Part 2 Matlab

a. Plot $z = x^2 - y^2$ adjust viewport so interesting + print

b) Plot $z = xye^{x^2 - y^2}$

* make sure to put the dots in *

why am I getting errors
that z is not a matrix - it is
- oh did not do mesh grid
- must do all steps in order

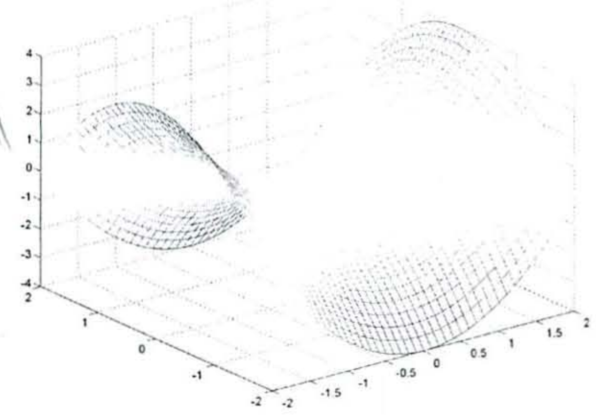
See printout

9/9

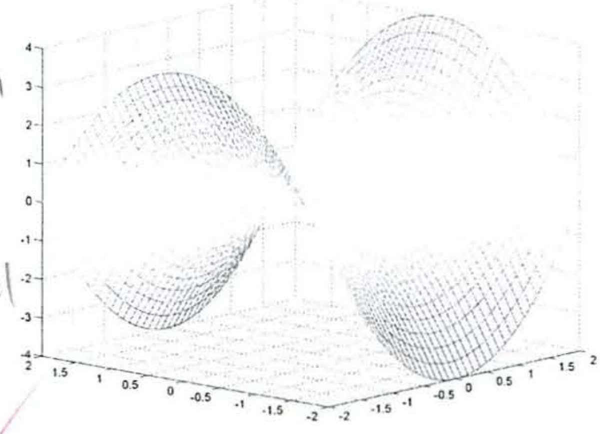
a)

Sorry print at so faint

1 interval

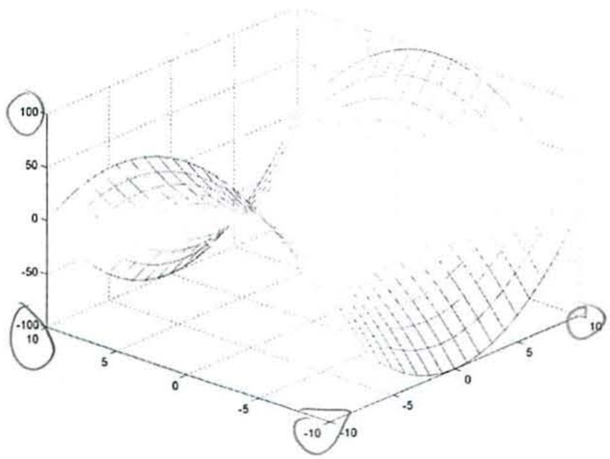


spread ↑ out more .15 interval

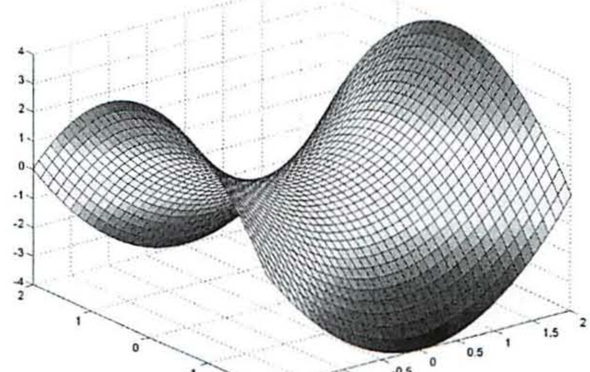


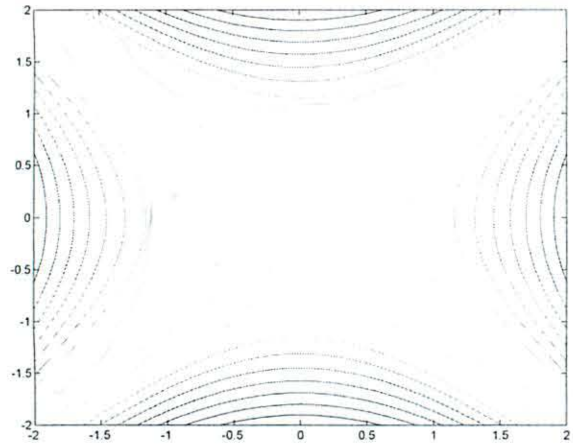
large axis

this is 100

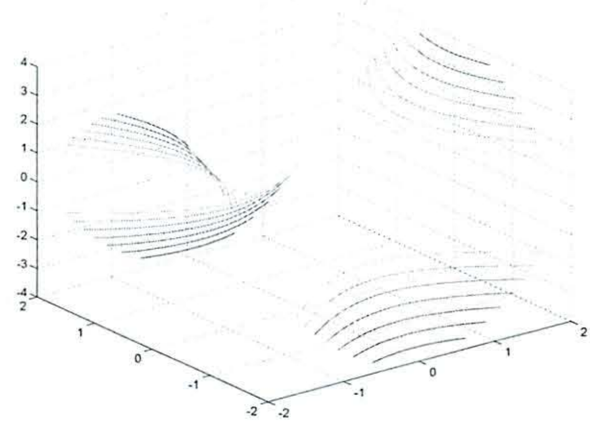


surf



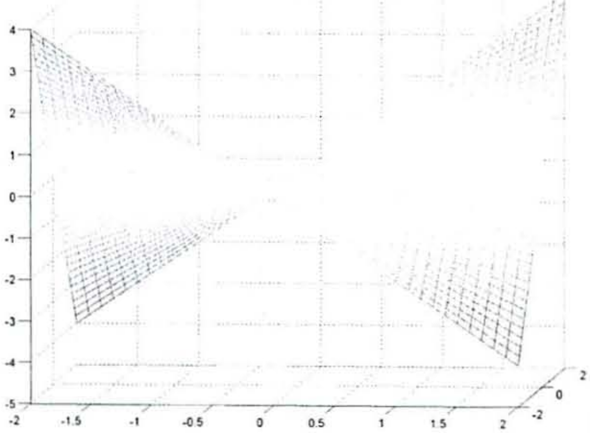


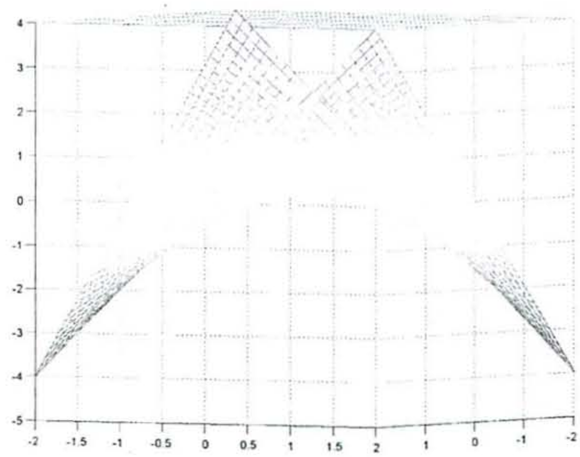
contour
from top



Contour 3d

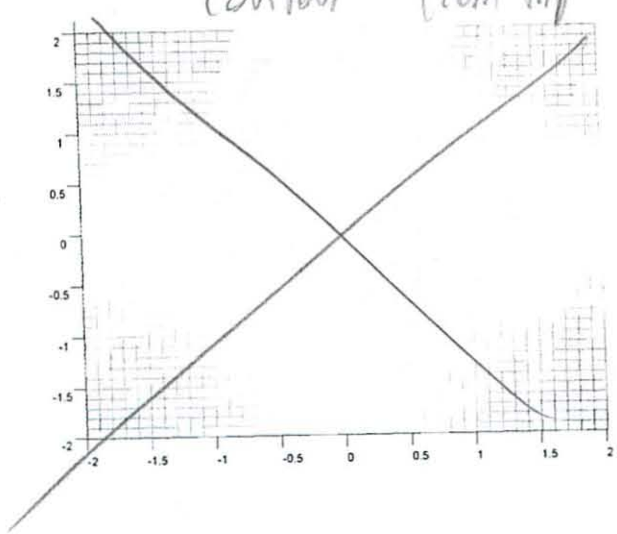
b+c)



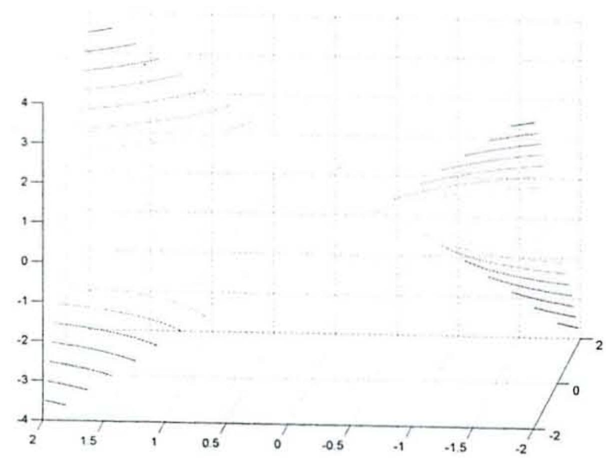


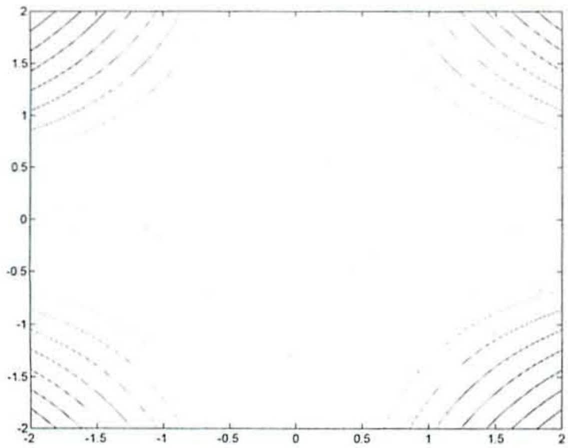
different views

contour from top



3D contour





Contour from top

2. Surface $z = x^2 - y^2$ saddle shape

- contains 2 Families of line which fill surface
  - ↳ can't see from MatLab
- 2 lines - one from each family - goes through every P_0
- on surface.
- spans tangent plane at P_0
- intersection of surface w/ tangent planes

Verify w/ $(3, 2, 5)$

a) Write the parametric equation for a general non-horiz line through P

- passes through P and parllel to $a\hat{i} + b\hat{j} + c\hat{k}$
- Are 2 sets of values for (a, b) for which the resulting line lies entirely on surface
- (lies on surface if every pt on line works)

Oliver OH

$S: f(x, y, z) = C$

C on curve $= \vec{r}(t) = \langle \ldots \rangle$) for every t belongs to surface

Is the curve on surface - plug in to

$A = \frac{a}{c}\hat{i} + \frac{b}{c}\hat{j} + \_\hat{k}$ ←same

↑just as specific

Yes for process but no answer

1/4

3.

2B-8 Two sides of of triangle a, b, θ in included angle
Third side is c

0/1.5 a) Give the approx for Δc in terms of $a, b, c, \theta, \Delta a, \Delta b, \Delta\theta$

b) If $a=1$, $b=2$, $\theta=\frac{\pi}{3}$ is c more sensitive to small changes in a or b ?

1/1.5

So this is error/approx formula

$$\Delta w = \left(\frac{\partial w}{\delta x}\right)_0 \Delta x + \left(\frac{\partial w}{\partial y}\right) \Delta y$$

-so need formula → trig

$$c = a^2+b^2-2ab\cos(\theta)$$

$$c = 2a-2b\cos\theta\ \Delta a + 2b-2a\cos\theta\ \Delta b$$
$$+ +2ab\sin(\theta)\ \Delta\theta$$

↑ need to do θ, yeah its a variable

$$c = 2(1)-2(2)\cos(\tfrac{\pi}{3})\ \Delta a + 2(2)-2(1)\cos(\tfrac{\pi}{3})\Delta b$$
$$+ 2(1)(2)\sin(\tfrac{\pi}{3})\ \Delta c$$

$$c = 0\Delta a + 3\Delta b + 2\sqrt{3}c$$

↑ largest change in volume due to c

1/1.5 b

but if this is not an option then b

total change $= 3+2\sqrt{3} = 6.46$

↑ correct
- don't have to do

4. Topo Map

level curve indicates h above sea level in ft

a. Estimate h_x, h_y at B

5/5

I guess h_x is gradiant in x dir ???

or is pt where tangent $=0$

- but pt given

did scale wrong should be fixed

change → / how far you go → $\frac{-100 \text{ ft}}{1000 \text{ ft}} = -.1$ ✓

h_y $\frac{-100 \text{ ft}}{500 \text{ ft}} = -.2$ ✓

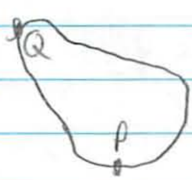
→ over

b. Now in dir $\vec{J} - \vec{i}$

$\frac{-100 \text{ ft}}{800 \text{ ft}} = -.125$ ✓

c Find P and Q having the smallest y coords such that $h(P) = h(Q) = 3000$ $h_x(P) = 0$ $h_y(P) = 0$

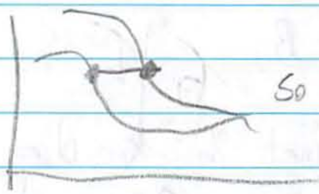
So are on the same 3000 line where ✓



d

Oliver
OH

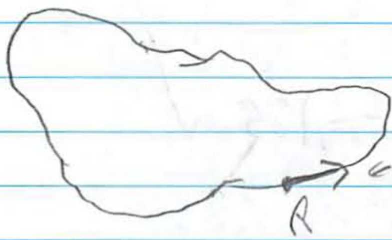
$h(x) = \frac{\partial h}{\partial x}$ = rate of change in d'r of x



So I did get it

d Find the R with the largest x-coord
such that $h(R) = 2900$
on that line

$\frac{dh}{ds} = 0$ at R in dir $\hat{\imath} + \hat{\jmath}$



← in 45° dir, no change in height since along line

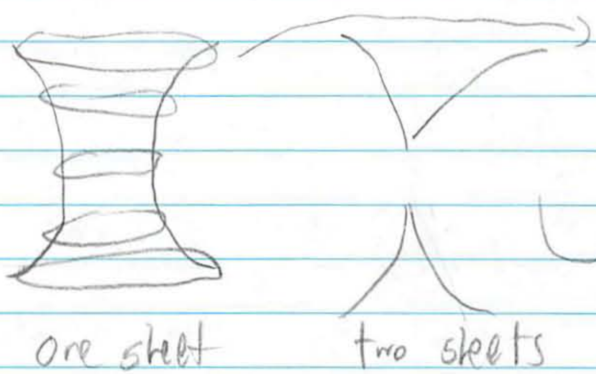
5. In the first octant of 3-space $xyz=a$
for a fixed constant $a>0$
Graphic surface is a hyperboloid
Has 3 coordinate planes as asymptotes
- gets closer + closer to a coord plane as variable $\Rightarrow \infty$

$P_0(x_0, y_0, z_0)$ tangent plane cuts off a tethedron from 1st octant, origin is 1 vertex, 3 coord planes + tangent plane are the 4 surfaces

As P_0 varries, shape varies, volume constant

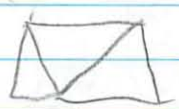
Prove + find volume

Wikipedia

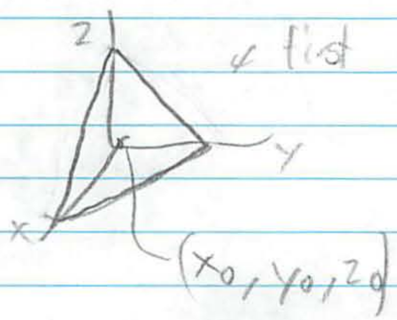


$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

$$-\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$



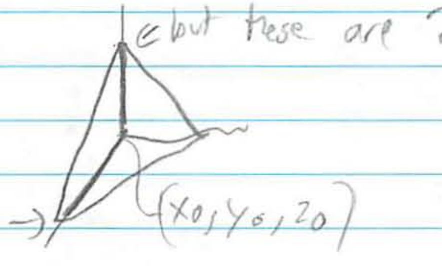
← worst tetrahedron ever
4 triangular faces
? but how can it do that
- sides are curved



← first octant

3 faces are coord plane
4th (face?) is from octant from tangent plane

And how can volume remain constant?
↳ volume = $\frac{1}{3}bh$



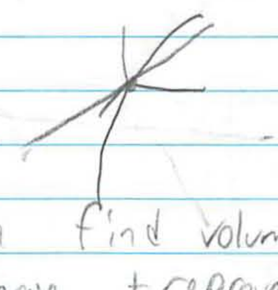
$w = xyz$

$\nabla w = \langle yz, xz, xy \rangle$

level surface - want dot product • with pt through

$\langle yz, xz, xy \rangle \cdot \langle x_0, y_0, z_0 \rangle$
$x_0(yz) + y_0(xz) + z_0(xy)$

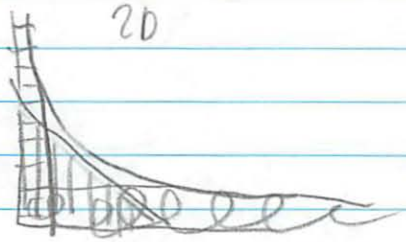
find where plane intersects coord axis
- did I draw it wrong



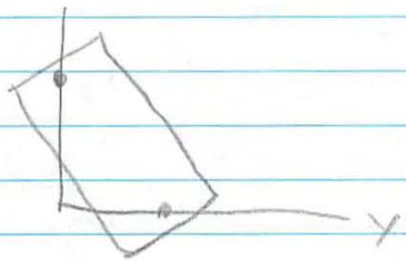
- so where do we put the axis?

then find volume
Change + reprove

Oliver



all 3 areas =
not obvious in 3D



partial derivitives

oliver OH

$(0, y, 0)$

$(0, 0, z)$ satisfy

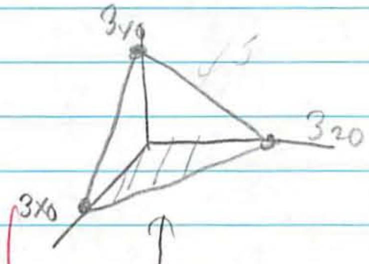
$[x_0 y_0 (z - z_0) + x_0 y_0 (y - y_0) + \quad = 0$

$y = 0$
$x = 0$

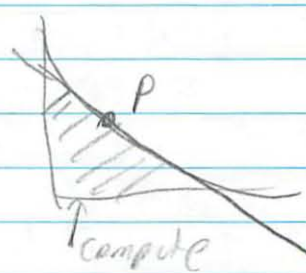
$\nabla w \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$

$S:$ $w(x, y, z) = c$

S: is a graph of $f(x, y) \Leftrightarrow z - f(x, y) = 0$



2D



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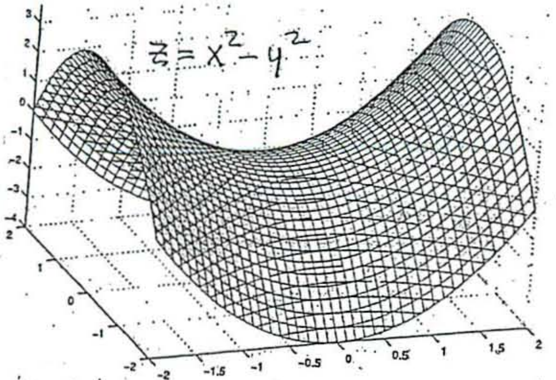
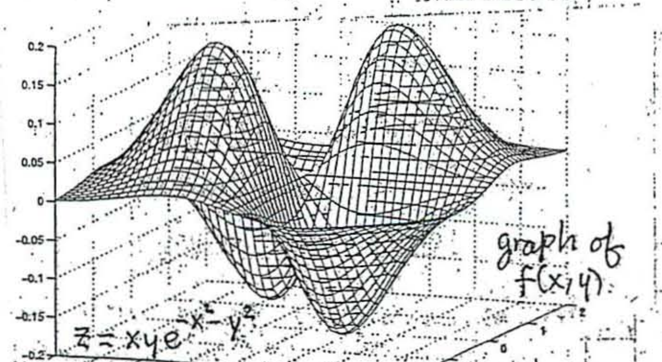
oh surface is just touching the tangent plane here

prove volume does not depend on x_0, y_0, z_0

Prob. Set 3 Solutions — Spring 2010

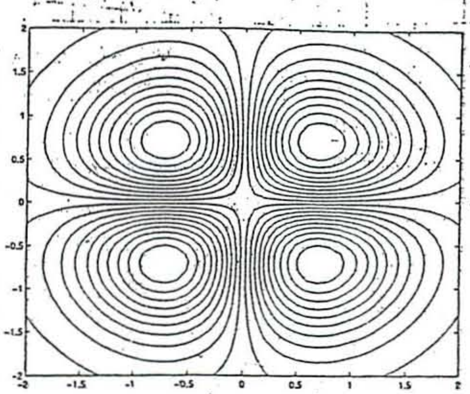
1





graph of $f(x,y)$

level curves of $f(x,y)$



2 $z = x^2 - y^2$ $P: (3,2,5)$ line is: L $x = 3 + at$, $y = 2 + bt$, $z = 5 + t$

a) L goes through P; it lies on the graph of z $\Leftrightarrow$

$$5 + t = (3+at)^2 - (2+bt)^2 \quad \text{for all } t.$$

$$5 + t = (a^2 - b^2)t^2 + (6a - 4b)t + 5$$

$\Leftrightarrow$ $\boxed{a^2 - b^2 = 0}$ and $\boxed{6a - 4b = 1}$ *

(so $b = a$ or $b = -a$) if $b = a$, $a = b = 1/2$; if $b = -a$, $a = 1/10$, $b = -1/10$

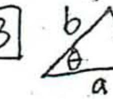
So these values of (a,b) give the two lines.

b) The tan. plane at $(3,2,5)$ is $z - 5 = 6(x-3) - 4(y-2)$, (since $\frac{\partial z}{\partial x} = 6$ at $(3,2)$; $\frac{\partial z}{\partial y} = -4$)

Subst. the eqns of the line in: line lies on plane if $t = 6at - 4bt$, for all t.

But this is true for our two lines, since a & b satisfy * for both pairs (a,b).

3



By law of cosines $c = \sqrt{a^2 + b^2 - 2ab\cos\theta}$

$$\frac{\partial c}{\partial a} = \frac{2a - 2b\cos\theta}{2c}, \quad \frac{\partial c}{\partial b} = \frac{2b - 2a\cos\theta}{2c}$$

$$\frac{\partial c}{\partial \theta} = \frac{+2ab\sin\theta}{2c}$$

a) $\Delta c \approx \left(\frac{a - b\cos\theta}{c}\right)\Delta a + \left(\frac{b - a\cos\theta}{c}\right)\Delta b + \left(\frac{ab}{c}\sin\theta\right)\Delta\theta$

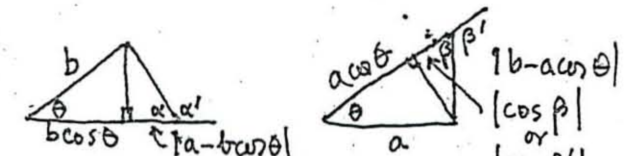
b) $a = 1$, $b = 2$, $\theta = \pi/3$, formula is:

$$\Delta c \approx \left(\frac{1 - 2\cdot 1/2}{c}\right)\Delta a + \left(\frac{2 - 1/2}{c}\right)\Delta b + \left(\frac{2\cdot\sqrt{3}/2}{c}\right)\Delta\theta$$

$$\Delta c \approx 0\cdot\Delta a + \frac{3}{2c}\Delta b + \frac{\sqrt{3}}{c}\Delta\theta$$

$\therefore$ more sensitive to b

c)



(since $\alpha' = \pi - \alpha$, $\cos\alpha' = -\cos\alpha$)

According to (a) c is most sensitive to changes in the side for which the angle α or β has the biggest cosine, i.e. is further from $\pi/2$.

Makes sense geometrically: c is more sensitive to a than to b:

4 a) $h_x \approx \frac{\Delta h}{\Delta x} \approx \frac{-100}{1000} = -.1$

$h_y \approx \frac{\Delta h}{\Delta y} \approx \frac{-100}{500} = -.2$

b) $\left.\frac{dh}{ds}\right|_{A,\,\hat{j}-\hat{i}} \approx \frac{\Delta h}{\Delta s} \approx \frac{-100}{500} = -.2$

c) $P \approx (1.8, 1.1)$ $Q: (2.6, 1.9)$

d) $R: (4.0, 1.7)$ or (maybe) $R: (4.2, 1.9)$

($\hat{i} + \hat{j}$ should have the tangent direction at R)

Lecture 12

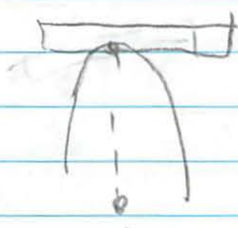
Min-max problems, least squares approx 3/4

Handout PS 4

In math, max and min are importantance
- how some problems can be solved

$w = f(x, y)$

(x_0, y_0) ← how do you find a max or min point
- where the tangent plane is horizontal



← max value

↑ max point (the independent variables)

$$w - w_0 = \left(\frac{\partial f}{\partial x}\right)_0 (x - x_0) + \left(\frac{\partial f}{\partial y}\right)_0 (y - y_0)$$

↑ equ. of tangent plane

horizontal when this $= 0$

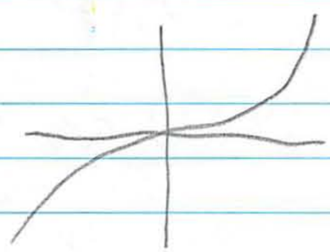
$\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y} = 0$ at P_0

Critical point

if $\begin{cases} \frac{\partial f}{\partial x} = 0 \\ \frac{\partial f}{\partial y} = 0 \end{cases}$ at P_0

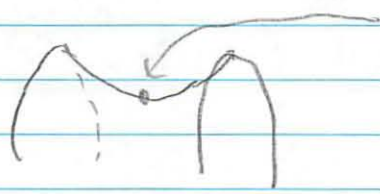
But it could be a max or min
which is it?
Usually obvious if physical
Will do tomorrow

But also can have issue



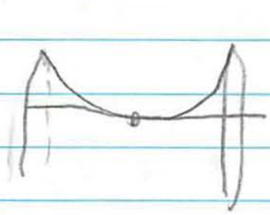
← deriv $=0$, no max or min

A saddle shaped surface is an example $x^2 - y^2$



saddle or minimax point

So the tangent line intersects the end of the saddle

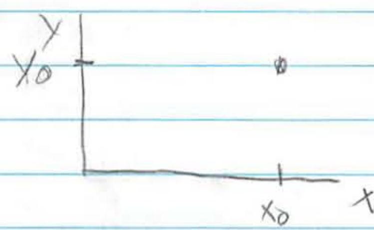


A cone as well

← minimum, obvious not a critical pt
does not have a tangent plane at that pt
so can't calc partial derivs don't exist there

If boundries, max and min can occur on boundry
- partial derivs $\neq 0$
- calculus will not help

If more than 3D
- can't draw tangent planes
- but its the same
- different reasonings



w = temp at x, y

$w = f(x, y_0)$ partial function

↳ has x_0 at max pt

P_0 = max point

$$\left(\frac{df(x, y_0)}{dx}\right)_0 = 0$$

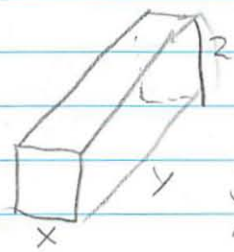
$$\left(\frac{\partial f}{\partial x}\right)_{P_0} = 0$$

Same reasoning for y

Can use identical reasoning for n-variables

$$\frac{d}{dx} f(x, y_0, z_0) = \left(\frac{\partial f}{\partial x}\right)_{P_0}$$

example



Vol must be 3 cuft

3 layers on bottom
2 layers on ends
1 layer on side rectangular
0 layers on top

Minimize cost → cardboard area C

$$C = 3(xy) + 2 \cdot 2(xz) + 2 \cdot 1(yz)$$

no gradiants + partial derivitives
- since variables depend on each other

need constraint $xyz = 3$

*remember how to do

$z = \frac{3}{xy}$ get into terms of 1 variable

$$C = 4 \cdot \frac{3}{y} + 2 \cdot \frac{3}{x} + 3xy \quad \text{now only 2 variables}$$

$$= \frac{12}{y} + \frac{6}{x} + 3xy$$

$$\frac{\partial C}{\partial x} = -\frac{6}{x^2} + 3y = 0$$

$$\frac{\partial C}{\partial y} = -\frac{12}{y^2} + 3x = 0$$

a) simltaneous
b) not linear

* must eliminate one variable (think about it)
* if not linear, not easy to solve, need to read a giagantic book on how
* guess for a solution

$$y = \frac{2}{x^2}$$

$$x = \frac{4}{y^2} = \frac{4x^4}{4}$$

reject $x = 0$

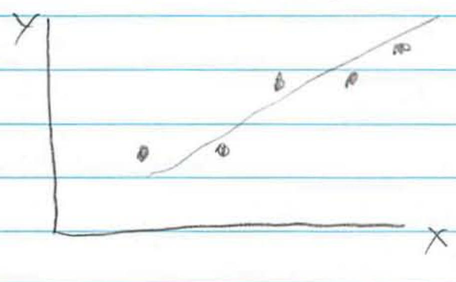
get $x = 1$

then $y = 2$

$z = \frac{3}{2}$

Method of Least Squares / Regression Line

Mattuck: explination cutting corners



trying to find general law connecting the lines

"line of best fit"

treated as a minimum problem

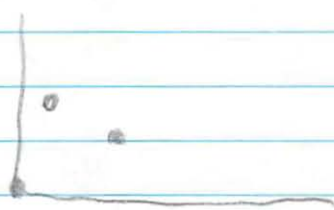
$y = \underbrace{mx + b}$ — which m, b should chose for line akward to calc perpendicular distance

So optimize vertical line
- all must be ⊕
- can't differenitate abs

deviation: $mx_i + b - y_i$

minimize $D = \sum_{i=1}^{n} (mx_i + b - y_i)^2$ m, b rational

example



(0,0)
(2,1)
(1,2)

$$b^2 + (2m + b - 1)^2 + (m + b - 2)^2 = D$$

do not expand
differenitate as tey are

$$\frac{\partial D}{\partial m} = 2(2m+b-1)\cdot 2 + 2(m+b-2)\cdot 1 = 0$$

$$\frac{\partial D}{\partial b} = 2b + 2(m+b-2) + 2(2m+b-1) = 0$$

Lecture 13

2nd Deriv Max Min 2
+ Langrarge multipliers

3/15

Review

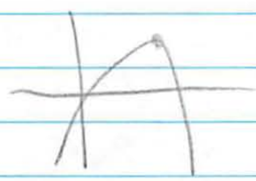
$w = f(x, y)$ max + min pts

critical pts ↳ solutions
$$\begin{cases} \frac{\partial f}{\partial x} = 0 \\ \frac{\partial f}{\partial y} = 0 \end{cases}$$

Usually non linear
- hard to solve
- easy in this class

2D

min ~~first deriv~~ ~~⊖ → ⊕~~
2nd deriv ⊕



max 2nd deriv ⊖

3D

Similar - but more partial derivitives

$$w = w_0 + (w_x)_0 (x - x_0) + (w_y)_0 (y - y_0)$$
$$+ \tfrac{1}{2}\left[W_{xx} (x - x_0)^2 + 2(W_{xy})_0 (x - x_0)(y - y_0) + W_{yy}(y - y_0) \right]$$

2nd order terms

$W_{xy} + W_{yx}$ same for nice functions

At a critical pt

$$(W_x)_0 = (W_y)_0 = 0$$

Must calc the 3 partial derivs

$$\left.\begin{aligned}(W_{xx})_0 &= A\\ (W_{xy})_0 &= B\\ (W_{yy})_0 &= C\end{aligned}\right\}\text{aliases}$$

↳ P_0 is the critical pt you are testing for min-max

$AC - B^2$ ← discriminant

if $< 0 \to$ saddle
$> 0 \to$ if A or $C > 0 \to$ min
↳ if A or $C < 0 \to$ max

<u>example</u> $w = x^3 + 3xy^2 - 3x$

$$\left.\begin{aligned}w_x &= 3x^2 + 3y^2 - 3\\ w_y &= 6xy\end{aligned}\right\}\text{ when both } = 0$$

$$6xy = 3x^2 + 3y^2 - 3 = 0$$

↳ know either x or y or both $= 0$

can be or $\begin{cases} x = 0 & y = \pm 1 \\ x = \pm 1 & y = 0 \end{cases}$

have 4 critical points

↓ do

the aliases

$W_{xx} = 6x \quad = A$
$W_{xy} = 6y \quad = B$
$W_{yy} = 6x \quad = C$

Critical pts to test

		$AC - B^2$		
$(0,1)$	→	$6\cdot0\cdot6\cdot1 - (6\cdot1)^2$	⊖	saddle
$(0,-1)$	→	$6\cdot0\cdot6\cdot-1 - (6\cdot-1)^2$	⊖	saddle
$(1,0)$	→	$6\cdot1\cdot6\cdot0 - (6\cdot0)^2$	⊕	→ A ⊕ min
$(-1,0)$	→	$6\cdot-1\cdot6\cdot0 - (6\cdot0)^2$	⊕	→ A ⊖ max

Why test works: algebra

at critical pt w
assume
$cp = (0,0) \quad w_0 = 0$

he did quickly

$$\cancel{\Delta} w \approx \frac{1}{2}\left[Ax^2 + 2Bxy + Cy^2\right]$$

when at $[0,0]$ min, max, or saddle?

Same as proving quadratic formula

* completing the square *

$$= A\left(x + \frac{B}{A}y\right)^2 + \left(\frac{AC - B^2}{A}\right)y^2$$

⊕ ⊕/⊖ ⊕

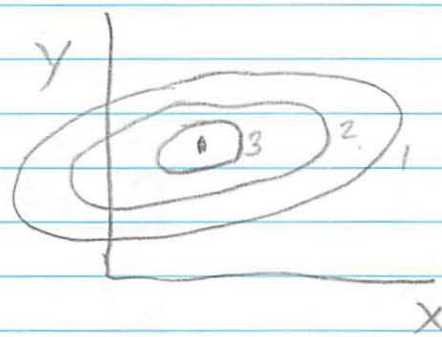
← did not really get what he said about this

Larange multiplier

$w = f(x, y, z)$
↳ variabes constrained by equation

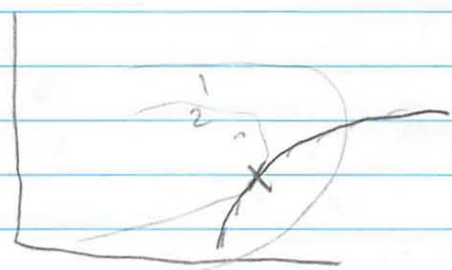
Find max or min
but variables are constraind to satisfy eq
$g(x, y, z) = 0$
↑ defines a surface
don't know what it looks like

$w = f(x, y)$ $g(x, y) = 0$



what do points satisfying constraint look like?
* Constraint curve *
- walk along the trail

$g(x,y) = 0$



What is special about this point?

- two curves tangent

* at P_0 level curves through P_0 and $G = 0$ *
have parallel Normals
- scalar multiple of the other

$\perp$ to level curve $\vec{\nabla w}$
$\perp$ to $g(x, y) = 0$ $\vec{\nabla g}$

P_0 is critical pt

$\vec{\nabla} w = \lambda \vec{\nabla} g$

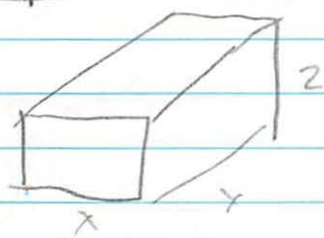
↑ some # (scalar)

$W_x = \lambda g_x$
$W_y = \lambda g_y$

) critical pt satisfies these equations for some value of λ

must now solve system in 3 variables
x, y, λ
↑ need to find, but then throw away

<u>example</u>



Same problem as befor

$\underset{min}{C} = 4xz + 2yz + 3xy$

$g: xyz - 3 = 0$

$$\left[\begin{array}{ll} C_x: 4z + 3y = \lambda yz & g = 0 \\ C_y: 2z + 3x = \lambda xz & \\ C_z: 4x + 2y = \lambda xy & \end{array}\right]$$

↑ don't forget

Don't hack
Think symetric
Stare at
Guided by form of equation

$\frac{4}{y} + \frac{3}{z} = \lambda$

$\frac{2}{x} + \frac{3}{z} = \lambda$ $\qquad g = 0$

$\frac{4}{y} + \frac{2}{x} = \lambda$

Set = to and solve $\qquad xyz = 3$

$\frac{4}{y} + \frac{3}{z} = \frac{2}{x} + \frac{3}{z} = \frac{4}{y} + \frac{2}{x}$

$\frac{4}{y} = \frac{2}{x} = \frac{3}{z}$

$\frac{24}{xyz} = 8$ ← where?

$\frac{4}{y} = 2 \qquad y = 2$

$\frac{3}{z} = 2 \qquad z = \frac{3}{2}$

$\frac{2}{x} = 2 \qquad x = 1$

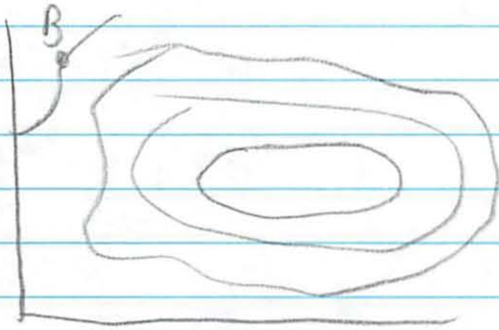
Recitation

3/8

Pset solutions - get

Mountain problem a lot of people missed

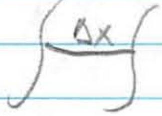
a)



$h(x, y)$

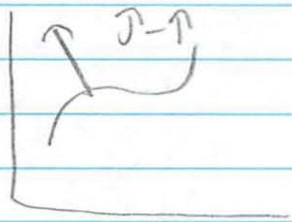
a) $h_x(B) = \frac{\partial h}{\partial x}(B) \approx \frac{\Delta h}{\Delta x}$

for small Δx (limit)



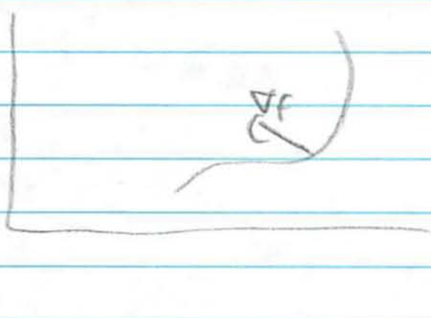
← parallel to x axis

b)



Same directional derivative

c) $h_x = 0$



∇ always perpendicular to level curve

$\nabla f = \langle 0, \_\_ \rangle$

↑ gradient is vertical
tangent is horizontal

all points inside tout

<u>Lectures</u> - optimization problems; min/max of $f(x, y)$

- without constraints
  - critical pts, etc
  - ex least ___ (could not read)
- w/ constraint $g(x, y) = 0$
  - ellimination or Lagrange multipliers

today
- do Past now
- then night for proposal
- german in spare time

ex) $f(x) = 2x^2 + y^4 - xy + 1$

a) What are the critical points?

b) Are they local mins, local maxes, or saddle pts?

a) When the 2 partial derivs each $= 0$

$$\frac{\partial}{\partial x} = 2 \cdot 2x - 1 \cdot y = f_x(x, y)$$

$$\frac{\partial}{\partial y} = 4y^3 - 1 \cdot x = f_y(x, y)$$

So set $=$ to 0

$$\begin{cases} 4x - y = 0 \\ 4y^3 - x = 0 \end{cases}$$

Solve system for x and y

Not linear - no matrix
Use substitution to get in terms of 1 equation

$$4y^3 - \frac{y}{4} = 0$$

Solve equation for y

$$\cancel{4y^3 = \frac{y}{4}}$$

$$\cancel{y^3 = \frac{y}{16}}$$

$$\cancel{y^4 = \frac{1}{16}}$$

also $y = 0$

$$yx - y = 0$$

$$y^2 - \frac{1}{16} = 0$$

$$y = \pm \frac{1}{4}$$

plug in to find x

$$(x, y) = (0, 0)$$
$$= \left(\frac{1}{16}, \frac{1}{4}\right)$$
$$\left(-\frac{1}{16}, -\frac{1}{4}\right)$$

These are the 3 posibilities

b) Which are max, min, or saddle?

Need the 2nd deriv test

$$\underset{A}{f_{xx}}\ \underset{C}{f_{yy}} - \underset{B}{f_{xy}}^2 > 0 \rightarrow f_{xx} (+) \text{ local min}$$
$$\rightarrow f_{xx} (-) \text{ local max}$$

$$f_{xx} f_{yy} - f_{xy}^2 < 0 \rightarrow \text{saddle pt}$$
$$= 0 \rightarrow \text{don't know}$$

often $= f_{yx}$

$f_{xx} = 4$ $\qquad f_{yy} = 12y^2$ $\qquad f_{xy} = -1$

$4 \cdot 12y^2 - (-1)^2$

$48y^2 - 1$

↳ at $(0,0)$ this is ⊖ so saddle pt

$(\frac{1}{16}, \frac{1}{4})$ ⊕ → f_{xx} ⊕ → minimum (local)

$(-\frac{1}{16}, -\frac{1}{4})$ ⊕ → f_{xx} ⊕ → local min

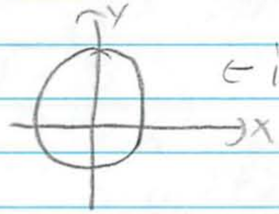
Ex2 Is a constraint

$f(x,y) = x^2 - y^2 - xy \qquad g(x,y) = x^2 + y^2 - 1$

What is the maximum of $f(x,y)$ subject to the constraint $g(x,y) = 0$

↑ needs to satisfy the equation

in this case it is on unit circle



← its somewhere on the line

2 methods

Method 1 - elimination

use constraint to get rid of 1 variable

$g(x,y) = 0 \longrightarrow y = \pm\sqrt{1-x^2}$

want to maximize

$h(x) = f(x,y) = x^2 - (1-x^2) \pm x\sqrt{1-x^2}$

± ↳ need to find max h⊕ min h⊖

Now eaiser to solve - only 1 variable
Sometimes this is easy - but not always

Method 2 - <u>Lagrange Multipliers</u>
-need to know this

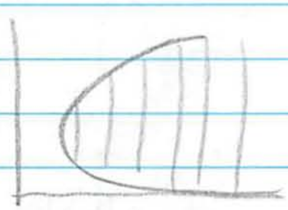
Solve $\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ g(x,y) = 0 \end{cases}$ ← don't forget constraint

Variables x, y, λ (3 equations w/ 3 unknowns ✓)

Here $\begin{cases} 2x - y = \lambda 2x \\ 2y - x = \lambda 2y \\ x^2 + y^2 = 1 \end{cases}$ ← did the partial derivitives

---

ex3 General situation → optimization problem



D = domain bounded by $g(x,y) = 0$
D: $g(x,y) > 0$

Find min + max of $\underset{f(x,y)}{f}$ in that $\underset{D}{\text{domain}}$

values of x and y constrained

Where can the maximum value be attained?
Either
- at critical pt inside domain D
- on the boundary $g(x,y) = 0$
  - must use ellimination / Lagrange multiplier to find

- can also find max nowhere (at ∞)
when $x \to \infty$ or $y \to \infty$
if ∞ is in the domain

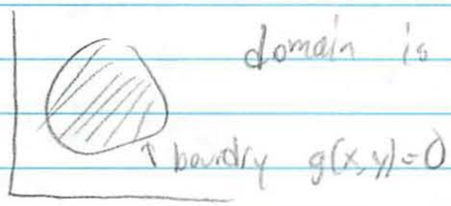
example

min of $f(x,y) = (x-1)^2 + y^2$

in domain $g(x,y) \leq 0$ (inside circle)

where $g(x,y) = x^2 + y^2 - 2$

there is no ∞



$\longrightarrow$ Look at these points + find max
↳ Compare the value of $f(x,y)$ for all of the above canidates

Lecture 14
Chain Rule

3/9

Chain rule
- important in functions that use several variables
- important in every subject
- may have wrong idea

Lvar
- 2 forms ; functions $f(g(x))$

$$\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$$

w/ respect to $g(x)$ considered as a variable

w/ respect to x

ex $\sin^2(\sqrt{2x+1})$

- variable form

$w = w(x)$ usually y

$x = x(t)$

$w(x(t))$

use libnet's form

$$\boxed{\frac{dw}{dt} = \frac{dw}{dx} \cdot \frac{dx}{dt}}$$

just cancle the xs (?right)

- theoriterical
- related rates

2 variables (n-variables same)

$w = w(x, y)$

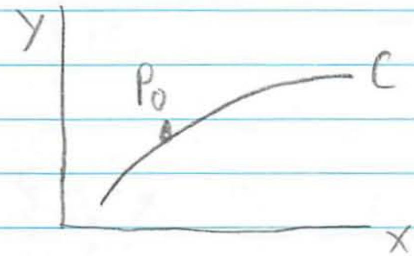
$x = x(t)$ } sub in and w becomes function of t

$y = y(t)$ } curve

$w = w(x(t), y(t))$

parametrized motion in a plane

$\frac{dw}{dt} = ?$



w = temp at (x, y)

What is being asked?

W is temp of ant at that pt at time t

What is rate of change of temp as ant walks? - chain rule gives you this

$\frac{dw}{dt} = \lim_{\Delta t \to 0} \frac{\Delta w}{\Delta t}$ ← new temp

all at P_0

calc Δw w/ approx formula

$\Delta w \approx \left(\frac{\partial w}{\partial x}\right)_0 \Delta x + \left(\frac{\partial w}{\partial y}\right)_0 \Delta y$

at Pt_0

Now / by Δt

$$\frac{\Delta w}{\Delta t} = \left(\frac{\partial w}{\partial x}\right)_0 \frac{\Delta x}{\Delta t} + \left(\frac{\partial w}{\partial y}\right)_0 \frac{\Delta y}{\Delta t}$$

$\downarrow$

$$\left(\frac{dw}{dt}\right)_0 = \left(\frac{\partial w}{\partial x}\right)_0 \left(\frac{dx}{dt}\right)_0 + \left(\frac{\partial w}{\partial y}\right)_0 \left(\frac{dy}{dt}\right) \quad \text{at } P_0$$

Can drop the P_0 to find general point
Get the chain rule

$$\boxed{\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} \qquad \text{chain rule (2 variables)}}$$

=

Now a form which does not specify # of variables

$$\boxed{\frac{dw}{dt} = \vec{\nabla} w \cdot \frac{d\vec{r}}{dt}}$$

$\vec{r} = \langle x(t), y(t) \rangle$

$\vec{v} = \frac{d\vec{r}}{dt}$

depends only on $w(x,y)$ — temp

the motion you are interested in $x(t), y(t)$ — motion

Works in nD (dimensions).

This is theoretical - not for actual calculation

Helix $\vec{r} = \langle a\cos t, a\sin t, t \rangle$

$w = x^2 + y^2$ ← temp function
$w(t) = (a\cos t)^2 + (a\sin t)^2$
$w(t) = a^2$
$\frac{dw}{dt} = 0$ easy, substitution, no need to do chain rule

$w = c$ cylinder in space
axis is on z-axis

As ant goes around path, its temp does not change, always C, so deriv = 0

Now with chain rule

$\frac{dw}{dt} = 2x(-a\sin t) + 2y(a\cos t)$

- can't abandon, need to simplify
- but how - still in terms of t
- sub in values for x, y, t for a pt
- otherwise sub in $x = x(t)$ $y = y(t)$

$= -2a\cos t\, a\sin t + 2a\sin t\, a\cos t$

$= 0$

↑ see how much more trouble
thats why theorotital

If can't remember 18.01

$$\frac{d}{dx} u(x) v(x)$$

$W = U \cdot V$
$U = U(x)$
$V = V(x)$

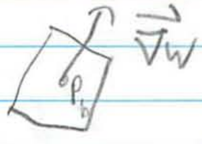
$$= V \cdot \frac{du}{dx} + U \cdot \frac{dv}{dx}$$ by the chain rule

3 functions

$$\frac{d}{dx}(uvw)$$

$$= vw \cdot \frac{du}{dx} + \_\_\_\_ + \_\_\_\_$$

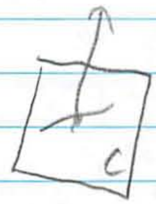
Works w/ n variables
will have n terms



$\uparrow$ level surface $w(x, y, z) = c$

For tangent vector of plane, take Normal vector to plane.

$(\vec{\nabla} w)_0 \perp$ surface at P_0



ie perpendicular to <u>any</u> curve on surface S through P_0

$C; \vec{r} = \langle x(t), y(t), z(t) \rangle$

lies on surface at $t=0$ at $P=0$

$\hookrightarrow w(x(t), y(t), z(t))$ when plug in must get back C

$\frac{dw}{dt} = 0$

constant - level surface, temp constant

<u>Now long version w/ chain rule</u>

$\frac{dw}{dt} = \vec{\nabla} w \cdot \frac{d\vec{r}}{dt} = 0$

<u>dot product</u>

Gradiant is perpendicular to tangent of curve

Functions that are functions of other variables

$w = w(x, y)$

$x = x(u, v)$

$y = w(u, v)$

$$w = w(x(u,v), y(u,v))$$

- What are the partial derivatives $\frac{\partial w}{\partial u}, \frac{\partial w}{\partial v}$

$\frac{\partial w}{\partial u}$ - hold v constant

$\frac{\partial w}{\partial u} = \frac{dw}{du}$ with v held constant)

Ok to use chain rule for 1 variable
chain rule from earlier today

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial u}$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial v}$$

↑ just another way of writing chainrule

Rect → Polar coords

$x = r\cos\theta$

$y = r\sin\theta$

$$w = \sqrt{x^2 + y^2}$$

$\frac{\partial w}{\partial r} = ?$ after changing to polar $= 1$ since $w = r$

$$\frac{\partial w}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} = \frac{x}{w}$$

$$\frac{\partial w}{\partial y} = \frac{y}{\sqrt{x^2+y^2}} = \frac{y}{w}$$

$$w = \sqrt{x^2+y^2}$$

$$\frac{\partial w}{\partial r} = \frac{x}{w} \cdot \cos\theta + \frac{y}{w} \cdot \sin\theta$$

Which $= 1$ since $w = r$
same hard to simplify
continue substitution

$$= \cos\theta \cdot \cos\theta + \sin\theta \cdot \sin\theta$$
$$= 1$$

Recitation

3/10

ex 1 Find the pt on the plane $x+2y+3z=7$ which is closest to origin

- optimization problem
- solve using La Grange

Need to minimize $f(x\ y z) \Rightarrow \sqrt{x^2+y^2+z^2}$
easier to do distance squared $\Rightarrow x^2+y^2+z^2$ function to be minimized

Constraint $g(x,y,z) = 0$
$\hookrightarrow g(x,y,z) = x+2y+3z-7$

Equations $\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ f_z = \lambda g_z \\ g = 0 \end{cases}$ ← don't forget

for variables x, y, z, λ

Compute partial derivatives

get it better now

$$\begin{cases} f_x = 2x = \lambda \\ f_y = 2y = \lambda 2 \\ f_z = 2z = \lambda 3 \\ x + 2y + 3z = 7 \end{cases}$$

Solve for x, y, z

$x = \frac{\lambda}{2}$
$y = \lambda$
$z = \frac{3}{2}\lambda$

$\frac{\lambda}{2} + 2\lambda + 3 \times \frac{3}{2}\lambda = 7$

$7\lambda = 7$

$\lambda = 1$

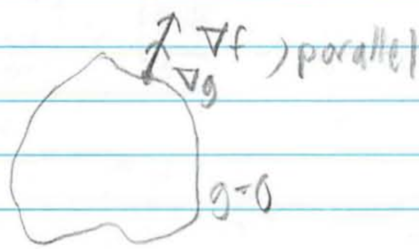
Now plug λ in

$x = \frac{\lambda}{2} = \frac{1}{2}$

$y = \lambda = 1$

$z = \frac{3}{2}\lambda = \frac{3}{2}$

Thus closet pt is $(x, y, z) = (\frac{1}{2}, 1, \frac{3}{2})$



Lecture review

$\frac{d}{dt} f(x(t), y(t)) = x(t) \frac{\partial f}{\partial x}(x, y) + y(t) \frac{\partial f}{\partial y}(x, y)$

$\frac{\partial}{\partial u} = w(x(u,v), y(u,v)) = \frac{\partial x}{\partial u}\frac{\partial w}{\partial x} + \frac{\partial y}{\partial u}\frac{\partial w}{\partial y}$

$= w_u = \frac{\partial x}{\partial u} w_x + \frac{\partial y}{\partial u} w_y$

Ex 2 $f(u,v) = u^v$

a) Find a general formula for $\frac{d}{dx} f\big(u(x), v(x)\big)$

$$\frac{d}{dx} f\big(u(x), v(x)\big) = u' \frac{\partial f}{\partial u} + v' \frac{\partial f}{\partial v}$$

plug in

$$\frac{\partial f}{\partial u} = \frac{\partial u^v}{\partial u} = u^c = c u^{c-1} = v u^{v-1}$$

c = constant

$$\frac{\partial f}{\partial v} = \frac{\partial u^v}{\partial v} = c^v = \frac{\partial}{\partial v} e^{(\ln u)v} =$$

$$= \ln u \, e^{(\ln u)v} = (\ln u) u^v$$

trick

$$\boxed{u^v = (e^{\ln u})^v = e^{(\ln u)v}}$$

know trick!

Thus $\frac{d}{dx} f\big(u(x), v(x)\big)$

$$\boxed{= u' v u^{v-1} + v' (\ln u) u^v}$$

b) $g(x) = x^{\sin x}$

$g'(x) = ?$

Plug in for these values

$$g(x) = x^{\sin x} = f(\underset{u(x)}{x}, \underset{v(x)}{\sin x})$$

recognize that it fits the pattern

calculate

$u = x \to u' = 1$
$v = \sin x \to v' = \cos x$

so $g'(x) = 1 x \sin x \cdot x^{\sin x - 1} + \cos x \ln x \cdot x^{\sin x}$

c) $h(x) = (\ln x)^x$ Plug in for these values
$h'(x) = ?$

$h(x) = f(\ln x, x)$

$u = \ln x \to u' = \frac{1}{x}$
$v = x \to v' = 1$

$h'(x) = \frac{1}{x} \cdot x \ln x^{x-1} + \ln(\ln(x))(\ln x)^x$

ex3 $x = u^2 - v^2$ $y = 2uv$
← a function of u and v

$w = f(x, y)$ ← a function of x and y

Show/Prove $(w_x)^2 + (w_y)^2 = \frac{(w_u)^2 + (w_v)^2}{4(u^2 + v^2)}$

$w_u = \frac{\partial x}{\partial u} w_x + \frac{\partial y}{\partial u} w_y$
$= 2u\, w_x + 2v\, w_y$

$w_v = \frac{\partial x}{\partial v} w_x + \frac{\partial y}{\partial v} w_y$
$= -2v\, w_x + 2u\, w_y$

Now put it in

$$W_u^2 + W_v^2 = (2uW_x + 2vW_y)^2 + (-2vW_x + 2uW_y)^2$$
$$= (4u^2 + 4v^2)W_x^2 + (4u^2 + 4v^2)W_y^2 + \cancel{(4uv - 4uv)}\, W_x W_y$$
$$= 4(u^2+v^2)(W_x^2 + W_y^2)$$

So proved now

$$\frac{W_u^2 + W_v^2}{4(u^2+v^2)} = W_x^2 + W_y^2$$

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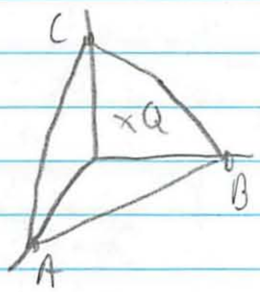
ex4 Optimization w/ Lagrange Multipliers

$Q = (1, 1, 2)$



Look at planes going through Q and crossing axis at point

$A = (a, 0, 0)$ with $a, b, c > 0$
$B = (0, b, 0)$
$C = (0, 0, c)$

I should realize this!

Find the minimum value of the tetrahedron OABC

Trying to optimize: volume
minimum

Variables are a, b, c

volume tetrahedron $= \dfrac{abc}{6}$ ← know this too!

Constraint: Q is in the plane ABC

(Thats setup - won't do today)

M'chael Plasmeier

$\boxed{30.5 / 35}$

18.02 Problem Set 4 due Thurs., Mar.11, 2-106

Part I (15 points)

Lecture 12. Thurs. Mar.4 Max-min problems; least squares approximation.
Read 19.7 to bottom p. 693; Notes LS Work: ~~2F-1b, 5; 2G-1c~~,4

Lecture 13. Fri. Mar.5 Second-derivative test; Lagrange multipliers.
Read: 19.7, pp.694-5; 19.8 (omit Example 3) Work: ~~2H-1ad; 2I-1a, 4a~~

Lecture 14. Tues. Mar.9 Chain rule.
Read 19.6 (omit Example 4) Work: ~~2E-1b, 2c, 3b, 5a, 7, 8a~~

Lecture 33. Thurs. Mar.11 Chain rule for non-independent vars. Read: Notes N.1-.3

Part II (20 points)

Directions. Try each problem alone for 15 minutes. If you subsequently collaborate, solutions must be written up independently. It is illegal to consult problem sets from previous semesters. With each problem is the day it can be done.

Problem 1. (Thurs. 3 pts: 1,2) (Problem 3b on Problem Set 2 done by another method)

Place a cube of edge length 1 in the corner of the first octant of xyz-space, so that one vertex is at the origin, and each of the three vertices adjacent to it lies on one of the coordinate axes.

As before, take two skew diagonals in two faces – the front-face diagonal containing $(1,0,0)$ and the right-side-face diagonal containing $(1,1,0)$; the problem is to find the length and position of the shortest line segment joining them.

a) Sketch the cube (use dashed lines for the coordinate axes, solid lines for the 12 cube edges). Write parametric equations for the two lines containing these two diagonals, using in each case the given point as the base point on the line. It will be necessary to use two different variables, t and u, as the parameters for the two lines.

b) Let $w(t,u)$ be the *square* of the length of a line segment AB joining a point A on the front diagonal with a point B on the side diagonal. Find the unique critical point (t_0, u_0) of the function $w(t,u)$, and from this determine the corresponding position and length of the minimal line segment A_0B_0. Draw it on your picture in dashed lines.

(Problem 3a below asks you to show the critical point is a minimum point.)

Problem 2. (Thurs. 7 pts: 1.5,1.5,3,1) Uses MatLab.

Before going to the terminal: Study pp. LS.1-3 and do the following:

a) Let $\mathbf{x} = [x_1\ x_2\ \dots\ x_n]$, $\mathbf{y} = [y_1\ y_2\ \dots\ y_n]$, $\mathbf{i} = [1\ 1\ \dots\ 1]$ (n $1's$) .
Let $Y = aX + b$ be the best-fitting line through the n points (x_i, y_i). Translate the equations (4) on p. LS.2 into a single 2×2 matrix equation $A\mathbf{z} = \mathbf{r}$, $\mathbf{z} = [a, b]'$.
Write the entries of A and $\mathbf{r}$ in terms of vector operations on $\mathbf{x}, \mathbf{y}$, and $\mathbf{i}$.
(Use dot products, not summation signs. For example, $\sum x_i = \mathbf{x} \cdot \mathbf{i}'$.)

At the terminal: Take with you the two-sided sheet "Directions for using MatLab" given out with Problem Set 3B (also posted on the 18.02 web page).

The cost of U.S. first class postage for an ordinary one ounce letter over the forty years from 1970 to 2010 is given by the following table, which shows for each postal rate the first year in which it appeared:

PC

$-\frac{6}{x^2} + 3y = 0$ $\frac{6}{x^2} = 3y$

$y = \frac{2}{x^2}$ got that

$-\frac{12}{y^2} + 3x = 0$

.08 (1971) .10 (1974) .13 (1975) .15 (1978) .20 (1981) .22 (1985) .25 (1988) .29 (1991)
.32 (1995) .33 (1999) .34 (2001) .37 (2002) .39 (2006) .41 (2007) .42 (2008) .44 (2009)

b) Using MatLab (or an equivalent program that gives printout, like Maple or Mathematica), make a scatter plot of this data (i.e., just plot the data points; put a * at each data point). To get a decent result, use as the scales: $x =$ number of years after 1970, $y =$ postage in cents.

c) In Matlab, enter A and $\mathbf{r}$, and calculate a and b, using part (a).

Then plot the line $Y = aX + b$, superimposed upon the scatter plot (use the "hold" command). To do this let $t = [0 : 1 : 45]$ and plot the function $a * t + b$. Note down the values of a and b. Get a printout and on it, write the equation of the line and label the two axes. On this printout, include the results of part (a).

d) What does the model predict for the postage rate in 2013, the year you graduate? Note it on the printout.

Problem 3. (Fri. 3 pts.: 1,2)

a) Verify using the second-derivative test that the critical point you found in Problem 1 is actually a minimum point for $w(t, u)$.

Dependent

b) Work 2H-2; you'll need to use the result in Exercise 1B-15, which you can accept without proof.

Problem 4. (Fri. 3 pts.: 1.5, 1.5)

A surface given by an equation $f(x, y, z) = c$ can be thought of as a level surface for the function $f(x, y, z)$.

Suppose the point $A : (a, b, c)$ is not on the surface, and we want to find the point $P_0 : (x_0, y_0, z_0)$ on the surface which is closest to A. The way to do this is (as in Problem 1 above) to find the point P_0 minimizing the *square* of the distance $w(x, y, z) = |AP|^2$, since this leads to easier computations than minimizing the distance $|AP|$ itself. (Here P is a general point on the surface.)

Suppose this minimum problem is solved by Lagrange multipliers.

a) Give the four equations of the Lagrange method.

b) Show they can be interpreted as saying that

the minimizing line segment AP_0 is perpendicular to the surface.

Problem 5. (Tues. 4 pts: 1, 1, 2)

Let $x = u \cos\phi - v \sin\phi, \quad y = u \sin\phi + v \cos\phi$.

a) Write this variable change in matrix form: $\langle x, y\rangle = \langle u, v\rangle\, M$, where M is a matrix.

b) The vectors $\mathbf{i}' = \langle 1, 0\rangle$ and $\mathbf{j}' = \langle 0, 1\rangle$ in the uv-coordinates correspond to what vectors in the xy-coordinates? Calculate and draw them in the xy-plane.

c) As part (b) suggests, the uv-axes are just the xy-axes rotated through the angle ϕ. Show using the chain rule that this change of variables does not change the length of the gradient vector. That is, if for any differentiable function $w(x, y)$ we calculate $\nabla w(x, y)$ in xy-coordinates, or make the above change of variables and calculate $\nabla w(x(u, v), y(u, v))$ in uv-coordinates, we get the same value for the length-squared of ∇w at every point P:

$$(w_u)^2 + (w_v)^2 = (w_x)^2 + (w_y)^2 .$$

(As we did in two previous problems, its suffices to work with the square of the length rather than the length itself.)

P-Set 4

Michael Plasmeier 3/9

Lecture 12 Min max problems, least squares approx

2F-1b Find the points on each of the following surfaces which is closet to the origin
- minimize the square of the distance not the distance itself

$$x^2 - yz = 1$$

Use the relation $x^2 = 1 + yz$ to eliminate x
- I guess just pick one

* Denote $x^2 + y^2 + z^2 = D$ as square of distance from point (x, y, z)

So $D = 1 + yz + y^2 + z^2$

$$\frac{\partial}{\partial y} = 0 + z + 2y + 0$$

$$\frac{\partial}{\partial z} = 0 + y + 0 + 2z$$

set $= 0$

$$z + 2y = 0$$
$$y + 2z = 0$$

solve - but get it in terms of 1 variable to make it easy.

$z + 2y = 0$

$y + 2z = 0$ → for both!

$z = -2y$ $\qquad$ $y = -2z$

plug 1 into other

$y = -2(-2y)$

$y = 4y$

now what solves that

$y = 0$

only solution

So find z $\qquad z = -2(0) = 0$

So find x $\qquad x^2 = 1 + (0)^2$

$x^2 = 1$

$x = \pm 1$

So 2 pts $(\pm 1, 0, 0)$ which are distance $1 + (0)(0) + (0)^2 + (0)^2 = 1$ from origin

get this better now

5. A drawer in a chest has open top bottom + back $\$1/ft^2$ sides $\$2/ft^2$ front $\$4/ft^2$ $V = 2.5\,ft^3$ Optimize.

$C = 1xz + 1xy + 2 \cdot 2yz + 4xy$

$xyz = 2.5$

Solve for 1: $z = \frac{2.5}{xy}$ and sub in

$$C = x \cdot \frac{2.5}{xy} + 5xy + 4y\frac{2.5}{xy}$$

$$= \frac{2.5}{y} + 5xy + \frac{10}{x}$$

$$\frac{\partial C}{\partial x} = 0 + 5y + -\frac{10}{x^2}$$

$$\frac{\partial C}{\partial y} = -\frac{2.5}{y^2} + 5x + 0$$

$$5y - \frac{10}{x^2} = 0 \qquad -\frac{2.5}{y^2} + 5x = 0$$

$$5y = \frac{10}{x^2} \qquad 5x = \frac{2.5}{y^2}$$

$$y = \frac{2}{x^2} \qquad x = \frac{1}{2y^2}$$

plug 1 into other

$$x = \frac{1}{2\left(\frac{2}{x^2}\right)^2} = \frac{1}{2\left(\frac{4}{x^4}\right)} = \frac{1}{\frac{8}{x^4}} = \frac{x^4}{8}$$

now what solves that?

$8x = x^4$

calc $x = 0$ or $x = 2$

↑reject, otherwise no box!

$$y = \frac{2}{(2)^2} = \frac{1}{2} \qquad z = \frac{2.5}{(2)\left(\frac{1}{2}\right)} = 2.5$$

$$\text{Cost} = \frac{2.5}{.5} + 5(2)\left(\tfrac{1}{2}\right) + \frac{10}{2} = 15$$

must have made math error - steps right (memorize)

26-1C Find by method of least squares the line which best fits the data points given

$(1,1)$ $(2,3)$ $(3,2)$

- trying to find general law connecting lines "line of best fit"

- want to minimize $D = \sum_{i=1}^{n} (m x_i + b - y_i)^2$ m, b rational

$(m + b - 1)^2 + (2m + b - 3)^2 + (3m + b - 2)^2$

Don't expand! Differentiate as is

$\frac{\partial D}{\partial m} = 2(m+b-1) \cdot 1 + 2(2m+b-3) \cdot 2 + 2(3m+b-2) \cdot 3 = 0$

$\frac{\partial D}{\partial b} = 2(m+b-1) \cdot 1 + 2(2m+b-3) + 2(3m+b-2) = 0$

~~always the same~~! no! copy error

lecture notes end!

$2m + 2b - 2 + 8m + 4b - 12 + 18m + 6b - 12 = 0$

I guess same — yeah

$28m + 16b = 26$ ← addition error

$14m + 8b = 13$

$2m + 2b - 2 + 2m + 2b - 2 + 2m + 2b - 2 = 0$

$6m + 3b = 6$

~~$2m + b = 1$~~

are kind fun even

$14m = 13 - 8b$

$14m = 13 - 8(2m - 2)$

$14m = 13 - 18m + 82$

$2\cancel{6}m = \cancel{21}$

$m = \cancel{3.5}$

$3b = \cancel{6m - 6}$ carried errors

$b = 2m - 2$

$b = 3.5 - 1 = 2.5$

~~$y = 3.5x + 2.5$~~ $\boxed{y = \frac{1}{2}x + 1}$

26-4 What linear equation in a, b, c does the method of least squares lead you to when you fit $z = a + bx + cy$ to (x_i, y_i, z_i), $i = 1, \dots, n$

So are trying to fit a line to a lot of points

I guess same $\sum_{i=1}^{n} (m x_i + b - y_i)^2 = 0$

what is this??

c w/ 3 variables

$$D = \sum (a + b x_i + c y_i - z_i)^2$$

so equations are

$$\frac{\partial D}{\partial a} = \sum 2 (a + b x_i + c y_i - z_i) = 0$$

$$\frac{\partial D}{\partial b} = \sum 2 x_i (a + b x_i + c y_i - z_i) = 0$$

$$\frac{\partial D}{\partial c} = \sum 2 y_i (a + b x_i + c y_i - z_i) = 0$$

2s cancle

like a matrix

$$n a + \left(\sum x_i\right) b + \left(\sum y_i\right) c = \sum z_i$$

$$\left(\sum x_i\right) a + \left(\sum x_i^2\right) b + \left(\sum x_i y_i\right) c = \sum x_i y_i$$

$$\left(\sum y_i\right) a + \left(\sum x_i y_i\right) b + \left(\sum y_i^2\right) c = \sum y_i z_i$$

would have never found this on own!

Lecture 13 Second Derivitive Test, La Grange multipliers

2H - 1a Find critical pts and tell if min or max

$$x^2 - xy - 2y^2 - 3x + 3y + 1 = W$$

So at critical pt when partial derivs each $= 0$

$$\frac{\partial}{\partial x} = 2x - y - 0 - 3 + 0 + 0 = 0$$

$$\frac{\partial}{\partial y} = 0 - x - 2 \cdot 2y - 0 + 3 + 0$$

$$2x - y - 3 = 0 \qquad -4y - x + 3 = 0$$

$$2x - y = 3 \qquad -4y - x = -3$$

$$y = -3 + 2x \qquad x = 3 - 4y$$

$$x = 3 - 4(-3 + 2x)$$

$$x = 3 + 12 - 8x$$

$$x = 15 - 8x$$

$$9x = 15$$

$$x = 5/3 \qquad y = -3 + 2(5/3) = 1/3$$

$(5/3, 1/3)$ is possibility

$$f_{xx} = A = \frac{\partial}{\partial x}(2x - y - 3)$$

$$= 2 - 0 - 0$$

$$= 2$$

$$f_{xy} = B = \frac{\partial}{\partial x}(-4y - x + 3)$$

$$= 0 - 1 + 0$$

$$= -1$$

$f_{yy} = C = \frac{\partial}{\partial y}(-4y + x + 3)$

$= -4 - 0 + 0$

$= -4$ ✓

$AC - B^2$

$2 \cdot (-4) - (-1)^2$

$-8 - 1 = -9$ ⊖ so saddle pt ✓ Correct!

d. $x^3 - 3xy + y^3$

$\frac{\partial}{\partial x} = 3x^2 - 3y + 0$ $\frac{\partial}{\partial y} = 0 - 3x + 3y^2$

$3x^2 - 3y = 0$ $3y^2 - 3x = 0$

$3x^2 = 3y$ $3y^2 = 3x$

$y = x^2$ $\longrightarrow$ $x = y^2$

$x = (x^2)^2 = x^4$

don't need ~~$y = (x^4)^2 = x^8$~~

calc solve

$x = 0$ $x = 1$

$y = (0)^2 = 0$ $y = (1)^2 = 1$

A) $f_{xx} = 6x$ B) $f_{xy} = 0 - 3 + 0$ C) $f_{yy} = 6y$

$AC - B^2$

$6(0) \cdot 6(0) - (-3)^2$ $6(1) \cdot 6(1) - (-3)^2$

-3 33 ⊕

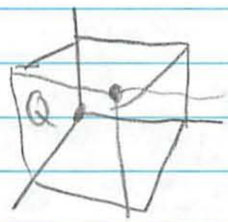
⊖ test $f_{xx} = 6(1)$ ⊕

saddle ✓ local min ✓

Lagrange

2I-1a A rectangular box in first octant

For what point P will the box have the largest volume



$f(x,y,z)=C$

plane $= x+2y+3z=18$

$\uparrow g(x,y,z)$ defines a surface

Solve system x,y,z,λ

These topics were presented clearly

$C_x = 1 = \lambda yz$

$C_y = 2 = \lambda xz \quad g=18$

$C_z = 3 = \lambda xy$

$\uparrow$ don't forget!

$xz = \lambda$

$xz = 2\lambda$

$xy = 3\lambda$

$\lambda x = 2\lambda y = 3\lambda z$

$x = 2y = 3z = 6$

sum 18

$x=6$

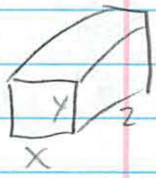
$y=3$

$z=2$

max pt - since if P lies on triangular boundary of the region over the 1st octant which it varries volume $=0$

- so Lagrange is alt method we need to know

4a Drawers open top 2 Sides + back \$2/ft^2
bottom \$1/ft^2 front \$4/ft^2
Largest w/ \$72



$$72 = 2xy + 2\cdot 2yz + 4xy + 1\cdot xz$$
$$72 = xz + 6xy + 4yz$$

$xyz = V$

$$C_x = 2 + 6y + 0 = \lambda yz$$
$$C_y = 0 + 6x + 4z = \lambda xz$$
$$C_z = x + 0 + 4y = \lambda xy$$

$C = 72$
right

now solve for λ for each

$$\lambda = \frac{z}{yz} + \frac{6y}{yz} = \lambda = \frac{6\cancel{x}}{\cancel{x}z} + \frac{4z}{x\cancel{z}} = \lambda = \frac{\cancel{x}}{\cancel{x}y} + \frac{4\cancel{y}}{x\cancel{y}}$$

they did not divide $\quad xy + 6xz = xy + 4yz = 4yz + 6xz = 72$

$$\lambda = \frac{1}{y} + \frac{6}{z} = \frac{6}{z} + \frac{4}{x} = \frac{1}{y} + \frac{4}{x} = g = 72$$

Solve

?? ask about

$$\frac{1}{y} = \frac{6}{z} = \frac{4}{x} = 72 \qquad = xy = 4yz = 6xz$$

$$\frac{24}{xyz} = 72 \rightarrow 24 = xyz\,72$$

by hypothesis
$xy = 24$ $\quad x = 4z$
$yz = 6$ $\quad y = 6z$
$xz = 4$

their letters may be diff

I don't really get what happened.

$x = 4$
$y = 6$
$z = 1$

Lecture 14 Chain Rule

2E-1b Find $\frac{df}{dt}$ for the composite function $f(x(t), y(t), z(t))$

$w = x^2 - y^2 \quad x = \cos t \quad y = \sin t$

i) Use the chain rule, express ans in terms of t by using $x = x(t)$

ii) express f in terms of t differenitate

Sort out what we should do b/c he said a lot was theoritical

i) $\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$

$= 2x(-\sin t) - 2y(\cos t)$

new function   deriv of what x is

$= -4 \sin t \cos t$

simplify (no x and y) why/how? * sub in values of a point

actually plug in

ii) $w = x^2 - y^2 = \cos^2 t - \sin^2 t$

$= \cos 2t$

now differenitate

$\frac{dw}{dt} = -2\sin 2t$

2c Function unknown $f(x,y)$
Gradiant is given

$$\nabla f = \langle 1, -1, 2 \rangle \text{ at } (1,1,1)$$
$$x = t \quad y = t^2 \quad z = t^3$$

Find $\frac{df}{dt}$ at $t=1$

no clue - like just doing a pattern

Use chain rule to find additional info about
$w = f(x(t), y(t))$ w/o finding f

$t = 1 \quad (x(1), y(1), z(1)) = (1,1,1)$

$$\left.\frac{df}{dt}\right|_1 = 1 \cdot \left.\frac{dx}{dt}\right|_1 - 1 \cdot \left.\frac{dy}{dt}\right|_1 + 2 \left.\frac{dz}{dt}\right|_1$$
$$= 1 \cdot 1 - 1 \cdot 2 + 2 \cdot 3$$
$$= 5$$

?

3b Using the chain rule for $f(u, v, w)$ derive a similar product rule for $\frac{d}{dt}(uvw)$ and use it to differentiate $te^{2t}\sin t$

~~?? have no clue~~

$$\frac{d(uvw)}{dt} = vw\frac{du}{dt} + uw\frac{dv}{dt}\ uv\frac{dw}{dt}$$

$$e^{2t}\sin t + 2te^{2t}\sin t + te^{2t}\cos t$$

~~? ? don't get all~~

Have more clue after recitation now

$u = t$

$v = e^{2t}$ then just apply formula

$w = \sin t$

and it will apply

6a. Let $w = f(x,y)$ change from rectangular $\to$ polar coords
$x = r\cos\theta$
$y = r\sin\theta$

Show that $(w_x)^2 + (w_y)^2 = (w_r)^2 + \frac{1}{r^2}(w_\theta)^2$

Well remember $r = \sqrt{x^2+y^2}$
$\theta = \tan^{-1}\left(\frac{y}{x}\right)$

¿But how to show?

①
$$w_r = w_x x_r + w_y y_r$$
$$= w_x \cos\theta + w_y \sin\theta$$
$$w_\theta = w_x x_\theta + w_y y_\theta$$
$$= w_x(-r\sin\theta) + w_y(r\cos\theta)$$

②
So $(w_r)^2 + \left(\frac{w_\theta}{r}\right)^2$
$$= (w_x)^2(\cos^2\theta + \sin^2\theta) + (w_y)^2(\sin^2\theta + \cos^2\theta) + 2w_x w_y \cos\theta\sin\theta - 2w_x w_y \sin\theta\cos\theta$$
$$= (w_x)^2 + (w_y)^2$$

Should watch Arnoux

So what does that have to do w/ chain rule
- like what did in recitation
  function of 2 sub function
- chain rule ①
- then plug it in and after lots of algebra it ends up right ②

Put in matrix form the chain rule

Lecture $\vec{\nabla} f \cdot$ vector

↑

Jacobian matrix

Oliver OH $\nabla f\big(x(u,v), y(u,v)\big)$

take gradiant

$\langle \frac{\partial}{\partial u}(\quad), \frac{\partial}{\partial v}(\quad) \rangle$

then write in matrix form

$$\frac{\partial x}{\partial u} \cdot \frac{\partial f}{\partial x} + \frac{\partial y}{\partial u} \frac{\partial f}{\partial y} = x_u f_x + y_u f_y$$

all 4 are partial derivs

$$\frac{d}{dt} f\big(x(t), y(t)\big)$$

$$= x_t^* f_x + y_t + f_y \qquad x_t = x'$$

$$\langle x_u f_x + y_u f_y, x_v f_x + y_v f_y \rangle$$

Put in matrix form

$$\langle f_x, f_y \rangle \begin{bmatrix} x_u & y_u \\ x_v & y_v \end{bmatrix}$$ translation

7. The Jacobian Maxtrix for change in variables

$x = x(u,v)$
$y = y(u,v)$

defined to be $J = \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix}$

Let $\nabla f(x,y)$ represent row vector $\langle f_x, f_y \rangle$

Show that $\nabla f(x(u,v), y(u,v)) = \nabla f(x,y) \cdot J$

matrix multiplication?

$$\frac{dw}{dt} = \vec{\nabla} w \cdot \frac{d\vec{r}}{dt} = 0$$

gradiant is perpendicular to tangent of curve

$$f_u = f_x x_u + f_y y_u$$

$$f_v = f_x x_v + f_y y_v$$

another function of something else

$$\langle f_u, f_v \rangle = \underbrace{\langle f_x f_y \rangle}_{d\vec{r}} \underbrace{\begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix}}_{?\ \text{something}}$$

Wikipedia: Matrix of all 1st order partial derivs of vector function

$$J = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial y_m}{\partial x_1} & \cdots & \frac{\partial y_m}{\partial x_n} \end{bmatrix}$$

8a Let $w = f\left(\frac{y}{x}\right)$ ie w is composite $w = f(v)$ $v = \frac{y}{x}$

Show that w satisfies the partial differential equation

$$x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = 0$$

Well isn't it

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$

? so integrate

$$\int\frac{dw}{dt} = \frac{\partial w}{\partial x}\int\frac{dx}{dt} + \frac{\partial w}{\partial y}\int\frac{dy}{dt}$$

* never integrate like this

$$w = \frac{\partial w}{\partial x}x + \frac{\partial w}{\partial y}y$$

? but how 0 ? and why is this interesting?

$$\frac{\partial w}{\partial x} = f'(v)\cdot\frac{\partial v}{\partial x}$$

$$= f'(v)\cdot\frac{-y}{x^2}$$

actually do it

$$\frac{\partial w}{\partial y} = f'(v)\cdot\frac{\partial v}{\partial y}$$

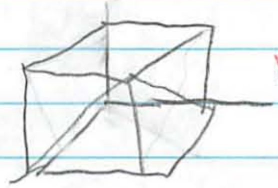
$$= f'(v)\cdot\frac{1}{x}$$

how does it change

? how does it change

$$\text{So it} = f'(v)\cdot\frac{-y}{x} + f'(v)\frac{y}{x} = 0$$

Part 2 1. Place cube of lenght 1



Find the lenght + position of shortest line segment joining them

a) Write parametric equations for 2 diagonals t and u

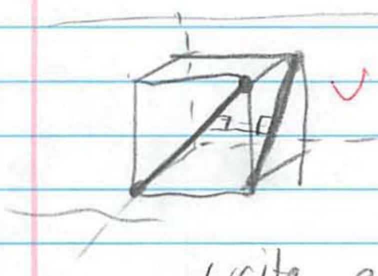
→ But can't I write regular equations ?

And don't the lines touch? so is no 2 lines bewtween the 2 lines, or do they want the 2 given pts

but in general $D = \sqrt{x^2+y^2+z^2}$

but $D^2 = x^2 + y^2 + z^2$ ← easier

O+l



Do as optimisation problem - did it before

retry afterwards

write as parameter

So from before $\overrightarrow{P_1P_2} = \langle 0, 1, 1 \rangle$

$\overrightarrow{Q_1Q_2} = \langle -1, 0, 1 \rangle$

and then find perp to find line

but this needs <u>parametrizing</u>

Jay (last semester) student

$\langle 1, 0, 0 \rangle + t \langle 0, 1, 1 \rangle$

so in t seconds it will have moved $\langle 1, t, t \rangle$

$\langle 1, 1, 0 \rangle + u \langle -1, 0, 1 \rangle$

^how did I overlook mistake so often!

so $d = \sqrt{(1 + 0t - (1 + -1u))^2 + (0 + 1t - (1 - 0u))^2 + (0 + 1t - (0 + 1 \cdot u))^2}$

On own

*remember $(a+b)^2 \neq a^2 + b^2$

$d = \sqrt{(0 + u)^2 + (t - 1)^2 + (t - u)^2}$

$d = \sqrt{u^2 + t^2 - 2t + 1 + t^2 - 2ut + u^2}$

$w(t, u) = d^2 = 2t^2 - 2ut + 2u^2 - 2t + 1$

What is the constraint?

-0.5

$t_0 = \langle 0, t, t \rangle$

$u_0 = \langle 0, -u, u \rangle$ ← are these constraints?

you use the right parametrization above, but give the wrong one here!?

Be more careful!

key *Note this as a thur problem - not a chain rule!

$\frac{\partial}{\partial t} = 4t - 2u + 0 - 2 + 0$

$\frac{\partial}{\partial u} = 0 - 2t + 4u - 0 + 0$

$4t - 2u - 2 = 0$ $\qquad$ $4u - 2t = 0$

$t = \frac{2u + 2}{4} = \frac{u + 1}{2}$ $\qquad$ $u = \frac{2t}{4} = \frac{1}{2} t$

plug 1 into other

$$t = \frac{\left(\frac{1}{2}t\right)+1}{2}$$

$$t = \frac{1}{4}t + \frac{1}{2}$$

$t = 2/3$ $\qquad$ $u = \frac{1}{2}t = \frac{1}{3}$

at pt

$\langle 1, 0, 0 \rangle + \frac{2}{3}\langle 0, 1, 1 \rangle$

pts $\langle 1, \frac{2}{3}, \frac{2}{3} \rangle$

$\langle 1, 1, 0 \rangle + \frac{1}{3}\langle -1, 0, 1 \rangle$

pts $\langle \frac{2}{3}, 1, \frac{1}{3} \rangle$

$$D = \sqrt{2\left(\frac{2}{3}\right)^2 - 2\left(\frac{2}{3}\right)\left(\frac{1}{3}\right) + 2\left(\frac{1}{3}\right)^2 - 2\left(\frac{2}{3}\right) + 1}$$

$$D = \sqrt{1/3}$$

$$D = \frac{\sqrt{3}}{3}$$ ✓

Dir $\langle 1 - \frac{2}{3}, \frac{2}{3} - 1, \frac{2}{3} - \frac{1}{3} \rangle$

$\langle \frac{1}{3}, -\frac{1}{3}, \frac{1}{3} \rangle$

I hope this is it - if it is I am glad I cracked this one

- Be able to know what problems need what

2. Uses MatLab, but before: (read Least Square Interpolation notes)

a) Let $\vec{x} = [x_1 \ x_2 \ \dots \ x_n]$
$\vec{y} = [y_1 \ y_2 \ \dots \ y_n]$
$\vec{1} = [1, 1, 1, 1 \dots 1]$

$y = ax + b$ is best fit line through n points (x_i, y_i)

Translate $(\sum x_i) a + nb = \sum y_i$
into a single 2x2 matrix $Az = r$
$z = [a, b]'$

Write entries of A and r interms of $\hat{x} \ \hat{y} \ \hat{1}$
(use dot products not sum signs
$\sum x_i = \hat{x} \cdot \hat{1}'$)

$A = \hat{x} \cdot \hat{1}'(a + nb) = \hat{y} \cdot \hat{1}'$ ← transpose

$\hat{x}\,(a\hat{1}' + nb\hat{1}') = \hat{y}\,\hat{1}'$

$1' = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$

$A \cdot [a, b]' = \begin{bmatrix} x \\ y \end{bmatrix}$

$\begin{pmatrix} x & x \\ y & y \end{pmatrix}$ or jacobian $\begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \dots & \frac{\partial y_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial y_m}{\partial x_1} & \dots & \frac{\partial y_m}{\partial x_n} \end{bmatrix}$

base thinking off lecture notes

have 2 eq w/ 2 unknowns a, b
write as matrix

$$\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \\ \\ \end{bmatrix}$$ ← each line gives equation

vector   vector

Coefficents
are sums
write as dot product
like $x \cdot \vec{1}'$
make it work w/ prime or not (means transpose)

? I really don't get this

Saher's help

Remember dot product

(1) $\begin{bmatrix} a \cdot b \end{bmatrix}_2 \begin{bmatrix} c \\ d \end{bmatrix}_2 = ac + bd$

(1)(1)

$$\begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = x_1 + x_2 + x_3 + x_4$$

so $A\begin{bmatrix} a \\ b \end{bmatrix} = \vec{r}$

* equation 4
are ~~2 seperate things~~ ~~X4~~

$$\left(\sum x_i^2\right) a + \left(\sum x_i\right) b = \sum x_i y_i$$

$$\left(\sum x_i\right) a + nb = \sum y_i$$

$$\underbrace{\begin{bmatrix} \sum (x_i)^2 & \sum x_i \\ \sum x_i & n \end{bmatrix}}_{A} \cdot \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{z} = \underbrace{\begin{bmatrix} \sum x_i y_i \\ \sum y_i \end{bmatrix}}_{\vec{r}}$$

$$\begin{bmatrix} x \cdot x' & x \cdot i' \\ x \cdot i' & n \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} x \cdot y' \\ y \cdot i' \end{bmatrix}$$

makes more sense now

At computer

Cost of post stamp for 1 oz envelope is given on chart

b) Use Matlab to make a scatter plot
(see page

c) Enter A and r
- but what is n?
↳ # of columns 2

Plot $Y = aX + b$
$t = [0:1:45]$
plot $at + b$

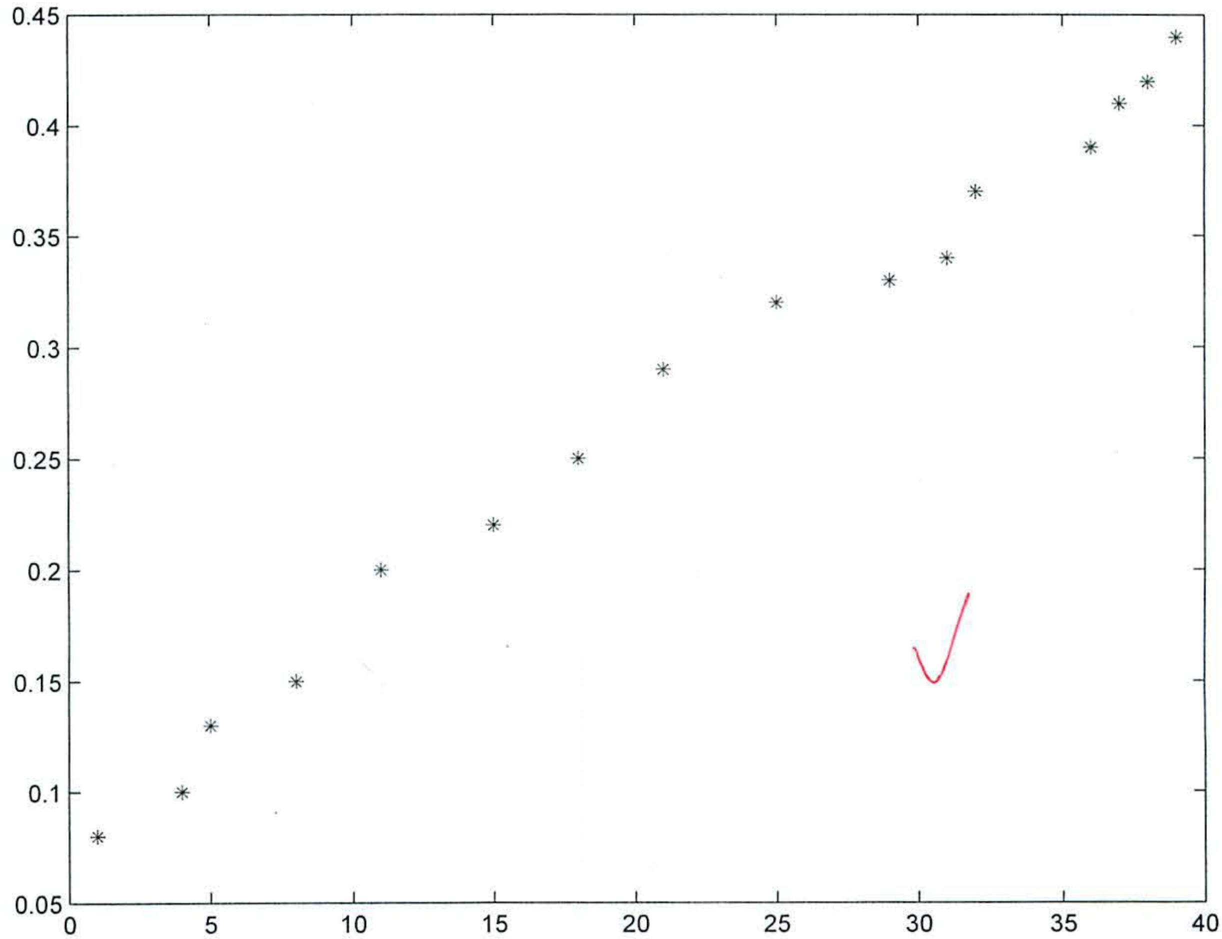
* # of row in first = # columns in second

* Recalculate a and b using above

* n is # of points = 16 (not 39)
* must enter i = [ ] or it will calculate

Still won't calculate, I give up, spent 30 min

-3


0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
5
10
15
20
25
30
35
40

```
>> a=[1 4 5 8 11 15 18 21 25 29 31 32 36 37 38 39];
>> b=[.08 .1 .13 .15 .20 .22 .25 .29 .32 .33 .34 .37 .39 .41 .42 .44];
>> plot(x,y)
??? Undefined function or variable 'x'.

>> plot(a,b)
>> plot(a,b'*')
??? plot(a,b'*')
            |
Error: A MATLAB string constant is not terminated properly.

>> plot(a,b,'*')
>> A=[x*x' x*i' x*i' n]
??? Undefined function or variable 'x'.

>> x=a

x =

     1     4     5     8    11    15    18    21    25    29    31    32    36    37    38 ↙
39

>> y=b

y =

  Columns 1 through 11

    0.0800    0.1000    0.1300    0.1500    0.2000    0.2200    0.2500    0.2900    0.3200 ↙
0.3300    0.3400

  Columns 12 through 16

    0.3700    0.3900    0.4100    0.4200    0.4400

>> A=[x*x' x*i' x*i' n]
??? Undefined function or variable 'n'.

>> A=[x*x' x*i' x*i' 2]

A =

  1.0e+004 *

  Columns 1 through 6

   1.0298                  0 - 0.0001i          0 - 0.0004i          0 - 0.0005i          0 - ↙
0.0008i          0 - 0.0011i

  Columns 7 through 12

        0 - 0.0015i          0 - 0.0018i          0 - 0.0021i          0 - 0.0025i          0 - ↙
```

Could not get what did I do wrong?

```
0.0029i          0 - 0.0031i

  Columns 13 through 18

        0 - 0.0032i        0 - 0.0036i        0 - 0.0037i        0 - 0.0038i        0 -↙
0.0039i          0 - 0.0001i

  Columns 19 through 24

        0 - 0.0004i        0 - 0.0005i        0 - 0.0008i        0 - 0.0011i        0 -↙
0.0015i          0 - 0.0018i

  Columns 25 through 30

        0 - 0.0021i        0 - 0.0025i        0 - 0.0029i        0 - 0.0031i        0 -↙
0.0032i          0 - 0.0036i

  Columns 31 through 34

        0 - 0.0037i        0 - 0.0038i        0 - 0.0039i   0.0002

>> r=[x*y' y*i']

r =

  1.0e+002 *

  Columns 1 through 5

   1.2070                  0 - 0.0008i        0 - 0.0010i        0 - 0.0013i        0 -↙
0.0015i

  Columns 6 through 10

        0 - 0.0020i        0 - 0.0022i        0 - 0.0025i        0 - 0.0029i        0 -↙
0.0032i

  Columns 11 through 15

        0 - 0.0033i        0 - 0.0034i        0 - 0.0037i        0 - 0.0039i        0 -↙
0.0041i

  Columns 16 through 17

        0 - 0.0042i        0 - 0.0044i

>> hold
Current plot held
>> t=[0:1:45]

t =
```

```
  Columns 1 through 18

     0     1     2     3     4     5     6     7     8     9    10    11    12    13    14 ↙
15    16    17

  Columns 19 through 36

    18    19    20    21    22    23    24    25    26    27    28    29    30    31    32 ↙
33    34    35

  Columns 37 through 46

    36    37    38    39    40    41    42    43    44    45

>> plot (a*t+b)
??? Error using ==> mtimes
Inner matrix dimensions must agree.

>> plot(a*t+b)
??? Error using ==> mtimes
Inner matrix dimensions must agree.

>> a*b
??? Error using ==> mtimes
Inner matrix dimensions must agree.

>> a*t
??? Error using ==> mtimes
Inner matrix dimensions must agree.

>> a.*t
??? Error using ==> times
Matrix dimensions must agree.

>> a.*t.
??? a.*t.
        |
Error: Expression or statement is incomplete or incorrect.

>> A*t
??? Error using ==> mtimes
Inner matrix dimensions must agree.

>> a.*t
??? Error using ==> times
Matrix dimensions must agree.

>> a

a =

     1     4     5     8    11    15    18    21    25    29    31    32    36    37    38 ↙
```

```
39

>> a = [1    4    5    8    11    15    18    21    25    29    31    32    36    37↙
38    39 45]

a =

     1     4     5     8    11    15    18    21    25    29    31    32    36    37    38↙
39    45

>> a.*t
??? Error using ==> times
Matrix dimensions must agree.

>> A.*r
??? Error using ==> times
Matrix dimensions must agree.

>> A=[x*x' x*i' x*i' 2]

A =

  1.0e+004 *

  Columns 1 through 5

   1.0298                  0 - 0.0001i        0 - 0.0004i        0 - 0.0005i        0 -↙
0.0008i

  Columns 6 through 10

        0 - 0.0011i        0 - 0.0015i        0 - 0.0018i        0 - 0.0021i        0 -↙
0.0025i

  Columns 11 through 15

        0 - 0.0029i        0 - 0.0031i        0 - 0.0032i        0 - 0.0036i        0 -↙
0.0037i

  Columns 16 through 20

        0 - 0.0038i        0 - 0.0039i        0 - 0.0001i        0 - 0.0004i        0 -↙
0.0005i

  Columns 21 through 25

        0 - 0.0008i        0 - 0.0011i        0 - 0.0015i        0 - 0.0018i        0 -↙
0.0021i

  Columns 26 through 30

        0 - 0.0025i        0 - 0.0029i        0 - 0.0031i        0 - 0.0032i        0 -↙
```

```
0.0036i

  Columns 31 through 34

        0 - 0.0037i          0 - 0.0038i          0 - 0.0039i   0.0002

>> r=[x*y' y*i']

r =

  1.0e+002 *

  Columns 1 through 5

   1.2070                    0 - 0.0008i          0 - 0.0010i          0 - 0.0013i          0 - ↙
0.0015i

  Columns 6 through 10

        0 - 0.0020i          0 - 0.0022i          0 - 0.0025i          0 - 0.0029i          0 - ↙
0.0032i

  Columns 11 through 15

        0 - 0.0033i          0 - 0.0034i          0 - 0.0037i          0 - 0.0039i          0 - ↙
0.0041i

  Columns 16 through 17

        0 - 0.0042i          0 - 0.0044i

>> A.*[a b]
??? Error using ==> times
Matrix dimensions must agree.

>> A.*[a b]'
??? Error using ==> times
Matrix dimensions must agree.

>> A

A =

  1.0e+004 *

  Columns 1 through 5

   1.0298                    0 - 0.0001i          0 - 0.0004i          0 - 0.0005i          0 - ↙
0.0008i

  Columns 6 through 10
```

```
        0 - 0.0011i        0 - 0.0015i        0 - 0.0018i        0 - 0.0021i        0 -
0.0025i

  Columns 11 through 15

        0 - 0.0029i        0 - 0.0031i        0 - 0.0032i        0 - 0.0036i        0 -
0.0037i

  Columns 16 through 20

        0 - 0.0038i        0 - 0.0039i        0 - 0.0001i        0 - 0.0004i        0 -
0.0005i

  Columns 21 through 25

        0 - 0.0008i        0 - 0.0011i        0 - 0.0015i        0 - 0.0018i        0 -
0.0021i

  Columns 26 through 30

        0 - 0.0025i        0 - 0.0029i        0 - 0.0031i        0 - 0.0032i        0 -
0.0036i

  Columns 31 through 34

        0 - 0.0037i        0 - 0.0038i        0 - 0.0039i   0.0002

>> A=[x*x' x*i' x*i' 2]

A =

  1.0e+004 *

  Columns 1 through 5

   1.0298                  0 - 0.0001i        0 - 0.0004i        0 - 0.0005i        0 -
0.0008i

  Columns 6 through 10

        0 - 0.0011i        0 - 0.0015i        0 - 0.0018i        0 - 0.0021i        0 -
0.0025i

  Columns 11 through 15

        0 - 0.0029i        0 - 0.0031i        0 - 0.0032i        0 - 0.0036i        0 -
0.0037i

  Columns 16 through 20

        0 - 0.0038i        0 - 0.0039i        0 - 0.0001i        0 - 0.0004i        0 -
0.0005i
```

```
  Columns 21 through 25

        0 - 0.0008i        0 - 0.0011i        0 - 0.0015i        0 - 0.0018i        0 - ↙
0.0021i

  Columns 26 through 30

        0 - 0.0025i        0 - 0.0029i        0 - 0.0031i        0 - 0.0032i        0 - ↙
0.0036i

  Columns 31 through 34

        0 - 0.0037i        0 - 0.0038i        0 - 0.0039i   0.0002

>> i = [1 1]

i =

     1     1

>> A=[x*x' x*i' x*i' 2]
??? Error using ==> mtimes
Inner matrix dimensions must agree.

>> n=39

n =

    39

>> i=[1:1:39]

i =

  Columns 1 through 18

     1     2     3     4     5     6     7     8     9    10    11    12    13    14    15 ↙
16    17    18

  Columns 19 through 36

    19    20    21    22    23    24    25    26    27    28    29    30    31    32    33 ↙
34    35    36

  Columns 37 through 39

    37    38    39

>> i[1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1]
??? i[1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1]
     |
```

```
Error: Unbalanced or unexpected parenthesis or bracket.

>> i=[1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1]

i =

  Columns 1 through 18

     1     1     1     1     1     1     1     1     1     1     1     1     1     1     1
1     1     1

  Columns 19 through 24

     1     1     1     1     1     1

>> i=[1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1]

i =

  Columns 1 through 18

     1     1     1     1     1     1     1     1     1     1     1     1     1     1     1
1     1     1

  Columns 19 through 36

     1     1     1     1     1     1     1     1     1     1     1     1     1     1     1
1     1     1

  Columns 37 through 40

     1     1     1     1

>> i=[1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 ]

i =

  Columns 1 through 18

     1     1     1     1     1     1     1     1     1     1     1     1     1     1     1
1     1     1

  Columns 19 through 36

     1     1     1     1     1     1     1     1     1     1     1     1     1     1     1
1     1     1

  Columns 37 through 39

     1     1     1

>> A=[x*x' x*i' x*i' n]
```

```
??? Error using ==> mtimes
Inner matrix dimensions must agree.

>> n=16

n =

    16

>> i=[1 1 1 1 1 1 1 1 1 1 1 1 1]

i =

     1     1     1     1     1     1     1     1     1     1     1     1     1

>> A=[x*x' x*i' x*i' n]
??? Error using ==> mtimes
Inner matrix dimensions must agree.

>>
```

3. Verify using 2nd deriv test that the critical pt from Problem 1 is actually a min for $w(t,v)$

The point was
$w\left(\frac{2}{3}, \frac{1}{3}\right)$

So basically do $AC - B^2$

that was easy
- memorize steps
* know when to use what

$A = W_{tt}$ | $B = W_{tv}$ | $C = W_{vv}$
$4 - 0 - 0$ | $0 - 2 - 0$ | $-0 + 4 - 0$
4 ✓ | -2 ✓ | 4 ✓

$4 \cdot 4 - (-2)^2$

$+$

↓

A is $+$ so min ✓

3b Do 2H2 → Use 2nd deriv criteria to verify (m_0, b_0) which is least sq of $y = m_0 x + b_0$ to minimize $D(m,b)$

Office Hrs

Will need inequality $A = \langle a_1, a_2, \dots a_n \rangle$

define $|A| = \sqrt{\sum a_i^2}$ $A \cdot B = \sum a_i b_i$

$$\sum_{i=1}^{m} (y_i - m_0 x_i - b_0)^2 = f(m_0, b_0)$$

- take deriv
- 0 for certain values
- 2nd deriv tells min

$$(f+g)' = f' + g'$$

← same sum ← deriv each part

sigh

$$f_{m_0} = \sum_{i=1}^{m} 2(-x_i)(y_i - m_0 x_i - b_0)$$

$$f_{b_0} = \sum 2 (y_i - m_0 x_i - b_0)$$

$$f_{m_0 m_0} = 2 \sum_{i=1}^{m} (-x_i)^2 \checkmark$$

$$f_{m_0 b_0} = 2 \sum x_i \checkmark$$

$$f_{b_0 b_0} = 2 \sum 1 = 2n \checkmark$$

$$AC - B^2 = 4n \sum (x_i)^2 - 4 \left(\sum_{i=1}^{n} x_i\right)^2 \checkmark$$

$$\boxed{\text{know} \to \left(\sum u_i v_i\right)^2 \le \left(\sum u_i^2\right)\left(\sum v_i^2\right)}$$

Want to prove $\left(\sum x_i\right)^2 \le n \sum x_i^2$

- find right u_i and v

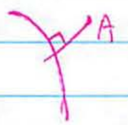
$\sum 1 \sum x_i^2$ $u_i = 1$ $v_i = x_i$

So it will be $+$

Now look at ~~M~~ C - eaiser

2n is $+$ so minimum

4. A surface is $F(x, y, z) = C$ which is level surface for $f(x,y,z)$



pt $A = (a, b, c)$ not on surface

want $P_0 (x_0, y_0, z_0)$ is on surface closest to A

Minimize the square of distance $w(x, y, z) = |AP|^2$

like 1st qu in recitation 3/10

Solve w/ Lagrange

C are not the same

$$w = x^2 + y^2 + z^2$$
$$= (a - x_0)^2 + (b - x_0)^2 + (c - z_0)^2 \checkmark$$

constraint $g = x_0 + y_0 + z_0 - c = 0 \leftarrow$ on surface

$$w_{x_0} = \lambda g_x = 2(a - x_0) \cdot -1 = \lambda 1$$
$$w_{y_0} = \lambda g_y = 2(b - y_0) \cdot -1 = \lambda 1 \quad \checkmark$$
$$w_{z_0} = \lambda g_z = 2(c - z_0) \cdot -1 = \lambda 1$$
$$x_0 + y_0 + z_0 - c = 0$$

?, where did go wrong?

Don't have to Solve

$$-2a + 2x_0 = -2b + 2y_0 = -2c + 2z_0 = \lambda$$
$$-a + x_0 = -b + y_0 = -c + z_0 = \frac{\lambda}{2}$$

solve for x, y, z

$$-a + x_0 = \frac{\lambda}{2} \qquad -b + y_0 = \frac{\lambda}{2} \qquad -c + z_0 = \frac{\lambda}{2}$$

$$x_0 = \frac{\lambda}{2} + a \qquad y_0 = \frac{\lambda}{2} + b \qquad z_0 = \frac{\lambda}{2} + c$$

? solve for λ by plugging in constraint eq

$$x_0 + y_0 + z_0 - c = 0$$
$$\frac{\lambda}{2} + a + \frac{\lambda}{2} + b + \frac{\lambda}{2} + \cancel{c} - \cancel{c} = 0$$

Now what?

$$\frac{3\lambda}{2} + a + b = 0$$

$$3\lambda + 2a + 2b = 0$$

$$\lambda = \frac{-2a - 2b}{3}$$

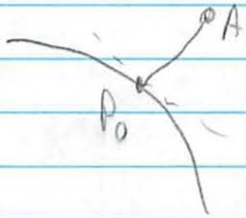
Now plug λ back in

$$x_0 = \frac{\left(\frac{-2a-2b}{3}\right)}{2} + a \quad y_0 = \frac{\left(\frac{-2a-2b}{3}\right)}{2} + b \quad z_0 = \frac{\left(\frac{-2a-2b}{3}\right)}{2} + c$$

That can't be right!,
Or were not suppose to actually find?

b) Show can that minimizing line AP_0 is $\perp$ to surface

Well that is the fastest way to something



P_0 will be on edge of surface as close as possible to A
This means that the line will make a line perpendicular to the tangent line at that surface at that point

not quite -1

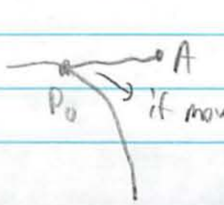
Also (lecture)

$$\frac{dw}{dt} = \vec{\nabla} w \cdot \frac{dr}{dt}$$

dot product $= 0$

always perp to surface

Anti example



$\nabla f = \langle f_x, f_y, f_z \rangle$
$\lambda \nabla f = \langle h_x, h_y, h_z \rangle$) parallel

5. Let $x = u\cos\phi - v\sin\phi$

$y = u\sin\phi + v\cos\phi$

a. Write this variable change in matrix form $\langle x, y\rangle = \langle u, v\rangle M$

Like the part A question on Jacobian Matrix

or from lecture $\vec{\nabla} W \cdot \frac{d\vec{r}}{dt}$

Or a constant way to adjust matrix values

Remember

$$\begin{bmatrix} x \\ y \end{bmatrix} = [u, v] \cdot \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$ ✓

? is this right?

b The vectors $\hat{i}' = \langle 1, 0\rangle$ in uv coords correspond

$\hat{j}' = \langle 0, 1\rangle$

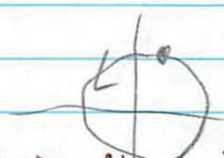
to what in xy coords? Calc + draw

$$\begin{bmatrix} x \\ y \end{bmatrix} = [1\ 0] \cdot \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$

$x = \cos\phi$

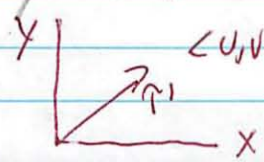
$y = \sin\phi$

? parametric



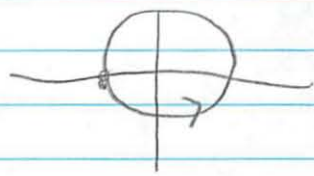
$\langle x, y\rangle = \cos\phi, \sin\phi\rangle$

$\langle u, v\rangle = \langle 1, 0\rangle = \hat{i}'$

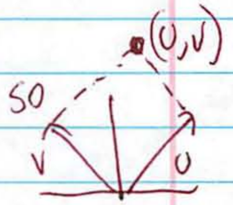


$$\begin{bmatrix} x \\ y \end{bmatrix} = [0,1] \begin{bmatrix} \cos\theta & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$ ✓

$x = -\sin\theta$
$y = \cos\theta$



will be perpendicular to other one



C As part b suggests the uv axis are just xy axis rotated by ϕ, Show using chain rule does not change lenght of $\vec{\nabla}$

← yeah if just rotating total lenght does not change

Aka For any differentiable Function $w(x,y)$ calculate $\nabla w(x,y)$ in xy coords or calc $\nabla w(x(u,v), y(u,v))$ in uv coords - get the same value for lenght squared of ∇w at every pt P

$$(w_u)^2 + (w_v)^2 = (w_x)^2 + (w_y)^2$$ ← as in the same

- one of these change variable comparisons 3/10 ~~#2~~ #3
← only 1 variable

$$\frac{d}{dx} w(x(u,v), y(u,v))$$

$$w_u = w_x \frac{\partial x}{\partial u} + w_y \frac{\partial y}{\partial u} \qquad w_v = w_x \frac{\partial x}{\partial v} + w_y \frac{\partial y}{\partial v}$$

$$W_u = \cos\theta\, w_x + \sin\theta\, w_y \qquad w_v = -\sin\theta\, w_x + \cos\theta\, w_y$$

Now plug it in

oh cool - that's how it works

$$(W_u)^2 + (w_v)^2 = (\cos\theta\, w_x + \sin\theta\, w_y)^2 + (-\sin\theta\, w_x + \cos\theta\, w_y)$$

$$(\cos^2\theta + \sin^2\theta)\, w_x^2 + (\cos^2\theta + \sin^2\theta)\, w_y^2$$

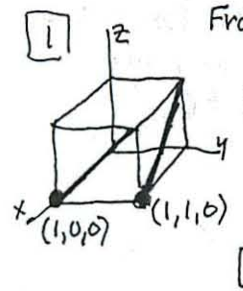
$$w_x^2 + w_y^2 \checkmark$$

☺ I did it!

well done :)

18.02 Prob Set 4 Solns - Spring 2010

1



Front diagonal:
$\vec{N} = \hat{j} + \hat{k} = \langle 0,1,1 \rangle$
$\boxed{x = 1,\ y = t,\ z = t}$

Side diagonal:
$\vec{N} = -\hat{i} + \hat{k} = \langle -1,0,1 \rangle$
$\boxed{x = 1-u,\ y = 1,\ z = u}$

$w = |\overrightarrow{AB}|^2$ A = pt on front diagonal, B = pt on side diagonal

$w = (-u)^2 + (1-t)^2 + (u-t)^2$ to be minimized

Find crit pts: $w_u = 2u + 2(u-t) = 4u - 2t = 0$
$w_t = -2(1-t) - 2(u-t) = -2 + 4t - 2u = 0$

Solving, $\begin{cases} 2u - t = 0 \\ 2t - u = 1 \end{cases} \Rightarrow t_0 = 2/3,\ u_0 = 1/3$

so $\boxed{\begin{array}{l} A_0 = (1, 2/3, 2/3) \\ B_0 = (2/3, 1, 1/3) \end{array}}$

min. distance is

$|\overrightarrow{A_0B_0}| = \sqrt{(\frac{-1}{3})^2 + (\frac{1}{3})^2 + (\frac{-1}{3})^2}$

$= \boxed{\dfrac{\sqrt{3}}{3}}$

2 a) Data vectors: $\vec{x} = \langle x_1\ x_2 \cdots x_{16} \rangle$

x_i = years after 1970

$\vec{\iota} = \langle 1\ 1 \cdots\cdots 1 \rangle$ (16 1's)

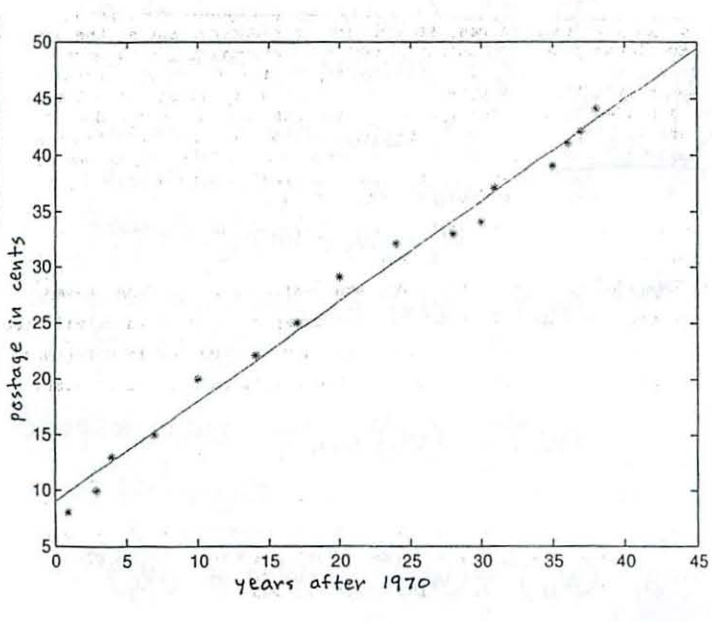
$\vec{y} = \langle y_1 \cdots\cdots y_{16} \rangle$

y_i = postage in ¢

$A = \begin{bmatrix} \vec{x}\cdot\vec{x} & \vec{x}\cdot\vec{\iota} \\ \vec{x}\cdot\vec{\iota} & 16 \end{bmatrix}$ $\vec{r} = \begin{bmatrix} \vec{x}\cdot\vec{y} \\ \vec{y}\cdot\vec{\iota} \end{bmatrix}$

(b, c) the line is $\boxed{y = .8988t + 8.9312}$

d) when $t = 43$ (=2013), $\boxed{y = .48}$ (.4758)



3 a) Using the expressions for $\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial t}$ in Problem 1 we get

$A = w_{uu} = 4$
$B = w_{ut} = -2$
$C = w_{tt} = 4$

$AC - B^2 = 12 > 0$
$A = 4 > 0$

$\therefore$ the point $(u,t) = (1/3, 2/3)$ is a <u>minimum point</u> for w.

b) Let $D = \sum (mx_i + b - y_i)^2$, to be minimized. Then we calculate

$A = D_{mm} = 2\sum_1^n x_i^2$
$B = D_{mb} = 2\sum_1^n x_i$
$C = D_{bb} = 2\sum_1^n 1 = 2n$

Thus: $AC - B^2 = 4n\sum x_i^2 - 4(\sum x_i)^2$

This is > 0, i.e., $\boxed{(\sum x_i)^2 < n \cdot \sum x_i^2}$ ⊛

(=0 if all x_i equal) not the case here

This follows from the Cauchy-Schwarz inequality for any 2 vectors $\vec{A}, \vec{B}$ in n-space: (1B-15)

$|\vec{A}\cdot\vec{B}| \leq |\vec{A}||\vec{B}|$

in 2D + 3D, it says: $|\cos\theta| \leq 1$
θ = angle between $\vec{A}$ + $\vec{B}$

using: $\vec{A} = \langle x_1, \ldots, x_n \rangle$
$\vec{B} = \langle 1, \ldots, 1 \rangle$

⊛ says $\left(\frac{\sum x_i}{n}\right)^2 \leq \frac{\sum x_i^2}{n}$

"square of the average of n numbers $\leq$ average of the squares"

$\boxed{4}$ a) The surface S is $\boxed{f(x,y,z)=c}$ ⊛

and $A:(a,b,c)$ is given

$P:(x,y,z)$ is a point on the surface.

We want to minimize

$$|AP|^2 = (x-a)^2+(y-b)^2+(z-c)^2 = w(x,y,z),$$

subject to the constraint ⊛.

The Lagrange multiplier equations are:

$$2(x-a) = \lambda f_x$$
$$2(y-b) = \lambda f_y \qquad f(x,y,z)=c$$
$$2(z-c) = \lambda f_z$$

b) If $P_0:(x_0,y_0,z_0)$ and λ_0 are a solution to the above equations,

then P_0 lies on the surface S (since $f(x_0,y_0,z_0)=c$)

and $\vec{AP_0} = \frac{\lambda_0}{2}(\vec{\nabla} f)_0$ (this is the 3 eqn's on the left above)

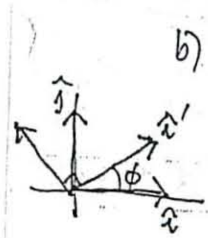
Since $(\vec{\nabla} f)_0$ is normal to the surface S at P_0, and $\vec{AP_0}$ is a non-zero scalar multiple of $(\vec{\nabla} f)_0$ (non-zero since A is not on the surface),

$\boxed{\vec{AP_0} \text{ is normal to } S \text{ at } P_0}$,

$\boxed{5}$ a) $(x,y) = (u,v)\begin{pmatrix}\cos\phi & \sin\phi \\ -\sin\phi & \cos\phi\end{pmatrix}$

b) $\hat{\imath}' = (1,0)M = (\cos\phi, \sin\phi)$

$\hat{\jmath}' = (0,1)M = (-\sin\phi, \cos\phi)$



c) Using the chain rule,

$$w_u = w_x\cos\phi + w_y\sin\phi$$
$$w_v = w_x(-\sin\phi) + w_y\cos\phi$$

Thus:

$$(w_u)^2 = (w_x)^2\cos^2\phi + 2w_xw_y\cos\phi\sin\phi + (w_y)^2\sin^2\phi$$

$$(w_v)^2 = (w_x)^2\sin^2\phi - 2w_xw_y\cos\phi\sin\phi + (w_y)^2\cos^2\phi$$

Adding, $(w_u)^2+(w_v)^2 = (w_x)^2 + (w_y)^2$.

Lecture 15

Chain Rule for non-independent variables 3/11

$w = w(x, y, z)$ $\frac{\partial w}{\partial x}$

To calc $\frac{\partial w}{\partial x}$ when x, y, z are constrained
$g(x, y, z) = 0$

Why is this a problem?
- will use specific formulas
- but in real life are hard to find

$w = x^2 + y^2 + z^2$
$1 = x + y + z$ constraint

Problem $\frac{\partial w}{\partial y}$

1. Substitute z in to remove it
$z = 1 - x - y$
$\downarrow$
$w = x^2 + y^2 + (1 - x - y)^2$

$\frac{\partial w}{\partial y} = 0 + 2y + 2(1 - x - y) \cdot -1$

$= -2x + 2x + 4y$

or $= 2y - 2z$

at $P_0 = -2$

Or can eliminate x instead
- by symmetry change z to x

$\frac{\partial w}{\partial y} = 2y - 2x$ at $P_0 = 0$

This is a problem, big problem!

- It matters

- Specify the independent variables if there is a constraint present.

$\left(\frac{\partial w}{\partial y}\right)_x \rightarrow x, y$ independent
z depends on this

$\left(\frac{\partial w}{\partial y}\right)_z \rightarrow y, z$ independent
x dependent

will be diff functions / notations

Must to calc $\left(\frac{\partial w}{\partial y}\right)_x$

1. eliminate dependent variable
   Var: $z = z(x, y)$

2. Use the chain rule

Alt method: Using differentials
- have not studied yet
- most people don't use

<u>How to use chain rule when variables constrained</u>

$w = w(x, y, z)$

$x = u$ } always independent $\quad z = z(u, v)$
$y = v$ } $\quad$ w/ constraint

Now use chain rule mechanically

$$\left(\frac{\partial w}{\partial y}\right)_x = \frac{\partial w}{\partial v}$$

don't have to hold u constant

u and v are independent variables

Calc w/ chain rule like before

could do

confusing

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial v} + \ldots$$

but what does this mean?

Convention

for constrained variables

$\frac{\partial w}{\partial x}$, w_x = take partial deriv as if no constraint

$\left(\frac{\partial w}{\partial x}\right)$ = then it is constrained

- before in 18.02 everything unconstrained but wrote in ( ) anyway

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x} \cdot \underbrace{\frac{\partial x}{\partial v}}_{=0} + \frac{\partial w}{\partial y} \cdot \underbrace{\frac{\partial y}{\partial v}}_{=1} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial v}$$

plain old chain rule, all independent so no parenthesis needed

=0 since x is not a function of v → u and v are independent

=1

$$\frac{\partial w}{\partial v} = 0 + \frac{\partial w}{\partial y} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v}$$

$$\frac{\partial z}{\partial v} = \left(\frac{\partial z}{\partial y}\right)_x$$

now can rename it back

must tell what ind variables are

$$\left(\frac{\partial w}{\partial y}\right)_x = 0 + w_y + w_z\left(\frac{\partial z}{\partial y}\right)_x$$

In the example $= 2y + 2z(-1)$

comes out to same answer

Have to get use to it → clumsy

- done same but forget u and v

$$\left(\frac{\partial w}{\partial y}\right)_x = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial y} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial y} + \frac{\partial w}{\partial z}\cdot\left(\frac{\partial z}{\partial y}\right)_z$$

$\frac{\partial x}{\partial y}$: 0 are both independant variables

$\frac{\partial y}{\partial y}$: 1

need () or else ambigeas

Examples in course notes

Differentials (bare minimum)

$$w = w(x, y, z)$$

$$dw = w_x\,dx + w_y\,dy + w_z\,dz$$

differential of w

total differential of w

Rules

1. You can add them
   Multiply by a scalar function
   Substitute (u-substitution)

2. $dw = f(x,y)\,dx + g(x,y)\,dy$

if x and y are independent variables

$\hookrightarrow f(x,y) = \frac{\partial w}{\partial x}$

$g(x,y) = \frac{\partial w}{\partial y}$

$w = x^2 + y^2 + z^2$ constraint $x + y + z = 1$

$dw = 2x\,dx + 2y\,dy + 2z\,dz$

want $\left(\frac{\partial w}{\partial y}\right)_x$ must eliminate term
Use constraint

$dx + dy + dz = 0 + 0 + 0$

so $dz = -dx - dy$

$dw = (2x - 2z)\,dx + (2y - 2z)\,dy$

want both partial derivs

expressed dw in terms of dx and dy

$\left(\frac{\partial w}{\partial x}\right)_y$ $\left(\frac{\partial w}{\partial y}\right)_x$

Cyclic Rule

$g(x, y, z) = 0$

1 dependent on the 2 others
which one ?

$$\left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y$$

3 partial derivs
each calculated differently

they cancle (libnets notation)

-1

why -? you don't want to know

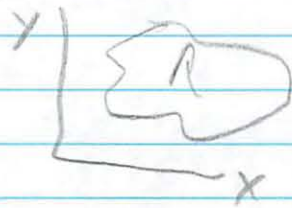
Lecture 16

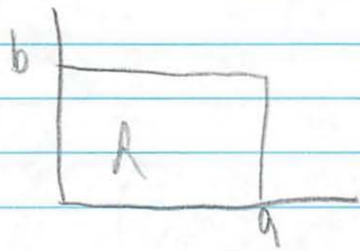
Double + Iterated Integrals in Rect Coordinates 3/12

1. $\iint dy\,dx$

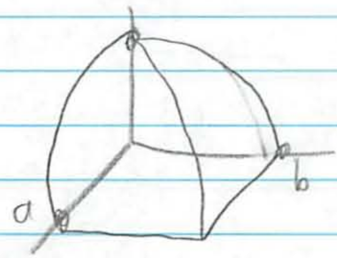
$w = f(x, y) > 0$

Volume under the graph over the region





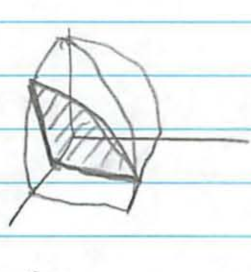
Uses nothing we learned this semester



By slicing

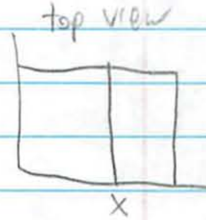
18.01 Slices by geometry
- circles
- rectangles

Here by integral

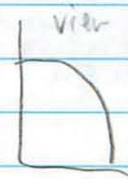


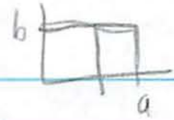
← slice at pt x

top view



Side view





Volume under graph $= \int_0^a A(x)$

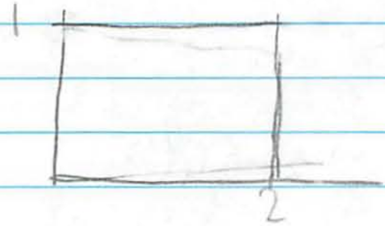
↑ area cross sectional slice at x

$A(x) = \int_0^b f(x, y)\, dy$ ← always ends at b in rectangle

↑ x held fixed

partial integration

think of Math as patterns

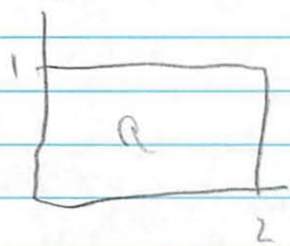


$w = x^2 + y^2$

Volume $= \underbrace{\int_a^a \int_a^b f(x,y)\, dy}_{A}\, dx$ iterated integral

order of integration

example



$w = x^2 + y^2$

Volume under graph x^2+y^2 $= \int_0^2 \int_0^1 (x^2 + y^2)\, dy\, dx$

*do the 2 integrations sepertly

Inner $\int$: $\int_0^1 (x^2+y^2)\, dy$

$$x^2y + \frac{y^3}{3}\Big]_{0=y}^{1=y}$$

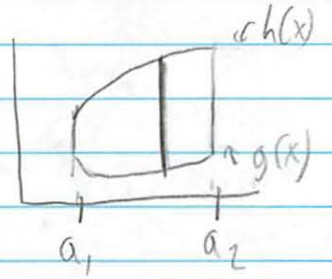
$$x^2 + \frac{1}{3} - 0$$

Outer $\int$: $\int_0^2 \left(x^2 + \frac{1}{3}\right) dx$

$$\frac{x^3}{3} + \frac{x}{3}\Big|_{0=x}^{2=x}$$

$$\frac{10}{3} = \text{Volume}$$

But what if no rectangle ?



could do circle
if start + ends were points

$$W = f(x,y)$$

$$A(x) = \int_{g(x)}^{h(x)} f(x,y)\, dy \qquad x \text{ fixed}$$

$$\text{Volume} = \int_{a_1}^{a_2}\int_{g(x)}^{h(x)} f(x,y)\, dy\, dx$$

$w = y$ ← just a plane

$y = \sqrt{1-x^2}$

$y = 1-x$

$$\text{vol} = \int_0^1 \int_{1-x}^{\sqrt{1-x^2}} y \, dy \, dx$$

Inner $\displaystyle\int_{1-x}^{\sqrt{1-x^2}} y \, dy$

$$= \frac{y^2}{2} \Bigg|_{1-x}^{\sqrt{1-x^2}}$$

$$= \frac{1-x^2}{2} - \frac{(1-x)^2}{2}$$

$$= x - x^2$$

Outer $\displaystyle\int_0^1 (x - x^2)\, dx$

$$\frac{x^2}{2} - \frac{x^3}{3} \Bigg|_0^1$$

$$\frac{1}{2} - \frac{1}{3}$$

$$\boxed{\text{volume} = \frac{1}{6}}$$

$$\int_0^{\pi/2}\int_x^{\pi/2} \frac{\sin y}{y}\, dy\, dx$$

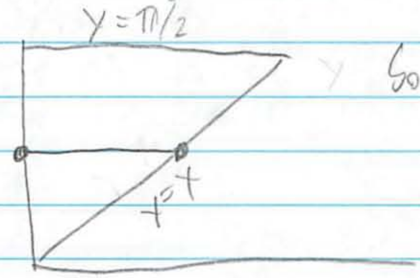
↑ but no elementary integral

use computer

but you can still get ans by changing the order of integration

but limits will change - so can't just change the $\frac{dy}{dx}$

1. Figure out region over which integrating
2. Redo integral



So now use horiz slices

$$\iint \frac{\sin y}{y}\, dx$$

↑ functions x of y

$$\int_0^{\pi/2}\int_{0;\ x=0}^{y} \frac{\sin x}{y}\, dx\, dy$$

solve for x as a function of y

Inner $\frac{\sin y}{y} \cdot x \Big|_{x=0}^{x=y}$

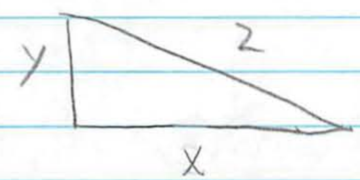
$= \frac{\sin y}{y} y$

$= \sin y$

Outer $\int_0^{\pi/2} \sin y \, dy$

1

More on Chain Rule



constraint $\rightarrow x^2 + y^2 - z^2 = 0$

$\left(\frac{\partial z}{\partial x}\right)_y$

say explicitly what ind. variables are

Method 1

- Solve for z

$z = \sqrt{x^2 + y^2}$

$\left(\frac{\partial z}{\partial x}\right)_y = \frac{x}{z}$

implicit differentiation = <u>kinda</u> like chain rule

Method 2

2. Chain Rule

$$\underbrace{x^2+y^2-z^2}_{w}=0$$

$$\left(\frac{\partial w}{\partial x}\right)_y = w_x = 2x - 2z\left(\frac{\partial z}{\partial x}\right)_y$$

think of z as f(x,y)

$= 0$ since $w = 0$

solve for

and it = what got befor

3. Abstractly

$w(x, y, z) = 0$

constant

$\left(\frac{\partial z}{\partial x}\right)_y$ use chain rule

$w = 0$ so ans comes out to 0

$$\left(\frac{\partial w}{\partial x}\right)_y = w_x + w_z \cdot \left(\frac{\partial z}{\partial x}\right)_y = 0$$

$\left(\frac{\partial w}{\partial x}\right)_y = 0$

w_z: can't hold constant

$$\boxed{\left(\frac{\partial z}{\partial x}\right)_y = \frac{-w_x}{w_z}}$$

Recitation

3/15

Lectures - Double Integrals

- Non independent variable
  - ↳ if constrained by an equation

Notation $\frac{\partial f}{\partial x} = f_x$ "formal derivitive"
↳ did up to now
↳ others independent

$\left(\frac{\partial f}{\partial x}\right)_y$ "real derivitive

ind variables

---

ex1 $f(x,y,z) = x^2 + e^y + yz \qquad xz + y\cos z = 0$

Compute $\left(\frac{\partial f}{\partial x}\right)_y$ - chain rule
- differential

Can not solve constraint for z
so derive implicitly

$\left(\frac{\partial f}{\partial x}\right)_y = \left(\frac{\partial}{\partial x} x^2 + e^y + yz\right)_y$

z function of x and y

$= 2x + 0 + y\left(\frac{\partial z}{\partial x}\right)_y$

but what is that?

$$\left(\frac{\partial}{\partial x}\, xz + y\cos z\right)_y = 0$$

↓product rule

z is like a function

$$= z + x\left(\frac{\partial z}{\partial x}\right)_y - y\left(\frac{\partial z}{\partial x}\right)_y \sin y = 0$$

$$(x - y\sin z)\left(\frac{\partial z}{\partial x}\right)_y = -z$$

$$\left(\frac{\partial z}{\partial x}\right)_y = \frac{-z}{x - y\sin z}$$

So plug it in

$$\left(\frac{\partial z}{\partial x}\right)_y = 2x + y\left(\frac{-z}{x - y\sin z}\right)$$

By Differentiation

$$df = f_x\,dx + f_y\,dy + f_z\,dz \quad \leftarrow \text{definition}$$

like approx formula

$$= 2x\,dx + (e^y + z)\,dy + y\,dz$$

We want an expression like $df = \underbrace{A}_{\text{coefficients}} dx + \underbrace{B}_{} dy$

$$\left(\frac{\partial f}{\partial x}\right)_y = A$$

$$\left(\frac{\partial f}{\partial y}\right)_x = B$$

Still need to find dz

$$dg = g_x\,dx + g_y\,dy + g_z\,dz = 0$$

$$z\,dx + \cos z\,dy + (x - y\sin z)\,dz = 0$$

$$dz = \frac{-z}{x - y\sin z}\,dx - \frac{\cos z}{x - y\sin z}\,dy$$

get as function of dx and dy only

$$df = \left(2x - \frac{yz}{x - y\sin z}\right)dx + \left(e^y + z - \frac{y\cos z}{x - y\sin z}\right)dy$$

$$\left(\frac{\partial f}{\partial x}\right)_y = 2x - \frac{yz}{x - y\sin z} \qquad \left(\frac{\partial f}{\partial y}\right)_x = e^y + z - \frac{y\cos z}{x - y\sin z}$$

* Must be able to work w/ both methods *

↳ if just want 1 partial deriv → chain rule
many → differentials

---

ex2 $\quad x^2y + yz = f \qquad x^2 + y^2 + z^2 = 1$

$\left(\frac{\partial f}{\partial x}\right)_y$ chain rule + differential

$$2x + \cancel{y^2} + 2z\left(\frac{\partial z}{\partial x}\right)_y = 0$$

constant

$\cancel{2z\left(\frac{\partial z}{\partial x}\right)_y = -2x - y^2}$

$$\left(\frac{\partial z}{\partial x}\right)_y = \frac{\cancel{-2x - y^2}}{\cancel{2z}} = -\frac{x}{2}$$

why am I being stupid and doing chain rule here

$$\left(\frac{\partial f}{\partial x}\right)_y = 2xy + \cancel{1}x^2 + \cancel{1z} + \left(\frac{\partial z}{\partial x}\right)_y y$$

$$\cancel{\left(\frac{\partial f}{\partial x}\right)_y = 2xy + x^2 + z + \left(\frac{-2x - y^2}{2z}\right) y}$$

$$\cancel{2xy + \left(2xy - \frac{xy}{2}\right) y}$$

$$\left(\frac{\partial f}{\partial x}\right)_y = 2xy + y\left(\frac{\partial z}{\partial x}\right)_y$$

$$= 2xy - \frac{xy}{2}$$

b) $df = 2xy\,dx + (x^2 z)\,dy + y\,dy$

$0 = 2x\,dx + 2y\,dy + 2z\,dz$ constraint

$$dz = -\frac{x}{y}\,dx - \frac{y}{2}\,dy$$

$$df = \left(2xy - \frac{xy}{2}\right) dx + \left(x^2 + z - \frac{y^2}{2}\right) dy$$

ex3 x, y, z constrained by $g(x,y,z)=0$

a) Show that $\left(\frac{\partial z}{\partial x}\right)_y = \frac{-g_x}{g_y}$

b) Show $\left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y = -1$

a) Differentiate g w/ respect to x, y constant

$$\left(\frac{\partial}{\partial x} g(x,y,z)\right)_y = 0$$

$$\underbrace{\left(\frac{\partial x}{\partial x}\right)_y}_{1} g_x + \underbrace{\left(\frac{\partial y}{\partial x}\right)_y}_{0} g_y + \left(\frac{\partial z}{\partial x}\right)_y g_z = 0$$

$\frac{\text{same}}{\text{same}}$ holding constant

$$g_x + \left(\frac{\partial z}{\partial x}\right)_y g_z = 0$$

$$\left(\frac{\partial z}{\partial x}\right)_y = \frac{-g_x}{g_z}$$

That was not hard - just was very unsure

Using part a

b) $\left(\frac{\partial x}{\partial y}\right)_z = -\frac{g_y}{g_x}$

$\left(\frac{\partial y}{\partial z}\right)_x = -\frac{g_z}{g_y}$

$\left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right) = \left(\frac{-g_y}{g_x}\right)\left(\frac{-g_z}{g_y}\right)\left(\frac{-g_x}{g_z}\right)$

$= -1$

Lecture 17

Double Integrals in Polar Coords 3/16

$\iint_R$ in polar coords

$\cancel{\iint_R}$

$$\int_a^b \int_{g(x)}^{h(x)} f(x,y)\, dy\, dx$$ iterated integral

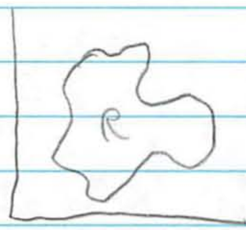
calculates volume under $f(x,y)=w$
by slicing

Back up: <u>Double Integral</u>

$$\iint_R f(x,y)\, dA$$ ← often not done in rect coords

↑ some region

double integral f over R

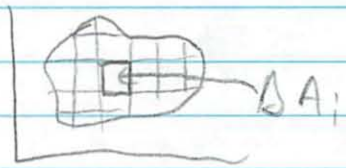


1. use the grid lines of coord system (u,v)
   "curves"
   $\begin{cases} u = c_1 \\ v = c_2 \end{cases}$
   aka level curves of coord functions

So rect $x = c_1$
$y = c_2$

2. Divide up R into little pieces

ΔA_i



3. Choose $P_i\,(u_i, v_i)$ in each ΔA_i

4. Form Riemann Sums

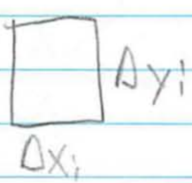
$$\lim_{n \to \infty} \sum_{i}^{n} f(u_i, v_i)\,\Delta A_i$$

$$= \iint_R f(u,v)\,dA$$

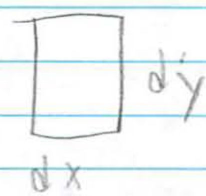
- but along boundry can be very complicated
- must estimate ← area
- try to divide into pieces which can be calculated ΔA_i

x, y coords

$$\iint_R f(x,y)\,dy\,dx$$

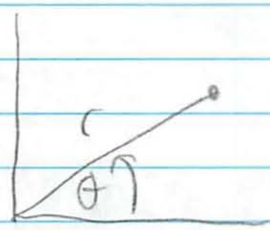


$dA =$



How to evaluate ....

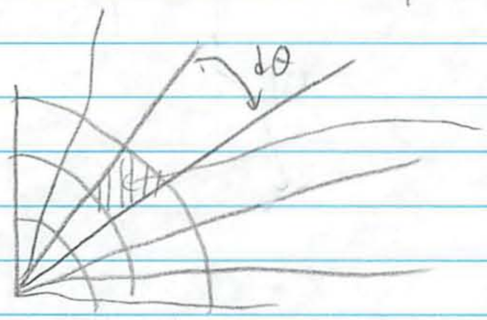
Polar Coordinates



What are the grid lines in polar

$r = c_1 \rightarrow$ circles with center at origin, radius r

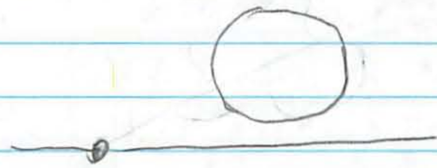
$\theta = c_2 \rightarrow$ rays from center out



depends on r

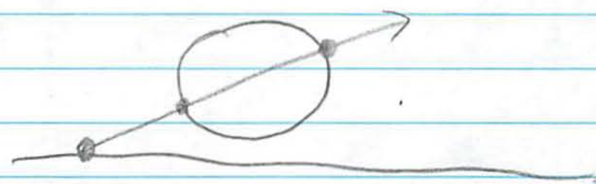
$$\boxed{dA = r\,dr\,d\theta}$$

How to put in limits of integration
- see notes I



$\iint f(r,\theta)\, r\, dr\, d\theta$

what interegrating 1st w/ respect to?
almost always r
(1) - so θ fixed, r increases



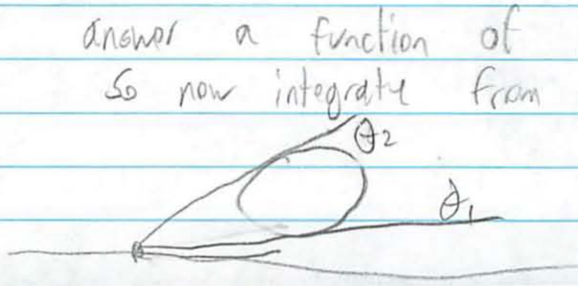
(2) inner limits → r value where enters region r
to r value where it leves

- depends on which ray you take (what θ is)

$r_1 = g(\theta)$ $\qquad$ $r_2 = h(\theta)$

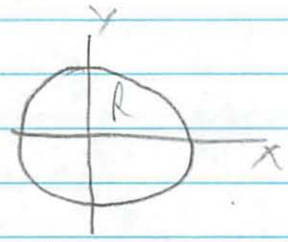
* you must know what region looks like

(3) answer a function of θ
so now integrate from where it just enters
to where
it just
leaves



example

volume of unit hemisphere



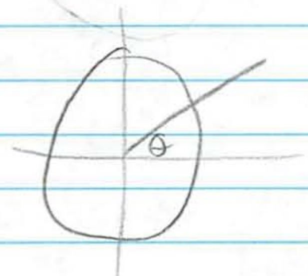
top view

$$f(x,y,z) = x^2+y^2+z^2=1$$

$$z=\sqrt{1-x^2-y^2}$$ solve for z

$$\text{volume} = \int\int \sqrt{1-x^2-z^2}$$

don't even bother
change to polar coords



$$r = x^2+y^2$$

$$\text{volume} = \int_0^{2\pi}\int_0^1 \sqrt{1-r^2}\, r\, dr\, d\theta$$

angle　　radius

easy to evaluate

$$\text{Inner} \rightarrow \int 1-r^2$$

$$-\frac{3}{2}(1-r^2)^{1/2} \cdot -2r \quad \text{so}$$

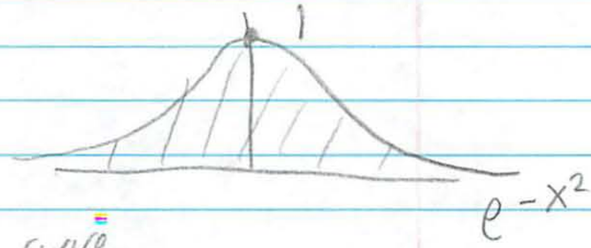
$$-\frac{1}{3}(1-r^2)^{3/2}\Big|_0^1$$

$= \frac{1}{3}$

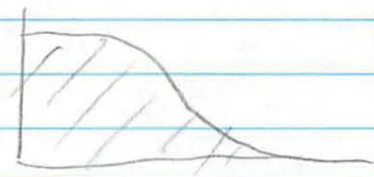
outer $= \frac{2\pi}{3}$

Example 2

$\int$



must know area under curve

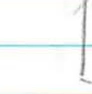


$$\int_0^\infty e^{-x^2}\,dx$$

↑ can't take deriv of "undoable" but oiler can do it

So trick call it $= I$

$$\iint e^{-x^2-y^2}\,dy\,dx$$



involves 2 diff coord systems

$$= \iint e^{-x^2} \cdot e^{-y^2}\,dy\,dx$$

Limits $\int_0^\infty \int_0^\infty dy\,dx$

Inner: $e^{-x^2} \int_0^\infty e^{-y^2}\, dy$

↑ don't know how to evaluate

but know it $= e^{-x^2} I$

Outer: $\int_0^\infty e^{-x^2}\, I$

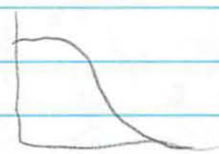
↑ don't know

$= I \cdot I$
$= I^2$

but what is I?

In polar coordinates

$\iint e^{-r^2}\, r\, dr\, d\theta$



Limits $\int_0^{\pi/2} \int_0^\infty e^{-r^2}\, r\, dr\, d\theta$

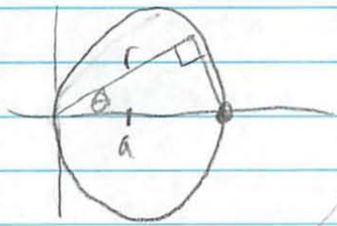
Inner $\rightarrow -\frac{1}{2} e^{-r^2} \Big|_0^\infty$

$= \frac{1}{2}$

Outer $\int_0^{\pi/2} \frac{1}{2}\, d\theta = \frac{\pi}{4}$ same answer in both coord system

$= I^2$

so $I = \frac{\sqrt{\pi}}{2}$



$r = 2a\cos\theta$

$$\int_{-\pi/2}^{\pi/2}\int_{0} f(r,\theta)\, r\, dr\, d\theta$$

Recitation

3/17

Lecture

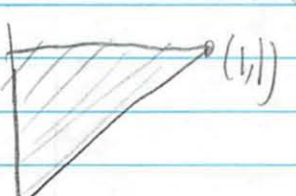
- definition
- interpertation of $\iint_R f(x,y)\,dx\,dy$

on volume under graph of f

Using polar coords

$$\iint_{R} \underset{\uparrow \text{ expressed in } x,y}{f}\, dx\,dy = \iint_{R} \underset{\uparrow \text{ expressed in } r,\theta}{f}\, r\,dr\,d\theta$$

ex1 Compute $I = \iint_R xy\,dx\,dy$

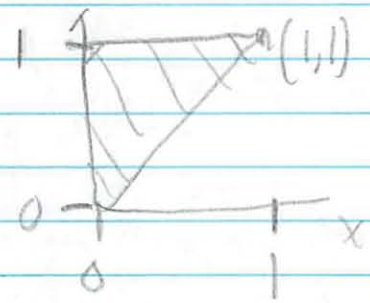


$I = \iint xy\,dy\,dx$

Now need to find limits

- from the triangle
  Not the function

- start w/ <u>Outside!</u> (calculation starts inside)

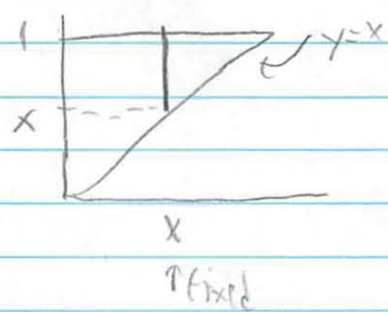


$$\int_0^1 \int \, dy\, dx$$

$\uparrow x$ do 1st

$\uparrow y$ do later

For a given x in the range $[0,1]$



$$\int_0^1 \int_x^1 xy \, dy\, dx$$

$$I = \int_0^1 \int_x^1 xy \, dy \, dx$$

Inside $\int_x^1 xy \, dy$

$$x\frac{y^2}{2}\Big|_x^1$$

$$\frac{x(1)^2}{2} - \frac{x(x)^2}{2}$$

$$\frac{x}{2} - \frac{x^3}{2}$$

Outside

$$\int_0^1 \frac{x}{2} - \frac{x^3}{2} \, dx$$

$$\frac{x^2}{4} - \frac{x^4}{8}\Big|_0^1$$

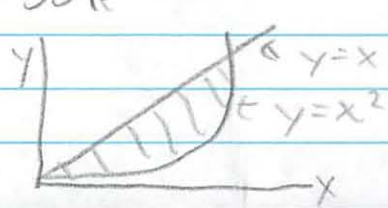
$$\frac{1}{4} - \frac{1}{8} = \frac{1}{8}$$

* at each point have a certain x, y value want to sum that, not just area *

---

ex 2 Find the limit of integration of

$$I = \iint_R f(x,y) \, dy \, dx$$

R 

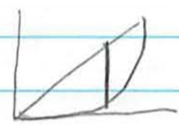
where $x = x^2$ $\longrightarrow$ where does $\frac{x}{x} = \frac{x^2}{x}$

$1 = x$

$$\int_0 \int \, dy \, dx$$

do 1st

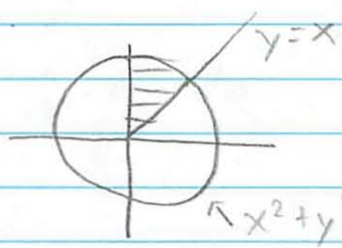
do second



$x^2 \to x$

$$\int_0^1 \int_x^{x^2} f(x,y) \, dy \, dx$$

b)



$$\int_0 \int f(x,y) \, dy \, dx$$

$\to x^2 + y^2 = x^2 + x^2 = 1$

$2x^2 = 1$

solve for x

$x^2 = \frac{1}{2}$

$x = \sqrt{\frac{1}{2}} = \frac{\sqrt{2}}{2}$

$= \cos \frac{\pi}{4}$

now y

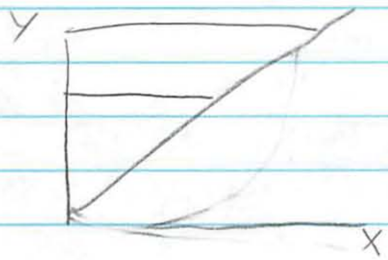
$x^2 + y^2 = 1$

solve for y

$$\int_x^{\sqrt{1-x^2}}$$

$$\int_0^{\sqrt{2}/2} \int_x^{\sqrt{1-x^2}} f(x,y) \, dy \, dx$$

What if wanted a $dx\,dy$

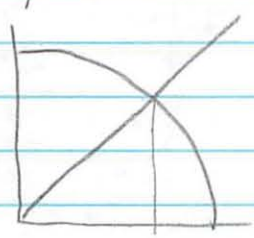


go horizontal

$$\int_{y=0}^{1}\int_{x=0}^{y} dx\,dy \quad ✓$$

picture is intermediate state

If you hat



$dy\,dx$

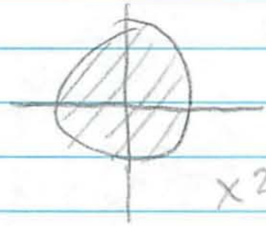
- could split into 2 parts
- or change order to $dx\,dy$

ex3 Polar Coordinates

Setup the integral

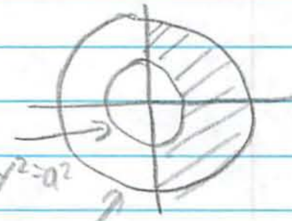
$I = \iint_R xy \, dx\, dy$ in polar coordinates

a)



$x^2+y^2 \leq 1$

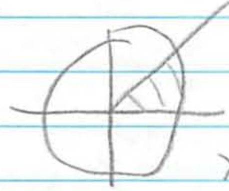
c)



$x^2+y^2=a^2$

$x^2+y^2=b^2$

b)



$y=x$

$x^2+y^2=1$

Find Setup

a) $\int_0^{2\pi}\int_0^1 xy \; r\,dr\,d\theta$ → have to replace

$\theta \quad r$ → fix θ and look at r θ

$x = r\cos\theta$

$y = r\sin\theta$

$\int_0^{2\pi}\int_0^1 r\cos\theta \; r\sin\theta \; r\,dr\,d\theta$

b) $\int_0^{\frac{\pi}{4}}\int_0^1 r\cos\theta, r\sin\theta \; r\,dr\,d\theta$

r do 1st!

$r^3\cos\theta\sin\theta \; dr\,d\theta$

these are actually really easy!

but don't make mistakes

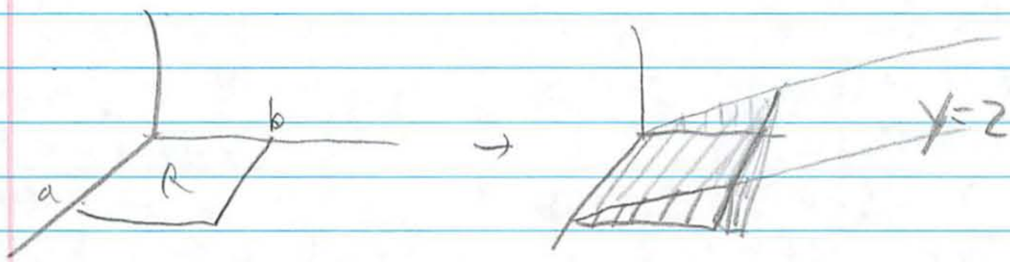
c
$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{a}^{b} r\cos\theta \, r\sin\theta \; r\,dr\,d\theta$$

r?? yes - the radius of each angle

↑ pay attention!

---

ex4 Computation of volume



$$\text{Vol} = \iint_R f(x,y)\,dx\,dy$$

$f = (x,y)$ ← need to find

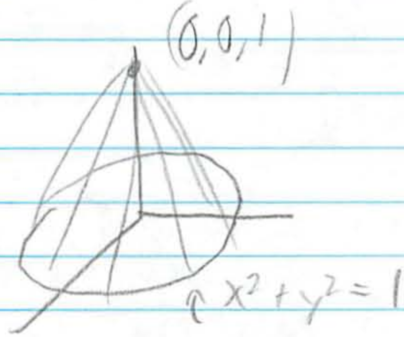
$= y$ since graph is a plane

R is rectangle $x \in [0,a]$, $y \in [0,b]$

$$\text{Vol} = \underbrace{\int_{x=0}^{a} \int_{y=0}^{b}}_{\text{is R rectangle}} \underbrace{y}_{\text{function}} \, dy\, dx$$

$$= \frac{ab^2}{2}$$ ← then do

b



$$\text{Vol} = \iint_R f \, dx \, dy$$

$$= \iint_R f \, r \, dr \, d\theta$$

R = unit disc

$$= \int_0^{2\pi} \int_0^1 f \, r \, dr \, d\theta$$

Michael Plasmeier

~~Get P-Set 4 solution~~

20.5 / 35

18.02 Problem Set 5 due Thurs. Mar.18, 10:45 2-106

Part I (15 points)

Lecture 15. Thurs. Mar. 1[illegible] Chain rule for non-independent variables.
Read: Notes N.1-.3 (N.4 optional) Work: 2J-1a, 2i,ii(for (a) only), 3a, 4a, 5a, 6, 7

Lecture 16. Fri. Mar. 12 Double and iterated integrals in rectangular coordinates.
Read: 20.1, 20.2, Notes I.1 Work: 3A-1ad, 2b, 3b, 4c, 5a

Lecture 17. Tues. Mar. ~~23~~ 16 Double integrals in polar coordinates
Read: 20.4, Notes I.2 Work: 3B-1ad, 2cd, 3bc

Part II (20 points)

Directions. Try each problem alone for 15 minutes. If you subsequently collaborate, solutions must be written up independently. It is illegal to consult problem sets from previous semesters. With each problem is the day it can be done.

Problem 1. (Thurs. 6 pts.: 1.5 each) Using the usual rectangular and polar coordinates, let w be the area of the right triangle in the first quadrant having its vertices at (0,0), (x,0), and (x, y).

Using the equation expressing w in terms of x and y, and the equation expressing y in terms of x and θ, calculate the two partial derivatives $\left(\frac{\partial w}{\partial x}\right)_\theta$ and $\left(\frac{\partial w}{\partial \theta}\right)_x$ in three different ways:

a) Directly, by first expressing w in terms of the independent variables x and θ;

b) by using the chain rule — for example, $\left(\frac{\partial w}{\partial x}\right)_\theta = w_x\left(\frac{\partial x}{\partial x}\right)_\theta + w_y\left(\frac{\partial y}{\partial x}\right)_\theta$, where w_x and w_y are the formal partial derivatives;

c) by using differentials: get two equations connecting dw, dx, dy, and $d\theta$, and use them to eliminate one of the differentials, getting a single equation connecting the remaining differentials, from which the answers can be read off.

d) Using the triangle picture and geometric intuition, estimate $\left(\frac{\Delta w}{\Delta x}\right)_\theta$ and $\left(\frac{\Delta w}{\Delta \theta}\right)_x$ from the picture, and show they agree with the two corresponding partial derivatives.

Problem 2. (Fri. 3 pts.) A rectangular prism is made by taking a long piece of wood with a rectangular cross-section, sawing off one end perpendicularly to the four sides, and the other end at an arbitrary angle (so that the four long edges have in general four different lengths).

As an exercise in using double integration, use it to show that the volume of the prism is the product of its cross-sectional area and the average of the lengths of the four long edges. Hint: place the prism so that one long edge lies along the z-axis, and the perpendicular end lies in the first quadrant of the xy-plane. So that everyone will use the same notation, let

$a =$ length of the edge lying along the x-axis;
$b =$ length of the edge lying along the y-axis;
$z = Ax + By + C$ be the equation of the plane forming the slanted top of the prism.

Problem 3. (Fri. 2 pts.) Evaluate by changing the order of integration:

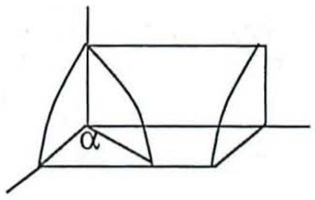
$$\int_0^1 \int_{\sqrt[3]{x}}^1 \sqrt{1+y^4}\, dy\, dx\ .$$

So far ahead this week - not using in b/w class time - which wastes it - using valuable time - but use class time for other purposes

Problem 4. (Fri. 3 pts.: 2.5, .5)

The picture shows a quarter of a cylindrical log, whose perpendicular cross-section is a quarter of a circular disc of radius a. A wedge-shaped piece has been cut out of the end, by a cutting plane making an angle α with the perpendicular end of the log.

Find the volume of this wedge-shaped piece by using double integration. Place the log and piece with respect to the coordinate axes as shown in the picture.



When $\alpha = \pi/4$, what is the volume? To check whether this is a reasonable answer, find the volume of the tetrahedron ($\frac{1}{3}$base $\times$ height) having the same four corners as the wedge.

Problem 5. (Tues. 6 pts: 2,2,2) Work the following problems:

a) 20.4/9 (cf. p. 564)

b) 20.4/15 (cf. formulas at top of Exercises 3B in the Notes)

c) 20.4/31a

P-Set 5

Michael Plasmeier 3/18

Lecture 15 Chain Rule for non independent variables

2J-1a Calc by direct substitution like N example 1 $\left(\frac{\partial w}{\partial y}\right)_z$

$w = x^2 + y^2 + z^2$
$z = x^2 + y^2$

So $w = x^2 + y^2 + (x^2+y^2)^2$
$x^2 + y^2 + x^4 + 2x^2y^2 + y^4$

$\frac{\partial w}{\partial y} = 0 + 2y + 0 + 4x^2y + 4y^3$

means x is the dependent variable
so get rid of that

$w = (z - y)^2 + y^2 + z^2$
$z + z^2$

$\left(\frac{\partial w}{\partial y}\right)_z = 0 + 0 = 0$

I kinda see - the 2 are the indep. variables which are changing, write the last as the dep. variable and then differenitate

2: Calc $\left(\frac{\partial w}{\partial y}\right)_z$ $\cancel{\left(\frac{\partial w}{\partial z}\right)_y}$ w/ chain rule
- differentiate $z = x^2 + y^2$

So w/ same formulas:

$$w = x^2 + y^2 + z^2$$

$$\left(\frac{\partial w}{\partial y}\right)_z = 2xx' + 2y + 2(x^2+y^2)\cdot(2xx'+2y)$$

$$= 2xx' + 2y + 4xx'(x^2+y^2) + 2y(x^2+y^2)$$

First differenitate $z = x^2 + y^2$ w/ respect to y

$$0 = 2x\left(\frac{\partial x}{\partial y}\right)_z + 2y$$

$$\left(\frac{\partial x}{\partial y}\right)_z = -\frac{y}{x}$$

Now by chain rule

$$\left(\frac{\partial w}{\partial y}\right)_z = 2x\left(\frac{\partial x}{\partial y}\right)_z + 2y$$

$$= 2x\left(\frac{-y}{x}\right) + 2y$$

$$= 0$$

2i b $\left(\frac{\partial w}{\partial z}\right)_y$ ← the dependent variable

diff w/ respect to z ✓

$z = x^2 + y^2$

$1 = \cancel{\partial y} + 2x\left(\frac{\partial x}{\partial z}\right)_y + 0^2 -$ ← what is left

$\left(\frac{\partial x}{\partial z}\right)_y = \frac{1}{2x}$

$\left(\frac{\partial w}{\partial z}\right)_y = 0 + \cancel{\partial y} + 2x\left(\frac{\partial x}{\partial z}\right)_y + 2z$ differentiate normally

$= 2\cancel{p} + 2x\left(\frac{1}{2x}\right)$

$\cancel{= 2y + \frac{2x}{2y}}$

$\cancel{= 2y + \frac{x}{y}}$

$= 1 + 2z$

Should practice more
Reviewed both w/o looking on seperate sheet

2ii a (a only) $\left(\frac{\partial w}{\partial y}\right)_z$ w/ differentials

$w = x^2 + y^2 + z^2$
$dw = 2x\,dx + 2y\,dy + 2z\,dz$

$z = x^2 + y^2$
$dz = 2x\,dx + 2y\,dy$

¿ are we supposed to have a constraint as well?

~~$dw = (2x$~~ Want to eliminate dx
subtract 2nd eq from 1st eq

$dw = 0\,dx + (1+2z)\,dz$

So what did they do
$2x\,dx = dz - 2y\,dy$

So $dw = (dz - 2y\,dy) + 2y\,dy + 2z\,dz$
$dw = dz + 2z\,dz$
$(1+2z)\,dz$ ✓ matches

Then D2 → If the variables x, y, ... are independent, 2 differentials are = if and only if their corresponding efficents are =

$Ax + By = A_1 x + B_1 y$
only if $A = A_1$
$B = B_1$

So what in all world does that mean?

$\left(\frac{\partial w}{\partial y}\right)_z = 0 \qquad \left(\frac{\partial w}{\partial z}\right)_y = 1+2z$

So basically coefficent of $dx = \frac{\partial w}{\partial x}$

" $dy = \frac{\partial w}{\partial y}$

" $dz = \frac{\partial w}{\partial z}$

3a Using chain rule calc $\left(\frac{\partial w}{\partial t}\right)_{x,z}$ in terms x, y, z, t
example 2

So now 4 variables

$w = x^3 y - z^2 t \qquad xy = zt$

So we want to get rid of y (the dep variable)

$y = \frac{zt}{x}$

$w = x^3 \left(\frac{zt}{x}\right) - z^2 t$

$w = x^2 zt - z^2 t$

$\left(\frac{\partial w}{\partial t}\right)_{x,z} = x^3 \left(\frac{\partial y}{\partial t}\right)_{x,z} - z^2$

differenitate original w → for y write $\left(\frac{\partial y}{\partial t}\right)_{xz}$

differenitate constraint

$$\left(\frac{\partial x}{\partial t}\right)_{xz} = \frac{2}{x}$$

That is chain rule seems simple - but write it all out

Now plug in

$$\left(\frac{\partial w}{\partial t}\right)_{x,z} = x^3\left(\frac{2}{x}\right) - z^2$$

$$\boxed{x^2 z - z^2}$$

4a. Again w/ differential

$\left(\frac{\partial w}{\partial t}\right)_{x,z}$ $\quad w = x^3 y - z^2 t \qquad xy = zt$

$$dw = 3x^2 dx\, dy - 2z\, dz\, dt \qquad dx\, dy = dz\, dt$$

$$dy = \frac{dz\, dt}{dx} = x\, dy$$

$$3x^2 dx\left(\frac{dz\, dt}{dx}\right) - 2z\, dz\, dt$$

$$3x^2 dz\, dt - 2z\, dz\, dt$$

$$(3x^2 - 2z)\, dz\, dt$$

coefficient $dt = 3x^2 - 2z = \left(\frac{\partial w}{\partial t}\right)_{xz}$

They get how in all world

$$dw = 2x^2 y\, dx + (x^2 z - z^2)\, dt + (x^2 t - 2zt)\, dz$$

$$\hookrightarrow x^2 z - z^2$$

5a Let $S = S(p, v, T)$ entropy of gas ~~pV~~ T

Give $\left(\frac{\partial S}{\partial p}\right)_v$ in terms of S_p, S_v, S_T

so what is constraint?

$pV = nRT$

n, R constants

$$\left(\frac{\partial S}{\partial p}\right)_v = S_p + S_T \cdot \left(\frac{\partial T}{\partial p}\right)_v$$

$$S_p + S_T \frac{V}{nR}$$

? I don't get what is so special about that

6. If $w = u^3 - uv^2$ $u = xy$ $v = u + x$

Find $\left(\frac{\partial w}{\partial u}\right)_x$ and $\left(\frac{\partial w}{\partial x}\right)_u$ using chain rule + differentials

will do two after recitations

so 2 constraints
- one for each one
- or will it work for both

$$\cancel{\left(\frac{\partial w}{\partial u}\right)_x = 3u^2\left(\frac{\partial u}{\partial u}\right)_x - \left(\frac{\partial u}{\partial u}\right)_x v^2}$$

5a
red.

office
hrs

$PV - nRT = 0$

$s(P, V, T)$

dont 't know S

$\left(\frac{\partial s}{\partial p}\right)_V$

$\cancel{\frac{\partial s}{\partial}}$

$\frac{\partial s}{\partial p}\left(\frac{\partial p}{\partial p}\right)_V + \frac{\partial s}{\partial V}\left(\frac{\partial V}{\partial p}\right)_V$

$+ \frac{\partial s}{\partial T} + \left(\frac{\partial T}{\partial p}\right)_V$

1

0 both
indp variable

$\cancel{p}$

$\frac{\partial s}{\partial p} + \frac{\partial s}{\partial T}\left(\frac{\partial T}{\partial P}\right)_V$

$T = \frac{PV}{nR}$

$\left(\frac{\partial T}{\partial P}\right)_V = \cancel{\quad} \frac{V}{nR}$

$\frac{\partial s}{\partial p} + \frac{\partial s}{\partial T}\left(\frac{V}{nR}\right)$

Or do they want us to sub it in?

$$\cancel{\begin{aligned} w &= (xy)^3 - (xy)(v+x)^2 \\ &= x^3y^3 - (xy)(xy+x)^2 \\ &= x^3y^3 - (xy)(x^2y^2 + 2x^2y + x^2) \\ &= x^3y^3 - x^3y^3 - 2x^3y^2 + x^3y \\ &= -2x^3y^2 + x^3y \end{aligned}}$$

~~But then how deal w/ v~~

Just do $\left(\frac{\partial w}{\partial u}\right)_x = 3u^2 - v^2 - \overbrace{u \cdot 2v \left(\frac{\partial v}{\partial u}\right)_x}^{\text{chain rule}}$

differentiate w/ respect to u

constraint

x constant

v should be $\left(\frac{\partial v}{\partial u}\right)_x$

which one? $\left(\frac{\partial v}{\partial u}\right)_x = 1 + 0$

plug in

$$\left(\frac{\partial w}{\partial u}\right)_x = 3u^2 - v^2 - 2vu$$ ✓

Now other one

$$\left(\frac{\partial w}{\partial x}\right)_v = 0 - u 2v \left(\frac{\partial v}{\partial x}\right)_v - 0$$

$v = u + x$

diff w/ respect x
u = constant
$v = \left(\frac{\partial v}{\partial x}\right)_u$

$$\left(\frac{\partial v}{\partial x}\right)_u = 0 + 1$$

$$\left(\frac{\partial w}{\partial x}\right)_u = -2uv$$

✓ got it no looking

b) Now lets do same w/ differentials

$$dw = 3u^2 du - u2v\,dv \cancel{+} v^2 du$$

$$dv = du + dx$$

$$\cancel{dw = 3u\,du - u \cdot 2(du + dx)\,dv + (du+dx)^2 dv}$$

$$dw = 3u^2 du - u2v(du + dx) \cancel{+} v^2 du$$
$$3u^2 du - 2uv\,du + 2vu\,dx \cancel{+} v^2 du$$
$$3u^2 \cancel{+} v^2 - 2uv\,du - 2vu\,dx$$

$$\left(\frac{\partial w}{\partial u}\right)_x = 3u^2 \cancel{+} v^2 - 2uv$$

almost right
product rule

$$\left(\frac{\partial w}{\partial x}\right)_u = -2vu$$ ✓

? and you do both at once → faster

Lecture 16 Double + iterated integrals in rect. coordinates

3A-1a $\int_0^2 \int_{-1}^1 (6x^2y+2y)\, dy\, dx$

↑ copy problem wrong

So this is volume under a surface
Integrate area in slice
Then Integrate all of the slices

So inner

$$\int_{-1}^1 6x^2y+2y\, dy$$

$$\frac{6x^2y^2}{2}+\frac{2y^2}{2}\Big|_{-1}^{1}$$

$$3x^2y^2+y^2\Big|_{-1}^{1}$$

$$3x^2(1)^2+(1)^2-6(x)^2(-1)^2(-1)^2$$

$$\cancel{6x^2+1-6x^2+1}$$

$$12x^2$$

$$\int_0^2 12x^2\, dx$$

$$\frac{12x^3}{3}\Big|_0^2$$

32

lame way to screw up

7. $P = (1,-1,1)$ $\quad z = x^2 + y + 1$

$f(x,y,z)$ differentiable
$\nabla f(x,y,z) = 2\hat{i} + \hat{j} - 3\hat{k}$ at that pt

$g(x,z) = f(x, y(x,z), z)$ Find ∇g at $(\underset{x}{1},\underset{z}{1})$

So x, z independent, y dependent ✓

So $\left(\frac{\partial g}{\partial x}\right)_z = 1 \quad \left(\frac{\partial g}{\partial z}\right)_x = 1$

but how work backwards w/o function ?

$dz = 2x\,dx + dy$ ✓ $\Rightarrow$ at $P = 2dx + dy$
$dy = dz - 2x\,dx$

$df = 2x\,dx + dy - 3\,dz$ at P

Oh yeah had f and points
- but its ∇f

eliminate dy ✓

$df = 0\,dx - 2\,dz$ at $(x,y) = (1,1)$

so $\nabla g = \langle 0, -2 \rangle$ at $(1,1)$

? don't get

7 redo

OH

$z = x^2 + y + 1$ $\qquad p = (1, -1, 1)$

$f(x, y, z) \leftarrow$ don't know

- he said what the too

$\nabla f(x, y, z)$ at $p = 2\hat{i} + \hat{j} - \hat{k}$

$\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$

$g(x, z) = f(x, y(x, z), z)$ $\qquad$ recognize pattern

Find ∇g at $x = 1$, $z = 1$

Can rewrite $y = z - x^2 - 1$

$\nabla g = \langle \frac{\partial g}{\partial x}\hat{i} + \frac{\partial g}{\partial z}\hat{j} \rangle$

$(\text{find})_z \qquad (\text{find})_z$

$\frac{\partial g}{\partial x} \quad \frac{\partial f}{\partial x}\underbrace{\left(\frac{\partial x}{\partial x}\right)_z}_{1} + \frac{\partial f}{\partial y}\left(\frac{\partial y}{\partial x}\right)_z + \frac{\partial f}{\partial z}\underbrace{\left(\frac{\partial z}{\partial x}\right)_z}_{0}$

because g is a function of x

$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}(-2x) + 0$

from $y = z - x^2 - 1$

$=$ at P given $2\hat{i} + \hat{j} - \hat{k}$

$2 + 1(-2(1))^{\text{at that pt}} = 0$ $\qquad \rightarrow$

$= 0$

that is for 1st coord of gradient

$\nabla g = \langle 0, \frac{\partial g}{\partial z} \rangle$

need to find (use differential)

$dg = \frac{\partial g}{\partial x} dx + \frac{\partial g}{\partial z} dz$ = definition

coefficent

$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$

know

try to cancle that

$dy = dz - 2x\,dx$)

← know

$df = 2\,dx + (dz - 2x\,dx) + -3\,dz$

$(2 - 2x)\,dx + 1 - 3\,dz$

at pt = 1

$(2 - 2(1))\,dx - 2\,dz$

$\frac{\partial g}{\partial x} = 0$ verified

$\frac{\partial g}{\partial z} = -2$

$\langle 0, -2 \rangle$

d.

Hope this goes better

$$\int_0^1 \int_0^u \sqrt{u^2+4}\ dv\ du$$

$$\int_0^u (u^2+4)^{1/2}\ dv$$

$$v(u^2+4)^{1/2} \Big|_0^u$$

$$u(u^2+4)^{1/2} - 0$$

$$\int_0^1 u(u^2+4)^{1/2}\ du$$

$$s = u^2+4$$
$$ds = 2\,du\,u$$

$$\frac{1}{2}\int \sqrt{s}\ ds$$

$$\frac{1}{2} \cdot \frac{2s^{3/2}}{3}$$

$$\frac{s^{3/2}}{3}$$

$$\frac{1}{3}(u^2+4)^{3/2} \Big|_0^1$$

$$\frac{1}{3}(1^2+4)^{3/2}$$

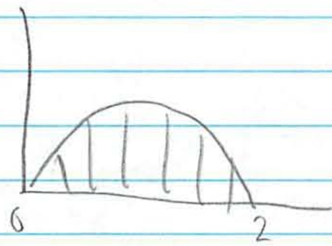
$$\frac{1}{3} \cdot 5^{3/2}$$

$$\frac{5\sqrt{5}}{3}$$

2b Express double integral as iterated integral

R is finite region by $y = 2x - x^2$ and x-axis

i) Express as $\iint_R dy\,dx$



what is iterated integral?
did we do that in lecture?

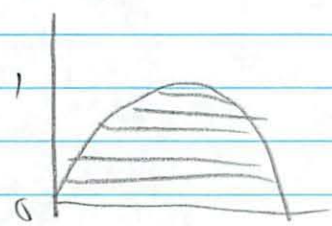
function y of x

Figure out bounds

$$\int_0^2 \int_0^{2x-x^2}$$? what here just dy dx

but how do we solve
don't think we have to - not in solution

b) $\iint_R dx\,dy$ now other way



$$\int_0^1 \int_{\cancel{2x-x^2}}^{\cancel{2x-x^2}}$$

^ not going to work

next →

Solve $y = 2x - x^2$ for x interms of y

$$-x^2 + 2x - y = 0$$

Solve w/ quadratic formula

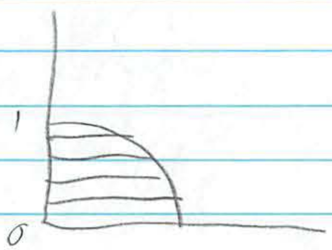
$$x = 1 \pm \sqrt{1-y}$$

$$\int_0^1 \int_{1-\sqrt{1-y}}^{1+\sqrt{1-y}} dx\,dy$$

3b. Evaluate $\iint_R (2x + y^2)\, dA$ $y^2 = 1 - x$ 1st quad

$dx\,dy$ easier

↑ means



$y = \sqrt{1-x}$

for dx

↑ need interms of x = something

$$\cancel{\int_0^1 \int_0^{\sqrt{1-x}} (2x+y^2)\, dx\,dy}$$

↑ why is this here?, and not before

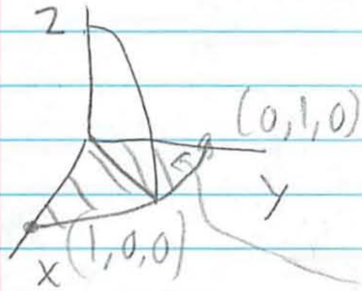
→

3b

$$\iint_R 2x + y^2 \, dA$$

$$y^2 = 1 - x$$

redo
Office
Hrs



R is the region

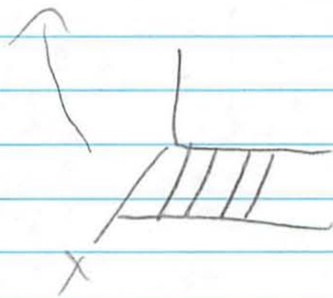
$$\int_0^1 \int^{\sqrt{1-y^2}} (2x + y^2) \, dx \, dy$$

find range

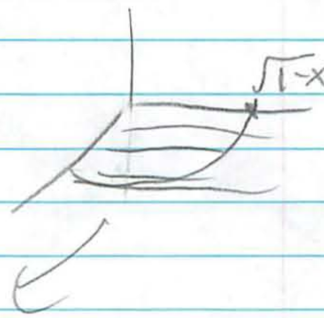
first part

first $\int$ over x

$1 - y^2 = x$ graphically how far it comes out of the page



dx



dy

$$\int_0^1 \int_0^{\sqrt{1-x}} \quad dy \, dx$$

Also need to actually do it

$$\int_0^1 \int_0^{1-y^2} (2x+y^2)\,dx\,dy$$

$$\int_0^{1-x^2} 2x+y^2\,dx$$

$$\frac{2x^2}{2} + y^2\frac{x}{1}\Big|_0^{1-y^2}$$

$$(1-y^2)^2 + (1-y^2)y^2 - 0$$

$$1^2 - 2y^2 \cancel{+y^4} + y^2 \cancel{-y^4}$$

$$-y^2+1$$

$$\int_0^1 -y^2+1\,dy$$

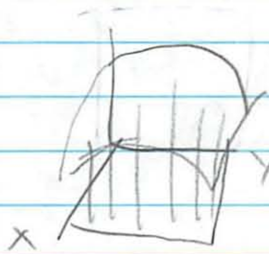
$$-\frac{y^3}{3}+y\Big|_0^1$$

$$-\frac{1^3}{3}+1-0$$

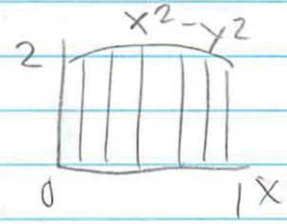
$$\frac{2}{3}$$ 0

4c Find by 2x integration volume of following solids

The finite solid lying underneath $x^2 - y^2$ above xy plane between $x=0$, $x=1$ planes



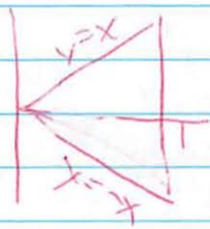
So lets do



$$\int_0^1 \int_0^{x^2-y^2} dy\,dx$$

what goes in here, if any thing?

function x^2-y^2 is 0 on $y=x$, $y=-x$
positive in region



what are axis of this graph

$$\iint_R (x^2-y^2)\,dA = \int_0^1 \int_{-x}^{x} (x^2-y^2)\,dy\,dx$$

Now solve

$$\int_{-x}^{x} x^2 - y^2 \, dy$$

$$x^2 y - \frac{x^3}{3} + C \Big|_{-x}^{x}$$

$$x^2(x) - \frac{(x)^3}{3} - \left[x^2(-x) - \frac{(-x)^3}{3} \right]$$

$$x^3 - \frac{x^3}{3} - \left[-x^3 + \frac{x^3}{3} \right]$$

$$x^3 - \frac{x^3}{3} + x^3 - \frac{x^3}{3}$$

$$2x^3 - \frac{2x^3}{3}$$

duh

$$\frac{6x^3}{3} - \frac{2x^3}{3}$$

$\frac{4x^3}{3}$ ← should have seen that

$$\int_0^1 \frac{4x^3}{3} \, dx$$

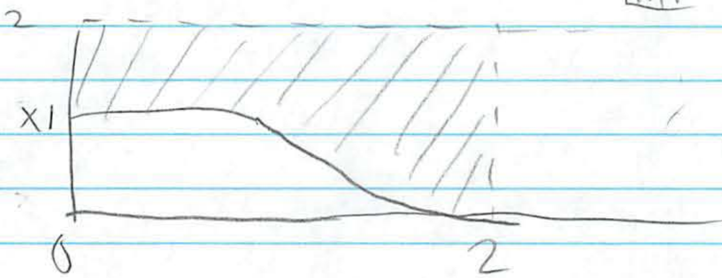
$$\frac{4x^4}{3 \cdot 4} \Big|_0^1 = \frac{x^4}{3}$$

$$\frac{(1)^4}{3} = \frac{1}{3}$$

5a. Evaluate each of the following interated integrals by changing the order of integration (draw R)

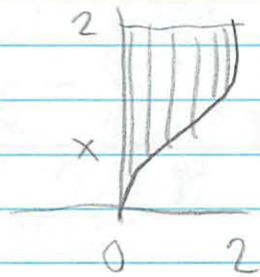
Once I get on a roll it gets fast

a) $\int_0^2 \int_x^2 e^{-y^2}\, dy\, dx$



So how do we flip - should be in notes

or is it



if set $y = e^{-y^2}$

??

don't really get

$= \int_0^2 \int_0^y e^{-y^2}\, dx\, dy$

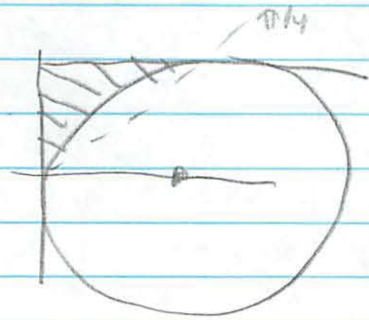
↑simple change

$= \int_0^2 e^{-y^2} \cdot y\, dy$

$= \frac{1}{2} e^{-y^2} \Big|_0^2 = \frac{1}{2}\left(1 - e^{-4}\right)$

36-1d

Office hrs



Go from origin - always from origin

$$\int_{\pi/4}^{\pi/2} \int^{a}$$

↳ fix θ

have to write lenght of r

Convert $y = a$

$r\sin\theta \Rightarrow y = a$ ← translate into polar do not forget

$r = \frac{a}{\sin\theta}$

Translate ~~...~~ circle function

$(x-a)^2 + y^2 = a^2$

$(r\cos\theta - a)^2 + (r\sin\theta)^2 = a^2$

$r^2 - 2ar\cos\theta = 0$

Solve for r

$r = 2a\cos\theta$

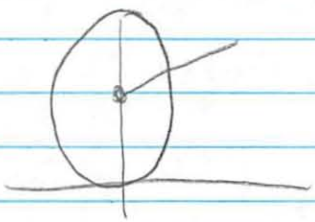
$$\int_{\pi/4}^{\pi/2} \int_{2a\cos\theta}^{a/\sin\theta}$$

Lecture 17 Double Integrals in Polar Coordinates

3B-1a Express each double integral over R as interated integral in polar coords.
May need to break up into parts

no ans on 1st problem stupid

b) Circle of radius 1 center at (0,1)

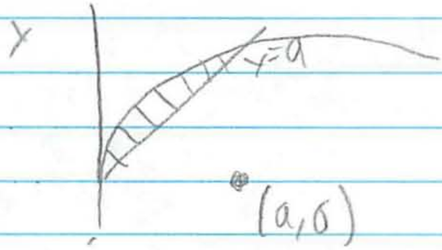


$r = x^2 + y^2$
↑lenght of r is that

$$\int_0^{2\pi}\int_0^{r} \cancel{x^2+y^2}\ \sqrt{1-r^2}\, r\,dr\,d\theta$$

well the +1 does not matter for volume, right

d Finite region



$$\int_{\frac{\pi}{2}}^{\pi}\int_a^{r} \cancel{x^2+y^2}\ \sqrt{1-r^2}\, r\,dr\,d\theta$$

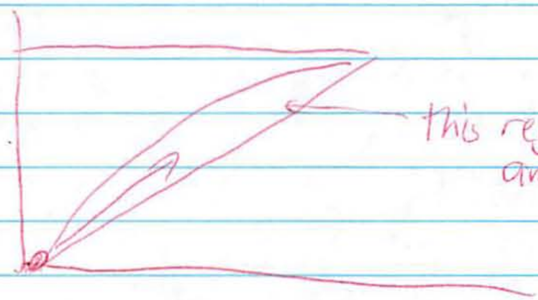
Well actually circle is $x^2+y^2+z^2=1$
$z = \sqrt{1-x^2-y^2}$ $\quad r = x^2+y^2$
$z = \sqrt{1-r^2}$

d

I don't get at all

$$\iint_R dr\,d\theta = \int_{\pi/4}^{\pi/2}\int_{2a\cos\theta}^{\csc\theta} dr\,d\theta$$

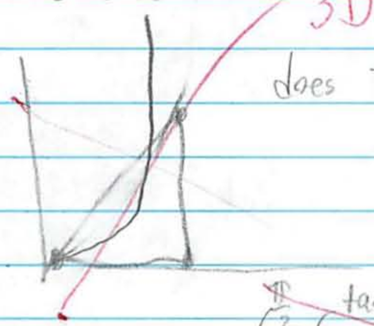
Oh they drew pic much differently



this region still I am guessing

2c Evaluate over the indicated region using polar coords

$$\iint_R \tan^2\theta \, dA \quad R \text{ is triangle } (0,0)(1,0)(1,1)$$



3D

does this not make more sense rectangular

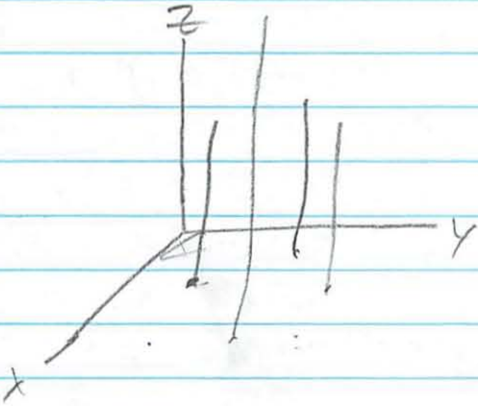
$$\cancel{\int_0^{\frac{\pi}{2}}\int_0^{\tan^2\theta} \tan^2\theta \; r\,dr\,d\theta}$$

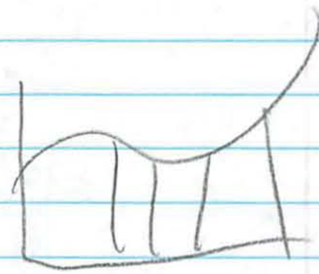
$$\int_0^{\frac{\pi}{4}}\int_0^{\sec\theta} \tan^2\theta \; r\,dr\,d\theta$$

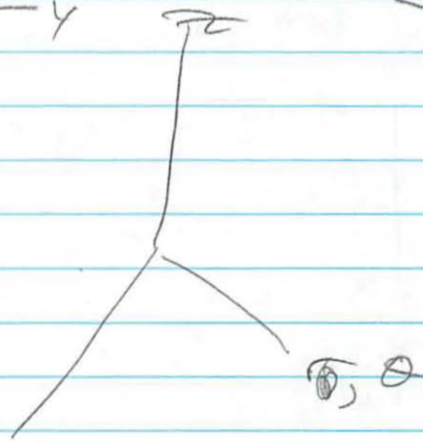
help from Jay →

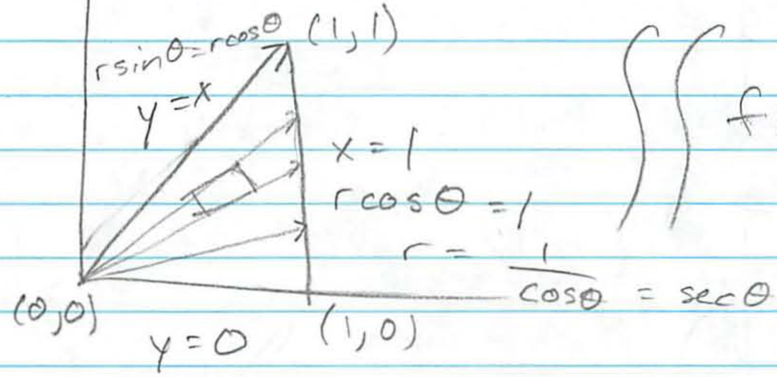
2c
help
from Jay

- last semester
student









$$\iint f(x,y)$$

$$\int_0^{\frac{\pi}{4}} \int_0^{\sec\theta} \tan^2\theta \, \underbrace{r\,dr\,d\theta}$$

+recitation 3/17

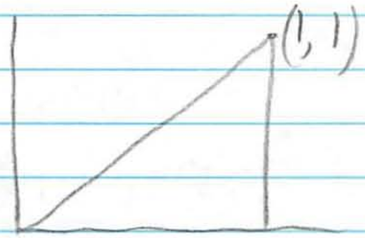
Yeah this makes much more sense

- ignore actual function for the boundries

- I was making it harder on myself

2c
redo
knowing
more

$\iint_R \tan^2\theta \, dA$



dA

choose

$\int_0^1 \int_0^x \, dy \, dx$

should be opposet letter

do 1st

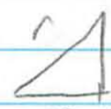
But want polar

in someways easier - other ways harder

this is the hard part here

$\int_0^{\frac{\pi}{4}} \int_0 \tan^2\theta \; r \, dr \, d\theta$

1st



$r = \tan^{-1}$

$\sec\theta$

why?

like Jay's picture

$x = 1$

$r\cos\theta = x = 1$ ← must know (used often)

$r = \frac{1}{\cos\theta} = \sec\theta$

Now need to actually do

$$\int_0^{\sec\theta} \tan^2\theta \; r\,dr$$

$$\frac{\tan^2\theta\, r^2}{2}\Bigg|_0^{\sec\theta}$$

$$\frac{\tan^2\theta \sec^2\theta}{2}$$

$$\int_0^{\pi/4} \frac{\tan^2\theta\sec^2\theta}{2}\, d\theta$$

$$\frac{1}{2}\int u^2\, du \qquad \begin{array}{l} u = \tan\theta \\ du = \sec^2\theta \end{array}$$

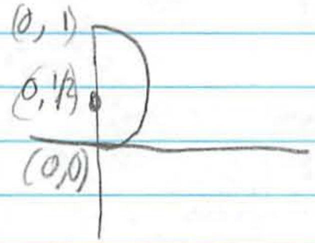
$$\frac{1}{2} \quad \frac{u^3}{3}$$

$$\frac{\tan^3\theta}{6}\Bigg|_0^{\pi/4}$$

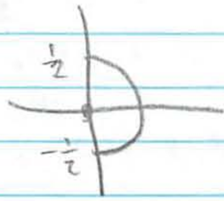
$$\frac{\tan^3\left(\frac{\pi}{4}\right)}{6} = \frac{1}{6}$$

2d $\displaystyle\iint_R \frac{dx\,dy}{\sqrt{1-x^2-y^2}}$

hopefully know now!



→ position does not matter



$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{1/2} ? \; r\,dr\,d\theta$$

$x = r\cos\theta$

* need to convert function $y = r\sin\theta$

$$\sqrt{1-r^2\cos^2\theta - r^2\sin^2\theta}$$

$$\sqrt{1-r^2(\cos^2\theta - \sin^2\theta)}$$

$$\sqrt{1-\cos(2\theta)}$$ double angle

now do

$$\int_0^{1/2} \sqrt{1-\cos(2\theta)}\; r\,dr$$

$$\left.\frac{\sqrt{1-\cos(2\theta)}\,r^2}{2}\right|_0^{1/2}$$

$$\frac{\sqrt{1-\cos(2\theta)}\left(\frac{1}{4}\right)}{2}$$

$$\int_{-\pi/2}^{\pi/2} \frac{\sqrt{1-\cos(2\theta)}}{8}\,d\theta$$

$$-\frac{\sqrt{1-\cos(2x)}\ \cot x}{8}\Bigg|_{-\pi/4}^{\pi/4}$$

$$=\frac{1}{4}\left(-1+\sqrt{2}\right)$$

$$\approx .10355$$

does not seem right

I went the wrong way w/ x and y

$$\rightarrow r=\sqrt{x^2+y^2}$$

So $\displaystyle\int_0^{\pi/2}\int_0^{\sin\theta}\frac{r}{1-r^2}\,dr\,d\theta$

$$\int_0^{\sin\theta}\frac{r}{\sqrt{1-r^2}}\,dr$$

$$-\frac{1}{2}\int\frac{1}{\sqrt{u}}\,du \qquad \begin{aligned}u&=1-r^2\\du&=-2r\,dr\end{aligned}$$

$$-\sqrt{u}$$

$$-\sqrt{1-r^2}\Big|_0^{\sin\theta}$$

$$\int_0^{\pi/2}-\sqrt{1-\sin^2\theta}\ d\theta$$

$$-\int_0^{\pi/2}\sqrt{\cos^2\theta}\ d\theta$$

$$-\int_0^{\pi/2}\cos\theta\ d\theta$$

$$\theta - \sin\theta \Big|_0^{\pi/2}$$

had to use pc to integrate - be able to do!, standard HS calc integration

$$\theta - \sin\frac{\pi}{2}$$

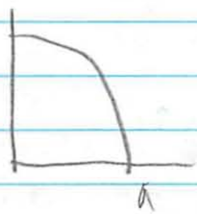
$$\frac{\pi}{4} \cdot -1$$

3b. Find volume by integrating polar coords

~~[illegible]~~

domain under xy over quarter disk region inside $x^2+y^2=a^2$

?? why can't they just rewrite this ??



$$\int_0^{\pi/2}\int_0^{a} xy \; r\,dr\,d\theta$$

this is the plane!

* convert x and y

$$\int_0^{\pi/2}\int_0^{a} r\cos\theta \cdot r\sin\theta \cdot r \; dr\;d\theta$$

$$\int_0^{\pi/2}\int_0^{a} r^3\cos\theta\sin\theta \; dr\;d\theta$$

$$\int_0^{a} r^3\cos\theta\sin\theta \; dr$$

$$\frac{r^4 \cos\theta \sin\theta}{4} \Big|_0^a$$

$$\int_0^{\pi/2} \frac{a^4 \cos\theta \sin\theta}{4} \, d\theta$$

$$\frac{a^4 \cancel{\cos^2\theta}\ \frac{1}{2}}{8} \Big|_0^{\pi/2}$$

keep making quick mistakes

$$\frac{a^4 \cancel{\cos^2\left(\frac{\pi}{2}\right)}}{\cancel{8}}$$ but $\cos\theta\left(\frac{\pi}{4} = 0\right)$

0 $\frac{a^4}{8}$

3c The domain lying under $z = \sqrt{x^2+y^2}$ over circle

cone

radius 1 center $(0,1)$

does not matter, right?

- did something like this in recitation

So base of a cone $\rightarrow x^2+y^2=1$

$$Vol = \iint_R f \, dx \, dy$$

$$= \iint_{R} f \, r \, dr \, d\theta$$

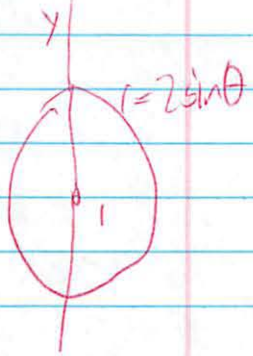
unit disk

$$\int_0^{2\pi} \int_0^1 f \, dr \, d\theta$$

- So what does circle's domain have to do w/ it?

So that was the formula

In order to use integral formula at 3B
use symmetry about y axis to compute
volume on right + double that answer



$$\iint_R \sqrt{x^2+y^2}\, dA$$

$$= 2\int_0^{\pi/2}\int_0^{2\sin\theta} r\, r\, dr\, d\theta$$

adding vertical slices? $= 2\int_0^{\pi/2} \frac{1}{3}(2\sin\theta)^3\, d\theta$

$$= 2\cdot\frac{8}{3}\cdot\frac{2}{3}$$

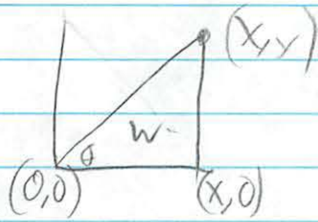
$$\frac{32}{9}$$

Oh there were formulas in the start of 3B question section
- just integration formulas

Think I visualized wrong

Part 2

1. Using the usual rectangular + polar coords



Express w in terms of x, y

$\downarrow$ $x\theta$

$w = \frac{1}{2} x y$ ✓

$\tan\theta = \frac{y}{x}$

$y = x\tan\theta$ ✓

Calc $\left(\frac{\partial w}{\partial x}\right)_\theta$ $\left(\frac{\partial w}{\partial \theta}\right)_x$ in 3 ways

a) Directly → express w in terms of x and θ

$w = \frac{1}{2}x(x\tan\theta)$

$= \frac{1}{2}x^2\tan\theta$

$\left(\frac{\partial w}{\partial x}\right)_\theta = \frac{1}{2}\cdot 2x\cdot\tan\theta$

diff x $\quad = x\tan\theta$ ✓

θ = constant

$y = \left(\frac{\partial y}{\partial x}\right)_\theta$ ← not here so no problems

$\left(\frac{\partial w}{\partial \theta}\right)_x = \frac{1}{2}\cdot\cancel{1}\cdot\sec^2\theta\, x^2$

$= \frac{1}{2}\sec^2\theta\, x^2$ -0.5

b By using chain rule

example $\left(\frac{\partial w}{\partial x}\right)_\theta = w_x \left(\frac{\partial x}{\partial x}\right)_\theta + w_y \left(\frac{\partial y}{\partial x}\right)_\theta$

formal partial deriv

isn't this 1

so are the 2 rules $w = \frac{1}{2} x y$

$\tan\theta = \frac{y}{x} \rightarrow y = x \tan\theta$

$\left(\frac{\partial w}{\partial x}\right)_\theta = \frac{1}{2} \cdot 1 \cdot \left(\frac{\partial y}{\partial x}\right)_\theta$

now I see a little clearer

$\left(\frac{\partial y}{\partial x}\right)_\theta = 1 \cdot \tan\theta + 0 \cdot x$

$\left(\frac{\partial w}{\partial x}\right)_\theta = \frac{1}{2} \cdot \tan\theta$ × -1.5

$= \frac{\tan\theta}{2}$ ← does not =

$\left(\frac{\partial w}{\partial \theta}\right)$

$$\left(\frac{\partial w}{\partial \theta}\right)_x = \text{ wait } \theta \text{ is not in original eq}$$

← does not matter

$$\frac{1}{2} \cdot x \left(\frac{\partial y}{\partial \theta}\right)_x$$

constant ↑ replace w

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$$\left(\frac{\partial y}{\partial \theta}\right)_x = 0$$

r,

0 ← ? again whats wrong

c) differentials

I actally like this better

$$dw = \frac{1}{2}\, dx\, dy \qquad dy = dx \quad d\theta$$ ← ? what is wrong

$$dw = \frac{1}{2}\, dx\, (dx\, d\theta)$$

? same different again

Shall really get this - what is missing ? !

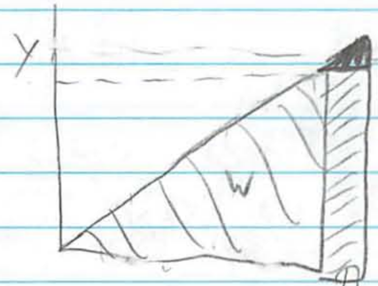
-1.5

d Using the triangular picture and geometric intuition

estimate $\left(\frac{\Delta w}{\Delta x}\right)_\theta$ and $\left(\frac{\Delta w}{\Delta \theta}\right)_x$

Show that agree w/ earlier results

office hrs Oliver



make x vary
keep θ fixed

$\Delta w \approx \underset{\substack{\uparrow \text{just } y \\ \text{- } y \text{ is not changing}}}{y \Delta x} + \underbrace{\frac{\Delta x \Delta y}{2}}$ ← small triangle on top, small 2nd order

$\frac{\Delta w}{\Delta x} \approx y + \frac{\Delta y}{2}$

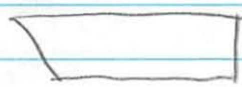
↳ $y = \left(\frac{\partial w}{\partial x}\right)_\theta$ ✓

ii)

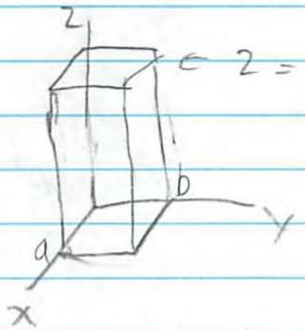
$\Delta w = \frac{r^2}{2}\Delta\theta$ + shape at end
↳ x held constant ✓

2. Rectangular prisim

but all 4 different



Show using double integration that volume is product of cross sectional area and average of all 4 edge lenghts



← $z = Ax + By + C$

but only 2 edges ????

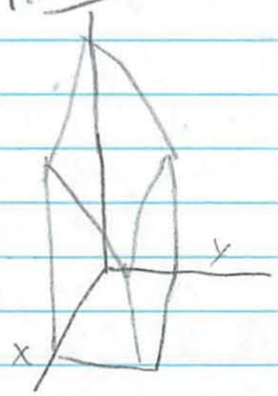
so $dy\,dx$

$$\int_0^a \int_0^b dy\,dx$$

$$\underbrace{\int_0^a \int_0^b}_{\text{of the region!}} Ax + By + C \; dy\,dx$$

Continued in method 2 →

Method 1



office
hrs
oliver

know x, y
plug in know height

$$\frac{C + aA + C + aA + bB + C + bB + C}{4}$$ average

$$C + \tfrac{1}{2}aA + \tfrac{1}{2}bB$$ reduces to

Vol = $ab\left(C + \frac{aA}{2} + \frac{bB}{2}\right)$
Method 1

self

Method 2

$$\int_0^a \int_0^b Ax + By + C \; dy\, dx$$

$$Axy + B\frac{y^2}{2} + Cy \Big|_0^b$$

$$\int_0^a Axb + B\frac{b^2}{2} + Cb \; dx$$

$$Ab\frac{x^2}{2} + B\frac{b^2}{2}x + Cbx \Big|_0^a$$

$$Ab\frac{a^2}{2} + B\frac{b^2}{2}a + Cba$$ ✓

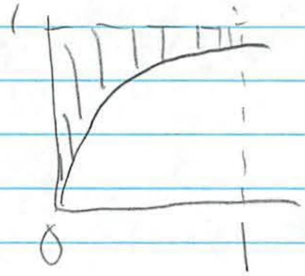
Cool how this works out

Vol = $ab\left(A\frac{a}{2} + B\frac{b}{2} + C\right)$ ① same will do after real recitation - when practice it more

3. Evaluate by changing order of integration

$$\int_0^1 \int_{\sqrt[3]{x}}^1 \sqrt{1+y^4}\, dy\, dx$$

So need to draw pic of what limit is



$\sqrt[3]{x} = y$

$x = y^3$

don't function - right ?!?

$$\int_0^1 \int_0^{y^3} \sqrt{1+y^4}\, dx\, dy \checkmark$$

y x

$$\int_0^{y^3} \sqrt{1+y^4}\, dx$$

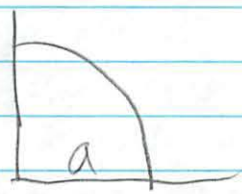
$$x\sqrt{1+y^4} \Big|_0^{y^3}$$

$$\int_0^1 y^3\sqrt{1+y^4}\, dy \checkmark$$

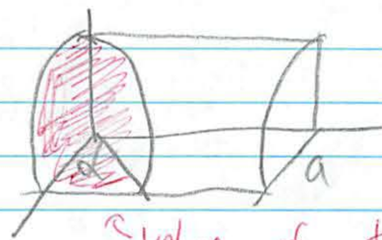
$$\frac{(y^4+1)^{3/2}}{6} \Big|_0^1 \checkmark$$

$$\frac{(1^4+1)^{3/2}}{6} = \frac{\sqrt{2}}{3} \times$$

4. Quarter of cylindrical log



weg shaped piece removed



volume of that piece

Find Volume of piece w/ 2x integration

Check by finding volume when $a = \frac{\pi}{4}$

---

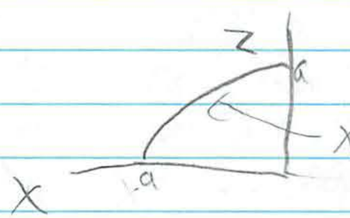
office hrs
oliver

volume → below graph of function

$$= \iint_R f \, dx \, dy$$

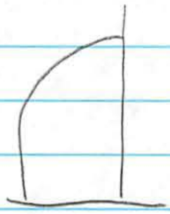
triangle on bottom

$f(x,y)$ = quarter of circle
= z
= cylinder around y-axis radius a



$= \sqrt{a^2 - x^2}$

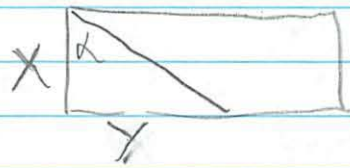
5. The base is that end of the log (it's so much ~~ea~~ eaiser than I first thought)



polar coords

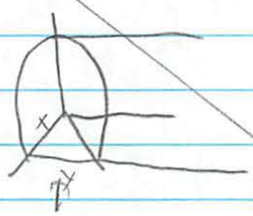
$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} \int_0^a f(x,y)\, r\, dr\, d\theta$$

↑ have to find a function for the slice



top view

$\tan\alpha = \frac{y}{x}$



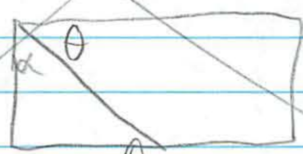
so x, y

from triangle

– but this is polar

$y = x\tan\alpha$

$x = \frac{1}{\tan\alpha}$

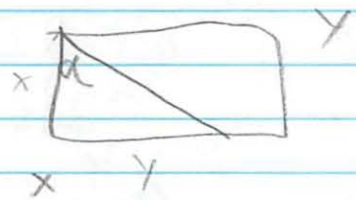


↑ note r lenght $= x^2 + y^2$

$$\int_{\frac{3\pi}{2}+\alpha}^{2\pi} \int$$

Don't want boundries

want plane equation!



What is equation of a plane
- normal vector

So $\searrow$ $x\hat{i} + y\hat{j}$

So $-y\hat{i} + x\hat{j}$

Plane eq $-yA + xB$

$-x\tan\alpha + xB$

or $x = r\cos\theta$

$y = r\sin\theta$

$-r\sin\theta\tan\alpha + r\cos\theta$

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} \int_0^a -r\sin\theta\tan\alpha + r\cos\theta \; r\,dr\,d\theta$$

$$\int_0^a -r^2\sin\theta\tan\alpha + r^2\cos\theta \; dr$$

$$\left.\frac{r^3(\cos\theta\cos\alpha - \sin\theta\sin\alpha)}{3\cos\alpha}\right|_0^a$$

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} \frac{a^3(\cos\theta\cos\alpha - \sin\theta\sin\alpha)}{3\cos\alpha}$$

$$\left.\frac{a^3(\cos\theta\sin\alpha + \sin\theta\cos\alpha)}{3\cos\alpha}\right|_{\pi/2}^{3\pi/4}$$

have calc do bounds :)

$$\frac{a^3\left(-\left(\left(\sqrt{2}-2\right)\cos\alpha-\sqrt{2}\sin\alpha\right)\right)}{6\cos\alpha}$$

So volume when $\alpha = \pi/4$

$$\underset{(\alpha = \frac{\pi}{4})}{\text{Volume}} = -\frac{a^3}{3}$$

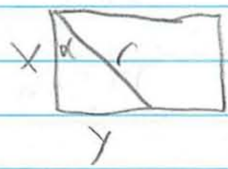
↑ why is volume ⊖?

Vol tetrahedron

$= \frac{1}{3}(\text{area base}) \cdot \text{height}$

↑ so we are going to do  up

area base =



so if $\alpha = \frac{\pi}{4}$

$\tan\frac{\pi}{4} = \frac{y}{x} = 1$ so $\frac{1}{1}$

$$\frac{1}{3} \cdot \left(\frac{1}{2} \cdot 1 \cdot 1\right) \cdot a$$

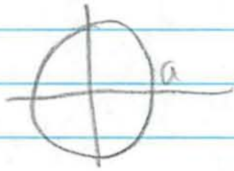
$\frac{a}{6}$ ← half my answer ✓

−2

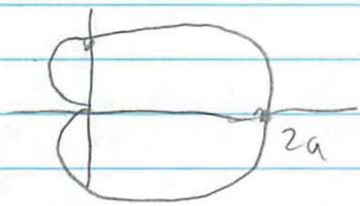
5a Exercise 20.4/9

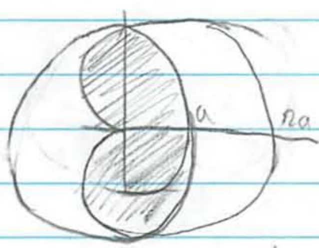
Use double integrals + polar coords

The region inside the cardioid $r = a(1+\cos\theta)$
and outside circle $r = a$



areas?
not volumes?





↑ So that is the only area that matters

just cortoid ↑ ↑ just half circle

$$\int_{-\frac{\pi}{4}}^{\pi/4} \int_0^a r\,dr\,d\theta + \int_{\pi/4}^{3\pi/4} \int_0^r a\left(1+\cos\theta\right) r\,dr\,d\theta$$

r is that for finding area of a circle

its when to use what that is confusing

Actually cartoid in book

$$= 2\int_0^{\pi}\int_0^{a(1+\cos\theta)} r\,dr\,d\theta$$

$$\cancel{= 2\int_0^{\pi}\left[\frac{1}{2}r^2\right]_0^{a(1+\cos\theta)} d\theta}$$

actualy finding but nothing there

$$\cancel{= 2\int_0^{\pi}\frac{1}{2}a^2(1+\cos\theta)^2\,d\theta}$$

$$\cancel{= a^2\int_0^{\pi}(1+2\cos\theta+\cos^2\theta)}$$

So

$$\int_{-\pi/4}^{\pi/4}\int_0^{a} r\,dr\,d\theta + \int_{\pi/4}^{3\pi/4}\int_0^{a(1+\cos\theta)} r\,dr\,d\theta$$

keep it like this - now need to actually do it

$$r^2\Big|_0^{a} \qquad + \qquad r^2\Big|_0^{a(1+\cos\theta)}$$

$$\int_{-\pi/4}^{\pi/4} a^2\,d\theta \qquad + \qquad \int_{\pi/4}^{3\pi/4}\left[a(1+\cos\theta)\right]^2$$

$$a^2\theta\Big|_{-\pi/4}^{\pi/4} \qquad + \qquad a^2\sin^2\theta\Big|_{\pi/4}^{3\pi/4}$$

$$a^2\frac{\pi}{4}+a^2\frac{\pi}{4}$$

$$\frac{\pi}{2}a^2 \qquad + \qquad \frac{\pi}{2}$$

$$\frac{\pi}{2}a^2+\frac{\pi}{2} \quad \times \quad -2$$

5b 20.4 #15

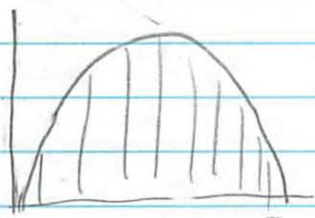
Evaluate by changing to polar

$$\int_0^{2a}\int_0^{\sqrt{2ax-x^2}} (x^2+y^2)\,dy\,dx$$

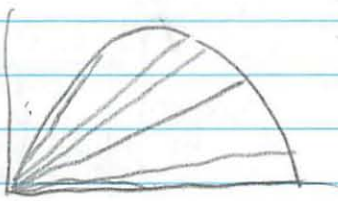
vertical

$x = r\cos\theta$
$y = r\sin\theta$

$r^2\cos^2\theta + r^2\sin^2\theta$
$r^2(1)$
r^2



0 2a

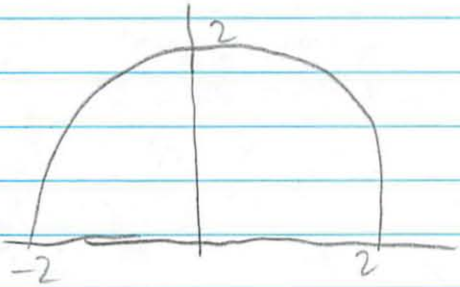


$$\int_0^{\pi/2}\int_0^{} r^2\, r\,dr\,d\theta$$

$$\sqrt{2a(r\cos\theta) - r^2\cos^2\theta}$$

$$\int_0^{\pi/2}\int_0^{\sqrt{2ar\cos\theta - r^2\cos^2\theta}} r^3\,dr\,d\theta$$

office
hrs
oliver
results



$r^2 - 2ar\cos\theta = 0$
$r = 2a\cos\theta$

$$\int_0^{\pi/2} \int_0^{2a\cos\theta}$$

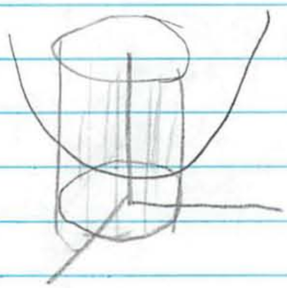
- but how does this fit w/ what I found
made me more confused :c

-2

5c 20.4 # 34a

For the solid bounded by the xy plane, the cylinder $x^2+y^2=a^2$, the parabolid $z=b(x^2+y^2)$ with $b>0$. Find the volume

office hrs
Oliver



$x^2+y^2=a^2$

$z=b(x^2+y^2)$

↑ parabola in cylinder
at some point it will cut it
want volume inside

$$Vol=\iint_R f(x,y)\,dx\,dy$$

↑ is a circle/disk

↑ is the parabola

if it was raised up

] g just subtract it out

$$\int_0^{2\pi}\int_0^a b(x^2+y^2)\,r\,dr\,d\theta$$

could also do in rectangular

self

Do I have to convert

$x = r\cos\theta$

$y = r\sin\theta$

$$\int_0^{2\pi}\int_0^{a} b\left(r^2\cos\theta^2 + r^2\sin\theta^2\right)$$

$$\int_0^{2\pi}\int_0^{a} b\ r^2\ r\ dr\ d\theta$$

$$\int_0^{a} b\ r^3\ dr\ d\theta$$

$$b\frac{r^4}{4}\Big|_0^a$$

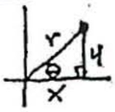
$$\int_0^{2\pi} b\frac{a^4}{4}\ d\theta$$

$$\frac{ba^4\theta}{4}\Big|_0^{2\pi}$$

Volume $= \boxed{2\pi\frac{a^4}{4}b}$ ✓

18.02 Sol'ns P.Set 5 Spring 2010

1 a) $\boxed{w = \frac{1}{2}xy, \quad y = x\tan\theta}$



a) using x, θ as ind. vars:

$$w = \frac{1}{2}x^2 \tan\theta$$

$$\left(\frac{\partial w}{\partial x}\right)_\theta = x\tan\theta, \quad \left(\frac{\partial w}{\partial \theta}\right)_x = \frac{1}{2}x^2\sec^2\theta$$

$$(= y) \qquad\qquad (= \tfrac{1}{2}r^2)$$

b) $$\left(\frac{\partial w}{\partial x}\right)_\theta = w_x + w_y\left(\frac{\partial y}{\partial x}\right)_\theta$$

$$= \frac{1}{2}y + \frac{1}{2}x\cdot\tan\theta = \begin{cases} y \\ x\tan\theta \end{cases}$$

$$\left(\frac{\partial w}{\partial \theta}\right)_x = w_x\underbrace{\left(\frac{\partial x}{\partial \theta}\right)_x}_{=0} + w_y\left(\frac{\partial y}{\partial \theta}\right)_x$$

$$= \frac{1}{2}x\cdot x\sec^2\theta = \begin{cases} \frac{1}{2}x^2\sec^2\theta \\ \frac{1}{2}r^2 \end{cases}$$

c) $$dw = \frac{1}{2}y\,dx + \frac{1}{2}x\,dy$$

$$dy = \tan\theta\cdot dx + x\sec^2\theta\,d\theta$$

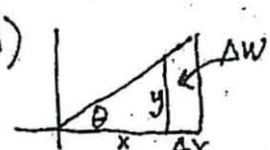
Eliminating dy:

$$dw = \left(\frac{1}{2}y\,dx\right) + \frac{1}{2}x\left(\tan\theta\cdot dx + x\sec^2\theta\cdot d\theta\right)$$

$$= \frac{1}{2}(y + x\tan\theta)\,dx + \frac{1}{2}(x^2\sec^2\theta)\,d\theta$$

coefficients are $\left(\frac{\partial w}{\partial x}\right)_\theta$ and $\left(\frac{\partial w}{\partial \theta}\right)_x$

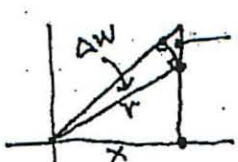
d)



$$\Delta w \approx y\,\Delta x$$

$$\therefore \frac{\Delta w}{\Delta x} \approx y = \left(\frac{\partial w}{\partial x}\right)_\theta$$

(θ held constant)



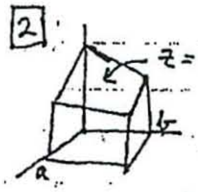
arc is $r\Delta\theta$

$$\therefore \Delta w \approx \frac{1}{2}r\cdot r\Delta\theta$$

$$\frac{\Delta w}{\Delta \theta} \approx \frac{1}{2}r^2 = \left(\frac{\partial w}{\partial \theta}\right)_x$$

(x held constant)

2



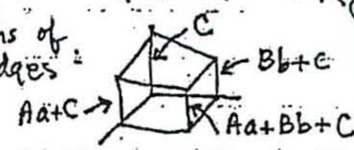
$$\text{vol} = \int_0^a\int_0^b (Ax+By+C)\,dy\,dx$$

Inner: $$Axy + \frac{By^2}{2} + Cy\Big]_0^b = Abx + \frac{Bb^2}{2} + Cb$$

Outer: $$Ab\frac{x^2}{2} + \left(\frac{Bb^2}{2} + Cb\right)x\Big]_0^a = Ab\,a^2/2 + \left(\frac{Bb^2}{2} + Cb\right)a$$

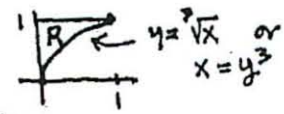
$$\text{volume} = ab\left(\frac{Aa}{2} + \frac{Bb}{2} + C\right) \leftarrow (*)$$

lengths of 4 edges:

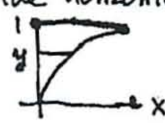


Quantity in parentheses $(*)$ = (sum of four edges)/4

3 The region is



Making slices in the horizontal direction:



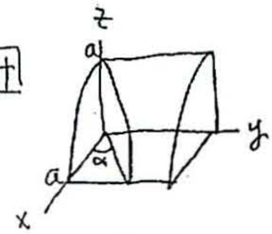
$$\int_0^1\int_0^{y^3} \sqrt{1+y^4}\,dx\,dy$$

Inner integral: $$\int_0^{y^3}\sqrt{1+y^4}\,dx = (1+y^4)^{1/2}x\Big]_0^{y^3} = y^3(1+y^4)^{1/2}$$

outer integral: $$\int_0^1 y^3(1+y^4)^{1/2}dy = \frac{1}{6}(1+y^4)^{3/2}\Big]_0^1$$

$$= \frac{1}{6}\left[2\sqrt{2} - 1\right]$$

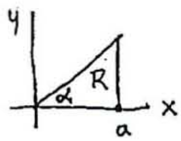
4



The equation of the cylindrical top surface is $x^2 + z^2 = a^2$

or $z = \sqrt{a^2 - x^2}$

Region in the xy-plane is



the line is $y = x \tan\alpha$

$\therefore V = \iint_R \sqrt{a^2-x^2}\, dy\,dx$

$= \int_0^a \int_0^{x\tan\alpha} \sqrt{a^2-x^2}\, dy\,dx$

Inner integral $= \sqrt{a^2-x^2}\, y \Big]_0^{x\tan\alpha}$

$= x\sqrt{a^2-x^2} \cdot \tan\alpha$

Outer integral:

$= \tan\alpha \cdot \int_0^a x(a^2-x^2)^{1/2}\, dx$

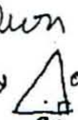
$= \tan\alpha \cdot -\frac{1}{3}(a^2-x^2)^{3/2} \Big]_0^a$

$= \frac{\tan\alpha}{3} \cdot a^3$

If $\alpha = \pi/4$, $\tan\alpha = 1$,

Volume $= a^3/3$

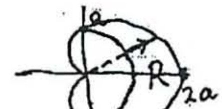
Volume of tetrahedron with height a & base



$= \frac{1}{3}$ base $\cdot$ height

$= \frac{1}{3} \cdot \frac{a^2}{2} \cdot a = \frac{a^3}{6}$, half the volume.

5 a) $\boxed{20.4/9}$ By symmetry,



$\frac{1}{2}$ area $R = \int_0^{\pi/2} \int_a^{a(1+\cos\theta)} r\,dr\,d\theta$

enters R where $r = a$, leaves where $r = a(1+\cos\theta)$

Inner: $\frac{r^2}{2}\Big]_a^{a(1+\cos\theta)}$

$= \frac{a^2}{2}\left[(1+\cos\theta)^2 - 1\right]$

$= \frac{a^2}{2}\left[2\cos\theta + \cos^2\theta\right]$

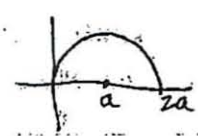
Outer: $\frac{a^2}{2}\int_0^{\pi/2}(2\cos\theta + \underbrace{\cos^2\theta}_{\frac{1+\cos 2\theta}{2}})\, d\theta = \frac{a^2}{2}\left[2\sin\theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^{\pi/2}$

$= \frac{a^2}{2}\left[2 + \frac{\pi}{4} + 0\right]$

$\therefore$ area $R = a^2\left(2 + \frac{\pi}{4}\right)$

(double the above)

b) $\boxed{20.4/15}$ $\int_0^{2a}\int_0^{\sqrt{2ax-x^2}} (x^2+y^2)\, dy\,dx.$



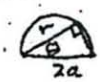
$y = \sqrt{2ax - x^2}$ is

$y^2 + x^2 = 2ax$

(completing the square) or $y^2 + (x-a)^2 = a^2$

upper circle shown

Polar eqn is $r = 2a\cos\theta$:



$\therefore$ integral becomes

$\int_0^{\pi/2}\int_0^{2a\cos\theta} r^2 \cdot r\,dr\,d\theta$

Inner: $\frac{r^4}{4}\Big]_0^{2a\cos\theta} = 4a^4\cos^4\theta$

Outer: $4a^4\int_0^{\pi/2}\cos^4\theta\, d\theta = 4a^4 \cdot \frac{3\cdot 1}{4\cdot 2}\cdot\frac{\pi}{2}$

$= \frac{3\pi}{4}a^4$

c) $\boxed{20.4/31a}$ $x^2 + y^2 = r^2$; in polar coords:

$\int_0^{2\pi}\int_0^a br^2 \cdot r\,dr\,d\theta$

Inner: $b\frac{r^4}{4}\Big]_0^a = \frac{ba^4}{4}$

Outer: $2\pi \cdot \frac{ba^4}{4} = \frac{\pi b a^4}{2}$

Lecture 18

3/18

Changing variables in a double $\iint$

Ex $\iint_R f(x,y)\, dy\, dx$

polar :	$x = r\cos\theta$ $y = r\sin\theta$	$r = \sqrt{x^2+y^2}$ $\theta = \arctan \frac{y}{x}$

inverse substitution
direct substitution
} sometimes use one sometimes other

change | limits | $r\,dr\,d\theta$
integrand

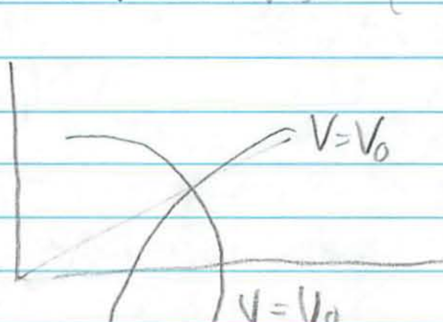
Must pratice!

$$\iint_R f(x,y)\, dA$$

$x = x(u,v)$
$y = y(u,v)$

in u-v coordinates (like polar coord)

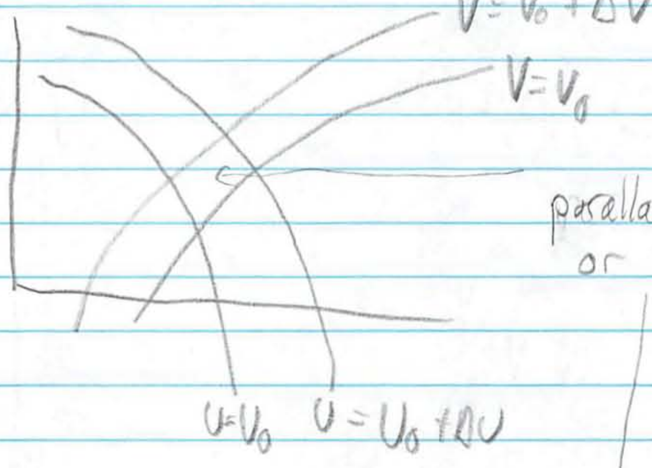
look for grid lines
(unit curves)
$u = u_0 = c_1$
$v = v_0 = c_2$



don't have to be $\perp$ to each other
makes integral nicer

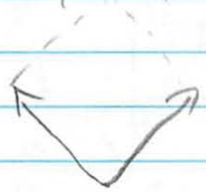
Find expression for typical little piece

- harder for this system



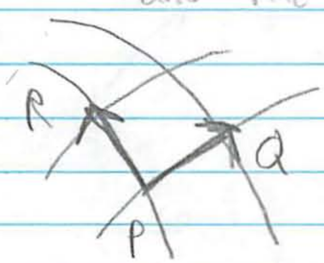
parallagram more or less

How to find area of a parallogram



$$\text{area} = \pm \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

- so draw the vectors
- and find out what they are explicitly in terms of $\hat{i}$ and $\hat{j}$ system



$\vec{PQ}$

$= \Delta x\,\hat{i} + \Delta y\,\hat{j}$

$$\Delta x \approx x_u \Delta u + x_v \Delta v$$

moved along level curve, v change

$$\Delta y \approx y_u \Delta u + y_v \Delta v$$

$\vec{PQ} = (x_u \hat{i} + y_u \hat{j})\,\Delta u$ ← moving along $v = v_0$

$\vec{PR} = (x_v \hat{i} + y_v \hat{j})\,\Delta v$ ← moving along $u = u_0$

so warm feels like 1st day vacation

$$\Delta A \approx \begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix} \Delta u\, \Delta v$$

its always written as transpose

$$dA = \Delta A \approx \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} du\, dv$$

called The Jacobian

$\dfrac{\partial(x,y)}{\partial(u,v)}$ ← scalar function

just the matrix = The Jacobian Matrix

$$\left[\frac{\partial(x,y)}{\partial(u,v)}\right]$$

Polar Coordinates

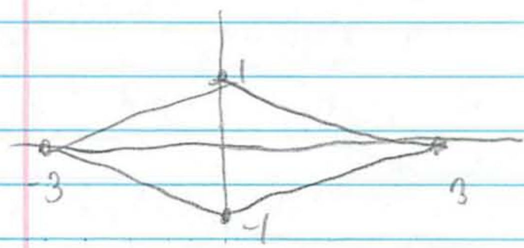
$$J = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$\cos\theta \cdot r\cos\theta - (-r\sin\theta \cdot \sin\theta)$$
$$r\cos^2\theta + r\sin^2\theta$$
$$r$$

so that is why it is $r\,dr\,d\theta$

example + Putting in Limits

$$\iint_R \underbrace{(x^2 - 9y^2)^4}_{\text{split}} \, dy\,dx$$



$$x^2 - 9y^2 = (\underset{\uparrow v}{x - 3y})(\underset{\uparrow u}{x + 3y})$$ telling us

normally region has something to do w/

$$\iint u^4 v^4$$ ↑ must calc Jacobean

$$\frac{\partial(x,y)}{\partial(u,v)}$$

$u = x + 3y$

$v = x - 3y$

nieave: solve for x, y calc in usual way

Shortcut instead

$$\frac{\partial(x,y)}{\partial(u,v)} \cdot \frac{\partial(u,v)}{\partial(x,y)} = 1$$

$$\begin{vmatrix} 1 & 3 \\ 1 & -3 \end{vmatrix}$$

$$-3-3$$

$$-6$$

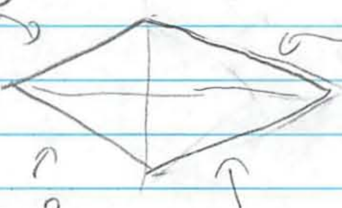
So that must be $-\frac{1}{6}$

what we actually want

(sometimes $\ominus$, but area always $\oplus$ so abs value)

$$\iint u^4 v^4 \cdot \underbrace{\tfrac{1}{6}\, dv\, du}_{dA}$$

$V=-3$



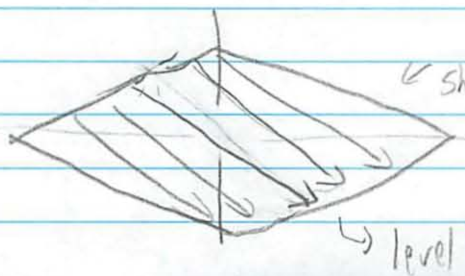
$\frac{x}{3} + \frac{y}{1} = 1$

so $U=3$

$U=-3$

$\frac{x}{3} + -\frac{y}{1} = 1$

$V=3$



should all be parallel

level curves of U

$$\int_{-3}^{3}\int_{-3}^{3} u^4 v^4 \cdot \frac{1}{6}\, dv\, du$$

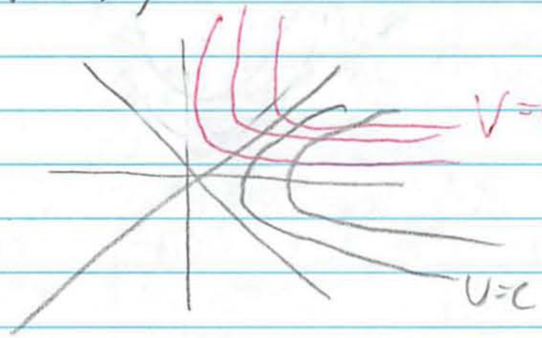
Can evaluate it yourslef

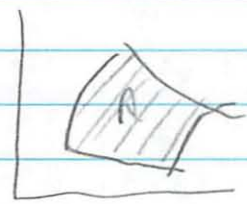
---

ex 2

$u = x^2 - y^2$

$v = xy$ ←





$J = \frac{\partial(x,y)}{\partial(u,v)}$ need

$$J^{-1} = \begin{vmatrix} 2x & -2y \\ y & x \end{vmatrix}$$

$$= 2(x^2+y^2)$$

$$= \frac{1}{2(x^2+y^2)}$$ ← when plug in integral in terms of x and y

Can get lucky + be able to solve

$$u^2 = x^4 + y^4 - 2x^2y^2$$
$$v^2 = x^2y^2$$

$$(x^2+y^2)^2 = u^2 + 4v^2$$

$$\frac{1}{2(x^2+y^2)} = \frac{1}{2\sqrt{u^2+4v^2}}$$

Notes Section CV

3/28

Sometime easier to work in polar coords

$$r = \sqrt{x^2+y^2} \qquad x = r\cos\theta$$
$$\theta = \tan^{-1}\left(\frac{y}{x}\right) \qquad y = r\sin\theta$$

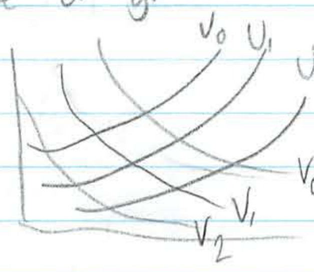
3 steps
1. Change $f(x,y)$ to $g(r,\theta)$
2. Supply area element $dA = r\,dr\,d\theta$
3. Use region to lock in limits of $\int$

Can also do for new coords $u\,v$

$$u = u(x,y) \qquad x = x(u,v)$$
$$v = v(x,y) \qquad y = y(u,v)$$

both directions
usually use one

Make a grid



for step 2 need a determinant called
The Jacobian

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$$

So we get

$$dA = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \, du \, dv$$

So change of variable formula

$$\iint_R f(x,y)\,dx\,dy = \iint_R g(u,v)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

So they as an example show it works w/ polar

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r(\cos^2\theta + \sin^2\theta) = r$$

what is this again???
partial deriv w/ respect to stuff

Ok lets start P-Set

Changing Limits

$$\iint_R f(x,y)\,dx\,dy = \iint_R g(u,v)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

$$g(u,v) = f(x(u,v), y(u,v))$$

$$x = x(u,v)$$
$$y = y(u,v)$$

$$\frac{\partial(x,y)}{\partial(u,v)} \cdot \frac{\partial(u,v)}{\partial(x,y)} = 1$$ ← Oh use this here

?? I really don't get what it is trying to say

Example

$$\iint_R \frac{y}{x}\,dx\,dy \qquad \begin{matrix} x^2-y^2=1 \\ x^2-y^2=4 \end{matrix} \qquad \begin{matrix} y=0 \\ y=\frac{x}{2} \end{matrix}$$

So this suggests $u = x^2 - y^2$, $v = \frac{y}{x}$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 2x & -2y \\ -\frac{y}{x^2} & \frac{1}{x} \end{vmatrix}$$

$$= 2 - 2\frac{y^2}{x^2}$$

$$= 2 - 2v^2$$

Flip $\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{2-2v^2} = \frac{1}{2(1-v^2)}$

Use this eq to put in limits + evaluate

$$\iint_R f(x,y)\,dx\,dy = \iint_R g(u,v) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$$

$$\iint_R \frac{y}{x}\,dx\,dy = \iint \frac{v}{2(1-v^2)}\,du\,dv$$

$$\int_0^{1/2} \int_1^4 \frac{v}{2(1-v^2)}\,du\,dv$$

$$-\frac{3}{4} \ln(1-v^2) \Big]_0^{1/2}$$

$$-\frac{3}{4} \ln\left(\frac{3}{4}\right)$$

Lecture 19

Physical applications of double integral

Makeup 3/27

Skipped lecture 3/19 due to Spring Break travel

Notes chap 20.3

$\iint_R f(x,y)\,dA$ gives volume of a certain solid if $f(x,y) \geq 0$

The way we've seen before is volume

$$dV = f(x,y)\,dA$$
$$V = \iint_R f(x,y)\,dA$$

dA sweeps over the reagon and we add up to the surface

We cut the quantity up into many small pieces and add the pieses together

Mass

If $\delta = \delta(x,y)$ is density

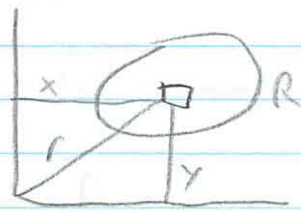
$\delta(x,y)\,dA$ is the mass contained in the element of dA

$$M = \iint_R \delta(x,y)\,dA$$ total mass of plate

Moment

Moment of element of mass $\delta(x,y)\,dA$ w/ respect to the x-axis is multiplied by the "lever arm" y namely $y\delta(x,y)\,dA$

$$M_x = \iint_R y\,\delta(x,y)\,dA$$



$$M_y = \iint_R x\,\delta(x,y)\,dA$$

Center of Mass

$$\bar{x} = \frac{M_y}{M} = \frac{\iint_R x\,\delta(x,y)\,dA}{\iint_R \delta(x,y)\,dA}$$

$$\bar{y} = \frac{M_x}{M} = \frac{\iint_R y\,\delta(x,y)\,dA}{\iint_R \delta(x,y)\,dA}$$

Where mass could be concentrated without changing momentum. When density δ is constant, the δ can be cancled away (since com is the geometric center)

Moment of Inertia

When square of lever arm used get moment of inertia of plate around corresponding axis

$$I_x = \iint_R y^2 \delta(x,y)\, dA$$

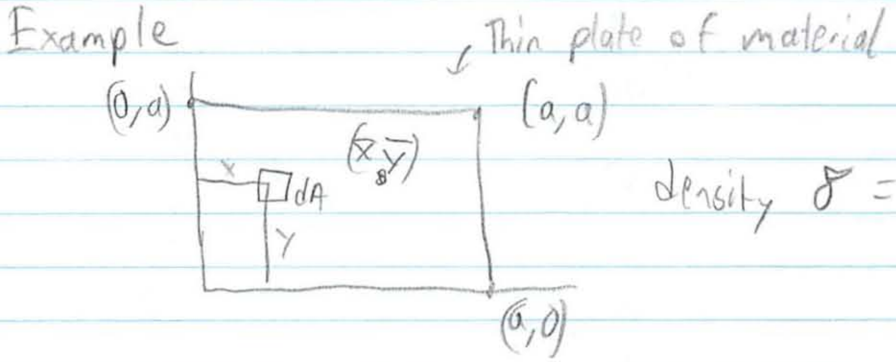
$$I_y = \iint_R x^2 \delta(x,y)\, dA$$

polar $\rightarrow$ $$I_z = \iint_R r^2 \delta(x,y)\, dA$$

$$r^2 = x^2 + y^2$$

Moment of inertia = capacity to resist angular acceleration

Example



density $\delta = xy$

$$M = \iint_R \delta\, dA$$

$$= \int_0^a \int_0^a xy\, dy\, dx$$

$$= \int_0^a \frac{1}{2} x y^2 \Big|_0^a dx$$

$$= \frac{1}{2} a^2 \int_0^a x \, dx$$

$$= \frac{1}{4} a^4$$

$$\bar{x} = \frac{M_y}{M} = \frac{4}{a^4} \iint_R x \, \delta \, dA$$

$$= \frac{4}{a^4} \int_0^a \int_0^a x^2 y \, dy \, dx$$

$$= \frac{4}{a^4} \int_0^a \left[\frac{1}{2} x^2 y^2 \right]_0^a dx$$

$$= \frac{2}{a^2} \int_0^a x^2 \, dx$$

$$= \frac{2}{3} a$$

By symmatry $\bar{x} = \bar{y} = \frac{2}{3} a$

$$I_x = \iint_R x^2 \, \delta \, dA$$

$$= \int_0^a \int_0^a x y^3 \, dy \, dx$$

$$= \int_0^a \left[\frac{1}{4} x y^4 \right] dx$$

$$= \frac{1}{4} a^4 \int_0^a x \, dx$$

$$= \frac{1}{8} a^6$$

And in terms of Mass

$$I_x = \frac{1}{2} Ma^2$$

Remarks

dA, dx, dy are not differentials

Euler's Formula

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

by doing a 2x integral

Alright lets see how P-set goes

Recitation

3/29

Lectures

Change of Variable $x, y \to u, v$

$$\iint_R f(x,y)\,dx\,dy = \iint_R f \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

def $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$ ← determinant

Useful $\dfrac{\partial(x,y)}{\partial(u,v)} = \dfrac{1}{\dfrac{\partial(u,v)}{\partial(x,y)}}$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = r$$

Pratical Applications

Averages of f over R

$$\frac{\iint_R f\,dx\,dy}{\iint_R dx\,dy}$$ ← area

Mass of 2D object

$$M = \iint_R dM$$

Center of Mass $(\bar{x}, \bar{y})$ $\qquad \bar{x} = \frac{1}{M} \iint_R x \, dr$

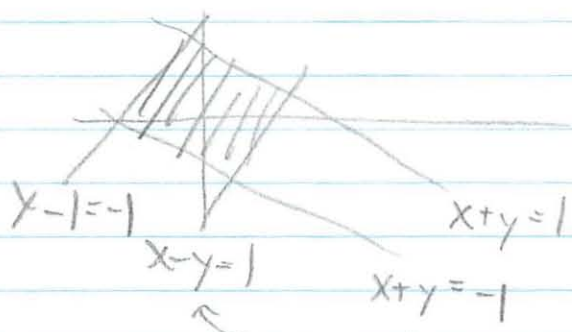
Moment of inertia $\qquad MI = \iint D^2 \, dM$

where D = distance to axis

$dM = \sigma \, dA = \delta \, dx \, dy$

small mass $\quad$ density $\quad$ small area

ex1 $\quad$ Compute $I = \iint_R (x+y) e^{x^2 - y^2} \, dx \, dy$



Use change of variables

So $\quad u = x + y$
$\quad\;\; v = x - y$

* We picked this because area easily expressed in terms of u and v

Can we express $f(x, y)$ in terms of u and v

$f(x,y) = ue^{uv}$

↳ Remember $x^2 - y^2 = (x+y)(x-y) = uv$

Now calculate Jacobian

↳ but use $\dfrac{\partial(x,y)}{\partial(u,v)} = \dfrac{1}{\frac{\partial(u,v)}{\partial(x,y)}}$

$$= \frac{1}{\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}} \qquad u_x = \frac{\partial u}{\partial x} \quad v_x = \frac{\partial v}{\partial x}$$

from $u = x+y$
$v = x-y$

$$= \frac{1}{\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}} = \frac{1}{-1-1} = -\frac{1}{2}$$

abs value $= \frac{1}{2}$

at this pt $\quad I = \iint_R ue^{uv} \dfrac{du\,dv}{2}$ "prefix"

Now express domain
- hardest part

$$R = \begin{cases} -1 \le x+y \le 1 \\ -1 \le x-y \le 1 \end{cases} = \begin{matrix} -1 \le u \le 1 \\ -1 \le v \le 1 \end{matrix}$$

$$\int_{-1}^{1}\int_{-1}^{1} u\,e^{uv} \frac{du\,dv}{2}$$

u v

du dv order

- just choose something
- depends on form of domain - sometimes one is eaiser

Now actually calculate (quickly)

$$= \int_{-1}^{1} \frac{e^{uv}}{2} \Big|_{-1}^{1} du$$

$$= \int_{-1}^{1} \frac{e^{u} - e^{-u}}{2} du$$

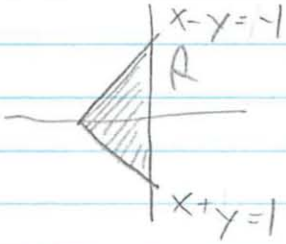
$$= \frac{e^{u} + e^{-u}}{2} \Big|_{1}^{-1}$$

$$= \frac{e^{1} + e^{-1}}{2}$$

... easy calculations

you will always be given a domain

b)



$$R \begin{cases} -1 \le x+y \le \\ -1 \le x-y \le \end{cases}$$

??? hardest part

well write   now express in u, v

$$\begin{cases} -1 \le x+y & \to & -1 \le u \\ -1 \le x-y & \to & -1 \le v \\ x \le 0 & \to & \dfrac{u+v}{2} \le 0 \end{cases}$$

$$\frac{x+y+x-y}{2} = \frac{2x}{2} = x$$

(*) Would never have thought of this! Manipulate u and v so matches the x and y you have (*)

$$\int_u \int_v u e^{uv} \frac{dv\,du}{2}$$

← for a given v, what is the range of u

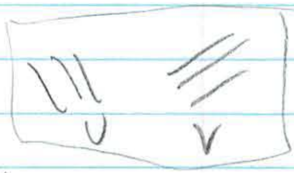
actually, this part is fairly hard too

$$\int_{-1}^{1} \int$$

$u \quad v$

still the same

constrained, depends on u → $\boxed{-1 \le v \le -u}$



its v that changes as u is fixed

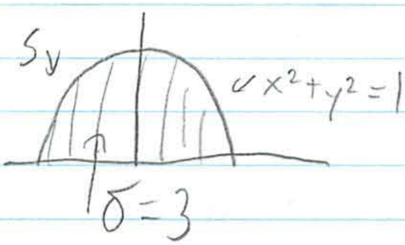
$$\int_{-1}^{1} \int_{-1}^{u} v\, e^{uv} \, \frac{dv\,du}{2}$$

Now solve as usual.

---

ex 2 Compute Moment of Inertia of S

was awake till 3 but awake now since got up at 10 sun + pre frosh here makes it diff



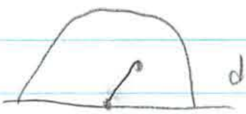
a) around $\hat{z}$ axis (out of board)

Want distance from axis^2

by definition

$$MI = \iint_R D^2 \, dm$$

$D = r = \sqrt{x^2+y^2}$



distance to any pt

Hint: Clearly polar coords

$$\cancel{\int_0^{\pi}\int_0^{r} \_\_\_\_ \, r\,dr\,d\theta}$$ ← almost had it forgot density did not write in function which is distance to axis^2

$$dM = \delta \, dx\,dy = 3r\,dr\,d\theta$$

↑ don't forget density

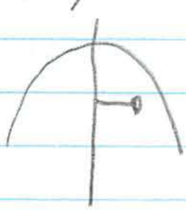
$$\iint_R r^2 \cdot 3r\,dr\,d\theta$$

$$\int_0^{\pi}\int_0^{r} 3r^3\,dr\,d\theta$$

Calculate .... $= \frac{3}{4}\pi$

On weekend alway try to do everything for week - of course does not get done

b) around y axis



← distance to point $= |x|$

↑ * key to these problems did not realize

Can you simplify using symmetry?

$$MI = 2 \iint_T D^2 \, dM$$

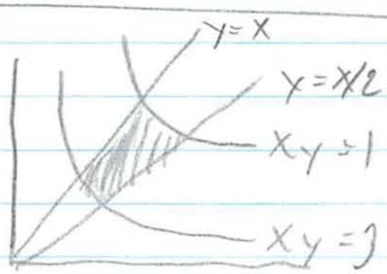
$$D = |x| = r\cos\theta$$

in polar!

$$MI = 2 \int_0^{\pi/2} \int_0^1 (r\cos\theta)^2 \cdot 3r\,dr\,d\theta$$

Calculate . . .

Ex 3



Find the average of function $F(x,y)$ over R

$$f(x,y) = (xy)^3$$

Change of variables

$u = xy$ ← pick these

$v = \dfrac{x}{y}$

- notice this is a good idea
- want to simplify
- notice can have $\dfrac{x}{y} = 1$ $\quad \dfrac{x}{y} = 2$

$\uparrow$

$y = \dfrac{x}{2}$

Now have nice region

$$R = \begin{cases} 1 \le u \le 3 \\ 1 \le v \le 2 \end{cases}$$

Still need to do Jacobean, right?

$$\hookrightarrow \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} y & x \\ \frac{1}{y} & -\frac{x}{y^2} \end{vmatrix} = -2\frac{x}{y}$$

Now want, in terms of u and v

$$\text{so} \quad -2\frac{x}{y} = -2v$$

Recitation / Review

3/31

Practice for exam 2

Questions

Ex1. Let $f(x, y) = x^2y^2 + x^2 + y^2$.
a) Find critical points of f and say wether they are local min, local max or saddle points.
b) Find where the minimum and maximum of f are attained on the unit circle $x^2 + y^2 = 1$ using Lagrange multipliers.
c) What are the minimimum and maximum values of f in the unit disc $x^2 + y^2 \leq 1$?

Ex2. Let $f = f(x, y)$ satisfying $\nabla f(1, 2) = < 3, 1 >$. Consider 2 new variables u, v.
a) Express $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ at the point $(x, y) = (1, 2)$ in terms of x_u, x_v, y_u, y_v.
b) Suppose $u = (x + y)/2$ and $v = (x - y)/2$ What are the value of $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ at $(x, y) = (1, 2)$?

Ex3. Find the volume above the graph $z = x^2 + y^2$ and below the plane $z = 4$.

Ex4. Evaluate $\int_0^1 \int_{x^2}^{x} e^y/(\sqrt{y} - y)\,dy\,dx$ by changing the order of integration.

Ex5. Let T be the triangle $(1, 0)$, $(0, 1)$, $(1, 1)$ with density $\delta = 1$.
a) Find the center of mass.
b) Find the moment of inertia around the x-axis.

Recitation

Review

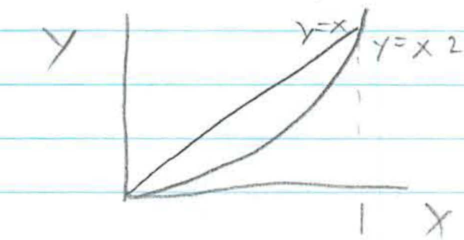
w/ question sheet

3/31

See sheet
ex 4

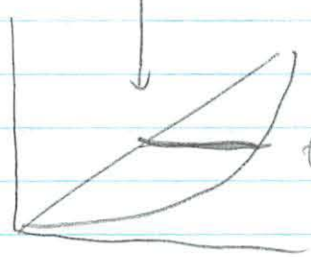
$$I = \int_0^1 \int_{x^2}^{x} \frac{e^y}{\sqrt{y} - y}\, dy\, dx$$

<u>Step 1</u> Draw graph of givens



Now Reverse Limits

$$\int_{y=0}^{1} \int_{x=} \frac{e^y}{\sqrt{y} - y}\, dx\, dy$$



fix a given y
see what going

$$\int_{y=0}^{1} \int_{x=y}^{\sqrt{y}} \frac{e^y}{\sqrt{y} - y}\, dx\, dy$$

Now actually calculate

$$I = \int_0^1 \frac{e^y}{\sqrt{y}-y} (\sqrt{y}-y)\, dy$$

$$= \int_0^1 e^y\, dy$$

$$e^y \Big|_0^1$$

$$e-1$$

---

ex2 a) Have x, y → want to go to new variables u, v

$\frac{\partial f}{\partial u}$ = "chain rule type thing"

$$= x_u f_x + y_u f_y$$

know $\nabla f(1,2) = \langle 3, 1 \rangle = \langle f_x, f_y \rangle$

$$\frac{\partial f}{\partial u} = 3x_u + 1 \cdot y_u$$

Similarly $\frac{\partial f}{\partial v} = 3x_v + 1 \cdot y_v$

b) $u = \frac{x+y}{2}$ $\quad v = \frac{x-y}{2}$

Solve for x, y

$u + v = x$
$u - v = y$

$\downarrow$

$x_u = 1 \quad x_v = 1$
$y_u = 1 \quad y_v = -1$

$$\frac{\partial f}{\partial u} = 3 \cdot 1 + 1 \cdot 1 = 4$$

$$\frac{\partial f}{\partial v} = 3 \cdot 1 \cdot 1 \cdot -1 = 2$$

---

ex 1 Optimization Problem
- find min + max

$$f(x, y) = x^2 y^2 + x^2 + y^2$$

a) We know we want $f_x = 0$
$f_y = 0$

$$f_x = 2xy^2 + 2x = 2x(y^2 + 1)$$
$$f_y = 2yx^2 + 2y = 2y(x^2 + 1)$$

so set = to

$$0 = 2x(y^2 + 1)$$
$$0 = 2y(x^2 + 1)$$

We know that

$y^2+1 \neq 0$ (can never $=0$)
$x^2+1 \neq 0$

So this means that

$x=0$
$y=0$

Only critical pt is $(x,y)=(0,0)$

Now Concavity or Saddle Point?

$f_{xx} f_{yy} - f_{xy}^2$

remember this →

$f_{xx} = 2(y^2+1)$
$f_{yy} = 2(x^2+1)$
$f_{xy} = 4xy$ ← take deriv of (f_x) w/ respect to y

$f_{xx} f_{yy} - f_{xy}^2$
$= 4(y^2+1)(x^2+1) - (4xy)^2$
$= 4$ at $(0,0)$

↳ this is (+) so look at f_{xx}
which is 2 at 0,0
(+) so local min

b) Let us denote $g(x,y) = x^2 + y^2 - 1$
so that the constraint is $g(x,y) = 0$

Lagrange system

remember this tool,

$$\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ x^2 + y^2 = 1 \end{cases}$$

$\downarrow$

$$\begin{cases} 2x(y^2+1) = \lambda\, 2x \\ 2y(x^2+1) = \lambda\, 2y \\ x^2 + y^2 = 1 \end{cases}$$

Case 1 $\sim$ $x \neq 0$, $y \neq 0$

$$\begin{cases} y^2 + 1 = \lambda \\ x^2 + 1 = \lambda \\ x^2 + y^2 = 1 \end{cases}$$

Now try to solve these equations

$$\begin{cases} x^2 = \lambda - 1 \\ y^2 = \lambda - 1 \\ x^2 + y^2 = 2\lambda - 2 = 1 \end{cases}$$

$$\begin{cases} \lambda = \frac{3}{2} \\ x^2 = \frac{1}{2} \\ y^2 = \frac{1}{2} \end{cases}$$

(x, y) can be

$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$
$\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$
$\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$
$\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

Case 2 :

$\boxed{x = 0}$

$2y = \lambda 2y$
$y^2 = 1$

(x, y) can be

$(0, 1)$
$(0, -1)$

Case 3 :

$\boxed{y = 0}$

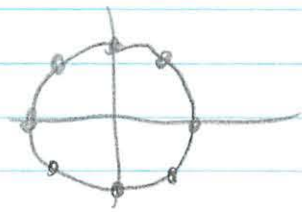
$2y = \lambda 2y$
$y^2 = 1$

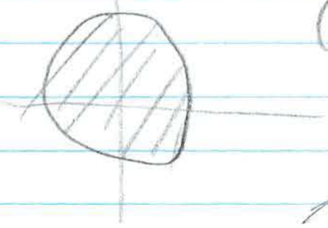
I think he did not write

(x, y) can be

$(1, 0)$
$(-1, 0)$

So 8 possible values



c) 

Canidates are from part b

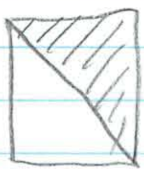
$f(0,0)$ critical pt on domain

plg into original eq to compute

(x,y)	$f(x,y)$	
$(0,0)$	0	← the minimum
$(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$	$\frac{5}{4}$	← maximum
$(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$	$\frac{5}{4}$	
⋮		
$(0,1)$	1	
$(1,0)$	1	

---

ex 5

 ← $\delta = 1$

need to memorize these formulas

a) Center of mass

$(\bar{x}, \bar{y})$

$$\bar{x} = \frac{1}{M}\iint_T x \, \underbrace{\delta \, dx \, dy}_{dM}$$

// Need to find mass first

$M = \text{Area} \cdot \delta$ (since δ is constant)

$\frac{1}{2} \cdot 1$

$\frac{1}{2}$

otherwise would have to $\int$ (this is what I really need to be clear on)

$$\iint_T x \, dx \, dy$$

$$\int_y \int_x x \, dx \, dy$$

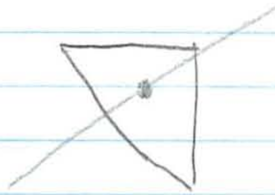
look at limit + be able to set it

$$\int_0^1 \int_{1-y}^1 x \, dx \, dy$$

$= \ldots, \frac{1}{3}$

$$\bar{x} = \frac{1}{1/2} \cdot \frac{1}{3} = \frac{2}{3}$$

$\bar{y} = \frac{2}{3}$ as well because of axis of symmetry

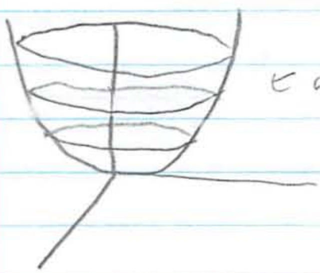


b) Compute Moment of Inertia

$$MI = \iint_T y^2 \delta \, dx \, dy$$

y^2 since y = distance from y axis

ex 3



← want volume inside domain

All of these qu seem kinda familiar - but can I do them from scratch? - prob not

$$Vol = \iint_R 4-(x^2+y^2) \, dx \, dy$$

$z=4$

What is the region?

Look from the top

$$x^2+y^2 \le z \le 4$$

R: $x^2 + y^2 \leq 4$

↳ Disk of Radius 2

Michael Plasmeier

Oliver 12 PM

PI 24/25
PII 14,5/20
38,5/45

18.02 Problem Set 6A due Thurs. Apr.8, 10:45 2-106

This problem set should be done before Exam 2, since the material will be included on the exam. It has Part I problems only, so you will have solutions available. Problem Set 6B will be given out Tuesday Mar. 30, with the later material not included in Exam 2. Hand in 6A and 6B stapled together, on Apr. 8.

Part I (10 points)

Lecture 18. Thurs. Mar. 18 Changing variables in a double integral.
Read: Notes CV.1-3 (through Example 3, top p. 5). Work: 3D-1,3,5,9.

(Hint for 3D-9, not solved in the Notes:

Since the equations (21) in Notes CV are obtained by solving (20) for u and v in terms of x and y, we can write

$$x(u(x,y),v(x,y)) = x, \qquad y(u(x,y),v(x,y)) = y\ .$$

Now use the chain rule to find the partial derivatives of both sides of these two equations with respect to x, and also find their partial derivatives with respect to y. This gives four equations in all involving partial derivatives.

Show that these four equations can be expressed by a single matrix equation

$$\begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix}\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(This is a good example of how matrices can be used to summarize and simplify a system of equations. The matrix on the left is called the *Jacobian matrix* of x and y with respect to u and v. There is no standard notation for it, but we will write it

$$\left[\frac{\partial(x,y)}{\partial(u,v)}\right] = \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix}$$

The determinant of the Jacobian matrix is called the *Jacobian* $\dfrac{\partial(x,y)}{\partial(u,v)}$ of x and y with respect to u and v. Note that the vertical lines are not used in the notation.)

Finally, finish the exercise, by using the determinant law for any two $n \times n$ matrices A and B: $|AB| = |A||B|$ as directed in the exercise.

Lecture 19. Fri. Mar. 19 Physical applications of double integration.
Read: 20.3 Work: 3C-1abc, 2ab, 4, 5 (cf. p. 565) .

Spring Break

Lecture 20. Tues. Mar. 30 Vector fields and line integrals in 2D.
Read: Notes V1; Textbook: 21.1
Problem Set 6B given out in lecture and posted.

Exam 2. Thurs. April 1 11:05-11:55 Walker 3rd floor

The exam will cover Lectures 9 - 19 on Partial Differentiation and Double Integration.
No books, notes, calculators; no use of cell-phones.

Practice Exam problems with solutions will be given out and posted Friday. Whatever formulas are given to you to use on the Practice Exam problems will also be given to you on the Exam 2 paper.

<u>18.02</u> P-Set 6A

Michael Plasmeier 3/28

Lecture 18 Changing Variables w/ a double integral

30-1 Evaluate $\iint_R \frac{x-3y}{2x+y}\,dx\,dy$ where R is bounded by $y=-2x+1$ $y=\frac{x}{3}$
$y=-2x+4$ $y=\frac{(x-7)}{3}$

$u=x-3y$
$v=2x+y$

Step 1 Convert function

$$dA = \Delta A \approx \begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix} du\,dv$$

$$\rightarrow \begin{vmatrix} 1 & -3 \\ 2 & 1 \end{vmatrix}$$

7

Now opposet for some reason
since we want $\frac{\partial(x,y)}{\partial(u,v)}$
but we found $\frac{\partial(u,v)}{\partial(x,y)}$ which was eaiser to calc

$\frac{1}{7}$

Step 2 Supply area element
Think I did that above

Step 3 Limits

So we had $u = x - 3y$
$v = 2x + y$

$2x + y = 1$ so v from $1 \to 4$
$2x + y = 4$

$x = 3y$ $\quad 0 = x - 3y$
$x - 7 = 3y$ $\quad 7 = x - 3y$

so u from $0 \to 7$

$\cancel{\int_1^4 \int_0^7}$ Since its $du\,dv$

v $\quad u$

Well they have $dv\,du$

$$\frac{1}{7} \int_0^7 \int_1^4 \frac{u}{v}\, dv\, du$$

realize thats what it is

modifier you find w/ Jacobian

But why this order??
Did they just pick this arbitrarily
- so you could switch and it would work just as well

Step 4 Solve
just like before

$$\int_1^4 \frac{u}{v}\, dv$$

$$\int_1^4 u v^{-1}\, dv$$

$$u \ln v \Big|_1^4$$

$$u \ln 4 - (u \ln 1)$$
$$u\,(2 \ln 2)$$

$$\int_0^7 u\,(2\ln 2)\, du$$

$$\frac{u^2 (\cancel{2} \ln 2)}{\cancel{2}} \Big|_0^7$$

$$7^2 \ln 2 - 0^2 \ln 2$$

$$\frac{1}{7} \cdot 49 \ln 2$$

$$7 \ln 2$$

don't forget prefix

✓

3. Find the volume under the surface
$z = 16 - x^2 - 4y^2$ and over the $x-y$ plane
simplify the integral w/ $u = x$
$v = 2y$

$$\iint_R 16 - x^2 - 4y^2 \, dA$$

$$\iint 16 - u^2 - v^2 \, du\,dv$$

Limits $x = x$ $x = 0$
$y = y$ $y = 0$
? should figure out what this is, right?

No first is Jacobian

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} = 2$$

so $\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{2}$ ✓

$$\frac{1}{2} \iint 16 - u^2 - v^2 \, du\,dv$$

R is <u>elliptical</u> region $x^2 + 4y^2 = 16$ in rect.
But in our coorss

so I did get it

But next line

$$\frac{1}{2}\int_0^{2\pi}\int_0^4 (16-r^2)\, r\, dr\, d\theta$$

?? How in all world does it jump to this ??
Why did they switch ~~again~~ to polar coords

Well it does make sense $v^2+v^2=16$
which is a circle of radius r
So I can see why it works
Did not know it was legal to switch twice

Now Solve

$$\int_0^4 (16-r^2)\, r\, dr$$

$$\int_0^4 16r - r^3\, dr$$

$$\frac{16r^2}{2} - \frac{r^4}{4}\Big|_0^4$$

$$128 - 64$$
$$64$$

$$\int_0^{2\pi} 64\, d\theta$$

$$64\theta \Big|_0^{2\pi}$$

$$\frac{1}{2} \cdot 128\pi$$

$$64\pi$$

Don't forget the "prefix" ✓

5. Set up the iterated integral for the polar moment of inertia of the finite "triangular" region R

$y = x \qquad xy = 3$

$y = 2x$

$$\iint_R \; ? \; dA$$

What is u, v ? Or should I change to r, θ ?

$u = xy \qquad y^2 = uv$

$v = \frac{y}{x} \qquad x^2 = \frac{u}{v}$

So first is the jacobean

$$\begin{vmatrix} 1 & 1 \\ -\frac{1}{2} & 1 \end{vmatrix} = \frac{1}{2} \quad \text{flipped} = 2$$

well do w/ variables

$$\begin{vmatrix} y & x \\ -\frac{y}{x^2} & \frac{1}{x} \end{vmatrix} = \frac{2y}{x} = 2v \rightarrow \text{flipped } \frac{1}{2v}$$

↑ convert

$$\iint_R (x^2+y^2)\,dx\,dy =$$

?? how did
you know this
-was that the reference
to polar ?

Now convert

$$\frac{1}{2v}\iint \frac{u}{v} + uv \; du\,dv$$

they do dv dU

$$\frac{1}{2v}\iint_R \frac{u}{v} + uv \; dv\,du$$

so now for this
we have $y = x$ $\quad xy = 3$
$y = 2x$

so $u = 3 \to 0$
$v = 1 \to 2$

$$\frac{1}{2v}\int_0^3\int_1^2 \frac{u}{v} + uv \; dv\,du$$

✓ slowly figuring out

Now solve

$$\int_1^2 \frac{u}{v} + uv \; dv$$

$$\int_1^2 uv^{-1} + uv \; du$$

$$u \ln v + \frac{u v^2}{2} \Big|_1^2$$

$$u \ln(2) + u2 - \left(u \ln(1) + \tfrac{1}{2}u\right)$$

$$u\left(\ln(2) - \ln(1) + \tfrac{3}{2}\right)$$

$$\int_0^3 u\left(\ln(2) \quad + \tfrac{3}{2}\right) du$$

$$u^2\left(\ln(2) \quad + \tfrac{3}{2}\right)\Big|_0^3$$

$$3^2\left(\ln(2) \quad + \tfrac{3}{2}\right)$$

* and you don't take prefix out - must integrate it *

$$\Rightarrow \tfrac{1}{2}\cancel{v}\left(9\ln(2) + 13.5\right)$$

No they have a simpler solution

Inner $\frac{-u}{2v} + \frac{u}{2}v \Big|_1^2$

$$u\left(-\frac{1}{4} + 1 + \frac{1}{2} - \frac{1}{2}\right) = \frac{3u}{4}$$

outer $\frac{3}{8}u^2 \Big|_0^3$

$$= \frac{27}{8}$$

9. Prove the relationship between Jacobeans

$$\frac{\partial(x,y)}{\partial(u,v)} \cdot \frac{\partial(u,v)}{\partial(x,y)} = 1$$

Using the chain rule for partial differentiation and the rules for multiplying determinants $|AB| = |A||B|$ where A and B are matricies of the same size

Hints from problem sheet

That equation is obtained by solving

$$x = x(u,v) \qquad y = y(u,v)$$

???

So can write

$$x(u(x,y), v(x,y)) = x$$
$$y(u(x,y), v(x,y))$$

Now use chain rule to find partial derivitives of both sides w/ respect to x and y

$$\frac{\partial x}{\partial y} = 1\left(u\left(1, \frac{\partial v}{\partial x}\right), v\left(1, \frac{\partial y}{\partial x}\right)\right)$$

I don't think thats what they want...

Now they want you to express it in a single matrix equation

$$\begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix}\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Good example of how a matrix can be used to simplify a system of equations

Jacobean matrix of x and y with respect to u and v

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{pmatrix} x_u & x_v \\ y_u & x_u \end{pmatrix}$$

Now finish by using determinant law for any 2 n x n matricies

$$|AB| = |A||B|$$

$$\det = x_u y_v - x_v y_u$$

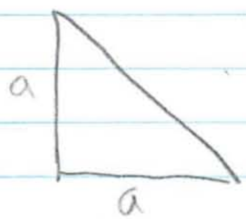
$$(x_u y_v - x_v y_u)(u_x v_y - u_y v_x) = 1 - 0$$

$$x_u y_v \cdot u_x v_y - x_u y_v \cdot u_y v_x - x_v y_u u_x v_y + x_v y_u \cdot u_y v_x$$

so adds to 1

Lecture 19 Physical applications of double integrations

3C-1a Let R be a right triangle



density = 1

Find Moment of inertia about a leg

$$\int_0^a \int_0^{a-x} x^2 \, dy \, dx$$

?? where is this from
Should prob go to recitation 1st

$$I_x = \iint_R y^2 \, \delta(x,y) \, dA$$

well they do I_y

$$I_y = \iint_R x^2 \, \underbrace{\delta(x,y)}_{1} \, dA$$

then its just figure out region

← don't forget normal how to find region rules

Now solve

$$\int_0^{a-x} x^2 \, dy$$

$$x^2 y \Big|_0^{a-x}$$

$$x^2(a-x) - 0$$

$$ax^2 - x^3$$

$$\int_0^a ax^2 - x^3 \, dx$$

$$\frac{ax^3}{3} - \frac{x^4}{4} \Big|_0^a$$

$$\frac{a \cdot a^3}{3} - \frac{a^4}{4}$$

$$\frac{a^4}{3} - \frac{a^4}{4}$$

$$I_y = \frac{a^4}{12} \checkmark \quad \checkmark$$

b Polar moment of inertia about right-angle vertex (shoud beable to deduce from a)

$$I_z = \iint_R r^2 \, \delta(x,y) \, dA$$

$$r^2 = x^2 + y^2$$

? Is that just the same thing?

$$\iint_R (x^2+y^2)\, dA$$

No - plug it in!

$$\int_0^a \int_0^{a-x} (x^2+y^2)\, dx\, dy$$

split x^2 and y^2

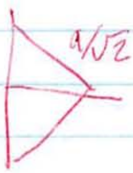
shortcut

$$\frac{a^4}{12} + \frac{a^4}{12}$$

$$\frac{a^4}{6}$$

c) Moment of inertia around hypotenuse

Divide triangle into smaller triangles



$$2 \cdot \frac{1}{12} \cdot \left(\frac{a}{\sqrt{2}}\right)^4$$

$$\frac{a^4}{24}$$

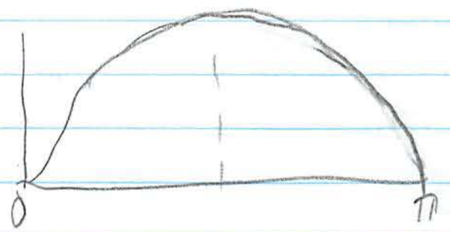
?? don't get what they did here

2a. Find the center of mass inside 1 arch of $\sin x$ if $\delta = 1$

$$\bar{x} = \frac{M_y}{M} = \frac{\iint_R x\,\delta(x,y)\,dA}{\iint_R \delta(x,y)\,dA}$$

; So what does that mean?
; do I actually need to solve for each?

Well $\bar{x}$ is clear by symmetry



$\bar{x}$ is at $\frac{\pi}{2}$

Find mass 1st (denominator what we need)

$$M = \iint_R \delta(x,y)\,dA$$

well just sum over the area

$$\int_0^{\pi}\int_0^{\sin x} 1 \; dy\,dx$$

they just have

$$\int_0^{\pi} \sin x\,dx = 2$$

I guess that is the simpler version of it

Then calculate the moment

$$\iint_R y\,\delta(x,y)\,dA$$

$$\int_0^{\pi}\int_0^{\sin x} y\,dy\,dx$$

↑ here they use both

$$\int_0^{\sin x} y\,dy$$

$$\frac{y^2}{2}\Big|_0^{\sin x}$$

$$\frac{1}{2}\int_0^{\pi} \sin^2 x\,dx$$

$$\frac{\pi}{4}$$

$$\text{so } \bar{y} = \frac{\frac{\pi}{4}}{2} = \frac{\pi}{8}$$ ✓

b Now $\delta = y$ (so density changes over shape)

Mass

$$= \iint_R y\,dA = \frac{\pi}{4}$$

↑ so they are hiding the

$\int_0^{\pi} \sin x\, y\, dx$ in there

Moment

$$\int_0^{\pi}\int_0^{\sin x} y \cdot y \, dy\, dx$$

$$\int_0^{\pi}\int_0^{\sin x} y^2 \, dy\, dx$$ ✓

$$\left.\frac{y^3}{3}\right|_0^{\sin x}$$

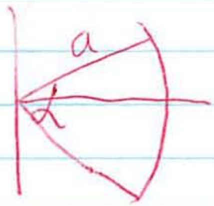
$$2 \cdot \int_0^{\pi} \frac{\sin^3 x}{3} \, dx$$

$$2 \cdot \frac{1}{3} \cdot \frac{2}{3} = \frac{4}{9}$$

So $$\frac{\frac{\pi}{4}}{\frac{4}{9}} = \frac{16}{9\pi}$$ ✓

I still don't clearly get why we are doing this

4. Find the center of gravity of a circular dish of radius a whose vertex angle is 2α
Take $\delta = 1$



so the trick on this one is to know the shape

By symmetry know $\bar{x}$
Need to find $\bar{y}$ ✓

Since constant density ⇒ area + mass are the same

~~3C-15~~
3C-5 ? -1

18.02 Exam 2 Thurs. Apr.1, 2010 11:05-11:55

Directions:

1. There are 3 sheets, printed on both sides: nine problems in all.
2. Do all the work on these sheets; use the blank part below if truly necessary. Write down enough to show you are not guessing.
3. No books, notes, calculators, use of cell-phones, etc.
4. Please don't start until the signal is given; stop at the end when asked to; don't talk until your paper is handed in.
5. When the exam starts, read through the exam and start with what you are surest of.
6. Fill out the information below now.

Name Michael Plasmeier e-mail@mit.edu theplaz

Recitation teacher Oliver Rec. hour 12

pg.1 19

pg.2 12

pg.3 8

pg.4 14

pg.5 19

Total. 72

Recitation
Mean 69
Median 74

Problem 1. (10) For the function $w = x^2y - xy^3$, find its directional derivative $\left.\frac{dw}{ds}\right|_{P,\hat{\mathbf{u}}}$ at the point $P:(1,1)$ in the direction $\hat{\mathbf{u}}$ of the vector $\mathbf{i}+\mathbf{j}$.

$= \nabla W \circ \hat{u}$

$W_x = 2xy - y^3$

$W_y = x^2 - x3y^2$

Plug pts in

$W_x = 2(1)(1) - (1)^3 = 1$ $\quad \nabla = \langle 1, -2 \rangle$ $\quad \sqrt{1^2+1^2} = \sqrt{2}$

$W_y = 1^2 - (1)\cdot 3(1)^2 = -2$

$\left.\frac{dw}{ds}\right|_{p,\vec{u}} = \langle 1, -2 \rangle \circ \frac{\langle 1, 1 \rangle}{\sqrt{2}}$

$\left(1 \cdot \frac{1}{\sqrt{2}}\right) + \left(-2 \circ \frac{1}{\sqrt{2}}\right)$

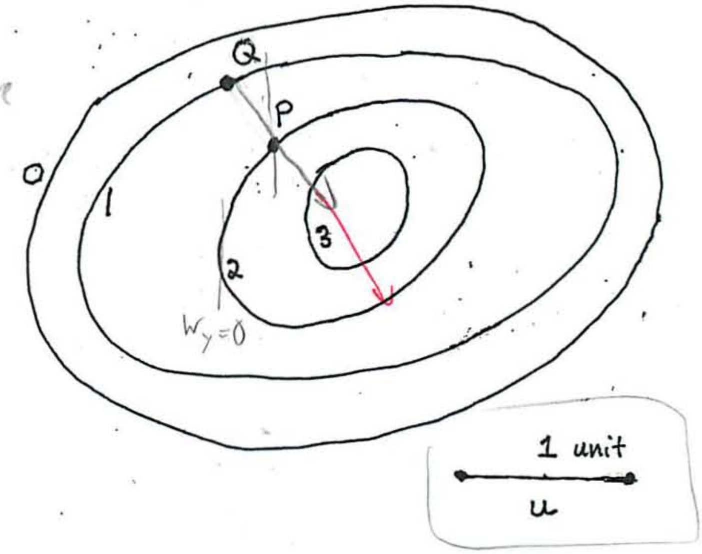
$\frac{1}{\sqrt{2}} - \frac{2}{\sqrt{2}}$

$\frac{-1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = -\frac{\sqrt{2}}{2}$

Problem 2 (10: 4, 6) Some level curves for $w = f(x,y)$ are shown; $\mathbf{u}$ is a unit distance.

a) At P, estimate the value of w_y.

b) At Q, draw the vector $(\nabla f)_Q$.



$W_y\Big|_p = \frac{\partial w}{\partial y} = \frac{1}{-1/2}$ change of curve / distance in y dir

$= -2$

$Q = (\nabla f)_Q$ draw gradient

towards increasing ground

$= \left\langle \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y} \right\rangle$

$= \left\langle -\frac{1}{-\frac{1}{2}}, \frac{-1}{\frac{1}{2}} \right\rangle$

$\langle 2, -2 \rangle$

5|6

Problem 3. (20: 3, 12, 5) Find the point P on the surface $x^2 + yz + 3z - 8 = 0$ which is closest to the origin, by following the steps below.

optimization

(a) It suffices to find the point P which minimizes the square of the distance to the origin. Show this leads to finding the point which minimizes $w(y,z) = y^2 + z^2 - yz - 3z + 8$.

rewrite constraint

?

$d = \sqrt{x^2 + y^2 + z^2}$

0

(b) Find the point (y_0, z_0) which minimizes $w(y,z)$, and use it to find P.
(You don't have to prove it is a minimum point.)

$W_y = 2y + 0 - z - 0 + 0$

$W_z = 0 + 2z - y - 3 + 0$

set = to 0

$2y - z = 0$ $\quad\quad 0 = 2z - y - 3$

find pts $\quad\quad 3 = 2z - y$

(y,z) $(1,2)$ $\quad\quad (1,2)$

$(-1,-2)$ $\quad\quad (-1,-2)$

P = ?

10

Try each

$1^2 + (2)^2 - 1\cdot 2 - 3\cdot 2 + 8 = 5$ ← minimum point $(1,2)$

$(-1)^2 + (-2)^2 - (-1)(-2) - 3(-2) + 8 = 15$

what is difference (y_0, x_0) and P

(c) If this problem is solved by Lagrange multipliers instead, give one of the equations involving the multiplier λ, and use it to determine the value of λ corresponding to the point P.

$2x + 0 + 0 - 0 = \lambda$

$0 + z + 0 - 0 = \lambda$

$0 + y + 3 - 0 = \lambda$

?

$$\boxed{\begin{aligned} 2x &= \lambda \\ z &= \lambda \\ y + 3 &= \lambda \end{aligned}}$$

2

$2x = z = y + 3$

Out of the

5

Problem 4 (10) Let $w = f(x,y)$, where in turn $x = 2u - v^2$ and $y = uv$.

If in xy-coordinates $\nabla f = \langle 2,3 \rangle$ at the point $P : (4,0)$, find the value of $\dfrac{\partial w}{\partial v}$ at the point in uv-coordinates corresponding to P.

point

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial v} + ?$$

$$= 2 \cdot 0 - 2v$$

$$= 2 \cdot -2v$$

$$= 2 \cdot -2(0)$$

$$= 0$$

$4 = 2u - v^2$ 2eq, 2 unknowns

$0 = uv$

$\cancel{u = \frac{v}{0}}$?

$4 = 2(2) - (0)^2$

$u = 2$

$v = 0$

3

can't picture graphically at all

Problem 5 (10: 5,5))

4

← they are not independent - have some relation

a) Suppose $f(x,y,z) = 0$. Derive a formula for $\left(\dfrac{\partial z}{\partial x}\right)_y$ in terms of the formal partial derivatives f_x, f_y, f_z, i.e., the derivatives taken as if x, y, z were independent; use the chain rule or differentials.

find formula for

$$\left(\frac{\partial f(x,y,z)}{\partial x}\right)_y = f_x \left(\frac{\partial x}{\partial x}\right)_y + f_y \left(\frac{\partial y}{\partial x}\right)_y + f_z \left(\frac{\partial z}{\partial x}\right)_y$$

z is function of x and y

$$= 0 \text{ ← by definition} \qquad \left(\frac{\partial x}{\partial x}\right)_y = 1 \qquad \left(\frac{\partial y}{\partial x}\right)_y = 0$$

what we want

$$0 = \left(\frac{\partial f}{\partial x}\right)_y = f_x + f_z \left(\frac{\partial z}{\partial x}\right)_y$$

$$\left(\frac{\partial z}{\partial x}\right)_y = \frac{\left(\frac{\partial f}{\partial x}\right)_y - f_x}{f_z} \quad \text{← don't think what they are asking for} \quad = \frac{-f_x}{f_z}$$

almost

b) Letting x, y, r, θ be the usual rectangular and polar coordinates, calculate $\left(\dfrac{\partial r}{\partial \theta}\right)_x$ in terms of r and θ.

has relation to part a

$x = r\cos\theta \qquad r = \sqrt{x^2+y^2}$

$y = r\sin\theta \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right)$

→ so $f(x,r,\theta) = 0$ for $f(x,r,\theta) = x - r\cos\theta$

∂f of this

$$\cancel{\frac{\partial r}{\partial x} \cdot \frac{\partial x}{\partial \theta}} \quad ?$$

$$\cancel{\sqrt{2x+y^2} \cdot \cos\theta}$$

$$\left(\frac{\partial r}{\partial \theta}\right)_x = \frac{-f_\theta}{f_r} = \frac{-r\sin\theta}{-\cos\theta} = r\tan\theta$$

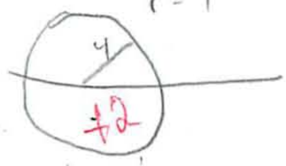
Problem 6 (10: 3,7) Set up a double iterated integral in polar coordinates which gives the volume of the solid lying under the graph of $z = 16 - x^2 - y^2$ and above the xy-plane, as follows.

a) Show the region of integration is the interior of the circle $x^2 + y^2 = 16$.

b) Then set up the integral. *Do not evaluate the integral.*

the region

$r=4$



+2

top view

$$\int_{0}^{2\pi}\int_{0}^{4} \underbrace{(16 - x^2 - y^2)}_{r^2} \, r\, dr\, d\theta$$

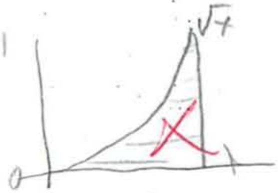
$\theta \quad r$

+6

Sid view

Problem 7 (10) By changing the order of integration, evaluate $\displaystyle\int_0^1 \int_{\sqrt{x}}^1 \cos(y^3)\, dy\, dx$.

don't change inside



$$\int_{0}^{1}\int_{0}^{y^2} \cos(y^3)\, dx\, dy$$

$y \quad x$

+3

$y = \sqrt{x}$

$y^2 = x$

$$\int_0^{y^2} \cos(y^3)\, dx$$

$$\cos(y^3)\, x \Big|_0^{y^2}$$

$$\int_0^1 y^2 \cos(y^3)\, dy$$

$u = \cos y^3$

$du = \sin y^3 \cdot 3y^2\, dy$

integration wrong

$$\frac{y^3 \cos(y^4)}{3 \cdot 4} \Big|_0^1$$

+3

$$\frac{1^3 \cos(1^4)}{12} = \frac{\cancel{x}\cos(1)}{12}$$

Problem 8 (10) A uniform metal plate has the form of an isosceles right triangle having its two legs both of length 1; find its moment of inertia about one of its legs L, taking the density $\delta = 1$.
(Place the triangle in the first quadrant so the right angle is at the origin, and L lies along the y-axis.)

Moment of inertia$_y$ = $\iint_R x^2 \delta$ Area dA

Mass = area $\cdot$ density

$\frac{1}{2} \cdot 1 \cdot 1 \cdot 1$

$= \frac{1}{2}$

$\int_0^1 \int_0^{1-x} \cdot x^2 \cdot \frac{1}{2}\ dy\ dx$ $\quad -1$

$\frac{1}{2} \cdot \int_0^{1-x} x^2\, dy$

$\frac{1}{2} \cdot x^2 y \Big|_0^{1-x}$

$\frac{1}{2} \int_0^1 x^2(1-x)\, dx$

$\frac{1}{2} \int_0^1 x^2 - x^3\, dx$

$\frac{1}{2} \left(\frac{x^3}{3} - \frac{x^4}{4} \right) \Big|_0^1$

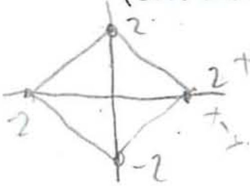
$\frac{1}{2} \left(\frac{1^3}{3} - \frac{1^4}{4} \right)$

$\frac{1}{2} \left(\frac{4}{12} - \frac{3}{12} \right)$

$\frac{1}{24}$

Problem 9. (10) Consider the double integral $\iint_R \sin(x-y)\cos(x+y)\, dy\, dx$,
where R is the square xy-region having its vertices at the four points ± 2 on the x- and y- axes.

Change it to a double iterated integral in uv-coordinates, where $u = x - y$ and $v = x + y$. (Give the new limits, integrand, and area element dA, but *do not evaluate*.)



$u = x - y$ $\qquad x = u + y \qquad y = x - u$

$v = x + y$ $\qquad x = v - y \qquad y = v - x$

want x in terms of u and v

~~$x = u + y = v - y$~~

~~$x = x - y + y$~~

~~$x = x$~~

~~$x = u + x - u$~~

look graphically

$\int_{-2}^{2} \int_{-2}^{2} \sin(u) \cos(v) \left| \begin{matrix} x_u & y_u \\ x_v & y_v \end{matrix} \right| du\ dv$

$v \quad u$

$\dfrac{1}{\left| \begin{matrix} u_x & u_y \\ v_x & v_y \end{matrix} \right|} = \dfrac{1}{\left| \begin{matrix} 1 & -1 \\ 1 & 1 \end{matrix} \right|} = \frac{1}{2}$ $+4$

18.02 Exam 2 Solns — Spring 2010

$\boxed{1}$ $w = x^2y - xy^3$

$\vec{\nabla} w = \langle 2xy - y^3,\ x^2 - 3xy^2 \rangle$

$(\vec{\nabla} w)_{(1,1)} = \langle 1, -2 \rangle$

$\left.\dfrac{dw}{ds}\right|_{P,\hat u} = \langle 1, -2 \rangle \cdot \dfrac{\langle 1, 1 \rangle}{\sqrt{2}} = \dfrac{-1}{\sqrt{2}}$

$\boxed{2}$ $(w_y)_P \approx \dfrac{\Delta w}{\Delta y} = \dfrac{-1}{1/2} = -2$

$(\vec{\nabla} f)_Q$: direction $\hat u$

$|\vec{\nabla} f|_Q \approx \left.\dfrac{\Delta w}{\Delta s}\right|_{\hat u} = \dfrac{1}{1/2} = 2$

so it should have length 2

$\boxed{3}$ $w = x^2 + y^2 + z^2$ (**)

a) $x^2 = -yz - 3z + 8$ (*)

$\therefore \boxed{w = y^2 + z^2 - yz - 3z + 8}$

b) $\dfrac{\partial w}{\partial y} = 2y - z = 0 \ :\ z = 2y$

$\dfrac{\partial w}{\partial z} = -y + 2z - 3 = 0 \ :\ 3y = 3$

solving: $y = 1,\ z = 2$

using (*) $x^2 = -2 - 6 + 8 = 0$

$\therefore P : (0, 1, 2)$

c) $\vec{\nabla} w = \langle 2x, 2y, 2z \rangle = \lambda \vec{\nabla} g$

(from **) $= \lambda \langle 2x, z, y+3 \rangle$

$g(x,y,z) = x^2 + yz + 3z - 8$

$2x = \lambda \cdot 2x$ useless (since $x = 0$)

$$\begin{cases} 2y = \lambda z & \text{— OK: } \lambda = 1 \\ 2z = \lambda(y+3) & \text{— OK: } \lambda = 1 \end{cases}$$

$\boxed{4}$ $\dfrac{\partial w}{\partial v} = \underbrace{\dfrac{\partial w}{\partial x}}_{=2} \cdot \dfrac{\partial x}{\partial v} + \underbrace{\dfrac{\partial w}{\partial y}}_{=3} \cdot \dfrac{\partial y}{\partial v}$ $\quad \begin{cases} x = 4 \\ y = 0 \end{cases}$

$x = 2u - v^2 \quad \dfrac{\partial x}{\partial v} = -2v$ corresponds to

$y = uv \quad \dfrac{\partial y}{\partial v} = u \quad \begin{cases} v = 0 \\ u = 2 \end{cases}$

($u = 0$ says $4 = -v^2$) — no sol'n

At $u = 2,\ v = 0$,

$\dfrac{\partial w}{\partial v} = 2 \cdot 0 + 3 \cdot 2 = \boxed{6}$

$\boxed{5}$ a) $f(x,y,z) = 0$

$f_x \underbrace{\left(\dfrac{\partial x}{\partial x}\right)_y}_{=1} + f_y \underbrace{\left(\dfrac{\partial y}{\partial x}\right)_y}_{=0} + f_z \left(\dfrac{\partial z}{\partial x}\right)_y = 0$

$\therefore \left(\dfrac{\partial z}{\partial x}\right)_y = \dfrac{-f_x}{f_z}$

b) $x - r\cos\theta = 0$

By formula: $\left(\dfrac{\partial r}{\partial \theta}\right)_x = \dfrac{-f_\theta}{f_r} = \dfrac{-r\sin\theta}{-\cos\theta}$

$\boxed{\text{OR}}$

Directly: $= r\tan\theta$

$r = \dfrac{x}{\cos\theta} = x\sec\theta \quad \left(\dfrac{\partial r}{\partial \theta}\right)_x = x\sec\theta\tan\theta = r\tan\theta$

[can give ans. in sin + cos also]

$\boxed{6}$ a) Graph intersects xy-plane where $z = 0$: $16 - x^2 - y^2 = 0$

$x^2 + y^2 = 16$ (R) radius 4

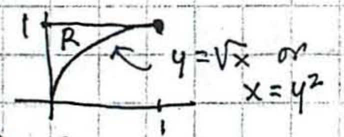
inside R, $16 - (x^2 + y^2) > 0$

(outside: < 0)

b) $\displaystyle\int_0^{2\pi}\int_0^4 (16 - r^2) \cdot r\,dr\,d\theta$

7 $\displaystyle\int_0^1 \int_{\sqrt{x}}^1 \cos(y^3)\,dy\,dx$

Region



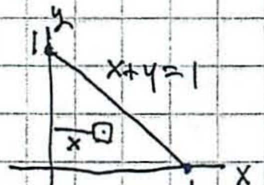
Changed order:

$$\int_0^1 \int_0^{y^2} \cos(y^3)\,dx\,dy$$

Inner: $\cos(y^3)\,x\Big]_0^{y^2} = y^2\cos(y^3)$

Outer: $\frac{1}{3}\sin(y^3)\Big]_0^1 = \frac{1}{3}\sin(1)$

8



Moment of inertia about y-axis

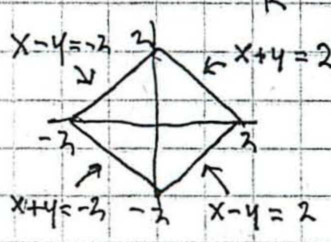
$$= \int_0^1 \int_0^{1-x} x^2\,dy\,dx$$

Inner $= x^2 y\Big]_0^{1-x} = x^2(1-x)$

Outer $= \frac{1}{3}x^3 - \frac{x^4}{4}\Big]_0^1 =$

$= \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$

9 $\displaystyle\iint_R \sin(x-y)\cos(x+y)\,dy\,dx$



$u = x-y$

$v = x+y$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2$$

$$\therefore \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{2}$$

$$\int_{-2}^{2} \int_{-2}^{2} \sin u \cos v \cdot \frac{1}{2}\,dv\,du$$