18.02 Calculus Spring 2010 *revised 2/1/10*

A syllabus is given below; specific readings and exercises are on the problem sets.)

Text: Simmons: Calculus with Analytic Geometry, second edition (McGraw-Hill)
18.02 Supplementary Notes (sold by Copy Tech, Basement Bldg. 11)

Lectures: T-Th 11, F 2 54-100
Recitations: M-W 10,11,12,1,2,3 change section on Stellar (cf. "18.02 Website" below)

Lecturer: A. Mattuck 2-241 Hours: W 3-5 or appt. ext. 3-4345 *mattuck@mit.edu*
Course Administrator: Galina Lastovkina 2-108 3-4977 *galina@math.mit.edu*

Problem Sets: Given out Thursdays in lecture; available afterwards on 18.02 website; due on Thursdays, 10:45 in 2-106; returned in recitation Monday with solutions; unclaimed sets are put in horizontal file in 2-108 (left wall) after recitation Monday.

18.02 Website: *http://math.mit.edu/~apm/1802.html*

Has corrections, occasional comments, timely course information, and a list of recitations; also links to problem sets, practice exams w/solutions, and this syllabus, all in *pdf* format, plus a link to the Stellar website, where you can check your record (grades on exams and problem sets).

Tutoring: 2-102 Mon-Tues-Wed-Thurs: 3-5 and 7:30-9:30 PM. (Starts second week.)

Exams: Three hour-exams in Walker, dates on syllabus; three-hour final.

Students who fail an hour-exam by more than 5 points receive e-mail same day; they can take a make-up exam once during certain hours the following week; maximum contribution of make-up to total score = passing score on hour-exam.

Grading: The final grade (ABC/NR for freshmen; ABCD/F for others) is based on a cumulative total score; in this, the relative weighting of problem sets, hour exams, and final will be around 2:3:2, respectively. In addition to this, to pass 18.02 (for freshmen, this means be at level C- or above, for others D- or above), you must pass at least two hour-exams and the final, and make a reasonable effort on at least seven of the nine problem sets. Exceptions to this for borderline fails will be considered on a case-by-case basis.

Questions:

About external problems affecting exams or homework (away games, illness, emergencies): see Course Administrator (Galina, cf. above);

About academic problems and difficulties: see Recitation Teacher.

18.02 Syllabus *revised 2/1/10*

Vectors

Tu	Feb. 2	1.	Vectors in 2D and 3D; scalar (dot) product	
Th	Feb. 4	2.	Determinants; cross product	
F	Feb. 5	3.	Matrices; inverse matrices	
Tu	Feb. 9	4.	Theorems about square systems; Cramer's rule, eqns. of planes	
Th	Feb. 11	5.	Parametric eqns: eqns. of lines and curves; polar coordinates	**P.Set 1 due**
F	Feb. 12	6.	Vector derivatives; tangent vector *Holiday Mon. Feb. 15*	
Tu	Feb. 16		*Recitation; Mon. schedule followed*	
Th	Feb. 18	7.	Acceleration vector **P.Set 2 due**	
F	Feb. 19	8.	Applications	
Tu	Feb. 23		**Exam 1**, covering Lectures 1-8	

Partial Differentiation

W	Feb. 24		*Recitation:* $f(x, y)$: partial derivs, graph, contour and level curves
Th	Feb. 25	9.	Tangent approx.; directional derivative and gradient in 2D
F	Feb. 26	10.	Gradient in 3D; contour surfaces, tangent planes
Tu	Mar. 2	11.	Max-min problems; method of least squares
Th	Mar. 4	12.	Second derivative criterion. Lagrange multipliers **PS. 3 due**
F	Mar. 5	13.	Chain rule and applications
Tu	Mar. 9	14.	Chain rule: non-indept. variables

Double Integration

Th	Mar. 11	15.	Double and iterated integrals in rectangular coordinates **PS. 4 due**
F	Mar. 12	16.	Polar coordinates; Double integrals in polar coords.
Tu	Mar. 16	17.	Continuation; Applications of double integration
Th	Mar. 18	18.	Change of variable in double integrals **PS. 5 due**
F	Mar. 19	19.	Continuation and review

Spring Break

Vector Calculus in the Plane

Tu	Mar. 30	20.	Vector fields; Line integrals in the plane.
Th	Apr. 1		**Exam 2**, covering Lectures 8-19
F	Apr. 2	21.	Gradient fields and conservative fields.
Tu	Apr. 6	22.	Potential functions.
Th	Apr. 8	23.	Green's theorem **PS. 6 due**
F	Apr. 9	24.	Flux; Normal form of Green's theorem; 2D curl
Tu	Apr. 13	25.	Extensions and applications of Green's theorem.

Vector Calculus in Space

Th	Apr. 15	26.	Triple integrals: rectangular, cylindrical coordinates **PS. 7 due**
F	Apr. 16	27.	Spherical coordinates; Gravitational attraction *(Holidays Apr. 20, 21).*
Th	Apr. 22		**Exam 3**, covering Lectures 20-27
F	Apr. 23	28.	Surface integrals.
Tu	Apr. 27	29.	Surface integrals continued; divergence theorem
Th	Apr. 29	30.	Divergence theorem: applications **PS. 8 due**
F	Apr. 30	31.	Vector fields and line integrals in 3-space
Tu	May 4	32.	Conservative fields, potential functions, curl F
Th	May 6	33.	Stokes' theorem **PS. 9 due**
F	May 7	34.	Extensions of Stokes' theorem; applications.
Tu	May 11	35.	Continuation
Th	May 13	36.	Last class

Three-hour final during finals week

18.02
Lecture 1

2/2/10

<u>Vector Review</u>

- does not know what we covered
- might be too much or too lityle

vector $\longrightarrow$ ← just one, straight, no varables

scalar, real constant, not a function

+, multip by scalar

· scalar (dot product)

- two ways to look at
  - coordinate free — physical laws
    "invarrient" — geometric problems
  - computational
    - will do later

- are same + equal
- move w/o rotation
- $\vec{A}$ capital, w/ arrow on top, bold

- multiplication by scalar $c\vec{A}$ ← multiples lenght

  $|c\vec{A}| = c|\vec{A}|$

  $c < 0 \rightarrow \ominus \rightarrow$ goes other way

- lenght of $\vec{A} = |\vec{A}|$

- directions  all of the possible directions

  $dir(\vec{A}) = \frac{\vec{A}}{|\vec{A}|}$ ← scalar, shrinks to unit vector, does not change direction

- Unit vector $\hat{u}$ → lenght = 1

  $\vec{A} \neq \vec{0}$

$$\vec{A} = \underbrace{|\vec{A}|}_{\text{value}} \ \underbrace{\frac{\vec{A}}{|\vec{A}|}}_{\text{direction}}$$

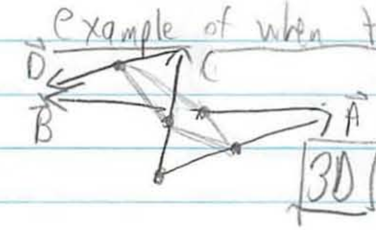
$\vec{A} + \vec{B}$ 2 ways of addition
1 usually easier

end result is same

parallagram
- forces
- velocities
- ≠ displacement

head to tail
displacement

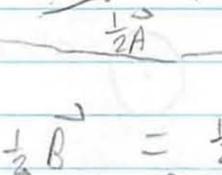
example of when to use



~ midpoints on each side form a parallagram

how to prove parallelagram
the 2 opposite sides are equal vectors

how to write



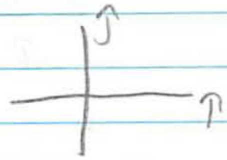
$\frac{1}{2}\vec{A} + \frac{1}{2}\vec{B}$

$$\frac{1}{2}\vec{A} + \frac{1}{2}\vec{B} = \frac{1}{2}\vec{C} + \frac{1}{2}\vec{D}$$

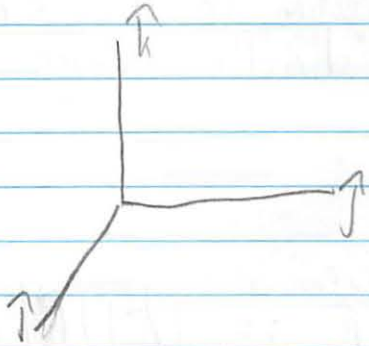
so $\vec{A} + \vec{B} = \vec{C} + \vec{D}$ obvious

Coordinates

$\hat{i}, \hat{j}$

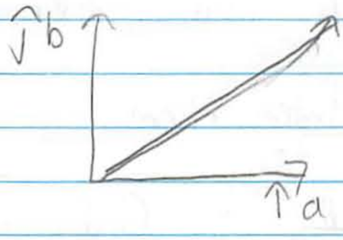


$\hat{i}, \hat{j}, \hat{k}$



Can draw diff ways
Will use this one in this class

favored when
- you have to calculate something
- economists, exlusivly
- arrays of #s,
- 6, 15
- n-space



$a\hat{i} + b\hat{j} = \langle a, b \rangle$

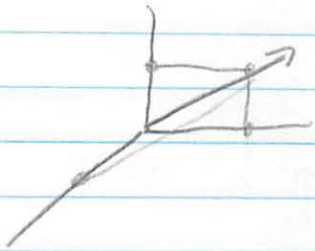
↑ the vector

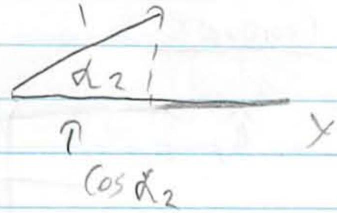
Advantages
- calculation $\vec{A} + \vec{B} = \langle a_1 + b_1, a_2 + b_2 \rangle$

$\langle a_1, a_2 \rangle + \langle b_1, b_2 \rangle$

$|\vec{A}| = \sqrt{a_1^2 + a_2^2}$

direction $\vec{A} = \frac{\langle a_1, a_2 \rangle}{\sqrt{a_1^2 + a_2^2}}$





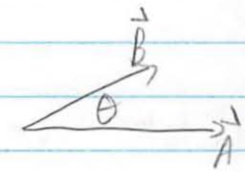
direction of $\vec{A}$ = $\langle \cos\alpha_1, \cos\alpha_2, \cos\alpha_3 \rangle$
"direction cosines"

Dot product

$:=$ a definition

- n-dimensional space

$\vec{A} \cdot \vec{B} := |\vec{A}||\vec{B}| \cos\theta$ = a scalar



direction of θ does not matter

- $a_1 b_1 + a_2 b_2 + a_3 b_3$

Can we prove that?

- either one can be used in n dimensional space
- lie in a plane
- use angle between them

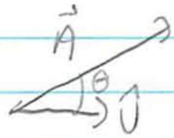
$\boxed{1}$ $\cos\theta = \dfrac{\vec{A} \cdot \vec{B}}{|\vec{A}||\vec{B}|}$: finding angles

$\boxed{2}$ $\vec{A} \cdot \vec{B} = 0$ one of the 3 things is 0
for example $\theta = \frac{\pi}{2}$
$\vec{A} \perp \vec{B}$ or $\vec{A} = 0$ or $\vec{B} = 0$
means $\vec{A}, \vec{B}$ are orthogonal

$\boxed{3}$ $\vec{A} \cdot \vec{A} = |\vec{A}|^2 = a_1^2 + a_2^2 + a_3^2$

$\left(\text{lenght of } \vec{A}\right)^2$

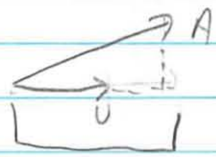
most important learn it → $\boxed{4}$ $\vec{A} \cdot \hat{u} = |\vec{A}| \cdot 1 \cdot \cos\theta$



project A perpendicularly on extension of projection) is wrong
= lenght of project

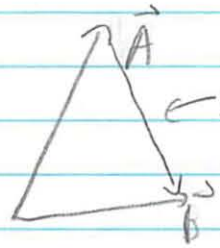
aka

scalar component of $\vec{A}$ in direction of $\hat{u}$



learn to think not copy

prove $|\vec{A}||\vec{B}|\cos\theta = \sum_{i=1}^{3} a_i b_i$



← vector if I add B I should get A

law of cosigns

$$|A - B|^2 = |A|^2 + |B|^2 - 2|A||B|\cos\theta$$

$$\sum_{i=1}^{3} (a_i - b_i)^2 = \sum a_i^2 + \sum b_i^2 - 2\sum_{i=1}^{3} a_i b_i$$

$$\sum_{i=1}^{3} (a_i - b_i)^2 = \sum a_i^2 + \sum b_i^2 - 2 \sum_{i=1}^{3} a_i b_i$$

18.02 Recitation

2/3

Olivier Bernardi French

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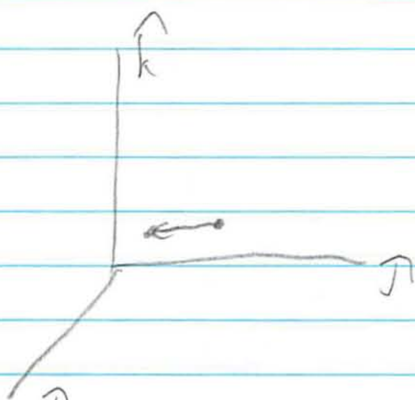
Mon 4:30 - 5:30
Wed 5:30 - 6:30

Ex1 $P = (1,3,-1)$ $Q = (0,1,1)$ $\vec{A} = \overrightarrow{PQ}$

Find $\vec{A}$

$|\vec{A}|$ = value

dir $\vec{A} = \dfrac{\vec{A}}{|\vec{A}|} =$



end - start

$(0,1,1) - (1,3,-1) =$

$\vec{A} = \langle -1, -2, 2 \rangle$

$|\vec{A}| = \sqrt{\text{sum of squares}}$

$= \sqrt{(-1)^2 + (-2)^2 + (2)^2}$

$\sqrt{1+4+4}$

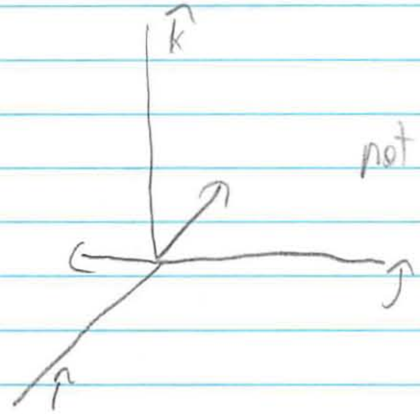
$\sqrt{9}$

3

$\dfrac{\vec{A}}{|\vec{A}|} = \dfrac{\langle -1, -2, 2 \rangle}{3} = \langle -\frac{1}{3}, -\frac{2}{3}, \frac{2}{3} \rangle$

Ex2 $\vec{u} = \langle 1, 2, 3 \rangle$ $\quad$ $\vec{v} = \langle 1, -1, 0 \rangle$

perpendicular?



not clear from picture

Memorize → know that perpendicular $\underline{\underline{\text{if}}}$ dot product $= 0$

$\checkmark$: $\vec{u} \cdot \vec{v} \stackrel{?}{=} 0$

multiply each part

$(1 \cdot 1) + (2 \cdot -1) + (3 \cdot 0)$

$1 + (-2) + 0$

Review dot product

-1

$\neq 0$ ⊗ Not perpendicular

Acute angle? $(\theta < 90°, \frac{\pi}{2})$

$\vec{u} \cdot \vec{v} = |\vec{u}|\,|\vec{v}| \cos\theta$

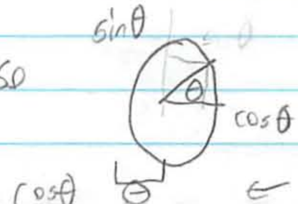
$-1 = \sqrt{1^2 + 2^2 + 3^2} \cdot \sqrt{1^2 + (-1)^2 + 0^2} \cos\theta$

$-1 = \underbrace{\sqrt{15} \cdot \sqrt{2}}\, \cos\theta$

* just notice $\cos\theta < 0$

→ did not have to find, will always be (+)

so



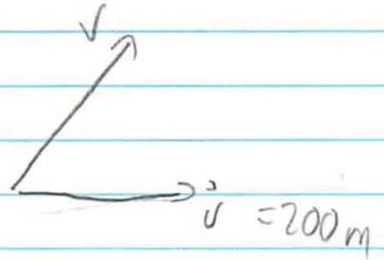
⊗ not acute

← no $\cos\theta$ is θ from $90 \to 270°$

Ex 3

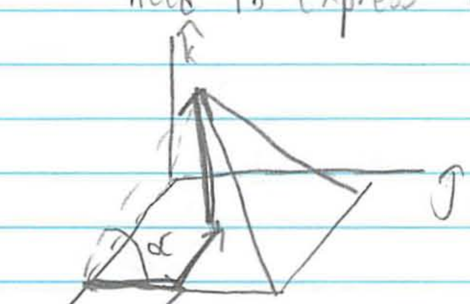
Find angle of pyrmid α

can we just bend up the side
- I think it will change ??
- or at least don't know height
- think about what we just learned



$\cos \alpha = \dfrac{\vec{u} \cdot \vec{v}}{|\vec{u}|\,|\vec{v}|}$ ← rewrite identity

But need to choose a coord system
need to express $\vec{v}$ in a system



$\vec{u} = \langle 0, 200, 0 \rangle$
$\vec{v} = \langle 100, -100, 100\sqrt{2} \rangle$ ← given
↑ think middle pt of base

$|\vec{u}| = \sqrt{200^2} = 200$

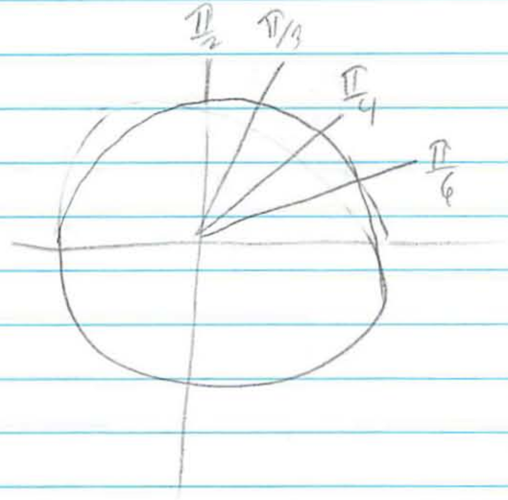
← factor out

$|\vec{v}| = 100\,|\langle -1, 1, \sqrt{2} \rangle|$

$100\sqrt{(-1)^2 + 1^2 + \sqrt{2}^2}$

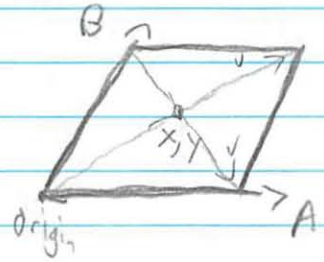
200

$\cos\alpha = \dfrac{\vec{u} \cdot \vec{v}}{|\vec{u}|\,|\vec{v}|} = \dfrac{200 \cdot 100}{200 \cdot 200} = \dfrac{1}{2} = \cos\alpha \rightarrow \boxed{\alpha = 60^\circ}$ $\frac{\pi}{3}$



$\cos 0 = 1$
$\cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$
$\cos \frac{\pi}{4} = \frac{\sqrt{2}}{2}$
$\cos \frac{\pi}{3} = \frac{1}{2}$
$\cos \frac{\pi}{2} = 0$

Ex 4 Show that diagonals of porralagrom intersect each other



Find midpoint of each diagonal
are =, ~~at center~~
not nessarcly at center

Express x and y in terms of A and B

So x) $\overrightarrow{Ox} = \frac{1}{2}\vec{u} = \frac{1}{2}\vec{A} + \vec{B}$
y) $\overrightarrow{Oy} = \vec{B} + \frac{\vec{v}}{2} = \vec{B} + \frac{\vec{A} - \vec{B}}{2} = \frac{1}{2}\vec{A} + \vec{B}$ are =

diagonal bisect

18.02 Lecture 2

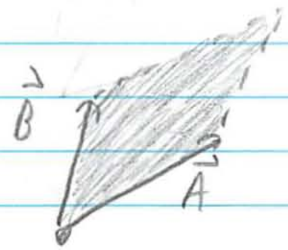
Determinants + Cross Products 2/4

Determinants — Cross Product

2x2

3x3

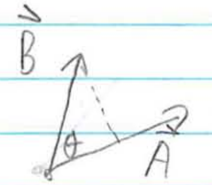
$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \quad \text{or} \quad \det\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} = a_1 b_2 - a_2 b_1$$



= signed area of a parallagram

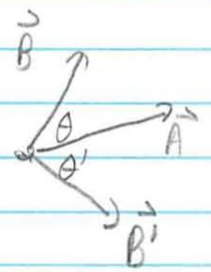
- usally disreguard sign

So why does this work?



$= \text{base} \cdot \text{height}$

$= |\vec{A}||\vec{B}| \sin\theta$



$\vec{B}' = \vec{B}$ rotated 90° clockwise

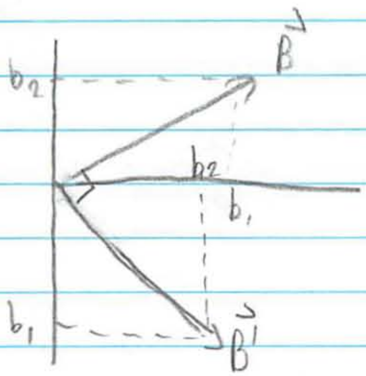
$\theta + \theta' = 90°$

$|\vec{A}||\vec{B}| \sin\theta = |\vec{A}||\vec{B}'| \cos\theta$

$= \vec{A} \cdot \vec{B}'$

So what is $\vec{B}'$?

↓



vertical becomes horizontal
horizontal becomes vertical

$|\vec{B}'| = \langle b_2, -b_1 \rangle$ ← always works

thus $\therefore$ signed area of parallagram $= \vec{A} \cdot \vec{B}'$

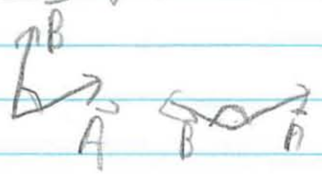
$= a_1 b_2 - a_2 b_1$

$=$ determinent

$= \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$

Determinent Sign

$\oplus$



acute or obtuse
Not Reflex

$\ominus$

Reflex angle

3 x 3 Determinent

$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ = signed value of a parallelepiped (3D)

rows $\vec{A}, \vec{B}, \vec{C}$ in 3D

coterminous tails at same point →

parallelagram in 3D

How to find (2 ways)

way 1 $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \cdot b_2 \cdot c_3 + a_2 \cdot b_3 \cdot c_1 + b_1 c_2 a_3 - (a_3 \cdot b_2 \cdot c_1 + a_2 b_1 c_3 + b_3 c_2 a_1)$

way 1 $\begin{vmatrix} 1 & 3 & 2 \\ 0 & -1 & 2 \\ 2 & -1 & 3 \end{vmatrix} = -3 + \underset{15}{12} + 0 - (-4 + -2 + 0)$

*be careful of minus signs - most common error

way 2 by the minors of the top row

Laplace expansion

$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$

left

minor of an element = 2 x 2 determinent when you delete the row + colum of the selected element

$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$ ← signs you use

can select any row ??? or just top ??

Cross Product $\qquad \vec{A} \times \vec{B}$ $\qquad$ - valid only in 3-space
- answer is a vector

geometric definition

$|\vec{A} \times \vec{B}|$ = area  (positive)

dir $(\vec{A} \times \vec{B})$ = unit vector perpendicular to plane of a and b
in the sense given by the right hand rule
fingers curl from A to B
directions thumb points

$\vec{B} \times \vec{A} = -\vec{A} \times \vec{B}$ (anticommunitive)

$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$ additive

$\vec{A} \times (\vec{B} \times \vec{C}) \neq (\vec{A} \times \vec{B}) \times \vec{C}$ not associtive

component-wise $\vec{A} \times \vec{B}$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

k non-deterinent

$$= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \hat{\imath} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \hat{\jmath} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \hat{k}$$

then determinant of each

18.02 Lecture 1. – Thu, Sept 6, 2007

Handouts: syllabus; PS1; flashcards.

Goal of multivariable calculus: tools to handle problems with several parameters – functions of several variables.

Vectors. A vector (notation: $\vec{A}$) has a direction, and a length ($|\vec{A}|$). It is represented by a directed line segment. In a coordinate system it's expressed by components: in space, $\vec{A} = \langle a_1, a_2, a_3 \rangle = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$. (Recall in space x-axis points to the lower-left, y to the right, z up).

Scalar multiplication

Formula for length? Showed picture of $\langle 3, 2, 1\rangle$ and used flashcards to ask for its length. Most students got the right answer ($\sqrt{14}$).

You can explain why $|\vec{A}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$ by reducing to the Pythagorean theorem in the plane (Draw a picture, showing $\vec{A}$ and its projection to the xy-plane, then derived $|\vec{A}|$ from length of projection + Pythagorean theorem).

Vector addition: $\vec{A}+\vec{B}$ by head-to-tail addition: Draw a picture in a parallelogram (showed how the diagonals are $\vec{A}+\vec{B}$ and $\vec{B}-\vec{A}$); addition works componentwise, and it is true that $\vec{A} = 3\hat{\imath} + 2\hat{\jmath} + \hat{k}$ on the displayed example.

Dot product.

Definition: $\vec{A}\cdot\vec{B} = a_1b_1 + a_2b_2 + a_3b_3$ (a scalar, not a vector).

Theorem: geometrically, $\vec{A}\cdot\vec{B} = |\vec{A}||\vec{B}|\cos\theta$.

Explained the theorem as follows: first, $\vec{A}\cdot\vec{A} = |\vec{A}|^2\cos 0 = |\vec{A}|^2$ is consistent with the definition. Next, consider a triangle with sides $\vec{A}$, $\vec{B}$, $\vec{C} = \vec{A}-\vec{B}$. Then the law of cosines gives $|\vec{C}|^2 = |\vec{A}|^2 + |\vec{B}|^2 - 2|\vec{A}||\vec{B}|\cos\theta$, while we get

$$|\vec{C}|^2 = \vec{C}\cdot\vec{C} = (\vec{A}-\vec{B})\cdot(\vec{A}-\vec{B}) = |\vec{A}|^2 + |\vec{B}|^2 - 2\vec{A}\cdot\vec{B}.$$

Hence the theorem is a vector formulation of the law of cosines.

Applications. 1) computing lengths and angles: $\cos\theta = \dfrac{\vec{A}\cdot\vec{B}}{|\vec{A}||\vec{B}|}$.

Example: triangle in space with vertices $P = (1,0,0)$, $Q = (0,1,0)$, $R = (0,0,2)$, find angle at P:

$$\cos\theta = \frac{\overrightarrow{PQ}\cdot\overrightarrow{PR}}{|\overrightarrow{PQ}||\overrightarrow{PR}|} = \frac{\langle -1,1,0\rangle\cdot\langle -1,0,2\rangle}{\sqrt{2}\sqrt{5}} = \frac{1}{\sqrt{10}}, \qquad \theta \approx 71.5^\circ.$$

Note the sign of dot product: positive if angle less than 90°, negative if angle more than 90°, zero if perpendicular.

2) detecting orthogonality.

Example: what is the set of points where $x+2y+3z=0$? (possible answers: empty set, a point, a line, a plane, a sphere, none of the above, I don't know).

Answer: plane; can see "by hand", but more geometrically use dot product: call $\vec{A} = \langle 1,2,3\rangle$, $P = (x,y,z)$, then $\vec{A}\cdot\overrightarrow{OP} = x+2y+3z = 0 \Leftrightarrow |\vec{A}||\overrightarrow{OP}|\cos\theta = 0 \Leftrightarrow \theta = \pi/2 \Leftrightarrow \vec{A}\perp\overrightarrow{OP}$. So we get the plane through O with normal vector $\vec{A}$.

18.02 Lecture 2. – Fri, Sept 7, 2007

We've seen two applications of dot product: finding lengths/angles, and detecting orthogonality. A third one: finding components of a vector. If $\hat{u}$ is a unit vector, $\vec{A}\cdot\hat{u} = |\vec{A}|\cos\theta$ is the component of $\vec{A}$ along the direction of $\hat{u}$. E.g., $\vec{A}\cdot\hat{\imath}$ = component of $\vec{A}$ along x-axis.

Example: pendulum making an angle with vertical, force = weight of pendulum $\vec{F}$ pointing downwards: then the physically important quantities are the components of $\vec{F}$ along tangential direction (causes pendulum's motion), and along normal direction (causes string tension).

Area. E.g. of a polygon in plane: break into triangles. Area of triangle = $\frac{1}{2}$ base × height = $\frac{1}{2}|\vec{A}||\vec{B}|\sin\theta$ (= 1/2 area of parallelogram). Could get $\sin\theta$ using dot product to compute $\cos\theta$ and $\sin^2+\cos^2=1$, but it gives an ugly formula. Instead, reduce to complementary angle $\theta' = \pi/2-\theta$ by considering $\vec{A}' = \vec{A}$ rotated 90° counterclockwise (drew a picture). Then area of parallelogram $= |\vec{A}||\vec{B}|\sin\theta = |\vec{A}'||\vec{B}|\cos\theta' = \vec{A}'\cdot\vec{B}$.

Q: if $\vec{A} = \langle a_1, a_2\rangle$, then what is $\vec{A}'$? (showed picture, used flashcards). Answer: $\vec{A}' = \langle -a_2, a_1\rangle$. (explained on picture). So area of parallelogram is $\langle b_1, b_2\rangle\cdot\langle -a_2, a_1\rangle = a_1b_2 - a_2b_1$.

Determinant. Definition: $\det(\vec{A},\vec{B}) = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$.

Geometrically: $\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = \pm$ area of parallelogram.

The sign of 2D determinant has to do with whether $\vec{B}$ is counterclockwise or clockwise from $\vec{A}$, without details.

Determinant in space: $\det(\vec{A},\vec{B},\vec{C}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$.

Geometrically: $\det(\vec{A},\vec{B},\vec{C}) = \pm$ volume of parallelepiped. Referred to the notes for more about determinants.

Cross-product. (only for 2 vectors in space); gives a vector, not a scalar (unlike dot-product).

Definition: $\vec{A}\times\vec{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \hat{\imath}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \hat{\jmath}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \hat{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$.

(the 3x3 determinant is a *symbolic* notation, the actual formula is the expansion).

Geometrically: $|\vec{A}\times\vec{B}|$ = area of space parallelogram with sides $\vec{A}$, $\vec{B}$; direction = normal to the plane containing $\vec{A}$ and $\vec{B}$.

How to decide between the two perpendicular directions = right-hand rule. 1) extend right hand in direction of $\vec{A}$; 2) curl fingers towards direction of $\vec{B}$; 3) thumb points in same direction as $\vec{A}\times\vec{B}$.

Flashcard Question: $\hat{\imath}\times\hat{\jmath}$ =? (answer: $\hat{k}$, checked both by geometric description and by calculation).

Triple product: volume of parallelepiped = area(base) · height = $|\vec{B}\times\vec{C}|\,(\vec{A}\cdot\hat{n})$, where $\hat{n} = \vec{B}\times\vec{C}/|\vec{B}\times\vec{C}|$. So volume $= \vec{A}\cdot(\vec{B}\times\vec{C}) = \det(\vec{A},\vec{B},\vec{C})$. The latter identity can also be checked directly using components.

18.02 Lecture 3
Matricies

2/5

Watching Fri lectures on OCW from 07 Arox

Last Time

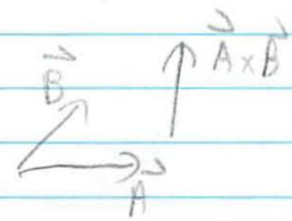
Cross Product

$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$ ← fake determinent

$= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \hat{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \hat{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \hat{k}$

$|\vec{A} \times \vec{B}|$ = area parallagram w/ sides $\vec{A}$ and $\vec{B}$

Direction $(\vec{A} \times \vec{B})$ = $\perp$ to $\vec{A}$ and $\vec{B}$ (right hand rule)



$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$

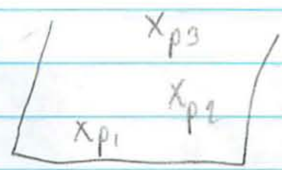
in particular $\vec{A} \times \vec{A} = 0$

- parallagram completely flat

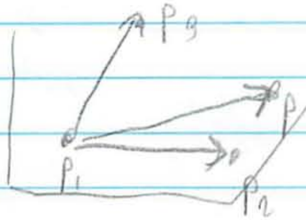
Application

3 pts in space

want plane that all are in (P_1, P_2, P_3)



$P(x, y, z)$ = condition on (x, y, z) telling us if P is in plane or not



is the parallelepiped completely flat? - then p is in the plane

$$\det\left(\overrightarrow{P_1P},\ \overrightarrow{P_1P_2},\ \overrightarrow{P_1P_3}\right) \stackrel{?}{=} 0$$

Get formula w/ x, y, z
That is equation of a plane

<u>Faster way</u>

rise to challenge

P is in the plane exactly when $\overrightarrow{P_1P} \perp N$ where N is the normal vector (perpendicular to the plane)

can tell by using dot product

$$\overrightarrow{P_1P} \cdot \vec{N} = 0$$

(multiply each part)

But how can find Normal vector?
We know 2 vectors in the plane $\overrightarrow{P_1P_2}$ $\overrightarrow{P_1P_3}$
Take cross product to find $\vec{N}$

$$\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_2} = \vec{N}$$ (it does not matter if ⊕ ⊖)

So have

$$\overrightarrow{P_1P} \cdot \left(\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}\right) \stackrel{\text{should}}{=} 0$$

is the triple product (aka the determinant)

Understand better b/c can pause

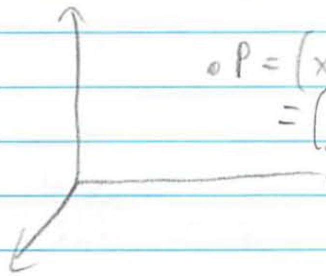
Matricies

(just an intro - take 18.06 for more)

not everything is a linear relation between variables

example

- change of coordinate systems



$\cdot P = (x_1, x_2\ x_3)$ then switch coord system

$= (v_1, v_2\ v_3)$

What is relation b/w old + new

- linear formulas

$$\begin{cases} v_1 = 2x_1 + 3x_2 + 3x_3 \\ v_2 = 2x_1 + 4x_2 + 5x_3 \\ v_3 = x_1 + x_2 + 2x_3 \end{cases}$$

$\boxed{\text{* A matrix is a table w/ \#s in it *}}$

Can express using matrix product

$$\begin{bmatrix} 2 & 3 & 3 \\ 2 & 4 & 5 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$$A \times X = U$$

Entries in matrix product (AX)

dot product between rows of A and columns of X

(A = 3 x 3 matrix)
(X = coloumn vector = 3 x 1 matrix)

Get the same thing back

$$2x_1 + 3x_2 + 3x_3 = v_1$$
$$2x_1 + 4x_2 + 5x_3 = v_2$$
$$x_1 + x_2 + 2x_3 = v_3$$

$\boxed{\text{* A vector is just a matrix w/ width 1 *}}$

Entries of product of 2 matricies (AB)

$$\underset{4 \cdot 3}{\underset{A}{\begin{bmatrix} 1 & 2 & 3 & 4 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}}} \cdot \underset{2 \cdot 4}{\underset{B}{\begin{bmatrix} \cdot & 0 \\ \cdot & 3 \\ \cdot & 0 \\ \cdot & 2 \end{bmatrix}}} = \underset{3 \cdot 2}{\underset{A \cdot B}{\begin{bmatrix} \cdot & 14 \\ \cdot & \cdot \\ \cdot & \cdot \end{bmatrix}}}$$

The 14 is $1 \cdot 0 + 2 \cdot 3 + 0 \cdot 3 + 4 \cdot 2$

Another way to set up

$$\begin{array}{cc} & [B] \\ [A] & [\quad] \leftarrow \text{answer} \end{array}$$

look left, look up
dot product of that
for each #

can know size ; height of A
as well width of B

There is a catch! Can not mulliply anything by anything

Must be same # of entries ?

* width of A must = height of B *

What AB represents doing first the transformation B then transformation A

counter intuitive
multiply from right to left

$(AB)X = A(BX)$ associative
- can multiply in any order

Identity Matrix

Matrix that does nothing
Not 0
Takes X → Gives X

$IX = X$

$$I_{3\times3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Example

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

In general $I_{n\times n} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ 0s everywhere else

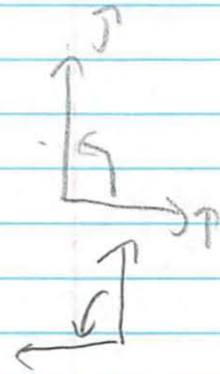
<u>Example</u>: In the plane, rotation by 90° counter-clockwise

why doing this stuff

$$R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$R\,i = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} j$$

$$R\,j = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} -i$$



$$R \begin{bmatrix} x \\ y \end{bmatrix} = -\begin{bmatrix} y \\ x \end{bmatrix}$$

We can do transformations eaiser w/ matrixes

$$R^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I_{2\times 2}$$

Rotate by 180° to $-x, -y$

<u>How find R</u> – just know it to rotate vector by 90° $\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} -y \\ x \end{bmatrix}$

Express transformation as a vector

(More on P-Set)

Invert Matricies

- Gives a nice way to think of transformations

- Relation is a linear matrix
- Is an inverse of a maxtrix

How to find the inverse of a matrix

Inverse of A matrix M such that

$AM = I$
$MA = I$) inverse of each other

$AA^{-1} = I$

Need A square matrix $n \times n$
same

$M = A^{-1}$

Solution of a linear system

$AX = B$ ← known vector
matrix ↑ ↑ Unknown vector

↓

$X = A^{-1}B$

$$AX = B$$

$\downarrow \quad \downarrow \cdot A^{-1}$ <u>on left</u>

$$A^{-1}(AX) = A^{-1}B$$

$$IX = A^{-1}B$$

$$X = A^{-1}B$$

Solve linear systems quickly

<u>Formula</u> to invert a matrix

- easy for small matricies

$$A^{-1} = \frac{1}{\det(A)} \cdot \operatorname{adj}(A)$$

$\uparrow$ adjoint

<u>Steps</u>

$$A = \begin{bmatrix} 2 & 3 & 3 \\ 2 & 4 & 5 \\ 1 & 1 & 2 \end{bmatrix}$$

1. Find the minors and the determinents left

$$\begin{pmatrix} 3 & -1 & -2 \\ 3 & 1 & -1 \\ 3 & 4 & 2 \end{pmatrix} \qquad \begin{vmatrix} 4 & 5 \\ 1 & 2 \end{vmatrix} \quad \begin{matrix} 4 \cdot 2 - 5 \cdot 1 \\ 3 \end{matrix}$$

2. Cofactors

flip signs according to checkerboard diagram

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

but + means leave alone
− means flip sign

$$\begin{pmatrix} 3 & -1 & -2 \\ -3 & 1 & 1 \\ 3 & -4 & 2 \end{pmatrix}$$

3. Tranpose

Switch rows and columns

$$\begin{pmatrix} 3 & -3 & 3 \\ -1 & 1 & -4 \\ -2 & 1 & 2 \end{pmatrix} = \text{adjoint matrix}$$

4. Divide A by determinant of a

determinant

$$\begin{vmatrix} 2 & 3 & 3 \\ 2 & 4 & 5 \\ 1 & 1 & 2 \end{vmatrix} = 3$$

So $A^{-1} = \frac{1}{3}\begin{bmatrix} 3 & -3 & 3 \\ -1 & 1 & -4 \\ -2 & 1 & 2 \end{bmatrix}$

how to find U interms of X

how to solve linear system $AX = ?$

Matrix Notes

2/7

4 types of operations

1. Scalar Multiplication
   - just multiply each #

2. Matrix Addition
   - add each position
   - must both be same size

3. Transposition
   - A' or A^T
   - flip

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \xrightarrow{T} \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

4. Multiplication

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^T$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad \begin{bmatrix} 1 & 4 & 9 \\ 4 & 10 & 18 \end{bmatrix}$$

$$A(B+C) = AB + AC$$
$$(AB)C = A(BC)$$

2/7

Recitation

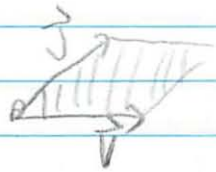
Lectures
- dot product, cross product
- matricies: determinant, multiplication, inverse

$$\vec{u}\cdot\vec{v} = |\vec{u}|\,|\vec{v}|\cos\theta$$
$$|\vec{u}\times\vec{v}| = |\vec{u}|\,|\vec{v}|\sin\theta$$

Ex 1

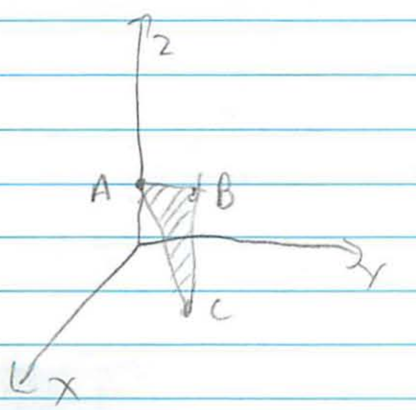
$A = (0,0,1)$
$B = (-1,1,2)$
$C = (3,2,1)$

a) Area of triangle ABC
b) Find the direction $\perp$ to plane ABC



$\text{Area} = |\vec{u}\times\vec{v}|$

$\text{Area} = \dfrac{|\vec{u}\times\vec{v}|}{2}$



Oh this is how $\vec{u}$ and $\vec{v}$ are the subtraction b/w points

$\vec{u} = \overrightarrow{AB} = \langle -1, 1, 1\rangle$
$\vec{v} = \overrightarrow{AC} = \langle 3, 2, 0\rangle$

Seems so obvious now

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & 1 \\ 3 & 2 & 0 \end{vmatrix}$$

$(1\cdot 0 - 1\cdot 2)\,\hat{i} - (-1\cdot 0 - 3\cdot 1)\,\hat{j} + (-1\cdot 2 - 3\cdot 1)\,\hat{k}$

$-2\hat{i} + 3\hat{j} - 5\hat{k}$

Now lenght of vector

$$\sqrt{2^2+3^2+5^2}$$

$$\sqrt{38}$$

a) $\dfrac{\sqrt{38}}{2}$

B) Now take ~~cross product w/ plane~~ ✓

If dot product is 0 is perpendicular

↑ harder

Want perpendicular to the 2 vectors w/ both vectors

No, I was right: take cross product of $\vec{u}$ ~~or~~ and $\vec{v}$

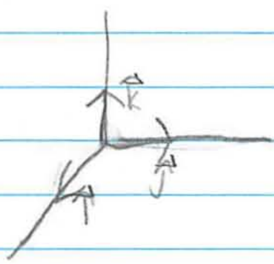
know the stuff clearly

$$\text{dir}(\vec{u}\times\vec{v})$$

$$\frac{\langle -2, 3, -5\rangle}{\sqrt{38}}$$ easy

* what does which *

Ex 2 $(\vec{i}\times\vec{i})\times\vec{j} \neq \vec{i}\times(\vec{i}\times\vec{j})$



$\vec{i} = \langle 1,0,0\rangle$
$\vec{j} = \langle 0,1,0\rangle$
$\vec{k} = \langle 0,0,1\rangle$

$$(\vec{i}\times\vec{i})\times\vec{j} = 0$$

↑ always 0 since $\vec{A}\times\vec{A}$ is always 0

$\vec{i} \times (\vec{i} \times \vec{j})$

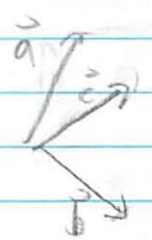
k - use the right hand rule + look at it

So the problem did put in #s for the problem - and drew it - at least it kept it to unit vectors

$\vec{i} \times \vec{k}$

$-\vec{j}$ so they are not =

Example 3 Parallelpiped w/ angles $\vec{a}, \vec{b}, \vec{c}$



a) Show $Vol = \pm \vec{a} \cdot (\vec{b} \times \vec{c})$

b) Show $\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

a) Take area of the base • height

$|\vec{b} \times \vec{c}|$ $\quad$ $|\vec{a}| \cos\theta$

θ is the angle b/w $\vec{a}$ and $\vec{b} \times \vec{c}$ so we recognize the formula for $\vec{a} \cdot (\vec{b} \times \vec{c})$

$Vol = \vec{a} \cdot (\vec{b} \times \vec{c})$

right hand side

b) RHS: $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

$$a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_2 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$a_1(b_2c_3 - b_3c_2) - a_2(b_1c_2 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$$

left side

LHS: $\vec{b} \times \vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \hat{i}x - \hat{j}y + \hat{k}z$

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = a_1x - a_2y + a_3z$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \quad \text{OK}$$

proved volume = ± determinant

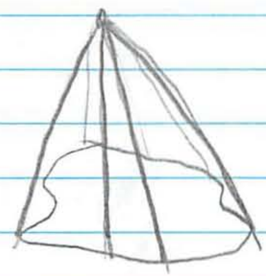
10-7 from notes

Ex 4 Find the volume of a tetrahedron PQRS

$P = (1, 0, 1)$
$Q = (-1, 1, 2)$
$R = (0, 0, 2)$
$S = (3, 1, -1)$

Need a formula for vol tetrahedron?

Volume of General cones



Vol $= \frac{1}{3}$ Area $\times$ height

Vol (Tetrahedron) $= \frac{1}{6}$ volume (parallelepiped)

1. Compute Vectors
2. Find determinant
3. Divide by 6

Ex 5 Matrix multiplication

$$A = \begin{pmatrix} 2 & -1 & 3 \\ 1 & 0 & 4 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & -1 \\ 2 & 3 \\ -1 & 2 \end{pmatrix}$$

Compute AB

First ✓ if makes sense

A: 2×3 B: 3×2

$=$

result size $= 2 \times 2$

$$\begin{pmatrix} 2\cdot 1 + -1\cdot 2 + 3\cdot -1 & 2\cdot -1 + -1\cdot 3 + 3\cdot 2 \\ 1\cdot 1 + 0\cdot 2 + 4\cdot -1 & 1\cdot -1 + 0\cdot 3 + 4\cdot 2 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 1 \\ -3 & 7 \end{pmatrix}$$

Lecture 4
Square Systems + Equ of Planes

2/8

Notes are a complete mess

__Last Time__

Solving Systems

$a_1 x + a_2 y + a_3 z = d_1$
$b_1 x + b_2 y + b_3 z = d_2$
$c_1 x + c_2 y + c_3 z = d_3$

square system
- # variables = ?
- ~~did not hear definition~~
  # of equations

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$$A \cdot \vec{x} = \vec{d}$$

Good in any number of dimensions

a) If A^{-1} exists, then system has a unique solution
$\vec{x} = A^{-1}\vec{d}$
- must be square

at least one / exists and is unique

b) A^{-1} exists if and only if $|A| \neq 0$

A invertable

non singular
- Square
- determinant $\neq 0$

"Proof" : $A^{-1} = \dfrac{\text{adjoint }(A)}{\text{determinant}} = \dfrac{(\text{Cofactor matrix})^T}{|A|}$

$A^{-1}A = I$
$|A^{-1}A| = |A^{-1}| \cdot |A| = 1$

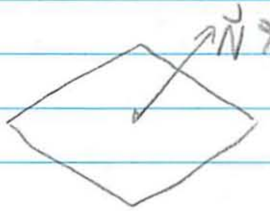
know determinant $\neq 0$ because it is 1 square matricies $|\vec{A}| \neq 0$

First Main Theorm

$|A| \neq 0 \rightarrow A\vec{x} = \vec{d}$ has the unique solution $\vec{x} = A^{-1}\vec{d}$

Geometric interpertation:

$a_1 x + a_2 y + a_3 z = d$ is equation of plane in 3 space (R^3) $(3D)$



* $P(x, y, z)$ lies on the plane if and only if x, y, z satisfy the equation *

~~We hope 3 points determine a plane~~

Or a point and a slope for 2 ways

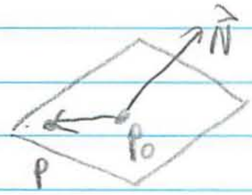
So $\hookrightarrow$ $P_0 (x_0\ y_0\ z_0)$

$\vec{N}$ = perpendicular to plane $\langle a_1\ a_2\ a_3 \rangle$

↑ are many options to use

Pick a point P. What condition to lie on plane?

When $\vec{P_0 P} \perp \vec{N}$



$\vec{P_0 P} = \langle x - x_0, y - y_0, z - z_0 \rangle$

(P lies on the plane if and only if $\vec{N} \perp$ to $\vec{P_0 P}$)

– when dot product = 0

Equation of plane form 1) $a_1(x-x_0)+a_2(y-y_0)+a_3(z-z_0)=0$

- fast for point + normal vector

Form 2) $a_1 x + a_2 y + a_3 z = d$

- gives normal vector
- make up your own point since does not specify P_0

Geometric interpertation of main theorm)

$$\begin{aligned} a_1 x + \dots &= d_1 \\ b_1 x + \dots &= d_2 \\ c_1 x + \dots &= d_3 \end{aligned}$$

3 planes
normal vectors $\vec{A}\ \vec{B}\ \vec{C}$

A solution = point that lies on all 3 planes
All solutions = intersection of the 3 planes

If $|A| \neq 0$, the 3 planes intersect in just 1 point
← what usually happens
unless in special position

If $|A| = 0$ - special case, last half 18.03

Second Main Theorm

- Square system of equations, homogeneous
← right hand side $=0$

- $\vec{A}\, x = \vec{0}$

$|\vec{A}| \neq 0$ has only trival solution $\vec{x} = \vec{0}$

obvious

$|A| = 0 \rightarrow$ has non-trival solutions

not 0

So homogeneous square system has non trival solutions only if $\det(A) = 0$

notes such a mess

$\vec{A} \cdot \vec{x} = 0$ each defines a plane

$\vec{B} \cdot \vec{x} = 0$

$\vec{C} \cdot \vec{x} = 0$ 3 planes through $(0,0,0)$

When do they also go through another plane?

- what special?

if the determinant is 0 - means volume parralelpiped spanned by $\vec{A}, \vec{B}, \vec{C}$ (origin) $= 0$

- not pointed in different direction
- must lie in single plane
- then not a parallelpiped

$\vec{A}, B, \vec{C}$ lie on one plane

- called "linear dependend"

$\rightarrow \vec{x} = \vec{B} \times \vec{C}$ is a solution to all 3 equations

(say $\vec{B} \times \vec{C} \neq 0$)

$\vec{B} \times \vec{C}$ is $\perp$ to $\vec{B}$ and $\vec{C}$

So $B \cdot x = 0$ and $C \cdot x = 0$

$\vec{x}$ is $\perp$ to A; $A \cdot \vec{B} \times \vec{C} = 0$

$A \cdot \vec{B} \times \vec{C} = 0$

$\shortparallel$

$|\vec{A}|$ ← original hypothsis

Recitation: go over te proof

Recitation

2/10

Ex 1 Find inverse of

$$A_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\boxed{B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad B^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}$$

$$A_\theta^{-1} = \frac{1}{\cos^2\theta + \sin^2\theta}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$A_\theta^{-1} = \frac{1}{1}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

(Note $A_\theta^{-1} = A_{-\theta}$
rotation in the plane)

Ex 1b Inverse?

$$\begin{bmatrix} 2 & 3 & 1 \\ 1 & 0 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

1. Compute determinant

$$2(0-2) - 3(2-1) + 1(2-0)$$
$$-4 \quad -3 \quad +2$$
$$-5 \quad ✓$$

2. Look at matrix of minors

$$\begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}$$

$\swarrow$ 2x2 determant when erase its row + column

$$\begin{bmatrix} -2 & 1 & 2 \\ 4 & 3 & 1 \\ 3 & 1 & -3 \end{bmatrix}$$

3. Signs

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix} \rightarrow \begin{bmatrix} -2 & -1 & 2 \\ -4 & 3 & -1 \\ 3 & -1 & -3 \end{bmatrix}$$

4. Transpose (flip on diagonal)

$$\begin{bmatrix} -2 & -4 & 3 \\ -1 & 3 & -1 \\ 2 & -1 & -3 \end{bmatrix}$$

memorize the steps + be able to do

5. $$\frac{1}{-5}\begin{bmatrix} -2 & -4 & 3 \\ -1 & 3 & -1 \\ 2 & -1 & -3 \end{bmatrix}$$

Ex2 System of Linear Equations using matricies

these recitations are doing examples

$$\begin{cases} 2x + 3y + z = 0 \\ x + 0 + z = 1 \\ x + 2y + 3z = 2 \end{cases}$$

$$A \quad x \quad = d$$

$$\begin{bmatrix} 2 & 3 & 1 \\ 1 & 0 & 1 \\ 1 & 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

1. Multiply both sides by the inverse (A^{-1}) of A

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

calc from 1b - remember really long

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{-5} \begin{bmatrix} -2 & -4 & 3 \\ -1 & 3 & -1 \\ 2 & -1 & -3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} =$$

$$= \frac{1}{-5} \begin{bmatrix} 2 \\ 1 \\ -7 \end{bmatrix}$$

$x = -\frac{2}{5} \quad y = -\frac{1}{5} \quad z = \frac{2}{5}$

Ex3 Equations of planes in the form $ax + by + cz = d$

a) Plane equation $\perp \vec{N}$ $\langle 1, 2, 3 \rangle$ $point_0$ $(1, 0, -1)$
a') " " " " " " $(2, 1, -2)$

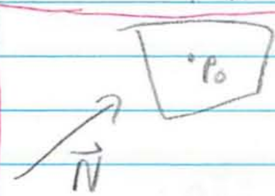
$\langle 1, 2, 3 \rangle \cdot \langle x-1, y-0, z--1 \rangle$

$\cancel{(1-1)x + (2-0)y + (3--1)z = 0}$ $\quad 1(x-1) + 2(y) + 3(z+1) = 0$

Close $\quad \cancel{2y + 4z = 0}$ $\quad x + 2y + 3z = -2$

$\cancel{(1-2)x + (2-1)y + (3--2)z = 0}$ $\quad 1(x-2) + 2(y-1) + 3(z-2)$

$\cancel{-1x + 1y + 5z = 0}$ $\quad x + 2y + 3z = -2$



P is in plane $\rightarrow \vec{P_0P} \perp \vec{N}$

$\vec{N} \cdot P_0P = 0$

actually ends up the same

b) Plane through $(1,0,2)$
$(-1,1,1)$
$(1,2,0)$

but need 2 vectors (take Δ)

So need to find $\vec{N}$ so cross multiply 2

$\langle 1--1, 0-1, 2-1\rangle = \langle 2 -1 1\rangle$
$\langle 1--1, 2-1, 0-1\rangle = \langle 2 1 -1\rangle$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & -1 & 1 \\ 2 & 1 & -1 \end{vmatrix}$$

$$(1-1)\hat{i} - (-2-2)\hat{j} + (-2-2)\hat{k}$$
$$4\hat{j} - 4\hat{k}$$

Then put it in

$$0(x-1) + 4(y-0) - 4(z-2)$$
$$4y - 4z + 8 = 0$$
$$4y - 4z = -8$$

Simplify

$$-y - z = -2$$
$$y + z = 2$$

Ex 4 Homogeneous System - For which value of c the system has non 0 solution

$$2x + cz = 0$$
$$x - y + 2z = 0$$
$$x - 2y + 2z = 0$$

↑ homogeneous - all 0

* If and only if $|A| = 0$
↑ determinant

$$\begin{bmatrix} 2 & 0 & c \\ 1 & -1 & 2 \\ 1 & -2 & 2 \end{bmatrix}$$

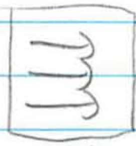
$$2(-2--4) + 0 + c(-2--1) = 0$$

$$4 + 0 + -c = 0$$

$$-c = -4$$

$$\boxed{c = 4}$$

* A is not invertable
$|A| = 0$
rows are vectors in the same plane



Volume $= 0$ (of the parallelpiped you would have)

Michael Plasmeier

33.5/40

18.02 Problem Set 1 due Thurs. 2/11/10, 2-106, 10:45 AM

18.02 Supplementary Notes On sale now at CopyTech, Bldg. 11.
(Same as the Fall 18.02 or .02A Notes or recent years' Notes; minor errors have been corrected in the solutions.)

Part I (20 points)

Hand in the exercises below, which are solved in the Notes. Do not include the exercises in parentheses, which are for more practice if you need it.

Notation:
17.3; 607/2 = section 17.3; exercise 2, p.607 of the text (Simmons 2nd ed.)
Notes D = section D of the 18.02 Supplementary Notes;
1A-2 = Exercise 2 in Section 1A of the Exercises section of the Notes, solved in the Solutions section.
Exercises marked in the Notes with an asterisk are not solved in the Notes.

Lecture 1. Tues. Feb.2 Vectors; addition, mult'n by scalar, dot (scalar) product.
Read: 17.3, 18.1-.2 Work: ~~1A-3b,4b,7bc,8ab~~ (1,2,6,11); ~~1B-1b,2a,3a,11~~ (5b, 13)
↑ what is this? Optional - yeah

Lecture 2. Thurs. Feb.4 Small determinants; cross (vector) product of in 3D
Read: Notes D, pp. 1-3; 18.3 ~~Work: 1C-2b,3b,4,9; 1D-1b,2,5,6~~

Lecture 3. Fri. Feb.5 Matrices; inverse matrices.
Read: Notes M.1, M.2 Work: ~~1F-5b, 8a; 1G-3, 4, 5~~

Lecture 4. Tues. Feb.9 Theorems about square systems; Equations of planes
Read: Notes M.3, M.4 Work: 1~~H-3abc~~, 7
Read: pp. 648,9 Work: 1E-1cd, 2

Lecture 5. Thurs. Feb. 11 Parametric equations for lines and plane curves; polar coordinates.
Read pp. 646-8; 17.1; 16.1; 16.2/exs.1,4; 16.3/ex.1

Part II (20 points)

Directions. *Try each problem alone for 15 minutes. If you subsequently collaborate, solutions must be written up independently. It is illegal to consult problem sets from previous semesters. With each problem is the day it can be done.*

Problem 1. (Tues., 3 pts: 1.5, 1.5) The *direction* of $\mathbf{A}$ is the unit vector dir $\mathbf{A} = \mathbf{A}/|\mathbf{A}|$.

a) Show that if $\mathbf{A}$ and $\mathbf{B}$ have the same tail, then $\frac{1}{2}(\text{dir }\mathbf{A} + \text{dir }\mathbf{B})$ bisects the angle between them (use 1A-4b and congruent triangles); deduce that $|\mathbf{B}|\mathbf{A} + |\mathbf{A}|\mathbf{B}$ also does.

b) Velocities in moving media like air or water are represented by vectors. For an airplane in flight, $\mathbf{v}_g = \mathbf{v}_a + \mathbf{v}_w$, where
$\mathbf{v}_w$ is the wind velocity;
$\mathbf{v}_a$ is the plane's air-velocity (as set and measured in the cockpit);
$\mathbf{v}_g$ is the plane's velocity as observed from the ground.

If the wind is blowing 50 mph from the southwest, and the plane is to travel 400 mph due north, how should the pilot set its air-velocity?
(Use coordinates: $\mathbf{i}$ = east, $\mathbf{j}$ = north.)

Please:
1) use only 1 color pen / pencil
2) do not include "extra thoughts"
3) avoid scratching

Problem 2. (Tues. 3 pts.) A molecule of methane CH_4 is modeled by a regular tetrahedron, with the carbon atom at its center and the four hydrogen atoms at its vertices. A carbon-hydrogen bond is modeled by the line segment joining the center with one of the vertices.

Find the *bond angle* — the angle between two of these C-H bonds — as follows.

a) Using solid lines, draw the xyz-axes in standard position, and on them draw the cube having as five of its vertices the points

$$O : (0,0,0),\ A : (2,0,0),\ B : (0,2,0),\ C : (0,0,2),\ \text{and}\ D : (2,2,2) \quad .$$

Draw using dashed lines the tetrahedron having A, B, C, D as its four vertices, then show without calculation that it is a regular tetrahedron, i.e., all its six edges have the same length.

b) Show by calculation that $P : (1,1,1)$ is the center of the tetrahedron, i.e., equidistant from its four vertices. (By symmetry you need only show that P is equidistant from two of the vertices – but not just any two.)

c) Find the bond angle in degrees by using vectors (calculator needed).

Problem 3. Thurs. (2 pts.) The important equation (A, B, ω are given constants):

$$A\cos\omega t + B\sin\omega t \;=\; C\cos(\omega t - \phi); \qquad C = \sqrt{A^2+B^2}, \quad \phi = \tan^{-1}(B/A)$$

expresses the sum of two oscillations with the same frequency as a single oscillation.

It can be proved by interpreting the two sides as the two different ways of calculating the scalar product of a constant vector $A\,\mathbf{i} + B\,\mathbf{j}$ with a vector depending on the time t. Prove it this way (and remember it!) – you can assume for ease in drawing that A and B are positive, though it is true in general.

Problem 4. (Thurs. 2 pts.) A right tetrahedron is one which can be placed so one vertex is at the origin, and the other three vertices lie on the three coordinate axes — say at the points where respectively $x = a$, $y = b$, and $z = c$.

Let A, B, C and D be the areas of the faces opposite to (i.e., not containing) the respective vertices at a, b, c, and the origin. (Note that three of these areas are "obvious"; only one of them has to be calculated.)

Prove the "Pythagorean theorem for a right tetrahedron": $A^2 + B^2 + C^2 = D^2$.

(Use the ideas of Lecture 2, not elementary geometry.)

Problem 5. (Thurs. 3 pts: 1,1,1)

Let $\mathbf{A} = 2\,\mathbf{i} + 3\,\mathbf{j} - 6\,\mathbf{k}$, and $\mathbf{B} = 6\,\mathbf{i} + 2\,\mathbf{j} + 3\,\mathbf{k}$. Use them to build up a right-handed coordinate system of unit origin vectors $\mathbf{i}'$, $\mathbf{j}'$, $\mathbf{k}'$ as follows:

a) Prove that $\mathbf{A}$ and $\mathbf{B}$ are orthogonal, and find unit vectors $\mathbf{i}'$ and $\mathbf{j}'$ in the directions of $\mathbf{A}$ and $\mathbf{B}$ respectively.

b) Using the cross product, find a third unit vector $\mathbf{k}'$ such that $\mathbf{i}'$, $\mathbf{j}'$, $\mathbf{k}'$ form a right-handed coordinate system. (It's easiest to work with $\mathbf{A}$ and $\mathbf{B}$.)

Check your work by verifying that $\mathbf{k}'$ is orthogonal to $\mathbf{i}'$ and $\mathbf{j}'$.

c) Let $\mathbf{F} = 3\,\mathbf{i} + \mathbf{j} - 2\,\mathbf{k}$.

To express $\mathbf{F}$ in terms of the primed coordinate system, one could solve for $\mathbf{i}, \mathbf{j}, \mathbf{k}$ in terms of $\mathbf{i}'$, $\mathbf{j}'$, $\mathbf{k}'$ and then substitute (see Problem 6).

An easier way is to observe that if $\mathbf{F} = a\,\mathbf{i}' + b\,\mathbf{j}' + c\,\mathbf{k}'$, then for example a is the component of $\mathbf{F}$ in the direction $\mathbf{i}'$. Use the dot product in this way to find a, b, and c.

Problem 6. (Fri. 3 pts; 2,1) Do part (c) of the previous problem as follows.

a) Using a suitable 3×3 matrix M, write symbolically (T denotes the transpose):

$$[\mathbf{i}'\ \mathbf{j}'\ \mathbf{k}']^T = M[\mathbf{i}\ \mathbf{j}\ \mathbf{k}]^T,$$

then calculate M^{-1}.

To make the calculations easier and better-looking, factor out of M the common denominators of its entries, writing $M = k\,N$, where N has integer entries and k is a constant. Then find N^{-1} and convert it to M^{-1}, using $M^{-1} = k^{-1}N^{-1}$.

b) To express $\mathbf{F}$ in terms of the primed unit vectors, do the substitution nicely by writing symbolically $\mathbf{F} = V[\mathbf{i}\ \mathbf{j}\ \mathbf{k}]^T$ for some suitable row vector $V = < v_1, v_2, v_3 >$, and then substitute for the column vector $[\mathbf{i}\ \mathbf{j}\ \mathbf{k}]^T$ using part (a), M^{-1}, and matrix multiplication.

Problem 7. (Fri. 4 pts: 2, 1, 1) (This problem and others like it on the next few problem sets give an introduction to MatLab, preferred by the MIT engineering program. Directions for using MatLab are at the end. If you prefer, you can use another system (Maple, Mathematica, etc.).

The competition in Brookline among HD TV suppliers centers on three cable suppliers: Verizon ("This is Fios, this is Expensive!"), Comcast, and RCN, and Other (non-cable cheapos like Dish and Satellite.)

Suppose the entries of the column vector $\mathbf{x} = [x_1, x_2, x_3, x_4]'$ (the $'$ denotes transpose) represent respectively the market share of each of the four suppliers; for example, x_1 is the fraction of all TV subscribers that use Verizon.

Suppose that after one year, as a result of consumer switching at the end of the initial year sign-up, 40% of the Verizon users have remained loyal, while 20% of the Comcast users have switched to Verizon, 30% of the RCN users, and 20% of the Others.

Then if y_1 represents the market share of Verizon after one year, we can write

$$y_1 = .4x_1 + .2x_2 + .3x_3 + .2x_4 .$$

Assuming various rates of switching to Comcast, RCN, and Other after a year, we get a matrix equation (y_2, y_3, y_4 are the new market shares of these other three suppliers)

$$\mathbf{y} = A\mathbf{x}$$

which we will write — changing the names of the column vectors — as

$$\mathbf{x1} = A\,\mathbf{x0}$$

(the original vector $\mathbf{x}$ is labeled as the starting vector $\mathbf{x0}$, and we change $\mathbf{y}$ to $\mathbf{x1}$ to show that it represents the new value of $\mathbf{x}$ after one year). Let's say that on the basis of data obtained by a market research firm, the matrix A is determined to have the value (to three significant figures, so $.4 = .400 = 40.0\%$)

$$\mathbf{A} = \begin{pmatrix} .4 & .2 & .3 & .2 \\ .2 & .3 & .1 & .2 \\ .1 & .4 & .4 & .3 \\ .3 & .1 & .2 & .3 \end{pmatrix}$$

a) Assume the switching matrix A remains the same year after year, so that the vector $\mathbf{x2} = A\,\mathbf{x1}$, $\mathbf{x3} = A\,\mathbf{x2}$, and so on; here the column vector $\mathbf{x}n$ gives the market shares after n years. Suppose the initial market shares are respectively (in percentages): 20.0, 30.0, 20.0, 30.0 . Using MatLab,

(i) calculate **x**1 and **x**2 to three significant figures, i.e. a % with one decimal place;
(ii) tell what the final market shares will be (to three significant figures) after several years have gone by, and the first year in which these final shares will appear.
(Use the operation which raises matrices to powers.)

b) Start instead with a set of initial market shares (percentages) of your own choosing, and find as before what the long-term market shares will be. (Give your choice for **x**0, and the value of n you used to find the long-term **x**n.)

c) Explain briefly why the columns in the matrix A all have 1 as their sum.

MatLab Directions

Access MatLab by selecting it from the Athena menu, or by typing:
% **add matlab** [return]   % **matlab** [return]
It may take a couple of minutes or more to appear, if there are a lot of Athena users.

MatLab calculates with matrices and vectors and draws graphs in 2D and 3D. Skip the Introduction and Help documents; as preliminary practice, just read and carry out the following. (Always hit [return] or [enter] to end a line or command.)

Entering matrices and vectors. Basically, in MatLab the variables represent matrices and vectors. The symbol = is used to assign values to the variables. In order, type each of these lines (proof-read very carefully to avoid error messages!), ending each with [return] and see what you get.

$A = [1\ 2\ 3;\ 4\ 5\ 6;\ 7\ 8\ 9]$ *(you can use commas instead of spaces: 1,2,3;)*
$b = [1\ 0\ 1]$
b'
eye(3)   *(eye = I, the identity matrix)*

Try a mistake: $C = [1,2,3;\ 4,5];$ to correct it, press any arrow key to get the line back.

Operations with matrices and vectors

Sum, difference	$A+B$, $A-B$	(matrices must be same size)
Product	$A*B$	(matrices must be compatibly sized)
Powers	. $A\hat{\ }n$	(A times itself n times; A must be square)
Quotient	left: $A\backslash b$	(the solution to $A\mathbf{x} = \mathbf{b}$)
	right: b/A	(the solution to $\mathbf{x}A = \mathbf{b}$)
Transpose	A'	
Inverse	inv(A)	

✓

Try typing (use the values of A and b above, and use [return] after each one):
$A + \text{eye}(3)$ $A*b$ $A*(b')$ $A*b'$ $3*b$

18.02 P-Set 1

2/4

Part 1 Day 1 — Vectors, addition, multiplication by scalar, dot (scalar) product

1A-3b A vector A has magnitude 6 and direction $\frac{\hat{i} + 2\hat{j} - 2\hat{k}}{3}$

Tail is at -2, 0, 1

Head ?

$\frac{1}{3}\hat{i} + \frac{2}{3}\hat{j} - \frac{2}{3}\hat{k} \cdot 6 = 2\hat{i} + 4\hat{j} - 4\hat{k}$ ← this is from head to tail

(-2, 0, 1)

(?, ?, ?) → ~~(-4, -4, 5)~~

Other way around (0, 4, -3)

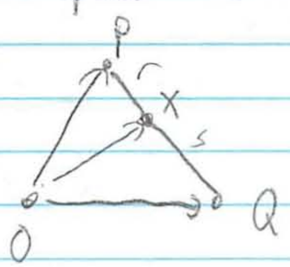
head = tail + A

↑ this makes more sense

Stop and think!

4b X divides PQ into ratio r:s where r + s = 1

Express OX in terms of OP and OQ



?? idk

OX = s OP + r OQ

OX = OP + r OQ ??

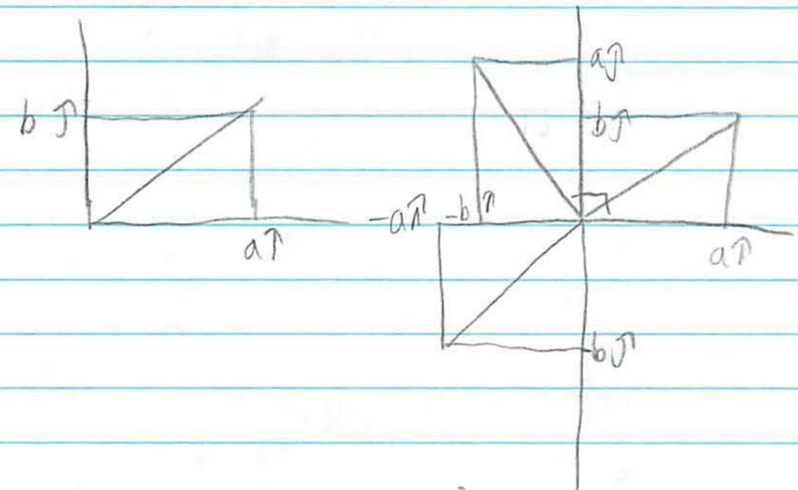
OX = (1-r) OP + r OQ ← ??

what do they want?

7b Let $a\hat{\imath} + b\hat{\jmath}$ be a plane vector. Find in terms of a and b the vectors $\vec{A}'$ and $\vec{A}''$ resulting by rotating $\vec{A}$ by 90° clockwise

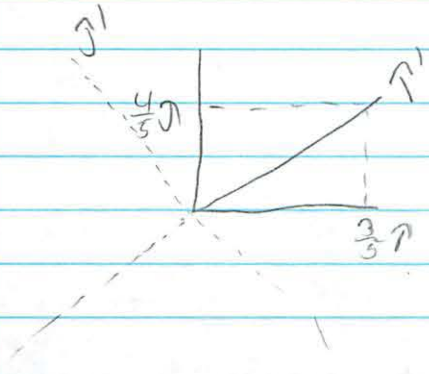
Hint: Make A the diagonal of a rectangle w/ sides on x and y axis and rotate the whole rectangle.

- Like we did in class



c Let $\hat{\imath}'$ be $\frac{(3\hat{\imath} + 4\hat{\jmath})}{5}$. Show that $\hat{\imath}'$ is a unit vector and use the first part of the exersise to find a vector $\hat{\jmath}'$ such that $\hat{\imath}'\hat{\jmath}'$ forms a right hand coord system

answer w/o looking



← coord system

$\frac{3}{5}^2 + \frac{4}{5}^2 = 1$

$\hat{\jmath} = -\frac{4}{5}\hat{\imath} + \frac{3}{5}\hat{\jmath}$

! h - that is just it rotated

8 The direction $\left(\frac{\vec{A}}{|\vec{A}|}\right)$ of a space vector is given by its direction cosines. Let $\vec{A} = a\hat{\imath} + b\hat{\jmath} + c\vec{k}$ represented as an origin vector and let α, β, γ be the 3 angles $(\leq \pi)$ that $\vec{A}$ makes respectivly with $\hat{\imath}, \hat{\jmath}, \vec{k}$

a Show that $\operatorname{dir}(\vec{A}) = \cos\alpha\,\hat{\imath} + \cos\beta\,\hat{\jmath} + \cos\gamma\,\vec{k}$

- did we do something like this in class or recitation?

this is 2 angles $\rightarrow$ $\cos\theta = \frac{\vec{A}\cdot\vec{B}}{|\vec{A}|\,|\vec{B}|}$

"is elementry trig"

$\frac{\vec{A}}{|\vec{A}|} = \frac{b\hat{\jmath}}{|\vec{A}|} + \frac{c\hat{k}}{|\vec{A}|} + \frac{a\hat{\imath}}{|\vec{A}|}$

$\cancel{\cos\alpha = \frac{a}{\sqrt{a^2+b^2+c^2}}}$

$\cos\alpha = \frac{b\hat{\jmath}}{\vec{A}}$

$\vec{A}\cos\alpha = b\hat{\jmath}$

$\hat{\jmath}$ component part

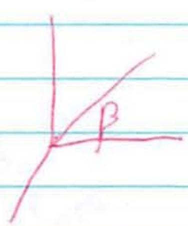
So how tie it all togeter?

$|\vec{u}|\cos\alpha \quad |\vec{u}|\cos\beta \quad |\vec{u}|\cos\gamma$

$\downarrow \quad \downarrow \quad \downarrow$

$\langle a, b, c\rangle$

?? don't really get



$\vec{u}\cdot\hat{\imath} = a$

$\vec{u}\cdot\hat{\jmath} = b = |\vec{u}|\,|\hat{\imath}|\cos\beta$

$u\cdot\hat{k} = c$

b Express the direction cosines in terms of a, b, c
Find the direction cosines of $-\hat{i}+2\hat{j}+2\hat{k}$

$$\frac{\vec{A}}{|\vec{A}|} = \frac{\langle a_1, a_2 \rangle}{\sqrt{a_1^2+a_2^2}}$$

$$\frac{\langle -1, 2, 2 \rangle}{\sqrt{1^2+4+4}} \qquad \frac{\langle 1, 2, 2 \rangle}{3}$$

✓ dir A

$$\cos\alpha = \frac{a}{a^2+b^2+c^2}$$

?oh 2 pt qu
don't get why this is

1B-1b ✓ Find the angle between the vectors

$\hat{i}+\hat{j}+2\hat{k}$ and $2\hat{i}-\hat{j}+\hat{k}$

$$\frac{\vec{A}\cdot\vec{B}}{|\vec{A}||\vec{B}|} = \cos\theta$$

$$\frac{1\cdot 2 + 1\cdot -1 + 2\cdot 1}{\sqrt{1^2+1^2+2^2}\sqrt{2^2+1^2+1^2}}$$ multiply each part

$$\frac{3}{6} = \frac{1}{2} = \cos\theta \qquad \theta = \frac{\pi}{3}$$

2a ✓ Tell for what values of c the vectors $c\hat{i} + 2\hat{j} - \hat{k}$ and $\hat{i} - \hat{j} + 2\hat{k}$ will be orthogonal (aka perpendicular)

will be orthogonal / perpendicular if dot product $= 0$

$$|\vec{A}||\vec{B}|\cos\theta = 0$$

$$\sqrt{c^2 + 2^2 + 1^2} \cdot \sqrt{4^2 + 1^2 + 2^2} \cos\theta$$

multiply each part

set $= 0$

$$(c \cdot 1) + (2 \cdot -1) + (-1 \cdot 2) = 0$$

$$1c + -2 + -2 = 0$$

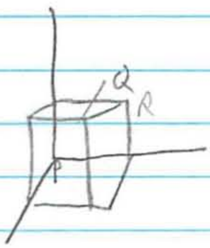
$$+2 \quad\quad +2$$

$$c = -4$$

✓

3a Using vectors, find the angle between a longest diagonal PQ of a cube and a diagonal PR of a face

(Hint: choose a size + position for cube)



$PQ = 1\hat{i} + 1\hat{j} + 1\hat{k}$

$PR = 1\hat{i} + 1\hat{j}$

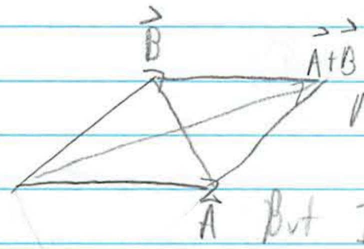
$$(1 \cdot 1) + (1 \cdot 1) + (1 \cdot 0) = \sqrt{1^2 + 1^2 + 1^2} \cdot \sqrt{1^2 + 1^2} \cos\theta$$

$$2 = \sqrt{3} \cdot \sqrt{2} \cos\theta$$

$$\cos\theta = \frac{2}{\sqrt{3}\sqrt{2}} = 35.26°$$

✓ Got it

11 Prove using Vector methods (w/o components) that the diagonals of a parallelogram have equal lenghts if and only if it is a rectangle.



Most certainly not =

But I am not good at writing proofs

Still don't really get vector stuff should write out cheat sheet w/ everything

$\vec{A}-\vec{B}$ $(\vec{A}+\vec{B})\cdot(\vec{A}-\vec{B}) = \underbrace{\vec{A}\cdot\vec{A} - \vec{B}\cdot\vec{B}}_{\text{how did they get this??}}$

therefore

$(\vec{A}+\vec{B})\cdot(\vec{A}-\vec{B}) = 0$

and

$\vec{A}\cdot\vec{A} = \vec{B}\cdot\vec{B}$

diagonals are = if and only if 2 adjecent edges have = lenght (is a rhombus)

Lecture 2 Small determinants, cross vector product in 3D

1C-2b Calculate determinent - Laplace 1st coloumn

$$\begin{vmatrix} + \\ - \\ + \end{vmatrix} \qquad \begin{vmatrix} -1 & 0 & 4 \\ 1 & 2 & 2 \\ 3 & -2 & -1 \end{vmatrix} \qquad -1\begin{vmatrix} 2 & 2 \\ -2 & -1 \end{vmatrix} - 1\begin{vmatrix} 0 & 4 \\ -2 & -1 \end{vmatrix} + 3\begin{vmatrix} 0 & 4 \\ 2 & 2 \end{vmatrix}$$

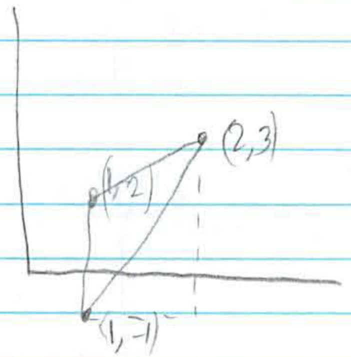
$$-1(-2+4) - 1(0+8) + 3(0-8)$$

$$-2 - 8 - 24$$

$$-34$$

3b ✓ Find the area of the plane triangle whose verticies lie at

$(1,2)\ (1,-1)\ (2,3)$



How does this have to do w/ vectors

$\frac{1}{2} bh$

$\frac{1}{2}(1)(4)$

2

Not what they had in mind

sides are $PQ = (0, -3)$
$PR = (1, 1)$,

$$\left|\begin{vmatrix} 0 & -3 \\ 1 & 1 \end{vmatrix}\right| = 3$$

area paralagram $= 3$
area triangle $= \frac{1}{2} \cdot 3 = \frac{3}{2}$

What is a plane triangle?

4. Show that $\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \end{vmatrix} = (x_1 - x_2)(x_2 - x_3)(x_3 - x_1)$

(This is a Vandermonde determinant)

$$1 \cdot x_2 \cdot x_3^2 + 1 \cdot x_3 \cdot x_1^2 + 1 \cdot x_1 \cdot x_2^2$$
$$-(1 \cdot x_2 \cdot x_1^2) - (1 \cdot x_1 \cdot x_3^2) - (1 \cdot x_3 \cdot x_1^2)$$

$$x_2 \cdot x_3^2 + x_3 x_1^2 + x_1 x_2^2 - x_2 \cdot x_1^2 - x_1 x_3^2 - x_3 \cdot x_1^2$$

? what next factor otter group

$$x_1^2 x_2 - x_1^2 x_3 - \cancel{x_1 x_3 x_2} + x_1 x_3^2 - x_2^2 x_1 + \cancel{x_2 x_1 x_3} + x_2^2 x^3 - x_2 x$$

So they are =

9. Use the formula in 1C-8 to calculate the volume of a tetrahedron w/ verticies $(0,0,0)$

PQ $(0,-1,2)$ Volume $= \frac{1}{3} bh$

PR $(0,1,-1)$

PS $(1,2,1)$

volume of a parallelpiped

$$= \pm \begin{vmatrix} 0 & -1 & 2 \\ 0 & 1 & -1 \\ 1 & 2 & 1 \end{vmatrix} = \pm(-1) = 1$$

volume tetrahedron $= \frac{1}{3} bh$ $\rightarrow$

$= \frac{1}{3} \cdot \frac{1}{2}$ parallepiped base $\cdot$ hignt

$= \frac{1}{6}$ volume parallelepiped

$= \frac{1}{6} \cdot 1 = \frac{1}{6}$

What is significene again?
- adding vectors = parallelagram
- determinant = volume "
- and in 3D some thing

So find determinant of 3 vectors, you have the volume par. piped
And ask yourself how does desired shape relate to a parallel piped

this is taking too long!

Cross Products

1D-1b ✓ Find $\vec{A} \times \vec{B}$

$\vec{A} = 2\hat{i} + 0\hat{j} - 3\hat{k}$
$\vec{B} = \hat{i} + \hat{j} - \hat{k}$

Its the area still
The determinent when written like this

-2--3

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 0 & -3 \\ 1 & 1 & -1 \end{vmatrix} = \begin{vmatrix} 0 & -3 \\ 1 & -1 \end{vmatrix} \hat{i} - \begin{vmatrix} 2 & -3 \\ 1 & -1 \end{vmatrix} \hat{j} + \begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} \hat{k}$$

$-3\hat{i} - \hat{j} + 2\hat{k}$

✓ correct

2. Find the area of the triangle in space having its vertices at the points

P $(2,0,1)$
Q $(3,1,0)$
R $(-1,1,-1)$

Is this like as before?

$$\frac{1}{2}\begin{vmatrix} 2 & 0 & 1 \\ 3 & 1 & 0 \\ -1 & 1 & -1 \end{vmatrix}$$

Remember origin does not matter.

$$\frac{1}{2}\left(2\begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix} - 0 + 1\begin{vmatrix} 3 & 1 \\ -1 & 1 \end{vmatrix}\right)$$

$$\frac{1}{2}(2\cdot -2 + 4)$$

$$\frac{1}{2}\cdot 0$$

$$0$$

No $PQ = \hat{i} + \hat{j} - \hat{k}$ } they have come from here
$PR = -3\hat{i} + \hat{j} - 2\hat{k}$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & -1 \\ -3 & 1 & -2 \end{vmatrix} = -1\hat{i} + 5\hat{j} + 4\hat{k}$$

area $= \frac{1}{2}|PQ \times PR| = \frac{1}{2}\sqrt{42}$

5. What can you conclude about $\vec{A}$ and $\vec{B}$

a) if $|\vec{A} \times \vec{B}| = |\vec{A}||\vec{B}|$

area of porralagram — magnitudes

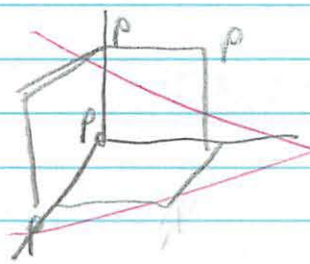
∴ they are 90° (orthagonal)?

b) if $|\vec{A} \times \vec{B}| = \vec{A} \cdot \vec{B}$
$= |A||B| \cos\theta$

That $\cos\theta = 0$ (at $\cancel{0}$, 90, $\cancel{180}$, 270, $\cancel{360}$°)

$\frac{\pi}{4}$

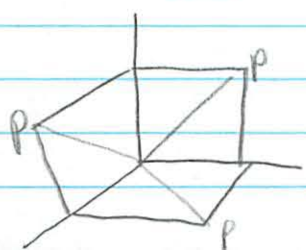
6. Take 3 faces of a unit cube having a common vertex P
each face has a diagonal ending at P
what is the volume of the parallel piped having these 3 diagonals as a coterminous edge?



call unit lengnt

$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$

want faces



$\hat{i} + \hat{j}$
$\hat{j} + \hat{k}$ $\rightarrow$ $\begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix}$ $\rightarrow$
$\hat{k} + \hat{i}$

starting to make sens,

$$1\begin{vmatrix}1 & 1\\0 & 1\end{vmatrix} - 1\begin{vmatrix}0 & 1\\1 & 1\end{vmatrix} + 0$$

$$1 \cdot 1 \quad - \quad 1 \cdot -1$$

2 ✓

Lecture 3 Matrices, inverse matrices

1F-5b ✓ $A = \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}$ Compute A^2, A^3, A^n

A squared?

A^2

$$\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix} = \begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}$$

$1 \cdot 1 + 1 \cdot 1$

← ~~same~~

Oh

$$\begin{bmatrix}(1\cdot 1 + 1 \cdot 0) & (1 \cdot 1 + 1 \cdot 1)\\(0 \cdot 1 + 1 \cdot 0) & (0 \cdot 1 + 1 \cdot 1)\end{bmatrix}$$

A^3

$$\begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix} = \begin{bmatrix}1 & 3\\0 & 1\end{bmatrix}$$

← ~~same~~ n

A^n $\begin{bmatrix}1 & n\\0 & 1\end{bmatrix}$

Proof via induction - Oh that's how its done - induction

$$\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}^{n+1} = \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}^{n}\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix} = \begin{bmatrix}1 & n\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix} = \begin{bmatrix}1 & n+1\\0 & 1\end{bmatrix}$$

↑ I did not know this property

$A = 3 \times 3$ matrix
what is A?

8a. If $A \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$ $\begin{bmatrix} x_1 \cdot 1 + x_2 0 + x_3 \cdot 0 = 2 \\ x_4 \cdot 1 \quad = 3 \\ x_7 \cdot 1 \quad = 1 \end{bmatrix}$

clever

$A \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix}$

? how

$\begin{bmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{bmatrix} A \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \leftarrow \begin{bmatrix} x_1 \cdot 0 + x_2 \cdot 0 + x_3 \cdot 1 = 1 \\ x_4 \cdot 0 + x_5 \cdot 0 + x_6 \cdot 1 = 1 \\ x_7 \cdot 0 + x_8 \cdot 0 + x_9 \cdot 1 = -1 \end{bmatrix}$

A B

ith row A
jth Column B

$$\begin{bmatrix} 2 & -1 & 1 \\ 3 & 0 & 1 \\ 1 & 4 & -1 \end{bmatrix}$$

✓ Very good q ✓

16-3 $A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ -1 & -1 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$ Solve $Ax = B$ by finding A^{-1}

? how does finding A^{-1} help?
x is the unknown vector

$$Ax = B$$
$$A^{-1}(Ax) = A^{-1}B$$
$$Ix = A^{-1}B$$

? why drop? I,

$$\boxed{Ix = A^{-1}B}$$

$$A^{-1} = \frac{1}{\det(A)} \cdot \text{adjoint}(A)$$

Step 1 = Determinant (w/ minors)

$$\begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & -\cancel{0}2 \\ -\cancel{0}2 & 1 & 1 \end{bmatrix} \quad \begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix} \quad 1 \cdot 2 - 1 \cdot -1$$

minor math mistakes

Step 2 = Cofactor (change signs)

$$\begin{bmatrix} 3 & -1 & 1 \\ +1 & 3 & +\cancel{0}2 \\ -\cancel{0}2 & -1 & 1 \end{bmatrix}$$

Step 3 = Transpose for adjoint

$$\begin{bmatrix} 3 & 1 & -\cancel{0}2 \\ -1 & 3 & -1 \\ 1 & +\cancel{0}2 & 1 \end{bmatrix}$$

Step 4 Find determinant of original

$$(1 \cdot 1 \cdot 2) + (-1 \cdot 1 \cdot -1) + (1 \cdot 0 \cdot -1)$$
$$2 + 1$$
$$\cancel{3}5 = \text{seems right}$$

$$A^{-1} = \frac{1}{\cancel{3}5} \begin{bmatrix} 3 & 1 & -2 \\ -1 & 3 & -1 \\ 1 & 2 & 1 \end{bmatrix}$$

Now multiply by B

last time pressure - however long it takes

$$\frac{1}{\cancel{3}\ 5}\begin{bmatrix} 3 & -1 & 1 \\ 1 & 3 & \cancel{2}\ 0 \\ -\cancel{2}\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 3\cdot 2 + -1\cdot 0 + 1\cdot 3 = 9 \\ 1\cdot 2 + 3\cdot 0 + 0\cdot 3 = 2 \\ 0\cdot 2 + -1\cdot 0 + 1\cdot 3 = 3 \end{bmatrix} \begin{matrix} 0 \\ -5 \\ 5 \end{matrix}$$

$$\frac{1}{\cancel{3}\ 5}\begin{bmatrix} 0 & 9 \\ -5 & 2 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} 3 \\ \frac{2}{3} \\ 1 \end{bmatrix} \quad \boxed{\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}}$$

goes wrong fast - lots of opportunity to screw up

4. Along with #3 ↑ solve

$$x_1 - x_2 + x_3 = y_1$$
$$x_2 + x_3 = y_2$$
$$-x_1 - x_2 + 2x_3 = y_3$$

for x as a function of y

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ -1 & -1 & 2 \end{bmatrix}$$

What is this type of problem again?

is $Ax = y$
solution $x = A^{-1}y$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{5}\underbrace{\begin{bmatrix} 3 & 1 & -2 \\ -1 & 3 & -1 \\ 1 & 2 & 1 \end{bmatrix}}_{\uparrow A^{-1}}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$x_1 = \frac{3}{5}y_1 + \frac{1}{5}y_2 - \frac{2}{5}y_3$$
$$x_2 = -\frac{1}{5}y_1 + \frac{3}{5}y_2 - \frac{1}{5}y_3$$
$$x_3 = \frac{1}{5}y_1 + \frac{2}{5}y_2 + \frac{1}{5}y_3$$

← vector x

5. Show that $(AB)^{-1} = B^{-1} A'$ using def. of inverse matrix

Use associtive, def of inverse, idenity law

$$(AB)C = A(BC) \qquad \frac{1}{B} = B^{-1} \qquad B B^{-1} = I$$
$$(C AB = C(AB)$$

$$(B^{-1}A^{-1}) AB \rightarrow B^{-1}(A^{-1}A) B$$

$B^{-1}B \circ A^{-1}A$

$I \quad I$

I^2

$B^{-1} I B$ how does I disappear?

$B^{-1}B$

I

$$(AB)(B^{-1}A^{-1}) = I$$

so $B^{-1}A^{-1}$ is inverse of AB

I don't really get what this proves

Lecture 4 Theorems about square systems, Equation of Planes

1A-3a For what c-values will
$$\begin{aligned} x_1 - x_2 + x_3 &= 0 \\ 2x_1 + x_2 + x_3 &= 0 \\ -x_1 + cx_2 + 2x_3 &= 0 \end{aligned}$$
have a non-trivial solution

Only if determinant of $A = 0$ $\quad A \cdot x = d$

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & 1 \\ -1 & c & 2 \end{bmatrix} \qquad \begin{aligned} 1(2-c) - -1(4 - -1) + 1(2c - -1) &= 0 \\ 2 - c + 5 + 2c + 1 &= 0 \\ c &= -8 \end{aligned}$$

good work

u, v, w are now in same plane
parralelpiped flat

3b For what c values will $\begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = c\begin{pmatrix} x \\ y \end{pmatrix}$

have a non-trival solution?

Write it as a system of homogeneous equations

When the determinant of $A = 0$

? but no c there, or variables

What are homogeneous equations?

- the right hand side $= 0$??

$A\vec{x} = 0$

$(2-c)x + y = 0$

$(-1-c)y = 0$

← where did they get that? when does c apply

$\begin{vmatrix} 2-c & 1 \\ 0 & -1-c \end{vmatrix} = 0$

$(2-c)(-1-c) - 1 = 0$

$-2 - 2c + c + c^2 = 1$

$c^2 - c - 3 = 0$

factor?

factors don't work w/ 1

~~just set each = 1~~

$2-c=1$ $\quad -1-c=-1$

$-c=+1$ $\quad -c=-2$

$c=-1$ $\quad c=+2$

c. For each value of c in part a find a non trival solution to the system

- asking for a vector orthogonal [cross product]

Want vector $(x_1\ \ x_2\ \ x_3)$

$(1\ -1\ \ 1)\quad (2\ 1\ 1)\quad (-1\ -8\ \ 2)$

Orthogonal to first two points

cross product to get orth vector

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 1 \\ 2 & 1 & 1 \end{vmatrix}$$

$$(-1-1)\hat{i} - (1-2)\hat{j} + (1--2)\hat{k}$$
$$-2\hat{i} + 1\hat{j} + 3\hat{k} \quad ✓$$

Orthogonal to this

dot product 2 orth vectors $=0$

$$\cancel{\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 1 & 3 \\ -1 & -8 & 2 \end{vmatrix}}$$

dot product

$$\cancel{(2-24)\hat{i} + (-4--3)\hat{j} + (-16--1)\hat{k}}$$
$$\cancel{-22\hat{i} + \hat{j} - 15\hat{k}}$$

$$(-2\cdot -1) + (1\cdot 8) + (3\cdot 2)$$
$$2 - 8 + 6$$
$$0$$

7. Suppose we wanted to find a pure oscillation (sine wave) of frequency 1 passing through 2 pts

$$f(x) = \overset{\text{constants}}{a} \cos x + b \sin x$$

$$f(x_1) = y_1$$
$$f(x_2) = y_2$$

a) Show this is possible in 1 and only 1 way if we assume $x_2 \neq x_1 + n\pi$ for every integer n

So unique solution $\vec{x} = A^{-1}\vec{d}$ for $A\vec{x} = \vec{d}$

← What is A ?

$$a \cos x_1 + b \sin x_1 = y_1$$
$$a \cos x_2 + b \sin x_2 = y_2$$

← duh $f(x)$ - should be able to decode notation

has unique solution $\begin{vmatrix} \cos x_1 & \sin x_1 \\ \cos x_2 & \sin x_2 \end{vmatrix} \neq 0$

? where did they get this from?

if $\cos x_1 \sin x_2 - \cos x_2 \sin x_1 \neq 0$

$\sin(x_2 - x_1) \neq 0$

only if $x_2 - x_1 \neq n\pi$ for any n

✓

1E - 1c Find the equation of the following plane

through $(1,0,1)$ $(2,-1,2)$ $(-1,3,2)$

Recall $ax+by+cz=d$ defines a plane

Find 2 vectors from Δ and then cross product? yes ✓

$\langle 1\ -1\ 1 \rangle$ $\langle 3,-4,0 \rangle$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 1 \\ 3 & -4 & 0 \end{vmatrix}$$

$(0--4)\hat{i} - (0-3)\hat{j} + (-4--3)\hat{k}$

$4\hat{i} + 3\hat{j} - 1\hat{j}$

$\cancel{4x+3y-z=d}$

Equation through a point

$4(x-1)+3(y-0)-1(z-1)=0$

$4x-4+3y-z+1=0$

$4x+3y-z=3$ ✓

does not matter that they $\cdot -1$ since its direction which does not matter, right?

d through the points on the x, y, z axes where

x = a
y = b $Ax + By + Cz = 1$
z = c

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

?, what is going on here
why $\frac{1}{a}$ $\frac{1}{b}$ $\frac{1}{c}$

✓

2. Find the dihedral angle between the planes

$$2x - y + z = 3$$
$$x + y + 2z = 1$$

dihedral angle b/w 2 planes

$$\cos\phi = N_A \cdot N_B$$

Find normal angle to each

$$\cancel{(a\ b\ c) \cdot (x\ y\ z) = 0}$$
$$\cancel{(2\ {-1}\ 1) \cdot (x\ y\ z) = 3}$$
$$\cancel{A \qquad X \qquad = d}$$
$$\cancel{(1\ 1\ 2)\ (x\ y\ z) = 1}$$

just (2, -1, 1)
(1 1 2)

$$\cos\theta = \frac{\overset{N_A \cdot N_B}{(2\cdot 1) + (-1\cdot 1) + (1\cdot 2)}}{\sqrt{2^2+1^2+1^2}\ \sqrt{1^2+1^2+2^2}} \Big/ |N_A||N_B|$$

$$2 \quad -1 \quad +2$$

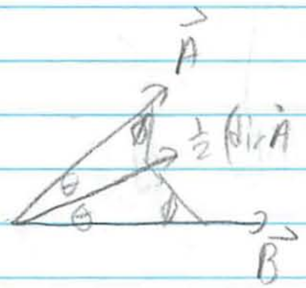
$$\cos\theta = \frac{3}{\sqrt{6}\ \sqrt{6}} \qquad \theta = 60° \quad \frac{\pi}{3}$$

Part 2

1. The direction of $\vec{A} = \dfrac{\vec{A}}{|\vec{A}|}$ ← lenght

a. Show that if $\vec{A}$ and $\vec{B}$ have the same tail, then $\frac{1}{2}(\text{dir } \vec{A} + \text{dir } \vec{B})$ bisects the angle between them (use 1A-4b and congruent triangles) deduce that $|\vec{B}|\vec{A} + |\vec{A}|\vec{B}$ also does

?



1A-4B $OX = sOP + rOQ$

$1 - r = s$

? still does not make much sense

congruent = same

$|\vec{B}|\vec{A} + |\vec{A}|\vec{B}$

? lenght B, ? lenght + dir $\vec{A}$, ? lenght A, ? lenght + dir $\vec{B}$

$= \text{dir } \vec{A} \circ \text{dir } \vec{B} \circ 2|\vec{B}| \circ 2|\vec{A}|$

? but no $\frac{1}{2}$ longer

so how can it be in middle

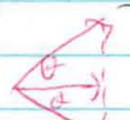
Oliver

How do dirs add?

- they are unit vectors
- or angles
- I guess angles add from 0°

does not matter dir()

half of parralagram



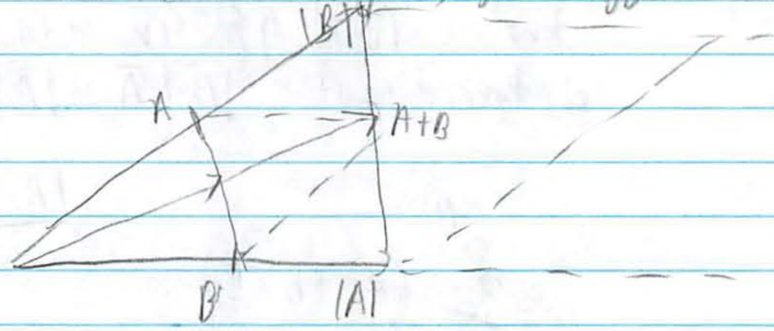
2 triangles are same

angles are same

(Over)

You want to relate to 2 vectors
Just change its lenght

Or for second part do they mean
use dir of A and lenght of B
– well double lenght, so bigger



Same deal

look @ the solutions

-1

b. Velocities in moving media like Air or water are represented by vectors

<u>Airplane</u> $V_g = V_a + V_w$ wind

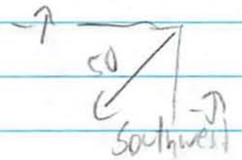
$\uparrow$ plane air velocity (in cockpit)

plane's ground velocity (observed)

If wind 50 mph southwest $= V_w$ $\qquad \hat{i}$ east

plane 400 mph north $= V_g$?? $\qquad \hat{j} =$ north

air velocity? $= V_a$



$= (-25\sqrt{2}\,\hat{i} + -25\sqrt{2}\,\hat{j})$

the wind is blowing from southwest to northeast

-0.5

$$400\,\hat{j} = -25\sqrt{2}\,\hat{i} + -25\sqrt{2}\,\hat{j} + V_a$$

easy to do it this way

$$0\,\hat{i} = V_a - 25\sqrt{2}\,\hat{i}$$

$$V_a = 25\sqrt{2}\,\hat{i}$$

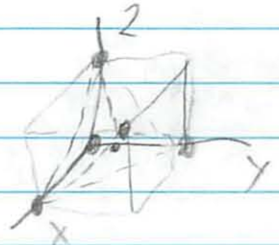
$$400\,\hat{j} = V_a - 25\sqrt{2}\,\hat{j}$$

$$V_a = -25\sqrt{2}\,\hat{j} - 400\,\hat{j}$$

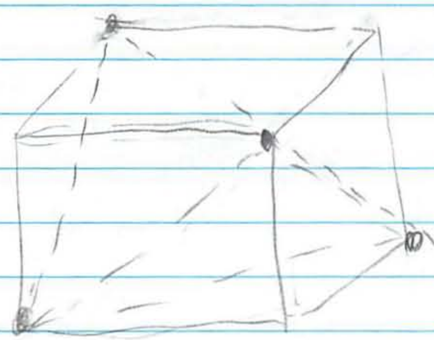
$$V_a = 425\sqrt{2}\,\hat{j}$$

$$V_a = 25\sqrt{2}\,\hat{i} + 425\sqrt{2}\,\hat{j}$$

2. A molecule of methane CH_4 is modeled by a regular tetrahedron w/ carbon at center and the 4 hydrogen atoms at its verticies. Find the bond angle



tetrahedron = pryrmid



? do all sides have same lenght
I'm really confused
I need a 3D model

b. Show that (1, 1, 1) is center

Sides (6)			Δ ← cool it works
	(2 0 0)	(0 2 0)	(2 2 0)
a) no calculations !	(2 0 0)	(0 0 2)	(2 0 2)
	(2 0 0)	(2 2 2)	(0 2 2)
-1	(0 2 0)	(0 0 2)	(0 2 2)
	(0 2 0)	(2 2 2)	(2 0 2)
	(0 0 2)	(2 2 2)	(2 2 0)

(0 0 2) (2 2 2)
(0 2 0) (2 0 0)

Sides don't go through center
lenght b/w $\vec{PA}$ ← $\frac{1}{2}$
do several check same →

How in all world do I show this?

Distance from Vortex

$(222) \to (111) \quad (-1\,-1\,-1)$
$(002) \to (111) \quad (1\,1\,-1)$
$(020) \to (111) \quad (1\,-1\,1)$
$(200) \to (111) \quad (-1\,1\,1)$

= distance change - vector

same in all cases

length $|\vec{PA}|\,|\vec{PB}|\,|\vec{PC}| = ?$ -1

C Find bond angle w/ calculator

How do you find bond angles again?
- think → 3.091
- was just given

Angle b/w edge + face $\arccos(\sqrt{2})$ = 54.736°
" " 2 faces $\arccos\left(\frac{1}{3}\right)$ = 70.529°
$= \arctan(2\sqrt{2})$
" " center + 2 vertecies $= \arccos\left(-\frac{1}{3}\right)$ = 109.47°

think this is right

Can look up CH_4 as well

Dot product gets you access

✓

$$\cos\theta = \frac{\vec{A}\cdot\vec{C}}{|A|\cdot|C|} = \frac{-1\cdot 0\cdot 0}{\sqrt{3}\,\sqrt{3}} = -\frac{1}{3} = 109.47^\circ$$ ✓

3. Important Equation (A, B, w constants; given)

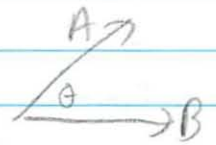
$$A \cos wt + B \sin wt = C \cos(wt - \phi)$$
$$C = \sqrt{A^2 + B2} \qquad \phi = \tan^{-1}\left(\frac{B}{A}\right)$$

Is the sum of 2 oscillations w/ same freq as a single ocillation,

Can be proved by interpreting the 2 sides as the 2 different ways of calculating the scalar product of a constant vector $A\hat{\imath} + B\hat{\jmath}$ with the vector depending on the time T

Prove it this way (and remember it)
Assume A and B (+)

Oh this is familiar from SHO from 8.01



Scalar Product $|\vec{A}|\,|\vec{B}| \cos\theta$
or $(a_1 \cdot b_1) + (a_2 b_2)$

$$(A\hat{\imath} + B\hat{\jmath}) \cdot \vec{T} = \sqrt{A^2+B^2}\,\sqrt{T^2}\,\cos\theta$$
$$\sqrt{A^2+B^2}\,\sqrt{T^2}\,\cos\left(\tan^{-1}\left(\frac{B}{a}\right)\right)$$
$$C \quad \sqrt{T^2} \quad \cos(\phi)$$

$\vec{T} = (\cos\theta, \sin\theta)$
? pick T as this

$\tan^{-1}\left(\frac{B}{A}\right) = \frac{1}{\tan\left(\frac{A}{A}\right)}$
$\cdot \tan\frac{B}{A}$
$= 1$
?

? ~~what next - how~~ $-\phi$? ✓

$$C \quad \sqrt{\cos^2 + \sin^2} \quad \cos\phi$$
$$C \circ 1 \circ \cos\phi$$

$\phi = wt - \theta$

$$C \cos(wt - \theta)$$

Still don't really get

4. A right tetrahedron can be placed so that one vertex is at the origin and the other 3 lie on the coordinate axes $x=a$ $y=b$ $z=c$

not another tetrahedron qv

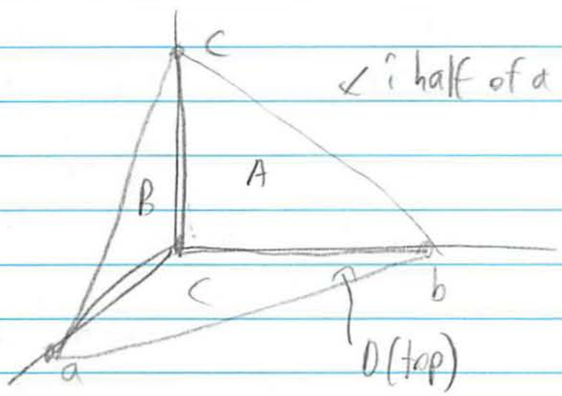
Let A, B, C, D be the faces opposite (NOT containing) the respective verticies $a, b, c, 0$

3 of the areas are obvious, I calculated

Prove $A^2+B^2+C^2=D^2$ w/ Lecture 2 (determinants and cross products)

↑ not relevant ↑ perhaps

better drawing of one



↙ half of a cube?

Relate area of different faces

↙ can not assume lenght = 1

area $A, B, C = \frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2}$

$\frac{ab}{2}$ not given length

area D = ? IDK

Cross product of 2 vectors is area parallelogram divide by 2

$$\frac{1}{2}^2 + \frac{1}{2}^2 + \frac{1}{2}^2 = D^2$$

$$\sqrt{.25 + .25 + .25} = D$$

$$\sqrt{.75} = D$$

$$.866025 = D$$

But I have to find D another way to prove

?

From Wikipedia: Base Plane Area $= \frac{\sqrt{3}}{4} a^2$

But how to prove using cross products

$\vec{A} \times \vec{B}$ = area of parrallepiped

oliver

take $\frac{1}{2}$ for triangle (2D)
trapizoid (3D)

$$\frac{|\vec{Pa} \times \vec{Pb}|}{2}$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -a & b & 0 \\ -a & 0 & c \end{vmatrix}$$ ✓

area $\frac{bc\,\hat{i} + ac\,\hat{j} + ab\,\hat{k}}{2}$ ✓ $\rightarrow \sqrt{b^2c^2 + a^2c^2 + a^2b^2}$

$$\left(\frac{ab}{2}\right)^2 + \left(\frac{bc}{2}\right)^2 + \left(\frac{ca}{2}\right)^2 = \left(\frac{\sqrt{b^2c^2 + a^2c^2 + a^2b^2}}{2}\right)^2$$ ✓

5. Let $A = 2\hat{\imath} + 3\hat{\jmath} - 6\hat{k}$ $\quad B = 6\hat{\imath} + 2\hat{\jmath} + 3\hat{k}$

Use them to build a right-handed coord system $\hat{\imath}'\,\hat{\jmath}'\,\hat{k}'$

a. Prove A and B are orthogonal. Find unit vectors $\hat{\imath}'$ and $\hat{\jmath}'$ in dir(A) and (B) respectivly

If orthogonal → dot product = 0

$$(2 \cdot 6) + (3 \cdot 2) + (-6 \cdot 3)$$

$$12 + 6 - 18$$

$$0 \checkmark$$

$$\text{dir}(A) = \frac{\vec{A}}{|\vec{A}|} = \frac{\langle 2, 3, -6\rangle}{\sqrt{2^2+3^2+6^2}} = \left\langle \tfrac{2}{7}, \tfrac{3}{7}, -\tfrac{6}{7} \right\rangle = \hat{\imath}' \checkmark$$

$$\text{dir}(B) = \frac{\vec{B}}{|\vec{B}|} = \frac{\langle 6, 2, 3\rangle}{\sqrt{6^2+2^2+3^2}} = \left\langle \tfrac{6}{7}, \tfrac{2}{7}, \tfrac{3}{7} \right\rangle \checkmark = \hat{\jmath}'$$

b. Using the cross product find k' so coord system
(Use $\vec{A}$ and $\vec{B}$)

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 3 & -6 \\ 6 & 2 & 3 \end{vmatrix}$$

$$\begin{vmatrix} 3 & -6 \\ 2 & 3 \end{vmatrix} \hat{\imath} - \begin{vmatrix} 2 & -6 \\ 6 & 3 \end{vmatrix} \hat{\jmath} + \begin{vmatrix} 2 & 3 \\ 6 & 2 \end{vmatrix} \hat{k}$$

$$(9 - -12)\hat{\imath} - (6 - -36)\hat{\jmath} + (4 - 18)\hat{k}$$

$$21\hat{\imath} - 42\hat{\jmath} - 14\hat{k} \checkmark$$

Check if orthogonal

$(2 \cdot 21) + (3 \cdot -42) + (-6 \cdot -14)$

$42 + -126 + 84$

0 ✓

$\text{dir}(\vec{C}) = \dfrac{\langle 21, -42, -14 \rangle}{\sqrt{21^2 + 42^2 + 14^2}} = \langle \frac{21}{49}, -\frac{42}{49}, -\frac{14}{49} \rangle = \hat{k}'$

$= \langle \frac{3}{7}, \frac{-6}{7}, -\frac{2}{7} \rangle$ ✓

c Let $F = 3\hat{\imath} + \hat{\jmath} - 2\hat{k}$

To express F in terms of primed coordinate system, solve for $\hat{\imath}, \hat{\jmath}, k$ in terms of $\hat{\imath}', \hat{\jmath}', \hat{k}'$ and substitute (see #6) Hint: $\vec{F} \cdot \hat{\imath}' = a$

An easier way is to observe that

$F = a\hat{\imath}' + b\hat{\jmath}' + c\hat{k}'$ so a is the $\hat{\imath}'$ component of F

Use the dot product this way to find a, b, c

right ??

$\cancel{\hat{\imath} \quad 3 \cdot \langle \frac{2}{7}, \frac{3}{7}, -\frac{6}{7} \rangle = \langle \frac{6}{7}, \frac{9}{7}, \frac{-18}{7} \rangle}$

$\cancel{\hat{\jmath} \quad 1 \cdot \langle \frac{6}{7}, \frac{2}{7}, \frac{3}{7} \rangle = \langle \frac{6}{7}, \frac{2}{7}, \frac{3}{7} \rangle}$

$\cancel{\hat{k} \quad -2 \cdot \langle \frac{3}{7}, -\frac{6}{7}, -\frac{2}{7} \rangle = \langle -\frac{6}{7}, \frac{+12}{7}, \frac{+4}{7} \rangle}$

Oliver

$\vec{F} \cdot \hat{\imath}' = a \qquad \underbrace{(a\hat{\imath}' + b\hat{\jmath}' + c\hat{k}')}_{F} \cdot \hat{\imath}' = ax$

Kimberly

$a = \vec{F}\,\hat{\imath}'$ dot multiply w/ the conversion factor

$= (3\hat{\imath} + \hat{\jmath} - 2\hat{k})(\frac{2}{7}\hat{\imath} + \frac{3}{7}\hat{\jmath} - 6/7\,\hat{k})$

$= \frac{6}{7} + \frac{3}{7} + \frac{12}{7}$

$= \frac{21}{7}$

$a = 3$

$$b = \vec{F} \cdot \hat{j}$$

$$(3\hat{i} + \hat{j} - 2\hat{k})\left(\frac{6}{7}\hat{i} + \frac{2}{7}\hat{j} + \frac{3}{7}\hat{k}\right)$$

$$\frac{18}{7} + \frac{2}{7} - \frac{6}{7}$$

$$\frac{14}{7}$$

$$2$$

$$c = \vec{F} \cdot \hat{k}$$

$$= (3\hat{i} + \hat{j} - 2\hat{k})\left(\frac{3}{7}\hat{i} - \frac{6}{7}\hat{j} - \frac{2}{7}\hat{k}\right)$$

$$= \frac{9}{7} - \frac{6}{7} + \frac{4}{7}$$

$$= \frac{7}{7}$$

$$= 1$$

$$F' = 3\hat{i}' + 2\hat{j}' + \hat{k}'$$ ✓

6. Do part C of previous problem as follows

a) Use a 3x3 matrix M

$$[\hat{i}' \hat{j}' \hat{k}']^T = M [\hat{i} \hat{j} \hat{k}]^T$$

Calculate M^{-1}. Factor out M of denominators

$M = kN$ ← integer
↑ konstant

Find N^{-1}

Convert $M^{-1} = k^{-1} N^{-1}$

?

$$\begin{bmatrix} 3 & 1 & -2 \end{bmatrix}$$ ← what constant to factor out

? M is a 3x3 matrix

$$\vec{i}' = \frac{2}{7}\vec{i} + \frac{3}{7}\vec{j} - \frac{6}{7}\vec{k}$$

Oliver OH

Find Matrix M

can factor out the $\frac{1}{7}$ so simpler to work

$$\begin{bmatrix} \vec{i}' \\ \vec{j}' \\ \vec{k}' \end{bmatrix} = \frac{1}{7}\underbrace{\begin{bmatrix} 2 & 3 & -6 \\ 6 & 2 & 3 \\ 3 & -6 & -2 \end{bmatrix}}_{M}\begin{bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{bmatrix}$$

$$F = \underbrace{\begin{bmatrix} \cdot & \cdot & \cdot \end{bmatrix}}\begin{bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{bmatrix}$$ use matricies

? from #5 V = row vector $\langle V_1, V_2, V_3 \rangle$

$$\begin{bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{bmatrix} = M^{-1}\begin{bmatrix} \vec{i}' \\ \vec{j}' \\ \vec{k}' \end{bmatrix}$$

$$F = [V] M^{-1}\begin{bmatrix} \vec{i}' \\ \vec{j}' \\ \vec{k}' \end{bmatrix}$$

← with $\frac{1}{7}$ when doing inverse

Matlab

$$\Pi^{-1} = \frac{1}{7}\begin{bmatrix} 2 & 6 & 3 \\ 3 & 2 & -6 \\ -6 & 3 & -2 \end{bmatrix}$$

b.) ↑ part b really starts here

$$F = V\begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix} = V\,\Pi^{-1}\begin{bmatrix} \hat{i}' \\ \hat{j}' \\ \hat{k}' \end{bmatrix}$$

$$F = \begin{bmatrix} 3 & 1 & 2 \end{bmatrix}\begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix} = \begin{bmatrix} 3 & 1 & 2 \end{bmatrix}\frac{1}{7}\begin{bmatrix} 2 & 6 & 3 \\ 3 & 2 & -6 \\ -6 & 3 & -2 \end{bmatrix}\begin{bmatrix} \hat{i}' \\ \hat{j}' \\ \hat{k}' \end{bmatrix}$$

$$F = 3\hat{i} + \hat{j} + 2\hat{k}$$

$$= \frac{1}{7}\begin{bmatrix} 21 \\ 14 \\ 7 \end{bmatrix}\begin{bmatrix} \hat{i}' \\ \hat{j}' \\ \hat{k}' \end{bmatrix}$$

Kimberly

$$= \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}\begin{bmatrix} \hat{i}' \\ \hat{j}' \\ \hat{k}' \end{bmatrix}$$

$$= 3\hat{i}' + 2\hat{j}' + \hat{k}'$$

b. To express F in terms of the primed unit vectors do the substitution nicely by writing

$F = V\,[\hat{i}\ \hat{j}\ \hat{k}]^{t}$ for some suitible row vector $V = \langle V_1, V_2\ V_3 \rangle$ and then substitute for the column vector $[\hat{i}\ \hat{j}\ \hat{k}]^{t}$ using part a M^{-1} and matrix multiplication

$$\begin{bmatrix} \vec{i}' \\ \vec{j}' \\ \hat{k}' \end{bmatrix} = M \begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix}$$

Can do same notation w/ V

$\boxed{\text{did as part A}}$ ✓

7, Matlab

Fios is not that expensive
And satilight is expensive

1 Verizon
2 Comcast
3 RCN
4 Other

$X = [x_1 \; x_2 \; x_3 \; x_4]'$

fraction verizon

$x = [20 \; 30 \; 20 \; 30]$

end of year 1

% switched to Verizon

$Y_1 = .4x_1 + .2x_2 + .3x_3 + .2x_4$

% remaining loyal

market share after 1 year

Y_2 = new market share Comcast

$y = Ax$

$X1 = Ax0$

original vector x

new version of x after 1 year (previously y)

$$A = \begin{pmatrix} .4 & .2 & .3 & .2 \\ .2 & .3 & .1 & .2 \\ .1 & .4 & .4 & .3 \\ .3 & .1 & .2 & .3 \end{pmatrix}$$

3 sig fig
40.0%

a) A stays the same every year

$x2 = Ax1$

$x3 = Ax2$

x_n = market share after n years

i) Calculate x_1 and x_2

(Had to do $x \cdot A$) ← b/c did not Transpose don't understand what A is

$x_1 = [25 \quad 24 \quad 23 \quad 25]$
$x_2 = [24.6 \quad 23.9 \quad 24.1 \quad 24.2]$ ×
$x_3 = [24.3 \quad 24.2 \quad 24.3 \quad 24.2]$
$x_4 = [24.2 \quad 24.2 \quad 24.2 \quad 24.2]$

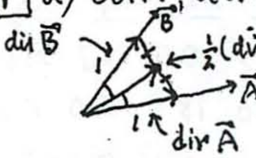
ii) In year 4 market share will flatten at 24.2 % ×

b) $x_0 = [70 \quad 20 \quad 10 \quad 0]$ −3
$x_1 = [33 \quad 24 \quad 27 \quad 21]$ ×
$x_2 = [27 \quad 26.7 \quad 27.3 \quad 25.8]$
$x_3 = [26.6 \quad 26.9 \quad 26.9 \quad 26.7]$
$x_4 = [26.7 \quad 26.8 \quad 26.7 \quad 26.7]$ ← why over 100%?

c) 1 = 100%, percentages must add to 100% ✓

18.02 Prob. Set 1 Solns (Part II) S-2010

1 a) dir $\vec{A}$, dir $\vec{B}$ are unit vectors



$\leftarrow \frac{1}{2}(\text{dir}\vec{A} + \text{dir}\vec{B})$ goes to the midpoint, by 1A-4b (since $r = s$).

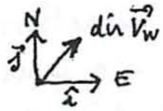
The triangles are congruent (s-s-s)

$\therefore$ the two angles are equal.

since $\frac{1}{2}\left(\frac{\vec{A}}{|A|} + \frac{\vec{B}}{|B|}\right)$ bisects the angle, so does every scalar multiple, since it has the same direction.

and $2|A|\cdot|B| \cdot \frac{1}{2}\left(\frac{\vec{A}}{|A|} + \frac{\vec{B}}{|B|}\right) = |B|\vec{A} + |A|\vec{B}$.

b) $\vec{V}_g = \vec{V}_a + \vec{V}_w$

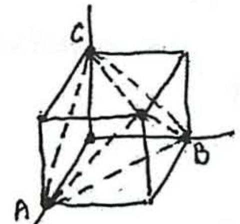


$\vec{V}_w = 50\frac{(\hat{i}+\hat{j})}{\sqrt{2}}$

$\vec{V}_g = 400\hat{j}$ $\qquad \vec{V}_a = \vec{V}_g - \vec{V}_w$

$\therefore \quad \vec{V}_a = -\frac{50}{\sqrt{2}}\hat{i} + \left(400 - \frac{50}{\sqrt{2}}\right)\hat{j}$

2 a)



All six edges are the diagonals of 2×2 squares, $\therefore$ have same length.

b) By symmetry, $|\vec{PA}| = |\vec{PB}| = |\vec{PC}|$

$\vec{PA} = \langle 1, -1, -1\rangle, \quad |\vec{PA}| = \sqrt{3}$

$\vec{PD} = \langle 1, 1, 1\rangle, \quad |\vec{PD}| = \sqrt{3}$

$\therefore$ P is equidistant from A, B, C, D.

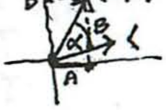
c) Cosine of the angle between say $\vec{PA}$ and $\vec{PD}$ $= \frac{\vec{PA}\cdot\vec{PD}}{|\vec{PA}|\,|\vec{PD}|}$

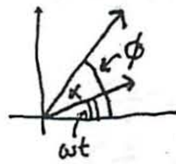
$\cos\alpha = \frac{\langle 1,-1,-1\rangle}{\sqrt{3}} \cdot \frac{\langle 1,1,1\rangle}{\sqrt{3}} = \frac{-1}{3}$

$\alpha \approx 109°$ (by calculator: $\cos^{-1}(-\frac{1}{3})$)

3 $A\cos\omega t + B\sin\omega t = \langle A, B\rangle \cdot \langle \cos\omega t, \sin\omega t\rangle$

$= \sqrt{A^2+B^2} \cdot 1 \cdot \cos\alpha$



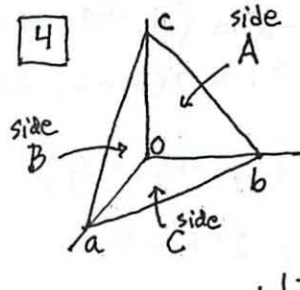


$\alpha = \phi - \omega t$

$\cos\alpha = \cos(\phi - \omega t) = \cos(\omega t - \phi)$

where $\tan\phi = \frac{B}{A}$, $\quad \phi = \tan^{-1}\frac{B}{A}$

4



top slanted face is D,

Areas of 3 sides:

$A = \frac{1}{2}bc$

$B = \frac{1}{2}ac$

$C = \frac{1}{2}ab$

Area of $D = \frac{1}{2}\left|\vec{ab} \times \vec{ac}\right|$

$\vec{ab} = -a\hat{i} + b\hat{j} = \langle -a, b, 0\rangle$

$\vec{ac} = -a\hat{i} + c\hat{j} = \langle -a, 0, c\rangle$

(using head-to-tail addition: $\vec{ab} = \vec{OB} - \vec{OA}$)

$\therefore \vec{ab} \times \vec{ac} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -a & b & 0 \\ -a & 0 & c \end{vmatrix} = \langle bc, ac, ab\rangle$

To verify that $A^2 + B^2 + C^2 = D^2$:

left side $= \frac{1}{4}(bc)^2 + \frac{1}{4}(ac)^2 + \frac{1}{4}(ab)^2$ ⊛

right side D^2 $= \Big[\underbrace{\tfrac{1}{2}\left|\vec{ab}\times\vec{ac}\right|}_{\text{area } D}\Big]^2$

$= \frac{1}{4}\left|\langle bc, ac, ab\rangle\right|^2$

$= \frac{1}{4}\left((bc)^2 + (ac)^2 + (ab)^2\right),$

which agrees with ⊛

$\boxed{5}$ $\vec{A} = \langle 2,3,-6\rangle$, $\vec{B} = \langle 6,2,3\rangle$

a) $\vec{A}, \vec{B}$ are orthogonal, since

$$\vec{A}\cdot\vec{B} = (12+6-18) = 0$$

$$|\vec{A}| = |\vec{B}| = 7$$

$$\therefore\ \hat{\imath}' = \text{dir}\,\vec{A} = \langle \tfrac{2}{7}, \tfrac{3}{7}, -\tfrac{6}{7}\rangle$$

$$\hat{\jmath}' = \text{dir}\,\vec{B} = \langle \tfrac{6}{7}, \tfrac{2}{7}, \tfrac{3}{7}\rangle$$

b) $$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 3 & -6 \\ 6 & 2 & 3 \end{vmatrix} = \langle 21, -42, -14\rangle = 7\langle 3,-6,-2\rangle$$

$$\hat{k}' = \text{dir}\,\vec{A}\times\vec{B} = \text{dir}\,\langle 3,-6,-2\rangle = \langle 3/7, -6/7, -2/7\rangle$$

(Check that $\hat{\imath}'\cdot\hat{k}' = 0$, $\hat{\jmath}'\cdot\hat{k}' = 0$.)

c) $\vec{F} = 3\hat{\imath} + \hat{\jmath} - 2\hat{k} = a\hat{\imath}' + b\hat{\jmath}' + c\hat{k}'$

Since $\hat{\imath}', \hat{\jmath}', \hat{k}'$ are unit vectors,

$$a = \vec{F}\cdot\hat{\imath}' = \langle 3,1,-2\rangle\cdot\tfrac{1}{7}\langle 2,3,-6\rangle = 3$$

$$b = \vec{F}\cdot\hat{\jmath}' = \langle 3,1,-2\rangle\cdot\tfrac{1}{7}\langle 6,2,3\rangle = 2$$

$$c = \vec{F}\cdot\hat{k}' = \langle 3,1,-2\rangle\cdot\tfrac{1}{7}\langle 3,-6,-2\rangle = 1$$

$$\therefore\ \vec{F} = 3\hat{\imath}' + 2\hat{\jmath}' + \hat{k}'$$

$\boxed{6}$ $$\begin{bmatrix}\hat{\imath}' \\ \hat{\jmath}' \\ \hat{k}'\end{bmatrix} = \frac{1}{7}\underbrace{\begin{bmatrix} 2 & 3 & -6 \\ 6 & 2 & 3 \\ 3 & -6 & -2\end{bmatrix}}_{N}\begin{bmatrix}\hat{\imath} \\ \hat{\jmath} \\ \hat{k}\end{bmatrix}$$ by 5a, 5b.

$(M = \tfrac{1}{7}N)$.

Find N^{-1}: $\det(N) = 343 = 7^3$ (by straight calculation)

$$\begin{bmatrix} 2 & 3 & -6 \\ 6 & 2 & 3 \\ 3 & -6 & -2\end{bmatrix} \to \underbrace{\begin{bmatrix} 14 & 21 & -42 \\ 42 & 14 & 21 \\ 21 & -42 & -14\end{bmatrix}}_{\text{matrix of cofactors}}$$

$$\to \underbrace{\begin{bmatrix} 14 & 42 & 21 \\ 21 & 14 & -42 \\ -42 & 21 & -14\end{bmatrix}}_{\text{adjoint mx.}} = \frac{7}{7^3}\begin{bmatrix} 2 & 6 & 3 \\ 3 & 2 & -6 \\ -6 & 3 & -2\end{bmatrix} = N^{-1}$$

$$\therefore\ M^{-1} = 7N^{-1} = \frac{1}{7}\begin{bmatrix} 2 & 6 & 3 \\ 3 & 2 & -6 \\ -6 & 3 & -2\end{bmatrix}$$

$$\vec{F} = \langle 3,1,-2\rangle\begin{bmatrix}\hat{\imath} \\ \hat{\jmath} \\ \hat{k}\end{bmatrix} = \langle 3,1,-2\rangle\cdot\underbrace{\frac{1}{7}\begin{bmatrix} 2 & 6 & 3 \\ 3 & 2 & -6 \\ -6 & 3 & -2\end{bmatrix}}_{M^{-1}}\begin{bmatrix}\hat{\imath}' \\ \hat{\jmath}' \\ \hat{k}'\end{bmatrix}$$

$$= \langle 3,2,1\rangle\begin{bmatrix}\hat{\imath}' \\ \hat{\jmath}' \\ \hat{k}'\end{bmatrix}$$ checks! with $\boxed{5}$c

$\boxed{7}$ (percentages shown: 20% = .2)

	X_0	X_1	X_2	X_3	X_4	X_5	X_∞
a)	20	26	28.3	28.7	28.7	28.6	29
	30	21	19.0	18.9	19.0	19.0	[illegible]
	20	31	30.0	29.2	29.1	29.1	29.1
	30	22	22.7	23.2	23.3	23.3	23.3

X_∞ is the long-term value, which thus first appears when $n=5$ (after 5 years have gone by).

b) The X_∞ should be what's above, regardless of the starting X_0 — give your X_0 and the value of n which you used to get the long-term X_∞.

c) Look at column 1, say (the reasoning is the same for the other columns.
Its entries show the % of Verizon subscribers who switched after 1 year to (respectively):

(40%) Verizon (i.e., stayed with Verizon)
(20%) Comcast
(10%) RCN
(30%) Other

These must total to 100% of the Verizon subscribers, i.e. to 1.

18.02 Lecture 3. – Tue, Sept 11, 2007

Remark: $\boldsymbol{A}\times\boldsymbol{B} = -\boldsymbol{B}\times\boldsymbol{A}$, $\boldsymbol{A}\times\boldsymbol{A} = 0$.

Application of cross product: equation of plane through P_1, P_2, P_3: $P = (x, y, z)$ is in the plane iff $\det(\overrightarrow{P_1P}, \overrightarrow{P_1P_2}, \overrightarrow{P_1P_3}) = 0$, or equivalently, $\overrightarrow{P_1P}\cdot\boldsymbol{N} = 0$, where $\boldsymbol{N}$ is the normal vector $\boldsymbol{N} = \overrightarrow{P_1P_2}\times\overrightarrow{P_1P_3}$. I explained this geometrically, and showed how we get the same equation both ways.

Matrices. Often quantities are related by linear transformations; e.g. changing coordinate systems, from $P = (x_1, x_2, x_3)$ to something more adapted to the problem, with new coordinates (u_1, u_2, u_3). For example

$$\begin{cases} u_1 = 2x_1 + 3x_2 + 3x_3 \\ u_2 = 2x_1 + 4x_2 + 5x_3 \\ u_3 = x_1 + x_2 + 2x_3 \end{cases}$$

Rewrite using matrix product: $\begin{bmatrix} 2 & 3 & 3 \\ 2 & 4 & 5 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$, i.e. $AX = U$.

Entries in the matrix product = dot product between rows of A and columns of X. (here we multiply a 3x3 matrix by a column vector = 3x1 matrix).

More generally, matrix multiplication AB:

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}\begin{bmatrix} \cdot & 0 \\ \cdot & 3 \\ \cdot & 0 \\ \cdot & 2 \end{bmatrix} = \begin{bmatrix} \cdot & 14 \\ \cdot & \cdot \\ \cdot & \cdot \end{bmatrix}$$

(Also explained one can set up A to the left, B to the top, then each entry of AB = dot product between row to its left and column above it).

Note: for this to make sense, width of A must equal height of B.

What AB means: BX = apply transformation B to vector X, so $(AB)X = A(BX)$ = apply first B then A. (so matrix multiplication is like composing transformations, but from right to left!)

(Remark: matrix product is not commutative, AB is in general not the same as BA – one of the two need not even make sense if sizes not compatible).

Identity matrix: identity transformation $IX = X$. $\quad I_{3\times3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Example: $R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ = plane rotation by 90 degrees counterclockwise.

$R\hat{\imath} = \hat{\jmath}$, $R\hat{\jmath} = -\hat{\imath}$, $R^2 = -I$.

Inverse matrix. Inverse of a matrix A (necessarily square) is a matrix $M = A^{-1}$ such that $AM = MA = I_n$.

A^{-1} corresponds to the reciprocal linear relation.

E.g., solution to linear system $AX = U$: can solve for X as function of U by $X = A^{-1}U$.

Cofactor method to find A^{-1} (efficient for small matrices; for large matrices computer software uses other algorithms): $A^{-1} = \frac{1}{\det(A)}\mathrm{adj}(A)$ ($\mathrm{adj}(A)$ = "adjoint matrix").

Illustration on example: starting from $A = \begin{bmatrix} 2 & 3 & 3 \\ 2 & 4 & 5 \\ 1 & 1 & 2 \end{bmatrix}$

1) matrix of minors (= determinants formed by deleting one row and one column from A): $\begin{bmatrix} 3 & -1 & -2 \\ 3 & 1 & -1 \\ 3 & 4 & 2 \end{bmatrix}$ (e.g. top-left is $\begin{vmatrix} 4 & 5 \\ 1 & 2 \end{vmatrix} = 3$).

2) cofactors = flip signs according to checkerboard diagram $\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$: get $\begin{bmatrix} 3 & +1 & -2 \\ -3 & 1 & +1 \\ 3 & -4 & 2 \end{bmatrix}$

3) transpose = exchange rows / columns (read horizontally, write vertically) get the adjoint matrix $M^T = adj(A) = \begin{bmatrix} 3 & -3 & 3 \\ 1 & 1 & -4 \\ -2 & 1 & 2 \end{bmatrix}$

4) divide by $\det(A)$ (here = 3): get $A^{-1} = \frac{1}{3}\begin{bmatrix} 3 & -3 & 3 \\ 1 & 1 & -4 \\ -2 & 1 & 2 \end{bmatrix}$.

18.02 Lecture 4. – Thu, Sept 13, 2007

Handouts: PS1 solutions; PS2.

A plane is defined by its normal vector

Equations of planes. Recall an equation of the form $ax + by + cz = d$ defines a plane.

1) plane through origin with normal vector $\boldsymbol{N} = \langle 1, 5, 10\rangle$: $P = (x, y, z)$ is in the plane $\Leftrightarrow$ $\boldsymbol{N} \cdot \overrightarrow{OP} = 0 \Leftrightarrow \langle 1, 5, 10\rangle \cdot \langle x, y, z\rangle = x + 5y + 10z = 0$. Coefficients of the equation are the components of the normal vector.

2) plane through $P_0 = (2, 1, -1)$ with same normal vector $\boldsymbol{N} = \langle 1, 5, 10\rangle$: parallel to the previous one! P is in the plane $\Leftrightarrow \boxed{\boldsymbol{N} \cdot \overrightarrow{P_0P} = 0} \Leftrightarrow (x - 2) + 5(y - 1) + 10(z + 1) = 0$, or $x + 5y + 10z = -3$. Again coefficients of equation = components of normal vector.

when not through (0 0 0)

(Note: the equation multiplied by a constant still defines the same plane).

So, to find the equation of a plane, we really need to look for the normal vector $\boldsymbol{N}$; we can e.g. find it by cross-product of 2 vectors that are in the plane.

Flashcard question: the vector $\boldsymbol{v} = \langle 1, 2, -1\rangle$ and the plane $x + y + 3z = 5$ are 1) parallel, 2) perpendicular, 3) neither?

<1 1 3> dot product = 0 it $\perp$

(A perpendicular vector would be proportional to the coefficients, i.e. to $\langle 1, 1, 3\rangle$; let's test if it's in the plane: equivalent to being $\perp$ $\boldsymbol{N}$. We have $\boldsymbol{v} \cdot \boldsymbol{N} = 1 + 2 - 3 = 0$ so $\boldsymbol{v}$ is parallel to the plane.)

wait - why parallel ?

Interpretation of 3x3 systems. A 3x3 system asks for the intersection of 3 planes. Two planes intersect in a line, and usually the third plane intersects it in a single point (picture shown). The unique solution to $AX = B$ is given by $X = A^{-1}B$.

iii parallel to?

think about more

Exception: if the 3rd plane is parallel to the line of intersection of the first two? What can happen? (asked on flashcards for possibilities).

If the line $\mathcal{P}_1 \cap \mathcal{P}_2$ is contained in $\mathcal{P}_3$ there are infinitely many solutions (the line); if it is parallel to $\mathcal{P}_3$ there are no solutions. (could also get a plane of solutions if all three equations are the same)

These special cases correspond to systems with $\det(A) = 0$. Then we can't invert A to solve the system: recall $A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A)$. Theorem: A is invertible $\Leftrightarrow$ $\det A \neq 0$.

Homogeneous systems: $AX = 0$. Then all 3 planes pass through the origin, so there is the obvious ("trivial") solution $X = 0$. If $\det A \neq 0$ then this solution is unique: $X = A^{-1}0 = 0$. Otherwise, if $\det A = 0$ there are infinitely many solutions (forming a line or a plane).

Note: $\det A = 0$ means $\det(\boldsymbol{N}_1, \boldsymbol{N}_2, \boldsymbol{N}_3) = 0$, where $\boldsymbol{N}_i$ are the normals to the planes $\mathcal{P}_i$. This means the parallelepiped formed by the $\boldsymbol{N}_i$ has no area, i.e. they are coplanar (showed picture of 3 planes intersecting in a line, and their coplanar normals). The line of solutions is then perpendicular to the plane containing $\boldsymbol{N}_i$. For example we can get a vector along the line of intersection by taking a cross-product $\boldsymbol{N}_1 \times \boldsymbol{N}_2$.

General systems: $AX = B$: compared to $AX = 0$, all the planes are shifted to parallel positions from their initial ones. If $\det A \neq 0$ then unique solution is $X = A^{-1}B$. If $\det A = 0$, either there are infinitely many solutions or there are no solutions.

(We don't have tools to decide whether it's infinitely many or none, although elimination will let us find out).

18.02 Lecture 5. – Fri, Sept 14, 2007

Lines. We've seen a line as intersection of 2 planes. Other representation = parametric equation = as trajectory of a moving point.

E.g. line through $Q_0 = (-1, 2, 2)$, $Q_1 = (1, 3, -1)$: moving point $Q(t)$ starts at Q_0 at $t = 0$, moves at constant speed along line, reaches Q_1 at $t = 1$: its "velocity" is $\vec{v} = \overrightarrow{Q_0Q_1}$; $\overrightarrow{Q_0Q(t)} = t\overrightarrow{Q_0Q_1}$. On example: $\langle x+1, y-2, z-2\rangle = t\langle 2, 1, -3\rangle$, i.e.

$$\begin{cases} x(t) = -1 + 2t, \\ y(t) = 2 + t, \\ z(t) = 2 - 3t \end{cases}$$

Lines and planes. Understand where lines and planes intersect.

Flashcard question: relative positions of Q_0, Q_1 with respect to plane $x + 2y + 4z = 7$? (same side, opposite sides, one is in the plane, can't tell).

(A sizeable number of students erroneously answered that one is in the plane.)

Answer: plug coordinates into equation of plane: at Q_0, $x+2y+4z = 11 > 7$; at Q_1, $x+2y+4z = 3 < 7$; so opposite sides.

Intersection of line Q_0Q_1 with plane? When does the moving point $Q(t)$ lie in the plane? Check: at $Q(t)$, $x + 2y + 4z = (-1 + 2t) + 2(2 + t) + 4(2 - 3t) = 11 - 8t$, so condition is $11 - 8t = 7$, or $t = 1/2$. Intersection point: $Q(t = \frac{1}{2}) = (0,\ 5/2,\ 1/2)$.

(What would happen if the line was parallel to the plane, or inside it. Answer: when plugging the coordinates of $Q(t)$ into the plane equation we'd get a constant, equal to 7 if the line is contained in the plane – so all values of t are solutions – or to something else if the line is parallel to the plane – so there are no solutions.)

General parametric curves.

Example: cycloid: wheel rolling on floor, motion of a point P on the rim. (Drew picture, then showed an applet illustrating the motion and plotting the cycloid).

Position of P? Choice of parameter: e.g., θ, the angle the wheel has turned since initial position. Distance wheel has travelled is equal to arclength on circumference of the circle $= a\theta$.

Setup: x-axis = floor, initial position of P = origin; introduce A = point of contact of wheel on floor, B = center of wheel. Decompose $\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BP}$.

$\overrightarrow{OA} = \langle a\theta, 0\rangle$; $\overrightarrow{AB} = \langle 0, a\rangle$. Length of $\overrightarrow{BP}$ is a, and direction is θ from the $(-y)$-axis, so $\overrightarrow{BP} = \langle -a\sin\theta, -a\cos\theta\rangle$. Hence the *position vector* is $\overrightarrow{OP} = \langle a\theta - a\sin\theta, a - a\cos\theta\rangle$.

Q: What happens near bottom point? (flashcards: corner point with finite slopes on left and right; looped curve; smooth graph with horizontal tangent; vertical tangent (cusp)).

Answer: use Taylor approximation: for $t \to 0$, $f(t) \approx f(0) + tf'(0) + \frac{1}{2}t^2 f''(0) + \frac{1}{6}t^3 f'''(0) + \ldots$. This gives $\sin\theta \approx \theta - \theta^3/6$ and $\cos\theta \approx 1 - \theta^2/2$. So $x(\theta) \simeq \theta^3/6$, $y(\theta) \simeq \theta^2/2$ Hence for $\theta \to 0$, $y/x \simeq (\frac{1}{2}\theta^2)/(\frac{1}{6}\theta^3) = 3/\theta \to \infty$: vertical tangent.

Lecture 5
Parametric Equations

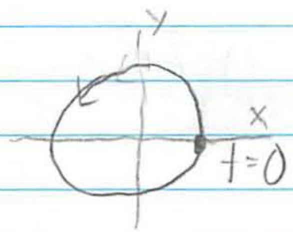
2/11

Parametric equation
↳ a variable less important than others
a letter that is really a constant

← or $f(t), g(t), h(t)$

Plane $\left(\begin{array}{l} x = x(t) \\ y = y(t) \\ z = z(t) \end{array}\right)$ in space

t = time in physics
otherwise can represent
something geometric
might not even know



never see time on graph

$$x = a\cos t$$
$$y = a\sin t$$

Now need to eliminate t
- can be very hard
or impossible

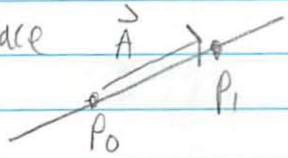
$$x^2 + y^2$$
$$a^2\cos^2 t + a^2\sin^2 t =$$
$$a^2(1)$$
$$a^2$$

Need to know the trick

← And loss of info

Often don't want to eliminate t

Space



Want parametric equation
for uniform motion on the line

data: point P_0 on line
$\vec{A}$ on line (having direction) $= \langle a_1\ a_2\ a_3 \rangle$

Is a Locus problem (:?)

$P(x, y, z)$ lies on the line $\Leftrightarrow$

$\boxed{\overrightarrow{P_0P} = t\vec{A}}$

scalar multiple of $t \cdot \vec{A}$

$$\langle x - x_0, y - y_0, z - z_0 \rangle = \langle a_1 t, a_2 t, a_3 t \rangle$$

$$= \begin{cases} x - x_0 = a_1 t \\ y - y_0 = a_2 t \\ z - z_0 = a_3 t \end{cases}$$

$$= \begin{cases} x = x_0 + a_1 t \\ y = y_0 + a_2 t \\ z = y_0 + a_3 t \end{cases}$$

Have a Plane $3x - 2y + z = 6$
Line P_0 $(1, 1, 3)$
P_1 $(2, 4, 4)$

Where does the line through these 2 points
intersect the plane?

$\vec{A} = \langle 1, 3, 1 \rangle$

$x = 1 + 1t$ 1. Pick P_0

$y = 1 + 3t$ 2. Add $A \cdot t$ for each component

$z = 3 + 1t$

↑ represents every possible pt on line

$3(1+t) - 2(1+3t) + 1(3+t) = 6$ 3. Plug into plane

$3 + 3t - 2 - 6t + 3 + t = 4$

$-3 \quad +2 \quad -3 \quad -4$

$3t - 6t + t = 2$

$-2t = 2$

$t = -1$ 4. Solve for t

5. Find pt intersection

$x = 1 + 1(-1)$

$y = 1 + 3(-1)$

$z = 3 + (-1)$

$(0, -2, 2)$ does lie on plane

↑ otherwise would not get ans

Steps Given Parametric Equation

1. Find curve
   - maybe eliminate t
2. Given geometry, find para equations
3. Given para. equations, get info

Vectors

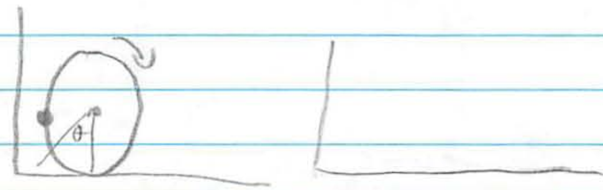
$$\vec{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j}$$

position vector (origin)

$\vec{r}$ is moving as t passes
interested where tip goes

Why the vector?
Complicated motions can be expressed as the sum of simpler ones.
- sphere rotating
- pt is moving on surface of sphere

For example a cycloid



the path of the point fixed to wheel as the wheel rolls w/o slipping on x-axis

Solved in notes
- Diff in Pset

Find the parametric equations of cycloid
Not too interesting in terms of time
Use something geometric
- like θ of how wheel rotated

$$\vec{r} = \vec{OP}$$

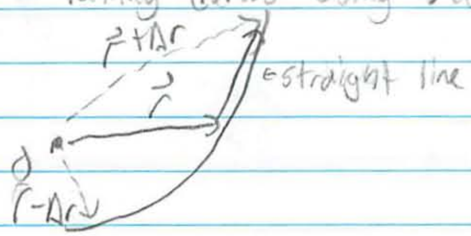
easy, easier, bit hard: cos and sin, strange

$$\vec{OP} = \vec{OA} + \vec{AB} + \vec{BP}$$

head - tail method Not parallelogram

2. $\vec{V}$ = velocity

taking derivs using vectors



Velocity $\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t}$ $\frac{\text{vector}}{\text{scalar}}$ $= \frac{d\vec{r}}{dt}$

to understand meaning, use chain rule

can see arc lenght along curve $= \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta s}$ ← arc lenght ($\Delta s \to 0$)

$= \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta s} \cdot \frac{\Delta s}{\Delta t} = \frac{d\vec{r}}{ds} \cdot \underbrace{\frac{ds}{dt}}_{\text{speed}} = \vec{V}$

$= \vec{T} \cdot \frac{ds}{dt}$

$\vec{T}$: unit vector, ratio 2 lenghts = 1; points same Δr tangent to curve; direction of v

$\frac{ds}{dt}$ ← the speed along curve, scalar

fundemental decomposition of velocity

Advice for HW2:
Good for 3 lectures + exam

Calculate $\vec{T}, \vec{a}, \vec{v}, \frac{ds}{dt} \ldots$

1. Avoid calculating s if possible

2. Instead differenitiate w/ respect to any param avalible

3. Use the chain rule to convert to s

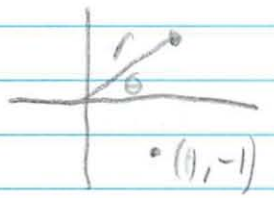
$$\vec{v} = \frac{d\vec{r}}{dt} \qquad \vec{T} = \frac{d\vec{r}}{ds} = \frac{d\vec{r}/dt}{ds/dt} = \frac{\vec{v}}{|\vec{v}|}$$

Lecture 6
Polar Coords

2/12

Polar Coordinate System

- vector + motions in polar coordinates



(r, θ)

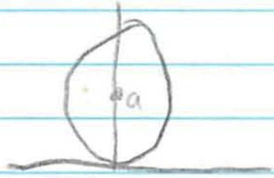
$\underset{\text{rect}}{(1,-1)} = (\sqrt{2}, -\frac{\pi}{4})$

$(\sqrt{2}, \frac{7\pi}{4})$ } best to do ⊕ angles

many ways to write 1 point

$(-\sqrt{2}, \frac{3\pi}{4})$
↑ could do, but most don't

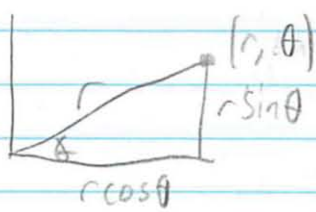
Can start w/ rect → convert to polar
or read from Geometry directly



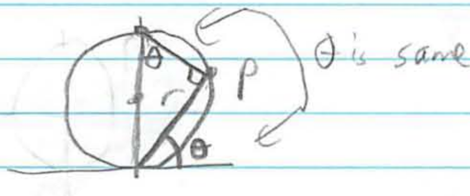
$r = f(\theta)$

$x^2 + (y-a)^2 = a^2$
$x^2 + y^2 - 2ay = 0$

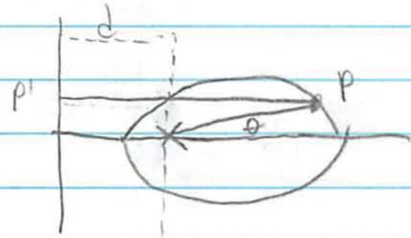
$\boxed{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}}$ transformation equations



$r^2 = 2ar\sin\theta$
$r = 2a\sin\theta$



What is polar equation of ellipse?



P lies on the ellipse

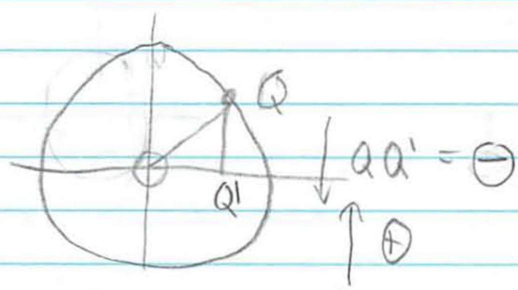
$\Rightarrow \dfrac{OP}{PP'} = e < 1$

size of constant = how flat ellipse is

$$\frac{r}{d + r\cos\theta} = e$$

$$r(1 - e\cos\theta) = ed$$

$$\boxed{r = \frac{ed}{1 - e\cos\theta}}$$ general eq of ellipse in polar coord.



For P on curve

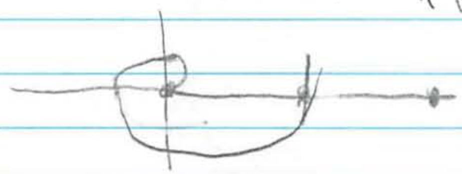
$r = a - a\sin\theta$

$\boxed{r = a(1 - \sin\theta)}$

$(r, \theta) = (r, \theta + 2k\pi)$

Spiral: $r = \frac{1}{2\pi}\theta$

↑ θ increases contineously



$r = \sqrt{x^2 + y^2} = \frac{1}{2\pi}$
$\theta = \arctan\left(\frac{y}{x}\right)$
) approx of that in rect
can not represent perfectly

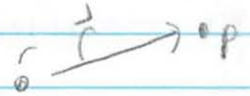
Vector Part

How motion is analized w/ polar coords,

$r = r(t)$
$\theta = \theta(t)$
) are functions of time

How to calculate $\vec{v}, \vec{T}, \frac{ds}{dt}$?

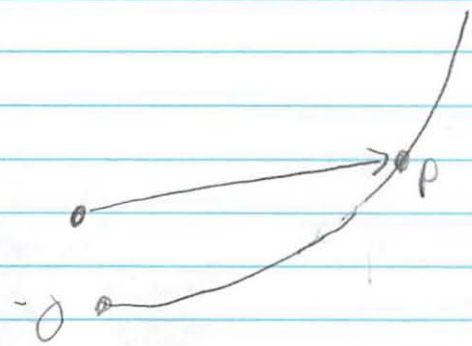
$\vec{r}$ = radius vector



$\vec{r} = x(t)\hat{i} + y(t)\hat{j}$

← can't do this in polar

Hard part



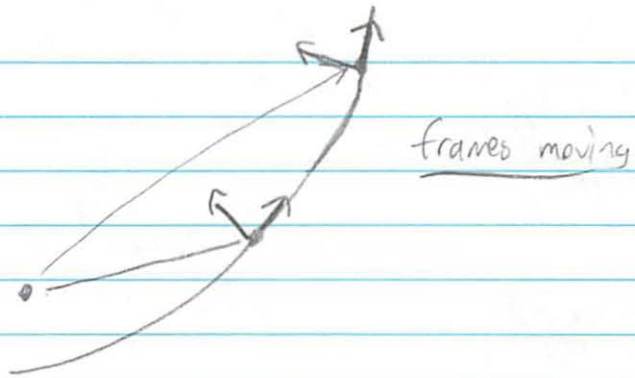
$\hat{i} + \hat{j}$ = fixed frame

polar = moving frame

← unit vector, radial

← perpendicular unit vector

← only θ changes



$\vec{U}_r = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}$

$\vec{U}_\theta = -\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}$

switch to 2 coords

know only what θ is, not motion

$$\boxed{\frac{d\hat{U}_r}{d\theta} = \hat{U}_\theta \quad \frac{d\hat{U}_\theta}{d\theta} = -\hat{U}_r}$$

$$\boxed{\frac{dr}{dt} = \dot{r}} \quad \boxed{\frac{d\theta}{dt} = \dot{\theta}}$$

Proof of this may be on exam

$\frac{d}{dt} w\vec{A}$ ← vector function of $t \Rightarrow \vec{A} = \vec{A}(t)$

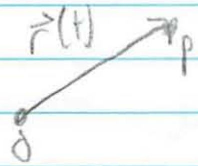
scalar function of $t \Rightarrow w = w(t)$

$$\frac{d}{dt} w\vec{A} = \frac{dw}{dt}\vec{A} + w\frac{d\vec{A}}{dt}$$

$$\frac{d}{dt}(\vec{A}\cdot\vec{B}) = \frac{d\vec{A}}{dt}\cdot\vec{B} + \vec{A}\cdot\frac{d\vec{B}}{dt}$$

Find $\vec{v}$ in polar coordinates

- express whole thing as a vector



$$\vec{r}(t) = r(t)\,\hat{u}_r$$

when differentiate, need $\hat{u}_\theta$

$$\frac{d\vec{r}}{dt} = \dot{r}\,\hat{u}_r + r\,\frac{du_r}{d\theta}\cdot\frac{d\theta}{dt}$$

$$\vec{v} = \dot{r}\,\hat{u}_r + r\cdot\dot{\theta}\,\hat{u}_\theta$$

$$|\vec{v}| = \frac{ds}{dt}$$

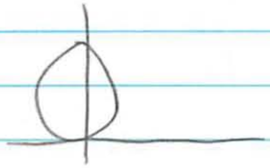
$$T = \frac{\vec{v}}{|\vec{v}|}$$

Lecture 5 Arcu

Part 2 Cycloid In Depth

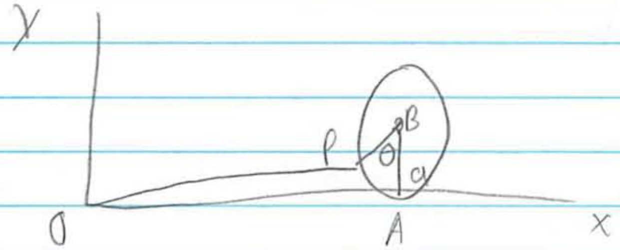
2/16

Wheel of radius R



position $x(t)$, $y(t)$ of point P?

$x(\theta)$ $y(\theta)$ ← simpler



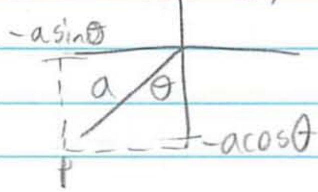
$$\vec{OP} = \vec{OA} + \vec{AB} + \vec{BP}$$

$\vec{OA} = \langle a(\theta), 0 \rangle$

↑ arc lenght of circumference of circle

$\vec{AB} = \langle 0, a \rangle$

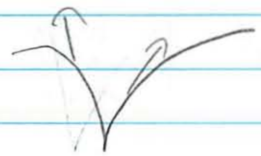
$\vec{BP} = \langle -a\sin\theta, -a\cos\theta \rangle$

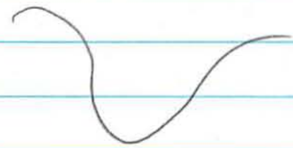


$$\vec{OP} = \langle \underbrace{a\theta - a\sin\theta}_{x(\theta)}, \underbrace{a - a\cos\theta}_{y(\theta)} \rangle$$

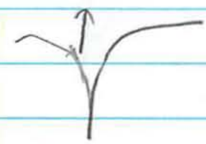
Seems so easy when he does it

What happens at the bottom?

1. 

2. 

3. 



Look at the formulas.

take lenght unit = radius

$$\begin{cases} x(\theta) = \theta - \sin\theta \\ y(\theta) = 1 - \cos\theta \end{cases}$$

Take some approximation for small θ

$$\sin(\theta) \sim \theta$$
$$\cos(\theta) \sim 1$$

Better approx?
Remember Taylor Expansion
- to get better and better approximations

For small $t \rightarrow f(t) \approx f(0) + t f'(0) + \frac{t^2}{2} f''(0) + \dots$

$$\sin(\theta) \approx \theta - \frac{\theta^3}{6}$$

$$\cos(\theta) \approx 1 - \frac{\theta^2}{2}$$

$$x(\theta) = \theta - \left(\theta - \frac{\theta^3}{6}\right) \approx \frac{\theta^3}{6}$$

$$y(\theta) = 1 - \left(1 - \frac{\theta^2}{2}\right) \approx \frac{\theta^2}{2}$$

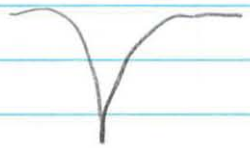
becomes very small

$$|x| \ll |y|$$

$$\frac{y}{x} \approx \frac{\theta^2/2}{\theta^3/6} = \frac{3}{\theta} \to \infty$$

when $\theta \to 0$

slope near origin is ∞

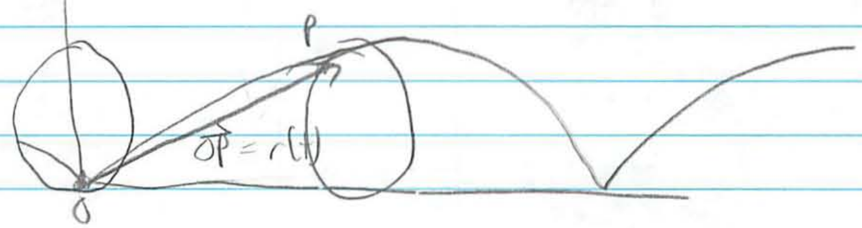


is right

Arovoux Lecture 6

Velocity, acc, keplers 2nd Law 2/17

Parametric Equations



$x(t) \quad y(t) \quad z(t)$ position of a point

Position vector $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

Example cycloid - wheel radius 1 at unit speed

$\vec{r}(t) = \langle t - \sin t, 1 - \cos t \rangle$

Velocity - speed + direction = vector

$\vec{v} = \frac{d\vec{r}}{dt}$ - take deriv of vector by taking deriv of each component

$\frac{d\vec{r}}{dt} = \langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \rangle$

Cycloid $\Rightarrow \vec{v} = \langle 1 - \cos t, \sin t \rangle$

(at $t=0$, $\vec{v} = 0$) at that particular instant
at bottom

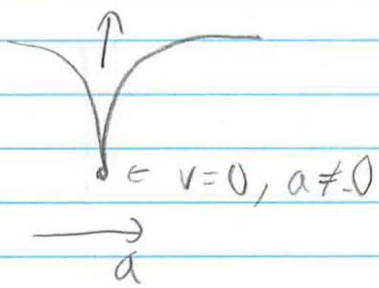
Speed $= |\vec{v}| = \sqrt{(1-\cos t)^2 + \sin^2 t}$
$= \sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t}$
$= \sqrt{2 - 2\cos t}$

$\downarrow$ vector

Acceleration (changing speed or direction)

cycloid $\rightarrow \langle \sin t, \cos t \rangle$

at $t=0$ $\vec{a} = \langle 0, 1 \rangle$



* Differentiate component by componant

$$\left| \frac{dr}{dt} \right| \neq \frac{d|\vec{r}|}{dt}$$ not the same

- hard to differenitate lenght of vector
- need to break it down into components
- no simple formula
- but not really relevant

Arc lenght - distance traveled along curve/trajectory

car odometer
- integrates speed over time over trajectory

variable is s

need to have fixed a reference point

s versus t ?

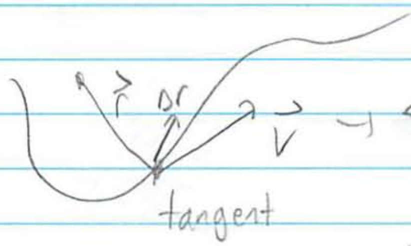
$$\frac{ds}{dt} = \int speed = \int |\vec{v}|$$

Cycloid

to get arclenght integrate speed from 0 to 2π

$$\int_0^{2\pi} \sqrt{2 - 2\cos t}\, dt$$

Unit Tangent Vector $\hat{T}$



$\to \hat{T}$ is unit vector for

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{dr}{ds}\frac{ds}{dt}$$

chain rule, speed

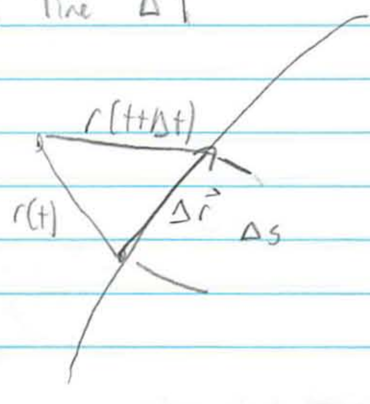
$$= \hat{T} \cdot |\vec{v}|$$

$$\hat{T} = \frac{\vec{v}}{|\vec{v}|}$$

Velocity has { direction tangent to trajectory $\hat{T}$; lengnt speed $\frac{ds}{dt}$ }

$$\vec{\Delta r} = \hat{T}\, \Delta s$$

In the ΔT

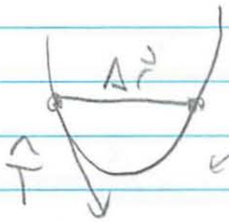


$$\frac{\Delta s}{\Delta t} = speed$$

$$\Delta\vec{r} = \hat{T}\, \Delta s$$

$$\frac{\Delta \vec{r}}{\Delta t} = \hat{T} \frac{\Delta s}{\Delta t}$$

gets better + better as go to smaller intervals



← not strictly parallel
and $\Delta r \neq s$
but as gets smaller + smaller then it becomes true) in the limit

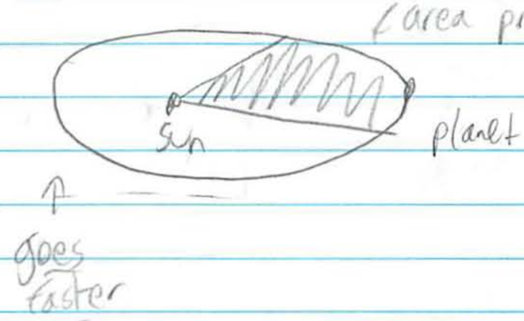
Kepler's 2nd Law (Example)

- motion of planets in the sky

- move in a plane
- the area swept out to planet is swept out at a constant rate

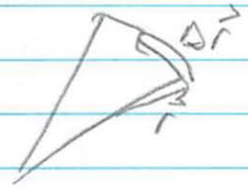
- tells you how fast planet will move on that orbit

← area proportional to time ellapsed



Newton later explained using laws of gravitational attraction

Kepler's law in terms of vectors:



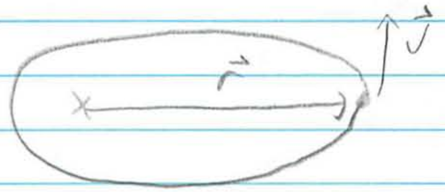
1. area $\approx \frac{1}{2}$ area of parallalogram of 2 vectors
swept $\approx \frac{1}{2} \left| \vec{r} \times \vec{R} \right|$
in time
Δt e small, lim

$\Delta r \approx \vec{v}\, \Delta t$

$\approx \frac{1}{2} \left| \vec{r} \times \vec{v} \right| \Delta t$

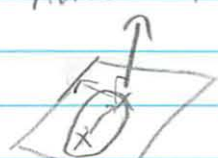
law says $\left| \vec{r} \times \vec{v} \right| = \text{constant}$

2. Plane of motion



contains $\vec{r}$ and $\vec{v}$

Direction of cross product $\vec{r} \times \vec{v} =$
normal to plane of motion



keplers 2nd Law $\Rightarrow \vec{r} \times \vec{v}$ = constant

$$\frac{d}{dt}\left(\vec{r} \times \vec{v}\right) = 0$$

make sure stays on right side

$$\frac{d\vec{r}}{dt} \times \vec{v} + \vec{r} \times \frac{d\vec{v}}{dt} = 0$$

$\vec{v} \times \vec{v} = 0$

$$\vec{v} \times \vec{v} + \vec{r} \times \vec{a} = 0$$

$$0 + \vec{r} \times \vec{a} = 0$$

$$\vec{a} \parallel \vec{r}$$

parrallel

(parrallag ram they form has no area)

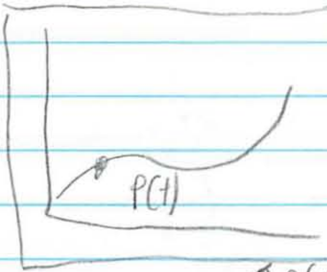
Gravitational force is parrallel to position vector

Recitation
Parametric Equations

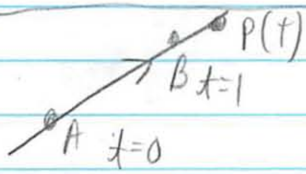
2/16

P-Set is only a small part of grade
- get help if don't understand

1. line going through $A = (0, -1)$ range of T?
$B = (1, 2)$



As t Δ → p will move along curve



$$P(t) = A + t\vec{AB}$$
$$P(0) = A$$
$$P(1) = A + t\vec{AB} = B$$

If want whole line $\quad -\infty < t < \infty$
$\qquad t \in (-\infty, \infty)$
But only want that one section

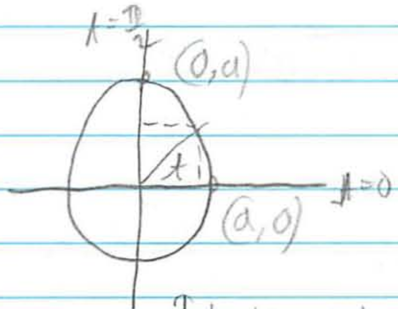
$$P(t) = A + t\vec{AB}$$
$$\rightarrow \quad = (0,-1) + t\langle 1, 3\rangle$$
$$= (t, -1+3t)$$

$$x(t) = t$$
$$y(t) = -1 + 3t$$

b Only segment [A, B]

$$P(t) = (t, -1+3t) \quad t \in [0,1]$$

c Circle of radius a around origin



$$x(t) = a\cos t \qquad t \in [0, 2\pi]$$
$$y(t) = a\sin t$$

↑ Looking at angle t

t does not identify time in this problem
represents angle
- could think that it represents time = 2π
- but find parameter that works best for you

d Part of the cycle satisfying $x \geq 0$

$$x(t) = a\cos t \qquad t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$
$$y(t) = a\sin t$$

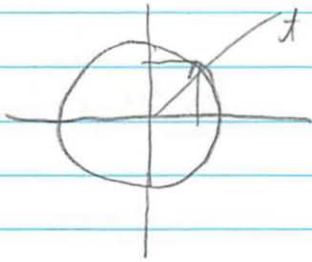
e Circle of radius a around B

$$\vec{OP} = \vec{OB} + \vec{BP}$$

$$P(t) = (1,2) + (a\cos t, a\sin t)$$
$$= (1 + a\cos t, 2 + a\sin t) \quad t \in [0, 2\pi]$$

Ex 2 Describe path traced by (eliminate t), relate x y

a) $x = a\sin t$
$y = a\cos t$



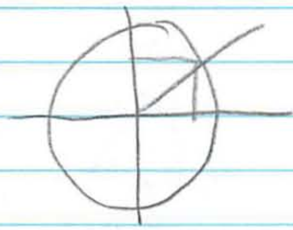
We can use idenitity $\cos^2 t + \sin^2 t = 1$

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{a}\right)^2 = 1$$

$\left(\frac{a\sin t}{a}\right)^2 = \sin t^2$ use x to get it to $=$

This is the circle of radius a.

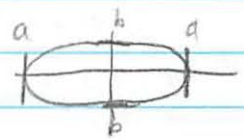
b) $x = a\cos t$
$y = a\sin t$



This is like the same (same idenity)

$$\left(\frac{y}{a}\right)^2 + \left(\frac{x}{a}\right)^2 = 1$$

↑ ellipse

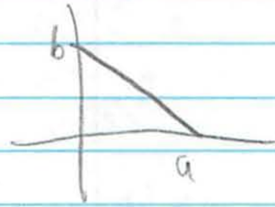


c) $x = a\cos^2 t$ ← between $0, a$
$y = a\sin^2 t$ ← between $0, b$

$$\frac{x}{a} + \frac{y}{a} = 1$$

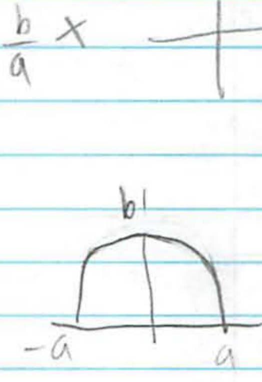
↑ line segment

$$y = b - \frac{b}{a}x$$



d) $x = a\cos t$
$y = b\sin t$

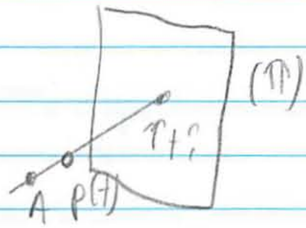
$$\left(\frac{x}{a}\right)^2 + \frac{y}{b} = 1$$



$$y = b - \frac{b}{a^2}x^2$$

Ex 3 Find intersection between (Π)

- plane $2x - y + z = 1$
- line going through $(1, 1, -1)$ and perpendicular to another plane $x + 2y - z = 3$



#1. Parametrise lines
#2 Find the t that corresponds to the intersection

$$\cancel{P(t) = A + t\,\overrightarrow{AB}}$$

$$\cancel{(1, 1, -1) + t \langle \tfrac{1}{3}, \tfrac{2}{3}, -\tfrac{1}{3} \rangle}$$ ← normal perpendicular to that plane

Problem: Need a direction

Take $\vec{N} = \langle 1, 2, -1 \rangle$ (perp to $x + 2y - z = 3$)

Parametrization

$$P(t) = A + \vec{N}$$

$$(1, 1, -1) + t\langle 1, 2, -1 \rangle$$

$$(1 + t, 1 + 2t, -1 - t)$$

Need to find t for intersection

$$2(1+t) - (1+2t) + (-1-t) = 1$$ ← plug into plane equation

$$t = -1$$

Intersection Plug $t = -1$ into parametric equation $P(-1)$

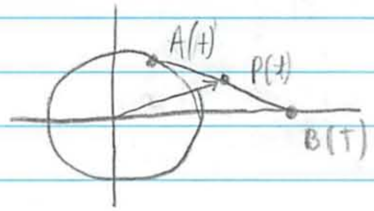
$$(0, -1, 0)$$

Recitation

Parametric Moving, Polar Coords

2/17

Ex 1 $A(t) = (2\cos t, 2\sin t)$



$P(T)$ = middle of segment

- Compute position vector $[r(t)] = \vec{OP}$
- " velocity $[v(t)] = \frac{d\vec{r}}{dt}$

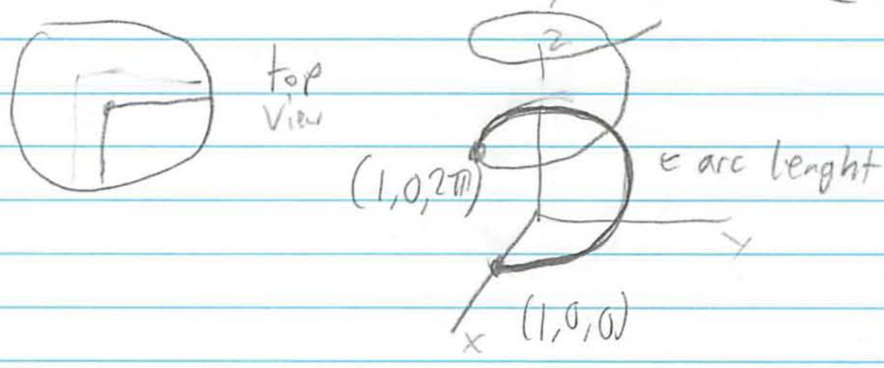
$$\vec{r}(t) = \vec{OP} = \vec{OA} + \vec{AP}$$
$$= OA + \tfrac{1}{2}\vec{AB}$$ ← know the shapes + deduce
$$= \langle 2\cos t, 2\sin t\rangle + \tfrac{1}{2}\langle 4 - 2\cos t, 0 - 2\sin t\rangle$$
$$= \langle 2\cos t + 2 - \cos t, 2\sin t - \sin t\rangle$$
$$= \langle \cos t + 2, \sin t\rangle$$

* Hard part: know what you want to compute

$$\vec{V} = \frac{d\vec{r}}{dt} = \frac{d}{dt}\langle \cos t + 2, \sin t\rangle$$ differentiate each component

$$\langle -\sin t, \cos t\rangle$$

Ex 2 Compute arc lenght of a curve (s) with parametric eq
$r(t) = \langle \cos t, \sin t, t\rangle$ $t \in [0, 2\pi]$



$$\frac{ds}{dt} = |\vec{v}|$$

$$S = \int_0^{2\pi} \frac{ds}{dt} \cdot dt$$

$$|\vec{v}|$$

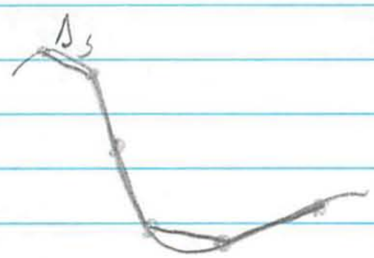
$$V = \langle -\sin t, \cos t, 1 \rangle$$

$$|\vec{v}| = \sqrt{\sin^2 t + \cos^2 t + 1^2}$$

$$\sqrt{2}$$

$$S = \int_0^{2\pi} \sqrt{2}\, dt = 2\pi\sqrt{2}$$

Where arc lenght comes from



$$\sum \Delta s$$

$$\sum r(t+\Delta t) - r(t)$$

$\hookrightarrow |\vec{v}|$

Ex 3 Polar Coordinates

Go from rectangle to polar for $x + 2y = 3$

a

$$\boxed{\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}\right\} \text{always do this}}$$

$$x + 2y = 3$$

$$(r\cos\theta) + 2(r\sin\theta) = 3$$

done!

6. Find rectangular for $r = 2(r\cos\theta + 1)$

$r = \sqrt{x^2+y^2}$

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$\cos\theta = \frac{x}{r} = \frac{x}{\sqrt{x^2+y^2}}$

$\sin\theta = \frac{y}{r} = \frac{y}{\sqrt{x^2+y^2}}$

$r = 2(r\cos\theta + 1)$

$\sqrt{x^2+y^2} = 2(x+1)$

Done! easy

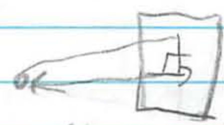
don't have to solve for something

Ex 4 Distance Problem

Distance between a point and a plane

point $A = (1, 2, 3)$

plane $x - y + z = 1$



Shortest distance

perpendicular

Find normal vector of plane

< duh - should have thought of this!

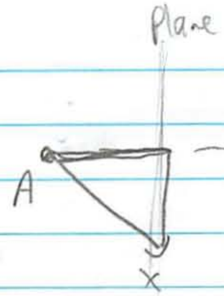
Can use to paramatrize line

Find intersection

$\vec{N} = \langle 1, 1, -1 \rangle$ look at coefficent of vector plane

Try to find a point in the plane

$(0, 0, 1)$ Put as many 0s as you can - until are stuck



$\vec{v} = \dfrac{\vec{N}}{|N|}$

distance = component of AX in direction $\dfrac{\vec{N}}{|N|}$

$= \vec{AX} \cdot \vec{v}$

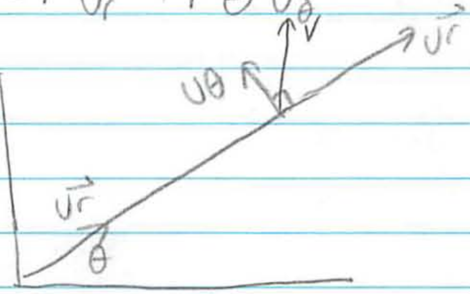
$= \langle -1, -2, -2 \rangle \cdot \dfrac{\langle 1, -1, 1 \rangle}{\sqrt{3}}$

Sign may be wrong if normal vector is wrong

derivitive w/ respect to time

Ex5 Prove $\vec{v} = \dot{r}\,\vec{U_r} + r\dot{\theta}\,\vec{U_\theta}$

$\dot{r} = \dfrac{dr(t)}{dt}$



Recall $\vec{v} = \dfrac{d}{dt}\left(\vec{r}(t)\right) = \dfrac{d}{dt}\left(r(t)\vec{U_r}(t)\right)$

$U_r = \dfrac{\vec{r}}{r}$

↑ direction

Look like product rule for differentiation.

Recall that $\vec{A}(t) = f(t)\,\vec{B}(t)$

$\dot{\vec{A}}(t) = \dot{f}(t)\,\vec{B}(t) + f(t)\,\dot{\vec{B}}(t)$

(Apply the product rule to each coordinate)

→

Here $\frac{d}{dt}\left(r\vec{u}_r\right) = \dot{r}\vec{u}_r + r\dot{\vec{u}}_r$

It remains to prove $\dot{\vec{u}}_r = \dot{\theta}\,\vec{u}_\theta$

$$\vec{u}_r = \frac{\vec{r}}{r} = \langle \cos\theta, \sin\theta \rangle$$

Differentiate that (chain rule θ depends on time)

$$\dot{\vec{u}}_r = \langle -\dot{\theta}\sin\theta, \dot{\theta}\cos\theta \rangle$$
$$= \dot{\theta}\underbrace{\langle -\sin\theta, \cos\theta \rangle}_{\vec{u}_\theta}$$

Michael Plasmeier

21/30

18.02 Problem Set 2 due Th. 2/18/10 10:45AM 2-106

Part I (10 points)

Lecture 5. Thurs. Feb. 11 Parametric equations; vector functions and their derivatives
Read 18.4, 17.1 (can omit Exs. 2,5); 18.4 (lines in 3D); 17.4
Work: ~~1I-2ab, 3bd, 4,~~ 7 (use hint); ~~1E-3bc, 4; 1J-1a,~~ 2, ~~4ab, 4c (don't use coordinates),~~ 9

Lecture 6. Fri. Feb. 12 Polar coordinates; polar curves; polar vector functions and their derivatives; arclength.
Read: 16.1, 16.2 Ex. 1, p. 575 (2); 17.7 to (7) p. 622
Work (in the 18.01 Notes): ~~4H-1dg, 2ac, 3af; 4I-1ab~~
(18.01 Notes are on 18.01 OCW site; the needed pages are on the 18.02 class website.)

Recitations. Tues. Feb. 16, Wed. Feb.17. *(Holiday Mon.; Tues. follows Mon. Schedule.)*

Lecture 7. Thurs. Feb. 18 Acceleration; the T-N frame; curvature. Read 17.5, 17.6.

Part II (20 points)

Directions. Try each problem alone for 15 minutes. If you subsequently collaborate, solutions must be written up independently. It is illegal to consult problem sets from previous semesters. With each problem is the day it can be done.

Problem 1. (Tues. 2 pts.) Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a matrix of constants, $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ a column vector, and λ a real number. If the matrix equation $M\mathbf{x} = \lambda\mathbf{x}$ has a non-trivial solution, λ is called an *eigenvalue* of M.

a) Show M has either 2, 1, or no real eigenvalues, and

b) give the respective inequalities connecting $a - d$ and bc that tell when each of the three cases in (a) holds.

(Proceed by writing the single matrix equation as a homogeneous system of two linear equations in x and y, and use the theorem which tells you when the system has a non-trivial solution; this leads to an polynomial equation for determining λ. When does it have real solutions, and how many? Use algebra to express the conditions on a, b, c, d in the form asked for in (b).)

Problem 2. (Thurs. 2 pts.) Imagine an eye at the point $E : (0, 0, 4)$ in xyz-space is looking down at the xy-plane, on which the coordinates axes are drawn.

A triangular plate of invisible glass has its vertices at the points on the three coordinate axes where $x = 1,\ y = 1,\ z = 1$ respectively. There is an ant on the glass plate which the eye thinks it sees at (x_0, y_0) in the xy-plane. Where is the ant actually located in space?

(Answer in terms of x_0, y_0. Check for gross errors by seeing if it gives the right answers for the points $(1, 0)$ and $(0, 1)$.)

Problem 3. (Thurs. 3 pts.: 2,1) Two lines in space are called *skew* if they don't intersect and are not parallel. Suppose two skew lines have the directions respectively of the vectors **A** and **B**.

a) Using **A**, **B**, and two arbitrarily chosen points P_1 and P_2 on the respective lines, give a formula for the distance between the two lines, which by definition is the shortest length of the vector P_1P_2 for any choice of the two points. (Use these two facts:

(i) If Q_1Q_2 is a shortest possible vector, it must be perpendicular to both lines, for if not perpendicular at Q_2 say, moving Q_2 a little in the right direction along its line would shorten Q_1Q_2.

(ii) The triangle $Q_1Q_2P_2$ has a right angle at Q_2.

Note that you don't know where Q_1 and Q_2 are, so they shouldn't appear in your formula. Give brief reasoning to justify your formula.

b) Use your formula to find the distance between two skew diagonals lying on two adjacent sides of a unit cube.

(For ease in calculation, place the unit cube in the corner of the first octant, so three of its adjacent square faces lie in the three coordinate planes. Use the two diagonals lying on the front face and the right face, and containing respectively (1,0,0) and (1,1,0).)

Problem 4. (Thurs. 2 pts.)

The diagram on p. 595 shows the *hypocycloid*, traced out by fixed point P on a circle of radius b which rolls around the inside of a circle of radius a.

Similarly, an *epicycloid* is traced out by a fixed point P on a circle of radius b which rolls counter-clockwise around a circle of radius a. Using the notation in the diagram on p. 595, suppose the starting point for P is at A on the x-axis. Using vector methods, derive the position vector function $\mathbf{r}(\theta)$ for the epicycloid.

(Use the positive angle ϕ through which the circle has rolled as an auxiliary variable (which helps the derivation, but should not appear in the final answer). "Using vector methods" means: express OP as a sum of simpler vector functions, determine each explicitly in terms of θ, and then add them up.)

The pre-Copernican Ptolomaic model of our solar system was based on observation, but had trouble finding simple curves which fit the data. The curves used for the planetary orbits were circles rolling on circles rolling on circles...I'm told one orbit needed five circles in all – an (epi)^{4}cycloid.

Problem 5. (Thurs. 3 pts: 1.5, 1, .5) Let $\mathbf{r} = -\ln\cos t\ \mathbf{i} + t\ \mathbf{j}$, for $0 \le t < \pi/2$.

a) Find the velocity vector $\mathbf{v}$, the unit tangent vector $\mathbf{T}$, and the speed ds/dt.

b) Sketch the curve of motion (pay attention to dir($\mathbf{v}$) at $t = 0$ and $t = \pi/4$).

c) Find the distance traveled along this curve over the the time interval $[0, \pi/4]$.

Problem 6. (Fri. 4 pts: 1, 2, 1)

a) Change the polar equation $r = 4a(\cos\theta + \sin\theta)$ to rectangular coordinates, putting it into a form where you can recognize the curve it represents; describe this curve.

b) A line segment having length $2k$ moves through the four quadrants so that one end always lies on the x-axis, and the other end is always on the y-axis. Let P be the head of that radius vector from the origin to the line segment which is perpendicular to the segment. Using elementary geometry (think area), find the polar equation of the locus of P.

c) Sketch the polar curve whose equation you found in (b).

Problem 7. (Fri. 4 pts.: 2,1,1) For the motion of a point P given by $r = e^{at}, \quad \theta = at$:

a) Find the velocity vector $\mathbf{v}$, the unit tangent vector $\mathbf{T}$, and the speed ds/dt, expressed in terms of the standard unit vectors $\mathbf{u}_r$ and $\mathbf{u}_\theta$ pointing respectively in the radial and (positive) perpendicular directions;

b) Find the length of its path from $t = 0$ to the next time P crosses the positive x-axis;

c) Show the curve of motion makes a constant angle with the radius vector $\mathbf{r}$.

4I. Area and arclength in polar coordinates

4I-1 $\sqrt{(dr/d\theta)^2 + r^2}d\theta$

a) $\sec^2\theta d\theta$

b) $2ad\theta$

c) $\sqrt{a^2+b^2+2ab\cos\theta}d\theta$

d) $\dfrac{a\sqrt{b^2+c^2+2bc\cos\theta}}{(b+c\cos\theta)^2}d\theta$

e) $a\sqrt{4\cos^2(2\theta)+\sin^2(2\theta)}d\theta$

f) $a\sqrt{4\sin^2(2\theta)+\cos^2(2\theta)}d\theta$.

g) Use implicit differentiation:

$$2rr' = 2a^2\cos(2\theta) \implies r' = a^2\cos(2\theta)/r \implies (r')^2 = a^2\cos^2(2\theta)/\sin(2\theta)$$

Hence, using a common denominator and $\cos^2 + \sin^2 = 1$,

$$ds = \sqrt{a^2\cos^2(2\theta)/\sin(2\theta) + a^2\sin(2\theta)}d\theta = \frac{a}{\sqrt{\sin(2\theta)}}d\theta$$

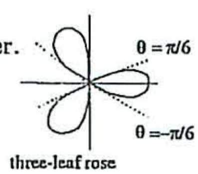
h) This is similar to (g):

$$ds = \frac{a}{\sqrt{\cos(2\theta)}}d\theta$$

i) $\sqrt{1+a^2}e^{a\theta}d\theta$

4I-2 $dA = (r^2/2)d\theta$. The main difficulty is to decide on the endpoints of integration. Endpoints are successive times when $r = 0$.

$$\cos(3\theta) = 0 \implies 3\theta = \pi/2 + k\pi \implies \theta = \pi/6 + k\pi/3, \quad k \text{ an integer.}$$

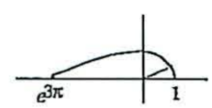


three-leaf rose
three empty sectors

$$\text{Thus,} \quad A = \int_{-\pi/6}^{\pi/6} (a^2\cos^2(3\theta)/2)d\theta = a^2\int_0^{\pi/6}\cos^2(3\theta)d\theta.$$

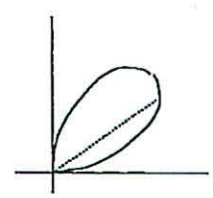
(Stop here in Unit 4. Evaluated in Unit 5.)

4I-3 $A = \int (r^2/2)d\theta = \int_0^{\pi}(e^{6\theta}/2)d\theta = (1/12)e^{6\theta}\Big|_0^{\pi} = (e^{6\pi} - 1)/12$



4I-4 Endpoints are successive time when $r = 0$.

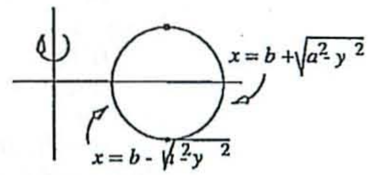
$$\sin(2\theta) = 0 \implies 2\theta = k\pi, \quad k \text{ an integer.}$$



Thus, $A = \int (r^2/2)d\theta = \int_0^{\pi/2}(a^2/2)\sin(2\theta)d\theta = -(a^2/4)\cos(2\theta)\Big|_0^{\pi/2} = a^2/2.$

b) We need to rotate two curves $x_2 = b + \sqrt{a^2 - y^2}$ and $x_1 = b - \sqrt{a^2 - y^2}$ around the y-axis. The value

$$dx_2/dy = -(dx_1/dy) = -y/\sqrt{a^2 - y^2}$$



inner and outer surfaces are not symmetrical and not equal

So in both cases,

$$ds = \sqrt{1 + y^2/(a^2 - y^2)}dy = (a/\sqrt{a^2 - y^2})dy$$

The integral is

$$A = \int 2\pi x_2 ds + \int 2\pi x_1 ds = \int_{-a}^{a} 2\pi(x_1 + x_2)\frac{a dy}{\sqrt{a^2 - y^2}}$$

But $x_1 + x_2 = 2b$, so

$$A = 4\pi ab \int_{-a}^{a} \frac{dy}{\sqrt{a^2 - y^2}}$$

c) Substitute $y = a\sin\theta$, $dy = a\cos\theta d\theta$ to get

$$A = 4\pi ab \int_{-\pi/2}^{\pi/2} \frac{a\cos\theta d\theta}{a\cos\theta} = 4\pi ab \int_{-\pi/2}^{\pi/2} d\theta = 4\pi^2 ab$$

4H. Polar coordinate graphs

4H-1 We give the polar coordinates in the form (r, θ):

a) $(3, \pi/2)$ b) $(2, \pi)$ c) $(2, \pi/3)$ d) $(2\sqrt{2}, 3\pi/4)$
e) $(\sqrt{2}, -\pi/4$ or $7\pi/4)$ f) $(2, -\pi/2$ or $3\pi/2)$
g) $(2, -\pi/6$ or $11\pi/6)$ h) $(2\sqrt{2}, -3\pi/4$ or $5\pi/4)$

4H-2 a) (i) $(x-a)^2+y^2 = a^2 \Rightarrow x^2-2ax+y^2 = 0 \Rightarrow r^2-2ar\cos\theta = 0 \Rightarrow r = 2a\cos\theta$.

(ii) $\angle OPQ = 90^o$, since it is an angle inscribed in a semicircle.
In the right triangle OPQ, $|OP| = |OQ|\cos\theta$, i.e., $r = 2a\cos\theta$.

b) (i) Analogous to 4H-2a(i); ans: $r = 2a\sin\theta$.
(ii) analogous to 4H-2a(ii); note that $\angle OQP = \theta$, since both angles are complements of $\angle POQ$.

c) (i) OQP is a right triangle, $|OP| = r$, and $\angle POQ = \alpha - \theta$.
The polar equation is $r\cos(\alpha - \theta) = a$, or in expanded form,

$$r(\cos\alpha\cos\theta + \sin\alpha\sin\theta) = a, \quad \text{or finally,}$$

$$\frac{x}{A} + \frac{y}{B} = 1,$$

since from the right triangles OAQ and OBQ, we have $\cos\alpha = \dfrac{a}{A}$, $\sin\alpha = \cos BOQ = \dfrac{a}{B}$.

d) Since $|OQ| = \sin\theta$, we have:
if P is above the x-axis, $\sin\theta > 0$, $OP| = |OQ| - |QR|$, or $r = a - a\sin\theta$;
if P is below the x-axis, $\sin\theta < 0$, $OP| = |OQ| + |QR|$, or $r = a + a|\sin\theta| = a - a\sin\theta$.
Thus the equation is $r = a(1 - \sin\theta)$.

e) Briefly, when $P = (0,0)$, $|PQ||PR| = a \cdot a = a^2$, the constant.
Using the law of cosines,

$$|PR|^2 = r^2 + a^2 - 2ar\cos\theta;$$
$$|PQ|^2 = r^2 + a^2 - 2ar\cos(\pi - \theta) = r^2 + a^2 + 2ar\cos\theta$$

Therefore

$$|PQ|^2|PR|^2 = (r^2 + a^2)^2 - (2ar\cos\theta)^2 = (a^2)^2$$

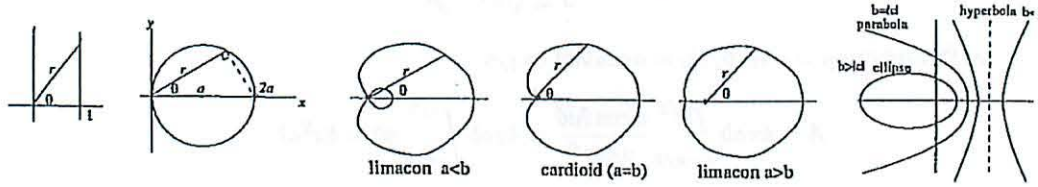
which simplifies to

$$r^2 = 2a^2\cos 2\theta.$$

4H-3 a) $r = \sec\theta \implies r\cos\theta = 1 \implies x = 1$ b) $r = 2a\cos\theta \implies r^2 = r \cdot 2a\cos\theta = 2ax \implies x^2 + y^2 = 2ax$

c) $r = (a + b\cos\theta)$ (This figure is a cardiod for $a = b$, a limaçon with a loop for $0 < a < b$, and a limaçon without a loop for $a > b > 0$.)

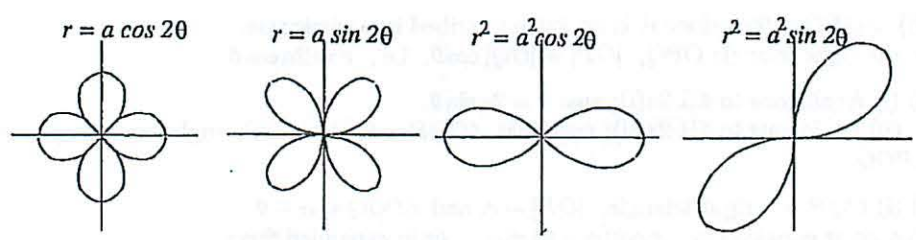
$$r^2 = ar + br \cdot \cos\theta = ar + bx \implies x^2 + y^2 = a\sqrt{x^2 + y^2} + bx$$



8a 8b 8c 8d

(d)
$$\begin{aligned} r = a/(b + c\cos\theta) &\implies r(b + c\cos\theta) = a \implies rb + cx = a \\ &\implies rb = a - cx \implies r^2b^2 = a^2 - 2acx + c^2x^2 \\ &\implies a^2 - 2acx + (c^2 - b^2)x^2 - b^2y^2 = 0 \end{aligned}$$

(e)
$$\begin{aligned} r = a\sin(2\theta) &\implies r = 2a\sin\theta\cos\theta = 2axy/r^2 \\ &\implies r^3 = 2axy \implies (x^2 + y^2)^{3/2} = 2axy \end{aligned}$$



f) $r = a\cos(2\theta) = a(2\cos^2\theta - 1) = a(\dfrac{2x^2}{x^2 + y^2} - 1) \implies (x^2 + y^2)^{3/2} = a(x^2 - y^2)$

g) $r^2 = a^2\sin(2\theta) = 2a^2\sin\theta\cos\theta = 2a^2\dfrac{xy}{r^2} \implies r^4 = 2a^2xy \implies (x^2 + y^2)^2 = 2axy$

h) $r^2 = a^2\cos(2\theta) = a^2(\dfrac{2x^2}{x^2 + y^2} - 1) \implies (x^2 + y^2)^2 = a^2(x^2 - y^2)$

i) $r = e^{a\theta} \implies \ln r = a\theta \implies \ln\sqrt{x^2 + y^2} = a\tan^{-1}\dfrac{y}{x}$

4H-3 For each of the following,

(i) give the corresponding equation in rectangular coordinates;

(ii) draw the graph; indicate the direction of increasing θ.

a) $r = \sec\theta$ b) $r = 2a\cos\theta$

c) $r = (a + b\cos\theta)$ (This figure is a cardioid for $a = b$, a limaçon with a loop for $0 < a < b$, and a limaçon without a loop for $a > b > 0$.)

d) $r = a/(b + c\cos\theta)$ (Assume the constants a and b are positive. This figure is an ellipse for $b > |c| > 0$, a circle for $c = 0$, a parabola for $b = |c|$, and a hyperbola for $b < |c|$.)

e) $r = a\sin(2\theta)$ (4-leaf rose) f) $r = a\cos(2\theta)$ (4-leaf rose)

g) $r^2 = a^2\sin(2\theta)$ (lemniscate) h) $r^2 = a^2\cos(2\theta)$ (lemniscate)

i) $r = e^{a\theta}$ (logarithmic spiral)

4I. Area and arclength in polar coordinates

4I-1 Find the arclength element $ds = w(\theta)d\theta$ for the curves of 4H-3.

4I-2 Find the area of one leaf of a three-leaf rose $r = a\cos(3\theta)$.

4I-3 Find the area of the region $0 \le r \le e^{3\theta}$ for $0 \le \theta \le \pi$

4I-4 Find the area of one loop of the lemniscate $r^2 = a^2\sin(2\theta)$

4I-5 What is the average distance of a point on a circle of radius a from a fixed point Q on the circle? (Place the circle so Q is at the origin and use polar coordinates.)

4I-6 What is the average distance from the x-axis of a point chosen at random on the cardioid $r = a(1 - \cos\theta)$, if the point is chosen

a) by letting a ray $\theta = c$ sweep around at uniform velocity, stopping at random and taking the point where it intersects the cardioid;

b) by letting a point P travel around the cardioid at uniform velocity, stopping at random; (the answers to (a) and (b) are different...)

4I-7 Calculate the area and arclength of a circle, parameterized by $x = a\cos\theta, y = a\sin\theta$.

4H. Polar coordinate graphs

4H-1 For each of the following points given in rectangular coordinates, give its polar coordinates. (For points below the x-axis, give two expressions for its polar coordinates, using respectively positive and negative values for θ.)

a) $(0,3)$ b) $(-2,0)$ c) $(1,\sqrt{3})$ d) $(-2,2)$
e) $(1,-1)$ f) $(0,-2)$ g) $(\sqrt{3},-1)$ h) $(-2,-2)$

4H-2

a) Find using two different methods the equation in polar coordinates for the circle of radius a with center at $(a,0)$ on the x-axis, as follows:

(i) write its equation in rectangular coordinates, and then change it to polar coordinates (substitute $x = r\cos\theta$ and $y = r\sin\theta$, and then simplify).

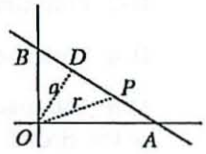
(ii) treat it as a locus problem: let OQ be the diameter lying along the x-axis, and $P:(r,\theta)$ a point on the circle; use ΔOPQ and trigonometry to find the relation connecting r and θ.

b) Carry out the analogue of 4H-2a for the circle of radius a with center at $(0,a)$ on the y-axis; OQ is now the diameter lying along the y-axis.

c) (i) Find the polar equation for the line intersecting the positive x- and y-axes respectively at A and B, and having perpendicular distance a from the origin.

(Let $\alpha = \angle DOA$; use the right triangle DOP to get the equation connecting r, θ, α and a.

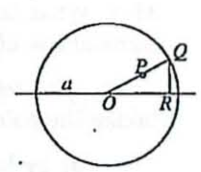
(ii) Convert your polar equation to the usual rectangular equation involving A and B, by using trigonometry.



d) In the accompanying figure, the point Q moves around the circle of radius a centered at the origin; QR is a perpendicular to the x-axis. P is a point on ray OQ such that $|QP| = |QR|$: P is the point inside the circle in the first two quadrants, but outside the circle in the last two quadrants.

(i) Sketch the locus of P; the locus is called a *cardioid* (cf. 4H-3c).

(ii) find the polar equation of this locus.



e) The point P moves in a locus so that the product of its distances from the two points $Q:(-a,0)$ and $R:(a,0)$ is constant. Assuming the locus of P goes through the origin, determine the value of the constant, and derive the polar equation of the locus of P.

(Work with the squares of the distances, rather than the distances themselves, and use the law of cosines; the identities $(A+B)(A-B) = A^2 - B^2$ and $\cos 2\theta = 2\cos^2\theta - 1$ simplify the algebra and produce a simple answer at the end. The resulting curve is a *lemniscate*, cf. 4H-3g.)

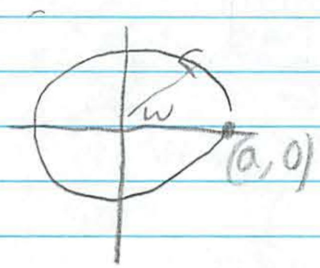
Michael Plasmeier

18.02 P-Set 2

2/16

Part 1
Lecture 5 — Parametric equations, vector functions, derivitives

II-2a A point moves w/ clockwise angular velocity ω (constant) on the circle radius r at origin. What is $r(t)$ if at $t=0$ at $(a,0)$



$x = a\cos\omega$
$y = a\sin\omega$

together $r = a\cos(\omega t)\,\hat{\imath} - a\sin(\omega t)\,\hat{\jmath}$
↑ not polar coords though

b. When $t=0$ is $(0,a)$?

$x = -a\sin\omega$
$y = -a\cos\omega$

$r = a\cos\theta\,\hat{\imath} + a\sin\theta\,\hat{\jmath}$

$t=0 \quad \theta = \frac{\pi}{2}$
decreases $\theta = \frac{\pi}{2} - \omega t$ ← does it not increase?

Sub in
use trig for $\cos(A+B)$, $\sin(A+B)$

$r = a\cos\left(\frac{\pi}{2}-\omega t\right)\hat{\imath} + a\sin\left(\frac{\pi}{2}-\omega t\right)\hat{\jmath}$
$a\sin\omega t\,\hat{\imath} + a\cos\omega t\,\hat{\jmath}$
what I wrote but no –

3b Describe the motion given by each of the following position vector functions as t goes from $-\infty$ to ∞. Give the xy equation and tell what part of p traced

b) $r = \cos 2t\,\hat{i} + \cos t\,\hat{j}$

Identity $\cos^2\theta + \sin^2\theta = 1$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

$x = \cos 2t$

$y = \cos t$ sub in for y

reduce other form for it

$\cos 2\theta = 2\cos^2\theta - 1$

$\underset{\cos 2t}{x} = 2\underset{\cos t}{y}^2 - 1$

confused about how you get this

need to know the magic

between $(1,1)$ $(1,-1)$ back + forth

think about it, ...

* in radian mode

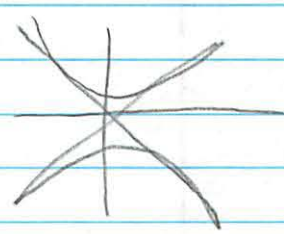
d. $r = \tan t\,\hat{i} + \sec t\,\hat{j}$

$1 + \tan^2 = \sec^2$

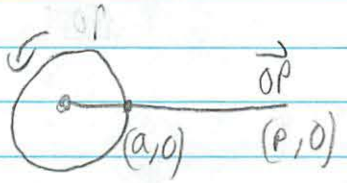
$\tan^2 = \sec^2 - 1$

$x = \tan t$ $\qquad$ $x^2 = y^2 - 1$

$y = \sec t$ $\qquad$ $x = \sqrt{y^2 - 1}$



4. A roll of plastic tape of outer radius a is held in a fixed position while being unrolled counter clockwise. End P held so ounward perp. to roll. Use center as orgin. $P = (a, 0)$



Parametric eq for P

$P(t) = A + t\,\overrightarrow{OP}$

$(a, 0) + t < P, 0 >$

$(a + Pt, 0)$

$OP = |OP| \cdot \text{dir } OP$ $\quad$ $\text{dir } OP = \cos\theta\, \hat{\imath} + \sin\theta\, \hat{\jmath}$

← that is just normal $\quad$ ↑ is this not fixed?

$|OP| = |OQ| + |QP|$ ← more duh

$a + a\theta$

$|QP| = \text{arc } QR = a\theta$

↑ what does arc mean?

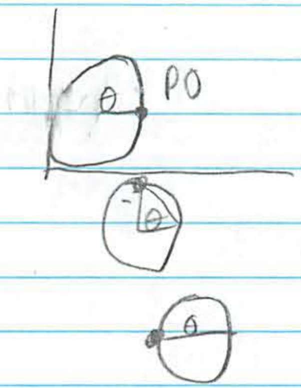
So $OP = a(1 + \theta)(\cos\theta \cdot \hat{\imath} + \sin\theta\, \hat{\jmath})$

$x = a(1 + \theta)\cos\theta$

$y = a(1 + \theta)\sin\theta$

/ I don't get why you need all this stuff...

7. The cycloid is the curve traced out by a fixed point P on a ~~c~~ -blah blah



+ use

$OP = 0\hat{i} + 0\hat{j}$ at $t = 0, \theta = 0$

$OP = \sin^2\theta + \cos^2\theta \quad \theta = \frac{\pi}{2}$

$\sqrt{1^2 + 1^2} = \sqrt{2}$

$OP = \sin^2\theta + \cos^2\theta \quad \theta = \pi$

$(-1)^2 \quad 0 = 1$

$OP = OA + AB + BP$

$OA = \text{arc } AP = a\theta$

↑ arc lenght

$AB = a\hat{j}$ ← definition

$BP = a(-\sin\theta\,\hat{i} - \cos\theta\,\hat{j})$ ← how are you supposed to find that?

$\underline{OP = a(\theta - \sin\theta)\hat{i} + a(1 - \cos\theta)\hat{j}}$

does that not depend on where you start ???

1E-3b Find the parametric form for

line (2, -1, -1) (through)
plane perpendicular $x - y + 2z = 3$

finally a problem that was modeled to us

1. Need a direction. Take $\vec{N} = \langle 1, -1, 2 \rangle$
write · drop 3

2. Parameterize

$$P(t) = A + \vec{N}$$
$$(2, -1, -1) + t\langle 1, -1, 2 \rangle$$
$$(2+t, -1-t, -1+2t)$$

stop here

3. Need to find a t for intersection
plug into plane equation & solve for t

$$(2+t) - (-1-t) + 2(-1+2t) = 3$$
$$2 + t + 1 + t - 2 + 4t = 3$$
$$-2 \quad -1 \quad +2 \quad -1$$

$$6t = 2$$
$$t = \frac{1}{3}$$

4. Plug $t = \frac{1}{3}$ into parametric equation $P(\frac{1}{3})$

$$\left(2 + \frac{1}{3}, -1 - \frac{1}{3}, -1 + \frac{2}{3}\right)$$
$$\left(\frac{7}{3}, -\frac{4}{3}, -\frac{1}{3}\right)$$

c All lines passing through (1, 1, 1) and lying in the plane $x + 2y - z = 2$

1. Direction $\langle 1, 2, -1 \rangle$
2. $P(t) = A + \vec{N}$

$(1, 1, 1) + t\langle 1, 2, -1 \rangle$

$(1 + 1t, 1 + 2t, 1 - t)$

what does all lines mean?

$(1 + at, 1 + bt, 1 + ct)$

where $a + 2b - c = 0$ ← just that it fits in

or $z = 1 + (a + 2b)t$

4. Where does the line going through (0, 1, 2) (2, 0, 3) intersect $x + 4y + z = 4$

? how get it w/ 2nd place

1. Direction $\langle 1, 4, 1 \rangle$
2. $P(t) = (0, 1, 2) + t\langle 1, 4, 1 \rangle$

$(t, 1 + 4t, 2 + t)$

3. ? Try other pt $(2, 0, 3) + t\langle 1, 4, 1 \rangle$

$(2 + t, 4t, 3 + t)$

when 2 pts find line b/w parametrize that

Well have 2 pts - find direction b/w them

$\langle 2, -1, 1 \rangle$

Then parametrize that w/ a point

$\langle 2t, 1 - t, 2 + t)$

plug in to find b/w line + plane

Sub into plane to find pt on line and plane

$2t + 4(1 - t) + (2 + t) = 4$

$t = 2$

Sub back in $(4, -1, 4)$

Differentiation of Vector Functions

1J-1a Calculate velocity, speed $\frac{ds}{dt}$, tangent, acc

$r = e^t \hat{\imath} + e^{-t} \hat{\jmath}$ ← vector function of time

Did 2nd half of class

$$\text{velocity } \vec{v} = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} \ \frac{\text{vector}}{\text{scalar}} = \frac{d\vec{r}}{dt}$$

$$= \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} \cdot \frac{\Delta s}{\Delta t} = \frac{dr}{ds} \cdot \frac{ds}{dt} = \vec{v}$$

$$= \underbrace{\vec{T}}_{\text{unit vector}} \cdot \underbrace{\frac{ds}{dt}}_{\text{speed along curve}}$$

$$\vec{v} = \frac{d\vec{r}}{dt} \qquad \vec{T} = \frac{d\vec{r}}{ds} = \frac{dr/dt}{ds/dt} = \frac{\vec{v}}{|\vec{v}|}$$

$$\vec{v} = \frac{d\ e^t \hat{\imath} + e^{-t}\hat{\jmath}}{dt} = e^t \hat{\imath} + e^{-t} \hat{\jmath}$$

derives to same thing

$$\text{speed} = |\vec{v}| = \sqrt{e^{2t} + e^{-2t}}$$

unit tangent vector ~~negative reciprocal~~ $= \frac{e^t \hat{\imath} - e^{-t} \hat{\jmath}}{\sqrt{2e^t + e^{-2t}}}$

$\frac{V}{||V||}$

$$acc = \frac{d^2 v}{dt} = e^t \hat{\imath} + e^{-t} \hat{\jmath}$$

2. $OP = \frac{1}{1+t^2}\hat{\imath} + \frac{t}{1+t^2}\hat{\jmath}$ position vector for motion

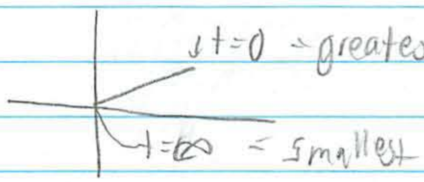
a. Calculate v, $|v|$, T

$$v = \frac{d\vec{r}}{dt} = \cancel{\frac{2}{(t^2+1)^2}\hat{\imath} + \frac{2}{(t^2+1)^2}\hat{\jmath}} \qquad \frac{-2t\hat{\imath} + (1-t^2)\hat{\jmath}}{(1+t^2)^2}$$

$$|v| = \cancel{\sqrt{\left(\frac{2}{(t^2+1)^2}\right)^2 + \left(\frac{2}{(t^2+1)^2}\right)}} \qquad \frac{1}{1+t^2}$$

$$T = \frac{dr/dt}{ds/dt} = \frac{-2t\hat{\imath} + (1-t^2)\hat{\jmath}}{(1+t^2)}$$

b. At what point is speed greatest / least?



✓ pt $(1,0)$

c. X, Y equation

$$y = tx \qquad (0,1)$$

$$t = \frac{y}{x}$$

$$x = \frac{1}{(1+t^2)}$$

$$x^2 + y^2 - x = 0$$

complete square

$$\left(x - \tfrac{1}{2}\right)^2 + y^2 = \tfrac{1}{4}$$

↳ why is TI graph nothing like this?

Circe at $(\tfrac{1}{2}, 0)$ $r = \tfrac{1}{2}$

4a Suppose a point P moves on the surface of a sphere at origin $OP = r(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$

Show that $\hat{v}$ is perpendicular to $\hat{r}$ in 2 ways

a) using x, y, z coord

Sphere of radius a: $x(t)^2 + y(t)^2 + z(t)^2 = a^2$

↓ differentiate

$$2xx' + 2yy' + 2zz' = 0$$

$$\underbrace{x\hat{\imath} + y\hat{\jmath} + z\hat{k} \cdot x'\hat{\imath} + y'\hat{\jmath} + z'\hat{k} = 0}_{\text{so perpendicular}}$$

$r \cdot r' = 0$

b. without coordinates

$$\frac{d\, r \cdot s}{dt} = \frac{dr}{dt} \cdot s + r \cdot \frac{ds}{dt}$$

? what is s - speed ?

$|r| = a \quad r \cdot r = a^2 \quad 2r \cdot \frac{dr}{dt} = 0 \quad r \cdot v = 0$

c. Prove the converse. If r and v are perpendicular, then the motion of p is on the surface of the sphere. (W/o coordinates)

$\xrightarrow{r\,v}$

$r \cdot v = 0$

$\frac{dr}{dt} \cdot r = 0$

$r \cdot r = \text{constant}$

$|r| = \sqrt{c}$ ← head of r moves on a sphere of radius $\sqrt{c}$

??

I don't get how does that prove anything

this section not introduced well in lecture + not covered in recitation

9. A point P is moving in space

$$r = OP = 3\cos t\,\hat{i} + 5\sin t\,\hat{j} + 4\cos t\,\hat{k}$$

a. Show it moves on the surface of a sphere

~~- can take deriv~~
~~- dot product it w/ original~~) if perp
~~- if 0 than parallel~~

parrallel - just show that is parallel
could say it has same coords.

b. Find speed

$$\sqrt{(3\cos t)^2 + (5\sin t)^2 + (4\cos t)^2}$$

5 - don't plug anything in for t
✓ works out on the calc
prob due to $\cos^2 + \sin^2 = 1$

c. ~~Acceleration~~ Velocity 1st
~~take deriv~~ deriv

$v = -3\sin t + 5\cos t - 4\sin t$

Now ACC - double deriv

$-3\cos t - 5\sin t - 4\cos t$ ✓

also $= -r$

d Show it moves in a plane through the origin

- could do perp $\perp$ test?

Coords satsify $4x-3z=0$

which is on plane through origin

visualize

e) Describe the motion of the point.

moves on surface of sphere

of radius 5

plane through origin

path is intersection of the 2 surfaces

Lecture 6 Polar coordinates
Notes from 18.01

4H-1D Give in polar coords

d $(-2,2)$

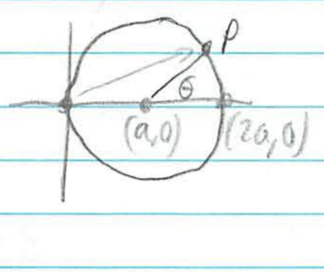
~~$-2 = r\cos\theta$ $2 = r\sin\theta$~~
Need (r,θ)

$r = \sqrt{x^2+y^2}$ $\theta = \tan^{-1}\left(\frac{y}{x}\right)$
$r = \sqrt{2^2+2^2}$ $\theta = \tan^{-1}\left(\frac{-2}{2}\right)$
$r = \sqrt{8}$ $\theta = \tan^{-1}(-1)$
$\theta = -45°, 315°$

g $(\sqrt{3},-1)$

$r = \sqrt{x^2+y^2}$ $\theta = \tan^{-1}\left(\frac{y}{x}\right)$
$r = \sqrt{3+1^2}$ $\theta = \tan^{-1}\left(\frac{2}{-2}\right)$
$r = \sqrt{4}$ $\theta = -45°, 315°$
$r = 2$

2a Find in polar coordinates for the circle of radius a w/ center at $(a,0)$



$\vec{OP} = \vec{OA} + \vec{AP}$ ← finally get this
$= \langle a, 0\rangle + \langle a\cos\theta, a\sin\theta\rangle$
$= \langle a + a\cos\theta, a\sin\theta\rangle$

$r = \sqrt{(a + a\cos\theta)^2 + (a\sin\theta)^2}$ $\quad \theta = \tan^{-1}\left(\frac{a\sin\theta}{a + a\cos\theta}\right)$

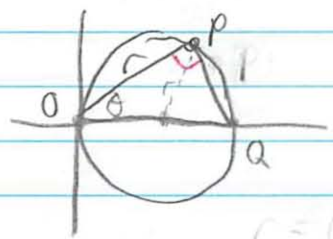
They wanted

$(x-a)^2 + y^2 = a^2$
$x - 2ax + y^2 = 0$
$r^2 - 2ar\cos\theta = 0$

$r = 2a\cos\theta$

ii) Treat it as a locust problem
???

let OQ be diameter on x axis, use trig



So θ and r are parts of the triangle

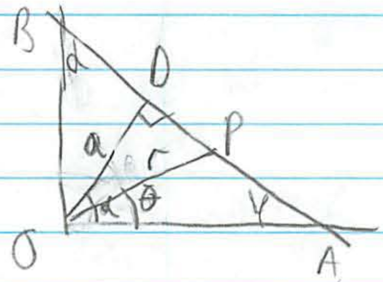
$\angle QOP = \theta$
$\overline{OP} = r$

we know $OQ = 2a$
But how does all this matter?
$\angle OPQ = 90°$

$|OP| = |OQ|\cos\theta$
$r = 2a\cos\theta$

Ok missed one fact - then can do trig but θ is still a variable in the answer for r - was not thinking that was allowed.

2. Find the polar equation for the line intersecting...



$\alpha = \angle DOA$

Find equation relating r, θ, α, a

Is it similar triangles, not really. But you have 2 angles α

$90 + \alpha + \gamma = 180$
$90 + \alpha_2 + \gamma = 180$

OQP is right Triangle
? there is no Q

$|OP| = r$
$\angle POQ = \alpha - \theta$

$r \cos(\alpha - \theta) = a$
$r(\cos\alpha \cos\theta + \sin\alpha \sin\theta) = a$
$\frac{x}{A} + \frac{y}{B} = 1$

I don't think this solution is correct - there is no Q on the diagram.

3a Give in rectangular coords

$r = \sec\theta$

try to get a $a\cos\theta$ in this

$x = \cancel{a}\cos\theta$ ✓ try $y = \cancel{a}\sin\theta$

$x = \sec\theta\cos\theta$ $y = \sec\theta\sin\theta$

$x = 1$

well do identity

Graph

this is a really weird result

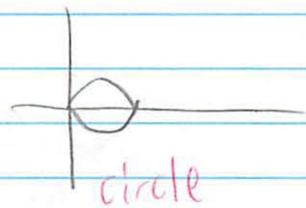
b. $r = 2a\cos\theta$

$x = 2a\cos\theta\cos\theta$ $y = 2a\cos\theta\sin\theta$

$2a\cos^2\theta$

what is a?

did by mistake



circle

$r = 2a\cos\theta$

$r^2 = r2a\cos\theta$

$r^2 = 2ax$

trying to get $x = a\cos\theta$

$x^2 + y^2 = 2ax$

$r =$

F) $r = a\cos(2\theta)$ (4 leaf rose)

$x = a\cos\theta$ $y = a\sin\theta$

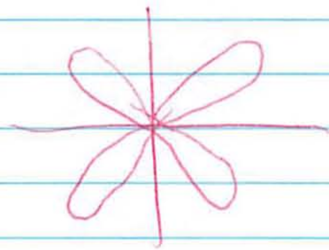
identity

$r = a\left(2\cos^2\theta - 1\right)$

$r = a\left(\frac{2x^2}{x^2+y^2} - 1\right)$

$= (x^2+y^2)^{3/2}$

$= a(x^2 - y^2)$



4I-1a Find the arc lenght $ds = w(\theta)\, d\theta$ for 4H-3

$r = a\sec\theta$

$\frac{ds}{dt} = |\vec{v}|$

$s = \int_0^{2\pi} |V|\, dt$

$\sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\, d\theta$

good way of writing

notice r → not defined here

$\vec{v} = \langle 1, \sec\theta\sin\theta \rangle$

$|\vec{v}|$ $\cancel{\sqrt{1^2 + (\sec\theta\sin\theta)^2}}$

$\sec^2\theta\, d\theta$

$2\pi\sqrt{1 + (\sec\theta\sin\theta)^2}$

I think I messed up y

b. $\sqrt{(2a\cos^2\theta)^2 + (2a\cos\theta\sin\theta)^2}$ or

$2a\,d\theta$

Formula I have converts to dt, not $d\theta$ -
- think this might be wrong
- but answer does not show how its solved

Part 2

1. Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a matrix of constants $x = \begin{pmatrix} x \\ y \end{pmatrix}$

a column vector and λ a real number

If $Mx = \lambda x$ has a non-trivial solution λ is called the eigenvalue

a Show M has either 2, 1, or no real eigenvalue

Wikipedia

- ie changes only magnitude - <u>not</u> direction

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix}$$

If $\lambda = 1$ then it does not change direction

det(M) for a non trivial solution

$$ax + by = \lambda x$$
$$cx + dy = \lambda x$$

$$\left.\begin{matrix} (a - \lambda)x + by = 0 \\ (d - \lambda)y + cx = 0 \end{matrix}\right] \; x(M - \lambda) = 0$$

$\det(M - \lambda) = 0$ for M to have a non-trivial solution

$$\det(M - \lambda) = \begin{vmatrix} a - \lambda & b \\ c & d - \lambda \end{vmatrix}$$

$$\det(M-\lambda) = (a-\lambda)(d-\lambda) - bc$$
$$= \lambda^2 - (a+d)\lambda + (ad-bc) ✓$$
$$= 0$$

So $\lambda = \dfrac{a+d}{2} \pm \sqrt{\dfrac{(a+d)^2 - 4(ad-bc)}{2}}$

$$= \frac{a+d}{2} \pm \sqrt{\frac{a^2+2ad+d^2-4ad+4bc}{2}}$$

$$= \frac{a+d}{2} \pm \sqrt{\frac{4bc+a^2-2ad+d^2}{2}}$$

$$= \frac{a+d}{2} \pm \sqrt{\frac{4bc+(a-d)^2}{2}}$$

✓ M can have 2 real eigenvalues if $4bc+(a-d)^2 > 0$

1 $\qquad = 0$

0 $\qquad < 0$

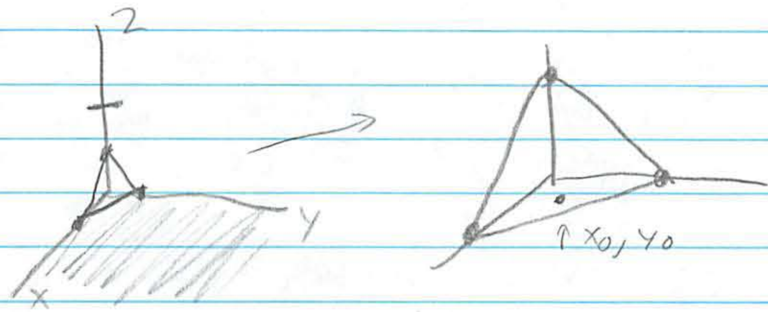
b. Give the respective inequalities connecting ad and bc that tell when each of the 3 cases in a hold.

$$4bc + (a-d)^2 > 0$$
$$4bc > -(a-d)^2$$
$$bc > -\frac{(a-d)^2}{4}$$

$$4bc + (a-d)^2 = 0$$
$$bc = -\frac{(a-d)^2}{4}$$

$$4bc + (a-d)^2 < 0$$
$$4bc < -(a-d)^2$$
$$-bc < -\frac{(a-d)^2}{4}$$

2. Imagine an eye at the point E $(0,0,4)$ looking down on xy plane



Where is the ant actually in space?

- answer in x_0, y_0

reduce to 2D and just x_0

wait - why are we answering in terms of x_0, y_0

$\leftarrow$ or how do you get the intersection here?

$x_0 = r \sin\theta$

$y_0 = r \cos\theta$? or w/ z-axis

Find normal vector of plane

~~$\langle 1, 1, 1 \rangle$~~ using diff

Find 2 lines and cross multiply

$AB = \langle -1, 1, 0 \rangle$

$AC = \langle -1, 0, 1 \rangle$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} i & j & k \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = \hat{i} - \hat{j} + \hat{k}$$

$N = \langle 1, -1, 1 \rangle$

P is a point on the plane

$\langle 1, -1, 1 \rangle \cdot \langle x-1, y, z \rangle = 0$ ← perpendicular

$x - y + z = 1$ × ← eq of plane

Find the parametric eq

$P = (x_0, y_0, 0)$

$Q = (0, 0, 4)$

$r(t) = P + t\,\vec{PQ}$

$= (x_0, y_0, 0) + (0, 0, 4t)$

$x(t) = x_0\ t$

$y(t) = y_0\ t$

$z(t)\ 4 - 4t$

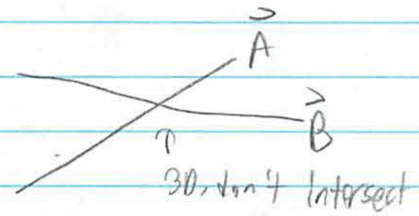
$t = \dfrac{1 - x_0 + y_0}{4}$

Find the intersection by plugging in for t

Ant located at $(x_0, y_0, 1 - x_0 + y_0)$ ×

−2

3. 2 lines in space are called skew if they do not intercept and are not parrallel



only works in 3D I guess

a) Using $\vec{A}$ $\vec{B}$ and 2 arbitrarily chosen points P_1 P_2 give a formula for the distance between 2 lines.

i) If $Q_1 Q_2$ is shortest must be perpendicular to both lines

ii) Triangle $Q_1 Q_2 P_2$ has right angle at Q_2

Ok this is operating in 3D.

But in 2D

shortest not many rules

Can see how it bends in 3d to just reach

= really ?!

On wikipedia

What is difference b/w p and Q?

(v_1, v_2) (v_3, v_4) ← 2 pts

$t(v_2 - v_1) + v_1$

$s(v_4 - v_3) + v_3$

s, t real valued parameters?

⟶

Apply a version of the Pythagorean Theorm

$$As^2 + Bst + Ct^2 + 2Ds + Et + F$$

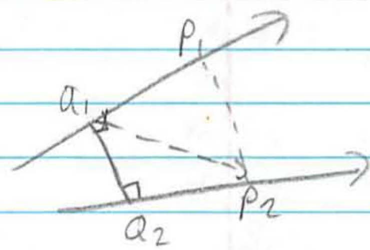
$$A = (v_4 - v_3)\cdot(v_4 - v_3)$$
$$B = (v_4 - v_3)\cdot(v_1 - v_2)$$
$$C = (v_1 - v_2)\cdot(v_1 - v_2)$$
$$D = (v_4 - v_3)\cdot(v_1 - v_2)$$
$$E = (v_1 - v_2)\cdot(v_3 - v_1)$$
$$F = (v_3 - v_4)\cdot(v_3 - v_1)$$

$$d^2 = \frac{ACF + 2BDE - AE^2 - CD^2 - FB^2}{AC - B^2} = \frac{\det R}{\det S}$$

$$R = \begin{bmatrix} A & B & D \\ B & C & E \\ D & E & F \end{bmatrix} \quad S \quad \begin{bmatrix} A & B \\ B & C \end{bmatrix}$$

Alternate way

$$Q Q_2 = \frac{A \times B}{|\vec{A} \times \vec{B}|}$$ cross products $\perp$



$Q_1 Q_2 P_2$ is right triangle

$$\vec{P_1 P_2} = P_1 Q_1 + \underbrace{Q_1 P_2}$$
$$= P_1 Q_1 + \underbrace{Q_1 Q_2 + Q_2 P_2}_{\text{components of } Q_1 P_2}$$

$$\sigma = \frac{\vec{Q_1 Q_2}}{|Q_1 Q_2|}$$) direction of skew

Shortest lenght of $\overrightarrow{P_1P_2}$

$= \overrightarrow{P_1P_2} \cdot \hat{u_r}$ = amt of $\overrightarrow{P_1P_2}$ in d'r $\overrightarrow{Q_1Q_2}$

$= \overrightarrow{P_1P_2} \cdot \dfrac{Q_1Q_2}{|\overrightarrow{Q_1Q_2}|}$

$= (\overrightarrow{P_1Q_1} + \overrightarrow{Q_1Q_2} + \overrightarrow{Q_2P_2}) \cdot \dfrac{Q_1Q_2}{|Q\,Q_2|}$

$= \dfrac{\overrightarrow{Q_1Q_2} \cdot \overrightarrow{Q_1Q_2}}{|Q_1Q_2|}$

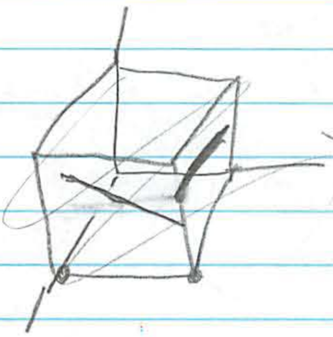
$= |Q_1Q_2|$

$= P_1P_2 \cdot \dfrac{\vec{A} \times \vec{B}}{|\vec{A} \times \vec{B}|}$ ✓

This works since $P_1P_2 = P_1Q_1 + Q_1Q_2 + Q_2Q_2 \cdot \dfrac{Q_1Q_2}{|Q_1Q_2|}$

Since $\overrightarrow{P_1Q_1} + \overrightarrow{Q_2P_2}$ will disapper once dot product is taken since both are orthogonal to $\overrightarrow{Q_1Q_2}$

b. Use the formula to find the distance between 2 skew diagonals lying on adjacent sides of a unit cube



$\langle 1, -\frac{1}{2}, \frac{1}{2} \rangle$ $\langle 1\ 1\ \frac{1}{2} \rangle$

I think

- no that is pts, lines are different

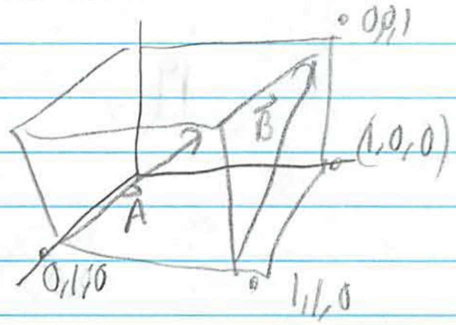
well just use these in calc

$\overset{v_1\ v_2}{(1, 0)}$ $\overset{v_3\ v_4}{(0, 1)}$ ← pts

$d^2 = (1-0)\cdot(1-0)(0-1)(0-1)(0-1)(0-1)+2$ - - - - (-1)

this is going to take forever + most likely wrong ✓

Alternate way



$\overrightarrow{P_1P_2} = \langle 0, -1, 0 \rangle$

$\vec{A} = \langle 1, 0, 0 \rangle$

$\vec{B} = \langle -1, -1, 1 \rangle$

distance b/w 2 diagonals

$$\langle 0, -1, 0 \rangle \cdot \frac{\langle 1, 0, 0 \rangle \times \langle -1, -1, 1 \rangle}{|\langle 1, 0, 0 \rangle \times \langle -1, -1, 1 \rangle|}$$

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 0 \\ -1 & -1 & 1 \end{vmatrix} = 0\hat{i} - \hat{j} - \hat{k}$$ ✗

$$|\vec{A} \times \vec{B}| = \frac{\sqrt{1^2 + 1^2}}{\sqrt{2}}$$

distance b/w 2 algorithms

$$= \langle 0, -1, 0 \rangle \cdot \frac{\langle 0, -1, -1 \rangle}{\sqrt{2}}$$

$$= 0 + \frac{1}{\sqrt{2}} \hat{j} + 0k$$

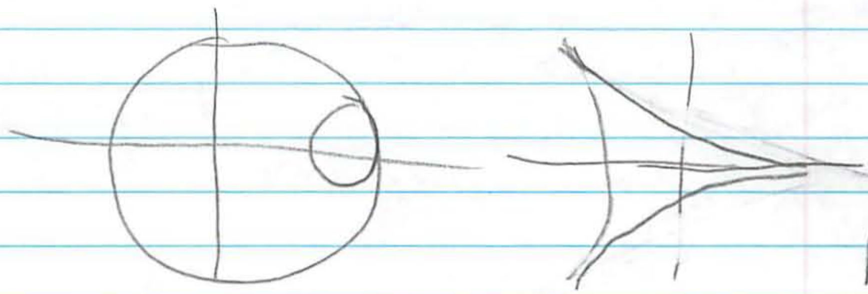
$$= \frac{1}{\sqrt{2}}$$

X

4. Diagram on p595 of some book : hypocycloid traced out by P on circle of radius b which rolls around on inside of circle of radius a

-2

rolls in circle not line



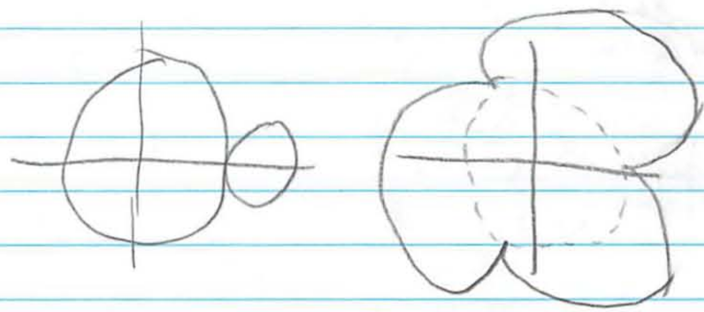
R=3

wikipedia

$$x(\theta) = (R-r)\cos\theta + r\cos\left(\frac{R-r}{r}\theta\right)$$

$$y(\theta) = (R-r)\sin\theta - r\sin\left(\frac{R-r}{r}\theta\right)$$

epicycloid circle b rolls counter clockwise circle a outside



R=3

wp

$$x(\theta) = (R+r)\cos\theta - r\cos\left(\frac{R+r}{r}\theta\right)$$

$$y(\theta) = (R+r)\sin\theta - r\sin\left(\frac{R+r}{r}\theta\right)$$

Derive $r(\theta)$

$$(R+r)\cos\theta - r\cos\left(\frac{R+r}{r}\theta\right)\hat{\imath} + (R+r)\sin\theta - r\sin\left(\frac{R+r}{r}\theta\right)\hat{\jmath}$$

5. $r = -\ln \cos t \; \hat{i} + t \; \hat{j}$ for $0 \le t < \frac{\pi}{2}$

a. Find v

$$\vec{v} = \frac{d\vec{r}}{dt} = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle$$

take deriv of each part seperaty

$$\vec{v} = \langle \tan\theta, 1 \rangle$$

Find $\hat{T}$

$$\frac{dr}{ds} \quad \text{or} \quad \frac{\vec{v}}{|\vec{v}|}$$

$d \ln = \frac{1}{x}$ chain rule

$$\frac{\langle \tan\theta, 1 \rangle}{\sqrt{\tan^2\theta + 1^2}} \leftarrow \sec\theta$$

$$\left\langle \frac{\tan\theta}{\sec\theta}, \cos\theta \right\rangle$$

Find speed $|\vec{v}|$

did above $\sqrt{\tan^2\theta + 1^2}$

$$\frac{1}{\cos\theta}$$

$\sec\theta$ ✓

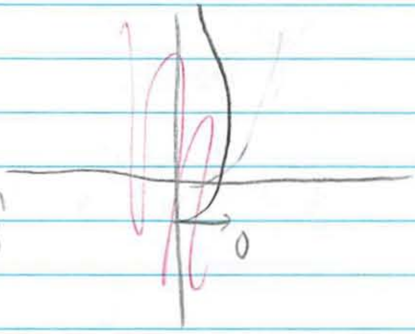
b. sketch

b. Sketch curve

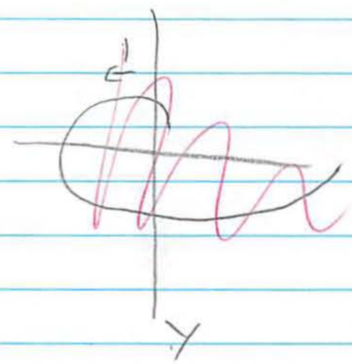
- pay attention to dir(v) at $t=0$
  $t=\frac{\pi}{4}$

really stupid

How does TI graph this?



X



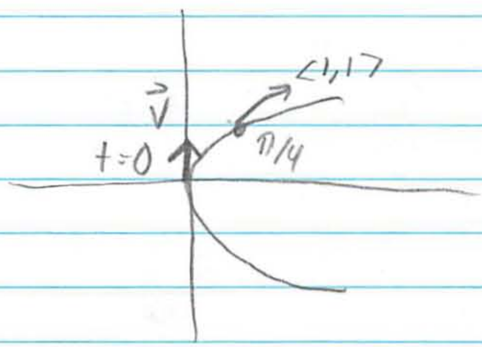
Y

Graph in Parametric

$\text{dir } \vec{v}(0) = \langle \tan 0, 1 \rangle$
$\langle 0, 1 \rangle$

$\text{dir } \vec{v}\left(\frac{\pi}{4}\right) = \langle \tan \frac{\pi}{4}, 1 \rangle$
$\langle 1, 1 \rangle$

makes much more sense now



md

$\times (-1)$

c. Find the distance traveled over the curve

$$\frac{ds}{dt} = \int |\vec{v}|$$

$$= \int_0^{\pi/4} \sec\theta \, d\theta$$

TT

$$= \ln(\sec t + \tan t) \Big]_0^{\pi/4}$$

$$= \ln(\sqrt{2}+1) - \ln(1+0)$$

$$= \ln(\sqrt{2}+1)$$

6. Change the polar to rectangular

$$r = 4a(\cos\theta + \sin\theta)$$

$x = r\cos\theta$

$= (4a[\cos\theta + \sin\theta])\cos\theta$

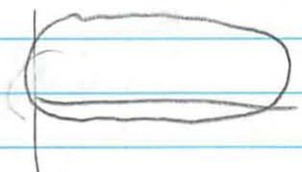
$4a\cos^2\theta + 4a\cos\theta\sin\theta$

does this simplify?

$y = r\sin\theta$

$= (4a[\cos\theta + \sin\theta])\sin\theta$

$4a\cos\theta\sin\theta + 4a\sin^2\theta$



oval

why do I it wrong every time

want to simplify

replace stuff

do other way

$\cos\theta = \frac{x}{r}$ $\sin\theta = \frac{y}{r}$

$\sqrt{x^2+y^2} = 4a\left(\frac{x}{\sqrt{x^2+y^2}} + \frac{y}{\sqrt{x^2+y^2}}\right)$

$x^2+y^2 = 4ax + 4ay$

$x^2 - 4ax + y^2 - 4ay = 0$

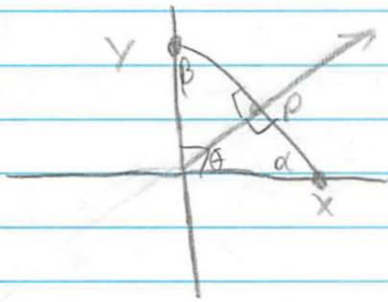
$(x-2a)^2 + (y-2a)^2 = 8a^2$

circle $(2a, 2a)$

r $\sqrt{8a^2}$ ✓

b. A line segment lenght 2 k

locus = set of lines that meet requirements



Area $= \frac{1}{2}XY$

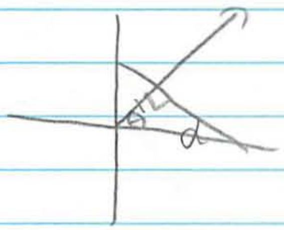
~~Area of smaller triangle $= \frac{1}{2} \cdot \frac{1}{2} XY$~~

not always true

$180 = 90 + \theta + \alpha$

$180 = 90 + \alpha + \beta$

so $\theta = \beta$ by similar triangles



$\langle a\cos\theta\,\hat{i} + a\sin\theta\,\hat{j}, \theta\rangle$

Need lenght of a - but how to find

will be $\frac{Y}{\sin\alpha} = \frac{X}{\sin\beta} \rightarrow \frac{r}{\sin\alpha} = \frac{X}{\sin 90}$

so

$$\langle \underset{?\,\alpha}{\underline{x \sin\alpha}} \cos\theta\, \hat{\imath} + x \sin\alpha \sin\theta\, \hat{\jmath}, \theta \rangle$$

is $\frac{y}{\sin\alpha} = \frac{x}{\sin\theta}$ ← defined it like this

$$y = \frac{x \sin\alpha}{\sin\theta}$$

$$y \sin\theta = x \sin\alpha$$

$$\sin\alpha = \frac{y \sin\theta}{x}$$

$$\alpha = \sin^{-1}\left(\frac{y\sin\theta}{x}\right)$$

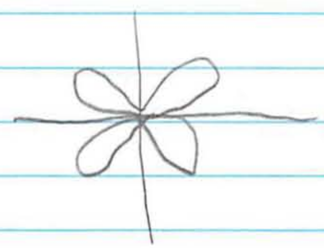
crazy, but might just work

-2

×

$$\langle x \sin^{-1}\left(\frac{y\sin\theta}{x}\right) \cos\theta\, \hat{\imath} + x \sin^{-1}\left(\frac{y \sin\theta}{x}\right) \sin\theta\, \hat{\jmath}, \theta \rangle$$

c. Sketch



is it a 4 leaf clover?

✓

7. For the motion of a point P given by $r = e^{at}$
$\theta = at$

don't think we did something like this

Find $\vec{V}$ $= \langle \frac{dr}{dt}, \frac{d\theta}{dt} \rangle$

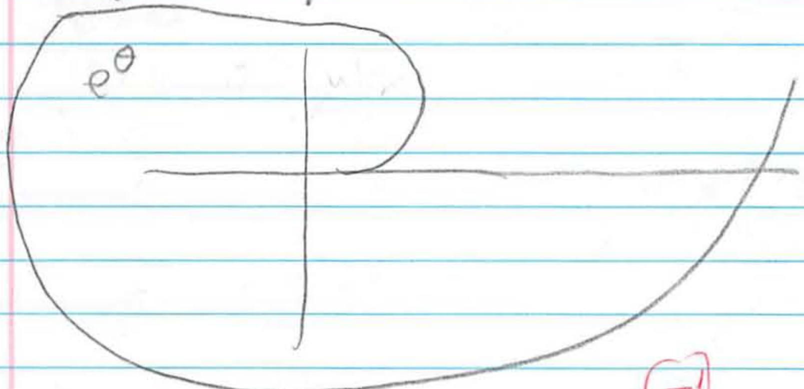
$\langle 5e^{5x}, a \rangle$

Find $\frac{ds}{dt} = |\vec{v}|$ $\sqrt{(ae^{ax})^2 + a^2}$

$a\sqrt{e^{2ax}+1}$

Find $\hat{T} = \frac{\langle 5e^{5x}, a \rangle}{a\sqrt{e^{2ax}+1}} = \frac{\vec{v}}{|\vec{v}|}$ × -1

b. Lenght of path from $t=0$ till P croses x axis

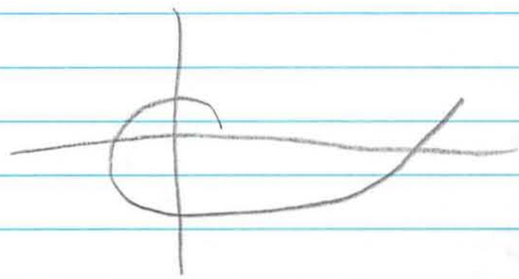


$\theta = 2\pi$
$x = 535$

-1

not to scale

c. Show that the curve of motion makes a constant angle w/ velocity r



-1

is curve of motion v?

or θ stays constant ← but that's not true

18.02 Prob Set 2 Solutions Spring 2010

1 $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \lambda x \\ \lambda y \end{pmatrix}$ becomes

a) $\begin{matrix} ax + by = \lambda x \\ cx + dy = \lambda y \end{matrix}$ or $\begin{matrix} (a-\lambda)x + by = 0 \\ cx + (d-\lambda)y = 0 \end{matrix}$ (homogeneous eqns)

By the theorem, these have a non trivial soln $\iff \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0$

or: $(a-\lambda)(d-\lambda) - bc = 0$

which we write

$$\lambda^2 - (a+d)\lambda + ad - bc = 0$$

a quadratic equation with at most 2 roots.

b) $A\lambda^2 + B\lambda + C = 0$ has soln $\lambda = \dfrac{-B \pm \sqrt{B^2 - 4AC}}{2A}$

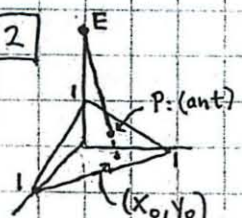
2 real roots: $B^2 > 4AC$ $\therefore (a-d)^2 > -4bc$

1 real root: $B^2 = 4AC$ $(a-d)^2 = -4bc$

no real roots: $B^2 < 4AC$ $(a-d)^2 < -4bc$

$(a+d)^2 > 4(ad-bc) \iff (a-d)^2 > -4bc$

2



The plane is $x + y + z = 1$

The line goes through $(0,0,4)$ and $(x_0, y_0, 0)$

dir vector of line: $\langle x_0, y_0, -4\rangle$

goes through $(0,0,4)$

line: $\begin{cases} x = x_0 t \\ y = y_0 t \\ z = 4 - 4t \end{cases}$

Substituting to find intersection point:

$x_0 t + y_0 t + 4 - 4t = 1$

$(x_0 + y_0 - 4)t = -3$

$t = \dfrac{-3}{x_0 + y_0 - 4}$

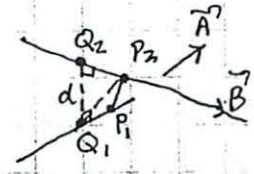
P: $\left(\dfrac{-3x_0}{x_0+y_0-4}, \dfrac{-3y_0}{x_0+y_0-4}, \dfrac{4x_0+4y_0-4}{x_0+y_0-4}\right)$

$(x_0, y_0) = (1, 0) \Rightarrow P = (1,0,0)$ checks.

$= (0,1) \Rightarrow P = (0,1,0)$

3

a)



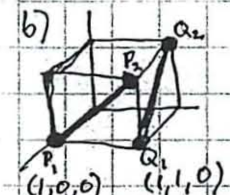
Since $\overrightarrow{Q_1Q_2}$ is $\perp$ to $\vec{A}$ and $\vec{B}$, it has the direction of $\pm \vec{A} \times \vec{B}$

$\overrightarrow{P_1P_2}$ has its scalar component in the direction of $\overrightarrow{Q_1Q_2} = d$ (⊛)

$\therefore \overrightarrow{P_1P_2} \cdot \dfrac{\vec{A}\times\vec{B}}{|\vec{A}\times\vec{B}|} = \pm d$ $d > 0$

[⊛ think of two ∥ planes through Q_1 + Q_2 and $\perp$ to $\overrightarrow{Q_1Q_2}$ — they are d distance apart, contain the two lines, and any points P_1 + P_2 on them obey ⊛, not just two points on the two lines.]

b)



[The P_i, Q_i are unrelated to the ones above]

P_2: $(1,1,1)$ Q_2: $(0,1,1)$

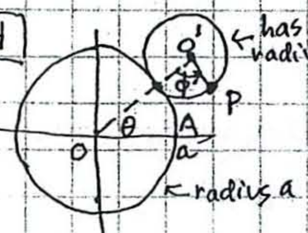
$\overrightarrow{P_1P_2} = \vec{A} = \langle 0,1,1\rangle$

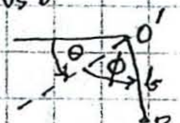
$\overrightarrow{Q_1Q_2} = \vec{B} = \langle -1,0,1\rangle$

$\vec{A}\times\vec{B} = \begin{vmatrix} i & j & k \\ 0 & 1 & 1 \\ -1 & 0 & 1 \end{vmatrix} = \langle 1, -1, 1\rangle$ $|A\times B| = \sqrt{3}$

$\overrightarrow{P_1Q_1} \cdot \dfrac{\vec{A}\times\vec{B}}{|\vec{A}\times\vec{B}|} = \dfrac{-1}{\sqrt{3}}$ so $d = \dfrac{1}{\sqrt{3}}$

4





$\overrightarrow{OP} = \overrightarrow{OO'} + \overrightarrow{O'P}$

$\overrightarrow{OO'} = (a+b)\langle\cos\theta, \sin\theta\rangle$

$\overrightarrow{O'P} = -b\langle\cos(\theta+\phi), \sin(\theta+\phi)\rangle$

[use the − sign to make $\hat{\imath}$-component > 0 according to the picture $\hat{\jmath}$-component < 0]

But since one circle rolls on the other:

$a\theta = b\phi$ (two arclengths are equal)

$\therefore \phi = \dfrac{a\theta}{b}$ $\theta + \phi = \left(\dfrac{a+b}{b}\right)\theta$

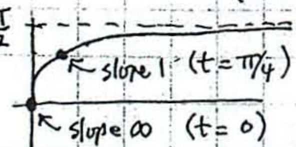
$\overrightarrow{OP} = \left[(a+b)\cos\theta - b\cos\left(\frac{a+b}{b}\theta\right)\right]\hat{\imath}$ + [same for $\hat{\jmath}$ replacing cos by sin]

5 $\vec{r} = -\ln\cos t\,\hat{\imath} + t\,\hat{\jmath}, \quad 0 \le t < \frac{\pi}{2}$

a) $\vec{v} = \frac{d\vec{r}}{dt} = \tan t\,\hat{\imath} + \hat{\jmath}$

$$\frac{ds}{dt} = |\vec{v}| = \sqrt{\tan^2 t + 1} = \sec t = \frac{1}{\cos t}$$

$$\vec{T} = \frac{\vec{v}}{|\vec{v}|} = \langle \tan t, 1\rangle \cdot \cos t = \langle \sin t, \cos t\rangle$$

b) 

since $\vec{T} = \langle \sin t, \cos t\rangle$

c) $$\int_0^{\pi/4} \sec t\,dt = \ln(\sec t + \tan t)\Big]_0^{\pi/4} = \ln(\sqrt{2}+1)$$

6 $r = 4a(\cos\theta + \sin\theta)$

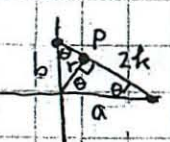
a) $= 4a\left(\frac{x}{r} + \frac{y}{r}\right)$

$x^2 + y^2 = 4ax + 4ay;$ completing squares:

$(x-2a)^2 + (y-2a)^2 = 8a^2$

circle with center at $(2a, 2a)$

radius $= 2a\sqrt{2}$ – goes through origin.

b) 

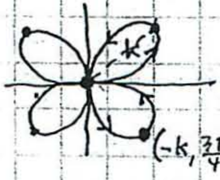
area $\triangle = \frac{1}{2}ab = \frac{1}{2}r\cdot 2k$

cancel the $\frac{1}{2}$

$\therefore \underbrace{2k\cos\theta}_{b} \cdot \underbrace{2k\sin\theta}_{a} = 2kr$

$\therefore r = k\cdot 2\sin\theta\cos\theta$

$\boxed{r = k\cdot\sin 2\theta}$

c) 

(use negative r also – when $\theta = \frac{3\pi}{4}$, $r = -k$ for example)

7 $r = e^{at}, \quad \theta = at$

a) using $\vec{r} = r\hat{u}_r$
$\vec{v} = \dot{r}\hat{u}_r + r\dot{\theta}\,\hat{u}_\theta$

we get
$\vec{v} = ar\,\hat{u}_r + ar\,\hat{u}_\theta$

$$\frac{ds}{dt} = |\vec{v}| = ar\cdot\sqrt{2}$$

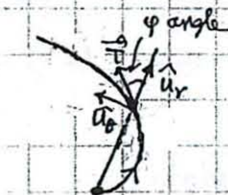
$$\vec{T} = \frac{\vec{v}}{|\vec{v}|} = \frac{\hat{u}_r + \hat{u}_\theta}{\sqrt{2}}$$

b) $\frac{ds}{dt} = ae^{at}\sqrt{2}$ $\quad t=0:\ r=0$

next crosses x-axis when $\theta = 2\pi$, $t = \frac{2\pi}{a}$

$\therefore$

$$s = \int_0^{2\pi/a} ae^{at}\sqrt{2}\,dt = e^{at}\sqrt{2}\Big]_0^{2\pi/a} = \sqrt{2}(e^{2\pi} - 1)$$

c) 

since $\vec{T} = \frac{\hat{u}_r + \hat{u}_\theta}{\sqrt{2}}$,

$\vec{T}\cdot\hat{u}_r = \frac{\sqrt{2}}{2} = \cos\varphi$

so $\varphi = \pi/4$ always

18.02 Lecture 6. – Tue, Sept 18, 2007

Handouts: Practice exams 1A and 1B.

Velocity and acceleration. Last time: position vector $\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath}$ $[+z(t)\hat{k}]$.

E.g., cycloid: $\vec{r}(t) = \langle t - \sin t, 1 - \cos t\rangle$.

Velocity vector: $\vec{v}(t) = \frac{d\vec{r}}{dt} = \langle \frac{dx}{dt}, \frac{dy}{dt}\rangle$. E.g., cycloid: $\vec{v}(t) = \langle 1 - \cos t, \sin t\rangle$. (at $t = 0$, $\vec{v} = \vec{0}$: translation and rotation motions cancel out, while at $t = \pi$ they add up and $\vec{v} = \langle 2, 0\rangle$).

Speed (scalar): $|\vec{v}|$. E.g., cycloid: $|\vec{v}| = \sqrt{(1-\cos t)^2 + \sin^2 t} = \sqrt{2 - 2\cos t}$. (smallest at $t = 0, 2\pi, \ldots$, largest at $t = \pi$).

Acceleration: $\vec{a}(t) = \frac{d\vec{v}}{dt}$. E.g., cycloid: $\vec{a}(t) = \langle \sin t, \cos t\rangle$ (at $t = 0$ $\vec{a} = \langle 0, 1\rangle$ is vertical).

Remark: the speed is $|\frac{d\vec{r}}{dt}|$, which is NOT the same as $\frac{d|\vec{r}|}{dt}$!

Arclength, unit tangent vector. s = distance travelled along trajectory. $\frac{ds}{dt}$ = speed $= |\vec{v}|$. Can recover length of trajectory by integrating ds/dt, but this is not always easy... e.g. the length of an arch of cycloid is $\int_0^{2\pi} \sqrt{2 - 2\cos t}\, dt$ (can't do).

Unit tangent vector to trajectory: $\hat{T} = \frac{\vec{v}}{|\vec{v}|}$. We have: $\frac{d\vec{r}}{dt} = \frac{d\vec{r}}{ds}\frac{ds}{dt} = \hat{T}\frac{ds}{dt} = \hat{T}|\vec{v}|$.

In interval Δt: $\Delta\vec{r} \approx \hat{T}\,\Delta s$, dividing both sides by Δt and taking the limit $\Delta t \to 0$ gives us the above identity.

Kepler's 2nd law. (illustration of efficiency of vector methods) Kepler 1609, laws of planetary motion: the motion of planets is in a plane, and area is swept out by the line from the sun to the planet at a constant rate. Newton (about 70 years later) explained this using laws of gravitational attraction.

Kepler's law in vector form: area swept out in Δt is area $\approx \frac{1}{2}|\vec{r} \times \Delta\vec{r}| \approx \frac{1}{2}|\vec{r} \times \vec{v}|\,\Delta t$
So $\frac{d}{dt}(\text{area}) = \frac{1}{2}|\vec{r} \times \vec{v}|$ is constant.

Also, $\vec{r} \times \vec{v}$ is perpendicular to plane of motion, so $\mathrm{dir}(\vec{r} \times \vec{v})$ = constant. Hence, Kepler's 2nd law says: $\vec{r} \times \vec{v}$ = constant.

The usual product rule can be used to differentiate vector functions: $\frac{d}{dt}(\vec{a} \cdot \vec{b})$, $\frac{d}{dt}(\vec{a} \times \vec{b})$, being careful about non-commutativity of cross-product.

$$\frac{d}{dt}(\vec{r} \times \vec{v}) = \frac{d\vec{r}}{dt} \times \vec{v} + \vec{r} \times \frac{d\vec{v}}{dt} = \vec{v} \times \vec{v} + \vec{r} \times \vec{a} = \vec{r} \times \vec{a}.$$

So Kepler's law $\Leftrightarrow$ $\vec{r} \times \vec{v}$ = constant $\Leftrightarrow$ $\vec{r} \times \vec{a} = 0$ $\Leftrightarrow$ $\vec{a} // \vec{r}$ $\Leftrightarrow$ the force $\vec{F}$ is central.

(so Kepler's law really means the force is directed $//\vec{r}$; it also applies to other central forces – e.g. electric charges.)

18.02 Lecture 7. – Thu, Sept 20, 2007

Handouts: PS2 solutions, PS3.

Review. Material on the test = everything seen in lecture. The exam is similar to the practice exams, or very slightly harder. The main topics are (Problem numbers refer to Practice 1A):

1) vectors, dot product. $\boldsymbol{A} \cdot \boldsymbol{B} = |\boldsymbol{A}||\boldsymbol{B}|\cos\theta = \sum a_i b_i$. Finding angles. (e.g. Problem 1.)

2) cross-product, area of space triangles $\frac{1}{2}|\boldsymbol{A} \times \boldsymbol{B}|$; equations of planes (coefficients of equation = components of normal vector) (e.g. Problem 5.)

3) matrices, inverse matrix, linear systems (e.g. Problem 3.)

4) finding parametric equations by decomposing position vector as a sum; velocity, acceleration; differentiating vector identities (e.g. Problems 2,4,6).

Lecture 7
Curvature and Acceleration

2/18

P-set 3A - Needed for exam

both involve 2nd derivitives

Review $\vec{r} = x\hat{i} + y\hat{j} = \langle x, y \rangle$ position vector

function of a variable

$\vec{v} = \langle x', y' \rangle$ velocity

$|\vec{v}| = v = \frac{ds}{dt}$ speed

s arc length

$\frac{dr}{dt} = \hat{T} \cdot \frac{ds}{dt}$

$\hat{T} = \frac{v}{|\vec{v}|}$ unit tangent vector

If $\vec{A}(t)$ has constant lenght as time varries,

then $\vec{A}(t) \perp \frac{d\vec{A}}{dt}$

$\vec{A} \cdot \vec{A} = C = |\vec{A}|^2$

↓ derivitive

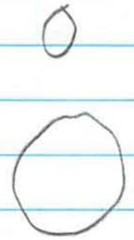
$2\vec{A} \cdot \frac{dA}{dt} = 0$

tip always moves in a circle

perpendicular is always tangent to circle

Curvature κ ← kappa, small curve, capital k

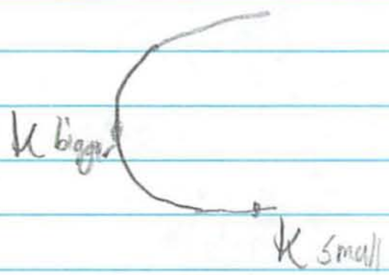
line → curvature $= 0$



← bigger k

← smaller k

$k_{circle} = \frac{1}{a}$



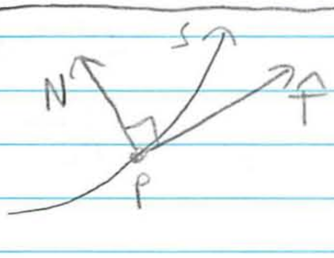
<u>Definition</u>

$\frac{d\hat{T}}{ds}$ ← w/ respect to arc lenght   want to look at the curve not how it is traced

when $\hat{T}$ changes direction - it has curve

$\frac{d\hat{T}}{ds}$ will give you $\hat{N}$ - perpendicular to tangent vector and a scalor factor k (its magnitude)

$$\boxed{\frac{d\hat{T}}{ds} = \hat{N} k}$$



$\hat{N}$ is $\perp$ to T so TN is right handed systen

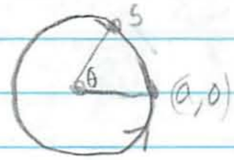
$\hat{N}$ is $\hat{T}$ 90° counter clock wise

$\kappa > 0$ curve going to left

- as s increases

$\kappa < 0$ " " " right

Example: Curvature of Circle



$\vec{r} = a$

$s = a\theta$

$\vec{r} = a\langle \cos\theta, \sin\theta \rangle$

$\vec{r} = a\langle \cos\frac{s}{a}, \sin\frac{s}{a} \rangle$

$\hat{T} = \frac{dr}{ds} = \langle -\sin\frac{s}{a}, \cos\frac{s}{a} \rangle$

$\theta = t$ $\qquad \frac{d\hat{T}}{ds} = \frac{1}{a}\langle -\cos\frac{s}{a}, -\sin\frac{s}{a} \rangle$ ← normal vector $\vec{N}$

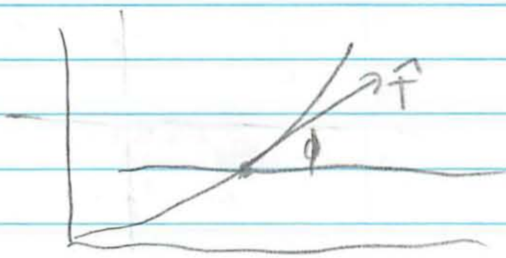
$\boxed{\kappa = \frac{1}{a}}$

3 ways

- from definition (as shown)
  - does not work well for parabola
- 18.01 method
  - no vectors
  - "nice" formula
  - only works on a plane $y = f(x)$

$\boxed{\kappa = \frac{d\phi}{ds}}$

proof →



$\hat{T} = \langle \cos\phi, \sin\phi \rangle$

$\frac{d\hat{T}}{ds} = \frac{d\hat{T}/d\phi}{ds/d\phi}$ Use avallible parameters

$= \frac{\langle -\sin\phi, \cos\phi \rangle}{ds/d\phi}$

↑ switch the 2 components put − before 1st one

$= \vec{N}\,\frac{d\phi}{ds}$

so $\kappa = \frac{d\phi}{ds}$

$\phi = \tan^{-1} y'$ (y' = slope of $\vec{T}$)

$\frac{d\phi}{ds} = \frac{d\phi/dx}{ds/dx}$ ← eaiser to calculate than finding s

$= \frac{\frac{1}{1+y'^2} \cdot y''}{\sqrt{1+y^2}}$

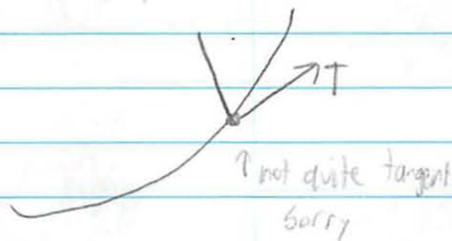
$$\boxed{= \frac{y''}{(1+y'^2)^{3/2}}}$$

$\vec{T}$-$\vec{N}$ Moving Frame, acceleration

- physics topic
- in book

$\vec{r} = \langle x(t), y(t) \rangle$ position vector
$\vec{a} = \langle x''(t), y''(t) \rangle$ acc
easy to calculate
but no meaning what acc is

must attach to coord system
Not any arbitrary choices of $\hat{\imath}$ and $\hat{\jmath}$



$\vec{r}(t)$ = given position vector
$\vec{v} = \vec{T} \cdot \frac{ds}{dt}$

differentiate $u(t) \cdot \vec{A}(t)$
- use product rule
$u'(t) \cdot \vec{A}(t) + u(t) \cdot \vec{A}'(t)$

$\vec{v} = \vec{T} \cdot \frac{ds}{dt}$

$$\frac{d\vec{v}}{dt} = \vec{a} = \frac{d^2s}{dt^2}\hat{T} + \frac{ds}{dt}\frac{d\vec{T}}{ds} \cdot \frac{ds}{dt}$$
$$= \frac{d^2s}{dt^2}\hat{T} + v^2 \kappa \vec{N}$$

$$\vec{a} = a_T \hat{T} + v^2$$

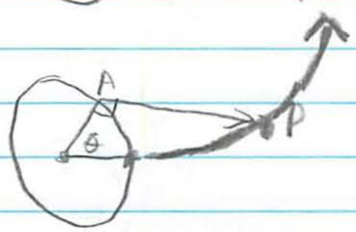
a_T tangental acceleration

$\rho = \frac{1}{\kappa}$ radius of curvature

$$\boxed{\vec{a} = \frac{d^2 s}{dt^2}\hat{T} + v^2 \frac{1}{\rho}\vec{N}}$$

<u>Example</u> Involute of circle

Start unwinding string
Pull keeping it taught so
tangent to circle



P was at $(a,0)$ now here
see arrow

$$\vec{r} = \vec{OA} + \vec{AP}$$
$$= a\langle \cos\theta, \sin\theta \rangle + a\theta\langle \sin\theta, -\cos\theta \rangle$$

↑ perpendicular, 90° clockwise

$$= a\langle \cos\theta + \theta\sin\theta, \sin\theta - \theta\cos\theta \rangle$$

$$\vec{V} = a\langle \theta\cos\theta, \theta\sin\theta \rangle$$
$$= a\theta \langle \cos\theta, \sin\theta \rangle$$

↑ $\vec{v}$ ↑ $\vec{T}$ decomposed itself

$$\frac{d\vec{T}}{ds} = \frac{d\vec{T}/d\theta}{ds/d\theta} = \frac{\langle -\sin\theta, \cos\theta \rangle}{a\theta}$$

$r = a\theta = v$

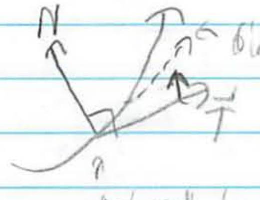
$\kappa = \frac{1}{a} \quad \rho = a\theta$

Lecture 8
Curvature in 3D

2/19

Plane Curvature

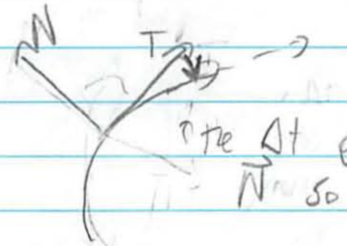
- 3 ways to calculate



arc lenght increasing

$k \oplus$

not really drawn tangently



the Δt goes opposite $\vec{N}$ so $\longrightarrow k \ominus$

$\frac{d\vec{T}}{ds} = N k$

$\perp$ to $\vec{T}$

Curvature in Space

in space don't know what is right or left
will be a whole circle of unit vectors

must use definition $\frac{d\vec{T}}{ds} = \vec{N} k$ scalar factor

normal vector - which one is the one

$\vec{T} \cdot \vec{T} = 1$ - constant lenght

$\frac{d\vec{T}}{ds} \cdot \vec{T} = 0$

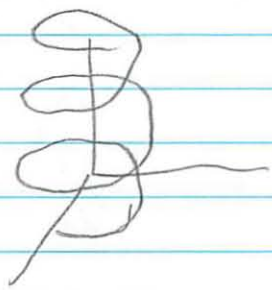
$\frac{d\vec{T}}{ds} \perp \vec{T}$ get a unique vector
make it unit vector $\vec{N}$
its lenght is k

$$d'r\left(\frac{d\vec{T}}{ds}\right) \underset{\uparrow \text{def}}{:=} \hat{N} \text{ unit normal vector}$$

$$\left|\frac{d\vec{T}}{ds}\right| = k$$

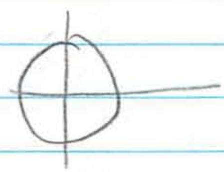
↑ curvature has to be ⊕
- no negitive in space since no left + right
- don't worry about it

ex: Helix / Corkscrew



$\vec{r}(t) = \langle a\cos t, a\sin t, bt \rangle$

↑ think about this and it seems very simple



top

curvature must be constant

$\frac{d\vec{T}}{ds}$ ← don't find s explicitly

$$= \frac{d\vec{T}/dt}{ds/dt \leftarrow |\vec{v}|}$$

↙ want ijk coord

$\vec{v} = \langle -a\sin t, a\cos t, b \rangle$

$v = |\vec{v}| = \sqrt{a^2+b^2} = \frac{ds}{dt}$

$\vec{T} = \frac{\vec{v}}{\sqrt{a^2+b^2}}$

$\frac{dT}{ds} = \dfrac{\left\langle \frac{-a\cos t, -a\sin t, 0}{\sqrt{a^2+b^2}} \right\rangle}{\sqrt{a^2+b^2}}$ ← derivitive of V

← ds/dt

$= \hat{N} \kappa$

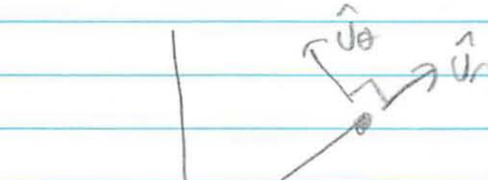
$= \underbrace{\langle -\cos t, -\sin t, 0 \rangle}_{\hat{N}} \underbrace{\left(\frac{a}{a^2+b^2}\right)}_{\kappa}$

↳ constant - like we said

↑ the negitive sign must stay here

<u>More on Polar Coordinates</u> (r, θ)

$\hat{u}_r, \hat{u}_\theta$



moving frame to look at velocity

$\vec{r} = r(t)\, \hat{u}_r$

$\vec{v} = \dot{r}\, \hat{u}_r + r\dot{\theta}\, \hat{u}_\theta$ } differentiate by Leibnitz rule

hard via chain rule

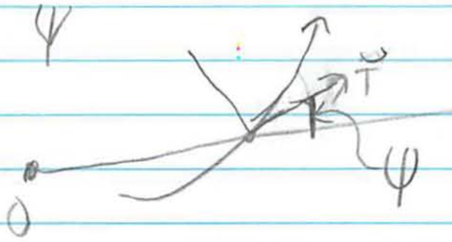
more ↓

what type of product does not matter

$\frac{d\vec{s}}{dt}$

$s = \int \frac{d\vec{s}}{dt}$

ψ



be able to calc in polar coord

-depend only on path of motion

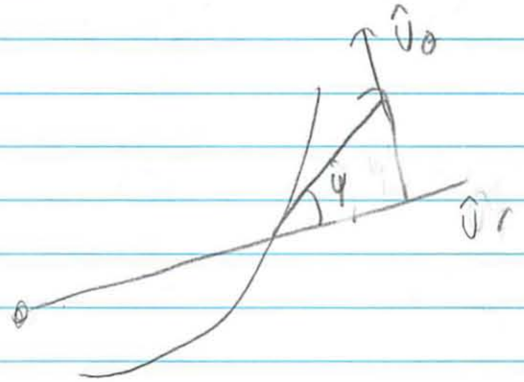
$$\boxed{\text{Remember} \quad \vec{v} = \dot{r}\,\hat{u}_r + r\dot{\theta}\,\hat{u}_\theta}$$

$$\frac{ds}{dt} = |\vec{v}| = \sqrt{\dot{r}^2 + (r\dot{\theta})^2}$$

any 2 perpendicular vectors can form a cord system.

$$s = \int_{t_1}^{t_2} \sqrt{\dot{r}^2 + (r\dot{\theta})^2}$$

arc lenght $t_1 \to t_2$

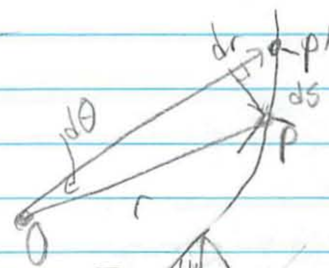


$$\text{tangent } \varphi = \frac{\text{component of } \vec{T} \text{ dir } \hat{u}_\theta}{\text{〃 〃 〃 〃 } \hat{u}_r}$$

$$= \frac{r \; d\theta/\cancel{dt}}{dr/\cancel{dt}}$$ cancelation! L'Hopitz/chain rule

$$\tan\psi = \frac{r}{dr/d\theta}$$

ds →



$$ds = \sqrt{(dr)^2 + (r\,d\theta)^2}$$

tan ψ

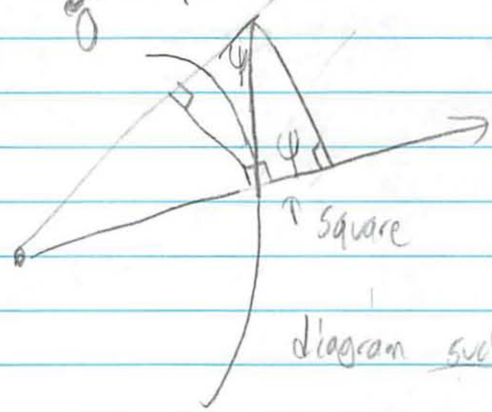
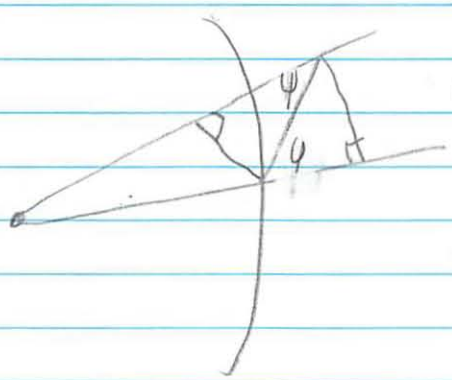


diagram sucks



extreemly small
-almost nothing
then its a square

to where line does not look like curve

Recitation
Before Test

2/22

Read the Edward + Penny Textbook
- kimberly says its helpful
- more helpful than this semester

Office Hrs 4:30 - 6:30

Last lecture

$$\vec{a}(r) = \frac{d\vec{v}(t)}{dt}$$

$\vec{T}\ \vec{N}$ frame

Curvature $\frac{d\vec{T}}{ds} = \kappa \vec{N}$

$\uparrow \frac{d\Delta}{ds}$

$\rho = \frac{1}{\kappa}$ radius of curvature

Curvature Problem

2D $\vec{r}(t) = \langle \frac{t^2}{2}, t \rangle$

a) Find the unique tangent vector $\vec{T}$
b) Find the curvature

$$\vec{T} = \frac{v}{|\vec{v}|} = \frac{\langle \frac{1}{2} \cdot 2t, 1 \rangle}{\sqrt{t^2+1}}$$ → write as $\frac{1}{\sqrt{t^2+1}} \langle t, 1 \rangle$

$k \rightarrow \frac{d\vec{T}}{ds} = \vec{N} k = \frac{d\vec{T}/dt}{ds/dt}$ ← have to calculate ← $|\vec{v}|$

$$\cancel{\frac{\frac{\langle t, 1 \rangle}{\sqrt{t^2+1}}}{\sqrt{t^2+1}}}$$

; how simplify ?

$$\cancel{\langle t, 1 \rangle \quad \frac{1}{t^2+1}}$$

$$\vec{T} = \frac{\langle t, 1 \rangle}{\sqrt{t^2+1}}$$

$$\frac{d\vec{T}}{dt} = \cancel{-\frac{1}{2}(t^2+1)^{-3/2} \langle 1, 0 \rangle}$$

$$\cancel{\frac{-\langle 1, 0 \rangle}{2\sqrt{t^2+1}^{\,3}}}$$

← did not do inside

↳ must do chain rule

$$= \frac{d}{dt}\left(\frac{1}{\sqrt{t^2+1}}\right)\langle t, 1 \rangle + \frac{1}{\sqrt{t^2+1}} \frac{d}{dt}\langle t, 1 \rangle$$

get common factor

$$= \overbrace{\left(-\frac{1}{2}\right)(2t)}^{-t}(t^2+1)^{-3/2} \langle t, 1 \rangle + \frac{1}{\sqrt{t^2+1}} \langle 1, 0 \rangle$$

$$= \overbrace{(t^2+1)^{-3/2}}^{\text{magnitude } k} \left(\langle -t^2, -t \rangle + \langle 1+t^2, 0 \rangle\right)$$

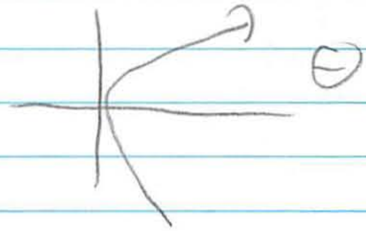
$$k = \pm\left|\frac{d\vec{T}}{ds}\right| = \pm\left|\frac{d\vec{T}/dt}{ds/dt}\right| = \frac{(t^2+1)^{-3/2}\cancel{\sqrt{1+t^2}}}{\cancel{\sqrt{1+t^2}}}$$

$$k = \pm (t^2+1)^{-3/2}$$

↳ is it ⊕ or ⊖

left ⊕ right ⊖

this one



$\kappa = -(t^2+1)^{-3/2}$

How to draw picture
- take pts $t=0$, $t=2$, $t=3$
- and find them + plot them

---

practice exam 7 $\vec{v} = 2\vec{u}_r + 2\vec{u}_\theta$ $r(0)=1$ $\theta(0)=0$

a) length of path between $t=0$ and $t=3$?

$s = \int \left|\frac{ds}{dt}\right| dt$

↑ lenght $|\vec{v}|$

$|\vec{v}| = \sqrt{2^2+2^2}$
$2\sqrt{2}$

$s = \int_0^3 2\sqrt{2}\, dt$

$s = 2\sqrt{2}\, t = 2\sqrt{2}(3) - 0 = \boxed{6\sqrt{2}}$

b) How far from origin at $t=3$?

$\vec{r}(3)$ we want

← integrate velocity to find position

$\int |v| = r$ — in polar coords

$$\boxed{\vec{v} = \dot{r}\,\vec{u}_r + r\dot{\theta}\,\vec{u}_\theta}$$ ← remember your formula

differentiation
• w/ respect to t

↳ $\dot{r} = 2$

$r\dot{\theta} = 2$

$\dot{r} = 2$

↓

$\int$

↓

$r(t) = 2t + c$
↑constant

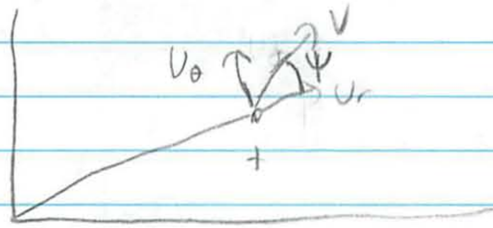
we know $r(0) = 1 = c$

$r(t) = 2t + 1$

now plug in

$r(3) = 7$

Now Draw it



Find ψ b/w $\vec{r}$ and $\vec{v}$

$$\boxed{\tan\psi = \frac{u_\theta}{u_r}}$$

$\tan\psi = \frac{2}{2}$ ← u_θ ← u_r

$\psi = \frac{\pi}{4}$

Where is the point at time t?

We know $r(t) = 2t + 1$

We need $\theta(t)$

$$\dot{\theta} = \frac{2}{r} = \frac{2}{2t+1}$$

$\downarrow \int$

$\downarrow$

review integration $\rightarrow$ $\theta(t) = \ln(2t+1) + c$

$\downarrow$

$\theta(0) = 0 = c$

$$x(t) = r\cos\theta$$
$$= (2t+1)\cos(\ln(2t+1))$$
$$y(t) = r\sin\theta$$
$$= (2t+1)\sin(\ln(2t+1))$$

Eaiser in polar than rectangular

Ex 3 Parametric equation of tangent

$$r(t) = \langle e^t, \cos t, t \rangle \qquad t = 0$$

derivitive of curve at that point

Direction $\vec{v}(0)$

Point $\vec{r}(0)$

Parameter $P(t) = \vec{r}(0) + t\,\vec{v}(0)$

Michael Plasmeier

46/50

3A = 10/10

18.02 Problem Set 3A

due Th. 3/4/10 10:45AM 2-106

This has just Part I exercises; the material is included on Exam 1 Tuesday.

Lecture 7. Thurs. Feb. 18 Curvature in 2D; the $\mathbf{T}-\mathbf{N}$ system; acceleration.
Read: Acceleration $\mathbf{a}$ (p. 608); 17.5, 17.6 to top of p. 618; Notes below, sections 1-3.
Work: 1J-3, 5, 7, 8; Exercises 1,2,3 below.

Lecture 8. Fri. Feb. 19 Curvature in 3D; More on the $\mathbf{u}_r - \mathbf{u}_\theta$ system.
Read: Notes below, sections 4,5; 18.02 Notes: Section K; 17.7 (end 622-mid 623)
Work: 1J-6,10; Exercises 4-7 below.

Exam 1. Tues. Feb. 23 during lecture hour; Walker 3rd floor.

Notes on Curvature

1. Calculating curvature. There are three ways to calculate curvature in the plane: sometimes one method is easier than another.

a) If the curve is given as the graph of a function $y = y(x)$, its curvature κ at (x, y) is

$$\kappa = \frac{y''}{(1+y'^2)^{3/2}}$$

This is (4) on p. 612; the derivation is in the book, and was given in lecture.

b) If the curve is given parametrically by the position vector $\mathbf{r}(t) = \langle x(t), y(t)\rangle$, then

$$\frac{d\mathbf{T}}{ds} = \mathbf{N}\,\kappa, \qquad \frac{d\mathbf{T}}{ds} = \frac{d\mathbf{T}/dt}{ds/dt},$$

where $\mathbf{T} = \mathrm{dir}(\mathbf{v})$ is the unit tangent vector, $ds/dt = v = |\mathbf{v}|$ is the speed, and $\mathbf{N}$ is the unit normal vector: $\mathbf{T}$ rotated counterclockwise by $\pi/2$.

c) If $\mathbf{a}$ is the acceleration vector, then (see (7), p. 617), the formula [(5) p. 616] for acceleration in the $\mathbf{T}-\mathbf{N}$ system shows that (with the above notation),

$$v^2\kappa = \sqrt{|\mathbf{a}|^2 - (dv/dt)^2},$$

where $|\mathbf{a}|^2$ and v can be calculated using the $\mathbf{i}-\mathbf{j}$-system.

(This was not covered in lecture; an exercise below illustrates its use.)

2. The sign of curvature. Look at the formula in (b) above, and move along the curve C in the direction of increasing arclength s.

If C bends to the left, then $d\mathbf{T}/ds$ and $\mathbf{N}$ point in the same direction, hence $\kappa > 0$;
if C bends to the right, then $d\mathbf{T}/ds$ and $\mathbf{N}$ point in opposite directions , hence $\kappa < 0$.

3. The radius of curvature ρ. The classical definition of curvature of C at a point P uses the circle that goes through P and "best fits" the curve C.

It's called the *osculating circle* or the *circle of curvature* at P (more Victorian).

Its radius ρ is called the *radius of curvature* and its curvature $\kappa = 1/\rho$ is defined to be the *curvature of C at P*. In the style of the now-gone Miller Analogies portion of the SAT, you can think of the osculating circle as the analog of the tangent line to C at P:

At the point P on C,

slope of C : slope of the tangent line : : curvature of C : curvature of osculating circle

4. Curvature for space curves. For space curves, it is not possible to prescribe a definite unit normal vector $\mathbf{N}$ at a given point P geometrically, since the possible choices for the heads of $\mathbf{N}$ fill out a circle of radius 1 in a plane perpendicular to the curve. Also, "curving to the left" or "to the right" makes no sense in 3D, so curvature cannot be negative.

However, in 3D it is still true that since $\mathbf{T}$ is a unit vector, $d\mathbf{T}/ds$ is perpendicular to $\mathbf{T}$. This gives us a way of picking out a unique vector $\mathbf{N}$:

Definition. Let $\mathbf{r}(t)$ be the position vector of a curve C in xyz-space, with arclength $s(t)$ and unit tangent vector $\mathbf{T}(t)$. We define the *unit normal vector* $\mathbf{N}$ and *curvature* κ by

$$\mathbf{N} = \mathrm{dir}\left(\frac{d\mathbf{T}}{ds}\right), \qquad \kappa = \left|\frac{d\mathbf{T}}{ds}\right|; \qquad \text{thus} \quad \frac{d\mathbf{T}}{ds} = \mathbf{N}\,\kappa.$$

(If the space curve lies in a plane in space, this definition of $\mathbf{N}$ can conflict with the one for a plane curve, and one has to specify which definition is being used.)

5. Polar formulas (cf. exercise 4 below; (3) need not be memorized or derived).

(1) $\mathbf{r} = r(t)\mathbf{u}_r$; $\quad d\mathbf{u}_r/d\theta = \mathbf{u}_\theta$, $\quad d\mathbf{u}_\theta/d\theta = -\mathbf{u}_r$

(2) $\mathbf{v} = r'\mathbf{u}_r + r\theta'\mathbf{u}_\theta$ $\quad$ (3) $\mathbf{a} = (r'' - r(\theta')^2)\,\mathbf{u}_r + (r\theta'' + 2r'\theta')\,\mathbf{u}_\theta$

(4) $\dfrac{ds}{dt} = \sqrt{(r')^2 + (r\theta')^2}$ $\quad$ (5) $\tan\psi = \dfrac{r}{(dr/d\theta)}$ $\quad$ (ψ is the angle between $\mathbf{T}$ and $\mathbf{u}_r$.)

Additional Part I Exercises for Lectures 7 and 8

Solutions will be posted on 18.02 website

1. (i) Sketch the curve of motion whose position vector is $\mathbf{r}(t) = t\,\mathbf{i}\; - \ln\cos t$, and find in order $\mathbf{v}$, v, $\mathbf{T}$, and κ by using method (b).

(Suggestion: Put common factors of the $\mathbf{i}$ and $\mathbf{j}$ components outside the angle brackets, and use 1J-8 where you can.)

(ii) Find just κ by using method (a), and sketch the curve, for $0 \le x < \pi/2$.

2. If for a motion $\mathbf{r}(t)$ we have $\mathbf{v} = at(\cos t\,\mathbf{i} + \sin t\,\mathbf{j})$, find $\mathbf{T}$ and $\mathbf{N}$, express the acceleration $\mathbf{a}$ in the $\mathbf{T} - \mathbf{N}$ system, and use this to find the radius of curvature.

3. For the parabolic motion whose position vector is $\mathbf{r} = \langle t, t^2/2\rangle$,

find $\mathbf{v}$, v, $\mathbf{a}$, the tangential acceleration $a_T = dv/dt$ and the normal acceleration $a_N = v^2\kappa$ (use method (c)), and from this the curvature κ.

Check your value for κ by using method (a).

4. a) Derive (2) from (1).

b) Derive (4) and (5) from (2); also derive them directly by a geometrical argument (you can use infinitesimal lengths).

5. For the curve $r = 2\cos\theta$, $0 \le \theta \le \pi$, find the arclength and angle ψ by using the above formulas, and check your answers by elementary geometry.

6. In polar coordinates, a lighthouse is at the origin, with a narrowly focused beam rotating counterclockwise at 1 radian/minute. At time $t = 0$, it shines on a smuggler boat 1 km away, at the point (1,0). The boat immediately takes off, following a path which always makes a 60^o angle with the beam. Find the equation $r = r(\theta)$ of the boat's path.

(Set up a simple differential equation that $r(\theta)$ satisfies, and solve it.)

P-Set 3A

2/21

Lecture 7 Curvature in 2D, T-N system

1J-3 Prove the rule for differentiating the scalar product of 2 plane vector functions

$$\frac{d}{dr} r \cdot s = \frac{dr}{dt} \cdot s + r \cdot \frac{ds}{dt}$$

by $r = x_1 \hat{i} + y_1 \hat{j}$
$s = x_2 \hat{i} + y_2 \hat{j}$

So isn't this just the chain rule?
but should calculate components

$$(x_1' \hat{i} + y_1' \hat{j}) \cdot (x_2 \hat{i} + y_2 \hat{j}) + (x_1 \hat{i} + y_1 \hat{j}) \cdot (x_2' \hat{i} + y_2' \hat{j})$$

Dot product $|A||B| \cos\theta$ or $(A_1 \cdot B_1) + (A_2 \cdot B_2)$

$$(x_1' x_2 \hat{i} + y_1' y_2 \hat{j}) + (x_1 x_2' \hat{i} + y_1 y_2' \hat{j})$$

$$x_1' y_1 + x_1 y_1' + x_2' y_2 + x_2 y_2'$$

$$(x_1, x_2)' \cdot (x_1, y_2) + (x_1, x_2) \cdot (\cancel{x_1}, y_2)'$$ / what I had

? why comma?

and how does this prove anything!

1J-5 Suppose a point moves w/ constant speed.
Show velocity + acc are perpendicular

$\vec{r} = \langle x(t), y(t) \rangle$
$\vec{v} = \langle x'(t), y'(t) \rangle$ ← easy to calculate
$\vec{a} = \langle x''(t), y''(t) \rangle$ but need a coord system

$\vec{v} = \vec{T} \cdot ds$

↳ differenitiate $v(t) \cdot \vec{A}(t)$
by using product rule

$$d\vec{V} = \vec{a} = \frac{d^2 s}{dt^2} \vec{T} + \frac{ds}{dt} \frac{dT}{ds} \circ \frac{ds}{dt}$$
$$= \frac{d^2 s}{dt^2} \vec{T} + V^2 \cancel{k} \vec{N}$$

$$\vec{a} = a_T \vec{T} + v^2$$

$$\vec{a} = \frac{d^2 s}{dt^2} \vec{T} + v^2 \frac{1}{\rho} \vec{N}$$

$|v| = c$
$v \cdot v = c^2$

? something to do w/ dot product ?

$\frac{dv}{dt} \circ v = 2v \cdot a = 0$ (1-J3)

So perpendicular

Some thing about Speed = 0

Si

b. Show the converse. If velocity and acceleration are perpendicular moves w/ constant speed

dot product review

$A \cdot B = |A||B|\cos\theta$

$|A|\cos\theta$

$|a| = \sqrt{a \cdot a}$

$v \cdot v = v^2$

$v \cdot a = 0$ ← yes if dot product = 0, perpendicular

$v \cdot \frac{dv}{dt} = 0$

$v \cdot v = a$ ← aka $v \cdot v = \frac{dv}{dt}$ $\quad \frac{dv}{dt} = v^2 = a$ I dk why - but true

$|v| = \sqrt{a}$ shows speed constant

1J-7 Suppose you have a differentiable vector function $r(t)$

How can you tell if parameter t is arc lenght s w/o calculating s?

$$\frac{dr}{dt} = \vec{T} \cdot \frac{ds}{dt}$$

↑ unit tangent vector

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{dr}{ds} \cdot \frac{ds}{dt}$$

$$= \vec{T} \cdot |\vec{v}|$$

↑ speed

$$\vec{T} = \frac{\vec{v}}{|\vec{v}|}$$

but this is not paramitrized

$$\frac{d\vec{T}}{ds} = N\kappa = \frac{dT/ds}{ds/dt} = \frac{dT/ds}{|\vec{v}|}$$

Criteria is $|v| = 1$

since $s(0) \to s(t)$

must increase at same rate as t

$$|v| = 1 = ds/dt$$

?

$$s = t + c$$

$$s = t$$

? what is c again;

constant → must $= 0$

b. How should a be chosen so that t is arclength

if $r(t) = (x_0 + at)\hat{\imath} + (y_0 + at)\hat{\jmath}$

What is a??

So we want $|v| = 1 = \frac{\Delta s}{\Delta t}$

$$\Delta r = \vec{T}\,\Delta s$$

$$\vec{v} = a(\hat{\imath} + \hat{\jmath})$$

$$|\vec{v}| = \sqrt{a^2 + a^2} = \sqrt{2}\,a$$

i don't get what asking

So chose $a = \frac{1}{\sqrt{2}}$

c How should a and b be chosen so that t is the arc length for helical motion $a\cos t\,\hat{i} + a\sin t\,\hat{j} + bt\,\hat{k}$

- curvature must be constant

$$\frac{d\vec{T}}{ds} = \frac{d\vec{T}/dt}{ds/dt} = \frac{d\vec{T}/dt}{|\vec{v}|}$$

$$\vec{v} = \langle -a\sin t, a\cos t, b \rangle$$

$$|\vec{v}| = \sqrt{a^2+b^2} = \frac{ds}{dt}$$

$$\vec{T} = \frac{\vec{v}}{\sqrt{a^2+b^2}} = \left\langle \frac{-a\cos t, -a\sin t}{\sqrt{a^2+b^2}}, 0 \right\rangle$$

$$= N k$$

$$= \langle -\cos t, -\sin t, 0 \rangle \left(\frac{a}{a^2+b^2} \right)$$

Choose a and b to be not negitive

$a^2+b^2=1$ $|v|=1$

want this

8. Proove $\frac{d}{dt} u(t) r(t) = \frac{du}{dt} r(t) + u(t) \frac{dr}{dt}$

Vectors in a plane

- This is same as 1st question
  except perhaps vectors

$r = x(t)\,\hat{i} + y(t)\,\hat{j}$

$u(t)\,\vec{r}(t) = ux\,\hat{i} + uy\,\hat{j}$

scalar times vector

$(u\vec{r})' = (ux)'\,\hat{i} + (uy)'\,\hat{j}$

$(u'x + ux')\,\hat{i} + (u'y + uy')\,\hat{j}$

$u'(x\hat{i} + y\hat{j}) + u(x'\hat{i} + y'\hat{j})$

$u'\vec{r} + u\vec{r}\,'$

b. Let $\vec{r}(t) = e^t \cos t\,\hat{i} + e^t \sin t\,\hat{j}$

- exponential spiral
- find speed

- So just take derivitive
- but have to use chain rule

$te^t \cos t\,\hat{i} + te^t \sin t\,\hat{j} +$

$e^t - \sin t\,\hat{i} + e^t \cos t\,\hat{j}$

also not vector $\sqrt{\hat{i} + \hat{j}}$

$|v| = e^t \,|\, (\cos t - \sin t)\,\hat{i} + (\sin t + \cos t)\,\hat{j} = 2e^t$

extra 1 Sketch curve of motion $r(t) = t\,\hat{i} - \ln\cos t\,\hat{j}$

Find $\vec{v}, |\vec{v}|, T, \kappa$

$t = 0 \quad 0\hat{i} - \ln(1)\hat{j} = 0 - 0$

$t = \frac{\pi}{4} \quad \frac{\pi}{4}\hat{i} - \frac{\ln(2)}{2}\hat{j} \rightarrow$

$\frac{\pi}{2}$ und

$\frac{3\pi}{4} \quad \frac{3\pi}{4}$ - non real $\hat{j}$

Hint put $\hat{i}\ \hat{j}$ just outside angle brackets

$\vec{v} = 1\hat{i} - \frac{1}{\cos t}\sin t\,\hat{j}$

$1\hat{i} - \tan x\,\hat{j}$ ✓

$v = \sqrt{1^2 + (\tan x)^2}$

$= \frac{1}{\cos x} = \sec t$

think they meant $-\ln\cos t\,\hat{j}$

$T = \frac{\vec{v}}{v} = \left\langle \frac{1\hat{i} - \tan x\,\hat{j}}{\cancel{1/\cos x}\ \sec t} \right\rangle = \langle -\sin t, \cos t \rangle$

Simplify

$\kappa = \frac{dT}{ds} = \frac{\left\langle \frac{\cancel{1\hat{i} - \tan x\,\hat{j}}\ -\sin t,\ \cos t}{1/\cos x} \right\rangle}{\cancel{1/\cos x}\ \sec t}$

$= \cancel{1 - \sin x \cos x}$

$\cancel{\kappa = t?}$

← or did I not keep in vectors here

did simplifying wrong

$N = \langle -\sin t, \cos t \rangle$

$\kappa = \frac{1}{\sec t} = \cos t$

) simplify, its much easier

< Oh from front

extra 1b Find k by using method a

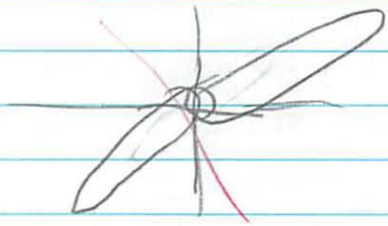
Method a $k = \frac{y''}{(1+y'^2)^{3/2}}$

$r = t\,\hat{i} - \ln\cos t\,\hat{j}$ $y = -\ln\cos x$
$r' = 1\,\hat{i} + \tan t\,\hat{j}$ $y' = \tan x$ } Use just y
$r'' = 0\,\hat{i} + \sec t^2\,\hat{j}$ $y'' = \sec^2 x$

$k = \frac{-\sec t^2\,\hat{j}}{(1+\cancel{(1-\tan t)^2})^{3/2}}$ $= \frac{\sec^2 x}{\sec^3 x} = \boxed{\cos x}$

simplify better

TI Graph



extra 2. If for motion $r(t)$ have $\vec{v} = a\,t(\cos t\,\hat{i} + \sin t\,\hat{j})$
Find T and N, express $\vec{a}$ in TN system

$\vec{T}$ is tangent
$\vec{N}$ is normal (90° ↺)

$\vec{V} = \vec{T} \cdot \frac{ds}{dt} = a\,t\cos t\,\hat{i} + a\,t\sin t\,\hat{j}$

Find the tangent to a curve
Take the deriv at that point → what the V is
So $\vec{T} = \frac{\vec{v}}{\frac{ds}{dt}}$ < split like

$|\vec{v}| = at$ $\vec{T} = \langle \cos t\ \sin t \rangle$

$$\vec{T} = \frac{at\cos t\ \hat{i} + at\sin t}{\frac{ds}{dt}}$$

$\vec{T} = \langle \cos t, \sin t \rangle$

c is often cross product

$\vec{N}$ is 90°

- flip coords + signs

$$\frac{-at\sin t + at\cos t}{\frac{ds}{dt}}$$

$N = \langle -\sin t\ \cos t \rangle$

$$\vec{a} = \frac{d^2s}{dt^2}\vec{T} + v^2 \frac{1}{\rho}\vec{N}$$

$\vec{a} = \frac{dv}{dt} = a\langle \cos t, \sin t \rangle + at\langle -\sin t, \cos t \rangle$

↑ why not here?

- using formula for differing products

$a\vec{T} + at\vec{N}$

$\frac{dv}{dt}\vec{T} + v^3 \kappa N$

$v^2 \kappa = at$

$v = at$

$\kappa = \frac{1}{at}$ $\boxed{\rho = at}$

↑ radius of curvature

3. For the parabolic motion whose position $r = \langle t, \frac{t^2}{2} \rangle$
find $\vec{v}, |\vec{v}|, \vec{a}, a_T\left(\frac{dv}{dt}\right)$ and $a_N(v^2 k)$, k

$$\vec{v} = \langle 1, \tfrac{1}{2}\cdot 2t \rangle = \langle 1, t \rangle$$

$$|\vec{v}| = \sqrt{1^2 + t^2} = \sqrt{t^2+1}$$

$$\vec{a} = \langle 0, 1 \rangle$$

$$a_T = \frac{dv}{dt} = \frac{1}{2}(t^2+1)^{-1/2}\cdot 2t$$

$$= \frac{t}{\sqrt{t^2+1}}$$

$$a_N = v^2 k$$

$$\cancel{= \sqrt{t^2+1}^{\,2}\, k}$$

$$\cancel{= (t^2+1)\, k}$$

method ( ) is right

$$= \sqrt{|a|^2 - \left(\frac{dv}{dt}\right)^2}$$

$$= \sqrt{\sqrt{1^2} - \left(\frac{t}{\sqrt{t^2+1}}\right)^2}$$

do top / bottom sepertly

$$\sqrt{1 - \frac{t^2}{t^2+1}}$$

$$|\vec{a}|^2 = 1 = a_T^2 + a_N^2$$

do they even use method (?

or

$$a_N^2 = 1 - a_T^2 = \frac{1}{1+t^2}$$

$$a_N = (1+t^2)\,k = \frac{1}{\sqrt{1+t^2}}$$

had that

$$k = \frac{1}{(1+t^2)^{3/2}}$$

Using method a

$y = x^2/2$

$y' = x$

$y'' = 1$

$$k = \frac{1}{(1+x^2)^{3/2}}$$

Lecture 8 Curvature in 3D, U_r U_θ system

1J-6 Similar to 1J-7c

Helical $r(t) = \langle a\cos t, a\sin t, bt \rangle$

$\vec{v} = \langle -a\sin t, a\cos t, b \rangle$ ✓

$\vec{a} = \langle -a\cos t, -a\sin t, 0 \rangle$ ✓

$$\vec{T} = \frac{\vec{v}}{|\vec{v}|} = \frac{\langle -a\sin t, a\cos t, b \rangle}{\sqrt{(-a)^2\sin^2 t + a^2\cos^2 t + b^2}}$$ ✓

$\sqrt{a^2(\sin^2 t + \cos^2 t) + b^2}$

$\sqrt{a^2+b^2}$

know all of these by heart

$$\left|\frac{ds}{dt}\right| = |\vec{v}| = \sqrt{a^2+b^2}$$ ✓

b. Show that $\vec{v}$ and $\vec{a}$ are perpendicular

$v \cdot v = v^2$

$|v| = \sqrt{a}$

$a = \frac{dv}{dt} \cdot v^2$

$$|v| = c$$

$$v \cdot v = c^2$$

$$\frac{dv}{dt} \cdot v = 2v \cdot a = 0$$

Or can dot product to show $\perp$

$$\langle -a\sin t, a\cos t, b \rangle \circ \langle -a\cos t, -a\sin t, 0 \rangle$$

$$(-a\sin t \cdot -a\cos t) + (a\cos t \cdot -a\sin t) + (b \cdot 0)$$

$$\cancel{(a^2 \cos t \sin t) + (-a^2 \cos t \sin t)} + 0$$

$$\cancel{a^2(\cos t \sin t - \cos t \sin t)}$$

$$\cancel{a^2}$$

screwed up here

keep x in front it all subtracts away

$$\cancel{\neq 0}$$

should $= 0$

Or speed $= 0$ (1J-5)

say that

1J-10 The positive curvature κ of $r(t) = \left| \frac{d\vec{T}}{ds} \right|$

a) Show that helix (1J-6) has constant curvature.

- Hint: calc $\frac{dT}{dt}$ relate to κ w/ chain rule

$$\frac{dT}{ds} = \frac{\left\langle \frac{-a\cos t, -a\sin t, 0}{\sqrt{a^2+b^2}} \right\rangle}{\sqrt{a^2+b^2}} \qquad \frac{\frac{\vec{v}}{|\vec{v}|}}{|\vec{v}|} \leftarrow \frac{ds}{dt}$$

$= N \quad k$

$\langle -\cos t, -\sin t, 0 \rangle \quad \frac{|a|}{a^2+b^2}$

$\vec{N} \qquad k$

they did diff way

[so basically everything sin, cos, t goes to N "vector"
everything constant scalar goes to k

b. What is the curvature of a helix if reduced to circle in x, y plane

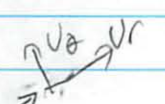
- Is this what is curvature of a circle?
- Or just drop z component

$b = 0 \qquad k = \frac{1}{|a|}$

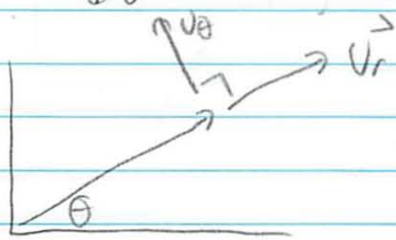
$\frac{|a|}{a^2+0^2} \to \frac{|a|}{a^2} = \frac{1}{|a|}$ I'm guessing

Extra 4 Derive 2 from 1

- what is 2 and what is 1? polar coords



2. $\frac{d\vec{u_r}}{d\theta} = \vec{u_\theta}$ 1. $\vec{r} = r(t)\,\vec{u_r}$



So basically $\frac{d\vec{u_r}}{d\theta}$ is u_θ

- but isn't it tangent?

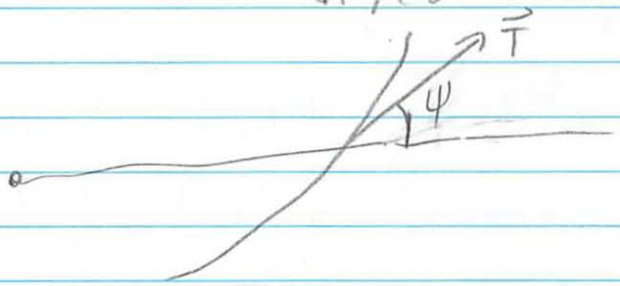
$$\vec{r} = r\,\hat{u}_r$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \dot{r}\,\hat{u}_r + r\dot{\theta}\,\hat{u}_\theta$$

b. Derive 4 from 5 from 2
and from geometric w/ infintismal lenghts

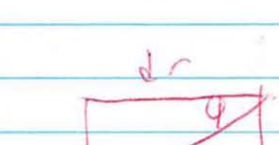
4. $\frac{ds}{dt} = \sqrt{(r')^2 + (r\theta')^2}$.

5. $\tan\psi = \frac{r}{dr/d\theta}$



$$\frac{ds}{dt} = v = |\vec{v}| = \sqrt{(\dot{r})^2 + (r\dot{\theta})^2}$$

$$\tan\varphi = \frac{\hat{u}_\theta - \text{comp of } \vec{v} \text{ or } \vec{T}}{\hat{u}_r - \text{comp of } \vec{v} \text{ or } \vec{T}} = \frac{r\,d\theta/dt}{dr/d\theta} = \frac{r}{dr/d\theta}$$



$$\tan\varphi = \frac{r\,d\theta}{dr} = \frac{r}{dr/d\theta}$$

extra 5 $\qquad \vec{r} = 2\cos\theta$ (↓ parameter) $\qquad 0 \le \theta \le \frac{\pi}{2}$ (c typo)

Find arc lenght and ψ

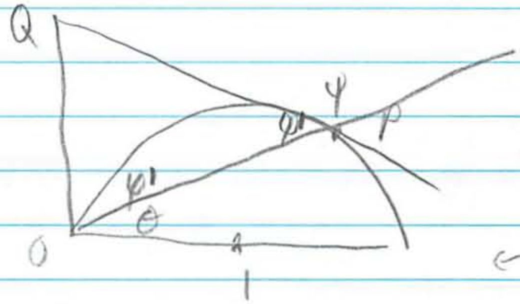
$$\frac{ds}{d\theta} = \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}$$

$$= \sqrt{(-2\sin\theta)^2 + (2\cos\theta)^2} = 2$$

$$S = \int_0^{2\pi} \frac{ds}{d\theta}\, d\theta = \int_0^{2\pi} 2\, d\theta = \pi$$

??? Have no clue at all Hard to visualize

$$\tan\psi = \frac{r}{dr/d\theta} = \frac{2\cos\theta}{-2\sin\theta} = -\cot\theta$$



← arc lenght $= \pi$

$\Delta\, OQP$ is isoscles

$$\psi + \psi' = \pi$$
$$\theta + \psi' = \frac{\pi}{2}$$

subtracting $\psi = \theta + \frac{\pi}{2}$

$$\tan\psi = \frac{\sin(\theta + \pi/2)}{\cos(\theta + \pi/2)} = \frac{\cos\theta}{-\sin\theta} = \boxed{-\cot\theta}$$

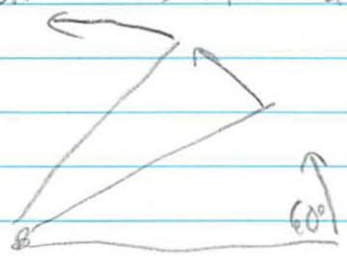
translate graps of sin + cos → by $\pi/2$

6. In polar coords lighthouse at origin

↶ 1 rad/min

$t = 0$ at smugglers boat at $(1,0)$

Boat takes off always 60° to beam



Find $r = r(\theta)$ of boat
- simple differential eq
solve

$$\tan\psi = \sqrt{3} = \frac{r}{dr/d\theta}$$

$$\frac{dr}{d\theta} = \frac{r}{\sqrt{3}}$$ seperate variables for differential eq

$$\frac{dr}{r} = \frac{d\theta}{\sqrt{3}}$$

?: what is ψ?

$$\ln r = \frac{\theta}{\sqrt{3}} + c_1$$

$$\boxed{r = e^{\theta/\sqrt{3}}}$$ exponential spiral path

18.02 SOLUTIONS TO P.set 3A – S-2010

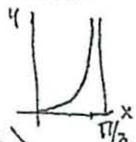
1 As on #5, Problem Set 2 (with $\hat{\imath}$ and $\hat{\jmath}$ components reversed):

$\vec{r} = \langle t, -\ln\cos t\rangle \qquad \vec{v} = \langle 1, \tan t\rangle$

$|\vec{v}| = \sec t = \frac{1}{\cos t} \qquad \vec{T} = \langle \cos t, \sin t\rangle$

To find the curvature:

sketch:



(i) using method (b):

$$\frac{d\vec{T}}{ds} = \frac{d\vec{T}/dt}{ds/dt} = \frac{\langle -\sin t, \cos t\rangle}{\sec t}$$

$$= \vec{N}\kappa, \quad \vec{N} = \langle -\sin t, \cos t\rangle$$

$$\kappa = \frac{1}{\sec t} = \boxed{\cos t}$$

(ii) method (a):

$$\left.\begin{aligned} y &= -\ln\cos x, \\ y' &= \tan x \end{aligned}\right\} \Rightarrow \kappa = \frac{y''}{(1+y'^2)^{3/2}} = \frac{\sec^2 x}{\sec^3 x} = \boxed{\cos x}$$

2 $\vec{v} = at\langle \cos t, \sin t\rangle$

$v = |\vec{v}| = at \qquad \vec{T} = \langle \cos t, \sin t\rangle$

$\vec{N} = \langle -\sin t, \cos t\rangle$

(since $\vec{N}$ = rotation of $\vec{T}$ by $\pi/2$).

$$\vec{a} = \frac{d\vec{v}}{dt} = a\langle \cos t, \sin t\rangle + at\langle -\sin t, \cos t\rangle$$

(using formula for diff'g product)

$$= a\vec{T} + at\,\vec{N} = \frac{dv}{dt}\vec{T} + v^2\kappa\vec{N}$$

Thus

$$\left.\begin{aligned} v^2\kappa &= at \\ \text{we know } v &= at \end{aligned}\right\} \Rightarrow \kappa = \frac{1}{at}, \quad \boxed{\rho = at}$$

radius of curvature

3 $\vec{r} = \langle t, t^2/2\rangle, \quad \vec{v} = \langle 1, t\rangle, \quad v = \sqrt{1+t^2}$

$\vec{a} = \langle 0, 1\rangle$

$a_T = \frac{dv}{dt} = \frac{t}{\sqrt{1+t^2}}$ using method (c) to find curvature:

$$|\vec{a}|^2 = 1 = a_T^2 + a_N^2 \qquad \begin{cases} a_T = \frac{dv}{dt} \\ a_N = v^2\kappa \end{cases}$$

Substituting:

$$a_N^2 = 1 - a_T^2 = \frac{1}{1+t^2}, \quad \text{using } a_T = \frac{t}{\sqrt{1+t^2}}$$

$$a_N = v^2\kappa = (1+t^2)\kappa$$

Therefore $(1+t^2)\kappa = \frac{1}{\sqrt{1+t^2}}$,

$$\kappa = \frac{1}{(1+t^2)^{3/2}}$$

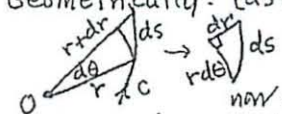
using method (a):
$y = x^2/2$
$y' = x, \quad y'' = 1$
$\kappa = \frac{1}{(1+x^2)^{3/2}}$

4 $\vec{r} = r\hat{u}_r \Rightarrow \vec{v} = \frac{d\vec{r}}{dt} = \dot{r}\hat{u}_r + r\dot{\theta}\hat{u}_\theta$ is (7) p.622 (using (5) and (6) p.622.

$$\frac{ds}{dt} = v = |\vec{v}| = \sqrt{\dot{r}^2 + (r\dot{\theta})^2}$$

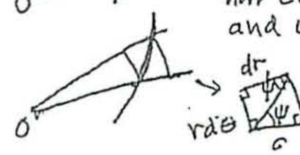
$$\tan\psi = \frac{\hat{u}_\theta\text{-comp. of } \vec{v} \text{ (or } \vec{T})}{\hat{u}_r\text{-comp. of } \vec{v} \text{ (or } \vec{T})} = \frac{r\,d\theta/dt}{dr/dt} = \frac{r}{dr/d\theta}$$

Geometrically: (as in Lecture 8)



$ds = \sqrt{(dr)^2 + (r\,d\theta)^2}$

now divide both sides by dt and use algebra on the right side



$$\tan\psi = \frac{r\,d\theta}{dr} = \frac{r}{dr/d\theta}$$

5 (should read $0 \le \theta \le \pi/2$)

$r = 2\cos\theta$
$\theta = \theta$
(use θ as the parameter: $t = \theta$.)

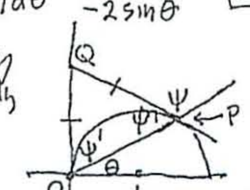
$$\frac{ds}{d\theta} = \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}$$

$$= \sqrt{(-2\sin\theta)^2 + (2\cos\theta)^2} = 2$$

$$s = \int_0^{\pi/2} \frac{ds}{d\theta}\cdot d\theta = \int_0^{\pi/2} 2\,d\theta = \boxed{\pi}$$

$$\tan\psi = \frac{r}{dr/d\theta} = \frac{2\cos\theta}{-2\sin\theta} = \boxed{-\cot\theta}$$

To verify geometrically



length of ⊙ $= 2\pi$, of ⌒ $= \boxed{\pi}$

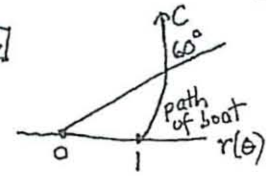
ΔOQP is isosceles, $\psi + \psi' = \pi$
$\theta + \psi' = \pi/2$

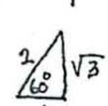
subtracting, $\psi = \theta + \pi/2$

$$\therefore \tan\psi = \frac{\sin(\theta+\pi/2)}{\cos(\theta+\pi/2)} = \frac{\cos\theta}{-\sin\theta} = \boxed{-\cot\theta}$$

[translate graphs of sin and cos to the <u>left</u> by $\pi/2$]

6





$$\tan\psi = \sqrt{3} = \frac{r}{dr/d\theta}$$

Therefore

$$\frac{dr}{d\theta} = \frac{r}{\sqrt{3}}; \text{ separating variables, get diffeq:}$$

$$\frac{dr}{r} = \frac{d\theta}{\sqrt{3}};$$

$$\ln r = \frac{\theta}{\sqrt{3}} + c_1 \qquad \begin{aligned} \theta &= 0 \\ r &= 1 \\ \therefore c_1 &= 0 \end{aligned}$$

$$\boxed{r = e^{\theta/\sqrt{3}}}$$

exponential spiral path

18.02 Exam 1 Tues. Feb. 23, 2010 11:05-11:55

Directions:

1. There are 3 sheets, printed on both sides: seven problems in all.
2. Do all the work on these sheets; use the blank part below if truly necessary. Write down enough to show you are not guessing.
3. No books, notes, calculators, use of cell-phones, etc.
4. Please don't start until the signal is given; stop at the end when asked to; don't talk until your paper is handed in.
5. When the exam starts, read through the exam and start with what you are surest of.
6. Fill out the information below now.

Name Michael Plasmeier e-mail@mit.edu theplaz

Recitation teacher Oliver Rec. hour 12

pg.1 16

pg.2 15

pg.3 15

pg.4 12 ✓

pg.5 8

Total. 66/90

$\frac{d\vec{r}}{dt} = \vec{v}$

$\frac{ds}{dt} = |\vec{v}|$

$\frac{dT}{ds} = kN$

$\vec{v} = \dot{r}\,\hat{u}_r + \dot{\theta} r\,\hat{u}_\theta$

16

Problem 1. (20) Three points in xyz-space are $P: (-1,1,2)$, $Q: (1,2,1)$, and $O: (0,0,0)$.

a) (5) Find angle POQ.

$\vec{PO} = \langle -1,1,2 \rangle$ $\quad PQ \cdot QO = |PQ|\,|QO| \cos\theta$

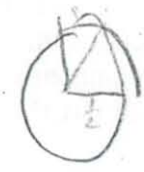
$\vec{QO} = \langle 1,2,1 \rangle$ $\quad \sqrt{(-1)^2+1^2+2^2} = \sqrt{6}$

$(-1\cdot1)+(1\cdot2)+(2\cdot1)$ $\quad \sqrt{1^2+2^2+1^2} = \sqrt{6}$

$-1+2+2$ $\quad 3 = \sqrt{6}\sqrt{6}\cos\theta$

3 $\quad 3 = 6\cos\theta$

$\frac{1}{2} = \cos\theta \quad \theta = 60^\circ$ (?)



5

why never remember this

b) (5) Find the scalar component of $\mathbf{i}+\mathbf{j}+\mathbf{k}$ in the direction of the vector PQ.

scalar component

$\vec{PQ} = \langle 2,1,-1 \rangle$ ← direction

magnitude

$2\hat{i} + \hat{j} - \hat{k}$

2

$|\vec{PQ}| = \sqrt{2^2+1^2+(-1)^2} = \sqrt{i^2+j^2+k^2}$

$\sqrt{6}$

" component of $\vec{A}$ in dir $\vec{B}$

Oliver they never told us this

$\vec{A} \cdot \frac{B}{|B|} = |\vec{A}| \cos\theta$ ← don't know θ

c) Find the equation of the plane through O, P, and Q.

5

eq of a plane

= Normal vector to it

– cross multiply

plug in pt

$-3(x--1)+3(y-1)-3(z-2)$

$-3x-3+3y-3-3z+6=0$

$+3 \quad +3 \quad -6$

$=6$

$-3x+3y-3z=0$

$\vec{PO} = \langle -1,1,2 \rangle$

$\vec{QO} = \langle 1,2,1 \rangle$

$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & 2 \\ 1 & 2 & 1 \end{vmatrix}$ $\quad (1-4)\hat{i} - (-1-2)\hat{j} + (-2-1)\hat{k}$

$-3\hat{i} + 3\hat{j} - 3\hat{k}$

d) Find the area of the space triangle OPQ.

4

$\frac{1}{2}\begin{vmatrix} i & j & k \\ -1 & 1 & 2 \\ 1 & 2 & 1 \end{vmatrix} = \frac{1}{2}(-3\hat{i} + 3\hat{j} - 3\hat{k})$

$\frac{1}{2}\sqrt{3^2+3^2+3^2}$

$\frac{1}{2}\sqrt{27}$

$\frac{1}{2}\,\cancel{\frac{1}{3}}\,3\sqrt{3} = \sqrt{3}$

– opps, why did I do that

Problem 2. (20) 15

Let $A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 2 \\ 1 & 0 & 2 \end{pmatrix}$. Its matrix of cofactors is (in part) $C = \begin{pmatrix} 2 & -2 & -1 \\ -4 & 2 & a \\ 4 & -2 & b \end{pmatrix}$.

10 a) (15) Confirm (mentally) the entry -4 in the first column of C, then fill in the last column of C and from this find A^{-1}.

$a = \begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} = (0-2) = -2$

$b = \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = (1-4) = -3$

det A

$1(2-0) - 2(4-2) + 0$

$2 - 8 + 4$

-2 ✓

$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix} \to \begin{pmatrix} 2 & 2 & -1 \\ 4 & 2 & 2 \\ 4 & 2 & -3 \end{pmatrix}$ ✓

Oh they already did it w/ signs (when you say cofactor)

(!) done twice! flip

$\begin{bmatrix} 2 & -4 & 4 \\ -2 & 2 & -2 \\ -1 & 2 & -3 \end{bmatrix} \to \boxed{-\frac{1}{2}\begin{bmatrix} 2 & -4 & 4 \\ -2 & 2 & -2 \\ -1 & 2 & -3 \end{bmatrix}}$

5 b) (5) Use the matrices of part (a) to solve the following system (no credit for solving the system by elimination):

$$x + 2y = 1, \qquad 2x + y + 2z = 0, \qquad x + 2z = 0\ .$$

$Ax = d$

$AA^{-1}x = dA^{-1}$

$x = dA^{-1}$

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} 2 & 4 & 4 \\ 2 & 2 & 2 \\ -1 & 2 & -3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$-\frac{1}{2}\begin{bmatrix} 2\cdot 1 + 2\cdot 0 + 0\cdot 0 \\ 2\cdot 1 + 1\cdot 0 + 2\cdot 0 \\ -1\cdot 1 + 0\cdot 0 + 2\cdot 0 \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ \frac{1}{2} \end{bmatrix}$

(✓) fixed at last min!

$x = -1 \quad y = -1 \quad z = \frac{1}{2}$

Problem 3. (5) Find the value(s) of c for which the system of homogeneous equations

$$cx + 2y + z = 0, \qquad 2x - y + z = 0, \qquad x + 3y - 2z = 0$$

has a solution other than $x = y = z = 0$. (No credit for solving by elimination.)

where det $= 0$

$$\begin{bmatrix} c & 2 & 1 \\ 2 & -1 & 1 \\ 1 & 3 & -2 \end{bmatrix}$$

$c(2-3) - 2(-4-1) + 1(6--1) = 0$

$2c - 3c + 8 + 2 + 6 + 1 = 0$

$-c = -17$

$c = 17$

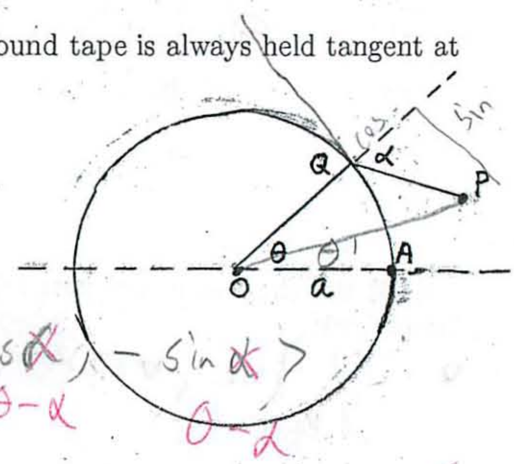
✓

Problem 4 (15) Scotch® tape is being unwound from a stationary circular spool having radius a. The end $P : (x, y)$ of the tape is initially at the point $A : (a, 0)$ on the x-axis; Q is the point on the circumference where the tape is leaving the spool. During the process, the unwound length of tape QP is held taut, and held so that it makes a constant negative angle $-\alpha$, $0 < \alpha < \pi/2$ with the radial vector OQ (as measured clockwise from OQ to QP).

Use vector methods to derive parametric equations for x and y in terms of the central angle θ and the constants a and α, for $0 \le \theta \le 2\pi$. Show work, indicating reasoning.

(If stuck, for 5 points less, you can take $\alpha = \pi/2$, so that the unwound tape is always held tangent at Q, in the direction where its sticky side faces the spool.)

$\vec{OP} = OQ + \vec{QP}$ + something w/ α when $\alpha \neq \frac{\pi}{2}$



$a\langle \cos\theta, \sin\theta \rangle + a\theta \langle + \cos\alpha, - \sin\alpha \rangle$

$\overset{\text{arclength}}{\uparrow}$ $\quad \theta - \alpha$

$\vec{r} = a\langle \cos\theta + \theta\cos\alpha, \sin\theta - \theta\sin\alpha \rangle$

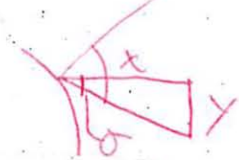
$\theta' = \tan^{-1}\left(\frac{y}{x}\right) = \left(\frac{a\sin\theta - a\theta\sin\alpha}{a\cos\theta + a\theta\cos\alpha}\right)$

$x = r\cos\theta' = a\cos\theta + a\theta\cos\alpha$

$y = r\sin\theta' = a\sin\theta - a\theta\sin\alpha$

note $\sin(-\alpha) = -\sin\alpha$

$\cos(-\alpha) = \cos\alpha$



create a new angle (was thinking about that)

$\sigma = \theta - \alpha$

10/15

Problem 5. (15) The path of a point P is a circular helix in space having position vector

$$OP = \mathbf{r}(t) = \langle 2\cos t,\ 2\sin t,\ t\rangle .$$

Find in order the following, in terms of t, giving enough calculation or reasoning to show you are not guessing or writing down answers from memory:

(3) a) the velocity vector $\mathbf{v}$

$d \sin = \cos$

$d \cos = -\sin$

derivitive

$\vec{v}(t) = \langle -2\sin t,\ 2\cos t,\ 1\rangle$

3/3

(4) b) the speed $|\mathbf{v}|$ and the length of one complete turn of the helix, i.e., the length between two successive points lying over the same point in the xy-plane.

$|\vec{v}| = \sqrt{(-2)^2 \sin^2 t + 2^2 \cos t + 1^2}$

$\sqrt{4\sin^2 t + 4\cos^2 t + 1}$

$\sqrt{4(\sin^2 t + \cos^2 t) + 1}$

$\sqrt{4+1}$

$\sqrt{5}$

$S = \int_0^{2\pi} \frac{ds}{dt}\, dt$

$S = \int_0^{2\pi} \sqrt{5}\, dt$

$S = \sqrt{5}\, t \Big|_0^{2\pi}$

4/4 $S = 2\pi\sqrt{5} - 0$

$\cancel{(5)^{1/2}}$

$\cancel{5^{1/2}}$

$\cancel{.3/2}$ Constan

(8) c) the unit tangent vector $\mathbf{T}$, the unit normal vector $\mathbf{N}$, and the curvature κ (k in the book), at time t.

$T = \frac{V}{|\vec{v}|} = \frac{\langle -2\sin t,\ 2\cos t,\ 1\rangle}{\sqrt{5}}$

5

$dT = \frac{dT/dt}{ds/dt} = \frac{? \text{ differentiate (product rule)}}{|\vec{v}|}$

$d\ \frac{1}{\sqrt{5}} \cdot \langle -2\sin t,\ 2\cos t,\ 1\rangle + \langle -2\cos t, -2\sin t, 0\rangle \cdot \frac{1}{\sqrt{5}}$

$0 + \frac{\left(\frac{\langle -2\cos t,\ -2\sin t,\ 0\rangle}{\sqrt{5}}\right)}{\sqrt{5}}$

$\frac{\langle -2\cos t,\ -2\sin t,\ 0\rangle}{\sqrt{5}\sqrt{5}}$

$\langle -2\cos t,\ -2\sin t,\ 0\rangle$

$N = \langle -2\cos t, -2\sin t, 0\rangle$

$k = \pm 1/5$

right?

so $\ominus$

$k = -1/5$

2

factor $\frac{\sqrt{2}}{5}$ out of normal vector

$\frac{\sqrt{2}}{5} / \sqrt{5} = \boxed{\frac{2}{5} = k}$

Problem 6. (5)

Find the length of the exponential spiral curve $r = e^{2\theta}$ in the plane, between the point on the curve where $r = 1$, $\theta = 0$, and the next point on the curve where it crosses the x axis as θ increases.

$S = \int_0^{2\pi} \frac{ds}{dt}\,dt$

$r = e^{2\theta}$

$\cancel{v = 2\theta e^{2\theta}}$?

$\cancel{v^2 = 2\theta e^{2\theta^2}} + ?$

$\frac{ds}{dt} = |\vec{v}| = \sqrt{\vec{v}_r^2 + \vec{v}_\theta^2}$

$S = \int_0^{2\pi} \cdots$

$\cancel{\vec{V} = \dot{r}\hat{u}_r + r\dot{\theta}\hat{u}_\theta}$

$\cancel{r = e^{2\theta}}$

$\cancel{\theta =}$

out of the

0

$S = \int_0^{2\pi} \sqrt{\vec{v}_r^2 + \vec{v}_\theta^2}\,dt$

$S = \sqrt{v_r^2 + v_\theta^2}\,t \Big|_0^{2\pi}$

$S = \sqrt{v_r^2 + v_\theta^2}\,2\pi$

$S = \sqrt{1^2 + 0^2}\,2\pi = 2\pi$

Problem 7. (10) The velocity vector of a moving point in the polar-coordinate $\mathbf{u}_r - \mathbf{u}_\theta$ system is given in general by $\mathbf{v} = r'\mathbf{u}_r + r\theta'\mathbf{u}_\theta$.

A point P moves with velocity vector $\mathbf{v} = -\sin t\,\mathbf{u}_r + \sin 2t\,\mathbf{u}_\theta$.

If it is at $r = 1$, $\theta = 0$ at time $t = 0$, what are the parametric equations $r = r(t)$, $\theta = \theta(t)$ that describe its motion?

Should have remembered better

Don't forget constant of integration

know: $\sin 2t = 2\cos t \sin t$

$\cos(a+b) = \cos a \cos b - \sin a \sin b$

$r(t) = \cos t + 1$

$\theta(t) = \frac{-\cos 2t}{\sin t + t} + \frac{\pi}{4}$ ✗

+4

$\sin(a+b) = \cos a \sin b + \sin a \cos b$

$\vec{V} = \dot{r}\hat{U}_r + r\dot{\theta}\hat{U}_\theta$

$\dot{r} = -\sin t$

$\downarrow \int$

$r = \cos t + C$

$r(1) = \cos(1) + ? = 1$

$r = \cos t + 1$ ✗ +4 ??? cos at 1?

$r\dot{\theta} = \sin 2t$

$\dot{\theta} = \frac{\sin 2t}{\cos t + 1}$

$\int$

$\theta = \frac{-\cos 2t}{\sin t + t} + C$

I did integral wrong prob

$0 = \frac{-\cos 2t}{\sin t + t} = \frac{-\cos 0}{\sin 0 + 0} \rightarrow \theta = \frac{-\cos 2t}{\sin t + t} + \pi/4$

6.

lenght $= \int_t |\vec{v}|\, dt$

need access to that

no t given

take $\theta = t$

$r(t) = r(\theta) = e^{2t}$

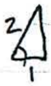
$\vec{v} = \dot{r}\,\hat{v_r} + r\dot{\theta}\,\hat{v_\theta}$

$|v| = \sqrt{\dot{r}^2 + (r\dot{\theta})^2}$ ← oh duh

⚠ $|\vec{v}| = \left|\frac{dr}{dt}\right|$ $\boxed{|\vec{v}| \neq \dot{r}}$ ← be careful

18.02 Exam 1 Sol'ns — Spring 2010

1 a) $\cos(POQ) = \dfrac{\vec{OP}\cdot\vec{OQ}}{|OP|\,|OQ|} = \dfrac{3}{\sqrt6\sqrt6} = \dfrac12$



$\angle POQ = \pi/3$ or $60°$

b) $\vec{PQ} = \langle 2,1,-1\rangle$

$\langle 1,1,1\rangle \cdot \dfrac{\langle 2,1,-1\rangle}{\sqrt6} = \dfrac{2}{\sqrt6} = \dfrac{\sqrt6}{3}$

c) $\vec{OP}\times\vec{OQ} = \begin{vmatrix} \hat\imath & \hat\jmath & \hat k \\ -1 & 1 & 2 \\ 1 & 2 & 1 \end{vmatrix} = \langle -3,3,-3\rangle = \vec N$ to plane

Plane through $(0,0,0)$: $-x+y-z=0$ or ($x - y + z = 0$ or a multiple)

d) $\frac12|\vec{OP}\times\vec{OQ}| = \frac32|\langle -1,1,-1\rangle| = \frac32\sqrt3$

---

2 a) $\begin{pmatrix} 2 & -2 & -1 \\ -4 & 2 & \boxed{2} \\ 4 & -2 & \boxed{-3} \end{pmatrix} = C \qquad \begin{vmatrix} 1 & 2 & 0 \\ 2 & 1 & 2 \\ 1 & 0 & 2 \end{vmatrix} = -2$

$C^T = \begin{pmatrix} 2 & -4 & 4 \\ -2 & 2 & -2 \\ -1 & 2 & -3 \end{pmatrix} \qquad A^{-1} = \begin{pmatrix} -1 & 2 & -2 \\ 1 & -1 & 1 \\ 1/2 & -1 & 3/2 \end{pmatrix}$

(or $\frac12 C^T$)

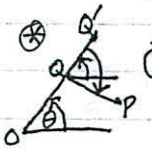
b) $A\vec x = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $\vec x = A^{-1}\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}-1\\1\\1/2\end{pmatrix} = \begin{pmatrix}x\\y\\z\end{pmatrix}$

3 $\begin{vmatrix} c & 2 & 1 \\ 2 & -1 & 1 \\ 1 & 3 & -2 \end{vmatrix} = \begin{matrix} 2c+2+6 \\ -(-1+3c-8) \end{matrix} = -c+17$

$= 0$ if $\boxed{c = 17}$

4 $\vec{OQ} = \langle a\cos\theta, a\sin\theta\rangle$

$\vec{QP} = a\theta\langle \cos(\theta-\alpha), \sin(\theta-\alpha)\rangle$



$\vec{OP} = a\langle \cos\theta + \theta\cos(\theta-\alpha),\ \sin\theta + \theta\sin(\theta-\alpha)\rangle$

---

if $\alpha = \pi/2$, $\vec{QP} = a\theta\langle \sin\theta, -\cos\theta\rangle$

$\vec{OP} = a\langle \cos\theta + \theta\sin\theta,\ \sin\theta - \theta\cos\theta\rangle$

⊛ $\angle PQQ' = \theta - \alpha$

5 $\vec r(t) = \langle 2\cos t, 2\sin t, t\rangle$

a) $\vec v = \langle -2\sin t, 2\cos t, 1\rangle$

b) $|\vec v| = \dfrac{ds}{dt} = \sqrt{4(\sin^2 t + \cos^2 t) + 1} = \sqrt5$

$s = \displaystyle\int_0^{2\pi} \frac{ds}{dt}\cdot dt = 2\sqrt5\,\pi$

c) $\vec T = \dfrac{\vec v}{|\vec v|} = \dfrac{1}{\sqrt5}\langle -2\sin t, 2\cos t, 1\rangle$

$\vec N = \mathrm{dir}\left(\dfrac{d\vec T}{dt}\right) = \langle -\cos t, -\sin t, 0\rangle$

$\kappa = \left|\dfrac{d\vec T}{ds}\right| = \left|\dfrac{d\vec T/dt}{ds/dt}\right| = \dfrac{\frac{2}{\sqrt5}|\vec N|}{\sqrt5} = \dfrac25$

6 Taking $t = \theta$ in the velocity formula (see prob. 7), or using

$\dfrac{ds}{d\theta} = \sqrt{r^2 + r'^2}$

$= \sqrt{(e^{2\theta})^2 + (2e^{2\theta})^2} = e^{2\theta}\sqrt5$

$\therefore\ s = \displaystyle\int_0^{2\pi} e^{2\theta}\sqrt5\, d\theta = \frac{e^{2\theta}}{2}\sqrt5\Big]_0^{2\pi} = \frac{\sqrt5}{2}(e^{4\pi}-1)$

[or $\displaystyle\int_0^{\pi} e^{2\theta}\sqrt5\,d\theta = \frac{\sqrt5}{2}(e^{2\pi}-1)$]

7 Comparing two formulas for $\vec v$,

$r' = -\sin t \qquad r\theta' = \sin 2t$

$\therefore\ r = \cos t + c_1;\quad r(0) = 1 \Rightarrow c_1 = 0$

$r\theta' = \sin 2t \Rightarrow \cos t\cdot\theta' = 2\sin t\cos t$

$\therefore\ \theta' = 2\sin t$

$\theta = -2\cos t + c_2 \qquad \theta(0) = 0 = -2 + c_2$

$\therefore\ c_2 = 2$

$\begin{cases} r(t) = \cos t \\ \theta(t) = 2 - 2\cos t \end{cases}$

$\sin 2t = 2\cos t\sin t$