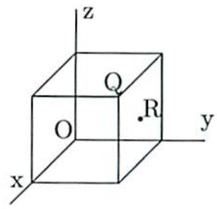
18.02 Practice Exam 1 A

Problem 1. (15 points)



A unit cube lies in the first octant, with a vertex at the origin (see figure).

a) Express the vectors $\overrightarrow{OQ}$ (a diagonal of the cube) and $\overrightarrow{OR}$ (joining O to the center of a face) in terms of $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$.

b) Find the cosine of the angle between OQ and OR.

Problem 2. (10 points)

The motion of a point P is given by the position vector $\vec{R} = 3\cos t\,\hat{\imath} + 3\sin t\,\hat{\jmath} + t\,\hat{k}$. Compute the velocity and the speed of P.

Problem 3. (15 points: 10, 5)

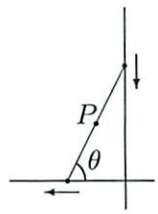
a) Let $A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix}$; then $\det(A) = 2$ and $A^{-1} = \frac{1}{2}\begin{bmatrix} 1 & a & b \\ -1 & -2 & 5 \\ 2 & 2 & -6 \end{bmatrix}$; find a and b.

b) Solve the system $AX = B$, where $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

c) In the matrix A, replace the entry 2 in the upper-right corner by c. Find a value of c for which the resulting matrix M is not invertible.

For this value of c the system $MX = 0$ has other solutions than the obvious one $X = 0$: find such a solution by using vector operations. (*Hint:* call U, V and W the three rows of M, and observe that $MX = 0$ if and only if X is orthogonal to the vectors U, V and W.)

Problem 4. (15 points)



The top extremity of a ladder of length L rests against a vertical wall, while the bottom is being pulled away. Find parametric equations for the midpoint P of the ladder, using as parameter the angle θ between the ladder and the horizontal ground.

Problem 5. (25 points: 10, 5, 10)

a) Find the area of the space triangle with vertices $P_0 : (2,1,0)$, $P_1 : (1,0,1)$, $P_2 : (2,-1,1)$.

b) Find the equation of the plane containing the three points P_0, P_1, P_2.

c) Find the intersection of this plane with the line parallel to the vector $\vec{V} = \langle 1,1,1\rangle$ and passing through the point $S : (-1,0,0)$.

Problem 6. (20 points: 5, 5, 10)

a) Let $\vec{R} = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$ be the position vector of a path. Give a simple intrinsic formula for $\frac{d}{dt}(\vec{R}\cdot\vec{R})$ in vector notation (not using coordinates).

b) Show that if $\vec{R}$ has constant length, then $\vec{R}$ and $\vec{V}$ are perpendicular.

c) let $\vec{A}$ be the acceleration: still assuming that $\vec{R}$ has constant length, and using vector differentiation, express the quantity $\vec{R}\cdot\vec{A}$ in terms of the velocity vector only.

18.02 Practice Exam 1 A – Solutions

Problem 1.

a) $\overrightarrow{OQ} = \hat{\imath} + \hat{\jmath} + \hat{k}$; $\overrightarrow{OR} = \frac{1}{2}\hat{\imath} + \hat{\jmath} + \frac{1}{2}\hat{k}$.

b) $\cos\theta = \dfrac{\overrightarrow{OQ} \cdot \overrightarrow{OR}}{|\overrightarrow{OQ}|\,|\overrightarrow{OR}|} = \dfrac{\langle 1,1,1\rangle \cdot \langle \frac{1}{2},1,\frac{1}{2}\rangle}{\sqrt{3}\sqrt{\frac{3}{2}}} = \dfrac{2\sqrt{2}}{3}$.

Problem 2.

Velocity: $\vec{V} = \dfrac{d\vec{R}}{dt} = \langle -3\sin t, 3\cos t, 1\rangle$. Speed: $|\vec{V}| = \sqrt{9\sin^2 t + 9\cos^2 t + 1} = \sqrt{10}$.

Problem 3.

a) Minors: $\begin{bmatrix} 1 & 1 & 2 \\ -2 & -2 & -2 \\ -3 & -5 & -6 \end{bmatrix}$. Cofactors: $\begin{bmatrix} 1 & -1 & 2 \\ 2 & -2 & 2 \\ -3 & 5 & -6 \end{bmatrix}$. Inverse: $\dfrac{1}{2}\begin{bmatrix} 1 & \boxed{2} & \boxed{-3} \\ -1 & -2 & 5 \\ 2 & 2 & -6 \end{bmatrix}$.

b) $X = A^{-1}B = \begin{bmatrix} -3 \\ 4 \\ -4 \end{bmatrix}$.

Problem 4.

Q = top of the ladder: $\overrightarrow{OQ} = \langle 0, L\sin\theta\rangle$; R = bottom of the ladder: $\overrightarrow{OR} = \langle -L\cos\theta, 0\rangle$.
Midpoint: $\overrightarrow{OP} = \frac{1}{2}(\overrightarrow{OQ} + \overrightarrow{OR}) = \langle -\frac{L}{2}\cos\theta, \frac{L}{2}\sin\theta\rangle$.
Parametric equations: $x = -\frac{L}{2}\cos\theta$, $y = \frac{L}{2}\sin\theta$.

Problem 5.

a) $\overrightarrow{P_0P_1} \times \overrightarrow{P_0P_2} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -1 & -1 & 1 \\ 0 & -2 & 1 \end{vmatrix} = \hat{\imath} + \hat{\jmath} + 2\hat{k}$. Area $= \frac{1}{2}|\overrightarrow{P_0P_1} \times \overrightarrow{P_0P_2}| = \frac{1}{2}\sqrt{6}$.

b) Normal vector: $\overrightarrow{P_0P_1} \times \overrightarrow{P_0P_2} = \hat{\imath} + \hat{\jmath} + 2\hat{k}$. Equation: $x + y + 2z = 3$.

c) Parametric equations for the line: $x = -1 + t$, $y = t$, $z = t$.
Substituting: $-1 + 4t = 3$, $t = 1$, intersection point $(0, 1, 1)$.

Problem 6.

a) $\dfrac{d}{dt}(\vec{R} \cdot \vec{R}) = \vec{V} \cdot \vec{R} + \vec{R} \cdot \vec{V} = 2\vec{R} \cdot \vec{V}$.

b) Assume $|\vec{R}|$ is constant: then $\dfrac{d}{dt}(\vec{R} \cdot \vec{R}) = 2\vec{R} \cdot \vec{V} = 0$, i.e. $\vec{R} \perp \vec{V}$.

c) $\vec{R} \cdot \vec{V} = 0$, so $\dfrac{d}{dt}(\vec{R} \cdot \vec{V}) = \vec{V} \cdot \vec{V} + \vec{R} \cdot \vec{A} = 0$. Therefore $\vec{R} \cdot \vec{A} = -|\vec{V}|^2$.

18.02 Practice Exam 1B

Problem 1.

Let P, Q and R be the points at 1 on the x-axis, 2 on the y-axis and 3 on the z-axis, respectively.

a) (6) Express $\overrightarrow{QP}$ and $\overrightarrow{QR}$ in terms of $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$.

b) (9) Find the cosine of the angle PQR.

Problem 2. Let $P = (1,1,1)$, $Q = (0,3,1)$ and $R = (0,1,4)$.

a) (10) Find the area of the triangle PQR.

b) (5) Find the plane through P, Q and R, expressed in the form $ax + by + cz = d$.

c) (5) Is the line through $(1,2,3)$ and $(2,2,0)$ parallel to the plane in part (b)? Explain why or why not.

Problem 3. A ladybug is climbing on a Volkswagen Bug (= VW). In its starting position, the the surface of the VW is represented by the unit semicircle $x^2 + y^2 = 1$, $y \geq 0$ in the xy-plane. The road is represented as the x-axis. At time $t = 0$ the ladybug starts at the front bumper, $(1,0)$, and walks counterclockwise around the VW at unit speed relative to the VW. At the same time the VW moves to the right at speed 10.

a) (15) Find the parametric formula for the trajectory of the ladybug, and find its position when it reaches the rear bumper. (At $t = 0$, the rear bumper is at $(-1,0)$.)

b) (10) Compute the speed of the bug, and find where it is largest and smallest. Hint: It is easier to work with the square of the speed.

Problem 4.

$$M = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & -1 & -1 \end{pmatrix} \qquad M^{-1} = \frac{1}{12}\begin{pmatrix} 1 & 1 & 4 \\ a & 7 & -8 \\ b & -5 & 4 \end{pmatrix}$$

(a) (5) Compute the determinant of M.

b) (10) Find the numbers a and b in the formula for the matrix M^{-1}.

c) (10) Find the solution $\vec{r} = \langle x, y, z \rangle$ to $\begin{aligned} x + 2y + 3z &= 0 \\ 3x + 2y + z &= t \\ 2x - y - z &= 3 \end{aligned}$ as a function of t.

d) (5) Compute $\dfrac{d\vec{r}}{dt}$.

Problem 5.

(a) (5) Let $P(t)$ be a point with position vector $\vec{r}(t)$. Express the property that $P(t)$ lies on the plane $4x - 3y - 2z = 6$ in vector notation as an equation involving $\vec{r}$ and the normal vector to the plane.

(b) (5) By differentiating your answer to (a), show that $\dfrac{d\vec{r}}{dt}$ is perpendicular to the normal vector to the plane.

18.02 Practice Exam 1B Solutions

Problem 1.

a) $P = (1,0,0)$, $Q = (0,2,0)$ and $R = (0,0,3)$. Therefore $\overrightarrow{QP} = \hat{\imath} - 2\hat{\jmath}$ and $\overrightarrow{QR} = -2\hat{\jmath} + 3\hat{k}$.

b) $\cos\theta = \dfrac{\overrightarrow{QP} \cdot \overrightarrow{QR}}{\left|\overrightarrow{QP}\right| \left|\overrightarrow{QP}\right|} = \dfrac{\langle 1,-2,0\rangle \cdot \langle 0,-2,3\rangle}{\sqrt{1^2+2^2}\sqrt{2^2+3^2}} = \dfrac{4}{\sqrt{65}}$

Problem 2.

a) $\overrightarrow{PQ} = \langle -1,2,0\rangle$, $\overrightarrow{PR} = \langle -1,0,3\rangle$.

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -1 & 2 & 0 \\ -1 & 0 & 3 \end{vmatrix} = 6\hat{\imath} + 3\hat{\jmath} + 2\hat{k}.$$

Then $\quad area(\Delta) = \dfrac{1}{2}\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right| = \dfrac{1}{2}\sqrt{6^2+3^2+2^2} = \dfrac{1}{2}\sqrt{49} = \dfrac{7}{2}.$

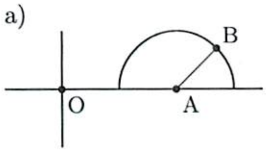
b) A normal to the plane is given by $\overrightarrow{N} = \overrightarrow{PQ} \times \overrightarrow{PR} = \langle 6,3,2\rangle$. Hence the equation has the form $6x+3y+2z = d$. Since P is on the plane $d = 6\cdot 1 + 3\cdot 1 + 2\cdot 1 = 11$. In conclusion the equation of the plane is

$$6x + 3y + 2z = 11.$$

c) The line is parallel to $\langle 2-1, 2-2, 0-3\rangle = \langle 1,0,-3\rangle$. Since $\overrightarrow{N} \cdot \langle 1,0,-3\rangle = 6-6 = 0$, the line is parallel to the plane.

Problem 3.

a)



$\overrightarrow{OA} = \langle 10t, 0\rangle$ and $\overrightarrow{AB} = \langle \cos t, \sin t\rangle$, hence

$$\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB} = \langle 10t + \cos t, \sin t\rangle.$$

The rear bumper is reached at time $t = \pi$ and the position of B is $(10\pi - 1, 0)$.

b) $\overrightarrow{V} = \langle 10 - \sin t, \cos t\rangle$, thus

$$|\overrightarrow{V}|^2 = (10-\sin t)^2 + \cos^2 t = 100 - 20\sin t + \sin^2 t + \cos^2 t = 101 - 20\sin t.$$

The speed is then given by $\sqrt{101 - 20\sin t}$. The speed is smallest when $\sin t$ is largest i.e. $\sin t = 1$. It occurs when $t = \pi/2$. At this time, the position of the bug is $(5\pi, 1)$. The speed is largest when $\sin t$ is smallest; that happens at the times $t = 0$ or π for which the position is then $(0,0)$ and $(10\pi - 1, 0)$.

Problem 4.

a) $|M| = -12$.

b) $a = -5$, $b = 7$.

c) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \dfrac{1}{12}\begin{bmatrix} 1 & 1 & 4 \\ -5 & 7 & -8 \\ 7 & -5 & 4 \end{bmatrix}\begin{bmatrix} 0 \\ t \\ 3 \end{bmatrix} = \begin{bmatrix} t/12 + 1 \\ 7t/12 - 2 \\ -5t/12 + 1 \end{bmatrix}$

d) $\dfrac{d\vec{r}}{dt} = \left\langle \dfrac{1}{12}, \dfrac{7}{12}, -\dfrac{5}{12}\right\rangle.$

Problem 5.

a) $\overrightarrow{N} \cdot \vec{r}(t) = 6$, where $\overrightarrow{N} = \langle 4,-3,-2\rangle$.

b) We differentiate $\overrightarrow{N} \cdot \vec{r}(t) = 6$:

$0 = \dfrac{d}{dt}\left(\overrightarrow{N} \cdot \vec{r}(t)\right) = \dfrac{d}{dt}\overrightarrow{N} \cdot \vec{r}(t) + \overrightarrow{N} \cdot \dfrac{d}{dt}\vec{r}(t) = \vec{0} \cdot \vec{r}(t) + \overrightarrow{N} \cdot \dfrac{d}{dt}\vec{r}(t)$ and hence $\overrightarrow{N} \perp \frac{d}{dt}\vec{r}(t)$.

18.02 Practice Exam 2 A

Problem 1. (10 points: 5, 5)

Let $f(x, y) = xy - x^4$.

a) Find the gradient of f at $P : (1, 1)$.

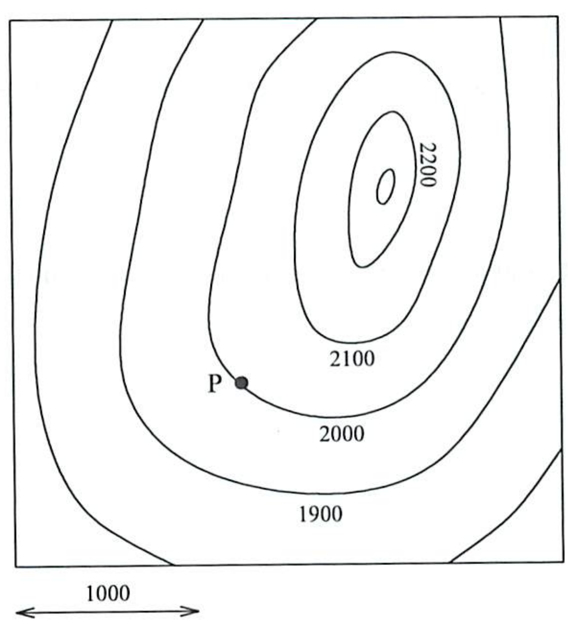
b) Give an approximate formula telling how small changes Δx and Δy produce a small change Δw in the value of $w = f(x, y)$ at the point $(x, y) = (1, 1)$.

Problem 2. (20 points)

On the topographical map below, the level curves for the height function $h(x, y)$ are marked (in feet); adjacent level curves represent a difference of 100 feet in height. A scale is given.

a) Estimate to the nearest .1 the value at the point P of the directional derivative $\left(\frac{dh}{ds}\right)_{\hat{u}}$, where $\hat{u}$ is the unit vector in the direction of $\hat{\imath} + \hat{\jmath}$.

b) Mark on the map a point Q at which $h = 2200$, $\frac{\partial h}{\partial x} = 0$ and $\frac{\partial h}{\partial y} < 0$. Estimate to the nearest .1 the value of $\frac{\partial h}{\partial y}$ at Q.

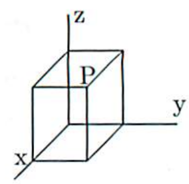


Problem 3. (10 points)

Find the equation of the tangent plane to the surface $x^3 y + z^2 = 3$ at the point $(-1, 1, 2)$.

Problem 4. (20 points: 5,5,5,5)

A rectangular box is placed in the first octant as shown, with one corner at the origin and the three adjacent faces in the coordinate planes. The opposite point $P : (x, y, z)$ is constrained to lie on the paraboloid $x^2 + y^2 + z = 1$. Which P gives the box of greatest volume?



a) Show that the problem leads one to maximize $f(x, y) = xy - x^3y - xy^3$, and write down the equations for the critical points of f.

b) Find a critical point of f which lies in the first quadrant ($x > 0$, $y > 0$).

c) Determine the nature of this critical point by using the second derivative test.

d) Find the maximum of f in the first quadrant (justify your answer).

Problem 5. (15 points)

In Problem 4 above, instead of substituting for z, one could also use Lagrange multipliers to maximize the volume $V = xyz$ with the same constraint $x^2 + y^2 + z = 1$.

a) Write down the Lagrange multiplier equations for this problem.

b) Solve the equations (still assuming $x > 0$, $y > 0$).

Problem 6. (10 points)

Let $w = f(u, v)$, where $u = xy$ and $v = x/y$. Using the chain rule, express $\dfrac{\partial w}{\partial x}$ and $\dfrac{\partial w}{\partial y}$ in terms of x, y, f_u and f_v.

Problem 7. (15 points)

Suppose that $x^2y + xz^2 = 5$, and let $w = x^3y$. Express $\left(\dfrac{\partial w}{\partial z}\right)_y$ as a function of x, y, z, and evaluate it numerically when $(x, y, z) = (1, 1, 2)$.

18.02 Practice Exam 2 A – Solutions

Problem 1.

a) $\nabla f = (y - 4x^3)\,\hat{\imath} + x\,\hat{\jmath}$; at P, $\nabla f = \langle -3, 1 \rangle$.

b) $\Delta w \simeq -3\,\Delta x + \Delta y$.

Problem 2.

a) By measuring, $\Delta h = 100$ for $\Delta s \simeq 500$, so $\left(\frac{dh}{ds}\right)_{\hat{u}} \simeq \frac{\Delta h}{\Delta s} \simeq .2$.

b) Q is the northernmost point on the curve $h = 2200$; the vertical distance between consecutive level curves is about $1/3$ of the given length unit, so $\frac{\partial h}{\partial y} \simeq \frac{\Delta h}{\Delta y} \simeq \frac{-100}{1000/3} \simeq -.3$.

Problem 3.

$f(x, y, z) = x^3y + z^2 = 3$: the normal vector is $\nabla f = \langle 3x^2y, x^3, 2z \rangle = \langle 3, -1, 4 \rangle$. The tangent plane is $3x - y + 4z = 4$.

Problem 4.

a) The volume is $xyz = xy(1 - x^2 - y^2) = xy - x^3y - xy^3$. Critical points: $f_x = y - 3x^2y - y^3 = 0$, $f_y = x - x^3 - 3xy^2 = 0$.

b) Assuming $x > 0$ and $y > 0$, the equations can be rewritten as $1 - 3x^2 - y^2 = 0$, $1 - x^2 - 3y^2 = 0$. Solution: $x^2 = y^2 = 1/4$, i.e. $(x, y) = (1/2, 1/2)$.

c) $f_{xx} = -6xy = -3/2$, $f_{yy} = -6xy = -3/2$, $f_{xy} = 1 - 3x^2 - 3y^2 = -1/2$. So $f_{xx}f_{yy} - f_{xy}^2 > 0$, and $f_{xx} < 0$, it is a local maximum.

d) The maximum of f lies either at $(1/2, 1/2)$, or on the boundary of the domain or at infinity. Since $f(x, y) = xy(1 - x^2 - y^2)$, $f = 0$ when either $x \to 0$ or $y \to 0$, and $f \to -\infty$ when $x \to \infty$ or $y \to \infty$ (since $x^2 + y^2 \to \infty$). So the maximum is at $(x, y) = (\frac{1}{2}, \frac{1}{2})$, where $f(\frac{1}{2}, \frac{1}{2}) = \frac{1}{8}$.

Problem 5.

a) $f(x, y, z) = xyz$, $g(x, y, z) = x^2 + y^2 + z = 1$: one must solve the Lagrange multiplier equation $\nabla f = \lambda \nabla g$, i.e. $yz = 2\lambda x$, $xz = 2\lambda y$, $xy = \lambda$, and the constraint equation $x^2 + y^2 + z = 1$.

b) Dividing the first two equations $yz = 2\lambda x$ and $xz = 2\lambda y$ by each other, we get $y/x = x/y$, so $x^2 = y^2$; since $x > 0$ and $y > 0$ we get $y = x$. Substituting this into the Lagrange multiplier equations, we get $z = 2\lambda$ and $x^2 = \lambda$. Hence $z = 2x^2$, and the constraint equation becomes $4x^2 = 1$, so $x = \frac{1}{2}$, $y = \frac{1}{2}$, $z = \frac{1}{2}$.

Problem 6.

$\frac{\partial w}{\partial x} = f_u u_x + f_v v_x = y\, f_u + \frac{1}{y}\, f_v$. $\frac{\partial w}{\partial y} = f_u u_y + f_v v_y = x\, f_u - \frac{x}{y^2}\, f_v$.

Problem 7.

Using the chain rule: $\left(\frac{\partial w}{\partial z}\right)_y = \frac{\partial w}{\partial x}\left(\frac{\partial x}{\partial z}\right)_y = 3x^2y\left(\frac{\partial x}{\partial z}\right)_y$. To find $\left(\frac{\partial x}{\partial z}\right)_y$, differentiate the relation $x^2y + xz^2 = 5$ w.r.t. z holding y constant: $(2xy + z^2)\left(\frac{\partial x}{\partial z}\right)_y + 2xz = 0$, so $\left(\frac{\partial x}{\partial z}\right)_y = \frac{-2xz}{2xy + z^2}$. Therefore $\left(\frac{\partial w}{\partial z}\right)_y = \frac{-6x^3yz}{2xy + z^2}$. At $(x, y, z) = (1, 1, 2)$ this is equal to -2.

18.02 Practice Exam 2 B

Problem 1. Let $f(x,y) = x^2y^2 - x$.

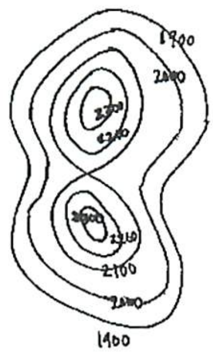
a) (5) Find ∇f at $(2,1)$

b) (5) Write the equation for the tangent plane to the graph of f at $(2,1,2)$.

c) (5) Use a linear approximation to find the approximate value of $f(1.9, 1.1)$.

d) (5) Find the directional derivative of f at $(2,1)$ in the direction of $-\hat{\imath} + \hat{\jmath}$.

Problem 2. (10) On the contour plot below, mark the portion of the level curve $f = 2000$ on which $\dfrac{\partial f}{\partial y} \geq 0$.



Problem 3. a) (10) Find the critical points of

$$w = -3x^2 - 4xy - y^2 - 12y + 16x$$

and say what type each critical point is.

b) (10) Find the point of the first quadrant $x \geq 0$, $y \geq 0$ at which w is largest. Justify your answer.

Problem 4. Let $u = y/x$, $v = x^2 + y^2$, $w = w(u,v)$.

a) (10) Express the partial derivatives w_x and w_y in terms of w_u and w_v (and x and y).

b) (7) Express $xw_x + yw_y$ in terms of w_u and w_v. Write the coefficients as functions of u and v.

c) (3) Find $xw_x + yw_y$ in case $w = v^5$.

Problem 5. a) (10) Find the Lagrange multiplier equations for the point of the surface

$$x^4 + y^4 + z^4 + xy + yz + zx = 6$$

at which x is largest. (Do not solve.)

b) (5) Given that x is largest at the point (x_0, y_0, z_0), find the equation for the tangent plane to the surface at that point.

Problem 6. Suppose that $x^2 + y^3 - z^4 = 1$ and $z^3 + zx + xy = 3$.

a) (8) Take the total differential of each of these equations.

b) (7) The two surfaces in part (a) intersect in a curve along which y is a function of x. Find dy/dx at $(x,y,z) = (1,1,1)$.

18.02 Practice Exam 2 B – Solutions

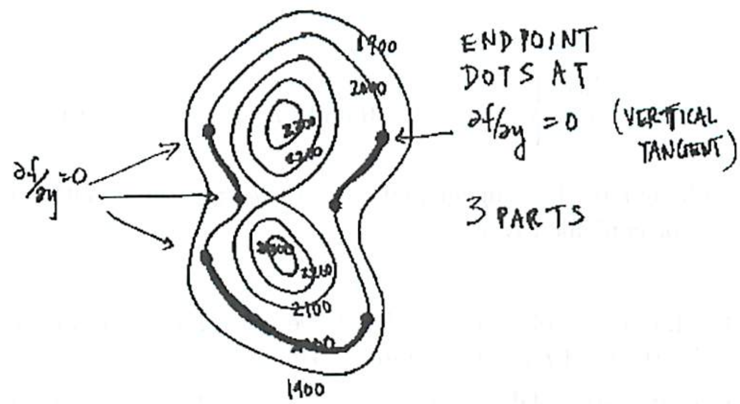
Problem 1. a) $\nabla f = \langle 2xy^2 - 1, 2x^2y\rangle = \langle 3, 8\rangle = 3\hat{\imath} - 8\hat{\jmath}$.

b) $z - 2 = 3(x-2) + 8(y-1)$ or $z = 3x + 8y - 12$.

c) $\Delta x = 1.9 - 2 = -1/10$ and $\Delta y = 1.1 - 1 = 1/10$. So $z \simeq 2 + 3\Delta x + 8\Delta y = 2 - 3/10 + 8/10 = 2.5$

d) $\left.\dfrac{df}{ds}\right|_{\hat{u}} = \nabla f \cdot \hat{u} = \langle 3, 8\rangle \cdot \dfrac{\langle -1, 1\rangle}{\sqrt{2}} = \dfrac{-3+8}{\sqrt{2}} = \dfrac{5}{\sqrt{2}}$

Problem 2.



Problem 3. a) $\begin{array}{lll} w_x = -6x - 4y + 16 = 0 & \Rightarrow & -3x - 2y + 8 = 0 \\ w_y = -4x - 2y - 12 = 0 & \Rightarrow & 4x + 2y + 12 = 0 \end{array} \Bigg\} \Rightarrow \Bigg\{ \begin{array}{l} x = -20 \\ y = 34 \end{array}$

Therefore there is just one critical point at $(-20, 34)$. Since

$$w_{xx}w_{yy} - w_{xy}^2 = (-6)(-2) - (-4)^2 = 12 - 16 = -4 < 0,$$

the critical point is a saddle point.

b) There is no critical point in the first quadrant, hence the maximum must be at infinity or on the boundary of the first quadrant.

The boundary is composed of two half-lines:

- $x = 0$ and $y \geq 0$ on which $w = -y^2 - 12y$. It has a maximum ($w = 0$) at $y = 0$.
- $y = 0$ and $x \geq 0$, where $w = -3x^2 + 16x$. (The graph is a parabola pointing downwards). Maximum: $w_x = -6x + 16 = 0 \quad \Rightarrow x = 8/3$. Hence w has a maximum at $(8/3, 0)$ and $w = -3(8/3)^2 + 16 \cdot 8/3 = 64/3 > 0$.

We now check that the maximum of w is not at infinity:

- If $y \geq 0$ and $x \to +\infty$ then $w \leq -3x^2 + 16x$, which tends to $-\infty$ as $x \to +\infty$.
- If $0 \leq x \leq C$ and $y \to +\infty$, then $w \leq -y^2 + 16C$, which tends to $-\infty$ as $y \to +\infty$.

We conclude that the maximum of w in the first quadrant is at $(8/3, 0)$.

Problem 4. a) $\left\{ \begin{array}{llll} w_x & = u_x w_u + v_x w_v & = & -\frac{y}{x^2} w_u + 2x w_v \\ w_y & = u_y w_u + v_y w_v & = & \frac{1}{x} w_u + 2y w_v \end{array} \right.$

b) $x w_x + y w_y = x(-\dfrac{y}{x^2} w_u + 2x w_v) + y(\dfrac{1}{x} w_u + 2y w_v) = (-\dfrac{y}{x} + \dfrac{y}{x}) w_u + (2x^2 + 2y^2) w_v = 2v w_v.$

c) $x w_x + y w_y = 2v w_v = 2v \cdot 5v^4 = 10v^5$.

Problem 5. a) $f(x,y,z) = x$; the constraint is $g(x,y,z) = x^4 + y^4 + z^4 + xy + yz + zx = 6$. The Lagrange multiplier equation is:

$$\nabla f = \lambda \nabla g \quad \Leftrightarrow \quad \begin{cases} 1 &= \lambda(4x^3 + y + z) \\ 0 &= \lambda(4y^3 + x + z) \\ 0 &= \lambda(4z^3 + x + y) \end{cases}$$

b) The level surfaces of f and g are tangent at (x_0, y_0, z_0), so they have the same tangent plane. The level surface of f is the plane $x = x_0$; hence this is also the tangent plane to the surface $g = 6$ at (x_0, y_0, z_0).

Second method: at (x_0, y_0, z_0), we have

$$\left.\begin{aligned} 1 &= \lambda g_x \\ 0 &= \lambda g_y \\ 0 &= \lambda g_z \end{aligned}\right\} \quad \Rightarrow \quad \lambda \neq 0 \text{ and } \langle g_x, g_y, g_z \rangle = \langle \frac{1}{\lambda}, 0, 0 \rangle.$$

So $\langle \frac{1}{\lambda}, 0, 0 \rangle$ is perpendicular to the tangent plane at (x_0, y_0, z_0); the equation of the tangent plane is then $\frac{1}{\lambda}(x - x_0) = 0$, or equivalently $x = x_0$.

Problem 6.

a) Taking the total differential of $x^2 + y^3 - z^4 = 1$, we get: $2x\,dx + 3y^2\,dy - 4z^3\,dz = 0$. Similarly, from $z^3 + zx + xy = 3$, we get: $(y + z)\,dx + x\,dy + (3z^2 + x)\,dz = 0$.

b) At $(1,1,1)$ we have: $2\,dx + 3\,dy - 4\,dz = 0$ and $2\,dx + dy + 4\,dz = 0$. We eliminate dz (by adding these two equations): $4\,dx + 4\,dy = 0$, so $dy = -dx$, and hence $dy/dx = -1$.

18.02 Practice Exam 3 A

1. Let $(\bar{x}, \bar{y})$ be the center of mass of the triangle with vertices at $(-2,0)$, $(0,1)$, $(2,0)$ and uniform density $\delta = 1$.

a) (10) Write an integral formula for $\bar{y}$. Do not evaluate the integral(s), but write explicitly the integrand and limits of integration.

b) (5) Find $\bar{x}$.

2. (15) Find the polar moment of inertia of the unit disk with density equal to the distance from the y-axis.

3. Let $\vec{F} = (ax^2y + y^3 + 1)\hat{\imath} + (2x^3 + bxy^2 + 2)\hat{\jmath}$ be a vector field, where a and b are constants.

a) (5) Find the values of a and b for which $\vec{F}$ is conservative.

b) (5) For these values of a and b, find $f(x,y)$ such that $\vec{F} = \nabla f$.

c) (5) Still using the values of a and b from part (a), compute $\displaystyle\int_C \vec{F} \cdot d\vec{r}$ along the curve C such that $x = e^t \cos t$, $y = e^t \sin t$, $0 \le t \le \pi$.

4. (10) For $\vec{F} = yx^3\hat{\imath} + y^2\hat{\jmath}$, find $\int_C \vec{F} \cdot d\vec{r}$ on the portion of the curve $y = x^2$ from $(0,0)$ to $(1,1)$.

5. Consider the region R in the first quadrant bounded by the curves $y = x^2$, $y = x^2/5$, $xy = 2$, and $xy = 4$.

a) (10) Compute $dxdy$ in terms of $dudv$ if $u = x^2/y$ and $v = xy$.

b) (10) Find a double integral for the area of R in uv coordinates and evaluate it.

6. a) (5) Let C be a simple closed curve going counterclockwise around a region R. Let $M = M(x,y)$. Express $\displaystyle\oint_C M dx$ as a double integral over R.

b) (5) Find M so that $\displaystyle\oint_C M dx$ is the mass of R with density $\delta(x,y) = (x+y)^2$.

7. Consider the region R enclosed by the x-axis, $x = 1$ and $y = x^3$.

a) (5) Use the normal form of Green's theorem to find the flux of $\vec{F} = (1 + y^2)\hat{\jmath}$ out of R.

b) (5) Find the flux out of R through the two sides C_1 (the horizontal segment) and C_2 (the vertical segment).

c) (5) Use parts (a) and (b) to find the flux out of the third side C_3.

18.02 Practice Exam 3 A – Solutions

1. a) The area of the triangle is 2, so $\bar{y} = \frac{1}{2}\int_0^1 \int_{2y-2}^{2-2y} y\, dxdy$.

b) By symmetry $\bar{x} = 0$.

2. $\delta = |x| = r|\cos\theta|$. $I_0 = \iint_D r^2\,\delta\, r drd\theta =$

$$\int_0^{2\pi}\int_0^1 r^2|r\cos\theta| rdrd\theta = 4\int_0^{\pi/2}\int_0^1 r^4\cos\theta drd\theta = 4\int_0^{\pi/2}\frac{1}{5}\cos\theta d\theta = \frac{4}{5}$$

3. a) $N_x = 6x^2 + by^2$, $M_y = ax^2 + 3y^2$. $N_x = M_y$ provided $a = 6$ and $b = 3$.

b) $f_x = 6x^2y + y^3 + 1 \implies f = 2x^3y + xy^3 + x + c(y)$. Therefore, $f_y = 2x^3 + 3xy^2 + c'(y)$. Setting this equal to N, we have $2x^3 + 3xy^2 + c'(y) = 2x^3 + 3xy^2 + 2$ so $c'(y) = 2$ and $c = 2y$. So

$$f = 2x^3y + xy^3 + x + 2y \quad (+\text{constant}).$$

c) C starts at $(1,0)$ and ends at $(-e^{\pi}, 0)$, so $\int_C \vec{F}\cdot d\vec{r} = f(-e^{\pi},0) - f(1,0) = -e^{-\pi} - 1$.

4. $\int_C yx^3dx + y^2dy = \int_0^1 x^2x^3dx + (x^2)^2(2xdx) = \int_0^1 3x^5dx = 1/2.$

5. a) $\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 2x/y & -x^2/y^2 \\ y & x \end{vmatrix} = 3x^2/y$. Therefore,

$$dudv = (3x^2/y)dxdy = 3u\; dxdy \quad\implies\quad dxdy = \frac{1}{3u}dudv.$$

b) $\int_2^4\int_1^5 \frac{1}{3u}dudv = \int_2^4 \frac{1}{3}\ln 5\; dv = \frac{2}{3}\ln 5.$

6. a) $\oint_C Mdx = \iint_R -M_y\, dA.$

b) We want M such that $-M_y = (x+y)^2$. Use $M = -\frac{1}{3}(x+y)^3$.

7. a) $\operatorname{div}\vec{F} = 2y$, so $\iint_R 2y\, dA = \int_0^1\int_0^{x^3} 2y\; dydx = \int_0^1 x^6 dx = \frac{1}{7}$.

b) For the flux through C_1, $\hat{\mathbf{n}} = -\hat{\mathbf{j}}$ implies $\vec{F}\cdot\hat{\mathbf{n}} = -(1+y^2) = -1$ where $y = 0$. The length of C_1 is 1, so the total flux through C_1 is -1.

The flux through C_2 is zero because $\hat{\mathbf{n}} = \hat{\mathbf{i}}$ and $\vec{F} \perp \hat{\mathbf{i}}$.

c) $\int_{C_3}\vec{F}\cdot\hat{\mathbf{n}}ds = \iint_R \operatorname{div}\vec{F}dA - \int_{C_1}\vec{F}\cdot\hat{\mathbf{n}}ds - \int_{C_2}\vec{F}\cdot\hat{\mathbf{n}}ds = \frac{1}{7} - (-1) - 0 = \frac{8}{7}.$

18.02 Practice Exam 3 B

Problem 1. a) Draw a picture of the region of integration of $\int_0^1 \int_x^{2x} dydx$.

b) Exchange the order of integration to express the integral in part (a) in terms of integration in the order $dxdy$. Warning: your answer will have two pieces.

Problem 2. a) Find the mass M of the upper half of the annulus $1 < x^2 + y^2 < 9$ $(y \geq 0)$ with density $\delta = \dfrac{y}{x^2 + y^2}$.

b) Express the x-coordinate of the center of mass, $\bar{x}$, as an iterated integral. (Write explicitly the integrand and limits of integration.) Without evaluating the integral, explain why $\bar{x} = 0$.

Problem 3. a) Show that $\mathbf{F} = (3x^2 - 6y^2)\hat{\imath} + (-12xy + 4y)\hat{\jmath}$ is conservative.

b) Find a potential function for $\mathbf{F}$.

c) Let C be the curve $x = 1 + y^3(1 - y)^3$, $0 \leq y \leq 1$. Calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$.

Problem 4. a) Express the work done by the force field $\mathbf{F} = (5x+3y)\hat{\imath}+(1+\cos y)\hat{\jmath}$ on a particle moving counterclockwise once around the unit circle centered at the origin in the form $\int_a^b f(t)dt$. (Do not evaluate the integral; don't even simplify $f(t)$.)

b) Evaluate the line integral using Green's theorem.

Problem 5. Consider the rectangle R with vertices $(0,0)$, $(1,0)$, $(1,4)$ and $(0,4)$. The boundary of R is the curve C, consisting of C_1, the segment from $(0,0)$ to $(1,0)$, C_2, the segment from $(1,0)$ to $(1,4)$, C_3 the segment from $(1,4)$ to $(0,4)$ and C_4 the segment from $(0,4)$ to $(0,0)$. Consider the vector field

$$\mathbf{F} = (xy + \sin x \cos y)\hat{\imath} - (\cos x \sin y)\hat{\jmath}$$

a) Find the flux of $\mathbf{F}$ out of R through C. Show your reasoning.

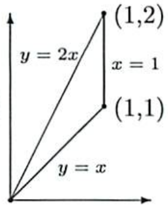
b) Is the total flux out of R through C_1, C_2 and C_3, more than, less than or equal to the flux out of R through C? Show your reasoning.

Problem 6. Find the volume of the region enclosed by the plane $z = 4$ and the surface

$$z = (2x - y)^2 + (x + y - 1)^2.$$

(Suggestion: change of variables.)

18.02 Practice Exam 3 B – Solutions

1. a) 

b) $\displaystyle\int_0^1\int_{y/2}^{y} dx\,dy + \int_1^2\int_{y/2}^{1} dx\,dy.$

(the first integral corresponds to the bottom half $0 \le y \le 1$, the second integral to the top half $1 \le y \le 2$.)

2. a) $\delta dA = \dfrac{r\sin\theta}{r^2} r\,dr\,d\theta = \sin\theta\,dr\,d\theta.$

$$M = \iint_R \delta\,dA = \int_0^\pi\int_1^3 \sin\theta\ dr\,d\theta = \int_0^\pi 2\sin\theta\,d\theta = \big[-2\cos\theta\big]_0^\pi = 4.$$

b) $\bar{x} = \dfrac{1}{M}\displaystyle\iint_R x\delta\,dA = \frac{1}{4}\int_0^\pi\int_1^3 r\cos\theta\sin\theta\,dr\,d\theta$

The reason why one knows that $\bar{x} = 0$ without computation is that the region **and the density** are symmetric with respect to the y-axis ($\delta(x,y) = \delta(-x,y)$).

3. a) $N_x = -12y = M_y$, hence $\mathbf{F}$ is conservative.

b) $f_x = 3x^2 - 6y^2 \Rightarrow f = x^3 - 6y^2x + c(y) \Rightarrow f_y = -12xy + c'(y) = -12xy + 4y$. So $c'(y) = 4y$, thus $c(y) = 2y^2$ (+ constant). In conclusion

$$f = x^3 - 6xy^2 + 2y^2 \quad (+\text{ constant}).$$

c) The curve C starts at $(1,0)$ and ends at $(1,1)$, therefore

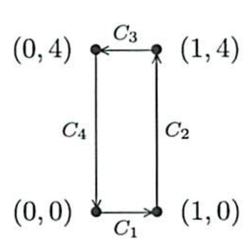
$$\int_C \mathbf{F}\cdot d\mathbf{r} = f(1,1) - f(1,0) = (1 - 6 + 2) - 1 = -4.$$

4. a) The parametrization of the circle C is $x = \cos t$, $y = \sin t$, for $0 \le t < 2\pi$; then $dx = -\sin t\,dt$, $dy = \cos t\,dt$ and

$$W = \int_0^{2\pi} (5\cos t + 3\sin t)(-\sin t)dt + (1 + \cos(\sin t))\cos t\,dt.$$

b) Let R be the unit disc inside C;

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_R (N_x - M_y)dA = \iint_R (0-3)dA = -3\ \text{area}(R) = -3\pi.$$

5. a) 

$$\oint_C \mathbf{F}\cdot\hat{\mathbf{n}}\,ds = \iint_R \operatorname{div}\mathbf{F}\,dx\,dy$$
$$= \iint_R (y + \cos x\cos y - \cos x\cos y)dx\,dy = \iint_R y\,dx\,dy$$
$$= \int_0^4\int_0^1 y\,dx\,dy = \int_0^4 y\,dy = [y^2/2]_0^4 = 8.$$

b) On C_4, $x = 0$, so $\mathbf{F} = -\sin y\,\hat{\mathbf{j}}$, whereas $\hat{\mathbf{n}} = -\hat{\mathbf{i}}$. Hence $\mathbf{F}\cdot\hat{\mathbf{n}} = 0$. Therefore the flux of $\mathbf{F}$ through C_4 equals 0. Thus

$$\int_{C_1+C_2+C_3} \mathbf{F}\cdot\hat{\mathbf{n}}\,ds = \oint_C \mathbf{F}\cdot\hat{\mathbf{n}}\,ds - \int_{C_4} \mathbf{F}\cdot\hat{\mathbf{n}}\,ds = \oint_C \mathbf{F}\cdot\hat{\mathbf{n}}\,ds\ ;$$

and the total flux through $C_1 + C_2 + C_3$ is equal to the flux through C.

6. Let $u = 2x - y$ and $v = x + y - 1$. The Jacobian $\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} = 3.$

Hence $dudv = 3dxdy$ and $dxdy = \frac{1}{3}dudv$, so that

$$\begin{aligned} V &= \iint\limits_{(2x-y)^2+(x+y-1)^2<4} (4 - (2x-y)^2 - (x+y-1)^2)\,dxdy \\ &= \iint\limits_{u^2+v^2<4} (4 - u^2 - v^2)\frac{1}{3}dudv \\ &= \int_0^{2\pi}\int_0^2 (4 - r^2)\frac{1}{3}rdrd\theta = \int_0^{2\pi}\left[\frac{2}{3}r^2 - \frac{1}{12}r^4\right]_0^2 d\theta \\ &= \int_0^{2\pi}\frac{4}{3}d\theta = \frac{8\pi}{3}. \end{aligned}$$

18.02 – Practice Exam 4A

Problem 1. (15)

a) Show that the vector field

$$\mathbf{F} = \langle e^x yz, e^x z + 2yz, e^x y + y^2 + 1 \rangle$$

is conservative.

b) By a systematic method, find a potential for $\mathbf{F}$.

c) Show that the vector field $\mathbf{G} = \langle y, x, y \rangle$ is not conservative.

Problem 2. (20) Let S be the part of the spherical surface $x^2 + y^2 + z^2 = 4$, lying in $x^2 + y^2 > 1$, which is to say outside the cylinder of radius one with axis the z-axis.

a) Compute the flux outward through S of the vector field $\mathbf{F} = y\mathbf{i} - x\mathbf{j} + z\mathbf{k}$.

b) Show that the flux of this vector field through any part of the cylindrical surface is zero.

c) Using the divergence theorem applied to $\mathbf{F}$, compute the volume of the region between S and the cylinder.

Problem 3. (20) Let S be the part of the spherical surface $x^2 + y^2 + z^2 = 2$ lying in $z > 1$. Orient S upwards and give its bounding circle, C, lying in $z = 1$ the compatible orientation.

a) Parametrize C and use the parametrization to evaluate the line integral

$$I = \oint_C xz\,dx + y\,dy + y\,dz.$$

b) Compute the curl of the vector field $\mathbf{F} = xz\mathbf{i} + y\mathbf{j} + y\mathbf{k}$.

c) Write down a flux integral through S which can be computed using the value of I.

Problem 4. (15) Use the divergence theorem to compute the flux of $\mathbf{F} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ outwards across the closed surface $x^4 + y^4 + z^4 = 1$.

Problem 5. (15) Consider the surface S given by the equation

$$z = (x^2 + y^2 + z^2)^2.$$

a) Show that S lies in the upper half space ($z \geq 0$).

b) Write out the equation for the surface in spherical polar coordinates.

c) Using the equation obtained in part b), give an iterated integral, with explicit integrand and limits of integration, which gives the volume of the region inside this surface. Do not evaluate the integral.

Problem 6. (15) Let S be the part of the surface $z = xy$ where $x^2 + y^2 < 1$. Compute the flux of $\mathbf{F} = y\mathbf{i} + x\mathbf{j} + z\mathbf{k}$ upward across S by reducing the surface integral to a double integral over the disk $x^2 + y^2 < 1$.

18.02 Practice Exam 4A - Solutions

1a)

$$M_y = e^x z = N_x$$
$$M_z = e^x y = P_x$$
$$N_z = e^x + 2y = P_y$$

1b) We begin with

$$\begin{cases} f_x = e^x yz \\ f_y = e^x z + 2yz \\ f_z = e^x y + y^2 + 1 \end{cases}$$

Integrating f_x we get $f = e^x yz + g(y, z)$. Differentiating and comparing with the above equations we get

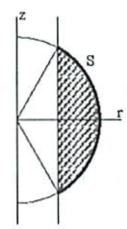
$$\begin{cases} f_y = e^x z + g_y \\ f_z = e^x y + g_z \end{cases} \rightarrow \begin{cases} g_y = 2yz \\ g_z = y^2 + 1 \end{cases}$$

Integrating g_y we get $g = y^2 z + h(z)$. Then $g_z = y^2 + h'(z)$ so comparing with the second equation above we get $h'(z) = 1$. Hence $h = z + C$. Putting everything together we get

$$f = e^x yz + y^2 z + z + C$$

1c) $N_z = 0$ and $P_y = 1$ hence the field is not conservative.

2a) Consider the figure



$\vec{n} = \frac{1}{2}(x, y, z)$ hence

$$\vec{F} \cdot \vec{n} = (y, -x, z) \cdot \frac{(x, y, z)}{2} = \frac{z^2}{2}$$

$z = 2\cos\phi$ and $dS = 2^2 \sin\phi \, d\phi \, d\theta$ hence we get

$$\int_0^{2\pi} \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \frac{4\cos^2\phi}{2} 4\sin\phi \, d\phi \, d\theta = 16\pi \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \cos^2\phi \sin\phi d\phi = -16\pi \left[\frac{\cos^3\phi}{3}\right]_{\frac{\pi}{6}}^{\frac{5\pi}{6}} = 4\sqrt{3}\pi$$

2b) $\vec{n} = \pm(x, y, 0)$ hence $\vec{F} \cdot \vec{n} = 0$. So the flux is 0.

2c) $div\vec{F} = 1$ hence

$$Vol(R) = \iiint_R 1 \, dV = \iiint_R div\vec{F} \, dV = \iint_S \vec{F} \cdot \vec{n} \, dS + \iint_{\text{Cylinder}} \vec{F} \cdot \vec{n} \, dS = 4\sqrt{3}\pi$$

3a) C is given by the equations $x^2 + y^2 + z^2 = 2$ and $z = 1$. So $x^2 + y^2 = 1$. Parametrization:

$$x = \cos t \qquad y = \sin t \qquad z = 1$$
$$dx = -\sin t \, dt \qquad dy = \cos t \, dt \qquad dz = 0$$

So

$$I = \int_0^{2\pi} (-\cos t \sin t + \sin t \cos t) dt = 0$$

3b)

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ xz & y & y \end{vmatrix} = \hat{\imath} + x\hat{\jmath}$$

3c) By Stokes theorem

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \vec{n} dS$$

$\vec{n}$ is the normal pointing upward hence

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S (1, x, 0) \cdot \frac{(x, y, z)}{\sqrt{2}} dS = \iint_S \frac{x + xy}{\sqrt{2}} dS$$

4) $div\vec{F} = 0$ hence

$$\iint_S \vec{F} \cdot \vec{n} \, dS = \iiint_R div\vec{F} \, dV = 0$$

5a)

$$z = (x^2 + y^2 + z^2)^2 \geq 0$$

5b) $z = \rho \cos\phi$ and $x^2 + y^2 + z^2 = \rho^2$ hence $\rho \cos\phi = \rho^4$. Canceling ρ we get $\cos\phi = \rho^3$.

5c)

$$\int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^{(\cos\phi)^{\frac{1}{3}}} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

6) The flux is upward so

$$\vec{n} \, dS = +(-f_x, -f_y, 1) \, dx \, dy = (-y, -x, 1) \, dx \, dy$$

$(f = xy)$. Hence

$$\iint_S \vec{F} \cdot \vec{n} \, dS = \iint_{x^2+y^2<1} (y, x, z) \cdot (-y, -x, 1) \, dx \, dy = \iint_{x^2+y^2<1} (-y^2 - x^2 + xy) \, dx \, dy$$

where we substituted $z = xy$. Using polar coordinates we get

$$\int_0^{2\pi} \int_0^1 (-r^2 + r^2 \cos\theta \sin\theta) r dr d\theta$$

- Inner: $\int_0^1 (-r^2 + r^2 \cos\theta \sin\theta) r dr = \frac{1}{4}(\cos\theta \sin\theta - 1)$
- Outer: $\int_0^{2\pi} \frac{1}{4}(\cos\theta \sin\theta - 1) d\theta = \frac{1}{4}\left[\frac{\sin^2\theta}{2} - \theta\right]_0^{2\pi} = -\frac{\pi}{2}$

18.02 Practice Exam 4B

Problem 1. (10 points)

Let C be the portion of the cylinder $x^2 + y^2 \leq 1$ lying in the first octant ($x \geq 0$, $y \geq 0$, $z \geq 0$) and below the plane $z = 1$. Set up a triple integral in *cylindrical coordinates* which gives the moment of inertia of C about the z-axis; assume the density to be $\delta = 1$.

(Give integrand and limits of integration, but *do not evaluate.*)

Problem 2. (20 points: 5, 5, 10)

a) A solid sphere S of radius a is placed above the xy-plane so it is tangent at the origin and its diameter lies along the z-axis. Give its equation in *spherical coordinates.*

b) Give the equation of the horizontal plane $z = a$ in spherical coordinates.

c) Set up a triple integral in spherical coordinates which gives the volume of the portion of the sphere S lying *above* the plane $z = a$. (Give integrand and limits of integration, but *do not evaluate.*)

Problem 3. (20 points: 5, 15)

Let $\vec{F} = (2xy + z^3)\,\hat{\imath} + (x^2 + 2yz)\,\hat{\jmath} + (y^2 + 3xz^2 - 1)\,\hat{k}$.

a) Show that $\vec{F}$ is conservative.

b) Using a systematic method, find a potential function $f(x, y, z)$ such that $\vec{F} = \vec{\nabla} f$. Show your work, even if you can do it mentally.

Problem 4. (25 points: 15, 10)

Let S be the surface formed by the part of the paraboloid $z = 1 - x^2 - y^2$ lying above the xy-plane, and let $\vec{F} = x\,\hat{\imath} + y\,\hat{\jmath} + 2(1 - z)\,\hat{k}$.

Calculate the flux of $\vec{F}$ across S, taking the upward direction as the one for which the flux is positive. Do this in two ways:

a) by direct calculation of $\iint_S \vec{F} \cdot \hat{n}\, dS$;

b) by computing the flux of $\vec{F}$ across a simpler surface and using the divergence theorem.

Problem 5. (25 points: 10, 8, 7)

Let $\vec{F} = -2xz\,\hat{\imath} + y^2\,\hat{k}$.

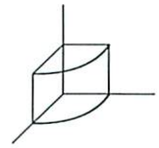
a) Calculate $\operatorname{curl} \vec{F}$.

b) Show that $\iint_R \operatorname{curl} \vec{F} \cdot \hat{n}\, dS = 0$ for any finite portion R of the unit sphere $x^2 + y^2 + z^2 = 1$. (take the normal vector $\hat{n}$ pointing outward).

c) Show that $\oint_C \vec{F} \cdot d\vec{r} = 0$ for any simple closed curve C on the unit sphere $x^2 + y^2 + z^2 = 1$.

18.02 Practice Exam 4B – Solutions

Problem 1.



$$\int_0^{\pi/2}\int_0^1\int_0^1 r^2\, r\, dz\, dr\, d\theta.$$

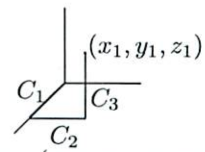
Problem 2.

a) sphere: $\rho = 2a\cos\phi$. b) plane: $\rho = a\sec\phi$.

c) $\displaystyle\int_0^{2\pi}\int_0^{\pi/4}\int_{a\sec\phi}^{2a\cos\phi} \rho^2\sin\phi\, d\rho\, d\phi\, d\theta.$

Problem 3.

a) $\frac{\partial}{\partial y}(2xy+z^3) = 2x = \frac{\partial}{\partial x}(x^2+2yz)$; $\frac{\partial}{\partial z}(2xy+z^3) = 3z^2 = \frac{\partial}{\partial x}(y^2+3xz^2-1)$;
$\frac{\partial}{\partial z}(x^2+2yz) = 2y = \frac{\partial}{\partial y}(y^2+3xz^2-1)$; so $\vec F$ is conservative.



b) <u>Method 1:</u> $f(x,y,z) = \int_{C_1+C_2+C_3}\vec F\cdot d\vec r$;
$\int_{C_1}\vec F\cdot d\vec r = \int_0^{x_1}(2xy+z^3)\,dx = \int_0^{x_1}0\,dx = 0 \quad (y=0,\ z=0)$
$\int_{C_2}\vec F\cdot d\vec r = \int_0^{y_1}(x^2+2yz)\,dy = \int_0^{y_1}x_1^2\,dy = x_1^2y_1 \quad (x=x_1,\ z=0)$
$\int_{C_3}\vec F\cdot d\vec r = \int_0^{z_1}(y^2+3xz^2-1)\,dz = \int_0^{z_1}(y_1^2+3x_1z^2-1)\,dz = y_1^2z_1+x_1z_1^3-z_1 \quad (x=x_1,\ y=y_1)$
So $f(x,y,z) = x^2y+y^2z+xz^3-z+c$.

<u>Method 2:</u> $\frac{\partial f}{\partial x} = 2xy+z^3$, so $f(x,y,z) = x^2y+xz^3+g(y,z)$.
$\frac{\partial f}{\partial y} = x^2+\frac{\partial g}{\partial y} = x^2+2yz$, so $\frac{\partial g}{\partial y} = 2yz$.
Therefore $g(y,z) = y^2z+h(z)$, and $f(x,y,z) = x^2y+xz^3+y^2z+h(z)$.
$\frac{\partial f}{\partial z} = 3xz^2+y^2+h'(z) = y^2+3xz^2-1$, so $h'(z) = -1$.
Therefore $h(z) = -z+c$, and $f(x,y,z) = x^2y+xz^3+y^2z-z+c$.

Problem 4.

a) S is the graph of $z = f(x,y) = 1-x^2-y^2$, so $\hat n\,dS = \langle -f_x,-f_y,1\rangle\,dA = \langle 2x,2y,1\rangle\,dA$.
Therefore $\iint_S \vec F\cdot\hat n\,dS = \iint_S\langle x,y,2(1-z)\rangle\cdot\langle 2x,2y,1\rangle\,dA = \iint_S 2x^2+2y^2+2(1-z)\,dA = \iint_S 4x^2+4y^2\,dA$ (since $z = 1-x^2-y^2$).
Shadow = unit disc $x^2+y^2\le 1$; switching to polar coordinates, we have
$\iint_S\vec F\cdot\hat n\,dS = \int_0^{2\pi}\int_0^1 4r^2\,r\,dr\,d\theta = \int_0^{2\pi}\left[r^4\right]_0^1 d\theta = 2\pi$.
b) Let T = unit disc in the xy-plane, with normal vector pointing down ($\hat n = -\hat k$). Then
$\iint_T\vec F\cdot\hat n\,dS = \iint_T\langle x,y,2\rangle\cdot(-\hat k)\,dS = \iint_T -2\,dS = -2\,\text{Area} = -2\pi$. By divergence theorem,
$\iint_{S+T}\vec F\cdot\hat n\,dS = \iiint_D \operatorname{div}\vec F\,dV = 0$, since $\operatorname{div}\vec F = 1+1-2 = 0$. Therefore $\iint_S = -\iint_T = +2\pi$.

Problem 5.

a) $\operatorname{curl}\vec F = \begin{vmatrix}\hat\imath & \hat\jmath & \hat k\\ \partial_x & \partial_y & \partial_z\\ -2xz & 0 & y^2\end{vmatrix} = 2y\hat\imath - 2x\hat\jmath.$

b) On the unit sphere, $\hat n = x\hat\imath+y\hat\jmath+z\hat k$, so $\operatorname{curl}\vec F\cdot\hat n = \langle 2y,-2x,0\rangle\cdot\langle x,y,z\rangle = 2xy-2xy = 0$; therefore $\iint_R\operatorname{curl}\vec F\cdot\hat n\,dS = 0$.

c) By Stokes, $\oint_C\vec F\cdot d\vec r = \iint_R\operatorname{curl}\vec F\cdot\hat n\,dS$, where R is the region delimited by C on the unit sphere. Using the result of b), we get $\oint_C\vec F\cdot d\vec r = \iint_R\operatorname{curl}\vec F\cdot\hat n\,dS = 0$.