Lecture 29

Flux surface S cont, Divergence Theorm 4/27

1. General surface integral
2. Divergence theorm

Flux of $\vec{F}$ over oriented surface S
(flow rate) in 3 space

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \hat{n}\, dS$$

geometry of surface

dS ← area, piece of surface

$$d\vec{S} = \hat{n}\, dS$$

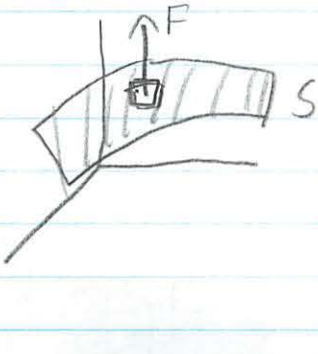
direction of area

shphers

cylinders

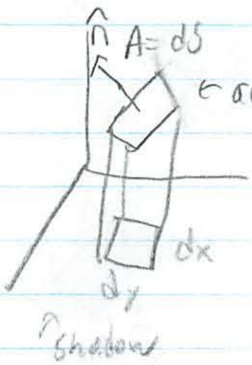
the surfaces it is easy to find (last lecture)

General surface = $z = f(x, y)$
graph of a function



reduce to shadow on xy plane
then integrate over that region r

Divide up region xy plane

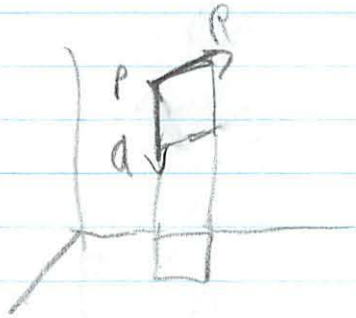


← actual surface, want $\hat{n} \cdot dS$(area) of top piece

← infatessimal rectange in plane

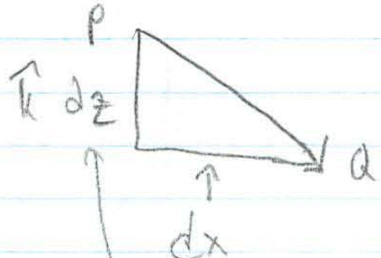
$$d\vec{S} = \hat{n}\, dS$$

shape made is a parallogram
Use cross product to simulateously find
(forgot)



$$d\vec{S} = \hat{n}\, dS = \vec{PQ} \times \vec{PR}$$

$\vec{PQ}$
- what are companants in $\hat{i}, \hat{j}, \hat{k}$ systen



Now just need lenght of each line

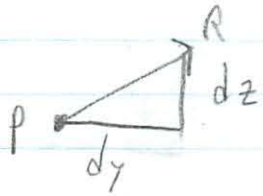
dz is amount low part of peak is from xy plane

$$\frac{\partial z}{\partial x} = \frac{dz}{dx} \qquad dz = \frac{dz}{dx} \cdot dx$$

y constant

$\vec{PQ} = \langle dx, 0, \frac{dz}{dx} dx \rangle$

(Used some knowledge of what partial deriv is)



$\vec{PR} = \langle 0, dy, \frac{dz}{dy} dy \rangle$

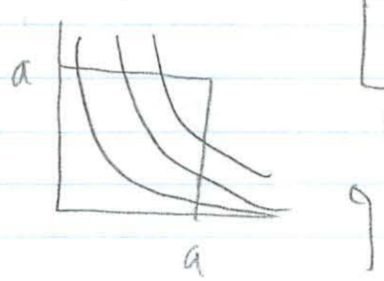
$$PQ \times PR = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ dx & 0 & z_x dx \\ 0 & dy & z_y dy \end{vmatrix}$$

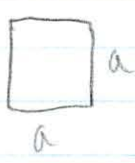
$\quad\parallel$

$d\vec{S} = \hat{n}\, dS$

$$= \boxed{\langle -z_x, -z_y, 1 \rangle dx\, dy \quad \underset{d\vec{S}}{\parallel} \quad (\hat{n} \text{ is up})}$$

$z = xy$





← part of S lying over R

Can draw some level curves

– only interested what is inside the square

Fr DB
Sa MITRE
Ju DB
FCC'S

$\vec{F} = \langle x, y, z \rangle$ flux of $\vec{F}$ over S

$$\iint_S \vec{F} \cdot d\vec{s} = \iint \langle x, y, z \rangle \cdot \langle -y, -x, 1 \rangle \, dy\,dx$$

$$\iint_S \vec{F} \cdot ds = \iint (-2xy + z) \, dy\,dx$$

flux $\square$

over the square in the xy plane
no longer over the shape

Sub out z

$$= \int_0^a \int_0^a -xy \cdot dy\,dx$$

$$= -\int_0^a x\,dx \cdot \int_0^a y\,dy = \frac{-a^4}{4}$$

Why is the flux minus?
- taking normal vector up
- have to change sign if normal vector ↓

Divergence Theorm
- 3 space Green's Flux "normal form"

Integrate over a closed surface

- ant will never fall off it (will walk upside down) closed surface

$\oiint_S \vec{F} \cdot d\vec{S}$ $\hat{n}$ points out by convention

$= \iiint_D (\text{div } \vec{F}) \, dV$

interior of surface

$\vec{F} = M\hat{i} + N\hat{j} + P\hat{k}$

$\text{div } \vec{F} = M_x + N_y + P_z$

solid hemisphere

$\vec{F} = \langle x, y, z \rangle$

$\oiint \vec{F} \cdot d\vec{S} = \iint_{S_1} + \iint_{S_2}$

field always outward
so always $\perp$
distance from origin $= a$
$|\vec{F}| = \rho = a$ (on hemisphere)

$= a \cdot 2\pi a^2 + 0$

flux at each pt

$\hat{n} = -k$

flow is in xy plane
so 0

flux $= 2\pi a^3$

$$\iiint_D 3 \, dV$$

divergence of $\vec{F}$

constant

$= 3 \cdot$ volume ← can only do if integrand is constant

$= 3 \cdot \frac{2}{3} \pi r^3$

$= 2\pi a^3$

Recitation

4/28

Lecture

Expression of $d\vec{S}$ where S is in the graph of function $f(x,y)$

$z = f(x,y)$

$dS = \langle -f_x, -f_y, 1 \rangle \, dx\, dy$

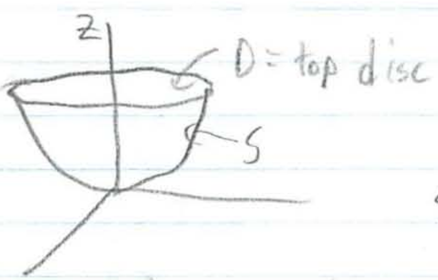
this is dS pointing ↑

Divergence Theorm

$$\iint_S \vec{F} \cdot d\vec{S} \underset{\text{Theorm}}{=} \iiint_V (\underbrace{\operatorname{div} \vec{F}}_{M_x + N_y + P_z})\, dV$$

---

ex 1 $\vec{F}(x,y,z) = \hat{\imath} + z\hat{k}$



$D = x^2 + y^2 \le a^2$ ds points ↑
$z = a^2$

$S =$ part of parabola ds points ↓
$z = x^2 + y^2$
below D

a) Compute $I = \iint_D \vec{F} \cdot d\vec{S}$

direction of dS $d\vec{S} = \hat{k}\, ds$ up

$\vec{F} \cdot \hat{n} = \hat{\imath} + z\hat{k} \cdot \cancel{1}\, \hat{k}$ ~~$z = x^2+y^2$~~

Could use very general formula - more work

z

$$\cancel{\iint_D \frac{z^2\hat{k}}{a}\, dA}$$

$$\cancel{\int_\theta \int_r}$$

~~convert z - but ⊥ to this flux so we want it~~

$$I = \iint_D \vec{F} \cdot d\vec{S} = \iint a^2 dS$$

$$\underset{\parallel}{}$$

$$(r + z\hat{k}) \cdot \hat{k} = z\, dS = a^2\, dS$$

$$I = a^2 \iint dS$$

$$= a^2 \cdot \text{area}$$
$$= \pi a^4$$

↑ Seems very Familiar

b) $J = \iint \vec{F} \cdot d\vec{s}$

$S =$ graph of $f(x,y) = x^2 + y^2$

$dS = -\langle -2x, -2y, 1 \rangle\, dx\, dy$

↑ minus so it points ↓

$$\vec{F} \cdot ds = \langle 1, 0, z \rangle \cdot \langle 2x, 2y, 1 \rangle\, dx\, dy$$
$$= (2x - z)\, dx\, dy$$

$$J = \iint_R (2x - z)\, dx\, dy$$

Will be a double integral

But what is R?
region of xy parameters
"shadow region"
in this problem same as the disc

$$\iint_R 2x - (x^2+y^2)\, dx\, dy$$

How to compute double integral?
- polar coordinates
- disc is radius a

$$\int_{\theta=0}^{2\pi} \int_{r=0}^{a} \left(2r\cos\theta - r^2\right) r\, dr\, d\theta$$

now integrate

$$= \int_0^{2\pi} \frac{2r^3}{3}\cos\theta - \frac{r^4}{4}\Big|_0^a \, d\theta$$

$$= \int_0^{2\pi} \frac{2a^3}{3}\cos\theta - \frac{a^4}{4}\, d\theta$$

$$= \frac{2a^3}{3}\sin\theta - \frac{a^4\theta}{4}\Big|_0^{2\pi}$$

$$= \frac{-a^4}{4} 2\pi = -\frac{a^4\pi}{2}$$

C Deduce the volume of region between S and D

Hint: observe that $Vol(V) = \iiint div(\vec{F})\,dV$

is true since $div(\vec{F}) = M_x + N_y + P_z = 1$

Can we use this to compute the volume
- combine parts A and B

$$Vol(V) = \iiint_V dV = \iiint div(\vec{F})\,dV$$
$$= \oiint_{S+D} \vec{F}\cdot d\vec{s}$$
$$= \iint_S \vec{F}\cdot d\vec{s} + \iint_D \vec{F}\cdot d\vec{s}$$
$$= \frac{-a^4\pi}{2} + a^4\pi$$
$$\boxed{= \frac{a^4\pi}{2} = Vol(V)}$$

Exersise can be presented in several ways
- reverse order
  a) compute flux over d
  b) compute $\iiint_V div(\vec{F})\,dV$
  c) Deduce $\iint_S \vec{F}\cdot d\vec{s}$

ex2 S is the surface defined by
$g(x,y,z) =$ level curve of g

The x, y, z are not independant on S
- if we take x, y as parameters
Can write $z = z(x,y)$

a) $d\vec{S} = \frac{\vec{\nabla} g}{g_z}\, dx\, dy$

Hint $z_x = \frac{-g_x}{g_z}$ $z_y = \frac{-g_y}{g_z}$

b) Application: S given by $x + y^2 + z^3 = 3$
$x \in [0,1]$
$y \in [0,1]$

$\vec{F} = \langle 2y, -1, x \rangle$ Compute $\iint_S \vec{F} \cdot d\vec{S}$

a) From lecture we know
$$d\vec{S} = \langle -z_x, -z_y, 1 \rangle\, dx\, dy$$
$$= \langle \frac{g_x}{g_z}, \frac{g_y}{g_z}, 1 \rangle\, dx\, dy$$
$$= \frac{\langle g_x, g_y, g_z \rangle}{g_z}\, dx\, dy$$
$$= \frac{\vec{\nabla} g}{g_z}\, dx\, dy$$

b) Here $g(x,y,z) = x + y^2 + z^3$

$\vec{\nabla} g = \langle 1, 2y, 3z^2 \rangle$

$d\vec{S} = \dfrac{\vec{\nabla} g}{g_z}\, dx\, dy$

$= \left\langle \dfrac{1}{3z^2}, \dfrac{2y}{3z^2}, 1 \right\rangle dx\, dy$

Now compute S

First dot product

$\vec{F} \circ d\vec{S} = \langle 2y, -1, x \rangle \circ \left\langle \dfrac{1}{3z^2}, \dfrac{2y}{3z^2}, 1 \right\rangle dx\, dy$

$= x\, dx\, dy$

$\iint_S \vec{F} \circ d\vec{S} = \iint_R x\, dx\, dy$

$= \int_0^1 \int_0^1 x\, dy\, dx$

x y

$= \int_0^1 x\, dx$

$= \dfrac{1}{2}$

Remark in ex 1, ex 2 we took 2 of the rectangular coords x, y as parameters
we expressed $d\vec{S} =$ ___ $dx\,dy$

In more general situation could use other parameters ...

Example $z, \theta \to d\vec{S} =$ ___ $dz\,d\theta$
$\phi, \rho \to d\vec{S} =$ ___ $d\phi\,d\rho$

18.02 Problem Set 8 (DUE THURS. APR. 29, 10:45 2-106)

Part I (20 pts.)

Lecture 26. Thurs. Apr. 15 Triple integrals: rectangular and cylindrical coordinates
Read: Notes I.3, 20.5, 20.6 Work: 5A-2ac, 4, 5

Lecture 27. Fri. Apr. 16 Spherical coordinates; gravitational attraction.
Read: 20.7, Notes G Work: 5B-1c, 3, 4; 5C-4 *Holidays Mon.-Tues. Apr. 19,20*

Thurs. Apr. 22 **Exam 3**, covering 20-27 **Walker 3rd floor** 11:05-11:55

Lecture 28. Fri. Apr. 23 Vector fields in 3-space; flux surface integrals.
Read: Notes V8, V9. Work: 6A-1,3,4; 6B-1,2,3; 6B-4,8

Lecture 29. Tues. Apr. 27 Flux surface integrals cont'd. Divergence Theorem.
Read: Notes V10 Work: 6B-5,6; 6C-3,5,7a,8

Lecture 30. Thurs. Apr. 29 Divergence theorem cont'd. Applications, interpretations
Read: Notes V15 sec. 1 for div in ∇ notation; V15 sec. 2 to middle p.3

Part II (15 pts.)

Directions: Try each problem alone for 20 minutes. If you subsequently collaborate, solutions must be written up independently. It is illegal to consult problem sets from previous semesters. With each problem is the day it can be done. In the actual Lectures 26 and 27 the material was rearranged somewhat, but the sum remained the same.

Problem 1. (Anytime 4 pts: 2,1,1) If you fix the volume V (and assume $\delta = 1$), it can be shown that the solid of that volume which exerts the strongest gravitational force in the $\mathbf{k}$ direction on a unit mass at the origin O is the spheroid of that volume whose boundary surface is given in spherical coordinates by

$$\frac{\cos\phi}{\rho^2} = k_1, \qquad \text{equivalently,} \qquad \rho = k\sqrt{\cos\phi}, \qquad k = \frac{1}{\sqrt{k_1}},$$

where the constant k is adjusted so the spheroid will have the desired volume V. This spheroid is symmetric about the z-axis, is tangent to the xy-plane at O, and looks like a somewhat flattened sphere. How does its gravitational attraction compare with that of the sphere?

a) Find the volume of the spheroid (in terms of k).

b) Find its gravitational attraction on a unit mass at O (in terms of k).
(This only gets 1 point because the integral is almost the same as the previous one.)

c) Take a solid sphere of radius a and $\delta = 1$, tangent to the xy-plane at O. Its gravitational attraction on a unit mass at O can be determined using Newton's theorem (cf. p. 743).

(i) Express k in terms of a, if the spheroid has the same volume as the sphere.

(ii) Calculate the ratio of the two gravitational attractions; how much bigger (percentage-wise) is the gravitational force exerted by the spheroid?

Problem 2. (2a, 2b Fri.; 2c Tues. 6 pts.: 2,2,2)

Take the finite domain D in space bounded below by the cone $z^2 = x^2 + y^2$, and above by the sphere of radius 2 centered at the origin.

These two surfaces intersect in a horizontal circle; let S_1 be the horizontal disc having this circle as boundary, S_2 the spherical cap forming the upper surface of D, and S_3 the part of the cone forming the lower surface of D. Orient S_1, S_2, and S_3 "upwards", i.e., in the direction giving the normal vector a positive $\mathbf{k}$-component.

Letting $\mathbf{F} = z\,\mathbf{k}$, calculate directly from the definition of surface integral the flux of $\mathbf{F}$ over the three surfaces:

a) Determine the radius of S_1, and calculate the flux of $\mathbf{F}$ over S_1.

b) Find the flux of $\mathbf{F}$ over S_2.

c) Find the flux of $\mathbf{F}$ over S_3.

Problem 3. (3a Fri; 3b Tues. 5 pts: 2,3)

Get the results of the preceding problem another way.

a) Find the volume of D by integration; then find the two volumes D is split up into by the horizontal disc: use the known volume of a cone: (base)x(height)/3.

b) Starting from the value you calculated for the flux over the disc S_1, use the divergence theorem to find the flux over S_2 and S_3. (Be careful about the orientations!)

P-Set 8 (20/35)

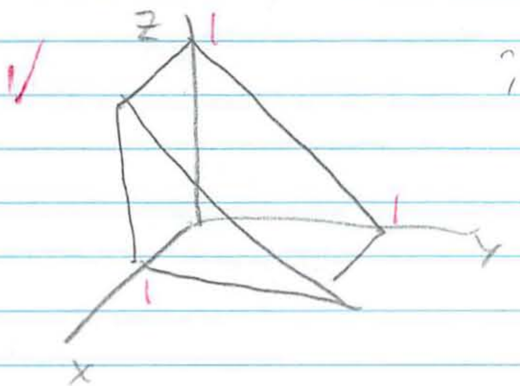
Michael Plasmeier 4/20

Lecture 26 Triple Integrals, rect + cyl coords

5A-2a Supply limits for triple integrals over 3-space

Did whole P-Set on own
no time for OH
first p-set I did that

The rectangular prism
triangles for its 2 bases
- in y, z plae from axis + $z+y=1$
corresponding triangle obtained by adding 1
to the x coord of each point in first triangle



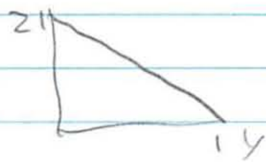
✓

? like a slice of bread in half?
- cake is more like

$$\iiint dz\,dy\,dx$$

$$\int_{x\,0}^{1}\int_{y\,1-z}^{0}\int_{z\,0}^{1} ?\ dz\,dy\,dx$$

? what put in middle?
nothing leave blank

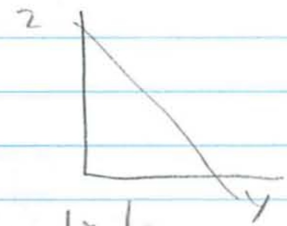


$$\int_0^1\int_0^1\int_0^{1-y}$$

Oh right functions of other stuff go inside

ii) $\iiint dx\,dz\,dy$

$$\underset{y}{\int_0^1}\ \underset{z}{\int_0^{1-y}}\ \underset{x}{\int_0^1} dx\,dz\,dy$$



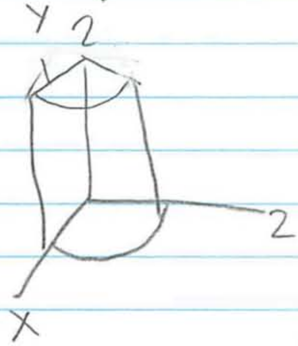
iii) $\iiint dy\,dx\,dz$

$$\underset{z}{\int_0^1}\ \underset{x}{\int_0^1}\ \underset{y}{\int_0^{1-z}} dy\,dx\,dz$$

c) Quarter of a solid circular cylinder

radius 1 — first octant

height 2 — axis $0 \le y \le 2$ on y axis



Note changing axis b/c drew it wrong

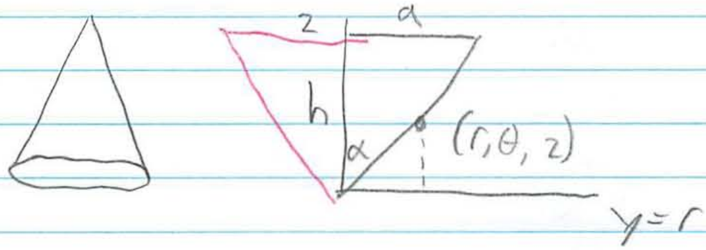
Use cylindrical coords

y axis first

$$\underset{\theta}{\int_{-\frac{\pi}{2}}^0}\ \underset{r}{\int_0^1}\ \underset{y}{\int_0^2} dy\,\cancel{r}\,dr\,d\theta$$

↳ is it b/c just setting up limits?

4. A solid right cylinderical cone
height h $90°$ vertex angle
density = distance to central axis



$dV = \rho^2 \sin \varphi \, d\rho \, d\varphi$ ← needs to be 3D

↑ α

Use cylindrical

~~$dV = r \, dr \, d\theta$~~ $\frac{z}{r} = \frac{h}{a} \rightarrow z = \frac{h}{a} r$

↑ what does this mean?

a Calculate mass

$\iiint_R \delta \, dA$

✓ $z = r$
$\delta = r$

$\int_{\theta=0}^{2\pi} \int_{z=0}^{h} \int_{r=0}^{h} \underset{\uparrow \delta}{r} \cdot r \, dr \, dz \, d\theta$

$\int_0^{2\pi} \int_0^{h} \int_h^{r} r \cdot r \, dz \, dr \, d\theta$

↑ (✓) ↑ they do this order

Yeah had it except $z = r$
and did not know how high it went
and flipped z + r –

but very close to correct

Calculated in b

b) Center of Mass

$$\bar{z} = \frac{\iiint z\sigma \, dA}{\iiint \sigma \, dA} \leftarrow \text{found}$$

do for z axis only one that matters! (since on z axis)

$$\frac{\int_0^{2\pi}\int_0^h\int_h^r z\, r\cdot r \, dz\, dr\, d\theta}{\int_0^{2\pi}\int_0^h\int_h^r r\cdot r \, dz\, dr\, d\theta}$$

Now need to actually find + calculate

Mass:

$$\int_h^r r^2 \, dz$$

(should have done in A)

$\leftrightarrow$ flipped ← come on integrate right

$$r^2 r - r^2 h$$

$$\int_0^h r^3 - r^2 h \, dr$$

$$\left.\frac{r^4}{4} - \frac{r^3}{3}h\right|_0^h$$

$$\int_0^{2\pi} \frac{h^4}{4} - \frac{h^4}{3} \, d\theta$$

$$\left(\frac{3h^4}{12} - \frac{4h^4}{12}\right)2\pi$$

$$\cancel{+}\ \frac{2\pi h^4}{12} = \boxed{\cancel{-\frac{h^4\pi}{6}}}$$

Calculate top part

$$\int_h^r r^3 z \, dz$$

$$\cancel{r^3 z \Big|_h^r} \qquad \frac{r^2 z^2}{2} \Big|_h^r$$

they flip for some reason

$$\cancel{r^3 h - r^3}$$

$$\int_0^h \frac{r^2 h^2}{2} - \frac{r^4}{2} \, dr$$

$$\frac{r^3 h^2}{2 \cdot 3} - \frac{r^5}{2 \cdot 5} \Big|_0^h$$

$$\frac{h^5}{6} - \frac{h^5}{10}$$

$$\frac{5h^5}{30} - \frac{3h^5}{30}$$

$$\frac{2h^5}{30} = \frac{h^5}{15}$$

Make sure solve correctly

$$\int_0^{2\pi} \frac{h^5}{15} \, d\theta$$

$$\frac{2\pi h^5}{15}$$

← don't forget last step

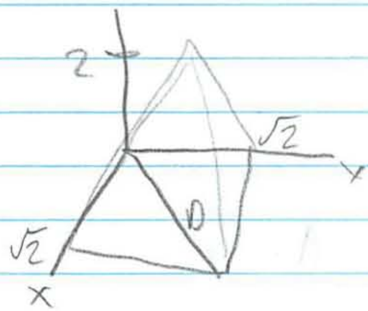
$$\bar{z} = \frac{2\pi h^5}{15} \Big/ \frac{2\pi h^4}{12} = \frac{\cancel{2\pi} h^5}{15} \cdot \frac{12}{\cancel{2\pi h^4}} = \frac{12h}{15} = \frac{4}{5}h$$

✓

5 An engine part is a solid S in the shape of an Egyptian-type pyramid having height 2 and square base w/ diagonal of 2

$\sigma = 1$

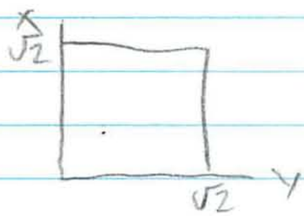
Set up (don't eval) integral of moment of inertia around D

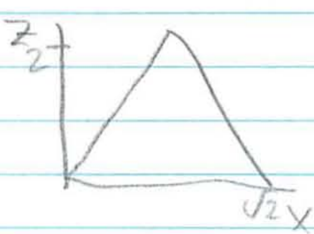


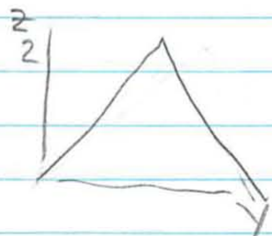
$$M_D = M_{xy} = \iiint_R z\,\sigma\,dA$$

$$Mass = \int_{2}^{2}\int_{0}^{\sqrt{2}}\int_{0}^{\sqrt{2}} dx\,dy\,dz$$

No what are the boundries







$z=0 \rightarrow y=\sqrt{2}$

$z=2 \rightarrow y=0$

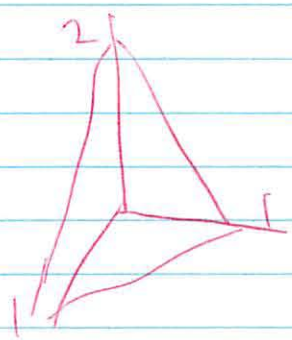
$z=1 \rightarrow y=\sqrt{2}-2z$

Is this better?

$$\int_0^2\int_0^{\sqrt{2}-2z}\int_0^{\sqrt{2}-2z} dx\,dy\,dz$$

They position the base so diagonal is along the x-axis

Divides S into 4 tetrahedrons
- Symmetry have same moment of inertia around x axis



$x + y + \frac{1}{2}z = 1$

really know this

Square of distance of a pt to axis of rotation (x-axis) = $y^2 + z^2$

Moment of inertia = $4\int_0^1 \int_0^{1-x} \int_0^{2(1-x-y)} (y^2+z^2)\, dz\, dy\, dx$

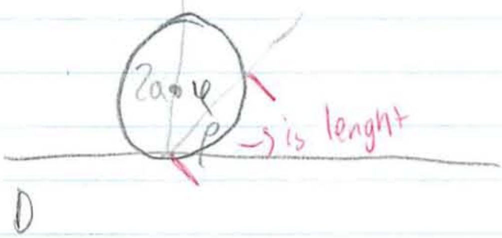
← think we did this in 26

Lecture 27 Spherical coordinates, gravitational attraction

5B -1c Supply limits in spherical coords

$$\iiint d\rho \, d\phi \, d\theta$$

The part of the sphere radius = 1
centered at $z = 1$ on z axis

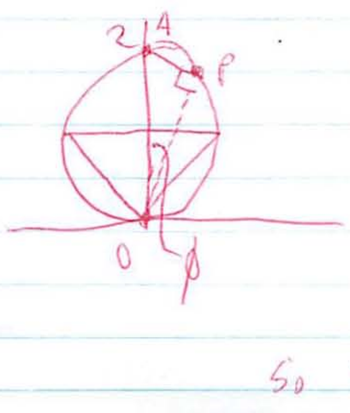


← did this in lecture

$$\rho = 2a\cos\varphi$$

$$G \iiint \cos\varphi \sin\varphi \, d\rho \, d\varphi \, d\theta$$

$$\int_0^{2\pi} \underbrace{\int_0^{\pi/2}}_{\varphi} \underbrace{\int_0^{2a\cos\varphi}}_{\rho} \cos\varphi \sin\varphi \, d\rho \, d\varphi \, d\theta$$

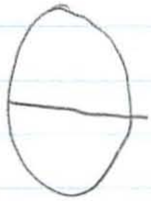


AOP is always a ⊾ for any position P
AO = 2
OP = ρ ← lenght of rod

so $\cos\phi = \rho/2 \qquad \rho = 2\cos\phi$

Plane $z=1$ in spherical eq

Oh I get it



but why do we need this?

$\rho \cos \phi = 1$

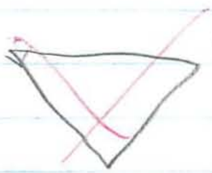
$\rho = \sec \phi$

$\pi/4$ is the max value for ϕ

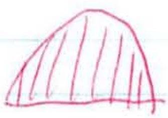
$$\int_0^{2\pi} \int_0^{\pi/4} \int_{\sec\phi}^{2\cos\phi} d\rho \, d\phi \, d\theta$$

So was the goal to get max line?

or area in triangle?



No area above it



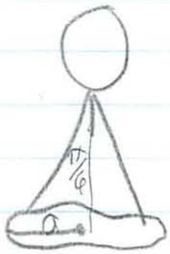
← but is this not just half the triangle?

And we just want limits - nothing to $\int$ over

3. A solid D is bounded below by a right circular cone whose generators have lenght a and make angle $\pi/6$ w/ the central axis Bounded above by a portion of the sphere radius a centered at vertex

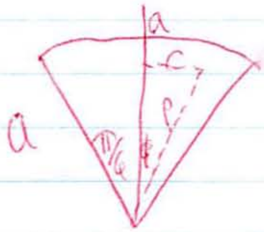
$\delta =$ is

V I height ; (very complex wording



ϵ is that the shape ?!

its very confusing



so ϕ, ρ, r is the small shape that changes

Cross Section $\quad \delta = z = \rho\cos\phi$

Find moment of Inertia around central axis

$$I_p = \iiint_D (\phi^2 + r^2)\,\sigma \; dV$$

r is that right
or does it change for polar

$$2\cdot \int_0^{2\pi} \int_{\pi/4}^{\pi/2} \int_0^{2\cos\phi} (\phi^2 + r^2)\,\rho\cos\phi \; d\rho\, d\phi\, d\theta$$

$\theta \quad \phi \quad \rho$

C_1

$$M \text{ of } I = \iiint_D r^2 \, z \, dV$$

↑ ↑ δ

they just did r^2
why - or is it equation?
but r is not the radius in
this case

$$\iiint_D \underbrace{(\rho\sin\phi)^2}_{r^2} \underbrace{(\rho\cos\phi)}_{\delta} \underbrace{\rho^2\sin\phi \, d\rho \, d\phi \, d\theta}_{\text{traditional}}$$

* know how all the parts fit together

$$\int_0^{2\pi}\int_0^{\pi/6}\int_0^{a} \rho^2\sin^2\phi \; \rho\cos\phi \; \rho^2\sin\phi \, d\rho \, d\phi \, d\theta$$

? don't need 2 since moment of inertia?

? oh it only opens $\pi/6$ (given)

? right want whole shape - not dotted outline which confused me

Now solve

$$\int_0^a \rho^5 \sin^3\phi\cos\phi \, d\rho$$

$$\frac{\rho^6}{6} \sin^3\phi \cos\phi \Big|_0^a$$

$$\int_0^{\pi/6} \frac{a^6}{6} \sin^3\phi \cos\phi \, d\phi$$

$$\frac{a^6}{6} \int_0^{\pi/6} \sin^3\phi \cos\phi \, d\phi$$

$$\frac{\sin^4\phi}{4} \frac{\cos^2\phi}{2} \Big|_0^{\pi/6}$$

$$\int \sin^3\phi \, d\phi = \frac{1}{12} \cos(3\phi) - \frac{3\cos(\phi)}{4}$$

~wolfram alpha - don't think this can be right

$$\int \cos\phi \, d\phi = \sin\phi$$

They say

They prob used this math

$$= 2\pi \cdot \frac{a^6}{6} \cdot \frac{1}{4} \sin^4\phi \Big|_0^{\pi/6}$$

$$2\pi \cdot \frac{a^6}{6} \cdot \frac{1}{4} \left(\frac{1}{2}\right)^4$$

$$\boxed{\frac{\pi a^6}{2^6 \cdot 3}}$$

Hard b/c I am not picturing everything

4. Find the average distance of a point in a Solid sphere of radius a from

a) the center

↳ just the radius a

But how using tripple integrals

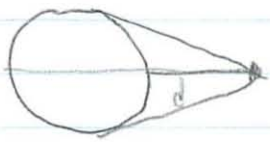
$$\int_0^{2a}\int_0^{\pi}\int_0^{a} \rho \cdot \underbrace{\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta} = 2\pi \cdot 2 \cdot \tfrac{1}{4}a^4 = \pi a^4$$

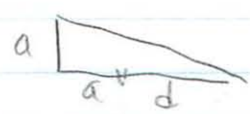
average $\dfrac{\pi a^4}{\pi a^3/3} = \boxed{\dfrac{3a}{4}}$

b) a fixed diameter

max = $d + 2a$

min = d





$\sqrt{a^2 + (a+d)^2}$

$\sqrt{a^2 + a^2 + 2ad + d^2}$

Now They want average distance

? average d, $d + 2a$, $\sqrt{2a^2 + 2ad + d^2}$

Must be some $\iiint$ way

Use z axis as a diameter distance from z axis = $\rho\sin\phi$

$$\int_0^{2a}\int_0^{\pi}\int_0^{a} \underbrace{\rho\sin\phi}_{\text{why}} \, \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

this is → distance from z axis

.vertical offset

Actually calculate

$$\int_0^a \rho^3 \sin^2\phi \, d\rho$$

$$\frac{\rho^4 \sin^2\phi}{4}\Big|_0^a$$

$$\int_0^{\pi} \frac{a^4 \sin^2\phi}{4} \, d\phi$$

$$\frac{a^4 \sin^3\phi}{4 \cdot 3}\Big|_0^{\pi} \leftarrow ??$$

$$\frac{a^4 \sin^3\pi}{12}$$

$$\int_0^{2\pi} 0$$

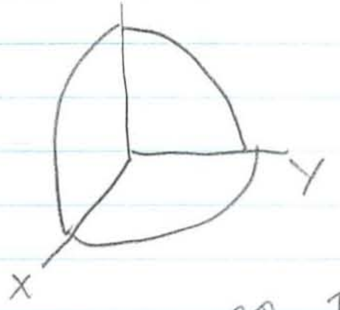
$$2\pi \cdot \frac{\pi}{2} \cdot \frac{1}{4} a^4$$

average $\quad \dfrac{\pi^2 a^4/4}{4\pi a^3/3} = \dfrac{3\pi a}{16}$

c) A fixed plane through the center

Use the xy plane
+ upper solid hemisphere

$z = \rho \cos \phi$



well hemisphere

$$\int_0^{2\pi} \int_0^{\pi/2} \int_0^{a} \_\_\_ \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$\int_0^{a} \rho^3 \sin\phi \cos\phi \, d\rho$$

$$\frac{\rho^4 \sin\phi \cos\phi}{4} \Big|_0^{a}$$

$$\int_0^{\pi/2} \frac{a^4 \sin\phi \cos\phi}{4} \, d\phi$$

$$\frac{a^4 \cos\phi \ -\sin\phi}{4} \Big|_0^{\pi/2}$$

$$\frac{1}{2} \frac{a^4}{4} (0 - 1)$$

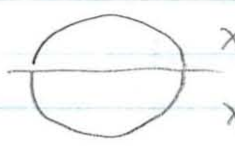
$$\int_0^{2\pi} -\frac{a^4}{4}\, d\theta$$

$$\frac{-a^4}{4} 2\pi$$

$$-\frac{\pi a^4}{2} \qquad \frac{\pi a^4}{4}$$

$$\frac{\pi a^4/4}{2\pi a^3/3} = \frac{3a}{8}$$

5C-4 Gravitational attraction - Find the gravatational attraction of the region which is bounded above by the sphere $x^2+y^2+z^2=1$
below $x^2+y^2+z^2=2z$
at origin $\delta=1$



$x^2+y^2+z^2=1$

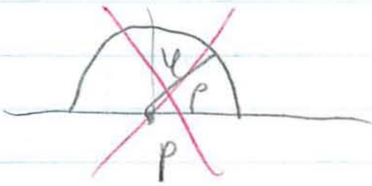
$x^2+y^2+z^2=2z$

$\vec{F} = \langle F_x, F_y, F_z \rangle$
want $\vec{F}_z$

$$|\vec{F}| = \frac{G \cdot \delta \cdot dm}{\rho^2}$$

$$F_z = \frac{G\, dm}{\rho^2} \cos \varphi$$

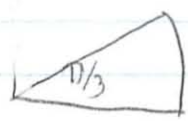
$$\text{Total } F_z = \iiint_D \frac{G\cos\varphi\,\sigma}{\rho^2}\,dV$$



Oh region is

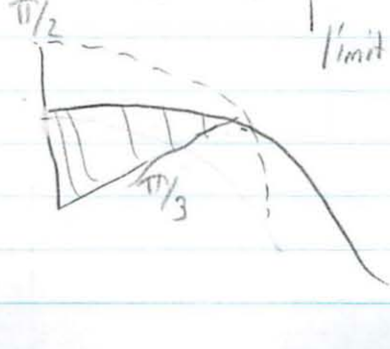
total gravitational attraction in $\hat{k}$ dir

$$G\int_0^{2\pi}\int_0^{\pi/3}\int_0^1 \cos\phi\sin\phi\,d\rho\,d\phi\,d\theta$$



$$+\;G\int_0^{2\pi}\int_{\pi/3}^{\pi/2}\int_0^{2\cos\phi} \cos\phi\sin\phi\,d\rho\,d\phi\,d\theta$$

limit of other circle



Now actually solve
-each part seperatly

$6 \int_0^1 \cos\phi \sin\phi \, dp$

$6 \int_0^{\pi/3} \cos\phi \sin\phi \, d\phi$

$-\sin\phi \cos\phi \Big|_0^{\pi/3}$

$-\sin \pi/3 \cos \pi/3$

$-\frac{\sqrt{3}}{2} \cdot \frac{1}{2}$

$6 \int_0^{2\pi} -\frac{\sqrt{3}}{4} \, d\theta$ $\qquad 2\pi 6 \cdot \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)^2$

$6 \quad -\frac{\sqrt{3} \cdot 2\pi}{4} = +\frac{\sqrt{3}\pi}{2}$ $\qquad \frac{3}{4}\pi 6$

↑work out algebra issues

$\int_0^{2\cos\phi} \cos\phi \sin\phi \, dp$

$\int_{\pi/3}^{\pi/2} 2\cos^2\phi \sin\phi \, d\phi$

$-\frac{2\sin\phi}{2} \cos\phi \Big|_{\pi/3}^{\pi/2}$

$-\sin\frac{\pi}{2} \cos\frac{\pi}{2} - \left[-\sin\frac{\pi}{3} \cos\frac{\pi}{3}\right]$

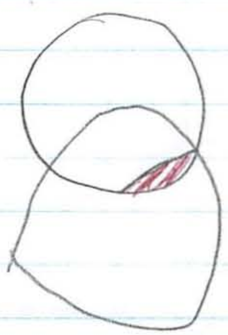
$$-0 \cdot 1 + \frac{\sqrt{3}}{2} \cdot \frac{1}{2}$$

$$\int_0^{2\pi} \frac{\sqrt{3}}{4} \, d\theta \qquad 2\pi 6 \cdot \frac{2}{3}\left(\frac{1}{8}\right)^3$$

$$\frac{2\pi\sqrt{3}}{4} \to \frac{\pi\sqrt{3}}{2} \qquad \frac{1}{6}\pi 6$$

$$\cancel{-\frac{\pi\sqrt{3}}{2}} + \cancel{\frac{\pi\sqrt{3}}{2}}$$

$$\cancel{0} \qquad \boxed{\frac{11}{12}\pi 6}$$



← this piece is curved
— then why did I do

← should only be shaded region

book wrong

Is book wrong?

5C-4
O11
redo

How arrived at shape

← sphere $= x^2 + y^2$

$x^2 + y^2 + z^2 = 2z$

↓ want only squares

$x^2 + y^2 + (z-a)^2 = b^2$

$z - 1$

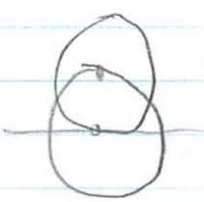
↑ because expand

$z^2 - 2z + 1$

1

↑ then balance what we added

↑ so we know centered at $(0,0,1)$



← what gravitational attraction on ~~center~~ origin

~~spherical coord~~ cylindrical is Oliver's choice

z is not symetrical - so axis of graditational attraction
- but symmatry when rotate around

← mass is up
- so direction is up

Know dir, so only need $|F|$, not vector

$\vec{F} = |\vec{F}|\hat{k} = \iiint$ ______

find function

$d\vec{F}\cdot\vec{k} = \frac{G m \, dM}{d^3} \vec{OP}\,\vec{k}$

Pick arbitrary point $P = (x, y, z)$
the points we are adding arbitrarily
little points we are summing
$\vec{OP}$ is distance from origin to point using at moment

$= \frac{G\sigma \, dV}{(x^2+y^2+z^2)^{3/2}} \; z$

$= \iiint_V \frac{G \sigma z \, dV}{(x^2+y^2+z^2)^{3/2}}$

ii) $\iiint dx\,dz\,dy$

$$\int_{0}^{1} \int_{0}^{1-y} \int_{0}^{1} dx\,dz\,dy$$

$y \quad z \quad x$

iii) $\iiint dy\,dx\,dz$

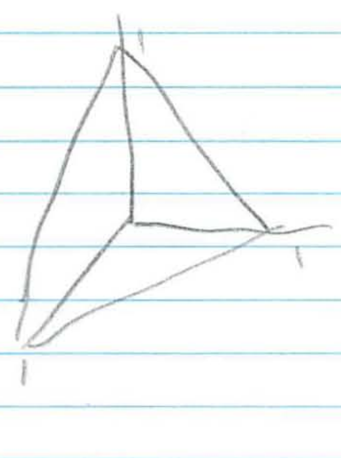
$$\int_{0}^{1} \int_{0}^{1} \int_{0}^{1-z} dy\,dx\,dz$$

$z \quad x \quad y$

Look at to study ↓

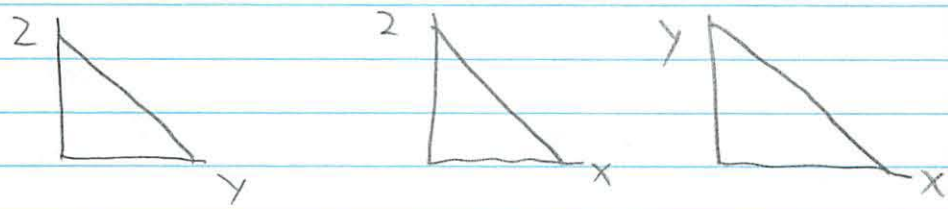
c) Find the COM of Tetrahedron D in the 1st octant formed by axis and $x + y + z = 1$

$\sigma = 1$



$$\bar{x} = \frac{M_{yz}}{M}$$

$$M = \iiint \sigma \, dV$$



$$M = \int_{z=0}^{1-y} \int_{y=0}^{1-x} \int_{x=0}^{1-z} dx\,dy\,dz$$

$$M_{yz} = \iiint x\sigma\, dV$$

wait $\dfrac{\iiint x\sigma\, dV}{\iiint \sigma\, dV}$ ← is that something special?

No literally calc both + see

$$\frac{\int_0^{1-y}\int_0^{1-x}\int_0^{1-z} x\,dx\,dy\,dz}{\int_0^{1-y}\int_0^{1-x}\int_0^{1-z} dx\,dy\,dz}$$

Lecture 28 Vector fields in 3-Space, flux surface integrals

6A-1 Describe geometrically the following vector field

$$\frac{x\hat{i} + y\hat{j} + z\hat{k}}{\rho}$$

Out of the surface radially ✓

all unit vectors

b) $-x\hat{i} - z\hat{k}$

vectorly in on $\hat{i}\hat{k}$ plane

Vector at P
head on x axis, perpendicular to y axis

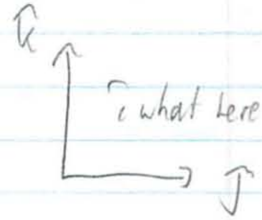
3. Write down the velocity field $\vec{F}$ representing a rotation
- about the x axis in dir of right hand rule
- having ω

But what is vector field for rotation?

$$\omega(-z\hat{j} + y\hat{k})$$

speed ✓

but why?
the - is for



no x-axis because that is what it is rotating around

4. Write down the most general $\vec{F}$ for vectors $\parallel$
$3x - 4y + z = 2$ ← plane

Mattuck never told us how to find these

Notes

$\vec{F}(x,y,z) = M(x,y,z) + N(x,y,z) + P(x,y,z)$

Vector tail at $(x_0\ y_0\ z_0)$ head (x, y, z)

force or flux

ans to last problem #3

- 2D problem
- same as V1 #4
- yeah same $\hat{n}$ ← normal as I know
  tangent

$x\hat{\imath} + y\hat{\jmath} \Rightarrow -y\hat{\imath} + x\hat{\jmath}$

So tail at

~~$\left(\frac{2}{3}, +\frac{1}{2}, 2\right)$~~ ?

$F = M\hat{\imath} + N\hat{\jmath} + P\hat{k}$ no one place that tail is

is parallel if vector field is $\perp$ to normal vector of plane

↳ $3\hat{\imath} - 4\hat{\jmath} + \hat{k}$ ← simple

So $3M - 4N + P = 0$

← how is that $\perp$

$P = 4N - 3M$

$F = M\hat{\imath} + N\hat{\jmath} + (4N - 3M)\hat{k}$

functions x, y, z

So I see how in terms of z

but how do we know this satisfies conditions

6B - 1 Find flux $x\hat{i} + y\hat{j} + z\hat{k}$ through sphere radius a w/o calculating

- So just a sphere

$$\vec{F} \cdot \hat{n} = \left(x\hat{i} + y\hat{j} + z\hat{k} \cdot \frac{x\hat{i} + y\hat{j} + z\hat{k}}{a} \right) ds$$

$= a$ ← so how do you find this $a = \sqrt{x^2+y^2+z^2}$ but still?

Flux $S = \iint_S \vec{F} \cdot \hat{n} \, ds$

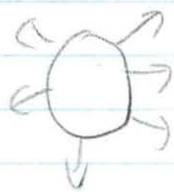
$= a(\overbrace{\text{area of } S}^{\text{integrate over}})$

$= 4\pi a^2 \cdot a$

$4\pi a^3$

2. Without calculating → flux of $\hat{k}$ through $x^2+y^2=1$

Think back to physics class



• height

but what is $\vec{F} \cdot \hat{n}$

but $\hat{k}$ is parallel to cylinder

flux $= 0$

3. Find flux of $\hat{i}$ through $x+y+z=1$ in 1st octant w/o calculations

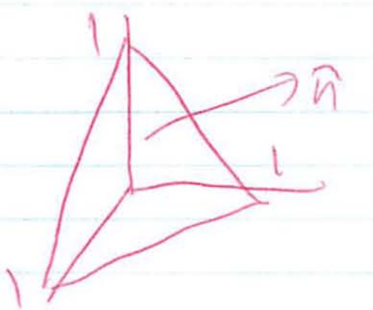
So from last problem
normal vector of plane $x+y+z=0$
$x = -y-z$

~~$\vec{F} = -y-z\,\hat{i} + N\hat{j} + P\hat{k}$~~

Normal vector to plane
$$\frac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}$$

$$\vec{F}\cdot\vec{n} = \frac{1}{\sqrt{3}}$$

$$\text{flux} = \iint_S \frac{1}{\sqrt{3}} = \text{area}\cdot\frac{1}{\sqrt{3}}$$
$$= \frac{1}{2}\,b\cdot h\cdot\frac{1}{\sqrt{3}}$$
$$= \frac{\frac{1}{2}\cdot\sqrt{2}\cdot\frac{\sqrt{3}}{2}\sqrt{2}}{\sqrt{3}} = \frac{1}{2}$$



I still don't get how this goes together
Recitation has some more info
W/ projection onto axis - "shadow"
$|\vec{F}| = m\,|\text{distance}|$
$\vec{F} = 1\ |\text{distance}|$
then check sign

6B-4 Find $\iint_S \vec{F}\, dS$ where $\vec{F} = y\,\hat{j}$

$S =$ half of sphere

$x^2 + y^2 + z^2 = a^2$

$y \geq 0$ so $\hat{n}$ away from origin

This is ~~similar~~ same to one in lecture I think

$n = \dfrac{x\hat{i} + y\hat{j} + z\hat{k}}{a}$ ← normally vertically out

$\vec{F} \cdot \hat{n} = \dfrac{y^2}{a}$

↳

So $y\hat{j} \cdot \dfrac{x\hat{i} + y\hat{j} + z\hat{k}}{a}$

- same dir is adding ??
- in lecture he just said that
- where does this come from
- Dot product

✗ got it
the solutions could be far more detailed

$(0 \cdot x)\hat{i} + (y \cdot \frac{y}{a})\hat{j} + (0 \cdot z)\hat{k}$

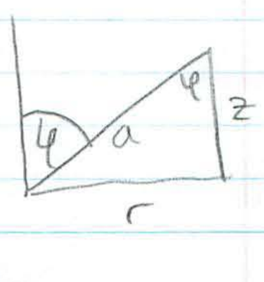
Here we go - just a simple dot product

Now we need to integrate
- sphere, so use spherical coords

$y = r \sin\theta$

$r = a \sin\varphi$

draw picture →



$$\vec{F} \cdot \hat{n} = \frac{y^2}{a} = \frac{(a \sin \varphi \sin \theta)^2}{a}$$

$$\text{flux} = \iint (\vec{F} \cdot \hat{n})\, ds$$

$$= \iint \underbrace{a \sin^2 \varphi \sin \theta}_{\vec{F} \cdot \hat{n}} \cdot \underbrace{a^2 \sin \varphi \, d\varphi \, d\theta}_{ds}$$

$$= a^3 \int_{\theta\,0}^{\pi} \int_{\varphi\,0}^{\pi} \sin^3 \varphi \sin^2 \theta \, d\varphi \, d\theta$$

Now can use shortcut
But even easier if $\frac{1}{4}$ hemisphere (1st octant)

$$= 4a^3 \int_0^{\pi} \int_0^{\pi} \sin^3 \varphi \sin^2 \theta \, d\varphi \, d\theta$$

$$= 4a^3 \int_0^{\pi} \sin^3 \varphi \, d\varphi \cdot \int_0^{\pi} \sin^2 \theta \, d\theta$$

$$= 4a^3 \cdot \frac{2}{3 \cdot 1} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

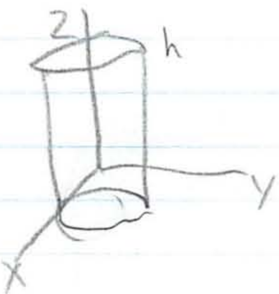
$$= \frac{a}{3} \pi a^3$$

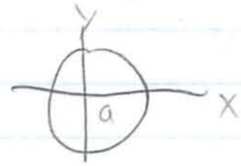
$$= \underbrace{2\pi a^2}_{\text{area hemisphere}} \cdot \underbrace{\frac{a}{3}}_{\text{velocity}}$$

That was very complex, solution glosses over

8. Find $\iint_S \vec{F} \cdot d\vec{S}$ where $\vec{F} = y\hat{j}$ and S is the portion of cylinder $x^2 + y^2 = a^2$ between $z=0$ $z=h$

$\hat{n}$ outward





so we have shadow

$$\hat{n} = \frac{x\hat{i} + y\hat{j}}{a}$$

$$y\hat{j} \cdot \frac{x\hat{i} + y\hat{j}}{a} \quad \checkmark$$

$$\vec{F} \cdot \hat{n} \quad \frac{y^2}{a} \quad \checkmark \text{ getting it}$$

$$\iint \frac{y^2}{a} \, dS$$

what area?
cylindrical coords
$d\theta, dr$

$$\cancel{h} \int_{\theta=0}^{2\pi} \int_{\cancel{0}}^{a} \underbrace{\frac{y^2}{a} \, r \, dr}_{a^2 \sin^2\theta \, dz \, d\theta} \, d\theta$$

need to convert y
$y = a \sin\theta$

z matters

$$\int_0^a \frac{y^2}{a} r \, dr$$

$$\frac{y^2}{a} \frac{r^2}{2} \Big|_0^a$$

$$\int_0^{2\pi} \frac{y^2 a}{2} \, d\theta$$

$$\int_{-\frac{\pi}{2}}^{\pi/2} \int_{0}^{h} a^2 \sin^2\theta \, dz \, d\theta$$

θ z

↑why $-\frac{\pi}{2} \to \frac{\pi}{2}$

$$\int_{0}^{h} a^2 \sin^2\theta \, dz$$

$$a^2 \sin^2\theta \, h$$

$$\int_{-\frac{\pi}{2}}^{\pi/2} a^2 \sin^2\theta \, h \, d\theta$$

$$a^2 h \int_{-\pi/2}^{\pi/2} \sin^2\theta \, d\theta$$

$$a^2 h \left(\frac{\theta}{2} - \frac{\sin 2\theta}{4} \right) \Bigg|_{-\pi/2}^{\pi/2}$$

↑ can never remember

$$\boxed{\frac{\pi}{2} a^2 h}$$

Lecture 22 — Flux surface integrals cont. Divergence Theorm

6B-5 — Find $\iint_S \vec{F} \cdot d\vec{S}$ $\quad F = z\hat{k}$

$S =$ plane $x + y + z = 1$

Normal vector to plane

$$\frac{\hat{i} + \hat{j} + \hat{z}}{\sqrt{3}}$$

$\vec{F} \cdot \hat{n} \quad \frac{z}{\sqrt{3}}$

$$\iint_S \frac{z}{\sqrt{3}}\, dS$$

$$\iint_S \frac{z}{\sqrt{3}} \frac{dx\,dy}{|\hat{n} \cdot \hat{k}|}$$

$$= \frac{1}{\sqrt{3}} \iint_S (1 - x - y) \frac{dx\,dy}{1/\sqrt{3}}$$

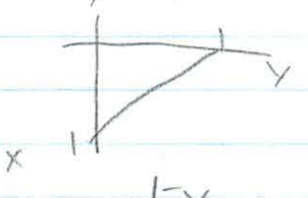
↑ why in all world?

- can't just · by area since variable in integrand (?) now
- so need to do shape

$$\int_0^1 \int_0^{1-y} \cancel{\frac{z^3}{\sqrt{3}}}\, dx\,dy$$

y x

$(1 - x - y)\, dx\, dy$



$$\int_0^{1-y} 1 - x - y\, dx$$

$$x - \frac{x^2}{2} - xy \Big|_0^{1-y}$$

$$(1-y) - \frac{(1-y)^2}{2} - (1-y)y$$

$1-2y+y^2$

$$\int_0^1 1-y-\frac{1}{2}+\cancel{2y}-\frac{y^2}{2}-\cancel{y}+y^2\,dy$$

$$\int_0^1 \frac{1}{2}-y+\frac{y^2}{2}\,dy$$

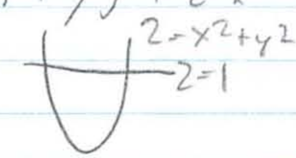
$$\frac{y}{2}-\frac{y^2}{2}+\frac{y^3}{6}\Big|_0^1$$

$$\frac{1}{2}-\frac{1}{2}+\frac{1}{6}$$

$$\frac{1}{6} \checkmark$$

6. Find $\iint_S \vec{F}\cdot dS$ where $F = x\hat{i}+y\hat{j}+z\hat{k}$

$S =$ parabola



n points upward

Why is answer negitive

* same as exam 3 + makeup

$\hat{n} = 2\hat{k}$

$\hat{F}\cdot\hat{n} = z^2$

$$\int_{y\,-1}^{1}\int_{x\,r^2}^{1} z^2\,dy\,dx$$

was it diff coord system?

or do I need to convert something?

$$d\vec{S} = \underbrace{-2x\,\hat{\imath} - 2y\,\hat{\jmath} + \hat{k}}\ dx\,dy$$

Oh this looks familiar - but from where

points up since $\hat{k}$ $+$

$$\iint_R -2x\,\hat{\imath} - 2y\,\hat{\jmath}\ k\,\hat{k}\ dx\,dy$$

unit circle in xy plane

$$= -\iint (x^2 + y^2)\,dx\,dy$$

here is what I had

$x^2 + y^2 = z^2$

But yes polar coords

$$-\int_0^{2\pi}\int_0^1 r^2\,r\,dr\,d\theta$$

θ $\quad r$

$$-2\pi \cdot \frac{1}{4} = -\frac{\pi}{2}$$

But why $\ominus$?

Because $\oplus$ flux is that of $\hat{n}$

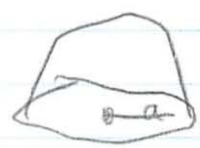
Here to inside of cup

Opposit $x\hat{\imath} + y\hat{\jmath} + 2\hat{k}$ which is in $\rightarrow$ out of cup

Divergence Theorm

6C-3 Verify divergence theorm when $F = x\hat{i} + y\hat{j} + z\hat{k}$

$S =$ ...



Memorize that $D = x^2 + y^2 \le a^2$

$z = a^2$

$S = z = x^2 + y^2$

above D

Notes

So what is the ~~divergence theorm?~~

- closed surface
- flux

D is the shadow

+ and − directions

$F = M\hat{i} + N\hat{j} + P\hat{k}$

$\operatorname{div} F = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}$

$$\iint_S F \cdot ds = \iiint_D \operatorname{div} \vec{F} \, dv$$

Guass's Theorm

Source rate - fluid added to flow

flux across S = source rate for D

* net flow outward across S = same rate fluid produced inside S *

- think of physics

div $\vec{F}$ = source rate at (x, y, z)

How does this differ from flux
-or is it just 3D flux?

So this problem is verify

What does that mean?

$$\iint_S F \cdot ds = \iiint_D \text{div } \vec{F} \, dV$$

$$\iiint \text{div } \vec{F} \cdot dV = 3 \, (\text{vol } D)$$

because non variable

$$= 3 \cdot \tfrac{2}{3} \pi a^3$$

$$2\pi a^3$$

Now: other side (like previous problems)

$$\hat{n}_1 = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{a}$$

$$\hat{n}_2 = -\hat{k} \quad \text{disc}$$

Since when are there 2 normal vectors?

$$\int_{S_1} x\hat{i} + y\hat{j} + z\hat{k} \cdot \frac{x\hat{i} + y\hat{j} + z\hat{k}}{a}$$

$$+ \int_2 -k \cdot \frac{x\hat{i} + y\hat{j} + z\hat{k}}{a}$$

$$\int_{S_1} \frac{x^2+y^2+z^2}{a}\, dS + \int_{S_2} \frac{-z}{a}\, dS$$

$$= \iint_{S_1} a\, dS$$

$x^2+y^2+z^2 = \rho^2 = a^2$

$z = 0$ on S_2

(how do we know

So $a \cdot$ area of S_1

$a(2\pi a^2)$

$2\pi a^3$

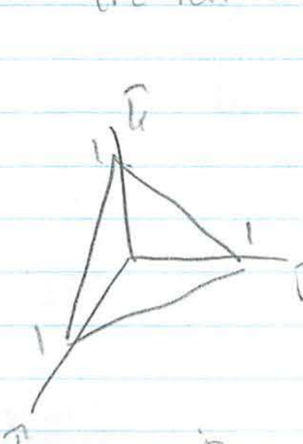
(✓) out b/w answers

Don't have full grasp of concept

5. By using divergence theorm, evaluate surface integral (the left side) $F = x\hat{i} + z^2\hat{j} + y^2\hat{k}$

why different?

-just different field



Working off of last problem

need normal

what is plane eq? $x+y+z=1$

$\hat{n} = \langle 1,1,1\rangle$???

$F\cdot n = \langle x, z^2, y^2\rangle \cdot \langle 1,1,1\rangle$

$= \langle x, z^2, y^2\rangle$

$$\int_{-1}^{1} x^4 - x^6 - x^4 y^2 + \frac{x - x^3 - x y^2}{4}\, dx$$

this seems wrong - too long

$$\frac{x^5}{5} - \frac{x^7}{7} - \frac{x^5}{5} y^2 + \frac{x^2}{8} - \frac{x^4}{16} - \frac{x^2 y^2}{8} \Big|_{-1}^{1}$$

-- My problem was not symplifying and writing y in terms of x

$$\iiint M_x + N_y + P_z$$

$$\iiint 2x + x + 0 \cdot dV$$

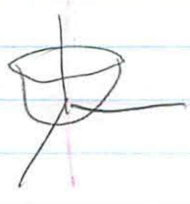
↑ does depend on variables
so no trick

solve - over the shadow

$$\iiint 3x\, dV = 0$$

since solid is symetrical w/ yz plane

knew axis weird
- no it was what I drew
←



assume $\delta = 1 \rightarrow$ integral has value
$\overline{x}$ = mass of D
↑ center of mass x-coord
(in yz plane due to symmatry)

There were 2 flat endcaps
and flux 0 through them (I got that)

$F \cdot n = x^3 + xy^2$
$= x^3 + x(1-x^2)$ add w/ out vectors
$= x$

$$\iint_S = \int_0^{2\pi} \int_0^1 \cos\theta \, dz \, d\theta$$

$\theta \quad z$

$$= \int_0^{2\pi} \cos\theta \, d\theta$$

$$= 0$$

8. Suppose $\text{div } \vec{F} = 0$

S_1 = upper hemisphere at origin
S_2 = lower
both unit normal is ↑ $\hat{k}$

↳ even S_2 → not out? ↙↓↘

What did he say was eaiser out or up?
And is it not always the same?

a) Show that $\iint_{S_1} \vec{F} \cdot ds = \iint_{S_2} \vec{F} \cdot ds$

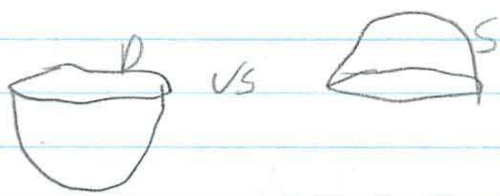
interprate in terms of flux

Well do not have a $\vec{F}$ but would be same for both

$\hat{n}$ is the same ↑
- looked at notes + this is standard

and surface is same just flipped

or is it



dish vs shell

Well $\iint$ is same in both cases
- since its based on its shadow

They reverse normal vector on down one to $-\hat{k}$

$S = S_1 + S_2$ = closed surface
* normal vector pointing out everywhere

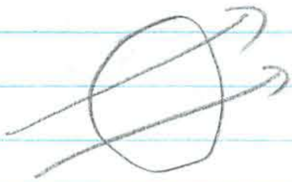
$$\iiint \text{div}\, F \cdot dV = 0$$

$$\iint F \cdot dS = -\iint F \cdot dS = 0$$

From physics: how do we know nothing enclosed?

b state a generalization to any arbitrary S and F that div $\vec{F} = 0$

Well this is from physics



field goes through in + out

so you add it in + subtract it out

their answer used a lot of words to say <u>nothing</u>

- same boundry curve → shadow

Part 2

1. If you fix the Volume V (assume $\sigma = 1$) can be shown that the solid of that volume which exerts the strongest gravitational force in the $\hat{k}$ direction on a unit mass at the origin O is spheroid whose boundry

$$\frac{\cos\phi}{\rho^2} = k_1 \quad \underline{\text{or}} \quad \rho = k\sqrt{\cos\theta} \qquad k = \frac{1}{\sqrt{k_1}}$$

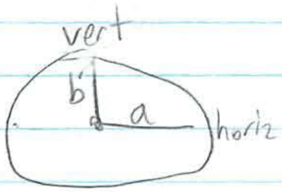
spheroid - eclipise revolved

Constant k is adjusted so it has desired volume V
Symmetric about z-axis
tangent to xy plane at O
- like a somewhat flattened sphere

How does gravitational attraction compare w/ that of a sphere?

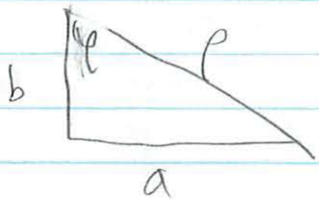
d) Find the volume of the spheroid

Volume of a spheroid $= \frac{4}{3}\pi a^2 b$



spherical coords

$$\rho = \frac{1}{\sqrt{k_1}}\sqrt{\cos\theta} = \frac{\sqrt{\cos\theta}}{\sqrt{k_1}}$$



$$\rho = \frac{\sqrt{\cos\phi}}{\sqrt{k_1}}$$

$$\sin\psi = \frac{a}{\rho}$$

$$a = \rho\sin\psi$$
$$b = \rho\cos\psi$$

$$\frac{4}{3}\pi\,\rho^2\sin^2\psi\;\rho\cos\psi$$

$$\frac{4}{3}\pi\,\frac{\cos\phi}{k_1}\cdot\frac{\sqrt{\cos\phi}}{\sqrt{k_1}}\,\sin^2\psi\cos\psi$$

I don't think that was the way we were supposed to find it

$$\text{vol} = \iiint_V dV$$

$$\text{mass} = \iiint_V \sigma\, dV$$

$$\int_{\theta=0}^{2\pi}\int_{\psi=\cancel{-\pi/2}\;0}^{\pi/2}\int_{\rho=0}^{\cancel{\frac{\sqrt{\cos\phi}}{\sqrt{k_1}}}\;k\sqrt{\cos\psi}} \rho^2\sin\psi\, d\rho\, d\psi\, d\theta$$

$$\rho^3\Big|_0^{\frac{\sqrt{\cos\phi}}{\sqrt{k_1}}}$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\frac{\sqrt{\cos\phi}}{\sqrt{k_1}}\right)^3\sin\psi\, d\psi$$

$$\left(\frac{\sqrt{\cos\phi}}{\sqrt{k_1}}\right)^3 \cos\varphi \Big|_{-\frac{\pi}{2}}^{\pi/2}$$

$$\left(\frac{\sqrt{\cos\phi}}{\sqrt{k_1}}\right)^3 \left(\underset{0}{\cos\frac{\pi}{2}} - \underset{0}{\cos\frac{-\pi}{2}}\right)$$

$$\int_0^{2\pi} \left(\frac{\sqrt{\cos\phi}}{\sqrt{k_1}}\right)^3 d\theta$$

? whats up w/ this ?

X -2

b) Find gravitational attraction on unit mass at 0

$$F = \frac{GMm}{\rho^2} \quad \text{c above} \quad \text{c } dm$$

$$\frac{G\, dm \cos\varphi}{\rho^2}$$

$$\iiint \frac{G\cos\varphi}{\rho^2} \delta \, dV$$

need help w/ gravitational attraction

$$\int_{\theta=0}^{2\pi} \int_{\varphi=\cancel{-\frac{\pi}{2}}\ 0}^{\frac{\pi}{2}} \int_{\rho=0}^{\rho = k\sqrt{\cos\varphi}} \frac{G\cos\varphi}{\cancel{\rho^2}} \cancel{\rho^2} \sin\varphi \, d\rho \, d\varphi \, d\theta$$

need to take another look here about from what to what here

-3

X -1

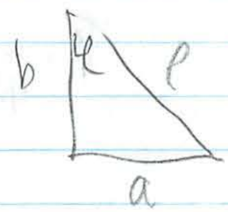
c) Take a solid sphere radius a

$\delta = 1$

tangent to xy plane at O

Use Newton's Theorm p 743

i) Express k in terms of a if sphoriod has same volume as sphere



sphere $a = b$

$\tan \frac{a}{b} = 45°$

$\rho = \frac{\sqrt{\cos\phi}}{\sqrt{k_1}}$

$\cos 45 = \frac{a}{\rho}$ $\quad$ $\sin 45 = \frac{b}{\rho}$

$a \frac{1}{\sqrt{2}} = \rho$ $\quad$ $b \frac{1}{\sqrt{2}} = \rho$

$a = \rho\sqrt{2} = b$

$a = \frac{\sqrt{\cos\phi}\,\sqrt{2}}{\sqrt{k_1}}$

ii) Calculate the ratio of the 2 gravitational attractions
How much bigger is spheroid?

$$F_{grav} = \iiint G \cos\varphi \sin\varphi \, d\rho \, d\varphi \, d\theta$$

$$\frac{\text{spheroid} \quad \int_0^{\pi/2} \int_{-\pi/2}^{\pi/2} \int_0^{\rho = \frac{\sqrt{\cos\phi}}{\sqrt{k_1}}} G \cos\varphi \sin\varphi \, d\rho \, d\varphi \, d\theta}{\text{sphere} \quad \int_0^{2\pi} \int_{-\pi/2}^{\pi/2} \int_0^{a} G \cos\varphi \sin\varphi \, d\rho \, d\varphi \, d\theta}$$

2. Take a finite Domain D bounded by $z^2 = x^2 + y^2$

$\hat{n} \uparrow$

Letting $F = z\hat{k}$ calculate directly from surface integral calc flux F over

a) Determine radius of S_1 and calc flux

So this looks like an achievable problem here for once

Is the radius just 2 - or am I missing something

$\vec{n} = \hat{k}$

$F \cdot n = z\hat{k} \cdot \hat{k} = z\hat{k}$

$\iint z k \, dS$

$\int_{\theta=0}^{2\pi} \int_{r=0}^{2} z \, r \, dr \, d\theta$

all perpendicular

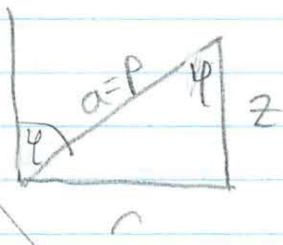
$z \cdot$ area

$z \cdot \pi r^2 = \pi r^2 = \boxed{4\pi}$ x ② b S_2

$\hat{n} = \dfrac{\langle x, y, z \rangle}{a}$ radially out

$F \cdot \hat{n} = z\hat{k} \cdot \dfrac{\langle x, y, z \rangle}{a}$

$= \dfrac{z^2}{a}$



$\cos\varphi = \dfrac{z}{a}$

$r = a\sin\varphi = 2$

$z = a\cos\varphi$

$a = \rho$

$= \dfrac{a^2\cos^2\varphi}{a} = a\cos^2\varphi$

$\int_\theta \int_\rho a\cos^2\varphi \; \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$

$\int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2} \rho^3 \cos^2\varphi \sin\varphi \, d\rho \, d\varphi \, d\theta$

$\dfrac{\rho^4}{4} \cos^2\varphi \sin\varphi \Big|_0^2$

$\int_{-\pi/2}^{\pi/2} 4\cos^2\varphi \sin\varphi \; d\varphi$

$$\cancel{\frac{4 - \cos^3\theta}{3}\Bigg|_{-\frac{\pi}{2}}^{\pi/2}}$$

$\cancel{0}$

Over shadow region

$a = r$

$$\int_0^{2\pi}\int_0^{2} \frac{z^2}{\cancel{a}}\,\cancel{r}\,dr\,d\theta$$

$\theta \quad r$

$$\int_0^{2} z^2\,dr$$

$$\int_0^{2\pi} 2z^2\,d\theta$$ X   -4

$$\boxed{4\pi z^2}$$

3. Flux over S_3

$n = \hat{i}$

cone $z^2 = x^2 + y^2$

$0 = x^2 + y^2 - z^2$

$F \cdot s \quad \langle x^2, y^2, -z^2 \rangle \langle z \rangle$

$-z^3\,\hat{k}$

Over the shadow region

$$\int_0^{2\pi}\int_0^2 -z^3 \, r \, dr \, d\theta$$

θ r

$$-z^3 \frac{r^2}{2} \Big|_0^2$$

$$\int_0^{2\pi} -2z^3 \, d\theta$$

$$\boxed{-4\pi z^3}$$

3. Get the results of the preceding problem another way.

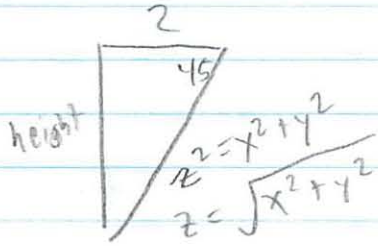
a) Find the volume of D by integrating, then find the two volumes D is split into by the horiz disc

$$V_{cone} = \frac{\text{base} \cdot \text{height}}{3}.$$

Lets do normal first

$$\text{top} \Rightarrow \frac{1}{2} \cdot \frac{4}{3} \pi r^3 = 16.75$$

$$\text{bottom} \Rightarrow \frac{\pi r^2 \cdot 2}{3} = 8.3775$$



← 45° angle assuming

$$\tan 45 = \frac{h}{2}$$

$$h = 2 \tan 45$$

Now by SSS

$$\int_{\theta=0}^{2\pi} \int_{x=-2}^{2} \int_{z=\sqrt{x^2+y^2}}^{x^2+y^2} dz \cdot dx \, d\theta$$

$$z \Big|_{\sqrt{x^2+y^2}}^{x^2+y^2}$$

$$(x^2+y^2)-\left(\sqrt{x^2+y^2}\right)$$

$$\int_{-2}^{2}(x^2+y^2)-\left(\sqrt{x^2+y^2}\right)\,dx$$

$$\left.\frac{x^3}{3}+xy^2-\left(\frac{y^2\cdot\ln(y)-y^2\ln\left(\sqrt{y^2+4}+2\right)}{2\sqrt{y^2+4}}\right)\right|_{-2}^{2}$$

$$=y^2\ln\frac{|y|}{\sqrt{y^2+4}+2}-2\sqrt{y^2+4}+4y^2+\frac{16}{3}$$

Seems wrong

$$\int_0^{2\pi}\text{ that mess }\uparrow$$

$$2\pi\cdot\text{ that mess }\uparrow$$

Does not deal w/ y
Should have used other coord system
? would it have worked

b) - Starting from the value that you calculated for the flux over S_1, use the divergence theorem to find flux over S_2 + S_3

$$\iint_S F \cdot ds = \iiint_V (\operatorname{div} \vec{F})\, dV$$

$M_x + N_y + P_z$

what I calculated 4π

$$4\pi = \iiint_V M_x + N_y + P_z \; dV$$

$$P_z = 1$$

$$4\pi = \iiint_V dV$$

So it should = the volume

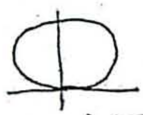
? That does not make sense!

-5

Remember this was no OH Should go over answers for once - but now Physics exam!

18.02 Problem Set 8 Solns   Spring 2010

1 a) Volume: $\displaystyle\int_0^{2\pi}\int_0^{\pi/2}\int_0^{k\sqrt{\cos\varphi}} \rho^2\sin\varphi \, d\rho\, d\varphi\, d\theta$



$$= \frac{4\pi k^3}{15}$$

$\rho = k\sqrt{\cos\varphi}$

Inner: $\frac{1}{3}\rho^3\sin\varphi\Big]_0^{k\sqrt{\cos\varphi}} = \frac{k^3}{3}\cos^{3/2}\varphi\sin\varphi$

Middle: $-\frac{k^3}{3}\cdot\frac{2}{5}\cos^{5/2}\varphi\Big]_0^{\pi/2} = \frac{2k^3}{15}$

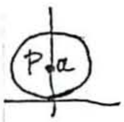
Outer: $2\pi\cdot\frac{2k^3}{15}$

b) Grav. attraction on origin $= G\displaystyle\int_0^{2\pi}\int_0^{\pi/2}\int_0^{k\sqrt{\cos\varphi}} \sin\varphi\cos\varphi\, d\rho\, d\varphi\, d\theta$

$$= G\cdot 2\pi\cdot\frac{2k}{5} = \frac{4\pi k}{5}G$$

Inner: $\sin\varphi\cos\varphi\cdot\rho\Big]_0^{k\sqrt{\cos\varphi}} = k\cos^{3/2}\varphi\sin\varphi$

rest is as above (replace $k^3/3$ by k).



c) Sphere of radius a: $(\delta = 1)$

Volume $= \frac{4}{3}\pi a^3$

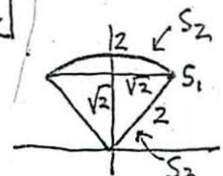
Grav. attraction on origin: (by Newton) $\dfrac{\frac{4}{3}\pi a^3 G}{a^2} = \frac{4}{3}\pi a G$

[acts as if all its mass were concentrated at P)

(i) $\dfrac{4\pi k^3}{15} = \frac{4}{3}\pi a^3 \Rightarrow k = a\sqrt[3]{5}$

(ii) $\dfrac{\text{Grav. of solid spheroid}}{\text{Grav. of sphere}} = \dfrac{\frac{4\pi}{5}\cdot a\sqrt[3]{5}\cdot G}{\frac{4}{3}\pi a G} = \dfrac{3\sqrt[3]{5}}{5} \approx 1.03$ $\therefore$ spheroid exerts $\approx 3\%$ more force.

2



$\vec{F} = z\hat{k}$

Eqn of cone: $z = r$

a) $\displaystyle\iint_{S_1} \vec{F}\cdot\hat{n}\, dS = \int_0^{2\pi}\int_0^{\sqrt{2}} z\, dS = \sqrt{2}\begin{pmatrix}\text{area}\\ \text{of } S_1\end{pmatrix}$

$\hat{n} = \hat{k}$ $z = \sqrt{2}$ on S_1 $dS = dA = r\,dr\,d\theta$

$$= \boxed{2\pi\sqrt{2}}$$

b) S_2: $0 \le \varphi \le \pi/4$

$\displaystyle\iint_{S_2} \vec{F}\cdot\hat{n}\, dS = \int_0^{2\pi}\int_0^{\pi/4} \frac{z^2}{2}\, dS$

$\boxed{\hat{n} = \frac{\langle x, y, z\rangle}{2} \text{ on } S_2}$ $= \displaystyle\int_0^{2\pi}\int_0^{\pi/4} \frac{4\cos^2\varphi\cdot 2^2\sin\varphi}{2}\, d\varphi\, d\theta$

$z = 2\cos\varphi$

Inner: $-\dfrac{8\cos^3\varphi}{3}\Big]_0^{\pi/4}$

$dS = 2^2\sin\varphi\, d\varphi\, d\theta$ $= \frac{8}{3}\left(1 - \frac{\sqrt{2}}{4}\right)$

Outer: $\boxed{2\pi\cdot\frac{8}{3}\left(1-\frac{\sqrt{2}}{4}\right)}$

c) cone: $x^2 + y^2 - z^2 = 0$ Call the left side g

$d\vec{S} = \dfrac{\nabla g}{g_z}\, dx\, dy$

$= \dfrac{\langle 2x, 2y, -2z\rangle}{-2z}\, dx\, dy$

$\therefore \vec{F}\cdot d\vec{S} = z\, dx\, dy$

On the cone, $z = r$

$\therefore \displaystyle\iint_{S_3} \vec{F}\cdot d\vec{S} = \int_0^{2\pi}\int_0^{\sqrt{2}} r\cdot r\, dr\, d\theta$

$= 2\pi\cdot\dfrac{r^3}{3}\Big]_0^{\sqrt{2}} = \boxed{\dfrac{4\pi\sqrt{2}}{3}}$

3 

$D = D_1 + D_2$

a) volume of $D = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{2} \rho^2 \sin\varphi \, d\rho\, d\phi\, d\theta$

$= \frac{16\pi}{3}\left[1 - \frac{\sqrt{2}}{2}\right] = \frac{16\pi}{3} - \frac{8}{3}\pi\sqrt{2}$

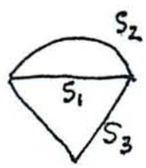
Inner: $\frac{1}{3}\rho^3 \sin\varphi\Big]_0^2$
$= \frac{8}{3}\sin\varphi$

Middle: $-\frac{8}{3}\cos\varphi\Big]_0^{\pi/4}$
$= \frac{8}{3}\left[-\frac{\sqrt{2}}{2} + 1\right]$

Outer: $\times 2\pi$

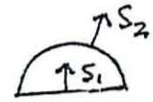
volume of $D_1 = \frac{1}{3}(\text{base})(\text{height})$
$= \frac{1}{3}\pi(\sqrt{2})^2 \cdot \sqrt{2}$
$= \frac{\pi}{3} \cdot 2\sqrt{2}$

volume of D_2 = vol D − vol D_1 $= \frac{16\pi}{3} - \frac{10}{3}\pi\sqrt{2}$
$= \frac{\pi}{3}\left(16 - 10\sqrt{2}\right)$

b) 

$\vec{F} = z\hat{k} \Rightarrow \operatorname{div}\vec{F} = 1$

Flux over $S_1 = 2\pi\sqrt{2}$

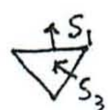
(i) Flux over S_2: Apply div. thm to 

:

$-\iint_{S_1} + \iint_{S_2} = \iiint_{D_2} 1 \, dV = \frac{16\pi}{3} - \frac{10}{3}\pi\sqrt{2}$

$\therefore \iint_{S_2} \vec{F}\cdot d\vec{S} = \frac{16\pi}{3} - \frac{10}{3}\pi\sqrt{2} + 2\pi\sqrt{2} = \boxed{\frac{16\pi}{3} - \frac{4}{3}\pi\sqrt{2}}$

agrees with earlier calculation

(ii) Flux over S_3: 

Apply div. thm:

$\iint_{S_1} \vec{F}\cdot d\vec{S} - \iint_{S_3} \vec{F}\cdot d\vec{S} = \iiint_{D_1} 1 \, dV = \frac{2\pi}{3}\sqrt{2}$

$\therefore \iint_{S_3} \vec{F}\cdot d\vec{S} = 2\pi\sqrt{2} - \frac{2}{3}\pi\sqrt{2} = \boxed{\frac{4}{3}\pi\sqrt{2}}$

agrees with problem 2

Lecture 30

Divergence Theorm Cont.

4/29

Last P-set coming out (#9)

Divergence theorm

$$\oiint_S \vec{F} \circ d\vec{S} = \iiint_D \text{Div } \vec{F} \cdot dV$$

must be closed — triple integral

Divergence $= M_x + N_y + P_z$ ← not vector anymore

Use one side to calculate other

Today: Understanding divergence theorm better

$$\nabla f(x,y,z) = \langle f_x, f_y, f_z \rangle$$
$$= f_x \hat{\imath} + f_y \hat{\jmath} + f_z \hat{k}$$
$$= \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial z}\hat{k}$$

∇ = symbolic vector
"Fake hocus pocus"

$$= \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$$
$$= \hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

? to avoid confusion
make clear that it is an operator

$\left[\frac{d}{dx} \text{ = operator on } f \rightarrow \frac{d}{dx} f \right]$

Net flux over top + bottom faces

$$\approx \left(\frac{\partial P}{\partial z}\right)_0 \Delta z \Delta x \Delta y$$

$\downarrow$ in the limit

net flux from the 2 faces

$$= \left(\frac{\partial P}{\partial z}\right)_0 dx\, dy\, dz$$

The same analysis for sides

M

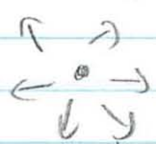
N

Net flux over all 6 sides

$$\underbrace{\left(\frac{\partial M}{\partial x}\right)_0 + \left(\frac{\partial N}{\partial y}\right)_0 + \left(\frac{\partial P}{\partial z}\right)_0}_{(\operatorname{div} \vec{F})_0} \cdot dx\, dy\, dz$$

Hints for Pset 4 part 2 #1

The problem you must know



at origin only (Point source)

Field $\vec{F}$ which is source – incompressable fluid

Operator "del" or "nabla" ∇

Maybe could apply symbolic vector operations to it

$$\text{div } \vec{F} = \frac{\partial}{\partial x} M + \frac{\partial}{\partial y} N + \frac{\partial}{\partial z} P$$

$$= \nabla \cdot \vec{F}$$

↑ dot product

compact divergence notation

Cross product Tuesday

$$\oiint_S \vec{F} \cdot d\vec{s} = \iiint_D \vec{\nabla} \cdot F \, dV$$

was wondering this

What is meaning of divergence?

interpert $\text{div } \vec{F}$

$\nabla \cdot F$

$\vec{F}$ velocity field flow of incompressible fluids

at point P_0

Function has a value at that point. What does that volume signify.

P_0 inside infatesimal box

$\hat{n}$ outward is $+$

$\oiint \vec{F} \cdot d\vec{s}$

Function inside is approx a constant

$= (\operatorname{div} \vec{F})_0 \cdot \underbrace{\Delta x \Delta y \Delta z}_{\text{vol box}}$

$$(\operatorname{div} F)_{P_0} = \frac{\left[\text{flux out of the}\ \underset{\text{infatemessial}}{\text{box}}\ \text{at } P_0\right]}{\text{Vol box}}$$

$\uparrow$ Source rate at P_0

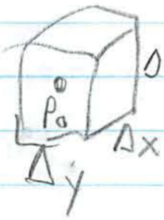
If fluid added from that box ...
But how can fluid come from nowhere
- kinda nonsense

rate is not meaning time-dependent
- $\frac{\text{amt added}}{\text{unit volume}}$ not time

Something unsatisfying ....
what does it have to do w/ 3 partial derivs
how it was defined in the 1st place

Now the "shady" way to explain:

Interpt div $\vec{F}$ as sum of diervitives



$\Delta z \leftarrow$ infatenessal box

net flux of $\vec{F}$ out

net flux of $\vec{F}$ over top + bottom faces

<u>top</u> $\langle M, N, P \rangle$
$\uparrow \hat{k}$

only interested in flow $\perp$ top + bottom
Only $\hat{k}$ direction matters

$P(x_0, y_0, z_0 + \Delta z) \underbrace{\Delta y \Delta x}_{\text{area of face}}$

↑ top face
not bottom

flux = velocity · area

Was very confused on that last P-set (8)

<u>bottom</u> $-P(x_0, y_0, z_0) \Delta y \Delta x$
↑ vector is down $(-\hat{k})$

<u>net flux</u> = add the top + bottom
x_0, y_0 held constant

$\Delta P \cdot \Delta y \Delta x$

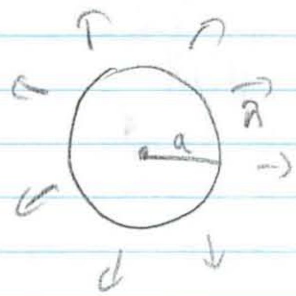
$$\left(\frac{\partial P}{\partial z}\right)_0 \approx \left(\frac{\Delta P}{\Delta z}\right)_0$$

$$\Delta P \approx \left(\frac{\partial P}{\partial z}\right)_0 \cdot \Delta z$$

$\vec{F}$ direction of $\langle x, y, z \rangle$

$|\vec{F}| = ?$

Flux over every sphere 
radius a, center origin
must be the same
- larger surface area means less flux through a small area



Harmless surface S:

Flux $= \oiint \vec{F} \cdot d\vec{S} = \oiint \vec{F} \cdot \hat{n} \, dS$

$$\hat{n} = \frac{\langle x, y, z \rangle}{a}$$

$$= \oiint \frac{\langle x, y, z \rangle}{c} \cdot \frac{\langle x, y, z \rangle}{a} \, dS$$

↑
$c = ?$

$x^2 + y^2 + z^2 = a^2$

$$= \oiint \frac{a^2}{ca} \, dS$$

$$= \frac{a}{c} \cdot \text{area}$$

$$= \frac{4\pi a^3}{c}$$

$\frac{a}{c} \cdot 4\pi a^2 = 1$

↑ unit point source at origin

Solve for c

$\frac{1}{c} = \frac{1}{4\pi a^3} = 4\pi a^3$

Field $\vec{F} = \frac{\langle x, y, z \rangle}{4\pi \rho^3}$

Where is it not defined

- at origin
- fluid added ∞ fast to be point source

So its fiction - but one everyone accepts

So divergence $= 0$

Lecture 31

Line Integrals in space, conservable fields, potential (functions) 4/30

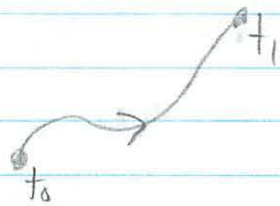
Work in 3D integrals

- easy
- extending from 2D → 3D
- its just longer
- lots of oppurtunity to mess up

$\vec{F} = \langle M, N, P \rangle \qquad d\vec{r} = \langle dx, dy, dz \rangle$

$$\int_C \vec{F} \cdot d\vec{r} = \int_C M\,dx + N\,dy + P\,dz$$

$C = \begin{array}{l} x = x(t) \\ y = y(t) \\ z = z(t) \end{array}$



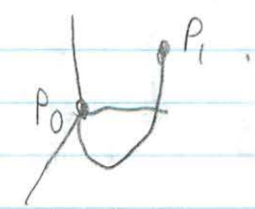
calculate $\displaystyle\int_{t_0}^{t_1} \left(F \cdot \frac{dr}{dt}\right) dt$

$$= \int_{t_0}^{t_1} M(x(t), y(t), z(t))\,\frac{dx}{dt} + \overset{N\ P}{\ldots\ldots}$$

example

$\vec{F} = yz\,\hat{\imath} + xz\,\hat{\jmath} + xy\,\hat{k}$

$C =$



twisted cubic

$x = t$
$y = t^2$
$z = t^3$

$P_0 \rightarrow t = 0$
$P_1 \rightarrow t = 1$

Work by $\vec{F}$ along C

$$= \int_C yz\,dx + xz\,dy + xy\,dz$$

$$dx = dt$$
$$dy = 2t\,dt$$
$$dz = 3t^2\,dt$$

$$= \int_0^1 6t^5\,dt$$

$$= t^6 \Big|_0^1$$

$$= 1$$

Imagine same calculation w/ C_1 which goes direct

$C_1 \ni x = t$
$y = t$
$z = t$

$$\int_{C_1} \text{differential} = \int_0^1 3t^2\,dt$$

$$= t^3 \Big|_0^1$$

$$= 1$$

Same → is it path independent?

FTC for line integrals still applies

$$\vec{F} = \vec{\nabla} f = \int_{P_0}^{P_1} \vec{F} \cdot d\vec{r} = f\Big|_{P_0}^{P_1}$$
$$= f(P_1) - f(P_0)$$

F = conservative
path independent
integral around any closed path $\oint \vec{F} \cdot dr = 0$

$\int_{P_0}^{P_1} \vec{F} \cdot d\vec{r}$ path independent

$\vec{F} = \vec{\nabla} f$
- differential form $\Rightarrow$
- $M\,dx + N\,dy + P\,dz = df$
"differential is exact"

General differential = "inexact" — chem thermo
don't know if it will work
diff'l = df

$M = f_x$
$N = f_y$ — Is there an f that makes these eq true?
$P = f_z$

$f_{xy} = f_{yx} \rightarrow M_y = N_x$ — knew this already
$f_{yz} = f_{zy} \rightarrow N_z = P_y$ — new, includes P
$M_z = P_x$

↑ 3 conditions <u>all</u> must be satisfied

Must remember the pattern of doing it

ex $\vec{F} = \langle 2xy^3, 3x^2y^2 + z^2, x^3 + 2yz \rangle$

Is this a gradient field / conservative?

$6xy^2 = 6xy^2$ ✓

$2z = 2z$ ✓

$0 = 0$ ✓ Yes

ex 2 $axy^2\,dx + (2x^2y + z^2)\,dy + (x^2 + bzy)\,dy$

What a + b will it work?

$2axy = 4xy$

$\hookrightarrow a = 2$

$2z = bz$

$\hookrightarrow b = 2$

$0 = 2x$ ⊗ So no values of a + b

$\boxed{\text{⁎} = \text{contridiction (symbol)}}$

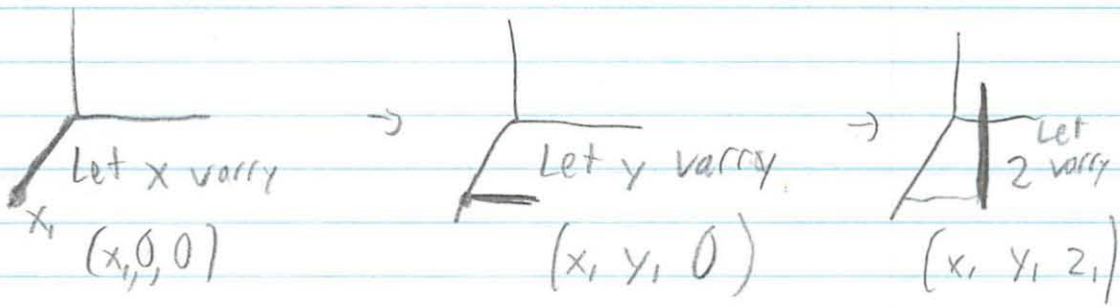
Now we need to find the function

- grungy part
- if exact
- ($\vec{F}$ conservative)

$\vec{F} = f(x, y, z)$

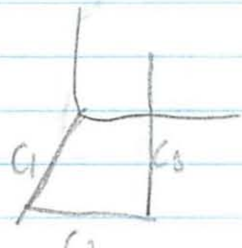
→

<u>Method 1</u> $f(x_1, y_1, z_1) = \int_{(0,0,0)}^{(x_1, y_1, z_1)} M\,dx + \ldots$



9 integrations to make
Use path simplifys dramatically

See general principals



$$\left(\int_{C_1} + \int_{C_2} + \int_{C_3}\right) M\,dx + N\,dy + P\,dz$$

$C_1 \to$ $y = 0$ $z = 0$ so only $M\,dx$
$dy = 0$ $dz = 0$

same reasoning for C_2 C_3

$$\int_{C_1} M\,dx + \int_{C_2} N\,dy + \int_{C_3} P\,dz$$

$$\int 2xy^3 + (3x^2y^2 + z^2)\,dy + (z^3 + 2yz)\,dz$$

$x = x_1$
$y = y_1$

Make integrals $\quad x = x_1$

$$\int_0^{x_1} 0\,dx + \int_0^{y_1} (3x_1y^2 + 0)\,dy + \int^{z_1} (z^3 + 2y_1z)\,dz$$

$C_1 \qquad C_2 \qquad C_3$

since $y=0$

$$0 + x_1^2y_1^3 + \frac{1}{4}z_1^4 + z_1^2y_1$$

$$= x^2y^3 + \frac{1}{4}z^4 + z^2y$$

Break it up into 3 integrals
Must use that path
- won't do anything more complex

Values of x + y

Method 2 $\quad \langle 2xy^3, 3x^2y^2 + z^2, z^3 + 2yz \rangle$

$f_x = 2xy^3$ remember
$f = x^2y^3 + g(y,z)$

$f_y = 3x^2y^2 + g_y$
$= 3x^2y^2 + z^2$
see

$g_y = z^2$ ← reverse now
$g = z^2y + h(z)$
g does not have x
we know

assemble what we know
$f = x^2y^3 + z^2y + h(z)$
$f_z = 2zy + h'(z)$

+4 etc

Recitation

5/3

<u>Lectures</u>

Work in 3D $\int_C \vec{F} \cdot d\vec{r}$

$d\vec{r} = \langle dx, dy, dz \rangle = (x'(t), y'(t), z'(t))$

- FTC: $\int_C \vec{F} \cdot d\vec{r} = f(B) - f(A)$ if $F = \vec{\nabla} f$

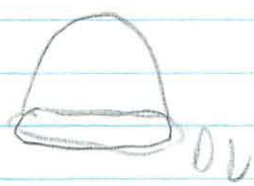
- Conditions for $\vec{F}$ to be conservative ($M_x = N_y$, ...) + methods to find functions such that $\vec{F} = \vec{\nabla} f$ in this case

---

ex1 a) Find flux of $\vec{F} = \langle 2, -1, 3 \rangle$ over $S = z = 1 - x^2 - y^2$, $z \geq 0$

We can avoid having to do paramitrization of S by using divergence theorm

We need to close the surface



flux over union = $\int$ divergence

divergence theorm

$$\oiint \vec{F} \cdot d\vec{S} = \iiint_V \operatorname{div} \vec{F} \, dV$$

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_V \operatorname{div} \vec{F}\, dV - \iint_D \vec{F} \cdot d\vec{S}$$

try to compute the 2 integrals

- $\operatorname{div} \vec{F} = 0$

  field constant
  compute derivitive
  see that it $= 0$

- $\iint_D \vec{F} \cdot d\vec{S} =$

  $d\vec{S}$ = vertical down $(-\hat{k})$

  $\vec{F} \cdot d\vec{S} = -3\, dS$

  $\iint_D \vec{F} \cdot d\vec{S} = \iint_D -3\, dS \to -3 \cdot (\text{area})$

  could only do this b/c field is constant

  $\iint dS$ = area
  small area

D: disc of radius 1
↳ satisfies $1 - x^2 - y^2 = z = 0$

$$\iint_S \vec{F} \cdot d\vec{S} = -\iint_D \vec{F} \cdot dS = 3\pi$$

b) What about $\vec{F} = \langle x, y, z \rangle$

Same method

$$\iint_S \vec{F} \cdot ds = \underbrace{\iiint}_{\substack{\text{costs} \\ \text{triple } \iiint}} \partial w \, \vec{F} \, \partial v - \iint_D \vec{F} \cdot d\vec{s}$$

$\text{div } \vec{F} = M_x + N_y + P_z = 3$

- number not vector

$$\iiint_V 3\, dV = \int_{\theta\, 0}^{2\pi} \int_{r\, 0}^{1} \int_{z\, 0}^{1-x^2-y^2 = 1-r^2} z\, r\, dz\, dr\, d\theta$$

$$= 3 \iint (1-r^2)\, r\, dr$$

$$= 3 \cdot 2\pi \cdot \left(\frac{1}{2} - \frac{1}{4}\right)$$

$$= \frac{3\pi}{2}$$

$$\iint_D \vec{F} \cdot d\vec{s} =$$

$d\vec{s} = -\hat{k}\, ds$

$\vec{F} \cdot d\vec{s} = -z\, ds = 0$

$\overset{\shortparallel}{0}$ over D

$$\iint_S \vec{F}\cdot d\vec{s} = \iiint \operatorname{div} F\, dV - \iint_D \vec{F}\cdot ds = \frac{3\pi}{2}$$

$$\operatorname{div} F\, dV = \frac{3\pi}{2}, \qquad \vec{F}\cdot ds = 0$$

---

ex2

2 surfaces - one inside the other

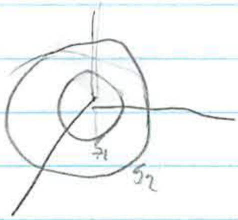
S_1, S_2 closed surfaces
S_1 inside S_2

~~(like a bubble)~~ V = volume b/w S_1 + S_2

We want to prove

$$\iint_{S_1+S_2} \vec{F}\cdot ds = \iiint_V \operatorname{div} \vec{F}\, dV$$

a) check it for $\vec{F} = \langle x, y, z\rangle$
S_1, S_2 are one sphere of radius a, b around origin



$$\iint_{S_1} \vec{F}\cdot d\vec{s} =$$

direction of $d\vec{s}$
- toward origin
- radially inward

$$d\vec{s} = \frac{\langle -x, -y, -z\rangle}{a}\, ds$$

$$a = \sqrt{x^2+y^2+z^2}$$

$$\vec{F}\cdot d\vec{s} = \langle x,y,z\rangle \cdot \frac{\langle -x,-y,-z\rangle}{\sqrt{x^2+y^2+z^2}}\, ds$$

$$= -\sqrt{x^2+y^2+z^2}\, ds$$

$$= -a\, ds$$

$$\iint_S \vec{F}\cdot d\vec{s} = \iint_S -a\, ds = -a \iint ds = -a\cdot \text{area}$$

$-a \cdot 4\pi a^2$

$-4\pi a^3$

Sphere surface area

Same thing for S_2

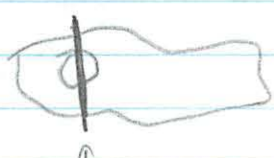
$$\iint_{S_2} \vec{F}\cdot d\vec{s} = \iint b\, ds = b(4\pi b^2)$$

$$\iiint \operatorname{div} \vec{F}\, dV = 3\iiint_V dV = 3\cdot \text{vol}$$

$$= 3\left(\frac{4\pi}{3}b^3 - \frac{4\pi}{3}a^3\right)$$

$= 1$ ✓ it works

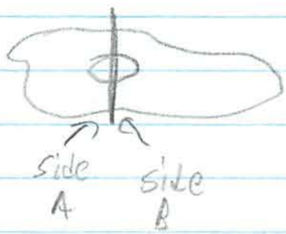
b) What is true in general



look at divergence theorm in 2 parts

apply div. theorm on each part

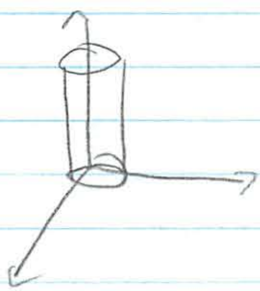
flux on slice region offset each other on each side



qu answered

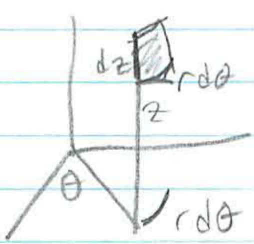
ex 3. r, θ, z

a) What is the surface $r = a$?



fix $r \Rightarrow \theta, z$ varrying

b) What is $\frac{d\vec{s}}{ds}$ in terms of $d\theta\, dz$?



← will get small square
$dS = r\, dz\, d\theta$

direction of $d\vec{s}$ (⊥ to cylinder) $= \frac{\langle x, y, 0 \rangle}{\sqrt{x^2+y^2}}$

$= \langle \cos\theta, \sin\theta, 0 \rangle$

$d\vec{s} = r \langle \cos\theta, \sin\theta, 0 \rangle\, dz\, d\theta$

Lecture 32

Stokes' Theorm

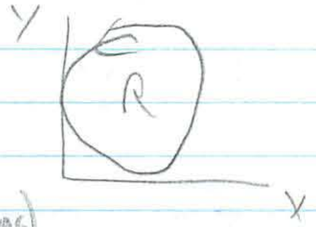
5/4

Generalization into space of green theorm's of work

2D (Green)

$$\oint \vec{F} \cdot d\vec{r} = \text{work (tangental)}$$

$$= \iint_R (N_x - M_y)\,dA$$



$\vec{F} = \langle M, N \rangle$

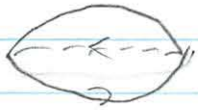
3D (stokes)

$$\oint_C \vec{F} \cdot d\vec{r} = \iint$$

$\vec{F} = \langle M, N, P \rangle$

C closed curve in 3D

No R - what is interior of 3D shape
What reduces to Green's if lies in plane



← cap

C is boundry of surface
oriented so cap is always on left
- left will have some material
- right will not

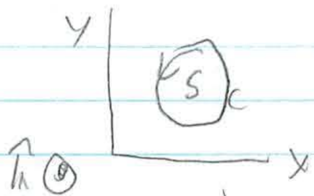
$\uparrow \hat{n}$

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \vec{G} \cdot d\vec{s}$$

↑ depends on $\vec{F}$
reduce to green's
in the coord plane

What $\vec{G}$ will work?

- consider coord planes one at a time

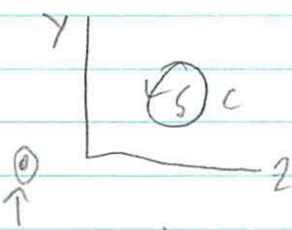


← surface is flat
in this coord plane

$d\vec{s} = \hat{n} \underbrace{ds}$ ↑ area element
$ds = \hat{k} \; dA$

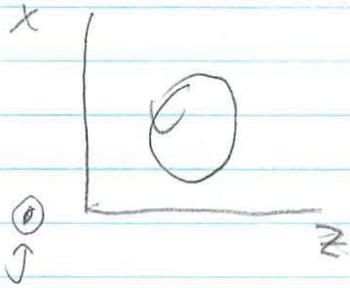
$\vec{G} = (N_x - M_y)\hat{k}$ neglecting P

exactly Green's Theorem! $d\vec{s} = n$



$d\vec{s} = \uparrow dA$
$\uparrow dy\,dz$

$\vec{G} = (P_y - N_z)\uparrow$ neglecting n



← make sure can visualize this + do it right

$d\vec{S} = \hat{j}\, dA$

$\vec{G} = (M_z - P_x)\hat{j}$ neglecting N

Simplest thing we can put on RHS

$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \langle P_y - N_z, M_z - P_x, N_x - M_y \rangle \cdot d\vec{S}$

$\hat{i}$ component $\hat{j}$ k

not in order we found in

$\underbrace{\qquad\qquad}_{\text{curl } \vec{F} \text{ (3D)}}$

$= \nabla \times \vec{F}$

(Remember $\nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$

$= \langle \partial_x, \partial_y, \partial_z \rangle$

$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ M & N & P \end{vmatrix}$

don't forget sign change

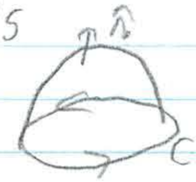
$$= \langle P_y - N_z, -P_x + M_z, N_x - M_y \rangle$$

~~Example~~

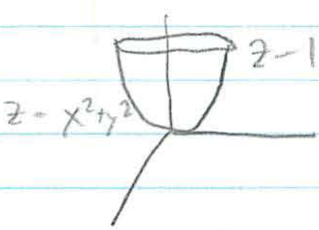
work by $\vec{F}$ going around $\vec{C}$ = flux of curl $\vec{F}$ area S

Stokes' Theorm

$$\oint_C \vec{F} \cdot d\vec{s} = \iint_S \nabla \times \vec{F} \cdot d\vec{S}$$



example



1. First choose a direction
   can choose either
   normal vector must be in conjunction

$\hat{n}$ toward the inside of the bowl

↳ inside would be to ant's left

$\vec{F} = \langle yz, -xz, xy \rangle$

$= yz\,\hat{\imath} - xz\,\hat{\jmath} + xy\,\hat{k}$

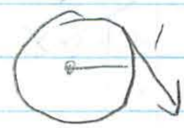
$\oint_C \vec{F} \cdot d\vec{r} = \oint_C y\,dx - x\,dy$

$\downarrow$

on plane

$z = 1 \quad dz = 0$

$yz\,dx$



$= -1 \cdot 2\pi \cdot 1$

$= -2\pi$

$\text{curl } F = \nabla \times F =$

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ M & N & P \end{vmatrix}$$

$= \langle 2x, 0, -2z \rangle$

$d\vec{S} = \hat{n}\,dS$

$\hat{n}$ up-ish

$= \langle -z_x, -z_y, 1 \rangle \; dx\,dy$

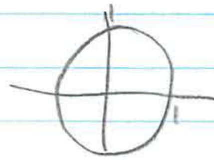
$= \langle -2x, -2y, 1 \rangle \; dx\,dy$

$\iint_S \nabla \times F \cdot dS$

$= \iint_R (-4x^2 - 2z)\,dx\,dy$

$z = x^2 + y^2$
sub in

$$= \iint_R \left(-6x^2 - 2y^2\right) dA$$



$\int$ of $x^2 + y^2$ same by symmetry

$$= \iint_R -4\underbrace{(x^2+y^2)}_{= r^2}\, r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 -4r^3\,dr\,d\theta$$

$$r^4 \Big|_0^1 \cdot 2\pi$$

$$-\boxed{2\pi}$$

↑
⊖ somehow
ran out of time
but its from somewhere

Recitation

5/5

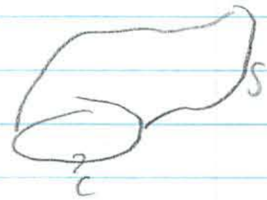
Lecture

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl}(\vec{F})$$

vector 3D

≠ 2D

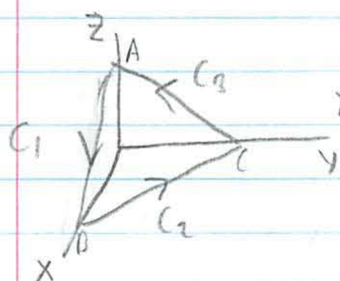
any surface w/ c on boundry



oriented w/ right

c = boundry of S

ex) $\vec{F} = \langle xy, yz, zx \rangle$



$x + y + z = 1$ $\quad C = C_1 + C_2 + C_3$

He did not clearly explain boundry
- is it given?
- largest side?
He claims its obvious
Rest of class confused too

a) Compute $\oint_{C_1} \vec{F} \cdot d\vec{r}$ directly

b) Compute $\oint_C \vec{F} \cdot d\vec{r}$ w/ Stokes

a) C_1 is a segment

$(x(t), y(t), z(t))$ How do you paramitrize?

First know $A = (0,0,1)$

$B = (1,0,0)$

$\vec{AB} = \langle 1, 0, -1 \rangle$

$(x(t), y(t), z(t)) = (t, 0, 1-t)$

$d\vec{r} = \langle dx, dy, dz \rangle = \langle x', y', z' \rangle dt$
$= \langle 1, 0, -1 \rangle dt$

$\vec{F} = \langle t \cdot 0,\ 0 \cdot 1-t,\ t(1-t) \rangle$
$= \langle 0,\ 0,\ t(1-t) \rangle$

$\vec{F} \cdot d\vec{r} = -t(1-t)\, dt$

$I = \int_0^1 -t(1-t)\, dt$

$+$

$= \frac{-t^2}{2} + \frac{t^3}{3} \Big|_0^1 = -\frac{1}{2} + \frac{1}{3} = -\frac{1}{6}$

b) w/ stokes theorm

$J = \oint \vec{F} \cdot d\vec{r} = \iint_T \text{curl}\, \vec{F} \cdot d\vec{S}$

$\text{curl}\, \vec{F} = \nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ xy & yz & zx \end{vmatrix} = \langle -y, -z, -x \rangle$

$d\vec{S} = \hat{n}\, dS =$
T is graph of $z = 1 - x - y$

$d\vec{S} = \langle -z_x, -z_y, 1 \rangle\, dx\, dy$
$= \langle 1, 1, 1 \rangle\, dx\, dy$

$\vec{F} \circ d\vec{S} = (-y - z - x)\, dxdy$

↳ remember not vector

know $= -1 xdy$

$$\iint_T \text{curl}\, \vec{F} \cdot d\vec{S} = \iint_R -dxdy$$

on the shadow
(not the C curve boundry)

$= -\text{Area}(R)$
$= -\frac{1}{2}$

So it does $=$

$$-\frac{1}{6} - \frac{1}{6} - \frac{1}{6} = \boxed{-\frac{1}{2}}$$

---

ex2 $\vec{F} = \langle y,\ x + \frac{z}{y},\ \ln y + 1 \rangle$

a) Is $\vec{F}$ conservative (justify)

b) Find a function f such that $\vec{F} = \nabla f$

c) What is the volume of $\int_C \vec{F} \circ d\vec{r}$ where C is paramitrized by

$x(t), y(t), z(t) = (t, 1, t^2)$

$t \in [0, 1]$

→

a) Yes since

$M_y = N_x \qquad f_{xy} = f_{yx}$

$M_z = P_x \qquad f_{xz} = f_{zx}$

$N_z = P_y \qquad f_{yz} = f_{zy}$

b) $f_x = y$

$f_y = x + \frac{z}{y}$

$f_z = \ln y + 1$

integrate each part

$f = xy + g(y, z)$

+ the constant - keep forgetting this!

method 2

$f_y = x + g_y(y, z)$

↳ use 2nd equation

$g_y(y, z) = \frac{z}{y}$

$g(y,z) = z\ln(y) + \ln(z)$

$f = xy + z\ln y + \ln z$

$f_z = \ln y + \ln' z$

↳ use 3rd eq

$\ln'(z) = 1$

$\ln(z) = z + c$

0

$f = xy + z\ln y + z$

c) FTC = $\int_C \vec{F} \cdot d\vec{r} = f(B) - f(A)$

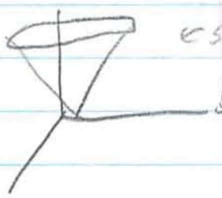
$A = (x(0), y(0), z(0)) = (0, 1, 0)$

$B = (x(1), y(1), z(1)) = (1, 1, 1)$

$f(B) - f(A) = 2 - 0 = 2$

---

ex 3 $\vec{G} = \langle 1, 1, 1 \rangle$

$z = a$  ← surface = cone S oriented up

S: $z = \sqrt{x^2+y^2}$

a) Compute $I = \iint_S \vec{G} \cdot d\vec{S}$ directly

b) Compute " w/ stokes Theorm

Hint $\vec{G} = \text{curl } \vec{F}$ for $F = \langle z, x, y \rangle$

a) Surface is graph of function, so know formula

$d\vec{S} = \langle -z_x, z_y, 1 \rangle \, dx\,dy$

$= \left\langle \frac{-x}{\sqrt{x^2+y^2}}, \frac{-y}{\sqrt{x^2+y^2}}, 1 \right\rangle$

$\int_S \vec{G} \cdot d\vec{S} = \iint \frac{-x-y}{\sqrt{x^2+y^2}} + 1 \rangle \, dx\,dy$

Where is region??

Can use polar coords

R is disc of radius a
↳ since intersection $\sqrt{x^2+y^2} = z = a$

Now switch to polar

$$\int_{\theta=0}^{2\pi} \int_{r=0}^{a} \left(\frac{-r\cos\theta - r\sin\theta}{r} + 1 \right) r\,dr\,d\theta$$

$$= a^2 \pi$$

18.02 Problem Set 9 (DUE THURS. MAY 6 10:45 2-106)

Part I (15 pts.)

Lecture 30. Thurs. Apr. 29 Divergence theorem continued.

Read: Notes V11 (skip proofs; read last paragraph); Notes V15, sec. 1 (just the first few lines for div in ∇ notation — skip Stokes' theorem references for now.)

Work: 6C-10, 11; 6H-1,3a

Lecture 31. Fri. Apr. 30 Line integrals in space; conservative fields, potential functions.

Read: Notes V11, V12 Work: 6D-1b,2,4,5; 6E-3(ii) (b: use method 1); 6E-5 (use method 1); 6E-6bc (use method 2)

Lecture 32. Tues. May 4 Stokes' theorem.

Read: Notes V13 Work: 6F-1b,2,5

Part II (15 pts.)

Try each problem alone for 15 minutes. If you subsequently collaborate, solutions must be written up independently. It is illegal to consult problem sets from previous years.

Problem 1. Thurs. (4 pts: 0,1,1.5,1.5) Work 6C-9.

For part (d) there are two cases. The first is easy; for the second, consider a tiny sphere S_0 centered at the origin and lying entirely inside S, and apply to the domain D lying inside S and outside S_0 the following extension of the divergence theorem:

If a domain D is bounded by two (or more) closed surfaces S_1 and S_2, each oriented so that the normal vector points away from D, then

$$\oiint_{S_1} \mathbf{F} \cdot d\mathbf{S} + \oiint_{S_2} \mathbf{F} \cdot d\mathbf{S} = \iiint_D \operatorname{div} \mathbf{F}\, dV .$$

(The proof uses the idea of making cuts in D given in the last paragraph of Notes V11.)

Problem 2. Fri. (3 pts.: 1,2)

a) For what value(s) of the constants a, b, c will the following differential be exact?

$$axy^2z\,dx + (bx^2yz + cz^2y)\,dy + y^2(x^2 - z) + 3z^2\,dz$$

b) Using these values, express it in the form df for an explicit function f.

(Do it both by Method 1 and Method 2.)

Problem 3. Tues. (2 pts) For $\oint_C -(y+z)dx + (2x-z)dy + (x-2y)dz$, show that the line integral is zero for all closed curves C lying in the plane $x - 2y - z = 2$.

Problem 4. Tues. (3 pts: 1,2) Suppose that in 3-space, $\mathbf{F} = \nabla \times \mathbf{G}$, where the components of $\mathbf{G}$ have continuous second partial derivatives. Prove in two different ways that if S is a closed positively-oriented surface, $\iint_S \mathbf{F} \cdot d\mathbf{S} = 0$:

a) use the divergence theorem;

b) divide S into two parts with a closed curve C and apply Stokes' theorem.

Problem 5. (Tues. 3 pts: 2,1) Let $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$. Show in two different ways that there cannot be a field $\mathbf{G}$ such that $\mathbf{F} = \nabla \times \mathbf{G}$:

a) Let S be a sphere of radius a centered at the origin, and C be a simple closed curve on S. Using Stokes' theorem, interpret the value of $\oint_C \mathbf{G} \cdot d\mathbf{r}$ geometrically, and show this leads to a contradiction.

b) Find another argument (look over the exercises in 6H for ideas.)

$\underline{18.02}$ P-Set 9 Part 1 15
Part 2 14,5
29,5

Michael Plasmeie, 5/1
Last P-Set!

Part 1 Lecture 30 Divergence Theorm in Depth

6C-10 A flow field $\vec{F}$ is said to be incompressable if $\iint_S \vec{F} \cdot d\vec{S} = 0$ for all closed surfaces. Assume $\vec{F}$ is contineously differentiable. Show that $\vec{F}$ is field of incompressible flow

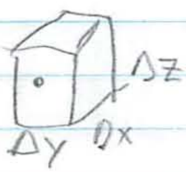
✓

no OH
work w/
others

$$\Updownarrow \operatorname{div} \vec{F} = 0$$

So prove what?
- $\vec{F}$ incompresible
- div F closed surface $= 0$

Is that just the infatessemal box proof?



Only flows ⊥ to top + bottom

$P(x_0, y_0, z_0 + \Delta z)\Delta y \Delta x$

bottom $- P(x_0, y_0, z_0 + \Delta z)\Delta y \Delta x$

$\Downarrow$ net

$\Delta P \cdot \Delta y \Delta x$

$$\left(\frac{\partial P}{\partial z}\right)_0 \approx \left(\frac{\Delta P}{\Delta z}\right)_0$$

$$\Delta P \approx \left(\frac{\partial P}{\partial z}\right)_0 \cdot \Delta z$$

in the limit $\left(\frac{\partial P}{\partial z}\right)_0 \Delta z \Delta x \Delta y$

$= \frac{\partial P}{\partial z} \, dx \, dy \, dz$

Same for sides

$$\operatorname{div} F = \left(\frac{\partial M}{\partial x}\right)_0 + \left(\frac{\partial N}{\partial y}\right)_0 + \left(\frac{\partial P}{\partial z}\right)_0 \, dx \, dy \, dz$$

If $\operatorname{div} \vec{F} = 0$, then for any closed surface S, we have have by the divergence theorm

$$\iint_S \vec{F} \cdot d\vec{s} = \iiint_D \operatorname{div} \vec{F} \, dV = 0$$

✓ yes that is just the divergence theorm

Conversely: $\iint \vec{F} \cdot d\vec{s} = 0$ for every closed surface $S \Rightarrow \operatorname{div} \vec{F} = 0$

✓ Yes that is the rule

For suppose there was a point P_0 at which $(\operatorname{div} F)_0 \neq 0$

$$\iint_S \vec{F} \cdot d\vec{s} = \iiint_D \operatorname{div} \vec{F} \, dV > 0$$

but this contradicts hypothesis

WTF How does that prove anything!

6C-11 Show that the flux of the position vector $\vec{F} = x\hat{i} + y\hat{j} + z\hat{k}$ outward through closed surface S is 3 times the volume contained in that surface.

✓

$$\hat{n} = \frac{\langle x, y, z \rangle}{a} \quad \text{sphere}$$

$$\iint \langle x, y, z \rangle \cdot \frac{\langle x, y, z \rangle}{a} \, dS$$

$$\iint \frac{a^2}{a} \, dS \quad a^2 = x^2 + y^2 + z^2$$

$$\iint a \cdot \text{area}$$

$$= 4\pi a^3$$

Volume of sphere $= \frac{4}{3}\pi r^3$

$3 \cdot \text{Vol} = \frac{4}{3}\pi r^3 \cdot 3 = 4\pi a^3$

✓ works

– weird

flux of $F = \iint_S \vec{F} \cdot d\vec{n} = \iiint_D \operatorname{div} \vec{F} \, dV =$

$\iiint 3 \, dV = 3(\text{vol of } D)$

but why does this work?,

applications to physics

GH-1 ✓

Prove that $\nabla \cdot \nabla \times \vec{F} = 0$

What are the appropriate hypothsis about F?

$$\text{div } F = \nabla \cdot F$$

$$\nabla \cdot F \, dV = \vec{F} \cdot d\vec{s}$$

$$\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$$

$$\text{div } \vec{F} = \frac{\partial}{\partial x} M + \frac{\partial}{\partial y} N + \frac{\partial}{\partial z} P$$

$$\left(\frac{\partial}{\partial x}\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\frac{\partial}{\partial y}, \frac{\partial}{\partial z}\frac{\partial}{\partial z} \right) \times \vec{F}$$

? why cross product

cross product $\perp = 0$
and it won't line up, so $= 0$

$$\vec{F} = M\hat{i} + N\hat{j} + P\hat{k}$$

? assumed that

$$\nabla \times \vec{F} = \text{curl } \vec{F} = (P_y - N_z)\hat{i} + (M_z - P_x)\hat{j} + (N_x - M_y)\hat{k}$$

forgot

$$\nabla(\nabla \times \vec{F}) = (P_{yx} - N_{zx}) + (M_{zy} - P_{xy}) + (N_{xz} - M_{yz})$$

$$P_{xy} = P_{yx} \quad \text{so} = 0$$

$$\text{div}(\text{curl } \vec{F}) = 0$$

Cool — seems reasonable

6H-3a ✓ Prove $\nabla \cdot (\phi F) = \phi \nabla \cdot F + F \cdot \nabla \phi$

ϕ = scalar function

No answer in back

Isn't ϕ just modify function
- well is a function itself
- but does in not just modify

$$\underbrace{x\hat{\imath} + y\hat{\jmath}}_{F} \cdot \underbrace{5x}_{\phi} = 5x^2\hat{\imath} + 5xy\hat{\jmath}$$

↑ vector function out

∇ = differentiate

$\nabla(\phi F) = 10x + 5x$

$15x$

$\nabla(F) = 1 + 1$

$\phi(\nabla F) = 5x + 5x$

$\nabla(\phi) = 5$

$F(\nabla\phi) = 5x + 5y$

$(5x + 5x) + (5x + 5y)$

$15x + 5y$

↑ not quite = due to this and I know # proof not accepted

Lecture 31 Line integrals in space, conservative fields, potential functions

6D-1b ✓ $\vec{F} = y\hat{i} + z\hat{j} - x\hat{k}$

C = line from $(0,0,0)$ to $(1,1,1)$

Step) paramitrize line
(did this problem in lecture 31)

$x = t \quad y = t \quad z = t$

$$\int_C y\,dx + z\,dy - x\,dz$$

$dx = 1 \quad dy = 1 \quad dz = 1$

$$\int_0^1 y + z - x \, dt$$

+

$$\int_0^1 t + t - t \, dt$$

$$\int_0^1 t$$

$$\left.\frac{t^2}{2}\right|_0^1$$

1

$\frac{1}{2}$

integration error
—almost, grrr

6D-2

$\vec{F} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$

Show $\int_C F \cdot dr = 0$ for any curve C

lying on a sphere of radius a at origin

Is this not proving the line integral around a closed shape $= 0$?

Field radially out θ

$\hat{T}$ is alway tangent

$\hat{n}$ always normal

$\vec{F} \cdot \hat{T} = 0$

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot \hat{T}\, ds = 0$$

So why is it tangent and not 0?

GD-4 Let $f(x,y,z) = x^2+y^2+z^2$ Calculate $F = \nabla f$

✓

So they want is to integrate back?
Or FTC?

differential form: $M\,dx + N\,dy + P\,dz = df$

$M = f_x \quad N = f_y \quad P = f_z$

$f_{xy} = f_{yx} \to M_y = M_x$
$f_{yz} = f_{zy} \to N_z = P_y$
$M_z = P_x$

Notes don't need

Yeah just integrat

$\vec{F} = \nabla f = 2x\,\hat{\imath} + 2y\,\hat{\jmath} + 2z\,\hat{k}$

b) Let C be helix: $x = \cos t$, $y = \sin t$, $z = t$, $t = 0 \to t = 2n\pi$

i) Calculate directly

$$\int_C 2x\,dx + 2y\,dy + 2z\,dz$$

$dx = -\sin t$
$dy = \cos t$
$dz = 1$

$$\int_0^{2n\pi} -2\cos t\sin t + 2\sin t\cos t + 2\cdot t\cdot 1\,dt$$

$$\int_0^{2n\pi} 2t \, dt$$

$$\frac{2t^2}{2}\Big|_0^{2n\pi}$$

$$(2n\pi)^2$$

$$\boxed{4n^2\pi^2}$$

ii) by using path independence to replace C w/ a simpler path

$x = t \qquad y = t \qquad z = t$

$dx = 1 \qquad dy = 1 \qquad dz = 1$

$$\int_0^{2n\pi} 2x \, dx + 2y \, dy + 2z \, dz$$

$$\int_0^{2\pi n} 2t + 2t + 2t \, dt$$

$$\int_0^{2\pi n} 6t \, dt$$

$$\frac{6t^2}{2}\Big|_0^{2\pi n}$$

$$3(2n\pi)^2$$

Why 3 times more?

Vertical path $x = 1$ $y = 0$ $z = t$

So you have to think about shortest path - not just 1 formula

$$\int_C M\,dx + N\,dy + P\,dz =$$

$$= \int_0^{2\pi n} 2t\, dt$$ ← why just that - guess that is how it works out

$$(2\pi n)^2$$

iii) By using 1st FTC line integrals

$$\int F \cdot dr = f(x, y, z)\Big|_{(x_0\, y_0\, z_0)}^{(x_1\, y_1\, z_1)}$$

$x_0 = \cos 0 = 1$ $x_1 = \cos 2\pi n = 1$

$y_0 = \sin 0 = 0$ $y_1 = \sin 2\pi n = 0$

$z_0 = 0$ $z_1 = 2\pi n$

$$f(1, 0, 2\pi n) - f(1, 0, 0)$$

$$1^2 + (2\pi n)^2 - (1^2)$$

$$(2\pi n)^2$$

oh this way is easy don't have to do part a - messed up on last exam + makeup

GD-5

✓

$F = \nabla f$

$f(x, y, z) = \sin(xyz)$

What is the maximum value of $\int_C \vec{F} \cdot d\vec{r}$ over all possible values of C?

-give a path

Isn't it path independence, so path does not matter?

Can use FTC

$\int \vec{F} \cdot d\vec{r} = f(x, y, z)\Big|_P^Q$

$f(x_1, y_1, z_1) - f(x_0\ y_0\ z_0)$

Remember 1 -1 is range of sin

$1 - (-1) = 2$

✓ any path works

like $(1, 1, -\frac{\pi}{2})$ $(1, 1, \frac{\pi}{2})$

$\sin(-\frac{\pi}{2}) = -1$ $\sin(\frac{\pi}{2}) = 1$

so fits

Gradiant Fields in Space

6E-3(ii) Fields defined for all x, y, z

✓

use method 1

Find a potential function $f(x,y,z)$ using method 1

$$f = \cancel{\vec{F}} = (2xy+z)\,\hat{i} + x^2\,\hat{j} + x\,\hat{k}$$

So potential function means take gradiant?

$\cancel{2y\,\hat{i} + 0 + 0}$

Want
$$f_x = 2xy + z$$
$$f_y = x^2$$
$$f_z = x$$

So integrate

not so simple

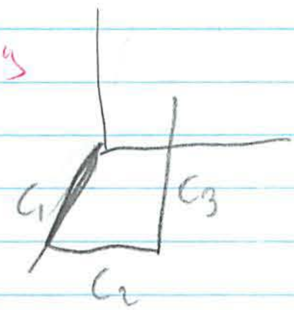
$\cancel{f = x^2 y + xz\,\hat{i} + x^2 y\,\hat{j} + xz\,\hat{j}}$

They use method 2

Supposed to use method 1 — looks complex

Why must I use this? above is wrong

must do like this



$$\int_{C_1} M\,dx + \int_{C_2} N\,dy + \int_{C_3} P\,dz$$

$y = 0$ $dy = 0$, $z = 0$ $dz = 0$ | $x = x_1$ $dx = 0$, $z = 0$ $dz = 0$ | $x = x_1$ $dx = 0$, $y = y_1$ $dy = 0$

$$0 + \int_0^{y_1} x_1^2\, dy + \int_0^{z_1} x_1\, dz$$

$$x_1^2 y_1 + x_1 z_1 + C$$ ✓

↑ dont forget

6E-5
method 1

✓

For what values of a and b will
$\vec{F} = yz^2\hat{\imath} + (xz^2 + ayz)\hat{\jmath} + (bxyz + y^2)\hat{k}$
be conservative?

Must be
$M_y = N_x$
$N_z = P_y$
$M_z = P_x$

$z^2 = (z^2 + 0)$ ✓

$(2zx + ay) = (bxz + 2y)$

$2zy = byz$

$\hookrightarrow b = 2$ ✓

$\hookrightarrow (2xz + ay) = (2xz + 2y)$

$\hookrightarrow a = 2$ ✓

Find the corresponing potential function $f(x, y, z)$ using a systimatic measure

? So differeinate again no (and why do I switch b/w d + $\int$ on this)

~~$f = 0\hat{\imath} + 2z\hat{\jmath} + 2xy\hat{k}$~~ both wrong

Must use method 1 or 2 →

$$\int_{C_1} M\,dx + \int_{C_2} N\,dy + \int_{C_3} P\,dz$$

$y=0$ $dy=0$ $\quad$ $x=x_1$ $dx=0$ $\quad$ $x=x_1$ $dx=0$
$z=0$ $dz=0$ $\quad$ $z=0$ $dz=0$ $\quad$ $y=y_1$ $dy=0$

$$0 + \int_0^{y_1} 0 + \int_0^{z_1} 2x_1y_1z + y_1^2\,dz$$

$$\frac{2x_1y_1z^2}{2} + y^2z \Big|_0^{z_1}$$

$$x_1y_1z_1^2 + y_1^2z_1$$

$$xyz^2 + y^2z + C$$ got in

6E-6b ✓ Find all of the values for a, b, c where exact
- guess that means conservative
- this is kinda fun

$$\vec{F} = (axyz + y^3z^2)\,dx + \left(\frac{a}{2}x^2z + 3xy^2z^2 + byz^3\right)dy + (3x^2y + cxy^3z + 6y^2z^2)\,dz$$

$$M_y = N_x \quad N_z = P_y \quad M_z = P_x$$

this is N_x

(1) $(axz + 3y^2z^2) = \left(\frac{a}{2}\cdot 2xz + 3\cdot y^2 z^2 + \cancel{bz^3}\right)$

$axz + 3y^2z^2 = (axz + \cancel{cxyz^2} + 3y^2z^2\ \cancel{bz^3})$

(2) $\left(\frac{a}{2}x^2 + 3\cdot 2zxy^2 + b\cdot 3yz^2\right) =$ ✓ working error

$(3x^2 + 3y^2cxz + 6\cdot 2yz^2)$

$\left(\frac{a}{2}x^2 + 6xy^2z + 3byz^2\right) = (3x^2 + 3cxy^2z + 12yz^2)$

This was P_x not P_z $\boxed{3 \cdot 2xy + cy^3z}$

(3) $(axy + 2zy^3) = (\cancel{cxy^3 + 6 \cdot 2y^2z})$

$axy + 2y^3z = \cancel{cxy^3 + 12y^2z}$

$6xy + cy^3z$

this is not as much fun...

~~(1) $3y^2z^2 = 6xyz^2 + bz^3$~~

screwed up each of the 3
forgot what I was partial deriving
must be more careful

(3) $axy + 2y^3z = 6xy + cy^3z$

$\hookrightarrow 6 = a$

$2 = c$

(1) $2xz + 3y^2z^2 = 2xz + 3y^2z^2$ ✓

(2) $\frac{6}{2}x^2 + 6xy^2z + 3byz^2 = 3x^2 + 3 \cdot 2xy^2z + 12yz^2$

$3byz^2 = 12yz^2$

$\hookrightarrow b = 4$

That was complex!

c) For those values express the differential as ds for a suitable $f(x,y,z)$ using method 2

Hopefully I can method 2 since Mattuck tried to do it after class ended
— well recitation — wait till after that

5/5

So I think naming the methods will be more helpful

1) The do each piece out method
2) Integrate w/ remainders method

from 5/5 recitation
- I actually tried to do this method earlier but forgot about the remainder thing

$F = \langle 6xyz + y^3z^2,\ 3x^2z + 3xy^2z^2 + 4yz^3,\ 3x^2y + 2xy^3z + 6y^2z^2 \rangle$

$f_x = 6xyz + y^3z^2$
$f_y = 3x^2z + 3xy^2z^2 + 4yz^3$
$f_z = 3x^2y + 2xy^3z + 6y^2z^2$

$f = 3x^2yz + xy^3z^2 + g(y,z)$

$\downarrow$

$f_y = 3x^2z + 3xy^2z^2 + g_y(y,z)$

↳ use 2nd equation

$g_y = 4yz^3$

$\downarrow$

$g = 2y^2z^3 + h(z)$

wow, cool how it works out

$f = 3x^2yz + xy^3z^2 + 2y^2z^3 + h(z)$

$\downarrow$

copy error

$f_z = 3x^2y + 2xy^3z + 6y^2z^2 + h_z(z)$

↳ use 3rd equation

$h_z = 0$
$h = c$ ✓

$f = 3x^2yz + 2xy^3z^2 + 6y^2z^3 + C$

did not $\int$ that last line
-opps

Lecture 32 - Stokes' Theorem

6F-1b ✓ Verify Stokes's Theorem when S = upper hemisphere

Calculate both $\int$ + show $=$

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \operatorname{curl}(F)$$

$$\oint_C F \cdot dr = 0$$ since closed curve

"differential exact"

$n = \hat{k}$

$$\iint_S \operatorname{curl}(F)$$

$$\downarrow$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ x & y & z \end{vmatrix}$$

i) $\vec{F} = x\hat{i} + y\hat{j} + z\hat{k}$

$0\hat{i} + 0\hat{j} + 0\hat{k} = 0$

ii) $\cancel{F = y\hat{i} + z\hat{j} + x\hat{k}}$

$$\cancel{\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ y & z & x \end{vmatrix}}$$

$$\cancel{0\hat{i} + 0\hat{j} + 0\hat{k} = 0}$$

Wrong, this is not the same

Line integral must be done again

Must still do!

And I don't think closed curve mattered

$$\oint y\,dx + z\,dy + x\,dz$$

$$\oint y\,dx \qquad \begin{matrix} z=0 \\ dz=0 \end{matrix}$$

Paramitize $\quad x = \cos t$; why not polar coord

$y = \sin t$

$$dx = -\sin t$$

$$\int_0^{2\pi} -\sin^2 t \; dt$$

$$\int_0^{2\pi} \frac{1-\cos 2t}{2}\, dt$$

$$= -\pi$$

Now surface integral

Had box right, but did it wrong

$$\begin{array}{l} \partial_y(x) - \partial_z(z)\ \hat{i} \\ - \partial_x(x) - \partial_z(y)\ \hat{j} \\ \partial_x(z) - \partial_y(y) \end{array}$$

$-1\hat{i} - 1\hat{j} - 1\hat{k}$

$-\hat{i} - \hat{j} - \hat{k}$

$n = x\hat{i} + y\hat{j} + z\hat{k}$

its not up, but radially out

Now need to dot them

$\iint -x\cancel{\hat{i}} - y\cancel{\hat{j}} - z\cancel{\hat{k}}\ dS$ once again its a ff

factor - out

$-\iint (x + y + z)\ dS$

Convert to spherical coords to eval

$x = r\cos\theta \qquad z = \rho\cos\phi$

$y = r\sin\theta \qquad r = \rho\sin\phi$

$$-\int_0^{2\pi}\int_0^{\frac{\pi}{2}} \left[\sin\phi(\cos\theta + \sin\theta) + \cos\phi\right]\sin\phi\ d\phi\, d\theta$$

$\theta \qquad \phi$

Now evaluate

$$-\left[(\cos\theta + \sin\theta)\left(\frac{\phi}{2} - \frac{\sin 2\phi}{4}\right) + \frac{1}{2}\sin^2\phi\right]_0^{\pi/2}$$

$$-\left[(\cos\theta + \sin\theta)\frac{\pi}{4} + \frac{1}{2}\right]$$

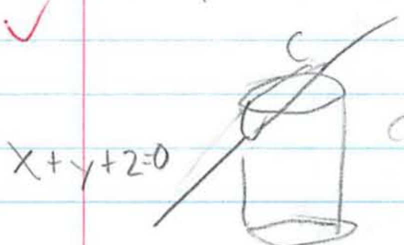
$$-\int_0^{2\pi} \left(-\left[(\cos\theta + \sin\theta)\frac{\pi}{4} + \frac{1}{2}\right]\right.$$

$-\pi$

Make sure you do it right!

2. Verify Stoke's Theorm $F = y\hat{i} + z\hat{j}\ x\hat{k}$

✓



cylinder $x^2+y^2=1$

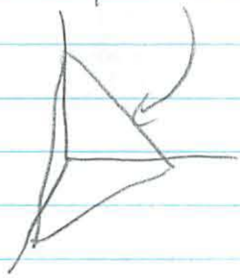
I think my drawling is wrong

Verify means do both sides

$$\oint_C y\,dx + z\,dy + x\,dz$$

same as last problem
but C different
need to paramitize I think

$x + y + z = 1$



$x + y + z = 0$??

- wolfram alpha
- is a slanted plane

What is paramitrization?

$x = \cos t$
$y = \sin t$
$z = -\cos t - \sin t$

Seems so obvious now!

$dx = -\sin t$
$dy = \cos t$
$dz = \sin t - \cos t$

$$\oint \sin t \cdot -\sin t + (-\cos t - \sin t)\cos t + \cos t(\sin t - \cos t)$$

$$\oint -\sin^2 t - \cos^2 t - \sin t \cos t + \cos t \sin t - \cos^2 t$$

$$\int_0^{2\pi} -1 - \cos^2 t \, dt$$

$$-\frac{3t}{2} - \frac{1}{4}\sin(2t) \Big|_0^{2\pi}$$

$\boxed{-3\pi}$

Now surface $\iint$

$\iint$ curl F ds

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \end{vmatrix}$$

same as last problem

but need normal vector from cylinder

~~$\hat{n} = x\hat{i} + y\hat{j}$~~

$z = -x-y$ ← is that the cyclinder or cut out - I thought it was cylinder

$f(x,y) = -x-y$

$n = \langle -f_x, -f_y, 1 \rangle$

$= \langle 1, 1, 1 \rangle$

↑ How do we get that?

$\iint \cancel{x} -1 \cdot 1 + -1 \cdot 1 + -1 \cdot 1 \; ds$

$\iint -3 \; ds$

So $-3 \cdot$ area

Interior of unit circle in xy plane

$= \boxed{-3\pi}$

Really need to learn all the rules + do - not end up like last test

just need to each of the shapes

working very relaxed - should be good

Understanding the rules this time, unlike last time

5. Let S be surface formed by cyclinder $x^2 + y^2 < a^2$ $0 \leq z \leq h$ circular disc on top facing out

✓

$F = -y\hat{i} + x\hat{j} + x^2\hat{k}$

Find flux of $\nabla \times F$ through S

a) Directly by caculating 2 surface SS

- So just like other questions
- or is it → before directly → line S

Both the tops and the sides and surface SS not line $\oint$

$$\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ -y & x & x^2 \end{vmatrix}$$

$$\partial_y(x^2) - \partial_z(x)\hat{i} + \partial_x(x^2) - \partial_z(-y) + \partial_x(x) - \partial_y(-y)$$

$$0 + 2x\hat{j} + 1 - -1\hat{k}$$
$$2x\hat{j} + 2\hat{k}$$

$\hat{n}$ top $= \bar{k}$

$$\iint F \cdot ds = \iint 2 \, dS$$

2 · area

$2 \cdot \pi (a)^2$

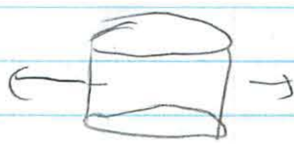
~~2π~~ top don't know a

$\bar{n}$ side $=$ $\cancel{x^2+y^2+z}$ $\frac{x\hat{i}+y\hat{j}}{a}$

$$\iint -\frac{2xy}{a} \, ds$$

$$\int_0^{2\pi}\int_0^{a} \frac{2r\cos\theta \, r\sin\theta}{a} \, r\,dr\,d\theta \qquad x = r\cos\theta,\ y = r\sin\theta$$

$\theta \quad r$



$r = a$

$$\frac{2r^3\cos\theta\sin\theta}{a} \, dr\,d\theta$$

$$\left.\frac{2r^4\cos\theta\sin\theta}{4a}\right|_0^a$$

$$\frac{\cancel{2}\, a^{\cancel{4}\,3}\cos\theta\sin\theta}{2\,\cancel{4}\cancel{a}}$$

$$\int_0^{2\pi} \cancel{a^3\cos\theta\sin\theta\,d\theta}$$
$$-2h(a\cos\theta)(a\sin\theta)\,d\theta$$

again I forget how to solve this
can move a^3 out

$= -h a^2 \sin^2\theta \Big|_0^{2\pi}$

$= 0$

add the 2 together

$= 2\pi a^2$

b) Using Stoke' Theorm

but what is about this
- still have to do curl F?

$\iint_S \text{curl } \vec{F} \cdot d\vec{S}$ ✓

$= \oint_C -y\,dx + x\,dy + x^2\,dz$

$= \int_0^{2\pi} (a^2 \sin^2\theta + a^2 \cos^2\theta)\,d\theta$

$= 2\pi a^2$

Oh so they did do the integral

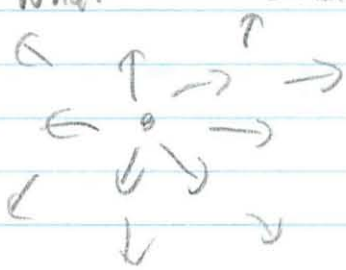
So Stoke's Theorm means do the other one

Part 2

1. Do 6C-9 - Let F be the vector field for which all vectors are aimed radialy away from the origin w/ magnitude $\frac{1}{\rho^2}$

In class Mattuck gave hints

a) What is domain $\vec{F}$



∞ except $x=0$ ✓
$[-\infty, 0)$ and $(0, \infty]$

b) Show that div $\vec{F} = 0$

div is the work

$\perp$ with the vectors so 0

$\int \text{div } F = \iint N_x - M_y \, dA$

$\vec{F} = x + y + z$

$= \int 0 - 0$

$= 0$ ✓

c) Evaluate $\iint_S \vec{F} \cdot d\vec{S}$ where S is sphere of radius a centered at the origin.

Does the fact it $\neq 0$ contridict divergence theorm?

$$\text{curl } F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ x & y & z \end{vmatrix}$$

$$= 0$$

problem says it should not be 0

but the reason it supposidly $\neq 0$ is because fluid is magically being generated at the origin

? math

or is it

$$\text{Flux} = \oiint \vec{F} \cdot d\vec{s} = \oiint (\vec{F} \cdot \hat{n}) \, ds$$

$$n = \frac{\langle x, y, z \rangle}{c} \cdot \frac{\langle x, y, z \rangle}{a} \, ds$$

$$= \oiint \frac{a^2}{ca}$$

$$= \frac{a}{c} \text{ area}$$

$$= \frac{4\pi a^3}{c}$$

$\frac{a}{c} \cdot 4\pi a^2 = 1$

↑ unit point source at origin

$\frac{a}{c} = \frac{1}{4\pi a^2}$

$\frac{c}{a} = 4\pi a^2$

$c = 4\pi a^3$

Solve for c

$c = \frac{4\pi a^3}{1}$

Field $\vec{F} = \frac{\langle x, y, z \rangle}{4\pi \left(\frac{1}{\rho^2}\right)^2}$

$\frac{a}{\frac{a}{b}} = \frac{a}{1} \cdot \frac{b}{a}$

$= \frac{\langle x, y, z \rangle \cdot \rho^4}{4\pi}$

Answer? -0.5

d) Prove the divergence theorm that $\iint \vec{F} \cdot d\vec{S}$ over a positivly oriented closed surface S has the value 0 if the surface does not contain the origin and the value 4π if it does

$\vec{F}$ is the vector field for the flow arising from a strenght of 4π at the origin

Hint For problem d there are 2 cases. The first is easy; for a second consider a tiny sphere S_0 centered at the origin and lying entirly inside S and apply to the domain lying inside S and outside S_0 the following extension of the divergence theorm

If a domain D is bounded by two or more closed surfaces S_1 and S_2, each oriented that the normal vector points away from D then

$$\oiint_{S_1} \vec{F} \cdot d\vec{s} + \oiint_{S_2} \vec{F} \cdot d\vec{s} = \iiint_D \operatorname{div} \vec{F} \, dV$$

Hint 2 This proof uses the idea of making cuts in D given in the last paragraph of Notes VII

So first method $\longrightarrow$

First method

Calculate somewhere other than origin

I can picture this from 8.02
the same flux that all goes in all comes back out

Where do we know that the source is at the origin?

So like last qu with?

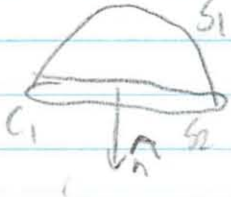
$\frac{a}{c} \cdot 4\pi a^2 = 0$

$c = 0$ ✓

?, right

Or could split sphere in half

$\vec{F} = M\hat{i} + N\hat{j} + P\hat{k}$
$\operatorname{div}\vec{F} = M_x + N_y + P_z$



$\iint \vec{F} \cdot d\vec{S}$

$\vec{F} = \frac{\langle x, y, z \rangle}{\rho^2}$

$\hat{n} = \frac{\langle x, y, z \rangle}{\rho} \, dS$

0th

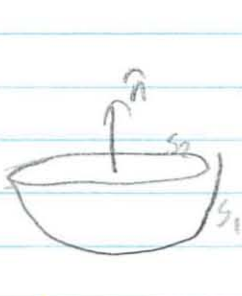
$\iint_{S_1} \vec{F} \cdot d\vec{S} + \iint_{S_2} \vec{F} \cdot d\vec{S}$

$\downarrow$ perpendicular to field

$\iint_{S_1} \vec{F} \cdot d\vec{S} = \iint_{S_1} \frac{\langle x, y, z \rangle}{\rho^2} \cdot \frac{\langle x, y, z \rangle}{\rho}$

$= \iint_{S_1} \frac{x^2 + y^2 + z^2}{\rho^4} \, dS$

$= \frac{2\pi\rho^2}{\rho^2} = 2\pi$



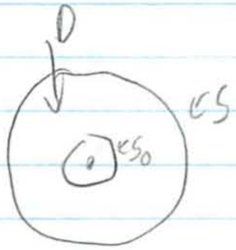
$\iint_{S_1} \vec{F} \cdot d\vec{S} + \iint_{S_2} \vec{F} \cdot d\vec{S}$

$\downarrow \quad \downarrow$

$-2\pi \quad 0$

$-2\pi + 2\pi = 0$

2nd Method



So calculate $\oiint_{S_1} + \oiint_{S_2}$

But they will = the same thing
We proved that in lecture

$$\frac{4\pi a^3}{4\pi a^3} = 1$$

confirm

$$\oiint_{S_0} = \oiint_{S} = 1$$

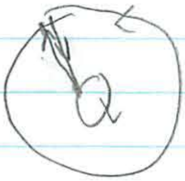
$$1 + 1 = \iiint_D \operatorname{div} \vec{F} \, dv$$

right book

Now what is it about VII
- that is fields being conservative
  + FTC
- or do I have an old book

Oth

Or could do



$$= \iint_{S_1} \frac{\langle x, y, z \rangle}{\rho^3} \cdot \frac{\langle x, y, z \rangle}{\rho} \, dS$$

$$= \iint_{S_1} \frac{x^2+y^2+z^2}{\rho^4} \, dS$$

$$= \iint_{S_1} \frac{1}{x^2+y^2+z^2} \, dS$$

$$= \frac{1}{\rho^2} \cdot 4\pi \rho^2 = 4\pi \checkmark$$

$$\iint_{S_2} = -\iint_{S_1} = -4\pi$$

$$4\pi - 4\pi = 0 \checkmark$$

2. For what values of the constants will this be exact?

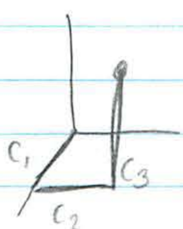
$$\vec{F} = axy^2z\,dx + (bx^2yz + cz^2y)\,dy + y^2(x^2 - z) + 3z^2\,dz$$

$M_y = N_x$
$M_z = P_x$
$N_z = P_y$

$2a\,xyz = 2bxyz$
$\hookrightarrow a = b$
$axy^2 = 2xy^2$
$\hookrightarrow a = 2 = b$ ✓
$bx^2y + 2czy = 2y(x^2 - z) + 0$
$2x^2y + 2cyz = 2x^2y - 2yz$
$\hookrightarrow c = -1$ ✓

b) Using these values express in f

1) Method 1



$$\int_{C_1} M\,dx + \int_{C_2} N\,dy + \int_{C_3} P\,dz$$

$y = 0$ $\quad x = x \quad$ $x = x$
$z = 0$ $\quad z = 0 \quad$ $y = y$

$$\int_{C_1} 2x\,y^2z\,dx + \int_{C_2} 2x^2yz + -1z^2y + \int_{C_3}$$

$0 \qquad 0 \qquad 0 + \int_{C_3}$

$$0 + 0 + \int_{C_3} x^2y^2 - y^2z + 3z^2 \, dz$$

$$\int_0^z x^2y^2 - y^2z + 3z^2 \, dz$$

$$x^2y^2z - \frac{y^2z^2}{2} + \frac{3z^3}{3} \Big|_0^z$$

$$x^2y^2z - \frac{y^2z^2}{2} + z^3 \quad \checkmark$$

ii) Method 2

$$f_x = 2xy^2z$$

$$\downarrow$$

$$f = x^2y^2z + g(y,z)$$

$$\downarrow$$

$$f_y = 2x^2yz + g'(y,z)$$

$$= 2x^2yz - yz^2 \quad \text{from above}$$

$$\downarrow \; g'(y,z) = -yz^2$$

$$g = -\tfrac{1}{2}y^2z^2 + h(z)$$

$$f = x^2y^2z - \tfrac{1}{2}y^2z^2 + h(z)$$

$$f_z = x^2y^2 - y^2z + h_z$$

$$\downarrow \; h_z = 3z^2$$

$$h = z^3$$

$$\checkmark$$

$$f = x^2y^2z - \tfrac{1}{2}y^2z^2 + z^3$$ ✓ worked out wow!

3. For $\oint_C -(y+z)\,dx + (2x-z)\,dy + (x-2y)\,dz$

show that the line integral is 0 for all closed curves lying in the plane
$x-2y-z=2$

But how does this work as the curve I thought was closed in one of these problems?

From the class
Stoke's Theorm

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \vec{G}\cdot d\vec{s}$$
$$= \iint \text{curl } F\, ds$$

Oh only 0 if conservative!

$M_y = N_x$
$M_z = P_x$
$N_z = P_y$

$-1 = 2$ ⊗
$-1 = 1$ ⊗
$-1 = -2$ ⊗

No - field is not conservative

Well what they give has normal vector in it already
- right?

oth

$$\oint \vec{F} \cdot d\vec{r} = \iint \underbrace{\langle P_y - N_z, M_z - P_x, N_x - M_y \rangle}_{\text{curl } F} \cdot d\vec{S}$$

$$= \iint \langle -2+1, -1-1, 2+1 \rangle \, dS$$

$$= \langle -1, -2, -3 \rangle \, \hat{n} \, dS$$

if plane is $x - 2y - z = 2$

$\langle 1, -2, -1 \rangle$ ✓

$F(x, y, z) = x - 2y - 1z - 2 = 0$

$$\hat{n} = \nabla f = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$\langle -1, -2, -3 \rangle \cdot \langle 1, -2, -1 \rangle$

$-1 + 4 - 3 = 0$

$f(x, y, z) = 2$

In the same plane, so line Integral will always $= 0$

✓

4 Suppose that in 3 space $\vec{F} = \nabla \times G$
where components of G have contineous 2nd derivitives.

Prove in two ways that if S is a closed
(+) oriented surface $\iint_S F \cdot ds = 0$

a) using divergence theorm

means smooth ?
just basic info I thing

So this one seems to flip the letters to confuse

So divergence = 0 if no inside source ✓

Or is this problem recursive

$$\iint \text{curl } F = \nabla \times F = \nabla \times (\nabla \times G)$$

So that is why they talked about
Contineous 2nd derivs

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ P_y - N_z & -P_x + M_z & N_x - M_y \end{vmatrix}$$

This is going to be long

$$\left(N_{xy} - M_{yy} + P_{xz} - M_{zz} \right) \hat{i} +$$
$$\left(N_{xx} - M_{yx} + P_{yz} + N_{zz} \right) \hat{j} +$$
$$\left(-P_{xx} + M_{zx} + P_{yy} + N_{zy} \right) \hat{k}$$

vectors don't matter

it all cancles
it does not now
- but if you expand it, I hope

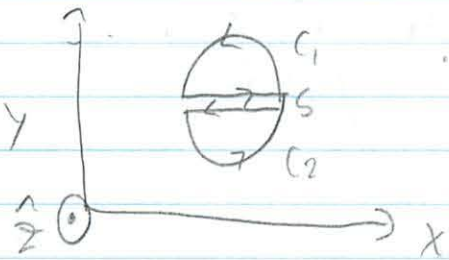
* Remember $N_{xy} = N_{yx}$ *

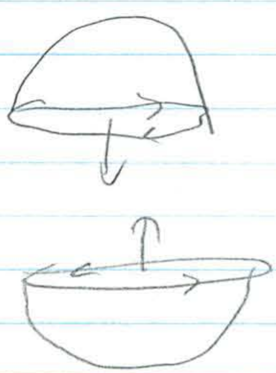
$= 0$ ✓

b) Divide S into 2 parts w/ a closed curve C
and apply Stokes' theorm

What 2 parts?
and doesn't stokes theorm mean do the other thing?
oth - But last time messed up divergence theorm

$$\oint \vec{F} \cdot d\vec{r} = \iint_S \text{curl } \vec{F} \cdot d\vec{S}$$





$F = \nabla \times G = \text{curl } \vec{G}$

$\hat{n}$ vectors cancle out

$$\oint \vec{F} \cdot d\vec{r} = \oint_C \nabla \times \vec{G} \; d\vec{r}$$

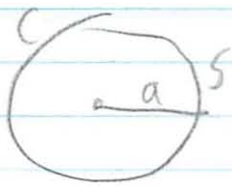
5. Let $F = x\hat{i} + y\hat{j} + z\hat{k}$

Show in 2 different ways that there can not be a field G such that $\vec{F} = \nabla \times G$

a) Let S be a sphere of radius a centered at the origin and C be a simple closed curve on S. Using Stokes Theorm, interpret the value of $\oint \vec{G} \cdot d\vec{r}$ geometrically, and show this leads to a contridiction.

So this is like the last problem, but how is it different ???

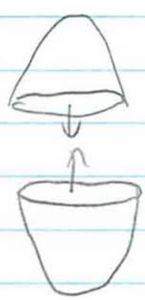
But in last problem didn't I prove that $\iint = 0$



Is this just saying $\oint$ closed curve conservative field $= 0$

oth

or Normal vectors are opposite + cancle at



$$\oint \vec{G} \cdot d\vec{r} = \iint_S \vec{F} \cdot \hat{n} \, ds =$$

$$= \iint_S \langle x, y, z \rangle \cdot \frac{\langle x, y, z \rangle}{a} ds$$

$$= \iint \frac{x^2 + y^2 + z^2}{a} dS = \iint_S a \, ds = a(\text{area})$$

Contridicition: ∞ # of boundry surfaces

b) Find a second argument,

Hint: Look at 6H exerciess

applications to physics

Not very helpful, not many answers there

oth $\vec{F} = \nabla \times G \rightarrow \operatorname{div} \vec{F} = 0$

$\operatorname{div} \vec{F} = M_x + N_y + P_z$

$= 1 + 1 + 1$

$= 3$

$\operatorname{div}(\vec{\nabla} \times \vec{G}) = \operatorname{div} \vec{F} = 3 \neq 0$

thus $\operatorname{div}(\nabla \times G) \neq \operatorname{div} F$ ✓

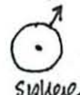
18.02 Problem Set 9 Sol'ns Spring 2010

1

a) $\vec{F} = \dfrac{x\hat{\imath}+y\hat{\jmath}+z\hat{k}}{\rho^3}$; domain = 3-space with $(0,0,0)$ excluded

b) Use: $\frac{\partial\rho}{\partial x}=\frac{x}{\rho}$, $\frac{\partial\rho}{\partial y}=\frac{y}{\rho}$, $\frac{\partial\rho}{\partial z}=\frac{z}{\rho}$. $\frac{\partial}{\partial x}\left(\frac{x}{\rho^3}\right) = \dfrac{\rho^3 - x\cdot 3\rho^2\cdot\frac{x}{\rho}}{\rho^6} = \dfrac{1}{\rho^3}-\dfrac{3x^2}{\rho^5}$
(by quotient rule)

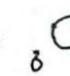
$\therefore$ by symmetry, $\frac{\partial}{\partial x}\left(\frac{x}{\rho^3}\right)+\frac{\partial}{\partial y}\left(\frac{y}{\rho^3}\right)+\frac{\partial}{\partial z}\left(\frac{z}{\rho^3}\right) = \dfrac{3}{\rho^3} - \dfrac{3(x^2+y^2+z^2)}{\rho^5}$
$= 0 \quad (x^2+y^2+z^2=\rho^2)$


sphere S

c) $\iint_S \vec{F}\cdot d\vec{S} = \iint_S \dfrac{\langle x,y,z\rangle}{a^3}\cdot\dfrac{\langle x,y,z\rangle}{a}\,dS$

$(\rho = a \text{ on sphere}) \quad = \iint_S \dfrac{x^2+y^2+z^2}{a^4}\,dS = \iint_S a^2/a^4\,dS = \dfrac{1}{a^2}(\text{area of } S)$, area of $S = 4\pi a^2$
$= \boxed{4\pi}$

. This does not contradict the div. thm since $\vec{F}$ is not defined everywhere inside S — it's not defined at $(0,0,0)$.

d) If  S does not surround the origin, $\iint_S \vec{F}\cdot d\vec{S} = \iiint_V \operatorname{div}\vec{F}\,dV = \iiint_V 0\,dV = 0$.

If it does  S draw a small positively oriented sphere S_1 inside S, centered at origin.
V = region between S and S_1. Let $S_1' = S_1$ oriented negatively (so its normal points

By the extension of the divergence thm, since $S+S_1'$ is the complete boundary surface for V,

$\iint_S \vec{F}\cdot d\vec{S} + \iint_{S_1'}\vec{F}\cdot d\vec{S} = \iiint_V \operatorname{div}\vec{F}\,dV = \iiint_V 0\,dV = 0$
(since $\operatorname{div}\vec{F}=0$ everywhere inside V)

$\therefore \iint_S \vec{F}\cdot d\vec{S} = -\iint_{S_1'}\vec{F}\cdot d\vec{S}$
$= -(-4\pi)$ by part c (regardless of the radius of S_1)
$= 4\pi$.

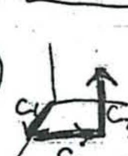
2

a) $M_y = N_x$: $2axyz = 2bxyz$ $\quad\boxed{a=b}$

$N_z = P_y$: $bx^2y + 2czy = 2x^2y - 2yz$ $\quad\boxed{b=2}$ $\boxed{c=-1}$

$M_z = P_x$: $axy^2 = 2xy^2$ checks, using $a=2$

$\therefore \boxed{a=2,\ b=2,\ c=-1}$

b) 

$\int 2xy^2z\,dx + (2x^2yz - z^2y)\,dy + [y^2(x^2-z)+3z^2]\,dz$

$\int_{C_1} = 0$ since $dy=0$, $dz=0$ and $z=0$

$\int_{C_2} = 0$ since $dx=0$, $dz=0$ and $z=0$

$\int_{C_3} = \int_0^{z_1} [y_1^2(x_1^2-z)+3z^2]\,dz$
$(dx=dy=0) \quad = y_1^2x_1^2z_1 - y_1^2\cdot\dfrac{z_1^2}{2} + z_1^3$

$\therefore f(x,y,z) = \boxed{x^2y^2z - \dfrac{y^2z^2}{2} + z^3}$

Method 2: $\begin{cases} f_x = 2xy^2z \\ f_y = 2x^2yz - z^2y \\ f_z = y^2(x^2-z)+3z^2 \end{cases}$

$f_x = 2xy^2z$

$f = x^2y^2z + g(y,z)$

$f_y = 2x^2yz + g_y = 2x^2yz - z^2y$

$\therefore g_y = -z^2y, \quad g = -\frac{1}{2}z^2y^2 + h(z)$

$f_z = x^2y^2 + g_z = x^2y^2 - zy^2 + h'(z)$
$= x^2y^2 - y^2z + 3z^2$

$\therefore h'(z) = 3z^2, \quad h(z) = z^3\ (+c)$

Putting it all together:

$f(x,y,z) = \boxed{x^2y^2z - \frac{1}{2}z^2y^2 + z^3}$
$(+c)$

$\boxed{3}$ $\vec{F} = \langle -(y+z),\ 2x-z,\ x-2y \rangle$

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ -(y+z) & 2x-z & x-2y \end{vmatrix}$$

$$= \langle -2+1,\ -(1+1),\ 2+1 \rangle$$
$$= \langle -1, -2, 3 \rangle$$

By Stokes' theorem, C in plane (closed), R its interior

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_R \nabla \times \vec{F} \cdot \hat{n}\, dS$$

$$\hat{n} = \frac{\langle 1, -2, -1 \rangle}{\sqrt{6}} \qquad \nabla \times \vec{F} \cdot \hat{n} = \frac{-1+4-3}{\sqrt{6}} = 0$$

$$\therefore \oint_C \vec{F} \cdot d\vec{r} = 0.$$

$\boxed{4}$

a) $\text{div}(\nabla \times \vec{G}) = 0$: (for any G with continuous 2nd partials)

$$\nabla \times \vec{G} = \langle P_y - N_z,\ M_z - P_x,\ N_x - M_y \rangle$$

$$\therefore \nabla \cdot (\nabla \times \vec{G}) = P_{yx} - N_{zx} + M_{zy} - P_{xy} + N_{xz} - M_{yz} = 0$$

$$\therefore \oiint_S \vec{F} \cdot d\vec{S} = \iiint_D \text{div } \vec{F}\, dV = \iiint_D \nabla \cdot \nabla \times \vec{G}\, dV$$
$$= \iiint_D 0\, dV = 0.$$

b) Using Stokes' theorem: draw C, oriented as shown.

$$\oint_C \vec{G} \cdot d\vec{r} = \iint_{S_1} \vec{F} \cdot d\vec{S}$$

$$\oint_{C'} \vec{G} \cdot d\vec{r} = \iint_{S_2} \vec{F} \cdot d\vec{S}$$

($C' = C$ reversed orient'n)

Adding: $\quad 0 = \iint_S \vec{F} \cdot d\vec{S}$

since the line integrals on C and C' sum to 0 $\left(\oint_{C'} = -\oint_C\right)$.

$\boxed{5}$ a) $\vec{F} = \langle x, y, z \rangle$

Assume $\vec{F} = \nabla \times \vec{G}$ for some G.

$$\oint_C \vec{G} \cdot d\vec{S} = \iint_R \nabla \times \vec{G} \cdot d\vec{S}$$

S = sphere radius a centered at origin

$$= \iint_R \langle x, y, z \rangle \cdot \hat{n}\, dS$$

$\hat{n} = \dfrac{\langle x, y, z \rangle}{a}$, therefore

$$\oint_C \vec{G} \cdot d\vec{S} = \iint_R \frac{a^2}{a}\, dS = a \cdot (\text{area of } R)$$

But similarly, letting $R' = \text{sphere} - R$

$$\oint_{C'} \vec{G} \cdot d\vec{S} = a \cdot (\text{area of } R')$$

this is impossible, since the two line integrals have opposite signs, whereas the two right-hand sides are both positive.

b) Using from Problem $\boxed{3}$:

$$\nabla \cdot \nabla \times \vec{G} = 0 \text{ always}$$

but $\nabla \cdot \langle x, y, z \rangle = 3$.

$\therefore \langle x, y, z \rangle \neq \nabla \times \vec{G}$ for any $\vec{G}$.

Lecture 33

Stoke's Theorm Subitilles 5/6

Read V14, 6G-1,2
not to hand in

S has a boundry Closed C] compadibly oriented

$$\boxed{\oint_C \underbrace{\vec{F}\cdot d\vec{r}}_{\text{work}} = \iint_S \nabla\times\vec{F}\cdot d\vec{S}}$$

curl $\vec{F}$
flux of curl $\vec{F}$
accross C

Stoke's Conservative Field

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ M & N & P \end{vmatrix} \leftarrow \vec{F} = \langle M, N, P \rangle$$

$$\langle P_y - N_z,\ M_z - P_x,\ N_x - M_y \rangle$$

When you say curl $\times$ F = 0
you say all 3 terms are 0

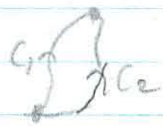
$$\boxed{\begin{aligned} N_x &= M_y \\ P_y &= N_z \\ M_z &= P_x \end{aligned}}$$

criteria for conservitave field

3 ways to say a field is conservitive

$\vec{F} = \vec{\nabla} f$

$\oint_C \vec{F} \cdot d\vec{r} = 0$ (conservitable)

$\int_{C_1} = \int_{C_2}$ path indpendence 

What does this have to do with?

$\nabla \times \vec{F} = \vec{0}$

- easy
- uses $f_{xy} = f_{yx}$, etc

But its not really what we want to know

- want to know other way
- different kinds must talk about all of them

wasn't this on the P-set

$\nabla \times \vec{F} = \vec{0} \overset{?}{\Rightarrow} \oint_C \vec{F} \cdot d\vec{r} = 0$

Problem: start w/ closed C

- C is the boundry
- Must find S having C as its boundry

Problem ① C is knot

- circle is unknotted
- something that can be easily fromed in a circle w/o crossing itself is unknotted

← simplest knot (trefoil knot)

always a closed loop
if not closed can be undone w/o crossing itself

How can prove ∮ around it is 0
Hard to find a surface to contain it
Dunk it into soap to see where bubble

~~like a mobius strip~~ ??

Stokes theorm <u>does</u> <u>not</u> apply in any sense
but it has only one side
can't choose normal vector
"non orientable"

It still makes sense to say work is being done but surface must be 2 sided for Stokes to work

One sided shapes have 0 flux

mobius strip

drawing
← came out really bad
- mobius strip

one sided

boundry

Is this boundry knotted?

A knot can always be drawn on paper by
indicating under/over
Must decide if it is a knot?
can it be deformed

Not a knot

Any closed C → knot or not → is the boundry
of a 2-sided surface (theory)
(in the notes)
knowing if it is a knot -
it is impossible to know
no computer program to write
provablky undecidable
- recent: last 20 years

An even worse/more significant problem

2. We're Ok if $\nabla \times \vec{F} = \vec{0}$ in all of 3 space
   - stokes theorm is applicable to any S you find
   - theorm guarentees there is one, so don't have to find it

In apps: $\vec{F}$ usually not defined in all of 3 space
   - just in some domain D

Property of D needed for using Stoke's Theorm (Curl F = 0 in D)
If a closed curve C lies in that domain, must be able to find a surface that spans it and also lives in D

D = 3-space / solid ball
remove something
= exterior of solid ball

= Simply Connected

Does D have this property?
yes

Can shrink C to a point w/o leaving D

D = 3 space / point
- works as well

$D = 3$ space / z-axis (∞ wire)
<u>Not</u> simply connected
- can't shrink everything to one point

$D = 3$ space / solid donut (torus)
exterior <u>not</u> simply connected

Lecture 34

More Stoke's Theorm

5/7

End of term stuff posed soon

Continue from yesterday

$\vec{F}$ Field
D simply connected

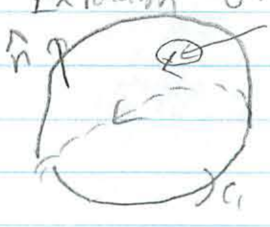
every closed curve can be shrunk to a point w/o leaving D

exterior of a solid ball
is simply connected

z axis → exterior of z axis
- is not simply connected
- point on z axis outside of D

$\vec{\nabla} \times F = 0$
→ $\vec{F}$ Conservative
?? if D is not simply connected

Extension of Stokes' Theorm



Closed curve on surface of cap C_2

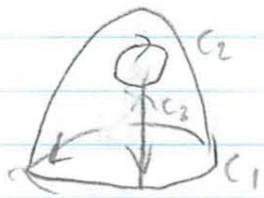
as you walk along surface
your head points to $\hat{n}$

Surface S is cap $\underset{\text{minus}}{-}$ area in C_2

Stokes Theorm still true
- with multiple boundries

$$\oint_{C_1} \vec{F} \cdot d\vec{r} + \oint_{C_2} \vec{F} \cdot d\vec{r} = \iint_S \nabla \times \vec{F} \cdot d\vec{S}$$

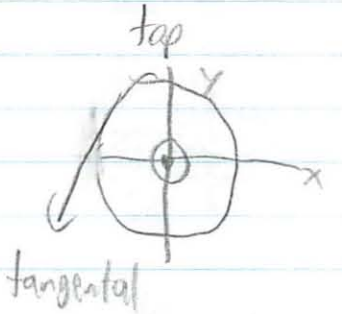
Make a cut



$$\oint_{C_1} + \int_{C_0} + \oint_{C_2} + \int_{C_3'}$$

$$= \iint_S \nabla \times \vec{F} \, d\vec{S}$$

D = exterior of z-axis (wire)
$\vec{F}$ = magnetic field produced by wire
wire has constants



$$\operatorname{dir} \vec{F} = \frac{\langle -y, x \rangle}{r}$$ ← to make unit vector

$$|\vec{F}| = \frac{1}{r}$$

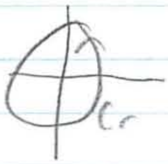
$$\vec{F} = \frac{\langle -y, x \rangle}{r^2}$$

$\nabla \times \vec{F} = 0$

← go back to proof
use chain rule
use $\frac{\partial r}{\partial x} = \frac{x}{r}$
good for final review

$$\oint_C \vec{F}\cdot d\vec{r} = \begin{cases} 0 & \text{if } C \text{ does not go around } z \text{ axis (Stokes)} \\ 2\pi & C_r \\ 2\pi & \text{any curve around wire once (closed)} \end{cases}$$

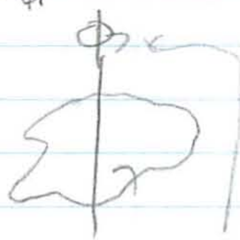
avoiding details



top

does not go around z axis

$$\oint_{C_r} F\cdot dr = \frac{1}{r}\cdot 2\pi r$$



Want work done to move along C

draw a circle up top

Make a lamp shade

$$\oint_C \vec{F}\cdot d\vec{r} + \oint_{C_r} \vec{F}\cdot dr =$$

$$= \iint_S \underbrace{\nabla\times\vec{F}}_{0}\cdot d\vec{s}$$

$$\oint_C \vec{F}\cdot dr = \oint_{C_r} \vec{F}\cdot d\vec{r}$$

Is $\vec{F}$ conservative
- sort of

Is $\vec{F}$ a gradiant field
- $\vec{F} = \vec{\nabla} f$
- sort of

$$\vec{F} = \nabla \theta = \nabla \tan^{-1}\left(\frac{y}{x}\right)_{x \neq 0}$$

check by calc

$\vec{F} = \nabla f$ yes if multi-valued

F pathind? yes - but must calculate

$$\int F \cdot dr = f(P_1) - f(P_0)$$

must use path in calc $f(P_1)$

Applications

$\vec{F}$ = when can I say "Flux of F through C"

Must mean $\iint_{S_1} \vec{F} \cdot d\vec{S} = \iint_{S*} \vec{F} \cdot d\vec{S}$

$\hat{n}$ must be ↑ For flux to be same "compatiable"

depends only on curve not surface

$* =$ any #

$$\oiint_{S_1 + S_2'} \vec{F} \cdot d\vec{S} = \iiint_D \nabla \cdot \vec{F} \, dV$$

Happens if $\mathrm{div}\, \vec{F} = 0$

Recitation

5/10

<u>Lecture</u>

- recall $\vec{F} = \vec{\nabla} f$ "conservative" $\Leftrightarrow$ path independent

- $\vec{F}$ conservative $\rightarrow$ $curl(\vec{F}) = \vec{0}$

$\left.\begin{cases} curl(\vec{F}) = 0 \\ \text{Domain simply connected} \end{cases}\right\} \vec{F}$ is conservative

+ In the case domain is not simply connected, one still has $curl(\vec{F}) = 0$

$\oint_C \vec{F} \cdot d\vec{r} = 0$ on curve C bounding a surface included in the domain (Apply Stokes)

- Extend Stoke's Theorm

$$\int_{C_1+C_2+C_3} \vec{F} \cdot d\vec{r} = \iint_S curl\ \vec{F} \cdot d\vec{S}$$

---

ex) Which are simply connected?
- Shrinkable to a pt

a) unit disc  ✓

b) toroid 2D  ✗

c) plane minus a point  ✗

<u>3D</u>

↓

Hint: if you can attach a lock than (X)

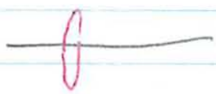
f) Half space $z \geq 0$ ✓

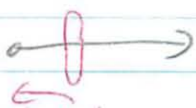
g) exterior of torus  (X)

h) interior of torus  (X)

i) exterior of disc $\begin{cases} z = 0 \\ x^2+y^2 \leq 1 \end{cases}$ ✓

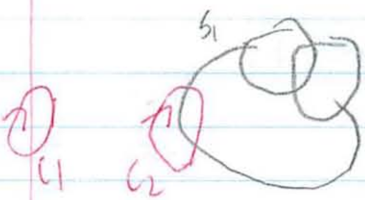
J " " circle $\begin{cases} z = 0 \\ x^2+y^2 = 1 \end{cases}$ (X)

e) exterior of lines  (X)

f) exterior of ray  ✓

stealable

ex2 Knot in 3D



a) Is space $-k$ simply connected?

b) Show that $\oint_{C_1} \vec{F} \cdot dr = 0$

Let $\vec{F}$ be defined everywhere except on k
$\text{curl}(\vec{F}) = 0$

c) Show that $\oint_{C_2} \vec{F} \cdot dr = 0$

a) No - can move ring C_2 out

b) $= 0$ because $\oint_{C_1} = \iint_{S_1} \text{curl}(\vec{F})\, d\vec{S}$

$\uparrow$ Stokes

Still conclude most curves have this property

c) $\oint_{C_2} = \iint_{S_2} = 0$

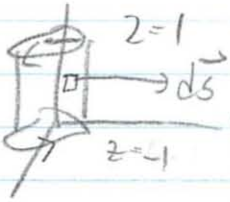
Think of k as wire
but plastic around
C_2 is only boundry

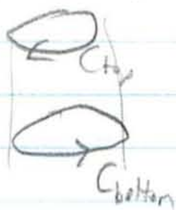
ex 3 $\vec{F} = \langle y - yz^2, xz^2 - x, 1 \rangle$ $\quad S = \begin{cases} x^2 + y^2 = a^2 \\ z^2 \leq 1 \end{cases}$

Show that $\iint_S \text{curl}(\vec{F}) \cdot dS = 0$

Can calculate $\iint$ or use Stoke's Theorm

- cylinder $\to$ z axis
  radius a





Must consider the 2 curves

each curve oriented independently w/
right hand rule
$\downarrow$
thumb points to $d\vec{S}$

$$\iint_S \text{curl}(F)\cdot d\vec{S} \underset{\text{Stokes}}{=} \int_{C_1+C_2} F\cdot dr$$

cancle out
Can we show this?

$\vec{F} = \langle ?, ?, 1 \rangle$ on $C_1 + C_2$

↳ know $z^2 = 1$ on curves
$= \langle 0, 0, 1 \rangle$

$F \cdot dr = 0$ since $\vec{F}$ vertical $\perp$ $d\vec{r}$ horizontal

ex4 Plane

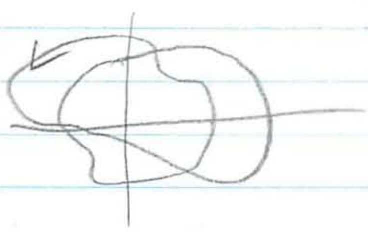
$f(x,y) = \tan^{-1}\left(\frac{y}{x}\right)$

$\vec{F} = \vec{\nabla} f$

a) Volume of $\vec{F}$, domain of definition (simply connected ?)

b) Geometrical interpelation of f ?

c) What is the volume $\oint \vec{F}\cdot d\vec{r}$ where C is closed curve turning k times around origin



$k = \mathbb{Z}$

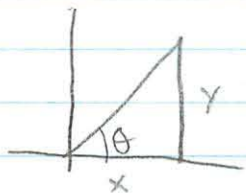
$(\tan^{-1}(x))' = \frac{1}{1+x^2}$

a) $\vec{F} = \left\langle \frac{-y}{x^2} \frac{1}{1+\left(\frac{y}{x}\right)^2}, \frac{1}{x} \frac{1}{1+\left(\frac{x}{y}\right)^2} \right\rangle$

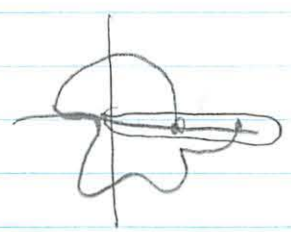
$= \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$

Defined everywhere except origin
not simply connected

b)



c) $\oint_C \vec{F} \cdot dr \neq 0$ because it encloses a point not in the domain
(cannot apply Green's Theorem)



← remove ray
Now simply connected
FTC can be applied

FTC $\int_C \vec{F} \cdot dr = f(B) - f(A)$

$= 2\pi$

Answer: $k2\pi$ (k pieces of integral 2π)

Lecture

Maxwell's Equations

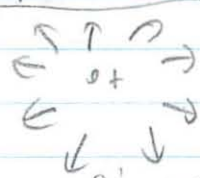
5/11

Maxwell's Eq 8.02

$\vec{E}$ = electrostatic field

from electrostatic charged particles ⊕⊖

Coulomb's Law



repels ⊕ particles
symettric in space in 3D

inverse square

$$\vec{E} = \frac{\langle x, y, z \rangle}{\rho^3} \cdot \underset{\text{constant}}{C_1}$$

basic inverse square law

If you have a bunch of them,
need to integrate them up

Guass' Law

$$\iint_S \vec{E} \cdot ds$$

the flux of E through S

$$= \frac{1}{\varepsilon_0} Q_D$$

total net charge in D

What connects the two is what we have
been studying

3 ways of doing this argument
- one in the notes is the worst

Guass' Law $=\iiint_D \operatorname{div}(E)\, dV = \frac{1}{\epsilon_0} Q_D$ $\qquad \frac{1}{\epsilon_0} = 4\pi G$

$\boxed{\text{or}}\ \nabla \cdot E\ dV$

divergence theorm

area of sphere

Interporate div by seeing what this eq means at a point

At a point P_0  B_0 (infatessimal box)

volume $= \Delta B_0$

Turn into general equation

Interporate at a point P_0

$(\vec{\nabla} \cdot E)_0$

pt P_0

Contineous function inside that infatessimal box

$$(\vec{\nabla} \cdot E)_0 \cdot \Delta B_0 = \frac{1}{\epsilon_0} Q_{B_0}$$

$$(\vec{\nabla} \cdot E)_0 \approx \frac{1}{\epsilon_0} \frac{Q_{B_0}}{\Delta B_0} \qquad \Delta B_0 \to 0$$

meaning of divergence

$$(\vec{\nabla} \cdot E)_0 = \frac{1}{\epsilon_0} \rho_0$$

charge density here (different than above!)

good intepertation of divergance

Big divergence theorm only used once
All of these equations are physics
- Statements could be false
- Just proven Expirientally

Reverse reasoning

$$\nabla \cdot \vec{E} = \frac{1}{\varepsilon_0} \rho$$ ← charge density

$$\oiint_S \vec{E} \cdot d\vec{S} \underset{\text{math}}{=} \iiint_D \nabla \cdot \vec{E} \, dV$$

$$= \iiint_D \frac{1}{\varepsilon_0} \rho \, dV$$

$$\overset{\text{math}}{=} \frac{1}{\varepsilon_0} Q_D$$ ← total charge in D

Curl is more subtile
fields that use it → physics more subtile

$\vec{B}$ = magnetic field changing w/ time
$B(x, y, z, t)$

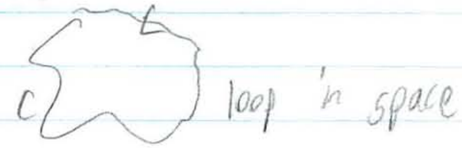
$$\frac{\partial}{\partial t} \vec{B}(x, y, z, t)$$

not as complicated to begin with

Produces a changing electromagnetic field $\vec{E}$

Work you get out of it

$$\oint_C \vec{E} \cdot dr$$



Unit positive charge being carried around C
= emf = electromotive force = form of work

$$\oint_C \vec{E} \cdot dr = -\underbrace{\frac{d}{dt} \underbrace{\iint_S \vec{B} \cdot d\vec{S}}_{\text{flux of } \vec{B} \text{ over } S}}_{\text{function of time alone}}$$

- ordinary, not partial derivitive

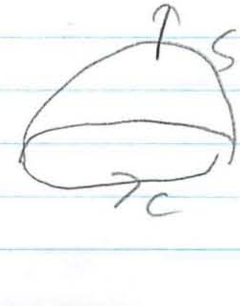
$$\oint \vec{E} \cdot dr \underset{\boxed{\text{math}}}{\overset{}{=}} \iint_S \nabla \times \vec{E} \cdot d\vec{S} \quad \left(\text{flux of curl E over surface}\right)$$

Stokes theorm

Curl E = $\nabla \times E$

$$\underset{\boxed{\text{math + physics}}}{=} \iint -\frac{\partial}{\partial t} \vec{B} \cdot d\vec{S}$$

Shady procedure



Since true for all surfaces

S infatessemal surface

P_0

$$\rightarrow \nabla \times \vec{E} = -\frac{\partial}{\partial t} \vec{B}(x, y, z, t)\, C_2$$

↑ constant dependent on units

Recitation

5/12

Lecture

Electrostatic field $\vec{E}$

Colomb $\vec{F} = c_1 q_1 q_2 \frac{\vec{P_1P_2}}{|P_1P_2|^3}$

$\vec{E} = c_1 \frac{\langle x, y, z \rangle}{r^3}$

if $P_1 = 0$, $P_2 = \langle x, y, z \rangle$

$q_1 = q_2$

Guass

$$\oiint_S \vec{E} \cdot d\vec{s} = \frac{1}{\epsilon_0} Q_{\text{charge inside}}$$

$$\text{div } \vec{E} = \frac{\delta}{\epsilon_0} \leftarrow \text{density of charge}$$

Electromagnetic

$\vec{B}$ = magnetic field

Faraday $\Rightarrow \oint_C \vec{E} \cdot d\vec{r} = -c_2 \frac{d}{dt} \iint_S \vec{B} \cdot d\vec{s}$

$$\text{curl } \vec{E} = -c_2 \frac{d\vec{B}}{dt}$$

ex 1 Maxwell $\text{curl}(\vec{B}) = \frac{1}{c}\frac{\partial E}{\partial t}$

What is the $\int$ form of this relation

Stokes $\iint_S \text{curl}(\vec{B}) d\vec{s} = \oint_C \vec{B} \cdot d\vec{r}$

$$\iint_S \frac{1}{c}\frac{\partial E}{\partial t} \cdot d\vec{s} = \oint \vec{B} \cdot d\vec{r}$$

some conditions on this

$$\frac{1}{c}\frac{\partial}{\partial t}\iint_S \vec{E} \cdot d\vec{s} = \oint_C \vec{B} \cdot d\vec{r}$$

ex2 Let R be a simply connected region of space, charge free and satisfying $\frac{\partial \vec{B}}{\partial t} = \vec{0}$

Show that there exists a function f such that $\begin{cases} f_{xx} + f_{yy} + f_{zz} = 0 \\ \vec{E} = \vec{\nabla} f \end{cases}$

Shows that divergence $\vec{E} = 0$

Faraday's $\rightarrow$ curl $\vec{E} = 0$

f must satisfy equation - can't just pick an arbitrary f

$\left[\begin{array}{l} \text{curl } \vec{E} = 0 \\ R \text{ simply connected} \end{array}\right.$ $\rightarrow$ there exists f such that $\vec{F} = \vec{\nabla} f$

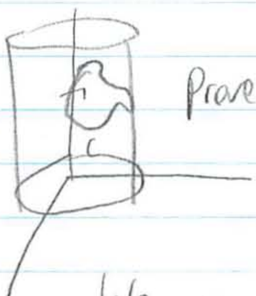
$\text{div}\,\vec{F} = \text{div}(\langle f_x, f_y, f_z \rangle) = f_{xx} + f_{yy} + f_{zz}$
$= 0$

So f satisifies the relations

<u>ex 3</u> [Like Ex B-10 pratice]

Let $\vec{F} = \langle x, xy, z \rangle$
$S = x^2 + 4y^2 = 1$

Show that $\oint_C \vec{F} \cdot d\vec{r} = 0$ for C closed curve



Prove work along curve $= 0$

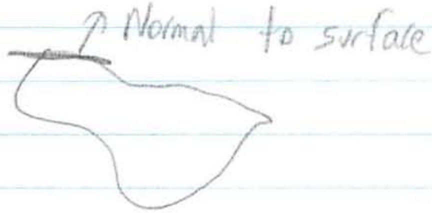
We can hope that Stokes will help

$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ x & xy & z \end{vmatrix} = \langle 0, 0, x \rangle$$

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl}\,\vec{F} \cdot d\vec{S}$$

Prove that $d\vec{S}$ is horizontal

S is level curve of the function
$g(x,y,z) = x^2 + 4y^2$



$d\vec{s} = \hat{n}\, ds$

$d\vec{s}$ is $\parallel$ to $\nabla g = \langle 2x, 8y, 0 \rangle$
horizontal

$\text{curl } \vec{F} \cdot d\vec{s} = 0 \rightarrow \oint_C \vec{F} \cdot d\vec{r} = 0$

ex 4 How to use Gauss' Theorm

We denote $\rho = \sqrt{x^2+y^2+z^2}$

Suppose the density of charge at (x, y, z) is $\sigma = 1$ if $a \le \rho \le b$
$= 0$ if otherwise

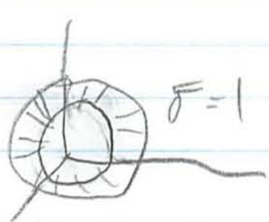
a) What is the total charge Q in space

b) What is the flux

$\phi_\rho = \iint_{S_\rho} \vec{F} \cdot d\vec{s}$

S_R is sphere of radius R

c) What is the value of $\vec{E}(x, y, z)$ if $\rho > b$?
$\rho < a$?



a) $Q = \iiint \delta \, dV = \iiint_R dV$

space between S_a, S_b

$= \text{Vol}(R)$

$= \frac{4\pi}{3}(b^3 - a^3)$

b) Guass $\Phi_R = \frac{1}{\epsilon_0}$ charge inside

$= \frac{Q}{\epsilon_0}$ if $R > b$

$= 0$ if $R < a$

c) Info on flux to info on field

direction of $\vec{E}$ field is radially outward

$\Phi_R = \iint_{S_R} \vec{E} \cdot d\vec{S} = \iint |E| \, dS = |\vec{E}_R| \iint dS$

parallel   constant

$= |\vec{E}_R| \, 4\pi R^2$

If $\rho > b$ $|\vec{E}| = \frac{\Phi_R}{4\pi\rho^2} = \frac{Q}{4\pi\rho^2\epsilon_0}$

$\rho < a$ $|\vec{E}| = 0$

Lecture

Last Lecture

5/13

$\vec{E}$ = electrostatic

$$\oiint_S \vec{E} \cdot d\vec{S}$$

flux of $\underline{E}$ through S

$$\underset{\boxed{\text{math}}}{=} \iiint_D \operatorname{div} \vec{E} \, dV$$

$$\underset{\boxed{\text{physics}}}{=} \iiint_D \frac{1}{\varepsilon_0} \rho \, dV \qquad \text{total charge in } D$$

charge density

$$\operatorname{div} \vec{E} = \frac{1}{\varepsilon_0} \rho \qquad \text{take out integrand}$$

true no matter the surface, D

differentiated form of Maxwell's Eq

$\vec{E}$ electromagnetic field

$$\oint_C \vec{E} \cdot d\vec{s} \underset{\boxed{\text{math}}}{=} \iint_S \operatorname{curl} \vec{E} \cdot d\vec{S}$$

$$\underset{\boxed{\text{physics}}}{=} \iint_S -c_1 \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S}$$

faraday

$\vec{B}$ = changing magnetic field

$$\text{curl } \vec{E} = -\frac{d}{dt}\begin{pmatrix} \text{flux of } \vec{B} \\ \text{over } \vec{S} \end{pmatrix} c_1$$

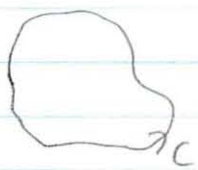
becomes ordinary derivative

say flux <u>through</u> surface

$\vec{F}$ field

div $\vec{F} = 0$

↳ flux over any S spanning C is independent of S



flux of $\vec{F}$ through $\vec{C}$

independent of which surface you choose

know mathmatically → div theorm

Intuitive: flow field of incompressable fluid
- b/c incompressible flow if flows in <u>and</u> out
- no sources b/w those 2 surfaces
  ↳ since div = 0

div $\vec{E}$ electrostatic

physics

where is div = 0, so can talk about flux through any surface
→ in a region of space where <u>no</u> charge
"flux of $\vec{E}$ through loop"

Physics

$\text{div } \vec{B} = 0$

- can speak of flux of $-\frac{\partial B}{\partial t}$ through loop

No sources of magnetic fields
↳ no magnetic monopoles

Calculations in this part of course
- use most topics from class

2D theorem (Green's)

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_R (N_x - M_y)\, dA \qquad F = \langle M, N \rangle$$

$$= \iint_R (\text{curl } \vec{F})\, dA$$

Work form

$$\oint_C \vec{F} \cdot \hat{n}\, ds = \iint (M_x + N_y)\, dA$$

Flux form

3D (Stokes)

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \nabla \times \vec{F} \cdot d\vec{S}$$

divergence

$$\oiint \vec{F} \cdot d\vec{S} = \iiint_D \nabla \cdot \vec{F}\, dV$$

Be able to verify
Think about it intuitively on simple surfaces

Use one side to calculate the other

$\frac{18.01}{18.03}$; $\frac{18.02}{\text{Partial Diff eq}}$

- only 3 or 4 importat ones

$\vec{F}$; $\nabla \cdot \vec{F} = 0$

$\nabla \times \vec{F} = \vec{0}$ conservation of energy

$\downarrow$ $\vec{F}$ is conservitive

$F = \vec{\nabla} \varphi = (\varphi_x, \varphi_y, \varphi_z)$

$\text{div } \vec{F} = 0 \rightarrow \varphi_{xx} + \varphi_{yy} + \varphi_{zz} = 0$

$\downarrow$

$\nabla^2 \varphi = 0$

$\nabla \cdot \nabla \varphi = 0$ = Laplace's eq

Tips

1. Learn to read
2. Read ahead
3. Questions you have yourself are the most exciting things to work on