{v}}{|\vec{v}|} = \left\langle \dfrac{-\sin t}{\sqrt{2}}, \cos t, \dfrac{-\sin t}{\sqrt{2}} \right\rangle$

b) $K = \left|\dfrac{d\vec{T}}{ds}\right| = \dfrac{|d\vec{T}/dt|}{|ds/dt|} = \dfrac{|d\vec{T}/dt|}{\sqrt{2}}$

$= \dfrac{1}{\sqrt{2}} \left| \left\langle -\dfrac{\cos t}{\sqrt{2}}, -\sin t, -\dfrac{\cos t}{\sqrt{2}} \right\rangle \right|$

$= \dfrac{1}{\sqrt{2}} \sqrt{\cos^2 t + \sin^2 t} = \dfrac{1}{\sqrt{2}}$.

c) Crosses xz-plane when $y = 0$

$\therefore$ when $\sqrt{2}\sin t = 0$: $t = 0, \pi$

The corresponding points are resp'y.:

$(1, 0, 1)$ and $(-1, 0, -1)$.

$t=0$

$\boxed{6}$

The line has dir'n vector $\vec{A} = \langle 1, -1, 2 \rangle$

$\vec{OP} = \langle 1+t, 3-t, 1+2t \rangle$

These are $\perp$ if $\vec{OP} \cdot \vec{A} = 0$:

$(1+t) - (3-t) + 2(1+2t) = 0$

or $6t = 0$

$\therefore t = 0$, corresponding to

$P = (1, 3, 1)$

$\boxed{7}$ $\vec{V} = 2\hat{u}_r + 2\hat{u}_\theta = \dot{r}\hat{u}_r + r\dot{\theta}\hat{u}_\theta$

a) $|\vec{V}| = \sqrt{4+4} = 2\sqrt{2} = v = \dfrac{ds}{dt}$

so $s = 2\sqrt{2} \cdot 3 = \boxed{6\sqrt{2}}$

b) $\dot{r} = 2 \Rightarrow r = 2t + 1$ $\therefore r(3) = 7$

$r(0) = 1$

c) $\tan\psi = \frac{2}{2} = 1$, $\therefore \psi = \pi/4$ (45°)

d) Find $\theta(t)$: $r\dot{\theta} = 2$, $r = 2t+1$ . .

$\therefore \dot{\theta} = \dfrac{2}{2t+1}$, $\begin{cases} \theta = \ln(2t+1) + \cancel{c} \\ r = 2t+1 \end{cases}$

($c = 0$ since $\theta(0) = 0$)

18.02 Practice Problems for Exam 2 (75 mins; Exam 2 is 50 mins.)

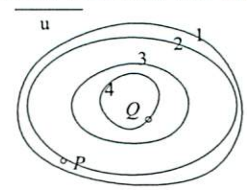
1. (20) For the function $w = y(1+x) + \sin(xy)$,

a) Write an approximate formula showing how Δw depends on Δx and Δy, at the point $(0,1)$.

b) At the point $(0,1)$, is w more sensitive to x or y? (give reason)

c) Find the directional derivative $\left.\frac{dw}{ds}\right|_{\mathbf{u}}$ at the point $(0,1)$ in the direction of the vector $3\,\mathbf{i} - 4\,\mathbf{j}$.

d) Starting at the point $(0,1)$, what is the minimal distance you could travel to increase the value of w by .2? (show work or indicate reasoning)



2. (15) Level curves for $w = f(x,y)$ are shown; $\mathbf{u}$ is a unit distance.

a) At P, draw in the vector $(\nabla f)_P$. b) At Q, estimate $\left(\frac{\partial w}{\partial x}\right)$.

c) Mark a point R where $f(R) = 3$ and $w_y = 0$.

3. (25) A wooden rectangular drawer with a capacity of one cubic foot is to be constructed. The wood costs \$1/sq.ft. for the bottom and the back, \$2/sq.ft. for the two sides, and \$3/sq.ft. for the front; there is no top. Let x be the end width, y the side width, and z the height, and C the total cost. What values for x, y, z minimize the total cost?

a) Show this leads to minimizing $C = xy + 2/x + 4/y$.

b) Find the minimizing values for x, y, z .

c) Use the second derivative test to show it is actually a minimum.

d) Give one of the equations for the Lagrange multiplier method, and use it to determine the value of the multiplier λ corresponding to the minimum.

4. (10) Where does the tangent plane to $x^2+2y^2+3z^2 = 12$ at the point $(1,2,-1)$ intersect the y-axis?

5. (15) Let $w = w(x,y)$, and let r, θ be the usual polar coordinates.

a) Express $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial \theta}$ in terms of w_x, w_y, r and θ.

b) $\nabla w = 2\,\mathbf{i} + 3\,\mathbf{j}$ at $(1,1)$; evaluate $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial \theta}$ at this point.

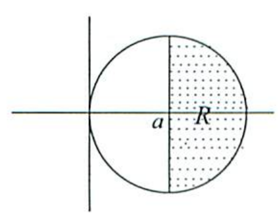
6. (15) Let $w = xy + xz + yz$, where the variables x, y, z are not independent, but constrained by a relation $y = f(x,z)$. Express $\left(\frac{\partial w}{\partial y}\right)_z$ in terms of x, y, z and the formal partial derivatives f_x and f_z. You can use either method: the chain rule or differentials.

7. (10) Find the volume of the region in space lying under the graph of $z = x^2+y^2$ and over the triangle in the xy-plane having vertices at $(0,0)$, $(1,0)$, $(0,1)$.

8. (20) Let R be the upper half of the circular disc of radius a centered at the origin. Express the average distance of a point in R from the x-axis by an iterated integral in

(a) rectangular coordinates (b) polar coordinates (c) evaluate the two integrals

9. (10) Evaluate $\displaystyle\int_0^1\int_x^1 \frac{dy\,dx}{\sqrt{1+y^2}}$ by changing the order of integration.



10. (15) Using polar coordinates and taking density $\delta = 1$,

a) set up an iterated integral giving the moment of inertia about the y-axis of the pictured shaded semicircular region R of radius a. *Don't evaluate it.*

b) Calculate the moment of inertia about the y-axis of the entire circular disc.

Definite integral formulas:

$$\int_0^{\frac{\pi}{2}} \sin^n x\,dx = \int_0^{\frac{\pi}{2}} \cos^n x\,dx = \frac{(n-1)!!}{n!!}A_n\,; \quad A_n = \begin{cases} 1, & n \text{ odd integer } \geq 3; \\ \pi/2, & n \text{ even integer } \geq 2; \end{cases} \qquad n!! = n(n-2)(n-4)\cdots.$$

Exam 2 Practice

1. $w = y(1+x) + \sin(xy)$

(a) $w_x = y + y\cos(xy)\,;\; = 2$ at $(0,1)$

$w_y = 1 + x + x\cos(xy), \; = 1$ at $(0,1)$

$\therefore \Delta w \approx 2\Delta x + \Delta y$

b) to x, since coeff. of Δx is bigger.

c) $\frac{dw}{ds}\Big|_{\hat u} = \nabla w \cdot \hat u = \langle 2,1\rangle \cdot \frac{\langle 3,-4\rangle}{5} = \frac{2}{5}$

d) $\frac{\Delta w}{\Delta s} \approx \frac{dw}{ds}\Big|_{\hat u} = |\langle 2,1\rangle| = \sqrt{5}$

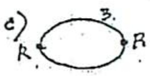
(go in dir $\hat u$ = dir ∇w to get most rapid increase) $\quad \therefore \Delta s = \frac{\Delta w}{\sqrt 5}$

Ans: $\approx .1$ or $\frac{.2}{\sqrt5} \approx \frac{.2}{2.2}$

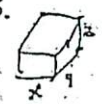
2. a) length $= 2$ $\quad |\nabla w|_P = \frac{\Delta w}{\Delta s} = \frac{1}{1/2} = 2$

$\perp$ to contour line $\quad$ (twice length of $\hat u$)

b) $\frac{\partial w}{\partial x}\Big|_Q \approx \frac{\Delta w}{\Delta x} = \frac{-1}{1/2} = -2$

c) 

either point (where tan. line is vertical)

3. 

$C = xy + xz + 2\cdot 2yz + 3\cdot xz$ $\qquad xyz = 1$

a) $\therefore C = xy + 4yz + 4xz$
$= xy + \frac{4}{x} + \frac{4}{y}$

or using: $\left[xy + \frac{2}{x} + \frac{4}{y}\right]$

$C_x = y - \frac{2}{x^2} = 0$

$C_y = x - \frac{4}{y^2} = 0$

$\therefore x = 1,\; y = 2,\; z = 1/2$

b) $C_x = y - \frac{4}{x^2} = 0$

$C_y = x - \frac{4}{y^2} = 0$

soln: $x = y \quad \therefore x^3 = 4$ (by symmetry)

$x = \sqrt[3]{4}$

$y = \sqrt[3]{4}$

$z = 1/4^{2/3}$

c) $C_{xx} = \frac{8}{x^3} = 2$

$C_{yy} = \frac{8}{y^3} = 2$

$C_{xy} = 1$

2nd der. test: In both cases, $AC - B^2 = 3$, $A > 0$ $\therefore$ minimum

$A = C_{xx} = \frac{4}{x^3} = 4$

$C = C_{yy} = 8/y^3 = 1$

$B = C_{xy} = 1$

$\rightarrow$

4. $x^2 + 2y^2 + 3z^2 = 12, \quad (1,2,-1)$

tan. plane has normal

$(\nabla w)_{(1,2,-1)} = \langle 2x, 4y, 6z\rangle_{(1,2,-1)}$

$= \langle 2, 8, -6\rangle$

or $\langle 1, 4, -3\rangle$

$\boxed{x + 4y - 3z = 12}$ (since it goes through $(1,2,-1)$)

intersects y-axis where $x = z = 0$, $\therefore \boxed{y = 3}$.

5. a) $w_r = w_x\cos\theta + w_y\sin\theta$ $\quad \left(\begin{matrix}x = r\cos\theta\\ y = r\sin\theta\end{matrix}\right)$

$w_\theta = -w_x r\sin\theta + w_y r\cos\theta$

b) $(x,y) = (1,1) \Rightarrow r = \sqrt2,\; \theta = \pi/4$

$\therefore w_r = 2\cdot\frac{\sqrt2}{2} + 3\cdot\frac{\sqrt2}{2} = \frac{5\sqrt2}{2}$

$w_\theta = -2\cdot 1 + 3\cdot 1 = 1$

6. $\left(\frac{\partial w}{\partial y}\right)_z = w_x\left(\frac{\partial x}{\partial y}\right)_z + w_y\underbrace{\left(\frac{\partial y}{\partial y}\right)_z}_{1} + w_z\underbrace{\left(\frac{\partial z}{\partial y}\right)_z}_{0}$

$1 = f_x\left(\frac{\partial x}{\partial y}\right)_z + f_z\underbrace{\left(\frac{\partial z}{\partial y}\right)_z}_{0}$

$\therefore \left(\frac{\partial w}{\partial y}\right)_z = w_x\cdot\frac{1}{f_x} + w_y = \frac{(y+z)}{f_x} + (x+z)$

Differentials:

$dw = w_x dx + w_y dy + w_z dz$

$dy = f_x dx + f_z dz \quad$ eliminate dx

$dw = w_x\left(\frac{dy}{f_x} - \frac{f_z}{f_x}dz\right) + w_y dy + w_z dz$

$= \underbrace{\left(\frac{w_x}{f_x} + w_y\right)}_{=\left(\frac{\partial w}{\partial y}\right)_z} dy + \left(-\frac{f_z}{f_x} + w_z\right)dz$

d) Lagrange:

$C_x = \lambda g_x : \; y + 4z = \lambda yz$

$C_y = \lambda g_y : \; x + 4z = \lambda xz$

$C_z = \lambda g_z : \; 4(x+y) = \lambda xy$

$x = y = \sqrt[3]{4} \quad 8\cdot 4^{1/3} = \lambda 4^{2/3}$

$\lambda = 2\cdot 4^{1/3}$

or:

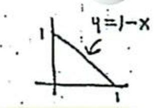
$y + 4z = \lambda yz$

$x + 2z = \lambda xz$

$4x + 2y = \lambda xy$

$x = 1,\; y = 2 \quad 8 = 2\lambda,\; \lambda = 4$

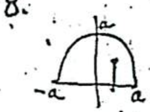
$\leftarrow$

7. $\int_0^1\int_0^{1-x} (x^2 + y^2)\,dy\,dx$ 

Inner: $x^2 y + \frac13 y^3\Big]_0^{1-x}$

$= x^2(1-x) + \frac13(1-x)^3$

Outer: $\frac13 x^3 - \frac14 x^4 + -\frac{1}{12}(1-x)^4\Big]_0^1 = \frac{1}{12} - \left(\frac{-1}{12}\right) = \frac16$

8. 

$\int_{-a}^{a}\int_0^{\sqrt{a^2-x^2}} y\,dy\,dx \Big/ \frac{\pi a^2}{2}$

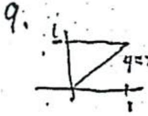
$\frac{1}{\pi a^2/2}\int_0^{\pi}\int_0^{a} r\sin\theta\cdot r\,dr\,d\theta$

Evaluating (2nd form): $= \frac{2}{\pi a^2}\cdot\int_0^{\pi}\sin\theta\,d\theta\cdot\int_0^a r^2 dr$

$= \frac{2}{\pi a^2}\cdot 2\cdot\frac13 a^3 = \frac{4a}{3\pi}$

1st: Inner: $\frac12 y^2\Big]_0^{\sqrt{a^2-x^2}} = \frac12(a^2 - x^2)$

outer: $\frac12\left(a^2 x - \frac13 x^3\right)\Big]_{-a}^{a} = \frac23 a^3$

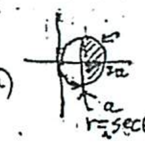
9. 

$\int_0^1\int_x^1 \frac{dy\,dx}{\sqrt{1+y^2}}$

$= \int_0^1\int_0^y \frac{dx\,dy}{\sqrt{1+y^2}}$

Inner: $\frac{y}{\sqrt{1+y^2}}$ $\quad$ Outer: $\sqrt{1+y^2}\Big]_0^1$

$= \sqrt2 - 1$

10. a) 

$\int_{-\pi/4}^{\pi/4}\int_{a\sec\theta}^{2a\cos\theta} r^2\cos^2\theta\cdot r\,dr\,d\theta$

b) $2\int_0^{\pi/2}\int_0^{2a\cos\theta} r^3\cos^2\theta\,dr\,d\theta$

Inner: $2\cdot\frac14 r^4\cos^2\theta\Big]_0^{2a\cos\theta} = 2\cdot\frac14\cdot 16a^4\cos^6\theta$

Outer: $8a^4\cdot\int_0^{\pi/2}\cos^6\theta\,d\theta = 8a^4\cdot\frac{5\cdot3}{6\cdot4\cdot2}\cdot\frac{\pi}{2}$

$= \frac{5\pi}{4}a^4$

1

18.02 Exam Pratice

3/31

Lets start reviewing topic ligt

Last exam 66/90

Vector
- find angle
- → matrix
- area of triangle
- unwinding tape

don't remember, but won't really be on

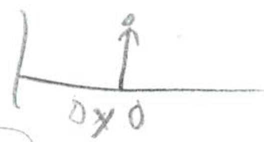
need for final

Last time got average, need to do this well again!

Function of several variables

$$f(x, y) = xy + x$$

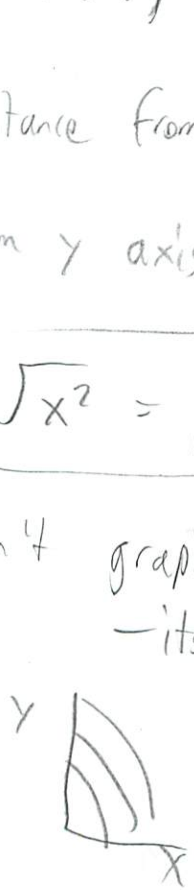
distance from origin $\sqrt{x^2+y^2+z^2}$

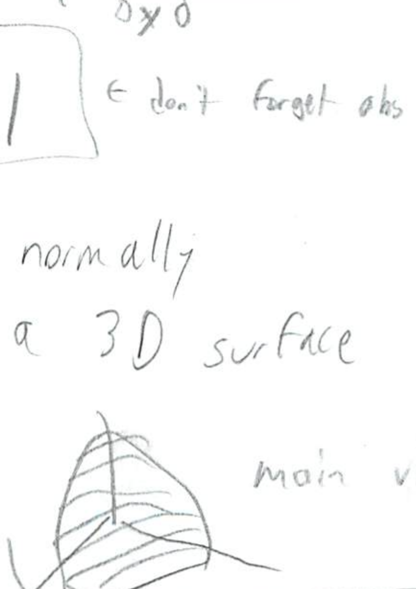
from y axis 

$\sqrt{x^2+z^2}$

$$\boxed{\sqrt{x^2} = |x|}$$ ← don't forget abs value

Can't graph normally
- its a 3D surface

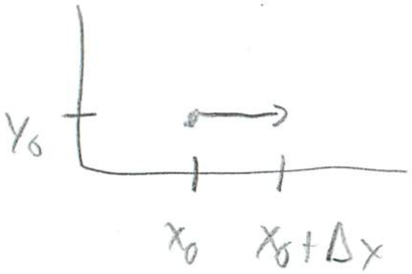




like a topo map

$^\circ f(x,y) = C$ side view

②



$\Delta w = w_1 - w_0$

$w = (x_0, y_0)$

$\left(\frac{\partial w}{\partial x}\right)_0 = \lim \frac{\Delta w}{\Delta x}$

at that pt

blah, blah, blah

basically keep one fixed

$\left(\frac{\partial}{\partial x}\right)_y x^4 y^2$

deriv of x keeping y constant ? or where evaling at ???

$4x^3 \; 2y \; \frac{\partial w}{\partial x}$

By how much does temp change as you go from one pt to other

$\Delta w = \frac{\partial w}{\partial x} \Delta x + \frac{\partial w}{\partial y} \Delta y$

? why both

- well $\Delta w \uparrow$ = $\frac{\Delta w}{\cancel{\Delta x}} \cdot \cancel{\Delta x} \uparrow$