18.03 Lecture Notes - Fall 2011

Handwritten notes will be posted here shortly after each lecture.

Lecture	Date	Topic
Lecture 1, and the slides	Wed. Sept. 7	Sep. of variables. Isoclines and fences.
Lecture 2, and the slides	Fri. Sept. 9	More on fences and funnels.
Lecture 3, and the slides	Mon. Sept. 12	Euler's method.
Lecture 4, and the slides	Wed. Sept. 14	Autonomous equations. Integrating factors.
Lecture 5, no slides today.	Fri. Sept. 16	Intro to IR models. Complex numbers.
Lecture 6, and the slides	Mon. Sept. 19	Complex numbers and ODEs.
Lecture 7, and the slides	Fri. Sept. 23	More complex numbers. Some input-response.
Lecture 8: Exam revision.	Mon. Sept. 26	
Exam I.	Wed. Sept. 28	
Lecture 9, and the slides	Fri. Sept. 30	Second order ODEs.

Lecture 10, and the slides	Mon. Oct. 3	Wronskians. Constant coefficients case.
Lecture 11, and the slides	Wed. Oct. 5	Operator notation. Example of pendulum.
Lecture 12, and the slides	Fri. Oct. 7	Inhomogeneous case.
Lecture 13, and the slides	Wed. Oct. 12	Some physical examples.
Lecture 14, no slides today.	Fri. Oct. 14	Springs. RLC Circuits.
Lecture 15, and the slides	Mon. Oct. 17	More on RLC. Resonance.
Lecture 16, and the slides	Wed. Oct. 19	Power series expansions.
Lecture 17, and the slides	Fri. Oct. 21	More power series.
Lecture 18, no slides today.	Mon. Oct. 24	Exam II revision.
Exam II.	Wed. Oct. 26	
Lecture 19, no slides today.	Fri. Oct. 28	Introduction to Fourier Series.
Lecture 20, and the slides.	Mon. Oct. 31	The Square Wave.

Lecture 21, and the slides.	Wed. Nov. 2	Fourier series and ODEs.
Lecture 22, no slides today.	Fri. Nov. 4	Heat equation. Wave equation.
Lecture 23, no slides today.	Mon. Nov. 7	Laplace Transform.
Lecture 24, and the slides.	Wed. Nov. 9	Solving ODEs with Laplace.
Lecture 25, no slides today.	Mon. Nov. 14	Impulses. The delta functions.
Lecture 26, and the slides.	Wed. Nov. 16	The weight function.
Lecture 27, no slides today.	Fri. Nov. 18	Pole diagrams.
Lecture 28, no slides today.	Mon. Nov. 21	Exam Revision.
Lecture 29, no slides today.	Mon. Nov. 28	Starting Linear Systems.
Lecture 30, no slides today.	Wed. Dec. 1	Solving linear systems.
Lecture 31, no slides today.	Fri. Dec. 2	Eigenvalues and eigenvector.
Lecture 32, no slides today.	Mon. Dec. 5	Repeated eigenvalues.
Lecture 33, no slides today.	Wed. Dec. 7	The exponential of a matrix.

Lecture 34, no slides today.	Fri. Dec. 9	Drawing solution curves.
Lecture 35, no slides today.	Mon. Dec. 12	Exam Review.
Lecture 36, no slides today.	Wed. Dec. 14	Exam Review II.

Back to the 18.03 home page.

18.03
Final Review

12/17

ODE = ordinary differential eq'n

only deriv of single variable

Not a ~~PDE~~

1st order ODE — higest deriv appearing is 1st order

Later

~~Sec~~ Second + Higher order ODEs

System of First order ODEs

We never did PDEs right?

1st order

$$\frac{dx}{dt} = f(x,t)$$

$$\frac{dx}{dt} = x^3 t + t^2 + \sin x$$

②

First is seperable

$$\frac{dx}{dt} = 2xt + x$$

So can I remember how to do this

want all x on one side and t on other

$dx \rightarrow dt$

So

$$\frac{dx}{dt} = x(2t+1)$$

$$\frac{1}{x}\,dx = 2t + 1\,dt$$

$$\int \frac{1}{x}\,dx = \int 2t + 1\,dt$$

need to be able to $\int$!

$$\cancel{e^x} + c \quad \ln(x) = \frac{2t^2}{2} + t + c$$

$$\cancel{e^x = t^2 + t}$$

$$\cancel{\ln \qquad \ln}$$

$$\cancel{x = \ln(t^2 + t)}$$

③

$$\ln(x) = \cancel{D}\, t^2 + t$$

$e^\wedge \qquad e^\wedge$

$\boxed{\int \frac{1}{x} = \ln(x)}$ $\quad e^{\ln(x)} = X$

$$X = e^{t^2+t}$$

$$x = Ce^{t^2+t}$$

↑ don't forget the constant $C = e^c$

Why are we here?

1. Predict the Future Model a real world situation
2. Analyze the Result
   a) exact sol
      - rare
   b) numerical sols
   c) qualatative sols
      - slope fields

4

So we could do slope field ~~for previous~~ for $\frac{dy}{dx} = y^2 - x$

So at $(0,0)$

$$\frac{dy}{dx} = \text{slope} = 0^2 - 0 = 0$$

At $(1,0)$

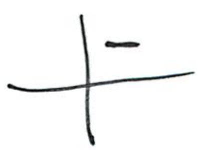
$0^2 - 1$

$(0,1)$

$1^2 - 0$

$(1,1)$

$1^2 - 1$



etc Draw a slope field like that

(5)

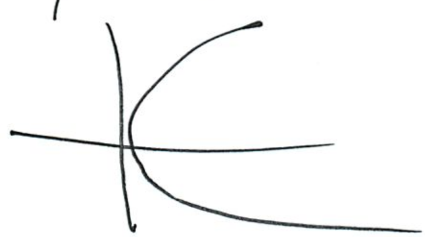
Isocline = fixed slope
" iso" "cline"

So this a line through that slope

Here $y^2 - x = C$

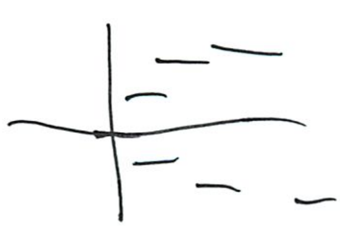
So 0-isocline $C=0$

$$y^2 - x = 0$$
$$y^2 = x$$



Says everywhere a slope [illegible] is 0
where sol has deriv $= 0$

Ie



Inside curve $\ominus$ derivs
Outside " $\oplus$ derivs

6

Once you get inside parabola you never leave

0-isoline is a fence for sols to diff. eqn

<u>lower fence</u> - sols never cross from above

$L'(x) < f(x, L(x))$

$\frac{dy}{dx} = f(x,y)$

<u>Upper fence</u> - sols never cross from below

<u>Verify</u> for $\frac{dy}{dx} = y^2 - x$

$L(x) = -\sqrt{x}$

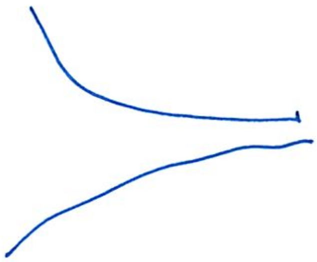
? where did we get that?

$L'(x) = \frac{-1}{2\sqrt{x}} = f(x, L(x))$

Now plug $-\sqrt{x}$ in for y

7a

If have 2 curves w/ same asy behavior → <u>funnel</u>



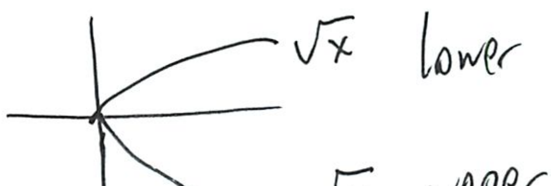
→ squeeze to same value

$\frac{dy}{dt} = f(x, t)$

Sep of variables $\frac{dx}{dt} = f(t) \cdot g(x)$

So $L(x)$ was a lower fence

Lok that we got from $y^2 - x = c = 0$



$L(x)$ lowers since $L'(x) < f(x, L(x))$ note $x^{-1/2} = \frac{1}{\sqrt{x}}$

find $L'(x)$ $\frac{d}{dx}\sqrt{x} = \frac{d}{dx} x^{1/2} = \frac{1}{2} x^{-1/2} = \cancel{t\sqrt{x}}$ $\frac{1}{2\sqrt{x}}$

7b

Now test

$\frac{1}{2\sqrt{x}} < 0$ ⊗ No So upper fence?

r'', since value we are comparing for

$U(x)$ $U'(x) > f(x, U(x))$ ✓ yes

So $U(x) = \sqrt{x}$
$L(x) = -\sqrt{x}$ ✓ Right

(I don't think I ever got that before)

Can then find -1 isocline to make a fence

No sol but try it

$y^2 - x = -1$

Solve for y

$y^2 = -1 + x$

$y = \sqrt{-1 + x}$

$y' = \frac{d}{dx}(-1 + x)^{1/2}$

(8)

Ahh chain rule

$$\frac{1}{2}(-1+x)^{-1/2} \cdot 1$$

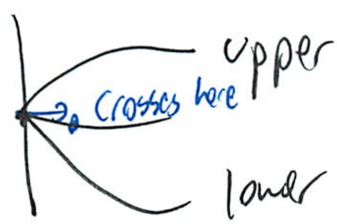
$$\frac{1}{\sqrt{-1+x}}$$

So is that $> < -1$

greater - so upper

but its actually $\pm$ when take $\sqrt{\ }$ something

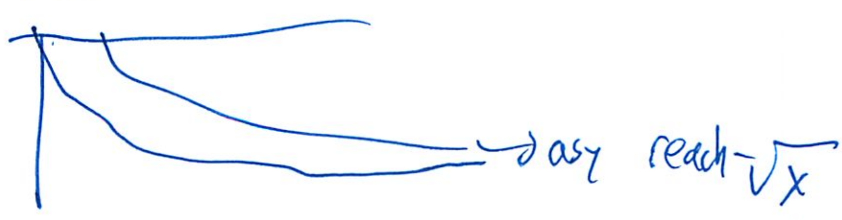
So same



They say $\sqrt{x-1}$ (v) same

but call

So how



Think about things graphically - what happens...

(9)

So what went wrong

$\frac{1}{\sqrt{x-1}}$ is what vs -1

when x is (+)

So $x=5$

$\frac{1}{\sqrt{4}} = \frac{1}{2}$ (>) -1

so upper

* actually do it!

Come on - should be able to do this!

Self review

Closed form - expression in terms of bounded well known fns ((+) (−) (×) (÷))

Not ∞ series, $\int$ int, lim, ∞ fractions

Linear any diff eq in form

$$a_n(t)\, y^{(n)}(t) + a_{n-1}(t)\, y^{(n-1)}(t) + \cdots + a_1(t)\, y'(t) + a_0(t)\, y(t) = g(t)$$

10

Solution on interval $\alpha < t < \beta$
is any fn $y(t)$ that satisfies
eqn in the interval

Can always verify by plugging in our $y(t)$ sol!
~~Should get back~~
get back original fn No!
get something that is =
like $0 = 0$

IVP - ODE w/ proper # of inital conditions

Interval of validity - where its valid

General Sol'n - no inital conditions

Actual sol - w/ " "

11

Explicit sol $y = y(t)$

all on one side

Implicit sol ys on both sides

Try another sep of variables

$$\frac{dx}{dt} = f(t)\, g(x)$$

$$\frac{dx}{dt} = x \cos t$$

$$\frac{1}{x}\, dx = \cos t \, dt$$

$$\int \qquad \int$$

$$\ln(x) = +\sin t + c$$

$$x = C e^{+\sin t}$$

(V) otherwise

$$\boxed{\int \cos = +\sin}$$

$$\boxed{\int \sin = -\cos}$$

(12) (missed)

$\boxed{L2}$

Use slope fields to prove long term behavior

└ Thought already did

Narrowing funnel

$$U(x) > L(x)$$

$$\lim_{x\to\infty} |U(x) - L(x)| = 0$$

So from last time

$$\lim_{x\to\infty} \left(\frac{1}{\sqrt{x-1}} - \frac{1}{\sqrt{x}}\right) = 0$$

Try it out

$x = 5$

$$\frac{1}{\sqrt{4}} - \frac{1}{\sqrt{5}}$$

$x = 100$

$$\frac{1}{\sqrt{99}} - \frac{1}{\sqrt{100}}$$

will go to 0

(13)

<u>Antifunnels</u> $L(x) > U(x)$

Always exists <u>a</u> sol inside antifunnel

~~a if $\frac{\partial f}{\partial y} \geq 0$~~

correction

<u>Unique</u> sol if $\frac{\partial f}{\partial y} \geq 0$

$\frac{\partial f}{\partial y}$ = dispersion rate of solutions
- if they remain close

Here $\frac{\partial f}{\partial y} = 2y \geq 0$ so unique sol $\sqrt{x}$ as $x \to \infty$

(I'm not really paying much attention to this proof stuff)

(14)

Eulers' Method (Numerical method)

So if start at $(x_0, y_0) = (0,0)$

$\llcorner$ given $y(x_0) = y_0$

Get a piecewise linear curve that approxs a sol

Also have a step size h

$\llcorner$ pick it, then fixed

So then

$$x_{n+1} = x_n + h \qquad y_{n+1} = y_n + h \cdot f(x_n, y_n)$$

So

$$x_1 = x_0 + h \qquad y_1 = y_0 + h \cdot f(x_0, y_0)$$

$$x_2 = x_1 + h \qquad y_2 = y_1 + h \cdot f(x_1, y_1)$$

$$\vdots \qquad \qquad \vdots$$

So lets do it

$$\frac{dy}{dx} = x + \frac{1}{5} y \qquad y(0) = -3$$

$$h = 1$$

15

X	Y
0	-3
1	$-3 + 1 \cdot \left(0 + \frac{1}{5}(-3)\right) = -3.6$
2	$-3.6 + 1 \cdot \left(1 + \underbrace{\frac{1}{5}(-3.6)}_{\text{need calc}}\right) = -4.32$
⋮	⋮ etc

just plug into diff eq

trying to get slope

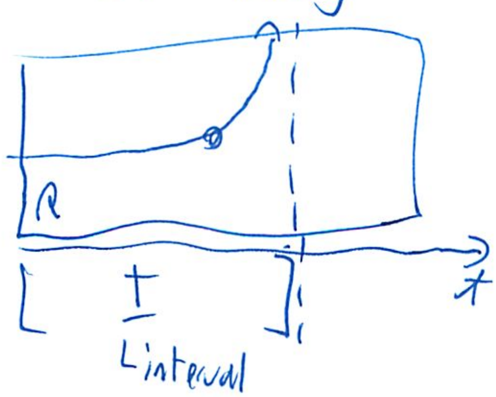
h aka Δx

Lots of possible error
1. Step size too large
2. Rounding error when calculating

Sol exists when f is "nice" near inital pt

"nice" = $f(t,x)$ and $\frac{\partial x}{\partial x} f(t,x)$ exist and are contiuous

Can do that box thing



(16)

Euler's Error

$$\left| \underbrace{y(x_n)}_{\text{actual}} - \underbrace{y_n}_{\text{estimate}} \right| \le C \cdot h$$

step size

can find C

$$\frac{\text{Error}}{h} = C$$

Can also use avg b/w 2 pts

more accurate

$$|y(x_n) - y_n| \le C \cdot h^2$$

L4 /

autonomous equations - independent

$$\frac{dy}{dx} = f(\cancel{x}\ y)$$

not of x

linear

$$\frac{dy}{dx} + P(x)\, y = Q(x)$$

polynomials

17

Autonomous

$$\frac{dy}{dt} = f(y) \cdot 1$$

$$\int \frac{dy}{f(y)} = t + C$$; seperation of variables ;

examples radioactive decay

$$\frac{dP}{dt} = -kP$$

The salt example qun

Can I figure out how to write it

$$\underset{?}{\frac{dS}{dt}} = \underbrace{\frac{1}{2}\ \text{lbs salt / gallon} \cdot 4\ \frac{\text{gallons}}{\text{min}}}_{\text{in}} - \underbrace{S \cdot 4\ \frac{\text{gal}}{\text{min}}}_{\text{out}}$$

S = conc of salt

$$\frac{dS}{dt} = 2 - \frac{4S}{200}$$

$$\underbrace{\frac{1\ \text{lbs}}{200\ \text{gal}}}$$

Concentration out

need to include current concentration somewhere

(18)

So can I solve this

We saw $\int \frac{dy}{f(y)} = t + C$

~~$\int$~~ $y = S$ here

So separate

$$\frac{200}{4S}\, dS = 2$$

$$50 \int \frac{1}{S}\, dS = \int 2\, dt$$

$$50 \ln(x) = 2t$$

$$\ln x = \frac{1}{25} t$$

$$x = C e^{1/25\, t}$$

But they did it some other way → qualatatively

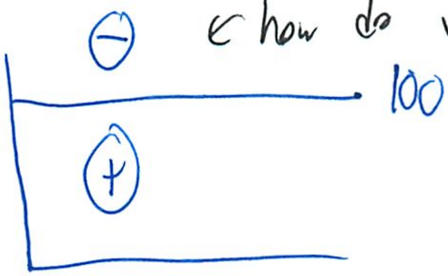
Find 0-isocline

$$0 = 2 - \frac{S(t)}{\cancel{200}\ 50}$$

Solve for $S(t)$

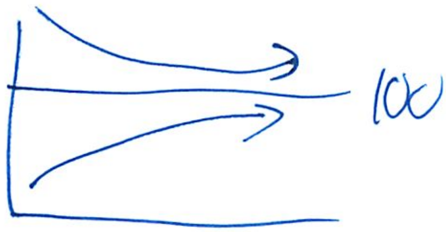
$$\frac{S(t)}{50} = 2 \qquad S(t) = 100$$

80



← how do we know + −

something w/ > or <?



phase line

basically root of diff eq

100 is a root of $f(x)$ in $\frac{dy}{dx} = f(x)$

↳ so equiv to what we found later

Another example

$$\frac{dy}{dx} = y^2 - 4y + 3$$

$$-y^2 + 4y \, dy = 3 \, dx$$

$$\int \qquad \int$$

$$-2\frac{y^3}{3} + 4\frac{y^2}{2} = 3x$$

(20)

↓ they had 4 but $\int y = \frac{y^2}{2} + C$ right?

Yes

$$-\frac{2}{3}y^3 + 2y^2 = 3x$$

? can get sol in just y

not easily

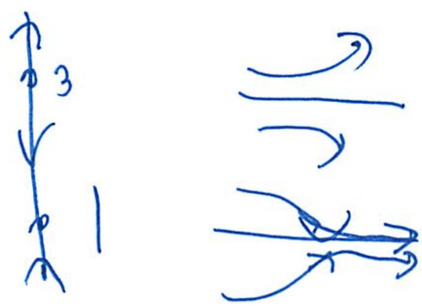
But can factor

$$(y-1)(y-3)$$

$f(y) > 0$ if $y > 3$

$f(y) < 0$ if $3 > y > 1$

$f(y) > 0$ if $y < 1$



"Source"

"Sink"

? but how did we get what was > or <?

(21)

Harvesting

$$\frac{dP}{dt} = RP(M - P) - h$$

max — pos. harvesting constant

So like fish in a pond

$$\frac{dP}{dt} = P(4 - P) - h$$

variables — constant

Find roots

$$P^2 - 4P + h$$

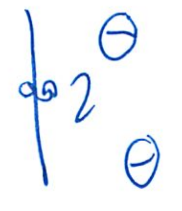
$$P = 2 \pm \sqrt{4-h}$$

So when $h > 4$

- imaginary
- no real roots
- no pts for phase line

When $h = 4$

- one real root

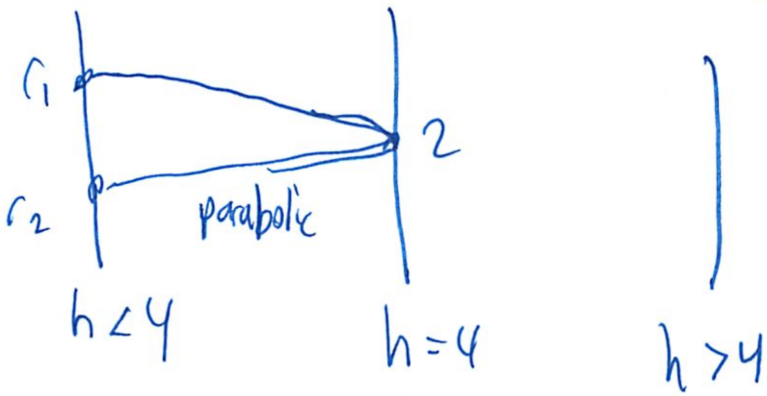


When $h < 4$

– two real rts

So put together → bifurcation diagram

↳ "like a movie"



↑ bifurcation pt – places on phase diagram where # of vertices on phase line Δ

here $h=4$

~~Linear Eqn~~

$$\cancel{\frac{dy}{dx} + P(x)\,y = Q(x)}$$

~~$\int$ both sides is hard~~

23

So found some answers on how + −
for the Salt qu $S(t) = 100$

it $s < 100$ then $s' > 0$ ~~So upper band?~~
flip $s > 100$ $s' < 0$ ~~So lower band?~~

No just think about
$S' > 0$ means ↗ ↗ ↗

$S' < 0$ means ↘ ↘ ↘
Slopes are neg

$\boxed{\frac{dy}{dx} \gtrless 0}$

~~Oh also I~~

Could you say upper or lower band?
└ it never does cross
I think I should ~~a~~ avoid

Can do this phase line stuff since
sols are translation invariant ~~&~~
└ + t_0 does not do anything!

(24)

Logistical model

L harvesting

$$\frac{dP}{dt} = kP(M-P) - h$$

(Should try a problem - like exam 1

L where forgot ~~[illegible]~~ the eqn)

Basic logistical model

$$\frac{dP}{dt} = aP - bP^2$$

So quiz or Exam 1 # 3

limiting pop = 400

grows at 75% p a year

"far from limit" means P small so P2 can be ignored"

L what does this mean?

(25)

$\frac{dP}{dt} = aP$

a so $.75$ ✓Which I had

So the limit ~~goes~~ ~~the~~ $= \frac{a}{b} = 400$

└Did we cover that in the review yet?

So $\frac{dP}{dt} = .75\left(P - \frac{1}{400}P^2\right)$

where did we get that from?

But can I find isoclines

where is slope $> <$ 0

Where $= 0$?

$0 = .75\left(P - \frac{1}{400}P^2\right)$

$0 = P - \frac{1}{400}P^2$

$P = \frac{1}{400}P^2$

So where does $P =$ curve ~~both~~ sides to $=$

$400P = P^2$

(26)

At $P=0$ $\quad 0=0$ ✓

$P=400$ $\quad 400^2=400^2$ ✓

Now pts in between

$P=500$

$$.75\left(500-\frac{2500}{400}\right)$$

$$.75(500-6)$$

$$370 \quad \oplus$$

$P=200$ Should be diff (!)

$$.75\left(200-\frac{4000}{200}\right)$$

$$\oplus$$

$P =$ ~~4000~~ 1

$$.75\left(1-\frac{1^2}{1}\right)$$

$$0 \quad (!)$$

$P=-1$ (? am I doing This wrong)

(27)

Suppose harvest r turtles

$$\frac{dP}{dt} = .75\left(P - \frac{1}{400}P^2\right) - r$$

Then find Biforcation Pt

¿How we do this again

Need to find where r causes it to $=0$

$$.75\left(P - \frac{1}{400}P^2\right) - r = 0 \quad \checkmark$$

$$P = \frac{-.75 \pm \sqrt{.75^2 - 4r\left(\frac{.75}{400}\right)}}{\frac{.75}{400}}$$

So find P where this is true

Now need

$$.75^2 = 4r\,\frac{.75}{400}$$

then find r

$$.75 = \frac{r}{100}$$

$$\boxed{r = 75}$$

Should be simple

∟ should actually try – when figure out limit thing

(28)

L5 $\int$

Linear Equations

$$\frac{dy}{dx} + P(x)\,y = Q(x)$$

$\int$ both sides w/r to x would be too hard here

Instead Multiply by fn(x) on both sides Then $\int$ w/r to x

$$\boxed{fn(x) = e^{\int P(x)\,dx}}$$

Urg I don't like this method

So

LHS $\frac{dy}{dx} \cdot e^{\int p(x)dx} + P(x)\,e^{\int p(x)\,dy} \cdot y$

is $\frac{d}{dx}$ of prod of 2 functions

$$\frac{d}{dx}\left(y \cdot e^{\int P(x)\,dx}\right)$$

$\downarrow$ using product rule
int w/ r.t. x

$$y \cdot e^{\int p(x)\,dx} + C$$

RHS

$$Q(x) \cdot e^{\int P(x)\,dx}$$

$\downarrow$ int w/ r.t. x

$$\int Q(x)\, e^{\int P(x)\,dx}$$

So Now example

$$\frac{dx}{dx} - 2xy = x$$

1. Identify P, Q

$P(x) = -2x$ $\qquad$ $Q(x) = x$

$\int P(x)\,dx = -x^2$ ← actually integrate

30

2. Now multiple everything by $e^{\int P(x)\,dx} = \underbrace{e^{-x^2}}_{\text{here}}$

$$\underbrace{\frac{dx}{dx}\underbrace{e^{-x^2}} - 2x\underbrace{e^{-x^2}}\cdot y} = x\underbrace{e^{-x^2}}$$

3. Apply derivitive ↳ product rule.

$$\frac{d}{dx}\left(y\cdot e^{-x^2}\right)$$; So by product rule, it all dissappears

$$\underbrace{ye^{-x^2}}_{\text{always } ye^{\int P(x)dx}\ ;} = \int x\, e^{-x^2}\,dx$$

; but where did $-2x$ go?

4. Do algebra for RHS

↳ Yes ingegral may be harder but other way harder

Try ; split like deriv

$$\int 1e^{-x^2} + -x^2 x e^{-x^2}$$

No! ~~int by ...~~ int by substitution

$$\int \frac{dy}{du}\cdot\frac{du}{dx}\,dx = \int \frac{dy}{du}\,du = y$$

31

I don't think thats what I am looking for...

Try WA

$U = -x^2$

$du = -2x$

$= -\frac{1}{2}\int e^u \, du$

$\boxed{\int e^u = e^u}$

$= -\frac{e^u}{2}$

$= -\frac{e^{-x^2}}{2}$ was by substitution

But answer not ~~ex~~ clear

5. Then divide

$$\boxed{y = \frac{-\frac{e^{x^2}}{2}}{e^{-x^2}} = -\frac{\cancel{e^{x^2}}}{2\cancel{e^{-x^2}}} = -\frac{1}{2}}$$

∴ answer not clearly given

(32)

~~Ther~~

Then solve for C when need it

¿ Then one more way?

Is this in textbook anywhere?

└ Chap 1.5

$$\frac{d}{dx}\left[\int P(x)\,dx\right] = P(x)$$

fund theorm of calc

$$\frac{d}{dx}\left[y(x)\cdot e^{\int P(x)\,dx}\right] = Q(x)\,e^{\int P(x)\,dx}$$

So integrating both sides

$$y(x)\,e^{\int P(x)\,dx} = \int \left(Q(x)\,e^{\int P(x)\,dx}\right)dx + C$$

$$y = e^{\overset{\text{+ since flipped}}{-\int P(x)\,dx}}\left[\int Q(x)\,e^{\int P(x)\,dx} + C\right]$$

(33)

Book claims you should not memorize formula

Instead

1. Find $e^{\int P(x)\,dx} = p(x)$
2. Multiply all by ?
3. Next recognize LHS of eq as deriv of product

$$D_x\left[p(x)\,y(x)\right] = p(x)\,Q(x)$$

4. Integrate

$$p(x)\,y(x) = \int p(x)\,Q(x)\,dx + C$$

5. Solve for y
6. (opt) Solve for value of C

So basically just $\int Q(x)\,p(x)\,dx$

Book example

$$e^{-x}y = \int \frac{11}{8} e^{-4x/3}\,dx$$

integrate

$$= \frac{11}{8}\cdot\frac{-4}{3}\,e^{-4x/3} + C$$

(34)

$$= -\frac{44}{24} e^{-4x/3} + C$$

$$= -\frac{22}{12} e^{-4x/3} + C$$

⊗ Not integrating right

Urg

$$\int e^{-\frac{4x}{3}} = -\frac{3}{4} e^{-4x/3}$$

Since

$$\int e^{-x} = -e^{-x}$$

$$\int e^{-2x} = -\frac{1}{2} e^{-x}$$

$$\frac{11}{8} \cdot -\frac{3}{4} e^{-4x/3}$$

$$-\frac{33}{34} e^{-4x/3} + C$$

But now divide by e^{-x}

↳ got e^{x}

$$y(x) = Ce^{x} - \frac{33}{32} e^{-x/3}$$ ← so the 4/3 got turned into $\frac{3}{3}$ somewhere?

38

I should do some pratice on that some time...

Input Response Models

$$\frac{dy}{dt} + ky = Q(t)$$

$y(0) = y_0$
k constant, > 0

Like linear eq but $P(t) = k$
 └ So integrating factor $\to e^{kt}$

General Sol

$$y(t) = \underbrace{e^{-kt}\int_0^t Q(s)\,e^{ks}\,ds}_{\text{steady state/ long term sol}} + \underbrace{y_0 e^{-kt}}_{\text{transient}}$$

Like for singnal processing
$Q(t)$ = the input signal
$Y(t)$ = the response

(still need to review word problems)

36

Complex #s

$\boxed{i^2 = -1}$

Cartesean coords $z = a + bi$

add + subtract component wise

multiplication $(a+bi)(c+di)$

$= (ac-bd) + (bc+ad)i$

division $\dfrac{a+bi}{c+di}$

rationalize denom

$= \dfrac{a+bi}{c+di} \cdot \dfrac{c-di}{c-di}$

$= \dfrac{(ac+bd)+(bc-ad)i}{c^2+d^2}$

$Re(a+bi) = a$

$Im(a+bi) = b$

differentate component wise

integrate "

37

Practice $Re\left(\frac{1}{2+5i}\right)$

Need to rationalize denom

$$= Re\left(\frac{1}{2+5i} \cdot \frac{2-5i}{2-5i}\right)$$

$$= Re\left(\frac{2-5i}{4+10i-10i-25(-1)}\right)$$

$$= Re\left(\frac{2-5i}{29}\right)$$

don't forget to actually ans qu

$$= \frac{2}{29}$$ ✓

$$Im() = \frac{5}{29}$$

$$z = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

I always forget this!

$$Re() = r\cos\theta$$

$$Im() = r\sin\theta$$

38

$r = x^2 + y^2$ if $z = x + iy$

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

If set $\theta = \pi$

$e^{i\pi} = -1 + 0$

$\boxed{e^{i\pi} + 1 = 0}$ ← famous identity

(skipping over stuff I prob shouldn't...)

(39)

Oh so have IR diff eq

$$\frac{dy}{dt} + y = \sin 3t$$

A week ago would IF

$$e^{\int P(t)\,dt} \quad \text{here } e^{\int 1\,dt} = e^t$$

$$\frac{d}{dt} e^t y = e^t \sin 3t = \text{Im}\left(e^{(1+3i)t}\right)$$

integrate both sides w/r to t

But now we can complexify

i do we need old model at all

Or only for cos/sin periodic functions?
new model good

$\tilde{y}$ = complex variable

Solve $\frac{d}{dt}\left(e^t \tilde{y}\right) = e^{(1+3i)t}$

$$\xleftarrow{\text{Im}}$$

$$\frac{d}{dt}\left(e^t y\right) = e^t \sin 3t$$

4c)

or Re

$$\frac{d}{dt}\left(e^{t}x\right) = e^{t}\cos 3t$$

2nd one is easy to ∞ w/r. to t

$e^{t}\tilde{y}$

I have no idea what doing here
- go slow + understand

Last step $\int$ both sides

$$e^{t}\tilde{y} = \frac{1}{1+3i}\, e^{(1+3i)t} + C$$

$$e^{t}y = Im(\qquad) + \underbrace{Im(c)}_{\text{solve for } C \text{ if IV problem}}$$

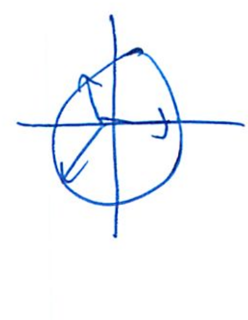
Going Back

Polar Qc) Find all sols to

$z^3 = 1$

$\hookrightarrow 1,\ e^{\frac{2\pi i}{3}},\ e^{\frac{4\pi i}{3}}$

$$e^{\frac{2\pi i}{3}} = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$$



40

Since sols $z^n = 1$

are $z = e^{\frac{2\pi i}{n}}, e^{\frac{2\pi i}{n}\cdot 2}, \cdots, e^{\left(\frac{2\pi i}{n}\right)\cdot n}, 1$

~~differentiate~~, ~~intergrate~~

like $\mathrm{Im}\left(e^{3xi}\right)$

$$= \mathrm{Im}\left(\overset{\cos}{\cancel{\sin}}\, 3x + i\, \overset{\sin}{\cancel{\cos}}\, 3x\right)$$

$$= \overset{\sin}{\cancel{\cos}}\, 3x$$

Fudge should know

$$\mathrm{Im}\left(e^{x(1+3i)}\right) = \mathrm{Im}\left(e^{x}\, e^{3ix}\right)$$

like before

$$= \mathrm{Im}\left(e^{x}\left(\cos 3x + i \sin 3x\right)\right)$$

$$= e^{x} \sin 3x$$

Now solving a diff eq (LTI w/ sinusoidal input)

L that $\frac{dy}{dt} + y = \sin 3t$ from p 39

Now try to understand

(12)

Ok can convert

So

$$\frac{d}{dt} e^t \cdot y = e^t \sin 3t$$

Now $$\mathrm{Im}\left(\frac{d}{dt}(e^t \cdot y)\right) = \mathrm{Im}\left(e^{(3i+1)t}\right)$$

since ~~Since~~ ~~$\mathrm{Im}\ e^t \sin 3t =$~~

$$\sin 3t = \mathrm{Im}\left(e^{3ti}\right)$$

✓

Cause

$$e^{3ti} = \cos 3t + i \sin 3t$$

So

$$e^t \sin 3t = \mathrm{Im}\left(e^{3ti + t}\right)$$

$$= \mathrm{Im}\left(e^{(1+3i)t}\right)$$

✓

Ohh I see this more clearly now

¿ So now we plug that in or something?

(43)

But what to we do next?

What ~~did~~ are/did they do w/ z?

$$Im(z) = y$$

Ohh I think they mean $z = \tilde{y}$

$$Im(\tilde{y}) = y$$

↑ yeah think that is true

So now basically S w/ $\tilde{y}$

$$\frac{d}{dt}\left(e^{t} \cdot \tilde{y}\right) =$$

$$= e^{(3i+1)t}$$

So I still don't understand how we got from here to there - that linear method we did before?

Then take Im parts

$$\tilde{y} = \frac{1}{1+3i} e^{3it} + C e^{-t}$$

↑ complex

So at begining multiplied both sides w/ $e^{t} = e^{\int P(t)\,dt}$

Then recognized $D_x\left[p(x)\, y(x)\right] = p(x)\, Q(x)$

↑ I still don't get this "recognized"

44

; So then we integrated other side ;

$$p(x)\cdot y(x) = \int p(x)\, Q(x)\, dx$$

Ahhh yes I belive we did that next

$$e^{t}\tilde{y} = \frac{1}{1+3i} e^{(1+3i)t} + C$$

now divide e^{t}

$$\tilde{y} = \frac{1}{1+3i} e^{(1+3i)t} + C + e^{-t}$$

So

$$= \frac{1}{1+3i} e^{3it} + Ce^{-t}$$

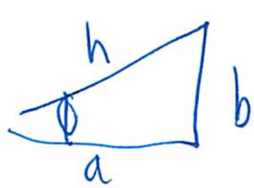
ok close

Thats as far as lecture notes went

But notes go further w/ ϕ

Ok I think that is just converting forms

$$a\cos\omega t + b\sin\omega t = h\cos(\omega t - \phi)$$



(45)

So $1+3i = \sqrt{10}\, e^{i\phi}$

$\angle \phi = \tan^{-1}(3) \approx \frac{\pi}{4}$

$(1+3i)^{-1} = \frac{1}{10} e^{-i\theta}$

$$\tilde{y} = \underbrace{\frac{1}{\sqrt{10}} e^{i(3t-\phi)}}_{\text{steady state}} + \underbrace{C e^{-t}}_{\text{transient}}$$

Now convert back

Take $\text{Im}(\tilde{y})$ to get y

$$y = \frac{1}{\sqrt{10}} \sin(3t-\phi) + Ce^{-t}$$

So what's all this converting

Lwas not mentioned in lecture

But I think I understand complexification better now

But still not enough to do it

(46)

Practice

roots of $x^2 + 3x + 5$

So quadratic formula

$$\frac{-3 \pm \sqrt{9 - 4 \cdot 1 \cdot 5}}{2}$$

$$\frac{-3 \pm \sqrt{-11}}{2}$$

$$-\frac{3}{2} \pm \frac{\sqrt{11}}{2} i$$

2 complex sols

OH example of the complex thing

(where I was explained everything I just refigured out

(47)

pratice integrating

$$\int e^{t+it}$$

Chain rule

$$\frac{d}{dt} e^{ct} \qquad f = e^x \quad f' = e^x$$
$$g = ct \quad g' = c$$

$$= c e^{ct}$$

So by FTC

$$\int \frac{d}{dt} e^{ct} = e^{ct}$$

$$\int c e^{ct}$$

$$c \int e^{ct}$$

$$\cancel{c} \frac{e^{ct}}{\cancel{c}}$$

$$= \operatorname{Im}\left(\frac{1}{1+i} e^{t+it}\right)$$ oh I just know the shortcut now

So basically looking over I understand it all now - but can I do w/o looking?

48

Unit 2 Higher order diff eqs

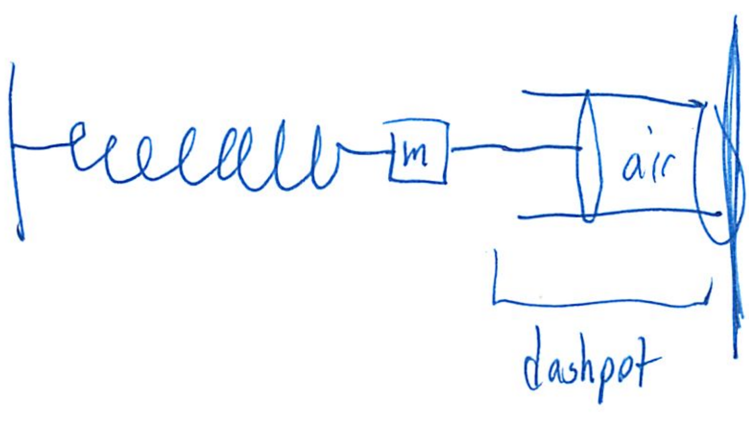
2nd order ODEs - highest order appearing = 2

linear case for all of unit 2

$$A(x)\, y'' + B(x)\, y' + C(x)\, y = F(x)$$

just fns of x

Example Spring - Mass - Dashpot



$F = ma$

$my'' = ma$

$y(t)$ = displacement (from rest pos) of spring

49

$F_{total} = F_{spring} + F_{dashpot} + F_{external}$

$F_{spring} = -ky$ ← depends on pos

$F_{dashpot} = -cy'$

← depends on the acc

$F_{external}$ = black box (will be given)

$$my'' + cy' + ky = F_{ext}(t)$$

↑ = 0 for 1st 2 lectures

← $F_{ext} = 0$
dashpot enabled

So of form

$$my'' + cy' + ky = 0$$

$$y'' + \frac{c}{m}y' + \frac{k}{m}y = 0$$

← if $m = 0$ bad → singularities
don't think we covered

Principle of Superposition

If $y_1(x), y_2(x)$ a sol

then $y_1(y) + y_2(x)$ is also a sol

Is this the first time we had more than 1 candidate sol to add together - no did earlier w/ liear

Except if

$$y_1(x) = \underset{\text{constant}}{k}\ y_2(x)$$

the dreaded repeated roots!

"linearly dependent"

(This stuff all makes more sense now!)

To test for linear dep → Wronskian

(can't you just tell when you do it?

$$\det\begin{pmatrix} f & g \\ f' & g' \end{pmatrix} = fg' - gf'$$

$= 0 \rightarrow$ lin dep (repeated roots)

$\neq 0 \rightarrow$ lin ind (we're good)

(56)

So this is the solving eqn method

when have IV problem only, right?

~~So before had~~ ~~x4~~

(know this stuff, skipping)

2 bad cases: repeated roots, complex

Complex (try myself)

$$y'' + 4y' + 8y = 0$$

$$(r^2 + 4r + 8)\, y = 0$$

$$\frac{-4 \pm \sqrt{16 - 4 \cdot 1 \cdot 8}}{2}$$

$$\frac{-4 \pm \sqrt{-16}}{2}$$

$$\frac{-4 \pm 4i}{2}$$

$$-2 \pm 2i \checkmark$$

So $A e^{(-2+2i)} + B e^{(-2-2i)}$

(52)

Or is it more complicated?

Oh right can take re() + Im() parts

$$Re(e^{(-2 \pm 2i)t})$$

$$= Re(e^{-2t} e^{2it}$$

$$= Re(e^{-2t}(\cos 2t + i \sin 2t))$$

$$= e^{-2t} \cos 2t$$

what I recently figured out

now that I figured this stuff out need to do!

Im()

$$= e^{-2t} \sin 2t$$

Then add those!

$$A e^{-2t} \cos 2t + B e^{-2t} \sin 2t$$

Better answer - for some reason...

If have IVs

$$y(0) = 1$$

$$y'(0) = 0$$

$$y(0) = A e^{-2 \cdot 0} \cos(0) + C_2 e^{-2 \cdot 0} \sin(0)$$

$$1 = A \cdot 1 \cdot 1$$

I know what these values are...

53

$A = 1$

$y'(0) =$ annoying

└ lots of product + chain rule

Shortcut Taylor Series

└ whats that again

wp: Add up lots of derivitives (a sum)

at a point - get close to function

Have some linear approximations

$e^x \approx 1 + x$ near 0

$\sin x \approx x$ "

$\cos x \approx 1$ "

So

$y'(t) \approx (1-2t) \cdot 1 \, c_1 + (1-2t)(2t) \, c_2$

$y'(0) =$ coeffs of t

└ ie linear t

$= -2c_1 + 2c_2 \rightarrow c_1 = 1$ we knew

$c_2 = 1 \leftarrow$ get

Ok, I see, but need practice w/

(54)

Repeated Roots Special Case

$$y'' - 6y' + 9y = 0$$

$$(y-3)^2_y = 0$$

$$Ae^{3t} + B\underline{t}e^{3t}$$

So why can we do that?

└ Something complicated about guessing I don't think we need to know

Higher Order ~~ODEs~~ ODEs (w/ constant coeffs)

└ Same stuff works ~~...~~ - just more of it...

L11 Linear Operators

Operator: functions → functions

$$D = \frac{d}{dx}$$

↑ Shortcut notation

Notation Fun: $D(y(x)) = \frac{d}{dx}(y(x)) = y'(x) = D(y) = y'$

(53)

$$y'' + 6y' + 17y = 0$$

$\downarrow$

$$D^2(y) + 6D(y) + 17y = 0$$

$$D^2 = \frac{d}{dx}\left(\frac{d}{dx}\left(y(x)\right)\right)$$

Combos (I don't really notice anything different here...)

$$(p(D) + q(D))y = p(D)y + q(D)y$$

$$p(D)(c_1 y_1 + c_2 y_2) = c_1 p(D) y_1 + c_2 p(D) y_2$$

$$(p(D)\, q(D))(y) = p(D)(q(D)y)$$

So if $p(D) = D+1$ $\quad q(D) = 7D$

$$p(D)q(D) = 7D^2 + 7D$$

So $(7D^2 + 7D)(y) = (D+1)(7D(y))$

$$7D^2(y) + 7D(y)$$

don't know if this notation is legal

me making stuff up

56

↙ I get this now!

So instead of having to do prod rule several times w/

$$\frac{d^4}{dx^4}\left(e^{-x}x^3\right) = D^4\left(e^{-x}x^3\right)$$

expand out

$$e^{-x}(D-1)^4\left(x^3\right)$$

↖ So thats the "pass over thing"

Then binomial expansion

$$e^{-x}\left(D^4-4D^3+6D^2-4D+1\right)X^3$$

Much easier

So back to ODEs

$$(D-r)^k y = 0 \quad \leftarrow k \text{ pos int } k>0$$

↖ if root is repeated many times

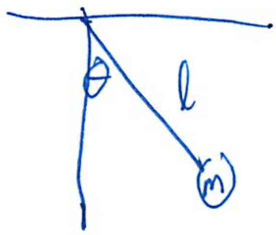
So if $\underline{k=2}$

$$C_1e^{rt}+C_2te^{rt}$$

(This does not seem very ground breaking to me...)

57

More modeling w/ ~~spring~~ ~~mass~~ ~~dashpot~~ pendulum



What is $\theta(t)$?

$$ma = m\frac{d^2s}{dt^2}$$

$$= ml\frac{d^2\theta}{dt^2}$$

$$F = F_{grav} = -mg\sin\theta$$

$$\frac{ml\,d^2\theta}{dt^2} = \underbrace{-mg\sin\theta}_{\text{external force}}$$

Cancel + clean

$$\frac{d^2\theta}{dt^2} + \underbrace{\frac{g}{l}}_{\text{constant}}\sin\theta = 0$$

↑ not linear - can't solve
So say $\sin\theta \approx \theta$ near 0
(linear approx)

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0$$

58

Then when solve

$$r^2 + \frac{g}{l} = 0$$

$$r = \pm i\sqrt{\frac{g}{l}} \quad \leftarrow \text{complex}$$

$$e^{i\sqrt{g/l}\,t}$$

$$\text{Re}(\,) = \cos\sqrt{\frac{g}{l}}\,t$$

$$\text{Im}(\,) = \sin\sqrt{\frac{g}{l}}\,t$$

General sol

$$C_1 \cos\sqrt{\frac{g}{l}}\,t + C_2 \sin\sqrt{\frac{g}{l}}\,t$$

$$= C\cos\left(\sqrt{\frac{g}{l}}\,t - \alpha\right)$$

Add Friction

$$\frac{d^2\theta}{dt^2} + \underbrace{c\frac{d\theta}{dt}}_{\text{based on speed}} + \frac{g}{l}\sin\theta = 0$$

$\uparrow$ rename $k = \frac{g}{l}$

Depending on c, can get real or imag roots

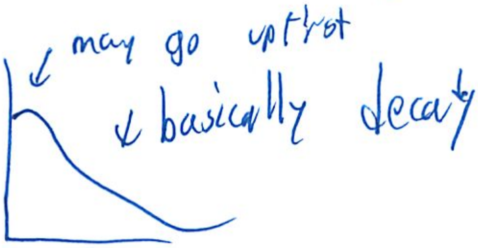
$c^2 > 4k \rightarrow$ dis $\oplus$ $\rightarrow$ real rts

$c^2 < 4k \rightarrow$ dis $\ominus$ $\rightarrow$ imag rts

$c = 0 \rightarrow$ purly imag roots

(59)

So when friction is big



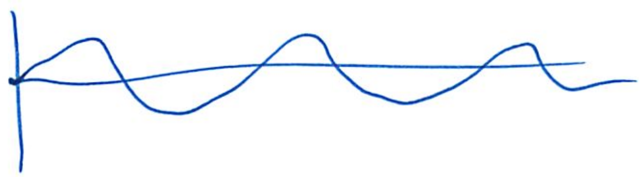
$c_1 e^{-r_1 t} + c_2 e^{-r_2 t}$

If imag rts → will ossilate

$e^{-at}(c_1 \cos bt + c_2 \sin bt)$

$c=0$

Ossilates forever



L12 | Method of Undetermined Coeffs

aka: lucky guess method

Today: non homogeneous

$$p(D)\circ y = \underbrace{F(x)}_{\text{new}}$$

Need to review this stuff

(60)

$p(D)\,y = F(x)$

Integrate both sides w/ c to x

$ay' + ky = F(x)$

Solve w/ complitication + integrating factor

<u>Ie</u> $(D-1)(D-2)\,y = e^{5x}$

guess $y = Ce^{5x}$

Then plug into ODE to check that it works

But need something more systematic

~~Need~~ ∗ → Need operator in D that kill RHS

$(D-5)$ here

Apply to both sides

$(D-5)(D-1)(D-2)\,y = (D-5)e^{5x}$

$= 0$

Now solve homogeneous eqn

$y = C_1 e^{5x} + C_2 e^{x} + C_3 e^{2x}$

How does $(D-5)$ eliminate e^{5x} again?
L What does it mean exactly?

So means $\frac{d}{dx} e^{5x} - 5e^{5x}$

oh ah $5e^{5x} - 5e^{5x} = 0$

Remember
$$\frac{d}{dx} e^{5x} = 5e^{5x}$$
$$\int e^{5x} = \frac{1}{5} e^{5x}$$

So we have our homogeneous sol and our particular sol
L for our nonhomogenous = something sols

$$Y = \underbrace{Y_h}_{\text{homogenous}} + \underbrace{Y_p}_{\text{particular}}$$

Then can find c_1 if want IVP

$$(D-1)(D-2)\, c_1 e^{5x} =$$
$$= c_1 (5-1)(5-2) e^{5x}$$

put const in front
Take deriv + subtract fn

62

ie $c_1 P(D) e^{5x} = c_1 P(5) e^{5x} = c_1 \cdot 12 e^{5x}$

where $P(D) = (D-1)(D-2)$

Solve $c_1 \cdot 12 e^{5x} = e^{5x}$

$c_1 = \frac{1}{12}$

$$\boxed{\text{So particular sol to } p(D)\, y = e^{ax} \qquad y = \frac{1}{p(a)} e^{ax}}$$

So this is a common pattern that shows up regularly

But $p(a) \neq 0$

L have to do something harder

$(D-5)^2 (D-1)\, y = 0$

$((c_1 + c_2 x)\, e^{5x} + c_3 e^x$

when in doubt, add!

Solve for c_2

$y = \dfrac{x^s \cdot e^{ax}}{p^{(s)}(a)}$

sth deriv of p at a

e3

Can also use Complex

└ like for $(D-1)(D-2)\,y = e^{5x}\sin x$

$$(D-1)(D-2)\,y = \mathrm{Im}\left(e^{(5+i)x}\right)$$

$$(D-1)(D-2)\,\tilde{y} = e^{(5+i)x}$$

$$\tilde{y} = \frac{1}{(4+i)(3+i)}\, e^{(5+i)x}$$

$$y = \mathrm{Im}(\text{above})$$

└ Do polar conversion + multiply it out

I guess I should try that now

$$\frac{1}{12+4i+3i+-1}\, e^{5}(\cos x + i\sin x)$$

$$\frac{1}{\cancel{12\ldots}\ 11+7i}\, e^{5}(\cos x + i\sin x)$$

?

$$\frac{1}{11+7i}\,\frac{11-7i}{11-7i} = \frac{11-7i}{121-49(-1)} = \frac{11-7i}{170}$$

Im part

$$-\frac{7}{170}\, e^{5}\sin x$$ — un checkable

$(D^2+1)\sin x = 0$

Basically we are taking derivs of RHS and tring to cancle it at ever break

---

Example

$(D-1)(D-2)y = x^3+4x+7$

$\underbrace{D^4}_{\text{cancles RHS}}(D-1)(D-2)y = 0$

$y = \underbrace{C_0 + C_1x + C_2x^2 + C_3x^3}_{\text{Specific}} + \underbrace{C_4e^x + C_5e^{2x}}_{\text{homogeneous}}$

I think they mean

$C_0 e^x \leftarrow$ oh $e^{0x} = 1$

right? I believe so

(65)

hardest part i factoring

Modeling

pendilum w/ external force

$$P(D)\,y = \underbrace{F(x)}_{\text{external black box}}$$

Same stuff as before...

Exponential Shift Review

$$(D-r)\left(e^{rx}u(x)\right) = e^{rx}D(u(x))$$

isn't that start
Should write 1st

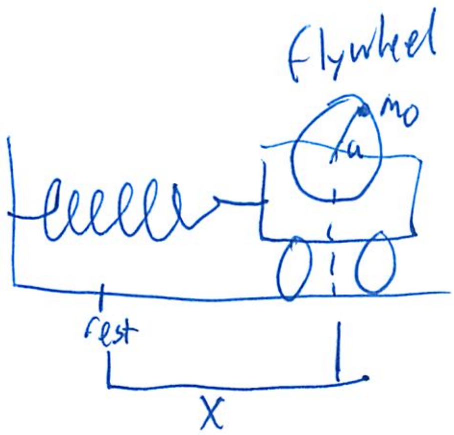
So basically move over e^{rx} → subtract r

$$e^{rx}D(\ ) \to (D-r)\,e^{rx}$$

What to do for $P(D)\,y = \tan x$

└ Org notes say do later

66



ω = angle of flywheel

mass of cart $m - m_0$

$$\bar{X} = \frac{(m-m_0)x + m_0(x + a\cos\omega t)}{m}$$

← Luckily this isn't Physics so shouldn't have to find

? means x average or center of cart

Or is it $\boxed{\text{COM}}$

but remember COM moves around!

↓ reduce

$$\bar{X} = x + \frac{m_0}{m} a\cos\omega t$$

$$\bar{X}'' = x'' - \frac{m_0}{m} a\omega^2\cos\omega t$$ take 2nd deriv

$$mx'' + kx = \underbrace{m_0 a\omega^2\cos\omega t}$$

RHS = some interesting

$= F_0\cos\omega t$ external force reduced to a constant

(67)

General sol

$$x(t) = c_1 \cos \underbrace{\sqrt{\tfrac{k}{m}}}_{\eta} t + c_2 \sin \sqrt{\tfrac{k}{m}}\, t + x_p(t)$$

Guess

$A\cos\omega t + B\sin\omega t$

$\omega \neq \eta$

Substitute in (Have we seen this before?)

$$-mA\omega^2 \cos\omega t + kA\cos\omega t = F_0 \cos\omega t$$

$$(-m\omega^2 + k)A = F_0$$

$$A = \frac{F_0}{k - m\omega^2} = \frac{F_0/m}{\eta^2 - \omega^2}$$

; So then is a sol given

I don't ~~th~~ something this complicated will be on the final

We didn't even do it in class - ran out of time

(68)

L14

Ah here problem is finished

Oh all that odd/even business ...

Yeah lecture skipped solving ← Yeah class in algebra

$A \cos \omega t$

$\hookrightarrow A = \dfrac{F_0/m}{\eta^2 - \omega^2}$

$\eta = \sqrt{\frac{k}{m}}$

Amplitude gain = $\dfrac{\text{output amplitude}}{\text{input amplitude}}$

here $= \dfrac{\frac{F_0/m}{\eta^2-\omega^2}}{F_0/m}$

$= \dfrac{1}{|\eta^2 - \omega^2|}$

↑ parameter

(64)

As $\omega \to \eta$ gain $\to \infty$

No Friction here

↳ if there was some would be exponential damping

↳ reduces amplitude gain

(I think we are getting close to resonance time)

If $\underline{\eta = \omega}$

$$x'' + \eta^2 x = \frac{F_0}{m} \cos \eta t$$

So guess
$$x_p(t) = t\,(A \cos \eta t + B \sin \eta t)$$

But better guess

even ∘ odd = odd
odd ∘ odd = even

$$x_p(t) = t\,(B \sin \eta t)$$

Then solve for B

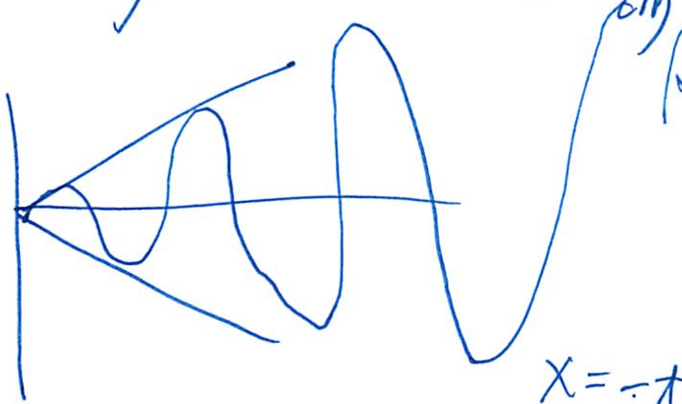
$$B = \frac{F_0}{2m\eta}$$

Never did much solving of eq'ns

20

Add the IV

$$= \frac{t}{2m}$$

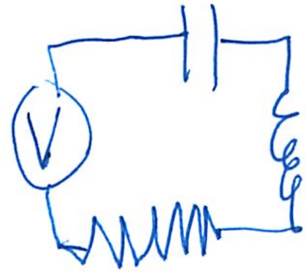


RLC circuits

R = resistor ohms

L - inductor henries

C = capicator farads



$$\frac{dQ}{dt} = I(t)$$

Kirchoff's Law

└ voltage drop over each piece

$$\sum V(t) = 0$$

$$V(t) = L\frac{dI}{dt} + IR + \frac{Q}{C}$$

involves both current + charge

charge
$$V(t) = LQ'' + RQ' + \frac{1}{C}Q$$

current
$$V'(t) = LQ''' + RQ'' + \frac{1}{C}Q'$$
$$= LI'' + RI' + \frac{1}{C}I \leftarrow \text{what we will use}$$

Looks like spring mass dashpot

└ but don't forget $\frac{1}{C}$

(should pratice sometime)

$$\omega_0 = \sqrt{\frac{1}{LC}}$$

72

[15] We can solve lin inhomog - now

Can add modeling
- 2nd order eq
- Spring
- RLC circuits

Review ~~sar~~

then Power Series Method ?
L is that Unit 2 ? - is unit 3

Only a few choices for $F(t)$
- polynomials in t
- $\cos / \sin(\omega t)$
- e^{rt}
- combos of ; $+$ $-$ $\times$

Table in book as what to guess
L could not find it

Practice the long example at some pt
This has one of the first solving systems examples

73

12/18

Long example from class

$R = 20\,\Omega$

$L = 10$ H

$C = .02$ F

$V = 85 \sin 2t$ V alternating voltage

Find general sol for current

$$LI'' + RI' + \frac{1}{c} I = V'(t)$$

↑ note V'

$+170 \cos 2t$

ugh got myself confused early on when I was doing ∫ of – this is $\frac{d}{dx}$

$$\overset{10}{\cancel{20}} I'' + \overset{20}{\cancel{10}} I' + 50 I = 170 \cos 2t$$

Reduce

it was not in order

$$\cancel{10}\cancel{2} I'' + \overset{2}{\cancel{10}} I' + 5 I = 17 \cos 2t$$

So general sol

homogeneous 1st

(74)

$$\cancel{2I'' + 10I' + 5I = 0}$$

$$\cancel{(2s^2 + 10s + 5)}$$

$$\cancel{\frac{-10 \pm \sqrt{100 - 4 \cdot 2 \cdot 5}}{4}}$$

$$\cancel{\frac{-10 \pm \sqrt{60}}{4}}$$

Copyed wrong prev pg

$$I'' + 2I' + 5I = 17\cos 2t$$

$$(1r^2 + 2r + 5)$$

$$\frac{-2 \pm \sqrt{4 - 4 \cdot 1 \cdot 5}}{2}$$

$$-1 \pm 2i \checkmark$$

So we write it in that other form

$$e^{(-1 \pm 2i)t}$$

$$= e^{-1t} e^{2it}$$

$$= e^{-t} (\cos 2t + i\sin 2t)$$

75

$A e^{-t} \cos 2t + B e^{-t} \sin 2t$ ✓

2nd order ODE should have 2 params

Now the Non-homogeneous part

have not tried in a while

Need to find a D for $17 \cos 2t$

$\underline{D} \mid -34 \sin 2t$

$\underline{D^2} \mid -68 \cos 2t$

So $D^2 + 4$

$-68 \cos 2t + 68 \cos 2t$

Can we just say?

~~$C e^{4t} + D t e^{4t}$~~

~~So2 They 2 did~~ wait (D^2+4) don't think you can just write it like that

roots

$\pm 2i$

$e^{\pm 2it} = \cos 2t \pm i \sin 2t$

$C \cos 2t + D \sin 2t$ — no und. constants

but need to fill in

have not done that solving yet

(26)

So they did it a very different way

They guessed $I_p(t) = A\cos 2t + B\sin 2t$

$\llcorner$ ok so that is what I had
still need to do what I did
but how we need this

Take derivitives

$I'_p(t) = 2A -\sin 2t + 2B\cos 2t$

$I''_p(t) = -4A\cos 2t - 4B\sin 2t$

Plug in (I remember doing this now)

$1(-4A\cos 2t - 4B\sin 2t) + 2(-2A\sin 2t + 2B\cos 2t)$
$+ 5(A\cos 2t + B\sin 2t) = 17\cos 2t$

Then match coeffs   Let me try to figure out on my own

$-4A\cos 2t - 4B\sin 2t$
$-4A\sin 2t + 4B\cos 2t$
$+ 5A\cos 2t + 5B\sin 2t = 17\cos 2t$

Add up all the coeffts cos and sin

(77)

$\underline{\cos}$ $-4A + 4B + 5A = 17$

$\underline{\sin}$ $-4B - 4A + 5B = 0$

Now ~~th~~ try to solve system

$$A + 4B = 17$$

$$B - 4A = 0$$

$$A = 17 - 4B$$

$$B - 4(17 - 4B) = 0$$

$$B - 68 + 16B = 0$$

$$17B = 68$$

$$B = 4$$

$$A + 4(4) = 17$$

$$A = 1$$

Then write answer

$$\underbrace{1\cos 2A + 4\sin 2t}_{y_p} + \underbrace{Ae^{-t}\cos 2t + Be^{-t}\sin 2t}_{y_h}$$

Next calc find IV

or find A, ϕ

78

Let me look up this A, ϕ thing

└ I remember asking in OH

So we have a trig identity

$$\cos(x-y) = \cos(x)\cos(y) + \sin(x)\sin(y)$$

use this trig identity

So we want this

$$A\cos(\omega t - \phi)$$

amp, freq, variable, phase shift

get

$$A\cos(\omega t)\cos(+\phi) + A\sin(\omega t)\sin(\phi)$$

combine

$$C\cos(\omega t) + D\sin(\omega t)$$

Comes from $Re, Im(e^{i\omega t})$

$C = A\cos\phi$

$D = A\sin\phi$

$\sqrt{C^2+D^2} = A$

$\phi = \tan^{-1}\left(\frac{D}{C}\right)$

Ok so what does this mean? (see above)

Now just need to be able to do

(79)

$I(0) = 0$
$I'(0) = 0$

So now try IV
↳ no answers though

p 104 textbook

What do we do again ↳ set the values

Just for homogeneous I think ← no ans on this Ohio State website; analogous to homogeneous state but that is still unclear

$0 = Ae^{-0} \cos 2(0) + Be^{-0} \sin 2(0)$ ← set $t=0$ (inside parenteses value

$0 = A \cos(0) + B \sin(0)$

$0 = A \cdot 1 + B \cdot 0$

$0 = A$

Take deriv of answer

$y' = \mp Ae^{-t} \sin 2t + -Be^{-t} \cos 2t$

$0 = Ae^{-(0)} \sin 2(0) + -Be^{-0} \cos 2(0)$

$0 = A \sin 0 - B \cos(0)$

$0 = -B$

$A, B = 0$

I think this makes sense

(80)

So the A, ϕ thing is to allow a pure cosine wave

So notes did $I(0) = Q(0) = 0$

$\uparrow I'(0)$

gets

$$c_1 + 1 = 0$$
$$c_1 = -1$$

hmm skips some steps

$$\left(e^{-t} \cos 2t\right)' \Big|_{t=0} = -e^{-t} \cos 2t = -1$$

why not here?

$$\left(e^{-t} \sin 2t'\right) \Big|_{t=0} = e^{-t} \cdot 2\cos 2t = 2$$

basically what I found

$$-c_1 + 2c_2 + \underbrace{8}_{\text{where did this come from}} = 0$$

General sol

$$\sqrt{17} \cos\left(2t - \tan^{-1}(4)\right) + \underbrace{c_1 e^{-t} \cos 2t}$$
$$\underbrace{+ c_2 e^{-t} \sin 2t}$$

goes to 0 as $t \to \infty$
so doesn't really matter

86

But that does not really answer what they did

└ Ask in OH

Unit 3: Power Series Methods

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

└ Identity of function

WP: Identity ~~of~~ Function: A function that always returns the same value that was used as an argument

└ ie $f(x) = x$

same value

So $e^1 = \sum_{n=0}^{\infty} \frac{1^n}{n!}$

82

You can't add the terms

but

$$\lim_{k\to\infty} \sum_{n=0}^{k} \frac{1}{n!}$$

Can only take limit if it converges

$$\lim_{k\to\infty} \left(\frac{1}{1} + 1 + \frac{1}{2} + \frac{1}{3\cdot 2} \dots + \frac{1}{k!}\right)$$

~~1 + 2 + 2 +~~

1, 2, 2.66, ...

$e \approx 2.718$ ← this seems wrong but w/e

Converges ~~bec~~ because new terms get smaller + smaller

└ $a_n \to 0$ fast enough in $\sum a_n x^n$

Test for Convergence

$$\lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = R$$

↑ radius of convergence

83

Try it out $\sum_{n=0}^{\infty} \frac{(-1)^n (n+1)}{5^n} x^n$

$$\lim_{n\to\infty} \frac{\frac{(-1)^n (n+1)}{5^n} \cancel{x^n}}{\frac{(-1)^{n+1} (n+2)}{5^{n+1}} \cancel{x^{n+1}}}$$

don't need

¿ now what

you do some algebra

$$= \lim_{n\to\infty} 5 \frac{n+1}{n+2}$$

$= 5$ = radius of convergence

let me try

$$\frac{\cancel{(-1)^n} (n+1)}{\cancel{5^n}} \cdot \frac{5^{n+1}}{(-1)^{n+1}(n+2)}$$

¿can you cross out like I did? $= \frac{(n+1)\, 5}{-1(n+2)}$

(84)

Remember we are taking lim $n \to \infty$

So $\lim \frac{\infty+1}{\infty+2} = \infty$

But the 5 stays there

?? So isn't it

$5 \cdot \infty = \infty$

That may not be correct

Oh or $\frac{\infty}{\infty} = 1$??

WA says indeterminant...

I am just not good at these type of qus

So this means we can only plug in values b/w [-5, 5] or else it blows up

Solving an ODE w/ Power Series

example $2(x+1)y' = y$

Generic form and first few terms

$$Y = \sum_{n=0}^{\infty} a_n x^n = a_0 + \cancel{a_0}\, a_1 x + a_2 x^2 + \cdots$$

$$Y' = \sum_{n=1}^{\infty} a_n n x^{n-1} = a_1 + a_2 x + a_3 x^2 + \cdots$$

I remeber this stuff

Plug into our generic forms

$$2(x+1)\sum_{n=1}^{\infty} a_n n x^{n-1} = \sum_{n=0}^{\infty} a_n x^n$$

by uniqueness of power series must be =

coeffs left = coeffs right

Expand LHS

$$\sum_{n=1}^{\infty} 2a_n n x^n + \sum_{n=1}^{\infty} 2a_n n x^{n-1} = \sum_{n=0}^{\infty} a_n x^n$$

$$\boxed{x^{n-1} \cdot x = x^n}$$

86

Move over terms

$$\sum_{n=0}^{\infty} 2\, a_{(n+1)}\, (n+1)\, x^n$$

Now can see left and right

$$\sum_{n=1}^{\infty} 2 a_n\, n\, x^n + \sum_{n=0}^{\infty} 2\, a_{(n+1)}\, (n+1)\, x^n =$$

$$= \sum_{n=0}^{\infty} a_n x^n$$

Compare terms

x^0: $2a_1 = a_0$

x^1: $2a_1 \cdot 1 + 2a_2 \cdot 2 = a_1$

So for $n \geq 1$

x^n: $2a_n\, n + 2\, a_{n+1}\, (n+1) = a_n$

there is a simpler way to actually do I remember

First order ODE - one undetermined constant

L have identity

- but can't solve for a_1 w/o a_0
- once have a_0, can find a_s

87

Suppose told

$$y(0) = 1 = \sum_{n=0}^{\infty} a_n 0^n = a_0$$

at 0   value of constant term

$$a_0 = 1$$

$$a_1 = \frac{1}{2}$$

$$a_{n+1} = \frac{(1-2n)}{2(n+1)} a_n$$

Can use to get more terms

$$a_2 = \frac{(1-2\cdot 1)}{2(1+1)} \cdot a_1 = -\frac{1}{4}\cdot\frac{1}{2} = -\frac{1}{8}$$

So we know

$$y(x) = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \cdots$$

So is this meaningful?

$$\lim \left|\frac{a_n}{a_{n+1}}\right| = \left|\frac{2(n+1)}{(1-2n)}\right| \to 1 \quad \leftarrow \frac{\text{polynomial in } n}{\text{polynomial in } n}$$

so valid $x \in (-1,1)$

(88)

¿Why are we doing this again?

└ what do we gain?

Can solve w/ <u>variable</u> (not just constant) coefficents

<u>Textbook</u>

table of powerseries representations

└ given on exam?

No does not seem to be

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \cdots$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n \, x^{2n+1}}{(2n+1)!} = 1 - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

That might be why I thought some of the conversions might be weird...

89

So $y' + 2y = 0$ book example

I could solve other ways I think

but sample

$\cancel{(x+1)}$

$$\sum_{n=1}^{\infty} n c_n x^{n-1} + 2\sum_{n=0}^{\infty} c_n x^n = 0$$

→ shift summation

$$\sum_{n=0}^{\infty} (n+1)\, c_{n+1}\, x^n + 2\sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+1)\, c_{n+1} + 2c_n\right] x^n = 0$$

Solve for c_{n+1} try

$$c_{n+1} = \frac{-2c_n}{n+1}$$ ✓

I think this is as much as ya need

Can write it at

$$c_1 = \frac{-2c_0}{1} \quad (n=0)$$

$$c_2 = -\frac{2c_1}{2} = \frac{2^2 c_0}{1\cdot 2} = \frac{2^2 c_0}{2!}$$ look for pattern don't just reduce

(95)

$$c_3 = \frac{-2c_2}{3} = \frac{2^3 c_0}{1 \cdot 2 \cdot 3} = \frac{2^3 c_0}{3!} \quad (n=2)$$

$$c_n = (-1)^n \frac{2^n c_0}{n!} \quad n \geq 1$$

← no often need to get this pattern but not on test? did he say?

So solution

$$y(x) = \underbrace{\sum_{n=0}^{\infty} c_n x^n}_{\text{Standard form}} = \sum_{n=0}^{\infty} (-1)^n \frac{2^n c_0}{n!} x^n$$

$$= c_0 \sum_{n=0}^{\infty} \frac{(-2x)^n}{n!}$$

$$= c_0 e^{-2x}$$

more pattern recognition I am pretty sure we don't have to do

So I think that is basically power series problem - need to pratice them

Radius of convergence (R) $\geq$ min distance from origin to a 0 for $A(x)$

Lagendre eq - heat flow in sperical form
└ can only do w/ power series since non constant

Get one sol ∞ series
other is polynomial degree n $P_n(x)$
└ Lagrende polynomials
¿how imporortant is this unit/section on Lagerndre?

$$\int_{-1}^{1} P_m(x) P_n(x)\, dx = 0$$
└ unless $m=n$ "super important"

Hw: Hermite polynomials
└ solve Warup's problem ← don't think important
└ Every pos integers can be written as a bounded sum of kth powers for given k

Fourier Series for Periodic Eqns

Periodic function $f(x+P) = f(x)$ for all x

shift by P = period
= same thing!

sin, cos → period $= 2\pi$

Constant function ~~only~~ small period
└ doesn't exist

eg $f(x) = \frac{a_0}{2} + a_1 \cos(x) + a_2 \cos(2x) + \cdots$
$+ b_1 \sin(x) + b_2 \sin(2x) + b_3 \sin(3x) + \cdots$

describe values for 1 period - a "window" $[-\pi, \pi)$

a_i → even functions
b_i → odd functions

(93)

Even  y axis

$\cos(x)$
$\cos(2x)$

$f(x) = f(-x)$

Odd  origin

$\sin(x)$
$\sin(2x)$

$f(+x) = -f(x)$

even + e = e
o + o = o
e + o = arbiary
e · e = e
e · o = o
o · o = e

Like Power series in some way
∟but w/ integrating, not diffeentating

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(mx)\,dx$$

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(mx)\,dx$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

Some useful preliminaries

1. $\int_{-\pi}^{\pi} \sin(mx)\,dx = \left.\frac{-\cos(mx)}{m}\right|_{-\pi}^{\pi} = 0$

2. $\int_{-\pi}^{\pi} \cos(mx)\,dx = \left.\frac{\sin(mx)}{m}\right|_{-\pi}^{\pi} = 0$

m = ~~integer multiples~~ integer

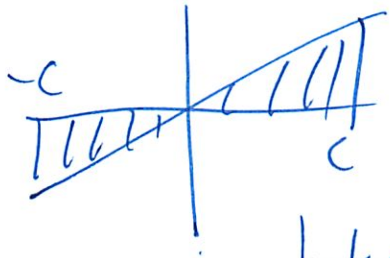
3. $\int_{-\pi}^{\pi} \cos(mx) \cos(nx)\,dx = \begin{cases} 2\pi & \text{if } m=n \\ 0 & \text{if } m\neq n \end{cases}$

4. $\int_{-\pi}^{\pi} \sin(mx) \sin(nx)\,dx = \begin{cases} \pi & \text{if } m=n \\ 0 & m\neq n \end{cases}$

5. $\int_{-\pi}^{\pi} \sin(mx) \cos(nx)\,dx = 0$

(45)

Oh right $\int_{-c}^{c}$ odd fn $= 0$



balances itself out $= 0$

Urg this chap had a lot of painful integration

L20 Like for the square wave

└ there are really only a few waves where this works

Add more terms till it gets closer + closer

$$Sq(x) = \begin{cases} 1 & x \in (0, \pi] \\ -1 & x \in (-\pi, 0) \end{cases}$$

piecewise

$$a_0 = \underbrace{\frac{1}{\pi}}_{\text{don't forget}} \int_{-\pi}^{\pi} Sq(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} (-1)\,dx + \frac{1}{\pi}\int_{0}^{\pi} (1)\,dx$$

$$= 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{\underbrace{Sq(x)}_{\text{odd}} \cdot \underbrace{\cos nx}_{\cancel{\text{odd}}\ \text{even}}}_{\cancel{\text{even}}\ \text{odd}}\,dx$$

← know thy shortcuts!

$\int_{-\pi}^{\pi}$ odd $= 0$

96

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} Sq(x)\ \sin n x$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-1)\sin n x\,dx + \int_{0}^{\pi}(1)\sin n x\,dx\right]$$

notice. Can I do it? I think so

$$= \frac{1}{\pi}\underbrace{\left[\frac{1}{n}\cos n x\Big|_{x=-\pi}^{x=0}\right]}_{\frac{1}{n}(1-\cos(-\pi n))} + \frac{1}{\pi}\left[-\frac{1}{n}\cos n x\Big|_{x=0}^{x=\pi}\right]$$

depends if
n odd $\to \frac{2}{n}$
or even $\to 0$

Visualize + picture

$= \frac{4}{n\pi}$ if n odd

0 if n even

So then put it all together

$$\frac{4}{\pi}\sin x + \frac{4}{3\pi}\sin 3x + \dots$$

$$= \sum_{n=1}^{\infty}\frac{4}{\pi}\,\frac{\sin(2n-1)}{2n-1}$$ ← is that similar to the table we had?

97

So when is a fn = to its fourier series?

L If $f(\cdot)$ is nice - they are = whenever f is contineous

L piecewise smooth

L f' is piecewise contineous

L finite # of holes

Oh if period $\neq 2\pi$

$\sin\left(\frac{2\pi x}{p}\right)$

$\cos\left(\frac{2\pi x}{p}\right)$

Then in sums $\sin\left(\frac{2\pi n x}{p}\right)$

$\cos\left(\frac{2\pi n x}{p}\right)$

Used w/o realizing

And if $p = 2\pi$ $\frac{\cancel{2\pi} x}{\cancel{2\pi}}$

Ok now need to practice

98

Applications Undamped spring w/ square wave input

if input is odd can throw out cos

wait so

$$\int \text{odd} \cdot \underset{}{\overset{\downarrow \cos}{\text{even}}}$$

$$\int \text{odd} = 0 \quad \checkmark$$

Amplitude Gain

- never correctly researched
- go back to L15

So $\text{Amp} = \sqrt{\underset{\substack{\uparrow \\ \text{coeff} \\ \cos}}{A}^2 + \underset{\substack{\uparrow \\ \text{coeff} \\ \sin}}{B}^2}$ ← make pure sin wave

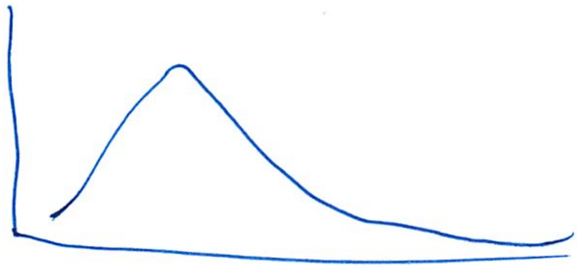
~~Can't then~~

$$= \frac{8.5\,w}{\sqrt{(5-w^2)^2+(2w)^2}}$$

Can then solve for w

(99)

And graph



And take deriv + set = 0 to find the max

Radios do this
∟ change capacitance so boost narrow peak

Find ω_0 resonance

Or Close to resonance
∟ has some almost-resonance name

Back to current lecture

Can graph 1st 10 terms
And ~~find~~ find max like before

(100)

Resonance

It is possible a guess on the LHS has included in the homogeneous sol
↳ causes resonance

roots $\pm\sqrt{\frac{k}{m}}$ ⇝ $\cos\sqrt{\frac{k}{m}}\,t$

$\sin\sqrt{\frac{k}{m}}\,t$

So I belive look for

$n\pi = \sqrt{\frac{k}{m}}$ for any n

↳ if it's there → resonance

So like for the Sq wave

if we had $mx'' + kx = Sq(t)$

then if $m=1$ $k=9$

$\sqrt{\frac{9}{1}} = 3$

So if sol has $\frac{\sin}{\cos} 3t$ we are in trouble

Put a t in front of sin and cos

("depending which one is there")

$$\underbrace{\underbrace{t}_{\text{odd}}\ \underbrace{\cos 3t}_{\text{even}}}_{\text{odd}}$$

Boundry Value Problems

Same $P(D)x = f(t)$ on a finite interval $t \in (0, L)$

but add boundry conditions

$$x(0) = x(L) = 0$$

is like heat eqn?

So extend $f(t)$ to periodic fn of $2L$

~~the boundry co~~ So

becomes

$$= \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin n\pi t$$ ← since we already figured this out?

Make a guess for undet. coeff

$$x_p(t) = \sum_{n=1}^{\infty} b_n \sin n\pi t$$

coeff for every n
not just odd ns

Plug into LHS + solve for b_n

$$x_p(t) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sin n\pi t}{n(9 - n^2\pi^2)}$$

skipping some steps
(which ones?)

So this one could have done before
But this method works w/ non linear + piecewise

L22 Heat Eqn I remember this lecture
was a complete mess
↳ But only need to remember how to set it up
Review my review f of exam 3

(103)

Oh the integration by parts derivation lol

$$d(uv) = u\,dv + v\,du$$

$$\int d(u \cdot v) = uv = \int u\,dv + \int v\,du$$

$$\int u\,dv = uv - \int v\,du$$

Oh from last lecture, remember for any single sine function

$$F(t) = F_0 \sin \omega t$$

the answer is

$$x_p(t) = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}} \sin(\omega t - \alpha)$$

$$\alpha = \tan^{-1}\left(\frac{c\omega}{k - m\omega^2}\right)$$

follows from ERF

Exponential Response Function

(took me a while to remember what this was called)

for example $\tilde{x}_p(t) = \dfrac{e^{i\omega t}}{p(i\omega)}$

w/ $p(r) = mr^2 + cr + k$

(104)

So that long formula seems familiar from somewhere - but where exactly?
Oh it continues

So by superposition if

$$F(t) = \sum b_n \sin \frac{n\pi t}{L}$$

then

$$F_{0,n} = b_n$$

$$\omega_n = \frac{n\pi}{L}$$

$\alpha = \ldots$ for each n given sol

So

$$X_p(t) = \sum_{n=1}^{\infty} \frac{b_n}{\sqrt{(k - m\omega^2)^2 + (c\omega_n)^2}} \sin(\omega t - \alpha_n)$$

Calc b_n as before $\frac{1}{L}\int_{-L}^{L} f(t) \sin \frac{n\pi t}{L}\, dt = b_n$

~~Reviewed~~ Marked to ask in OH
TA confused too
Marked to forget it

When studying for exam 3

105

Now review Heat eqn

$$T = T(x, t)$$

temp, pos., time

↳ function of 2 things

C = heat capacity

F - heat flux

Heat change

$$\frac{d}{dt}\int_a^b CT\,dx\,A = F_a A - F_b B$$

$$= -A\int_a^b F_x\,dx$$

$$\int_a^b \left(C\frac{\partial T}{\partial t} + \frac{\partial F}{\partial x}\right)dx = 0 \quad \text{for any } a, b$$

But this is not the relevant stuff...

Fick's law $F = -\mu T_x$

$$CT_t - \mu T_{xx} = 0$$

$$T_t = \mu T_{xx}$$

μ ← heat difference

(106)

Neet to solve for T periodic

Want ODE

$$T = \varphi(x)\, e^{rt}$$

$$r\varphi = \varphi''$$

r find r periodic

$$r = \lambda^2 \qquad \varphi = b_n \sin \lambda x + a_n \cos \lambda x$$

$$\lambda_n = \frac{n\pi}{L}$$

$$r_n = -\lambda_n^2 = -\frac{n^2\pi^2}{L^2}$$

Also need $\varphi = \frac{1}{2}a_0 \qquad r_0 = 0$

So now have tons of sols to Heat eqn

$$T = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{L} x\, e^{r_n t} + \left(b_n \sin \frac{n\pi}{L} x\right) e^{r_n x}$$

When set $t = 0$

$$T(x,0) = T_0(x) = \frac{1}{2}a_0 + \sum \left(a_n \cos \frac{n\pi}{L} x + b_n \sin \frac{n\pi}{L} x\right)$$

Skip all this, don't need to know...

(10) I think I found the relevant stuff

So Rod's temp is given by

$$u(x,t) = \frac{4u_0}{\pi} \sum_{n\,odd} \frac{1}{n} e^{\frac{-n^2\pi kt}{L^2}} \sin \frac{n\pi x}{L}$$

So to answer question

(I should also just look at exam)

$$u(25, 1800) = \frac{400}{\pi} \sum_{n\,odd} \frac{(-1)^{n+1}}{n} e^{\frac{-18n^2\pi^2 k}{25}}$$

half of rodd

30 min in Seconds

$$= 43.85\,^\circ C$$

Here we go:

Basically 2 types of problems

<u>ends frozen</u> $u(0,t) = u(L,t) = 0$

Sine sol

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{\frac{-n^2\pi^2 kt}{L}} \sin \frac{n\pi x}{L}$$

fourier sine coeffs of rods' inital temp function $f(x) = u(x,0)$

108

Ends insulate $\rightarrow \frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t)$

derivitive

Cos sol

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{\frac{-n^2\pi^2 k t}{L}} \cos\frac{n\pi x}{L}$$

Fourier cosine solution

rods' inital temp fn

$f(x) = u(x,0)$

what does this mean exactly

$$\lim_{t\to\infty} u(x,t) = \frac{a_0}{2} = \frac{1}{L}\int_0^L f(x)\,dx$$

Also similar Wave eqn - string vibrating under tension

So where get a_n, b_n?

$$b_n = \begin{cases} \frac{4u_0}{n\pi} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

But where is that from?

And is it specific for a certain set of problems

(109)

Found in textbook

$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$

(zero/ice endpoints)

$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$

$\to$ w/ $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \cos\frac{n\pi x}{L}$

So in our numeric example (50 cm rod) (ends insulated)

$f(x) = 50 - \frac{400}{\pi^2}\sum_{n\,odd}\frac{1}{n^2}\cos\frac{n\pi x}{25}$

Rewrite slightly so $\cos\frac{n\pi x}{50}$

$f(x) = 50 - \frac{1600}{\pi^2}\sum_{n=2,6,10\ldots}\frac{1}{n^2}\cos\frac{n\pi x}{50}$

So $u(x) = 50 - \frac{1600}{\pi^2}\sum_{n=2,6,10}\frac{1}{n^2}e^{\frac{-n^2\pi^2 kt}{2500}}\cos\frac{n\pi x}{50}$

Review quiz problem later

(110)

L23 LaPlace Transforms | (2nd big thing of unit 3)

Somewhat like the operator method

$D = \frac{d}{dx}$

s is a real variable $s > 0$ so integral converges

$$\mathcal{L}(f) = \int_0^\infty f(t)\, e^{-st}$$

$$\mathcal{L}(f)(s) = F(s)$$

notation

$$\mathcal{L}(1) = \int_0^\infty 1\, e^{-st}\, dt$$

$$= \lim_{A \to \infty} \int_0^A e^{-st}\, dt$$

fund therom of calculus

$$= \lim_{A \to \infty} \left(-\frac{1}{s} e^{-st}\right) \Big|_{0=t}^{A=t}$$

$$= \frac{1}{s}$$

since $e^{-sA} \to 0$ as $A \to \infty$ if $s > 0$ $= \frac{1}{s}$

I don't think we have to know how to do this stuff

So how to solve ODEs w/ ?

1. Start w/ ODE
2. LaPlace transform both sides
   - converts deriv to algebra
3. Do algebra to simplify problem
4. Take inverse of LaPlace transform

$$\mathcal{L}(e^{at}) = \frac{1}{s-a}$$

skipping derivation

$$\mathcal{L}(\cos bt) = \frac{s}{s^2+b^2}$$

$$\mathcal{L}(\sin bt) = \frac{b}{s^2+b^2}$$

this stuff on exam

Properties

$\mathcal{L}(f) = F(s)$ then

(A) $\mathcal{L}(e^{at} \cdot f(t)) = F(s-a)$

112

B $\mathcal{L}(t \cdot f(t)) = -F'(s)$

C $\mathcal{L}(f'(t)) = sF(s) - f(0)$

D $\mathcal{L}(f''(t)) = s^2 F(s) - sf(0) - f'(0)$

$$\mathcal{L}(t) = \frac{1}{s^2}$$

$$\mathcal{L}(t^2) = \mathcal{L}(t) \cdot \mathcal{L}(t)$$

$$= -\frac{d}{ds} \cdot \left(\frac{1}{s^2}\right) = \frac{2}{s^3}$$

$$\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$$

Then inverse LaPlace Transforms

$$\mathcal{L}^{-1}\left(\frac{1}{s^2+5s+4}\right)$$

factor

$$\frac{1}{(s+4)(s+1)}$$

(113)

Split using Coverup method

$$\frac{1}{(s+4)(s+1)} = \frac{A}{s+4} + \frac{B}{s+1}$$

Pratice

~~A~~ so set $s = -4$

~~A~~ No don't think so

So I think I need a more interesting numerator

Notes example

$$\frac{x-7}{(x-1)(x+2)} = \frac{A}{X-1} + \frac{B}{x-2}$$

→ Set $x = 1$

└ so was right

$$\frac{1-7}{(\quad)(\cancel{}\,1+2)} = -2 = A$$

I do it slightly differenty

Multiply both sides by $x - 1$

$$\frac{(x-7)\cancel{(x-1)}}{\cancel{(x-1)}(x+2)} = \frac{A\cancel{(x-1)}}{\cancel{(x-1)}} + \frac{B(x-1)}{x-2}$$

(114)

Then set $x = 1$ I think

$$\frac{1-7}{1+2} = A + \frac{B(0)}{x-2}$$

$$-2 = A$$

Ah same thing ✓ Nice

↳ I think we are basically doing that...

So back to our question

$$\frac{1}{(s+4)(s+1)} = \frac{A}{s+4} + \frac{B}{s+1}$$

$$\frac{\cancel{(s+4)}}{\cancel{(s+4)}(s+1)} = \frac{A\cancel{(s+4)}}{\cancel{s+4}} + \frac{B(s+4)}{s+1}$$

$$s = -4$$

$$-\frac{1}{3} = A$$ Matches

$$\frac{1}{s+1} = \frac{A(s+1)}{(s+4)} + \frac{B\cancel{(s+4)}}{\cancel{(s+4)}}$$ $s = -1$

(115)

$$\frac{1}{0} = B$$

rutro...

Can also use do alt method I believe

$$\frac{1}{(s+4)(s+1)} = \frac{-\frac{1}{3}}{s+4} + \frac{B}{s+1}$$

They have

$$A(s+1) + B(s+4) = 1$$

that looks easier

but I can use above I think

Set s = something - anything that is not weird (-1, -4)

like s = 0

$$\frac{1}{4} = \frac{-\frac{1}{3}}{4} + \frac{B}{1}$$

$$\frac{1}{4} = -\frac{1}{12} + B$$

$$\frac{4}{12} = B$$

$= 1/3$ ☺ Matches

And their method

$$A(s+1) + B(s+4) = 1$$

$$As + A + Bs + B4 = 1$$

$$(A+B)s + A + 4B = 1$$

$\therefore$ Need s term to disappear

↳ so $A+B=0$

Then

$$A + 4B = 1$$

$$A + B = 0$$

$$A = -B$$

$$-B + 4B = 1$$

$$3B = 1$$

$$B = \frac{1}{3}$$ ✓ Matches

$$A + \frac{1}{3} = 0$$

$$A = -\frac{1}{3}$$ ✓

So what was original problem ...

oh inverse Laplace $\mathcal{L}^{-1}\left(\frac{-1/3}{s+4}\right) + \mathcal{L}^{-1}\left(\frac{1/3}{s+1}\right)$

117

Pull out constants

$$-\frac{1}{3}\mathcal{L}^{-1}\left(\frac{1}{s+4}\right) + \frac{1}{3}\mathcal{L}^{-1}\left(\frac{1}{s+1}\right)$$

L $\mathcal{L}^{-1}\left(\frac{1}{s-a}\right) = e^{at}$

$$-\frac{1}{3}e^{-4t} + \frac{1}{3}e^{-t}$$

Can apply to spring Model

? What is the special thing about LaPlace?

Textbook: More convienient when discontinuous

L But for linear diff eqs w/ constant coeffs

? when you use each should have been emphized more...

Fourier of odd fn L> only b_n terms

even fn L> " a_n "

neither L> both a_n, b_n terms

Should pratice this more

L good verbose example 11/9 OH

(118)

(¿ am I even doing the right thing reviewing all this?)

So then using it to solve

$$y'' - 4y' + 3y = e^{3t}$$

└ can already solve

$$\mathcal{L}(y'') - 4\mathcal{L}(y') + 3\mathcal{L}(y) = \mathcal{L}(e^{3t})$$

$$\mathcal{L}(y)(s^2 - 4s + 3) - \underset{\substack{\uparrow \\ y(0) \\ s+1}}{s} - \underset{\substack{\uparrow \\ y'(0)}}{2} - \underset{\substack{\uparrow \\ y(0)}}{1} = \frac{1}{s-3}$$

$$\mathcal{L}(y) = \frac{\frac{1}{s-3} + s + 2}{s^2 - 4s + 3}$$

This I remember well
Should try, not read

Make proper function

$$\cancel{\frac{1}{(s-3) + (s+2) + s^2 - 4s + 3}}$$

$$\frac{1 + (s-2)(s+3)}{(s-3)^2(s-1)}$$

how did they do that?

119

$$\frac{\frac{1}{x}}{x} = \frac{1}{x^2}$$ right? Yeah says WA

So factor $s^2 - 4s + 3$

$$\frac{+4 \pm \sqrt{16 - 4 \cdot 3 \cdot 1}}{2}$$

2 ± 1

$3, 1$

So

$$\frac{\frac{1}{(s-3) + (s+2)}}{(s-3)(s-1)}$$

is $\cancel{\frac{1}{(s-3)t}}$ $\frac{1}{(s-3+s+2)(s-3)(s-1)}$

Ahh back of ~~the~~ book explains it
I did previous step too fast

(120)

$\cancel{\mathcal{L}(y) = \frac{1}{(s-3)(s}}$

Let me try

$\mathcal{L}(y)(s^2-4s+3) = \frac{1}{s-3} + s-2$ ← You normally don't have these (rest conditions)

$$\mathcal{L}(y) = \frac{1}{(s-3)(s^2-4s+3)} + \underbrace{\frac{s}{s^2-4s+3} - \frac{2}{s^2-4s+3}}_{\text{can connect - common denom (duh)}}$$

$$= \frac{1}{(s-3)(s^2-4s+3)} + \frac{s-2}{s^2-4s+3}$$

Ah so they want common denom

$\cancel{\times (s-3)}$

$$\underbrace{\frac{1}{(s-3)(s^2-4s+3)}}_{\text{↑ don't touch}} + \underbrace{\frac{s-2\,(s-3)}{(s-3)(s^2-4s+3)}}_{\text{↑ add}}$$

↓ add

merge

$$\frac{1+(s-2)(s-3)}{(s-3)(s-3)(s-1)}$$ ✓ got it now

(121)

Now partial fractions

$$\frac{s^2-5s+7}{(s-3)^2(s-1)} = \frac{A}{(s-3)^2} + \frac{B}{s-3} + \frac{C}{s-1}$$

note need 2nd term!

Now try this

$$\frac{(s^2-5s+7)\cancel{(s-3)^2}}{\cancel{(s-3)^2}\, s-1} = A + \frac{B(s-3)^{\cancel{2}}}{\cancel{s-3}} + \frac{C(s-3)^2}{s-1}$$

$s=3$

$$\frac{9-15+7}{2} = A + 0B + 0C$$

$$A = \frac{1}{2}$$

~~(s-2)2t~~

$$\frac{s^2-5s+7}{(s-3)(s-1)} = \frac{A}{s-3} + B + \frac{C(s-3)}{s-1}$$

$s=3$

problem — need to do later

(122)

$$\frac{s^2-5s+7}{(s-3)^2} = \frac{A(s-1)}{(s-3)^2} + \frac{B(s-1)}{s-3} + C$$

$s=1$

$$\frac{1-5+7}{4} = 0 + 0 + C$$

$C = \frac{3}{4}$ ✓

this method is a lot of work!

$$\frac{s^2-5s+7}{(s-3)^2(s-1)} = \frac{\frac{1}{2}\cancel{(s-3)}}{(s-3)^2} + \frac{B\cancel{(s}}{s-3} + \frac{\frac{3}{4}}{s-1}$$

$s=0$

✓ should be 1?

$$-\frac{7}{9} = \frac{\frac{1}{2}}{9} + \frac{B}{-3} - \frac{3}{4}$$

$$-\frac{7}{9} = \frac{1}{18} - \frac{B}{3} - \frac{3}{4}$$

$$-\frac{53}{36} = -\frac{B}{3}$$

$\frac{53}{12} = B$ ✗ $\frac{1}{4}$

(123)

Prof did it in far fewer pages :)

Ok back to our regularly scheduled problem

$$y = \mathcal{L}^{-1}\left(\frac{\frac{1}{2}}{(s-3)^2} + \frac{1/4}{s-3} + \frac{3/4}{s-1}\right)$$

then I can do first to

$$\underline{\qquad\quad} + \frac{1}{4}e^{3t} + \frac{3}{4}e^{t}$$

↑ but need

We know $(s-2)^2$ is a derivitive

know $\mathcal{L}(e^{3t}) = \frac{1}{s-3}$

So $\mathcal{L}(t e^{3t}) = -\left(\frac{1}{s-3}\right)'$

$$= +\left(\frac{1}{s-3}\right)^2$$

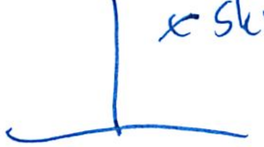
So first term is

$$\frac{1}{2} t e^{3t}$$

Makes sense

Good for impulse functions

└ a lot less math

delta function  ← sky scraper fake function $= \delta(t)$

So say $\int_{0^-}^{1} \delta(t)\,dt = 1$

$\int_{0^+}^{1}$ " $= 0$

$$\mathcal{L}(\delta) = e^{-st}\Big|_{t=0}$$

$$= 1$$

$$\mathcal{L}(\delta') = s$$

Pratice actually using it

└ Don't convert it I believe

125

So can use this to model an impulse force

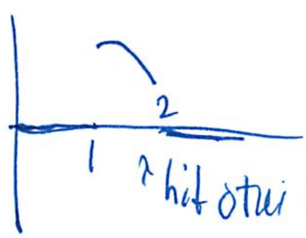
when hammer hits

$\delta(t-2) = \delta_2(t)$

notation

They seem to convert
but in Recitation we did not do that...

Can add these to go up and then back down



hit other direction

$u_1(t) - u_2(t)$

(26)

Convolution

$$\frac{2s}{(s^2+4)^2} = \frac{2}{s^2+4} \cdot \frac{s}{s^2+4}$$

$$= \mathcal{L}(\sin 2t)\ \mathcal{L}(\cos 2t)$$

but then what?

$$= \mathcal{L}(\sin 2t * \cos 2t)$$

↑ convolution!

$$f * g(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau$$

I Don't think we ever have to use

$$f * \delta = f(t) \cdot 1$$

$$= f(t)$$

But note

$$f * 1 = 1 * f$$

$$\neq f$$

Actually I think you do have to S

127

$f * g = g * f$

Note that

$$\mathcal{L}(u(t-a) f(t-a)) = e^{-as} \mathcal{L}(f(t))$$

$$\mathcal{L}(\delta(t-7)) = \int_0^\infty e^{-st} \delta(t-7)\, dt = e^{-7s}$$

(but I still think you can save it like that?

Like $\delta(t-7) * e^{-t}$

~~Means~~ Means

$$= \begin{cases} 0 & t < 7 \\ e^{(t-7)} & t \geq 7 \end{cases}$$

$$= u(t-7)\, e^{-(t-7)}$$

In general $\mathcal{L}^{-1}(e^{-cs} F)$

$$\delta(t-c) * \mathcal{L}^{-1}(F)$$

$$= \int_0^t \delta(z-c) \mathcal{L}^{-1}(F)(t-z)\, dz$$

$$= \begin{cases} 0 & t < c \\ e^{-(t-c)} & t \geq c \end{cases}$$

Need to pratice that

L26 Weight Function

Lots of ways to think about δ function

(Oh this is what I did not get on the exam)

$P(D)x = f$

$\mathcal{L}(p(D)x) = \mathcal{L}(f)$

$P(s)\,\mathcal{L}(x) = \mathcal{L}(f)$

Char eqn

$\mathcal{L}(x) = \frac{1}{P(s)}\mathcal{L}(f)$

So this is just like normal

Define $w(t) = \mathcal{L}^{-1}\left(\frac{1}{P(s)}\right)$ weight fn

Then $\mathcal{L}(x) = \underbrace{\mathcal{L}(w)\,\mathcal{L}(f)}_{\mathcal{L}(w * f)}$

By uniqueness of Lap. $x = w * f$

(29)

Input Response Models

Lwhat does this mean exactly?

$f(t) \to \int_0^t$ convolves w $w(t-v)\,dv \to x(t)$

? Green's function

Don't need to solve diff eq again + again for every f (oh now I might see motivation of this)

Can code up integration into this box

Record response

(Examples confusing...)

Ask in OH

Ah more in next lecture

Useful b/c solution

$x(t) = w * f(t)$ ← magic ans w/ input response formula

130

$$= \int_0^t w(t-u)\, f(u)\, du$$

In our example $\quad x'' + x = \delta + \delta_\pi$

$$w(t) = \mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) = \sin t\,(u(t))$$

No matter fn on right

$$x = \sin t \cdot u(t) \cdot f$$

here $f = \delta + \delta_\pi$

$$= \sin t\, u(t) * (\delta + \delta_\pi)$$

$$= \sin t\, u(t) * \delta + \sin t\, u(t) * \delta_\pi$$

acts like identity in convo

$$= \sin t\, u(t) + \int_0^t \sin z\, u(z)\, \delta_\pi(t-z)\, dz$$

since $\delta(t-z-\pi)$

$$= \sin t\, u(t) + \sin(t-\pi)\, u(t-\pi)$$

what ?!

don't get

Smoothes stuff out in a circuit

↳ Can treat it as a black box

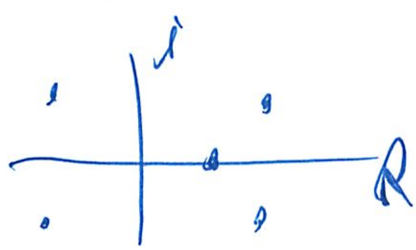
(looks the same to me...)

Poles

~~Need $\mathcal{L}^{-1}$ (rational functions)~~

0s of denoms $\frac{1}{P(s)}$

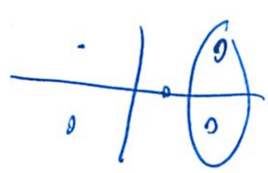
Called poles



So denom $= s(s^2+4s+5)\cdot(s^2-2s+2)$

inverse Laplace $= 1,\ e^{2t}\sin t,\ e^{-t}\sin t,$
$e^{2t}\cos t,\ e^{-t}\cos t$

What dominates in long term?

 the far right on diagram

(132)

So poles can be used to help make guess

what guess?

How is this any different then what we did before?

I don't get this method!

Go to office hrs

18.03 OH

Jethro

12/19

Agenda: Inhomogeneous IVP
Weight Fn

Weight function

$P(D)\,y = 0$

$y'' - 2y + 2y = 0$

$P(D) = D^2 - 2D + 2$

Weight Fn $= \mathcal{L}^{-1}\left(\dfrac{1}{s^2 - 2s + 2}\right)$

$y'' - 5y' + 6$ ← weight Fn - it doesn't matter what is on RHS till you consider it later on

$P(D) = D^2 - 5D + 6$

Weight fn $= \mathcal{L}^{-1}\left(\dfrac{1}{P(s)}\right)$

$= \mathcal{L}^{-1}\left(\dfrac{1}{s^2 - 5s + 6}\right)$

Now factorize - Heavysides

- Complete the square

$(s-2)(s-3)$

2

$$= \mathcal{L}^{-1}\left(\frac{?}{s-2} + \frac{?}{s-3}\right)$$

Heavysides cover up method

$$= \mathcal{L}^{-1}\left(\frac{-1}{s-2} + \frac{1}{s-3}\right)$$

$$= \mathcal{L}^{-1}\left(\frac{-1}{s-2}\right) + \mathcal{L}^{-1}\left(\frac{1}{s-3}\right)$$

$$Y = \frac{1}{p(s)}$$

$$w(t) = -e^{2t} + e^{3t} \checkmark$$

$p(s)\,Y = 1$

$\leftarrow$ $w(t)$ = solution to $p(D)\,y = \delta(t)$ w/ rest conditions

Isn't anything really that special about Laplace transform

(I was thinking ahead while he was doing)

Key fact about $w(t)$:

$$p(D)\,y = f(t)$$

then $y(t) = w(t) * f(t)$ is a particular solution

(3)

For example

$$y'' - 5y + 6y = e^t$$

We already know the Guess + Check method
└ undetermined coeffs

If found weight fn

Could convolve

(TA: It is harder)

└ More important for theoritical reason
Or can use it to compute raw data

$$W(t) = e^{3t} - e^{2t}$$

So
$$y = (e^{3t} - e^{2t}) * e^t$$
$$= \int_0^t e^{t-s}(e^{3s} - e^{2s})\,ds$$
$$= e^t \int_0^t (e^{2s} - e^s)\,ds$$
$$= e^t \left(\tfrac{1}{2}e^{2s} - e^s\right)\Big|_0^t$$

(9)

$$= e^t\left(\frac{1}{2}e^{2t} - e^t - \frac{1}{2} + 1\right)$$

$$= \frac{1}{2}e^t + \frac{1}{2}e^{3t} - e^{2t}$$

↑ particular sol (happened to give all components but not always but not general sol $\frac{1}{2}e^t + Ae^{3t} + Be^{2t}$)

(Can check by doing it the easy way

Since had rest conditions $x(0)=0$, $x'(0)=0$
(particular sol w/ those)

Convolving w/ δ is easy
↳ just the same thing

Note this is all w/ rest conditions
↳ if not - could still do something
- but formula not as neat
- so will always start w/ rest conditions

Inhomogeneous IVP

Try Ae^t substitution

$$A - 5A + 6A = 2A$$

$$2Ae^t = e^t$$

$$A = \frac{1}{2}$$

5

$y_p(t) = \frac{1}{2} e^{t}$

↑ different particular soltions from before
other included homogeneous as well

So if wanted to find y w/ $y(0) = 0 = y'(0)$

1. Find homogeneous eqn + solution
2. Find a particular soln
3. Substitute inital conds to fix constants
   ↳ you do need the particular soln first

1. $y'' - 5y' + 6y = 0$

   $r^2 - 5r + 6 = (r-2)(r-3)$

   $r = 2, 3$

   $y_h = Ae^{2t} + Be^{3t}$

2. RHS is e^t

   So guess $y_p = Ce^t$

(6)

$$Ce^{t} - 5Ce^{t} + 6Ce^{t} = e^{t}$$

$$2Ce^{t} = e^{t}$$

$$C = \frac{1}{2}$$

$$y_p = \frac{1}{2} e^{t}$$

3. $$y(t) = \frac{1}{2} e^{t} + Ae^{2t} + Be^{3t}$$

general sol

Find

$$y'(t) = \frac{1}{2} e^{t} + 2Ae^{2t} + 3Be^{3t}$$

$y]$ $$0 = \frac{1}{2} e^{0} + Ae^{2 \cdot 0} + Be^{3 \cdot 0}$$

$y]$ $$0 = \frac{1}{2} + A + B$$

$y']$ $$0 = \frac{1}{2} e^{0} + 2Ae^{2 \cdot 0} + 3Be^{3 \cdot 0}$$

$y']$ $$0 = \frac{1}{2} + 2A + 3B$$

(7)

$A = -B - \frac{1}{2}$

$0 = \frac{1}{2} + 2\left(-B - \frac{1}{2}\right) + 3B$

$0 = \frac{1}{2} - 2B - 1 + 3B$

$= -\frac{1}{2} + B$

$B = \frac{1}{2}$ ✓

~~Am~~ $0 = \frac{1}{2} + A + \frac{1}{2}$

$A = -1$ ✓

So essentially as before – but particular sol included in calculations

$Y_p(t) = \frac{1}{2} e^{t} - e^{2t} + \frac{1}{2} e^{3t}$

133

12/19

Unit 4

Systems

~~If~~ you have many dep variables
often only 1 ind variable
L like t = time

So if you wanted to model several
separate, but connected aspects

like position in 3D

$x(t)$
$y(t)$
$z(t)$

$$\underline{F}(t, \vec{s}(t)) = m\,\underline{s}''(t)$$

vector in bold

$$s = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} \qquad s'' = \begin{bmatrix} x''(t) \\ y''(t) \\ z''(t) \end{bmatrix}$$

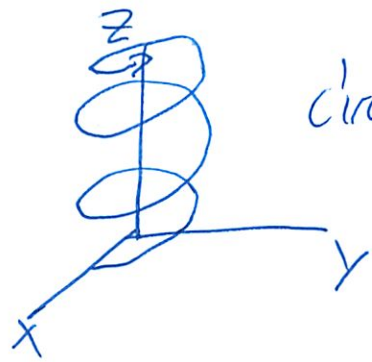
134

$\underline{s}(t)$ is really 3 equations

$$F_1(t, s(t)) = m x''(t)$$
$$F_2(\quad " \quad) = m y''(t)$$
$$F_3(\quad " \quad) = m z''(t)$$

So sol could be like

$$\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} \cos t \\ \sin t \\ t \end{bmatrix}$$



circle helix

Systems only deal w/ 1st order ODEs

Can introduce dummy variables

└ this always felt a bit weird

$$x(t) = x_1(t)$$
$$x'(t) = x_1'(t) = x_2$$
$$x''(t) = x_2'(t) = x_1''(t)$$

Introduce as many variables as you need

135

So 2 circuits

Left) $1\frac{dI_1}{dt} + 5(I_1 - I_2) - 10$

Right) $4Q_2 + 2I_2 + 5(I_2 - I_1) = 0$

$$4I_2 + 2I_2' + 5(I_2' - I_1') = 0$$

So system

$$\frac{dI_1}{dt} = 5I_2 - 5I_1 + 10$$

$$\frac{7dI_2}{dt} = 5I_1' - 4I_2$$

rewrite

$$= 5(2I_2 - 5I_1 + 10) - 4I_2$$

$$= 21I_2 - 25I_1 + 50$$

Can solve by eliminating a variable
↳ the backwards / shortcut way

Or linear algebra
↳ The right way

136

Eliminating a variable

$$I_2 = \frac{I_1' + 5I_1 - 10}{5} = \frac{I_1'}{5} + I_1 - 2$$

Then 2 ways to ~~[illegible]~~ express I_2'

1. From bottom pair of eqn
2. From taking deiv of I_2

$$I_2' = \frac{21I_2 - 25I_1 + 50}{7} = \frac{I_1''}{5} + I_1'$$

Sub in $I_2 = \frac{I_1'}{5} + I_1 - 2$

(I don't think we have to do this method...)

$$\frac{7}{5}I_1'' + \frac{14}{5}I_1' + 4I_1 = 8$$

↳ can solve w/ our normal methods

Then find I_1

Can find I_2

(137)

In general we'll deal w/ systems

$$x_1' = a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n + f_1(t)$$

$$\vdots$$

$$x_n' = a_{n1} x_1 + \cdots + a_{nn} x_n + f_n(t)$$

Basically

$$\begin{bmatrix} x_1' \\ \vdots \\ x_n' \end{bmatrix} = A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{bmatrix}$$

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

138

Homogeneous if $f(t) = 0$

then f vector is all 0s $\begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix}$

Rewrite things into system of equations

~~Old~~

So answer is

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = C_1 \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-4t} + C_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t}$$

eigenvectors eigenvalues

learn the terms!

Oh asked about w() in OH

L forgot about it since so simple

Solving w/ Eigenvalues w/ Linear Algebra

- Doing 2x 2 systems
- Ie Nuclear proliferation

$$x_1' = -3x_1 + x_2$$
$$x_2' = x_1 - 3x_2$$

(seems like a long + complicated explination way)

Guess

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} e^{\lambda t}$$

$$x' = \begin{bmatrix} (v_1 e^{\lambda t})' \\ (v_2 e^{\lambda t})' \end{bmatrix} = \lambda \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} e^{\lambda t}$$

$$\underline{x'} = A\underline{x}$$

$$A = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix}$$

Substitute in guess

(140)

$$\lambda \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = A \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$\hookrightarrow$ $\lambda v_1 = -3v_1 + v_2$

$\lambda v_2 = v_1 - 3v_2$

Now solve
└ but 2 eq, 3 unknowns!

Declare λ a fixed constant
Solve for v_1, v_2. Try.

$$\lambda v_1 + 3v_1 = v_2$$

$$(\lambda + 3) v_1 = v_2$$

$$v_1 = \frac{v_2}{\lambda + 3}$$

$$\lambda v_2 = \frac{v_2}{\lambda + 3} - 3v_2$$

$$(\lambda + 3)\lambda v_2 = v_2 - 3v_2(\lambda + 3)$$

$$\lambda^2 v_2 + 3\lambda v_2 = v_2 - 3\lambda v_2 - 9v_2$$

141

ii this does not seem to be working

$$\lambda^2 v_2 + 3\lambda v_2 - v_2 + 3\lambda v_2 + 9 v_2 = 0$$

$$\lambda^2 v_2 + 6\lambda v_2 + 8 v_2 = 0$$

$$(\lambda^2 + 6\lambda + 8) v_2 = 0$$

$$v_2 = 0$$

$$v_1 = \frac{0}{\lambda + 3}$$

$$= 0$$

ii wrong

No we care when this has non-zero sol

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

* This system has non-zero vector sol iff det of matrix is 0

$$\begin{bmatrix} -3-\lambda & 1 \\ 1 & -3-\lambda \end{bmatrix}$$

So $(-3-\lambda)(-3-\lambda) - (1)\cdot(1) = 0$

this is more familiar now

$$\lambda^2 + 6\lambda + 9 - 1 = 0$$

$$\lambda^2 + 6\lambda + 8 = 0$$ ← had something like this earlier

But now set λ so this is true??

$\lambda = -2, -4$

Same as factoring - yea?

$$\frac{-6 \pm \sqrt{36 - 4\cdot 1\cdot 8}}{2}$$

-3 ± 1 ✓

Now try each value

$\lambda = -2$

$$\begin{bmatrix} -1 & 1 \\ +1 & -1 \end{bmatrix}$$

} rows are multiples of each other

This means something

what??

(143)

So it does mean we can solve top or bottom

└ But I think it means something else too

$$-V_1 + V_2 = 0$$
$$V_1 - V_2 = 0$$ } either same

$$V_1 = V_2$$

Then lots of multiples that will solve this

Like $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ $\begin{bmatrix} 3 \\ 3 \end{bmatrix}$ since $V_1 = V_2$

↑ use

So we have

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t}$$

<u>2nd one</u>

$\boxed{\lambda = -4}$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

144

$$V_1 = -V_2$$

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-4t}$$

Put it together

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-4t}$$

So

$$Y_1 = 1e^{-2t} + 1e^{-4t}$$

$$Y_2 = 1e^{-2t} - 1e^{-4t}$$ ✓

Now some shortcuts to the method

$$\det(A - \lambda I) = 0$$

$$\lambda I = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

is a polynomial in λ
"charastic polynomial"

So in the 2x2 case

$$\det(A - \lambda I) = \lambda^2 - \underbrace{(a+d)}_{\text{trace}(A)}\lambda + \underbrace{(ad-bc)}_{\det(A)}$$

So about the dummy variables

If get

$$x_1' = x_2$$
$$x_2' = -6\underbrace{x_1'}_{\text{can't have - so sub in}} - 8x_1$$

$$x_1' = x_2$$
$$x_2' = -6x_2 - 8x_1$$

Much better :)

$$\begin{array}{rcc l} & & & \\ A = & \begin{pmatrix} 0 & 1 \\ -8 & -6 \end{pmatrix} & & \begin{array}{l} \leftarrow x_1' \\ \leftarrow x_2' \end{array} \\ & \begin{array}{cc} \uparrow & \uparrow \\ x_1 & x_2 \end{array} & & \end{array}$$

Complex Eigenvalues

Just solve the complex values

Take the $Re()$, $Im()$ parts

Example

$$\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} = A$$

$$\det(A - \lambda I) = \lambda^2 - 4\lambda + 5$$

$$\frac{4 \pm \sqrt{16 - 4 \cdot 1 \cdot 5}}{2}$$

$$\lambda = 2 \pm i$$

Now do it like before

λ $\underline{2 + i}$

$$\begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix} \quad \text{bottom} = i \cdot \text{top}$$

$$i v_1 - v_2 = 0$$
$$v_1 + i v_2 = 0$$

So now what - like before?

$$iv_1 = v_2$$

$$v_1 + i(iv_1) = 0$$

$$v_1 - v_1 = 0$$

helpful...

$$v_1 = \frac{v_2}{i}$$

$$\frac{v_2}{i} + iv_2 = 0$$

$\cdot i$

$$v_2 + -1v_2 = 0$$

0...

Sol is a complex mutiple of

they did $\ominus$ minus

$$\begin{bmatrix} 1 \\ i \end{bmatrix} e^{(2-i)t}$$

How did they get that

L guess + check?

I think the prof said that

(148)

Then you only need 1 since the other is the same

$$= \begin{bmatrix} 1 \\ i \end{bmatrix} e^{2t} \left(\cos(-t) + i\sin(-t) \right)$$

So this is like before

$$= \begin{bmatrix} 1 \\ i \end{bmatrix} e^{2t} \left(\cos t - i\sin t \right)$$

$$= \begin{bmatrix} e^{2t}\cos t - e^{2t} i\sin t \\ i e^{2t}\cos t + e^{2t}\sin t \end{bmatrix}$$

Take Re(), Im() of each part

$$Re = \begin{bmatrix} e^{2t}\cos t \\ e^{2t}\sin t \end{bmatrix}$$

$$Im = \begin{bmatrix} -e^{2t}\sin t \\ e^{2t}\cos t \end{bmatrix}$$ ①

Put it together

$$= A e^{2t} \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} + B e^{2t} \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix}$$

(4.9)

MATH L31 Complex Eigenvalues

didn't we just do this?

(stuff makes so much more sence then it did in lecture)..

So usually that solving methods works for complex values

Repeated Eigenvalues

Like $(\lambda-1)^2(\lambda+3)$

Need each eigenvalue to make an eigenvector

$\boxed{\lambda}$ $\boxed{\underline{v}}$

- Complex case - have enough what??
- incomplete case - dont " "

└ linearly ind eigenvectors (not multiples of each other - can check w/ Wronskian)

(150) Complete example

On 3×3 case

$$\begin{pmatrix} -2-\lambda & 1 & 1 \\ 1 & -2-\lambda & 1 \\ 1 & 1 & -2-\lambda \end{pmatrix}$$

expand by minors

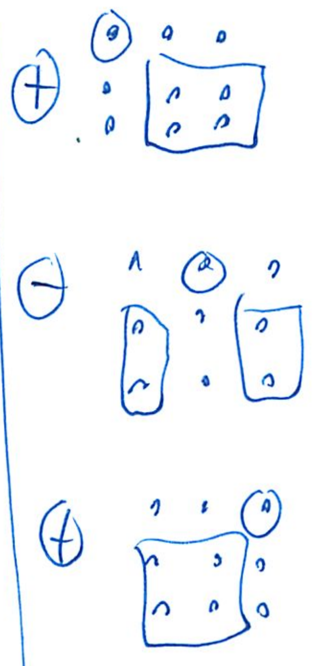
$$-2-\lambda \det\begin{bmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{bmatrix}$$

$$- 1 \cdot \det\begin{bmatrix} 1 & 1 \\ 1 & -2-\lambda \end{bmatrix}$$

$$+ 1 \cdot \det\begin{bmatrix} 1 & -2-\lambda \\ 1 & 1 \end{bmatrix}$$

$$= \lambda(\lambda+3)^2$$

repeated



151

So

$\boxed{\lambda = -3}$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

Complete example

since 1 diff eq

3 variables

2 degrees of freedom

└ can pick anything for 2 of them - 3rd one falls in place

$v_1 + v_2 + v_3 = 0$

possible $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

So have 2 of them

$$x(t) = \underbrace{C_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{0t}}_{\text{from the } \lambda = 0 \text{ one I skipped over}} + C_2 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} e^{-3t} + C_3 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{-3t}$$

152

(3 more lectures!)

L32 Repeated Eigen Values

$$(\lambda - \lambda_1)^2 (\lambda - \lambda_2)$$

Might not be able to find enough eigenvectors

we can always ~~need~~ find 1

but need enough for to = multiplicity

What's this?

WP1

= Multiplicity of corresponding root of polynomial

So like the largest power of

$(\lambda - 4)^{4}$ = multiplicity (here = 4)

Incomplete example

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$x_1' = x_1 + x_2$$

$$x_2' = x_2$$

$$\lambda^2 - 2\lambda + 1$$

So $$\frac{2 \pm \sqrt{4-4\cdot1\cdot1}}{2}$$

$$\lambda = 1$$

$\underline{\lambda = 1}$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$V_2 = 0$$

$$0 v_1 + 0 v_2 = 0$$

So any multiple of

$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an Eigenvector

?? how get that? guess

- well bottom must be 0
- but top can be anything
- simplest is 1
- but only 1 linear ind since ?? only 1 degree of freedom?

But thats' it - only 1 linearly ind eigenvector

$$x(t) = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{1t} + \underbrace{\qquad\qquad}_{\text{need 2nd sol}}$$

does not work

Could it be

$$c_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \underbrace{t}_{\text{add our ol' favorite trick}} e^{t}$$

Can check

↳ No!

So instead

$$x(t) = \left(\underline{u} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} t\right) e^{t}$$

$$x'(t) = u e^{t} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} (1+t) e^{t}$$

$$A \underline{x} = A \underline{u} e^{t} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} t e^{t}$$

Compare the 2 sides

Cancle $\quad t e^{t} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} t e^{t}$

Left w/

$$1 \cdot \underline{u} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} = A \underline{u}$$

155

$(A - 1 I)\,\underline{u} = \begin{pmatrix}1\\0\end{pmatrix}$

Need to solve for a u that does that

$$(A - I)\,u = \begin{pmatrix}1\\0\end{pmatrix}$$

$$\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}\begin{pmatrix}u_1\\u_2\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$$

So pick

$$\begin{pmatrix}u_1\\u_2\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}$$

(but can pick any # on top $\begin{pmatrix}78\\1\end{pmatrix}$)

So Now we have our 2 sols

$$\underline{x}(t) = c_1\begin{bmatrix}1\\0\end{bmatrix}e^{t} + c_2\left(\begin{bmatrix}0\\1\end{bmatrix} + \begin{bmatrix}1\\0\end{bmatrix}t\right)e^{t}$$

Fundamental matrix F

this confused me somewhat
but it might just be other notation

matrix whose column vectors are lin ind sols
to $\underline{x}' = A\,\underline{x}$

156

So it had $A = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix}$

Get $\underline{x} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-4t}$

So F is (is or is a good ~~a~~ ~~ca~~ choice i.e.)

$$F = \begin{pmatrix} e^{-2t} & e^{-4t} \\ e^{-2t} & e^{-4t} \end{pmatrix}$$

So all sols $\underline{X} = F \cdot \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

F satisfies $\underline{F}' = A F$

on the right

Can use to solve IVPs

$\underline{X}(t_0) = X_0$

$\underline{X_0} = F(t_0) \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

157

So if $t_0 = 0$

$$\cancel{F(0)}\ F(t_0) = F(0) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Take inverse (will always be invertable)

$$F(t_0)^{-1} = \frac{-1}{2} \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}$$

Inverse of a matrix

$$A^{-1} = \underbrace{\frac{1}{|A|}}_{\text{det}} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

swap

multiply each by -1

So sol to IVP

$$X(t) = F(t)\, F(0)^{-1} X_0$$

$$= \begin{pmatrix} e^{-2t} & e^{-4t} \\ e^{-2t} & e^{-4t} \end{pmatrix} \cdot -\frac{1}{2} \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_{1,0} \\ x_{2,0} \end{pmatrix}$$

plug in IVs

Do matrix multiplication

Something w/ e^{At} I didn't get end

↳ didn't get in last few recitations either

(158)

Exponential Matrix

Recall Taylor series for e^x is (at $x=0$)

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Similar def for matricies

$$e^A = \underline{I} + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \dots$$

$\llcorner A^3 = A \cdot A \cdot A$

that looks very annoying...

more later

$$e^{At} = I + At + \frac{A^2 t^2}{2!} + \dots$$

So $e^{At} \underline{x}_0$

(more in seperate lecture

$\llcorner$ Prof always trys to cranch stuff in last few min)

(33)

$$e^A = \underset{\begin{pmatrix}1&0\\0&1\end{pmatrix}}{I} + \begin{pmatrix}1&0\\0&0\end{pmatrix} + \frac{\begin{pmatrix}1&0\\0&0\end{pmatrix}}{2!} + \dots$$

$$= \begin{pmatrix} 1+1+\frac{1}{2}+\frac{1}{3!}+\dots & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} e & 0 \\ 0 & 1 \end{pmatrix}$$

159

In general, can't compute e^A from power series expansion for an arbitrary matrix (? since its a lot of math?)

But usually we look at e^{At}

$$e^{At} = I + \frac{At}{1} + \frac{A^2t^2}{2!} + \cdots$$

eg $At = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} t = \lambda t \cdot I$

So $e^{at} = \begin{pmatrix} e^{\lambda t} & 0 \\ 0 & e^{\lambda t} \end{pmatrix}$

But ~~a~~ can only compute e^{At} if it is really easy

Simplest

$A = 1 \cdot 1$ matrix $=$ a # called a

$e^{At} = e^{at}$

So it solves the ODE

$\underline{x}' = a\underline{x}$

~~a~~

Want $e^{At} \cdot \underline{c}$ to solve $\underline{x}' = A\underline{x}$

↑ constant vector

So this works w/ any linear system w/ constant coeffs

↳ inc repeated roots

Even works for $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

But computing e^{At} is hard

So we need n linearly ind sols

What are good choices for c?

↳ Pick $\underline{c}$ so that $e^{At} \cdot \underline{c}$ has only finetly many non-zero terms in the power series.

So choose $\underline{c}$ to be an eigenvector for a nice eigenvalue

$$e^{At} \cdot \underline{c} = \underbrace{e^{(A-\lambda I)t} e^{\lambda I t}}_{e^{At}} \underline{c}$$

(161)

$$= e^{(A-\lambda I)t} \cdot e^{\lambda t} \underline{c}$$

$$= e^{\lambda t} e^{(A-\lambda I)t} \underline{c}$$

(I don't get this at all ...)

$$= e^{\lambda t}\left(I + (A-\lambda I)t + \frac{(A-\lambda I)^2 t^2}{2!} + \dots\right)\underline{c}$$

What happens if c is an eigenvector w/ eigenvalue λ?

$(A-\lambda I)\underline{c} = 0$

$(A-\lambda I)^2\underline{c} = 0$ ← since 1st term killed it

$(A-\lambda I)^n\underline{c} = 0$

Most terms now 0

↳ think saw in OH

$$= e^{\lambda t}(I + 0 + 0 + 0 + 0 + \dots)\underline{c}$$

$$= e^{\lambda t}\underline{c}$$

But works in even more ~~sp~~ sophisticated cases

If $\underline{c}$ is killed by $(A-\lambda I)^2$

but not $(A-\lambda I)$

Then soln

$$e^{At} \cdot \underline{c} = e^{\lambda t}\left(I + (A-\lambda I)t\right)\underline{c}$$

$\llcorner$ first 2 terms survive

Example

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Want $e^{At} \cdot c$

$$\det(A - \lambda I) = (\lambda - 1)^2$$

So one possibility

$$A - \underline{1}I = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

So anything $\begin{pmatrix} c_1 \\ 0 \end{pmatrix}$ works

$\llcorner$ pick $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

163

But need 2nd ind sol

$\llcorner$ find one killed by $(A - \lambda I)^2$

$\boxed{e^{At} \underline{c}}$

$$(A - \lambda I)^2 \underline{c} = \underline{0}$$

$\llcorner$ but $(A - \lambda I)\underline{c} \neq 0$

since want to find a new one

we found before $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

So can choose $\underline{c}$ such that

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

can pick anything

(just not a multiple of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$)

So $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

So now plug c into eqn

$$e^{At}\begin{pmatrix}0\\1\end{pmatrix} = e^{It}\underbrace{\underbrace{\left(I + \begin{pmatrix}0&1\\0&0\end{pmatrix}t\right)\begin{pmatrix}0\\1\end{pmatrix}}_{\text{matrix multiplication}}}$$

$$\underbrace{\begin{pmatrix}1&t\\0&1\end{pmatrix}\qquad\begin{pmatrix}0\\1\end{pmatrix}}_{\text{matrix multiplication}}$$

$$\begin{pmatrix}t\\1\end{pmatrix}$$

$$= e^{t}\begin{bmatrix}t\\1\end{bmatrix} \text{ is a sol}$$

Then other ind sol from other version of formula

? which one ?

So all together

$$x(t) = Ae^{t}\begin{bmatrix}t\\1\end{bmatrix} + Be^{t}\begin{bmatrix}1\\0\end{bmatrix}$$

(165)

So $e^{At} = F(t) \cdot F(0)^{-1}$

fundamental matrix

Same because both are sols to same IVP

L34 Drawing Pics of Sol (last lecture)

Mostly 2×2 of the form

$$\frac{dx_1}{dt} = F(x_1, x_2)$$

$$\frac{dx_2}{dt} = G(x_1, x_2)$$

includes $\underline{x}' = A x$

Critical pts

one of the most important things

a pt (a,b) where $(x_1', x_2') = (0,0)$

ie (a,b) s.t. $F(a,b) = 0$

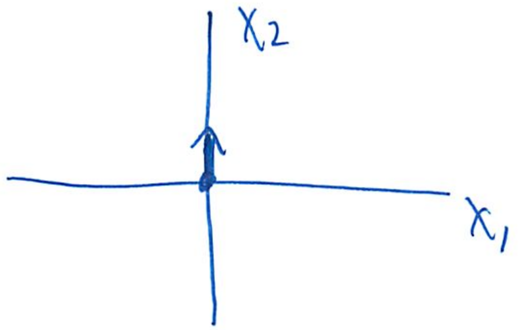
$G(a,b) = 0$

(166)

If a sol arrives to one of these points
it will never leave!

Ex
$$x_1' = x_1 - \tfrac{1}{2}x_2$$
$$x_2' = 1 - x_1^2 \leftarrow \text{not linear}$$
So can't solve systematically

But can plot phase plane



1. Pick pt
2. Plug value in
3. Draw unit vector

Etc do for all pts

(167)

Now find critical pts

$x_2' = 0$

$1 - x_1^2 = 0$

$\boxed{x_1 = \pm 1}$

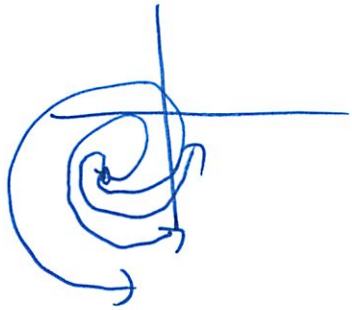
$x_1^2 = 0$

$\boxed{2x_1 = x_2}$

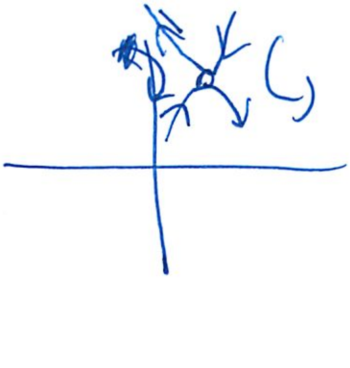
So that means

$(1,2)$, $(-1,-2)$

interesting stuff around these



"unstable spiral"



"saddle"

168

; So can review by looking at linear sols ...

Plot on PPlane

Can ~~plot off~~ build a dictionary of common pt behaviors

Sols occur when eigenvalues of A are
complex of form $a+bi$

Then can apply patterns to non linear systems
L linearize - linear approximate

Can change variables ~~to~~ so center around origin
L so have homogeneous sol

w/ Jacobian Matrix $A = \begin{bmatrix} F_{x_1}(a,b) & F_{x_2}(a,b) \\ G_{x_1}(a,b) & G_{x_2}(a,b) \end{bmatrix}$

Ok will be one nonlinear eq on exam
So basically say at a certain pt
its behavior is "close enough" to
some other expression

Done!!!!!!!

(169)

Review Notes

$-y^3 + 3y + x$

↑ non linear

So can ask long term behavior

- direction fields
- fences / funnels

Or numerical approx

- Euler's

but not exact ans!

$C'(x) < f(x, C(x))$

Find 0-isoline

Try to factor

Now find upper fence

└ differentiate

Find other fence

└ 1-isocline

(170)

Remember $\lim \left(\frac{\text{one}}{\text{other}}\right)$ → highest power dominates

; I guess see if funnel ;

Modeling

4 types of first order qv
(I should review one of each)

1. Exponential Growth / decay

$$\frac{dx}{dt} = kx$$

Population growth

β = births per individual per unit of time
δ = deaths "

$$\Delta P = (\beta - \delta) P(t) \Delta t$$ makes sense...

$$\frac{dP}{dt} = \lim_{t \to 0} \frac{\Delta P}{\Delta t} = kP$$

$k = \beta - \delta$

(171)

Compounded Interest

$$\frac{dA}{dt} = \lim_{\Delta t \to 0} \frac{\Delta A}{\Delta t} = rA$$

↑ annual interest rate r
compounded continuously

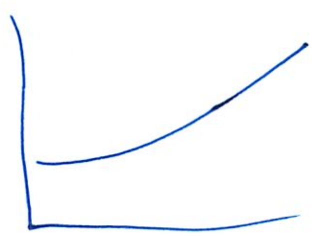
Radioactive decay

$$\frac{dN}{dt} = -kN$$

All solve by seperation

$A e^{kt}$

↑ e^{c}



Logistic Equation

- bounded max pop level
  - ie a fixed food supply

Birth rate is linear decreasing fn of population size P

so $\beta = \beta_0 - \beta_1 P$

$\oplus$ constants

if δ = death rate remains constant

$$\frac{dP}{dt} = (\beta_0 - \beta_1 P - \delta)P$$ ← but that is

$$= \underbrace{a}_{\beta_0 - \delta_0} P - \underbrace{b}_{\beta_1} P^2$$ ← Thats not intuitive

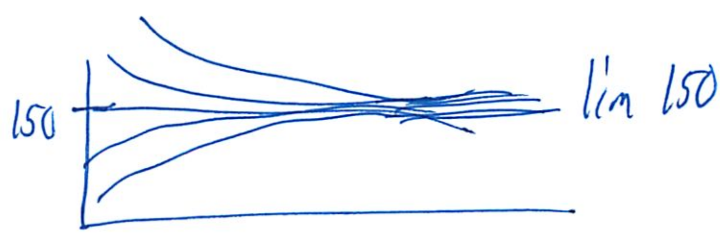
¿But how do we account for max pop?

If a, b $\oplus$ then logistic eq

$$\frac{dP}{dt} = \underbrace{k}_{k=b} P(\underbrace{M}_{M=\frac{a}{b}} - P)$$

Limiting Population

So if its 150



173

$$\lim_{t \to \infty} P(t) = \frac{MP_0}{P_0 + 0} = M$$

carrying / limiting population

Anyway in the notes:

$$aP - bP^2$$

natural growth

$\frac{a}{b}$ = limiting pop

Look at this qu on exam

Harveesting

add

$$aP - bP^2 - h$$

bifurcation diagrams

---

3. Mixing salt problems

$$\frac{d\,\text{Salt}}{dt} = \text{rate in} - \text{rate out}$$

as a variable against S

don't forget make into concentration w/ $\frac{1}{\text{capacity}}$

174

Try example

30 is there

$$\frac{dS}{dt} = 40\frac{g}{L} \cdot 20\frac{L}{min} - 20\frac{L}{min} \cdot \frac{\cancel{30}\,1}{10,000}\frac{g}{L}\,S$$

when does $S(t) = 35$

$S(0) = 30$ I think

Try $\frac{dS}{dt} = 800 - \frac{S}{500}$

↑ Move S ↑ t

← no +

$\frac{S}{500}\ \frac{dS}{dt} = 800$

$S\,dS = 400,000$

I think wrong...

Book

$$\int \frac{dS}{800 - \frac{S}{500}} = \int dt$$

Move it all over - don't split the addition

? Now how to integrate??

(175)

$-500 \ln\left(800 - \frac{s}{500}\right)$

divide by 500, raise e^

$800 - \frac{s}{500} = e^{-t/500} + C$

WA

since $u = 800 - \frac{s}{500}$

$du = -\frac{1}{500}\, ds$

right $\frac{1}{\frac{1}{-500}} = -500$

$= -500 \int \frac{1}{u}\, du$

$= -500 \ln(u)$

Oh that makes sense

Stop + think!

How would you actually do it

L do it more myself on p-sets

$-\frac{s}{500} = e^{-t/500} - 800 + C$

$\cdot -500 \qquad -500$

$s = -500\, e^{-t/500} + 400\,000 + C$ ✓

176

4. Newton's Law of Cooling

$$\frac{\partial T}{dt} = k(\underbrace{T_e}_{\text{temp external}} - \underbrace{T}_{\text{temp of object}})$$

$$\frac{dt}{dt} + kT = k\underbrace{T_e}_{\text{Sinusoidal usually}}$$

Can complexify + solve

Integrating factor

linear + non-constant coefficients

$$x^3 \frac{dy}{dx} + \underbrace{x^2}_{P(x)} y = \underbrace{\cos x}_{Q(x)}$$

Integrating factor

$$e^{\int P(x)}$$

$$e^{x^3/3}$$

$$Y(x)\, e^{x^3/3} = \int \left(\cos x\, e^{x^3/3}\right) dx$$

↑ integrate

→ divide by

(1767)

?? They did it a diff way?

└ why is $P(x) = \frac{1}{x}$

Ohh I think you have to ~~add~~ divide initial factor!

$$\frac{dy}{dx} + \frac{x^2}{x^3} y = \frac{\cos x}{x^3}$$

$$\frac{dy}{dx} + \frac{1}{x} y = \frac{\cos x}{x^3}$$

$\int \frac{1}{x} = e^x$
first thing on formula sheet!

$$e^{\int P(x)\,dx} = e^{\int 1/x\,dx} = e^{\ln x} = x$$

$$y \cdot x = \int \frac{\cos x}{x^3} \cdot x \, dx$$

$$y = \frac{1}{x} \int \frac{\cos x}{x^2} \, dx \quad ✓$$

Prof says we can leave it at that (on exam - or just here?)

This is not a calculus class...

178

In notes they show an IVP

$y(1) = 4$

$$y = \frac{1}{x}\left(4 + \int_1^x \frac{\cos t}{x^2}\,dt\right)$$

They don't show where they got it

└ oh well

18.03 Practice Final Exam 1

1. This problem concerns the differential equation

$$\frac{dy}{dx} = x^2 - y^2 \qquad (*)$$

Let $y = f(x)$ be the solution with $f(-2) = 0$.

(a) Sketch the isoclines for slopes -2, 0, and 2, and sketch the direction field along them.

(c) On the same diagram, sketch the graph of the solution $f(x)$. What is its slope at $x = -2$?

(d) Estimate $f(100)$.

(e) Suppose that the function $f(x)$ reaches a maximum at $x = a$. What is $f(a)$?

(f) Use two steps of Euler's method to estimate $f(-1)$.

2. In **(a)–(b)** we consider the autonomous equation $\dot{x} = 2x - 3x^2 + x^3$.

(a) Sketch the phase line of this equation.

(b) Sketch the graphs of some solutions. Be sure to include at least one solution with values in each interval above, below, and between the critical points.

(c) Some solutions have points of inflection. What are the possible values of $x(a)$ if a non-constant solution $x(t)$ has a point of inflection at $t = a$?

(d) A radioactive isotope of the element Bostonium, Bo, decays with half life of two years to an isotope of Cantabrigium, Ct. The MIT reactor is loaded with this material. At $t = 0$ there is no Ct in it, but starting at $t = 0$ Ct is added in such a way that the cumulative total amount inserted by time t is t moles.

Write down a differential equation for the number of moles of Ct in the reactor as a function of time. What is the initial condition?

(e) Solve the initial value problem $x\dfrac{dy}{dx} + 3y = x^2$, $y(1) = 1$.

3. **(a)** Find non-negative real numbers A, ω, and ϕ such that $\mathrm{Re}\left(\dfrac{ie^{2it}}{1+i}\right) = A\cos(\omega t - \phi)$.

(b) Sketch the trajectory of $e^{(1-\pi i)t}$.

(c) Express the cube roots of $8i$ in the form $a + bi$ (with a and b real).

4. **(a)–(c)** Find one solution to $\ddot{x} + 2\dot{x} + 2x = q(t)$ for

(a) $q(t) = t^2 + 1$.

(b) $q(t) = e^{-2t} + 1$.

(c) $q(t) = \sin t$. What is the amplitude of the sinusoidal solution?

In **(d)** and **(e)**, suppose that that t^3 is a solution to $\ddot{x} + 2\dot{x} + 2x = q(t)$.

(d) What is $q(t)$?

(e) What is the general solution to $\ddot{x} + 2\dot{x} + 2x = q(t)$?

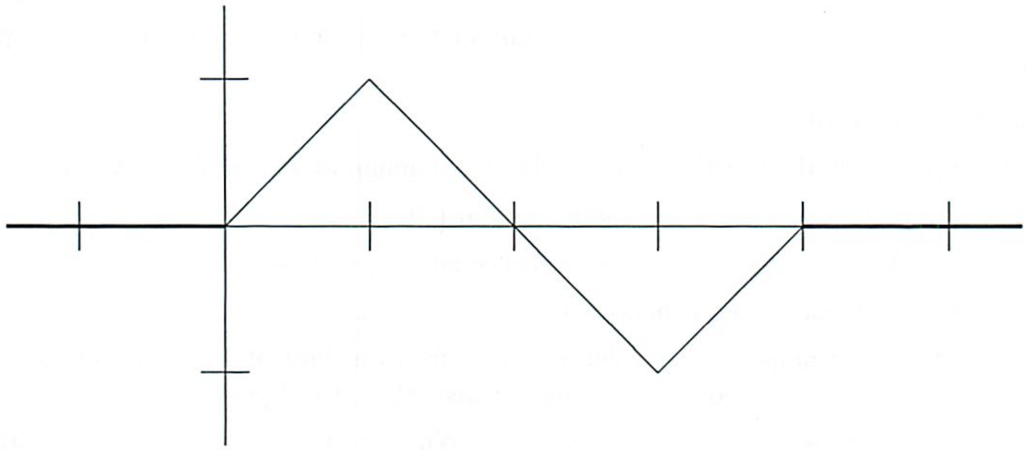
5. (a)–(b) concern the function $f(t) = \mathrm{sq}(t + \frac{\pi}{2})$ [1].

(a) Graph $f(t)$.

(b) What is its Fourier series? (Simplify the trig functions.)

(c) Find a solution to $\ddot{x} + x = \mathrm{sq}(t)$.

6. (a)–(d) In a recent game of Capture the Flag, a certain student was observed to move according to the following graph, in which the hashmarks are at unit spacing.



(a) Graph the generalized derivative $v(t)$.

(b) Write a formula for $v(t)$ in terms of the unit step and (if necessary) the delta function.

(c) Still with the same function as in (a): Graph the generalized derivative $\dot{v}(t)$.

(d) Write a formula for the acceleration $\dot{v}(t)$ in terms of the unit step and (if necessary) the delta function.

(e) Suppose that the unit impulse response of a certain operator $p(D)$ is $w(t)$. Let $q(t) = 0$ for $t < 0$ and $t > 1$, and $q(t) = 1$ for $0 < t < 1$. Please find functions $a(t)$, $b(t)$ so that the solution $x(t)$ to $p(D)x = q(t)$, with rest initial conditions, is given by

$$x(t) = \int_{a(t)}^{b(t)} w(\tau)\, d\tau$$

7. This problem concerns the operator $p(D) = 2D^2 + 8D + 16I$.

(a) What is the transfer function of the operator $p(D)$?

(b) What is the unit impulse response of this operator?

(c) What is the Laplace transform of the solution to $p(D)x = \sin(t)$ with rest initial conditions?

[1] If $\mathrm{sq}(t)$ is the odd function of period 2π which has value 1 between 0 and π, then

$$\mathrm{sq}(t) = \frac{4}{\pi}\left(\sin(t) + \frac{\sin(3t)}{3} + \frac{\sin(5t)}{5} + \cdots\right)$$

8. In (a) and (b), $A = \begin{bmatrix} 2 & 12 \\ 3 & 2 \end{bmatrix}$.

(a) What are the eigenvalues of A?

(b) For each eigenvalue, find a nonzero eigenvector.

(c) Suppose that the matrix B has eigenvalues 1 and 2, with eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ respectively. Calculate e^{Bt}.

(d) What is the solution to $\dot{\mathbf{u}} = B\mathbf{u}$ with $\mathbf{u}(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$?

9. (a) Suppose again that the matrix B has eigenvalues 1 and 2, with eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ respectively. Sketch the phase portrait on the graph below.

(b) Let $A = \begin{bmatrix} a & -2 \\ 2 & 1 \end{bmatrix}$, and consider the homogeneous linear system $\dot{\mathbf{u}} = A\mathbf{u}$. For each of the following conditions, determine all values of a (if any) which are such that the system satisfies the condition.

(i) Saddle
(ii) Nodal Sink
(iii) Nodal Source
(iv) Spiral Sink (Indicate Direction)
(v) Spiral Source (Indicate Direction).
(vi) Unstable

10. Parts (a)–(c) deal with the nonlinear autonomous system $\begin{cases} \dot{x} = x^2 - y^2 \\ \dot{y} = x^2 + y^2 - 8 \end{cases}$.

(a) Find the equilibria of this system.

(b) There is one equilibrium in the south-west quadrant. Find the Jacobian at this equilibrium.

(c) The equilibrium you found in (b) is a stable spiral. For large t, the solutions which converge to this equilibrium have x-coordinate which are well-approximated by the function $Ae^{at}\cos(\omega t - \phi)$ for some constants A, ϕ, a, and ω. Some of these constants depend upon the particular solution, and some are common to all solutions of this type. Find the values of the ones which are common to all such solutions.

(d) Finally, return to the autonomous equation $\dot{x} = 2x - 3x^2 + x^3$ that you studied in problem 2. Write down a formula approximating the solutions converging to the stable equilibrium when t is large.

18.03 Pactice Final Exam 2

1. (15) Find the general solution to $xy' - 2y = x^3$.

2. (15) Find the general solution to $xy' = y + \sqrt{x^2 - y^2}$.

3. (10) Sketch a direction field for $y' = x^2 - y$, showing three isoclines, corresponding to the slopes $1, 0, -1$. Draw in some integral curves.

4. (10) A population is modeled by the ODE $\frac{dy}{dt} = y^3 - 2y^2 + y$. Find the long-term constant populations, sketch roughly some of the solutions, and tell what happens as $t \to \infty$ to the two populations whose starting values at the time $t = 0$ are respectively $y_1 = .5$ and $y_2 = 1.5$.

5. (10) A three-liter tank contains a salt solution, continuously mixed. Salt solution with a declining concentration $3e^{-t}$ flows into the tank at a rate r, and the solution in the tank flows out at the same rate. Write two linear ODE's, one for the amount $x(t)$ of salt in the tank at time t, and one for its concentration $C(t)$.

6. (15) a) Find by using polar representation the value of $(-1 + i\sqrt{3})^9$.
b) Use this to find the value of $D^9(e^{-t} \sin \sqrt{3}t)$.

7. (15) Find the solution $y(x)$ to $y'' + 4y' + 4y = 0$ satisfying $y(0) = 1$, $y(1) = 0$.

8. (15) Find by using complex exponentials a particular solution to $y'' - y' + 2y = e^t \sin t$.

9. (10) The ODE $x'' + 4x' + cx = 0$ represents a mass-spring-dashpot system. For what values of the spring constant c will the mass oscillate, and what will be its pseudoperiod, i.e., how much time will be needed for a single back-and-forth oscillation?
(Answer in terms of c.)

10. (25: 10, 5, 8, 2)
a) Find the Fourier sine series $\sum b_n \sin n\pi t$ for $1 - t$, on the interval $(0, 1]$.
b) Sketch over the interval $[-3, 3]$ the function $f(t)$ to which the Fourier series you found in (a) converges.
c) Find a Fourier sine series solution $y(t)$ satisfying on $(0, 1)$ the ODE (k constant) $y'' + ky = 1 - t$, and the conditions $y(0) = 0$, $y(1) = 0$.
d) Which term in the Fourier series solution in (c) would dominate (i.e., be resonant) if $k = 90$?

11. (10) Derive a formula expressing $\mathcal{L}(f(at))$ in terms of $\mathcal{L}(f(t))$. Assume $a > 0$ and $f(t)$ is of exponential type; let $F(s) = \mathcal{L}(f(t))$.

12. (15) Solve the IVP $y'' - y = e^t$, $y(0) = 0$, $y'(0) = 1$ by using the Laplace transform.

13. (10) Find $\mathcal{L}^{-1}\left(\frac{1}{s} - \frac{e^{-s}}{s(s+1)}\right)$; express your answer in the "cases" format, and sketch its graph for $t \geq 0$.

14. (10) Starting at time $t = 0$, a biotech plant dumps continuously into a lake radioactive waste having the decay constant k (i.e., with the usual notation, $A = A_0 e^{-kt}$).
Let $f(t)$ be the dumping rate; this means that over the time interval $t_0 \leq t \leq t_1$,

$$\text{the amount dumped} \approx f(t_0)(t_1 - t_0).$$

Derive a formula for the amount of radioactive material in the lake at time $t = x$.

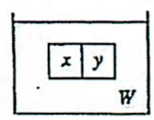
15. (15) For the linear system $\mathbf{x}' = \begin{pmatrix} a & 1 \\ 2 & a \end{pmatrix} \mathbf{x}$, draw an a-axis, and indicate on it for which values of a the trajectories will form a node, a spiral, or a saddle, and indicate whether these will be unstable or asymptotically stable. (Show work or indicate reason.)

16. (20) For the system $x' = xy - y^2, \;\; y' = x^2 - 2y + 1$,
 a) show it has just one critical point;
 b) linearize it at the critical point, and show you get a borderline case (i.e., one not structurally stable);
 c) draw the trajectories of the linearized system;
 d) draw sketches showing the different possible geometric types and stability for the trajectories of the non-linear system in the vicinity of the critical point.
 e) Why is America's Cowboy Poet a real degenerate case?

17. (10) Show that $x'' + x' \sin x + (x^2 + 1) = 0$ has no periodic solutions by converting it to a system and showing that the system has no closed trajectories.

18. (5, 10, 20) The rate at which heat flows by conduction between two adjacent bodies is proportional to the difference in their temperatures.

Two identical metal cubes having temperatures respectively $x(t)$ and $y(t)$ have two adjacent faces separated by a partially insulating thin film; the system is immersed in a water bath whose temperature $W(t)$ is falling exponentially: $W(t) = e^{-t}$.



 a) Taking the constants of proportionality to be 1, show the system is modeled by
$$\mathbf{x}' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}.$$
 b) Find the complementary solution (the gen. soln: to the assoc. homog. equation).
 c) Find the general solution to the system by variation of parameters.

19. (10) A linear system $\mathbf{x}' = A\mathbf{x}$ has the general solution $\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t}$. Find e^{At} (you can leave it in factored form) and find a matrix D such that the change of variable $\mathbf{u} = D\mathbf{x}$ decouples the system.

20. (10) A lake has bass and minnows; let the sizes of the two populations be respectively $B(t)$ and $M(t)$. The bass eat the minnows; assume the two populations fluctuate around a long-term equilibrium point according to the Volterra model:
 $B' = -k_1 B + c_1 BM, \qquad M' = k_2 M - c_2 BM, \qquad c_i$ and k_i positive constants.
 Starting at time $t = 0$, fishermen catch the bass at the rate of 10% per year. How does this change the long-term equilibrium point? What happens to the two populations?

Exam formulas: those on exams 2 and 3, plus var'n of par's formula $\mathbf{x}_p = X \int X^{-1} \mathbf{r}\, dt$.

Other Practice Questions: review the four hour exams and four practice hour exams; for further practice:
1. First-order equations: 1A-4a,5b; 1B-3b,9a; 1C-1e, 4; 1D-1a,2b, 3abc; 1E-1cd
2. Cx. nos., 2nd order equations: 2E-7b,10,15; 2A-1b,4a,7ab; 2C-1bce,3, 9; 2F-6cd; 2G-1
3. Fourier series: 7A-2b; 7B-1b,2a,4; 7C-1; 3D-6
4. Laplace transform: 3A-1,2,3d; 3B-1ce,3c,5a; 3D-3b,4b,7; 3D-1,2,8
5. Linear systems: 4B-2,6b; 4C-1a,2,6; 4D-2,4; 5B-4cd; 4F-1; 4G-2a; 4H-2,7abc; 4E-1
6. Non-linear systems: 5C-4,5; 5D-2ab,3; 5E-1a,2a,3

18.03 Pactice Final Exam 1 Solutions

1. (a) The isocline for slope 0 is the pair of straight lines $y = \pm x$. The direction field along these lines is flat.

   The isocline for slope 2 is the hyperbola on the left and right of the straight lines. The direction field along this hyperbola has slope 2.

   The isocline for slope -2 is the hyperbola above and below the straight lines. The direction field along this hyperbola has slope -2.

   (b) The sketch should have the following features:

   The curve passes through $(-2,0)$. The slope at $(-2,0)$ is $(-2)^2 - (0)^2 = 4$.

   Going backward from $(-2,0)$, the curve goes down ($dy/dx > 0$), crosses the left branch of the hyperbola $x^2 - y^2 = 2$ with slope 2, and gets closer and closer to the line $y = x$ but never touches it.

   Going forward from $(-2,0)$, the curve first goes up, crosses the left branch of the hyperbola $x^2 - y^2 = 2$ with slope 2, and becomes flat when it intersects with $y = -x$. Then the curve goes down and stays between $y = -x$ and the upper branch of the hyperbola $x^2 - y^2 = -2$, until it becomes flat as it crosses $y = x$. Finally, the curve goes up again and stays between $y = x$ and the right branch of the hyperbola $x^2 - y^2 = 2$ until it leaves the box.

   (c) $f(100) \approx 100$

   (d) It follows from the picture in (b) that $f(x)$ reaches a local maximum on the line $y = -x$. Therefore $f(a) = -a$.

   (e) Since we know $f(-2) = 0$, to estimate $f(-1)$ with two steps, the step size is 0.5. At each step, we calculate

$$x_n = x_{n-1} + 0.5, \qquad y_n = y_{n-1} + 0.5(x_{n-1}^2 - y_{n-1}^2)$$

   The calculation is displayed in the following table.

n	x_n	y_n	$0.5(x_n^2 - y_n^2)$
0	-2	0	2
1	-1.5	2	-0.875
2	-1	1.125	

   The estimate of $f(-1)$ is $y_2 = 1.125$.

2. (a) The equation is $\dot{x} = x(x-1)(x-2)$. The phase line has three equilibria $x = 0, 1, 2$.

   For $x < 0$, the arrow points down.

   For $0 < x < 1$, the arrow points up.

   For $1 < x < 2$, the arrow points down.

   For $x > 2$, the arrow points up.

   (b) The horizontal axis is t and the vertical axis is x. There are three constant solutions $x(t) \equiv 0, 1, 2$. Their graphs are horizontal.

   Below $x = 0$, all solutions are decreasing and they tend to $-\infty$.

   Between $x = 0$ and $x = 1$, all solutions are increasing and they approach $x = 1$.

   Between $x = 1$ and $x = 2$, all solutions are decreasing and they approach $x = 1$.

   Above $x = 2$, all solutions are increasing and they tend to $+\infty$.

   (c) A point of inflection $(a, x(a))$ is where $\ddot{x}$ changes sign. In particular, $\ddot{x}(a)$ must be zero. Differentiating the given equation with respect to t, we have

$$\ddot{x} = 2\dot{x} - 6x\dot{x} + 3x^2\dot{x} = \dot{x}(2 - 6x + 3x^2)$$

   If $x(t)$ is not a constant solution, $\dot{x}(a) \neq 0$ so that $x(a)$ must satisfy

$$2 - 6x(a) + 3x(a)^2 = 0 \quad \Leftrightarrow \quad x(a) = 1 \pm \frac{1}{\sqrt{3}}.$$

   (d) Typo in the original version: The material being added into the reactor should be Bo instead of Ct.

   Let $x(t)$ be the number of moles of Bo in the reactor at time t. The rate of loading is 2 moles per year. Hence $x(t)$ satisfies $\dot{x} = -kx + 2$, where k is the decay rate of Bo. Since the half life of Bo is 2 years, $e^{-k \cdot 2} = 1/2$ so that $k = (\ln 2)/2$. Therefore we have

$$\dot{x} = -\frac{\ln 2}{2} x + 2.$$

   The initial condition is $x(0) = 0$.

   (e) The differential equation is linear. Since we have

$$y' + \left(\frac{3}{x}\right) y = x$$

   an integrating factor is given by

$$\exp\left(\int \frac{3}{x}\, dx\right) = \exp(3 \ln x) = x^3.$$

   Multiply the above equation by x^3 and integrate:

$$(x^3 y)' = x^3 y' + 3x^2 y = x^4 \quad \Rightarrow \quad x^3 y = \frac{1}{5} x^5 + c$$

   Since $y(1) = 1$, we have $c = 4/5$ and

$$y = \frac{1}{5} x^2 + \frac{4}{5} x^{-3}.$$

3. (a) Express all complex numbers in polar form:

$$\frac{ie^{2it}}{1+i} = \frac{e^{i\pi/2} e^{2it}}{\sqrt{2} e^{i\pi/4}} = \frac{1}{\sqrt{2}} e^{i(2t + \pi/2 - \pi/4)} = \frac{1}{\sqrt{2}} e^{i(2t + \pi/4)}$$

   The real part is

$$\mathrm{Re}\left(\frac{ie^{2it}}{1+i}\right) = \frac{1}{\sqrt{2}} \cos\left(2t + \frac{\pi}{4}\right).$$

(b) The trajectory is an outgoing, clockwise spiral that passes through 1.

(c) The polar form of $8i$ is $8e^{i\pi/2}$. Its three cubic roots are

$$\begin{aligned} 2e^{i\pi/6} &= 2\cos\frac{\pi}{6} + 2i\sin\frac{\pi}{6} = \sqrt{3} + i, \\ 2e^{i(\pi/6+2\pi/3)} &= 2\cos\frac{5\pi}{6} + 2i\sin\frac{5\pi}{6} = -\sqrt{3} + i, \\ 2e^{i(\pi/6+4\pi/3)} &= 2e^{3i\pi/2} = -2i. \end{aligned}$$

4. (a) Let $x_p(t) = at^2 + bt + c$. Plug it into the left hand side of the equation

$$\begin{aligned} \ddot{x} + 2\dot{x} + 2x &= (2a) + 2(2at + b) + 2(at^2 + bt + c) \\ &= 2at^2 + (4a + 2b)t + (2a + 2b + 2c) \end{aligned}$$

and compare coefficients

$$2a = 1, \qquad 4a + 2b = 0, \qquad 2a + 2b + 2c = 1.$$

The solution is $a = 1/2$, $b = -1$, $c = 1$. Therefore $x_p(t) = \frac{1}{2}t^2 - t + 1$.

(b) The characteristic polynomial is $p(s) = s^2 + 2s + 2$. Using the ERF and linearity,

$$x_p(t) = \frac{e^{-2t}}{p(-2)} + \frac{1}{p(0)} = \frac{e^{-2t}}{2} + \frac{1}{2}$$

(c) Consider the complex equation

$$\ddot{z} + 2\dot{z} + 2z = e^{it}.$$

For any solution z_p, its imaginary part $x_p = \operatorname{Im} z_p$ satisfies the real equation

$$\ddot{x} + 2\dot{x} + 2x = \sin t.$$

The ERF provides a particular solution of the complex equation

$$z_p(t) = \frac{e^{it}}{p(i)} = \frac{e^{it}}{1+2i} = \frac{e^{it}}{\sqrt{5}\,e^{i\phi}} = \frac{1}{\sqrt{5}}e^{i(t-\phi)}$$

where ϕ is the polar angle of $1 + 2i$. Take the imaginary part of z_p

$$x_p(t) = \operatorname{Im} z_p(t) = \frac{1}{\sqrt{5}}\sin(t - \phi)$$

This is a sinusoidal solution of the real equation. Its amplitude is $1/\sqrt{5}$.

(d) If $x(t) = t^3$ is a solution, then $q(t) = \ddot{x} + 2\dot{x} + 2x = 6t + 6t^2 + t^3$.

(e) The general solution is $x(t) = t^3 + x_h(t)$, where $x_h(t)$ is a solution of the associated homogeneous equation. Since the characteristic polynomial $s^2 + 2s + 2$ has roots $-1 \pm i$,

$$x(t) = t^3 + x_h(t) = t^3 + c_1 e^{-t}\cos t + c_2 e^{-t}\sin t.$$

5. (a) See the formula sheet for the definition of $\operatorname{sq}(t)$. The graph of $f(t)$ is a square wave of period 2π. It has a horizontal line segment of height 1 in the range $-\pi/2 < t < \pi/2$ and a horizontal line segment of height -1 in the range $\pi/2 < t < 3\pi/2$.

(b) Replace t by $t + \pi/2$ in the definition of $\operatorname{sq}(t)$

$$\begin{aligned} f(t) = \operatorname{sq}\left(t + \frac{\pi}{2}\right) &= \frac{4}{\pi}\left[\sin\left(t + \frac{\pi}{2}\right) + \frac{1}{3}\sin\left(3t + \frac{3\pi}{2}\right) + \frac{1}{5}\sin\left(5t + \frac{5\pi}{2}\right) + \ldots\right] \\ &= \frac{4}{\pi}\left(\cos t - \frac{1}{3}\cos 3t + \frac{1}{5}\cos 5t + \ldots\right) \end{aligned}$$

(c) First consider the complex equation

$$\ddot{z} + z = e^{int} \qquad \text{for a positive integer } n.$$

The characteristic polynomial is $p(s) = s^2 + 1$. One of the ERFs provides a particular solution of the complex equation

$$\begin{aligned} z_p(t) &= \frac{e^{int}}{p(in)} = \frac{e^{int}}{1 - n^2}, \qquad n \neq 1 \\ z_p(t) &= \frac{te^{it}}{p'(i)} = \frac{te^{int}}{2i}, \qquad n = 1 \end{aligned}$$

The imaginary parts of these functions

$$\begin{aligned} u_p(t) &= \operatorname{Im}\left(\frac{e^{int}}{1 - n^2}\right) = \frac{\sin nt}{1 - n^2}, \qquad n \neq 1 \\ u_p(t) &= \operatorname{Im}\left(\frac{te^{it}}{2i}\right) = -\frac{1}{2}t\cos t, \qquad n = 1 \end{aligned}$$

satisfy the imaginary part of the above complex equation, namely

$$\ddot{u} + u = \sin nt.$$

By linearity, a solution of $\ddot{x} + x = \operatorname{sq}(t)$ is given by

$$x_p(t) = \frac{4}{\pi}\left(-\frac{1}{2}t\cos t + \frac{1}{3}\cdot\frac{\sin 3t}{1 - 3^2} + \frac{1}{5}\cdot\frac{\sin 5t}{1 - 5^2} + \ldots\right).$$

6. (a) For $t < 0$, the graph is flat on t-axis.
For $0 < t < 1$, the graph is flat at 1 unit above t-axis.
For $1 < t < 3$, the graph is flat at 1 unit below t-axis.
For $3 < t < 4$, the graph is flat at 1 unit above t-axis.
For $t > 4$, the graph is flat on t-axis.

(b)
$$\begin{aligned} v(t) &= [u(t) - u(t-1)] - [u(t-1) - u(t-3)] + [u(t-3) - u(t-4)] \\ &= u(t) - 2u(t-1) + 2u(t-3) - u(t-4) \end{aligned}$$

(c) The graph coincides with t-axis for all t, except for two upward spikes at $t = 0, 3$ and two downward spikes at $t = 1, 4$.

(d) $\dot{v}(t) = \delta(t) - 2\delta(t-1) + 2\delta(t-3) - \delta(t-4)$

(e) By the fundamental solution theorem (a.k.a. Green's formula),

$$x(t) = (q * w)(t) = \int_0^t q(t-\tau)w(\tau)\,d\tau = \int_{a(t)}^{b(t)} w(\tau)\,d\tau.$$

Now $q(t-\tau) = 1$ only for $0 < t - \tau < 1$, or $t - 1 < \tau < t$, and it is zero elsewhere. Therefore the upper limit $b(t)$ equals t. The lower limit $a(t)$ is $t-1$ if $t-1>0$, or 0 if $t-1<0$. In other words, $a(t) = (t-1)u(t-1)$.

7. (a) The transfer function is $W(s) = \dfrac{1}{p(s)} = \dfrac{1}{2s^2+8s+16}$.

(b) The unit impulse response $w(t)$ is the inverse Laplace transform of $W(s)$. In other words,

$$\begin{aligned} \mathcal{L}(w(t)) &= \frac{1}{2s^2+8s+16} = \frac{1}{2[(s+2)^2+4]} \\ \Rightarrow \quad \mathcal{L}(e^{2t}w(t)) &= \frac{1}{2(s^2+4)} = \frac{1}{4}\mathcal{L}(\sin 2t) \end{aligned}$$

Therefore $e^{2t}w(t) = \frac{1}{4}\sin 2t$, and $w(t) = \frac{1}{4}e^{-2t}\sin 2t$.

(c) Take the Laplace transform of

$$p(D)x = 2\ddot{x}(t) + 8\dot{x}(t) + 16x(t) = \sin t$$

with the initial conditions $x(0+) = 1$, $\dot{x}(0+) = 2$.[1] This yields

$$\begin{aligned} 2[s^2X(s) - s - 2] + 8[sX(s) - 1] + 16X(s) &= \frac{1}{s^2+1} \\ \Rightarrow \quad X(s) &= \frac{1}{2s^2+8s+16}\left(\frac{1}{s^2+1} + 2s + 12\right) \end{aligned}$$

8. (a) The characteristic polynomial of A is

$$\det(A - \lambda I) = \det\begin{bmatrix} 2-\lambda & 12 \\ 3 & 2-\lambda \end{bmatrix} = (2-\lambda)^2 - 36 = (\lambda-8)(\lambda+4).$$

Therefore the eigenvalues are $\lambda = 8, -4$.

(b) For $\lambda = 8$, solve $(A - 8I)\mathbf{v} = 0$. Since $A - 8I = \begin{bmatrix} -6 & 12 \\ 3 & -6 \end{bmatrix}$, a solution is $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

For $\lambda = -4$, solve $(A + 4I)\mathbf{v} = 0$. Since $A + 4I = \begin{bmatrix} 6 & 12 \\ 3 & 6 \end{bmatrix}$, a solution is $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

[1] With rest initial conditions, the answer would simply be $X(s) = \dfrac{1}{2s^2+8s+16}\left(\dfrac{1}{s^2+1}\right)$

(c) The following is a fundamental matrix for $\dot{\mathbf{u}} = B\mathbf{u}$

$$F(t) = \begin{bmatrix} e^t & -e^{2t} \\ e^t & e^{2t} \end{bmatrix}$$

Then e^{tB} can be computed as $F(t)F(0)^{-1}$.

$$F(0) = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \qquad F(0)^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$e^{tB} = F(t)F(0)^{-1} = \begin{bmatrix} e^t & -e^{2t} \\ e^t & e^{2t} \end{bmatrix} \cdot \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} e^t + e^{2t} & e^t - e^{2t} \\ e^t - e^{2t} & e^t + e^{2t} \end{bmatrix}$$

(d) The general solution of $\dot{\mathbf{u}} = B\mathbf{u}$ is

$$\mathbf{u}(t) = c_1\begin{bmatrix} e^t \\ e^t \end{bmatrix} + c_2\begin{bmatrix} -e^{2t} \\ e^{2t} \end{bmatrix} = F(t)\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

The given initial condition implies

$$\begin{aligned} \begin{bmatrix} 2 \\ 1 \end{bmatrix} &= F(0)\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\ \Rightarrow \quad \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} &= F(0)^{-1}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3/2 \\ -1/2 \end{bmatrix} \end{aligned}$$

Therefore the solution of the initial value problem is $\mathbf{u}(t) = \dfrac{1}{2}\begin{bmatrix} 3e^t + e^{2t} \\ 3e^t - e^{2t} \end{bmatrix}$.

9. (a) The phase portrait has the following features:
- All trajectories start at $(0,0)$ and run off to infinity.
- There are straight line trajectories along the lines $y = \pm x$.
- All other trajectories are tangent to $y = x$ at $(0,0)$.
- No two trajectories cross each other.

9. (b) $\mathrm{Tr}(A) = a+1$, $D = \det(A) = a+4$. Therefore the eigenvalues $\{\lambda_1, \lambda_2\}$ are given by $\frac{a+1}{2} \pm \frac{\sqrt{(a-5)(a+3)}}{2}$.

i) Saddle $D < 0 \iff a < -4$.
ii) Nodal Sink $\lambda_1 < \lambda_2 < 0 \iff -4 < a < -3$
iii) Nodal Source $\lambda_1 > \lambda_2 > 0 \iff a > 5$.
iv) Spiral Sink $\lambda_1, \lambda_2 = r \pm i\omega$, $r < 0 \iff -3 < a < -1$; counterclockwise.
v) Spiral Source $\lambda_1, \lambda_2 = r \pm i\omega$, $r > 0 \iff a > -1$; counterclockwise.
vi) Unstable If either eigenvalue has a positive real part; $a < -4$, $a > -1$. (Put another way, all saddles are unstable. So is anything in the $T > 0$ range. Stability is the case in which *both* $D > 0$ and $T < 0$, that is when the characteristic polynomial $\lambda^2 - T\lambda + D$ has positive coefficients.)

The geometric way of seeing this is to draw the line in the (T, D) plane given by $T = a+1$, $D = a+4$ and see where it crosses the dividers, $D = 0$, $T^2 = 4D$, $T = 0$, and $T^2 = 4D$ the second time. One can see that this will happen at $a = -4, -3, -1, 5$.

10. (a) The equilibria are the solutions of

$$\dot{x} = x^2 - y^2 = 0, \qquad \dot{y} = x^2 + y^2 - 8 = 0$$

This implies $(x^2, y^2) = (4, 4)$, so that $(x, y) = (2,2), (2,-2), (-2,2), (-2,-2)$.

(b) The Jacobian is $J(x,y) = \begin{bmatrix} 2x & -2y \\ 2x & 2y \end{bmatrix}$. In particular, $J(-2,-2) = \begin{bmatrix} -4 & 4 \\ -4 & -4 \end{bmatrix}$.

(c) The linearization of the nonlinear system at $(-2,-2)$ is the linear system $\dot{\mathbf{u}} = J(-2,-2)\mathbf{u}$. A computation shows that the eigenvalues of $J(-2,-2)$ are $-4 \pm 4i$. The first component of $\mathbf{u}(t)$ is of the form

$$c_1 e^{-4t}\cos(4t) + c_2 e^{-4t}\sin(4t) = Ae^{-4t}\cos(4t - \phi)$$

This means $x(t) \approx -2 + Ae^{-4t}\cos(4t - \phi)$ near $(-2,-2)$.

(d) Let $f(x) = 2x - 3x^2 + x^3$. The phase line in problem 2(a) shows that $\dot{x} = f(x)$ has a stable equilibrium at $x = 1$.

The linearization of the nonlinear equation at $x = 1$ is the linear equation $\dot{u} = f'(1)u = -u$. Its solutions are $u(t) = Ae^{-t}$. This means $x(t) \approx 1 + Ae^{-t}$ near $x = 1$.

18.03 Pactice Final Exam 2 Solutions

1. Write $y' - \frac{2}{x}y = x^2$

Integrating factor: $e^{-2\int \frac{dx}{x}} = e^{-2\ln x} = x^{-2}$

$\frac{1}{x^2}y' - \frac{2}{x^3}y = 1$

$\left(\frac{1}{x^2}y\right)' = 1$

$y/x^2 = x + c$

$y = x^3 + cx^2$

2. $y' = \frac{y}{x} + \sqrt{1 - \left(\frac{y}{x}\right)^2}$

Put $z = y/x$; $y = xz$, $y' = xz' + z$

$xz' + z = z + \sqrt{1 - z^2}$

$\frac{dz}{\sqrt{1-z^2}} = \frac{dx}{x}$

$\sin^{-1} z = \ln x + c$

$z = \sin(\ln x + c)$

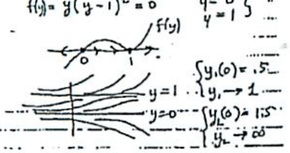
$y = x\sin(\ln x + c)$

3. $y' = x^2 - y$ Isoclines: $x^2 - y = c$ or $y = x^2 - c$

4. Critical pts = long-term const pops.

$y^3 - 2y^2 + y = 0$

$f(y) = y(y-1)^2 = 0$ $\left.\begin{matrix} y = 0 \\ y = 1 \end{matrix}\right\}$



$\begin{cases} y_1(0) = .5 \\ y_1 \to 1 \end{cases}$

$\begin{cases} y_2(0) = 1.5 \\ y_2 \to \infty \end{cases}$

5. $\frac{dx}{dt} = \text{rate of inflow} - \text{rate of outflow}$

$= r \cdot 3e^{-t} - r\cdot\frac{x}{3}$

$\therefore \frac{dx}{dt} + \frac{r}{3}x = 3re^{-t}$

$c = \frac{x}{3}$ $\therefore \frac{dc}{dt} + \frac{r}{3}c = re^{-t}$

6. a) $-1 + i\sqrt{3} = 2e^{2\pi i/3}$

$(-1 + i\sqrt{3})^9 = 2^9 \cdot e^{6\pi i} = 2^9$

b) $D^9\left(e^{(-1+i\sqrt{3})t}\right) = (-1+i\sqrt{3})^9 e^{(-1+i\sqrt{3})t}$

$= 2^9\left(e^{-t}(\cos\sqrt{3}t + i\sin\sqrt{3}t)\right)$

$\therefore$ Imag. part $= 2^9 e^{-t}\sin\sqrt{3}t$

(since we want $D^9(\mathrm{Im}(e^{(-1+i\sqrt{3})t}))$).

7. $y'' + 4y' + 4y = 0$

$r^2 + 4r + 4 = 0$, $(r+2)^2 = 0$

$\therefore y = e^{-2t}(c_1 + c_2 t)$

At 0: $\boxed{1 = c_1}$

At 1: $0 = e^{-2}(c_1 + c_2)$

$0 = \frac{1 + c_2}{e^2}$ $\therefore \boxed{c_2 = -1}$

$y = e^{-2t}(1 - t)$

8. $y'' + y' + 2y = e^{(1+i)t}$ Solv.

Try $\tilde{y}_p = ce^{(1+i)t}$ (c complex) ex. part. soln

$[(1+i)^2 + (1+i) + 2]c = 1$

$(1+i)c = 1$ $\therefore c = \frac{1}{1+i} = \frac{1-i}{2}$

$y_p = \mathrm{Im}\left(\frac{1-i}{2} e^t(\cos t + i\sin t)\right)$

$y_p = -\frac{1}{2}e^t\cos t + \frac{1}{2}e^t\sin t$

9.

$x'' + 4x' + cx = 0$

$r^2 + 4r + c = 0$ char. eq'n

$r = \frac{-4 \pm \sqrt{16 - 4c}}{2} = -2 \pm \sqrt{4-c}$

Get oscillation if r is complex,

$\therefore$ if $\boxed{c > 4}$

Its (angular) pseudofrequency is then $\sqrt{c-4}$

(since a soln is like $e^{-2t}(c_1\cos\sqrt{c-4}\,t + c_2\sin\sqrt{c-4}\,t)$)

So its pseudoperiod $= \frac{2\pi}{\sqrt{c-4}}$

10. $f(t) = \sum_1^\infty a_n \sin n\pi t$

a) $a_n = \frac{2}{1}\int_0^1 (1-t)\sin n\pi t\,dt$

Using integration by parts, the integral (leave out 2)

$= -(1-t)\frac{\cos n\pi t}{n\pi}\Big|_0^1 - \int_0^1 (-1)\left(\frac{-\cos n\pi t}{n\pi}\right)dt$

$= 0 - (-1)\cdot\frac{1}{n\pi} - \frac{\sin n\pi t}{(n\pi)^2}\Big|_0^1 = \frac{1}{n\pi}$

$\therefore f(t) = \sum_1^\infty \frac{2}{n\pi}\sin n\pi t$

b)

c) Use Fourier sine series since bdry values are 0 at 0, 1.

Soln: $y = \sum B_n \sin n\pi t$

$y'' = \sum -B_n(n\pi)^2 \sin n\pi t$

Substituting, and comparing coeffs. of $\sin n\pi t$ on both sides:

$-B_n(n\pi)^2 + kB_n = \frac{2}{n\pi}$

$\therefore B_n = \frac{2}{n\pi(k - (n\pi)^2)}$

d) If $k = 90$ the dominant term is $B_3 \sin 3\pi t$, since

$90 - (3\pi)^2 \approx 2$

($B_1 \approx \frac{1}{120}$, $B_2 \approx \frac{1}{150}$, $B_3 \approx \frac{1}{9}$, $B_4 \approx \frac{1}{360}$, etc.)

11. $F(s) = \int_0^\infty e^{-st}f(t)\,dt$

$\int_0^\infty e^{-st}f(at)\,dt = \int_0^\infty e^{-su/a}f(u)\frac{du}{a}$

$u = at$

$\mathcal{L}(f(at)) = \frac{1}{a}F(s/a)$

12. $y'' - y = e^t$, $y(0) = 0$, $y'(0) = 1$

$s^2Y - 1 - Y = \frac{1}{s-1}$

$(s^2 - 1)Y = 1 + \frac{1}{s-1} = \frac{s}{s-1}$

$Y = \frac{s}{(s+1)(s-1)^2} = \frac{1/2}{(s-1)^2} + \frac{1/4}{s-1} + \frac{-1/4}{s+1}$

$y = \frac{1}{2}te^t + \frac{1}{4}e^t - \frac{1}{4}e^{-t}$

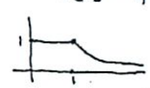
13. $\mathcal{L}^{-1}(1/s) = u(t)$, $\mathcal{L}^{-1}\left(\frac{1}{s+1}\right) = u(t)e^{-t}$

$\frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1} \xrightarrow{\mathcal{L}^{-1}} u(t)(1 - e^{-t})$

$\therefore \mathcal{L}^{-1}\left(\frac{e^{-s}}{s(s+1)}\right) = u(t-1)(1 - e^{-(t-1)})$

$\mathcal{L}^{-1}\left(\frac{1}{s} - \frac{e^{-s}}{s(s+1)}\right) = u(t) - u(t-1)(1 - e^{-(t-1)})$

$= \begin{cases} 1 & \text{if } 0 \le t \le 1 \\ e \cdot e^{-t} & \text{if } t \ge 1 \end{cases}$



14. Divide up $[0, x]$ into small time-intervals: $t_0 = 0, t_1, t_2, \dots, t_n = x$ of length Δt.

Amt $f(t_i)\Delta t$ dumped in ith interval decays to $\approx f(t_i)\Delta t \cdot e^{-k(x - t_i)}$ by time x.

Total amt at time x: $\approx \sum f(t_i)$

$\approx \sum_{i=0}^{n-1} f(t_i)\Delta t \cdot e^{-k(x-t_i)}$

As $n \to \infty$, Total $= \int_0^x f(t)e^{-k(x-t)}dt$.

(This is $f(x) * e^{-kx}$)

15. $\begin{bmatrix} a & 1 \\ 2 & a \end{bmatrix}$ Char. eqn: $\lambda^2 - 2a\lambda + a^2 - 2 = 0$

Roots: $\lambda = \frac{2a \pm \sqrt{4a^2 - 4(a^2 - 2)}}{2}$

$\lambda = a \pm \sqrt{2}$

asy. stable	$-\sqrt{2}$	unstable	$-\sqrt{2}$	unstable
node		saddle		node
$\lambda_1 < 0$, $\lambda_2 < 0$		$\lambda_1 > 0$, $\lambda_2 < 0$		$(\lambda_1, \lambda_2 > 0)$

16. $x' = y(x - y)$ Crit. pt.: $y = 0$ or $y = x$

(a) $y' = x^2 - 2y + 1$ $\begin{cases} x^2 + 1 = 0 & \text{no soln} \\ (x-1)^2 = 0 & x = 1, y = 1 \end{cases}$

$\therefore$ has $(1, 1)$ as its only critical pt.

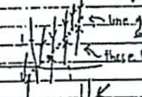
(b) $J = \begin{bmatrix} y & x - 2y \\ 2x & -2 \end{bmatrix}_{(1,1)} = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix}$

$\lambda^2 + \lambda = 0$ $\lambda_1 = 0$: $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda_2 = -1$: $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\lambda_1 = 0$ is a borderline case — (when perturbed gives $\lambda_1 > 0$ or $\lambda_1 < 0$) (or $\lambda_1 = 0$)

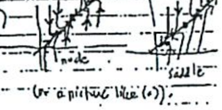
(c)



line of constant sol'ns

these have slope 2.

(d)



node

saddle

(or a picture like (c)).

(e) At critical points he can't decide between an node or a saddle.

17. $\begin{cases} x' = y \\ y' = -y\sin x - (x^2 + 1) \end{cases}$

No critical points, since if $y = 0$, $x^2 + 1 = 0$ (no solns)

$\therefore$ No closed trajectories

$\therefore x(t)$ cannot be periodic soln.

18. a) $x' = \underbrace{(y - x)}_{\text{heat from y-cube}} + \underbrace{(e^{-t} - x)}_{\text{heat from ext}}$

$y' = (x - y) + (e^{-t} - y)$

$\begin{cases} x' = -2x + y + e^{-t} \\ y' = x - 2y + e^{-t} \end{cases}$

b) $\lambda^2 + 4\lambda + 3 = 0$ $\lambda = -1$: $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$(\lambda + 3)(\lambda + 1) = 0$ $\lambda = -3$: $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\therefore X' = c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{-t} + c_2\begin{bmatrix} 1 \\ -1 \end{bmatrix}e^{-3t}$

c) $X = \begin{bmatrix} e^{-t} & e^{-3t} \\ e^{-t} & -e^{-3t} \end{bmatrix}$ $X^{-1} = \begin{bmatrix} -e^{-3t} & -e^{-3t} \\ -e^{-t} & e^{-t} \end{bmatrix}(-e^{4t})\cdot\frac{1}{2}$

(1)

$X^{-1}r = \frac{1}{2}\begin{bmatrix} e^{t} & e^{t} \\ e^{3t} & -e^{3t} \end{bmatrix}\begin{bmatrix} e^{-t} \\ e^{-t} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$X\int X^{-1}r(t)\,dt = \begin{bmatrix} e^{-t} & e^{-3t} \\ e^{-t} & -e^{-3t} \end{bmatrix}\begin{bmatrix} t \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}te^{-t}$

(Add this to (b) to get gen'l. soln)

19. $e^{At} = X(t)X(0)^{-1}$

$= \begin{bmatrix} e^t & 2e^{2t} \\ 2e^t & e^{2t} \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}^{-1}$

$\therefore e^{At} = \begin{bmatrix} e^t & 2e^{2t} \\ 2e^t & e^{2t} \end{bmatrix}\begin{bmatrix} -1/3 & 2/3 \\ 2/3 & -1/3 \end{bmatrix}$

$D = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -1/3 & 2/3 \\ 2/3 & -1/3 \end{bmatrix}$

(Also OK. to omit the denoms. — i.e., multiply all entries by 3.)

20. $B(-k_1 + c_1M) = 0$ $\therefore M = k_1/c_1$

$M(k_2 - c_2B) = 0$ $B = k_2/c_2$

This is the crit. pt, corresponding to long-term equilibrium populations.

With bass-fishing, get new ODE

$B' = (-k_1 - 1/10)B + c_1BM$

So new equil. is $B = k_2/c_2$, $M = \frac{k_1 + 1/10}{c_1}$

— bass pop'n stays same,

— minnow pop'n rises.

Private 18.03 Formula Sheet

12/17

$\int \frac{1}{x} = \ln(x) + C$

$dx \leftrightarrow dt$

$\frac{dy}{dx}$ = slope Can make slope field

Isocline = fixed slope
iso cline

Chain rule
$(f \circ g)'(t) = f'(g(t))\, g'(t)$

Opposite: Int by substitution

$\int_a^b f(g(x))\, g'(x)\, dx = \int_{g(a)}^{g(b)} f(x)\, dx$

Int by parts

$\int f(x)\, g'(x)\, dx = f(x)g(x) - \int f'(x)\, g(x)\, dx$

$\int \sin = -\cos$ $\frac{d}{dx} \sin = \cos$

$\int \cos = \sin$ $\frac{d}{dx} \cos = -\sin$

$L'(x) < f(x, L(x))$ ← just c
found w/ $\frac{dy}{dx} = C$ - isocline

$U'(x) > f(x, U(x))$

funnel

$U(x) > L(x)$

$\lim_{x \to \infty} |U(x) - L(x)| = 0$

antifunnel

$L(x) > U(x)$

Euler's Method

$x_{n+1} = x_0 + h$

$y_{n+1} = y_n + h \cdot f(x_n, y_n)$
just the diff eq'n

$\frac{dy}{dx} = \frac{x}{n} + \frac{1}{5} \cancel{y}$
plain

Phase line

Where does $\frac{dy}{dt} < 0$ ⊖

$\frac{dy}{dt} > 0$ ⊕

(autonomous)
Phase line when t_0 does not matter

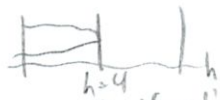
Harvesting/(Logistical)

$\frac{dP}{dt} = \underbrace{kP(M-P)}_{\text{logistical}} - \underbrace{h}_{\text{harvesting}}$

Get roots a = growth rate a/b = limiting pop

$aP - bP^2 - h$

# of roots depends on h
L make bifurcation diagram



Logistical $\frac{dP}{dt} = aP - bP^2$

Product Rule

$(f \cdot g)' = f'g + fg'$

Linear eqn Must divide 1st factor!

$\frac{dy}{dx} + P(x)\, y = Q(x)$

L multiply all by $e^{\int P(x)dx}$

$D_x\left[\int P(x)\, dx\right] = P(x)$

$D_x\left[y(x) \cdot e^{\int P(x)dx}\right] = Q(x)\, e^{\int P(x) dx}$

So

$y(x) \cdot e^{\int P(x)dx} = \int \left(Q(x) e^{\int P(x)dx}\right) dx + C$

$y(x) = e^{-\int P(x)dx} \int \left(Q(x) e^{\int P(x)dx}\right) dx + C$

$\int e^x = e^x$ $\frac{d}{dx} e^x = e^x$

$\int e^{-x} = -e^{-x}$ $\frac{d}{dx} e^{-x} = -e^{-x}$

$\int e^{-2x} = -\frac{1}{2} e^{-2x}$ $\frac{d}{dx} e^{-2x} = -2e^{-2x}$

Input Response Model

$\frac{dy}{dt} + ky = Q(t)$ $y(0) = y_0$
k const, > 0

L Int factor e^{kt}

Gen sol
$y(t) = e^{-kt} \int_0^t Q(s)\, e^{ks}\, ds + y_0 e^{-kt}$

$i^2 = -1$

$r(\cos\theta + i\sin\theta) = re^{i\theta}$

$Re() = r\cos\theta$

$Im() = r\sin\theta$

$e^{i\theta} = \cos\theta + i\sin\theta$

$e^{i\pi} + 1 = 0$

Ex Find sol $z^3 = 1$

L $1,\ e^{2\pi i/3},\ e^{4\pi i/3}$

$\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$

$= -\frac{1}{2} + i\frac{\sqrt{3}}{2}$

$z^n = 1$ sols $1, e^{\frac{2\pi}{n}i}, e^{\frac{2\pi}{n}i \cdot 2}, \ldots, e^{\frac{2\pi}{n}i(n-1)}$

$e^{x+iy} = e^x e^{iy} = e^x(\cos y + i\sin y)$

$\frac{d}{d\theta}(e^{i\theta}) = ie^{i\theta}$

$\frac{d}{d\theta} e^{(a+bi)\theta} = (a+bi)\, e^{(a+bi)\theta}$

$\int e^{(a+bi)\theta} d\theta = \frac{1}{a+bi} e^{(a+bi)\theta} + C_1 + C_2 i$

Wronskian $\det\begin{pmatrix} f & g \\ f' & g' \end{pmatrix} = fg' - gf'$

$= 0 \to$ lin dep (repeated roots)

$\neq 0 \to$ lin ind (good)

Linear Approxs

$e^x \approx 1+x$ near 0

$\sin x = x$ near 0

$\cos x = 1$ near 0

Pendulum

$c_1 \cos\sqrt{\frac{g}{l}}\,t + c_2 \sin\sqrt{\frac{g}{l}}\,t$

w/ friction

$$\frac{d^2\theta}{dt} + c\frac{d\theta}{dt} + \frac{g}{l}\sin\theta = 0$$

1. $c^2 > 4k \to$ dis $\oplus$ $\to$ real rts
   dies fast
2. $c^2 < 4k \to$ dis $\ominus$ $\to$ img rts
   ossilates + dies
3. $c = 0$ purly imag rts
   ossilates forever

$Re(e^{ix}) = \cos(x)$

$Re(e^{-ix}) = \cos(-x) = \cos(x)$

$Im(e^{ix}) = \sin(x)$

$Im(e^{-ix}) = \sin(-x) = -\sin(x)$

Exponential Shift

$$e^{rx} D(u(x)) = (D-r)\, e^{rx} u(x)$$

$0! = 1$

Test for Convergence

$$\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = R \sim \text{radius of convergence}$$

Power Series

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots$$

$$y' = \sum_{n=1}^{\infty} a_n n x^{n-1} = a_1 + a_2 x + a_3 x^2 + \dots$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

Even — y-axis — $f(x) = f(-x)$ cos

Odd — origin — $f(x) = -f(x)$ sin

$e + e = e$ $\quad e \cdot e = e$

$o + o = o$ $\quad o \cdot o = e$

$e + o =$ arbitrary $\quad e \cdot o = o$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(mx)\,dx$$

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(mx)\,dx$$

Preliminaries

1. $\int_{-\pi}^{\pi}\sin(mx)\,dx = \left.\frac{-\cos(mx)}{m}\right|_{-\pi}^{\pi} = 0$
2. $\int_{-\pi}^{\pi}\cos(mx)\,dx = \left.\frac{\sin(mx)}{m}\right|_{-\pi}^{\pi} = 0$
3. $\int_{-\pi}^{\pi}\cos(mx)\cos(nx)\,dx = \begin{cases} 2\pi & m=n \\ 0 & m\neq n\end{cases}$
4. $\int_{-\pi}^{\pi}\sin(mx)\sin(nx)\,dx = \begin{cases} \pi & m=n \\ 0 & m\neq n\end{cases}$
5. $\int_{-\pi}^{\pi}\sin(mx)\cos(nx) = 0$

$\int_{-c}^{c}$ odd $= 0$

$\sin\left(\frac{2\pi x}{p}\right)$ $\quad \sin\left(\frac{2\pi n x}{p}\right)$

$\cos\left(\frac{2\pi x}{p}\right)$ $\quad \cos\left(\frac{2\pi n x}{p}\right)$

Fourier Expansion (odd) $\to$ only b_n terms

" " (even) $\to$ " a_n "

" " (neither) $\to$ both a_n, b_n "

Integr by Parts Derivation

$d(uv) = u\,dv + v\,du$

$\int d(uv) = uv = \int u\,dv + \int v\,du$

$\int u\,dv = uv - \int v\,du$

Convolution $\mathcal{L}(\cos 2t)\cdot\mathcal{L}(\sin 2t) = \mathcal{L}(\cos 2t * \sin 2t)$

$$f*g = \int_0^t f(\tau)\,g(t-\tau)\,d\tau$$

$\mathcal{L}^{-1}(e^{-cs}F) =$

$= \delta(t-c) * \mathcal{L}^{-1}(F)$

$= \int_0^t \delta(z-c)\,\mathcal{L}^{-1}(F)(t-z)$

$= \begin{cases} 0 & t<c \\ e^{-(t-c)} & t\geq c\end{cases}$

Weight fn

$P(D)x = f$

$\mathcal{L}(P(D)x) = \mathcal{L}(f)$

$P(s)\mathcal{L}(x) = \mathcal{L}(f)$

$\mathcal{L}(x) = \frac{1}{P(s)}\mathcal{L}(f)$

$w(t) = \mathcal{L}^{-1}\left(\frac{1}{P(s)}\right)$ weight fn

---

$\det(A - \lambda I) = 0$

in 2x2

$$\lambda^2 - \underbrace{(a+d)}_{trace(A)}\lambda + \underbrace{(ad-bc)}_{\det(A)}$$

λ = eigenvalues

$\underline{v}$ = eigenvectors

Inverse of Matrix

$$A^{-1} = \frac{1}{\det(A)}\begin{bmatrix} d & -b \\ -c & a\end{bmatrix}$$

Population Model

$$\frac{dP}{dt} = (B-D)P$$

18.03 Do Pratice Exams

4/12

(I wish dept would release new exams...)

L annoying dept!

Exam 1

1. Solve $(1+x)^2 \frac{dy}{dx} - y^2 = 0 \quad y(0) = 1$

Try sep of variables

$$(1+x)^2 \frac{dy}{dx} = y^2$$

$$(1+x)^2 dy = y^2 dx$$

$$\frac{1}{y^2} dy = \frac{1}{(1+x)^2} dx$$

$\cdot -1 \quad \cdot -1$; can you do that - dont

$$\int y^{-2} = \frac{y^{-1}}{\frac{1}{-1}} = -\frac{1}{y}$$

$$\int (1+x)^{-2} = \frac{(1+x)^{-1}}{\left(\frac{1}{-1}\right)}$$

↓ plus? No multiply I think

$$-\frac{1}{y} = -\frac{1}{(1+x)} + C \checkmark$$

$\cancel{\cdot} -1 \quad \cancel{\cdot} -1$

$$y = 1+x \checkmark$$

(2)

I don't think that is right...

So did same thing as last time

No is correct!

2a. $\frac{dy}{dt} + 4y = \sin 3t$

Use complex methods

Solve homogeneous first

$\frac{dy}{dt} = -4y$

$\frac{1}{-4y}\,dy = 1\,dx$

$-\frac{1}{4}\ln(y) = x$

$-\frac{1}{4}y = e^{x}$

$y = -4e^{x}$

← do we have this value already

Or alt method

$(y' + 4y)$

$D + 4$

Ae^{-4t} ← Don't flip sign I think

← but root is -4

(3)

To get rid of $\sin 3t$

$D|$ $3\cos 3t$

$D^2|$ $-9\sin 3t$

So D^2+9

$B e^{-9t} + C t e^{-9t}$

Now can find particular sol for B, C

Plug in?

$y = Ae^{-4t} + Be^{-9t} + Ct e^{-9t}$

$y' = -4Ae^{-4t} + -9Be^{-9t} + -9Ct e^{-9t}$

$(-4Ae^{-4t} - 9Be^{-9t} - Ct e^{-9t}) + 4(Ae^{-4t} + Be^{-9t}$
$+ Cte^{-9t}) = \sin 3t$

I don't think this is right...

Didn't use complex methods

WA) $y' + 4y = 0$

$y(x) = C e^{-4x}$ ✓

(4)

$$y' + 4y = \sin 3t$$

$\hookrightarrow y(t) = Ae^{-4t} + \frac{1}{25}(4\sin 3t - 3\cos 3t))$

But where did I go wrong?

Is technically an input response problem

$$y(t) = \underbrace{e^{-kt}\int_0^t Q(s)e^{ks}\,ds}_{\text{steady state}} + \underbrace{y_0 e^{-kt}}_{\text{transient}}$$

$Q(s)$ = input

$y(t)$ = response

From the end of this chap

Where is this solving qv from?

where plug in

Can't easily find from where...

I think plugging in is just for particular

$\hookrightarrow$ but should come at same??

Ohh I realized what I did wrong

$D^2+9 \hookrightarrow$ need to factor $\pm 3i$

(5)

Be^{3it}
$\hookrightarrow$ which is

$B(\cos 3t + i\sin 3t)$

$y_p = B\cos 3t + C\sin 3t$

Then still need to find B, C

$y_p' = -3B\sin 3t + 3C\cos 3t$

Pretty sure plug only in particlar sol

$(-3B\sin 3t + 3C\cos 3t) + 4(B\cos 3t + C\sin 3t) = \sin 3t$

Sin terms $-3B + 4C = 1$

Cos $3C + 4B = 0$

So
$4C = 1 + 3B$

$C = \frac{1}{4} + \frac{3}{4}B$

$3(\frac{1}{4} + \frac{3}{4}B) + 4B = 0$

$\frac{3}{4} + 4\frac{3}{4}B = 0 \Leftarrow$ wrong

$4\frac{3}{4}B = -\frac{3}{4}$

(6)

$B = -\frac{3}{19}$

$A \quad -3\left(-\frac{3}{19}\right) + 4C = 1$

$\frac{9}{19} + 4C = 1$

$4C = 1 - \frac{9}{19}$

$C = \dfrac{1 - \frac{9}{19}}{4} = \frac{5}{38}$

I think I did the math wrong somewhere

Yeah I see

$\frac{3}{4} + \frac{9}{4}B + 4B = 0$

$-\frac{3}{4} = \frac{25}{4}B$

$-3 = 25B$

$B = -\frac{3}{25}$

$-3\left(-\frac{3}{25}\right) + 4C = 1$

$\frac{9}{25} + 4C = 1$

$4C = 1 - \frac{9}{25}$

$C = \frac{4}{25}$

7

$y = Ae^{-4t} + -\frac{3}{25}\cos 3t + \frac{4}{25}\sin 3t$

✓ that actually matches

~~Rdd~~ Review what I did + what I did wrong

Did not factor D^2+9

Then split to sin, cos

Then plug in y_p and y_p'

↑ particular only!

Solve system for Particular A, B

Plug in values

Other way $P(x) = 4$ $Q(x) = \sin 3t$

Integrating factor e^{4t}

$$ye^{4t} = \int \sin 3t \, e^{4t} \, dt$$

$$= \int \operatorname{Im}(e^{3ti}) e^{4t}$$

$$= \frac{1}{3i+4} e^{3it+4t} \cancel{dt}$$

(8) Divide by e^{4t}
Rationalize denom

(Darn added $-9-16=25$ wrong on exam)

Then seperate out Im

b) What is amplitude of resulting wave?

how do you start this one

Amp of $A\cos\theta + B\sin\theta = \sqrt{A^2+B^2}$

So $\sqrt{\left(\frac{4}{25}\right)^2 + \left(\frac{-3}{25}\right)^2} = \frac{5}{25} = \frac{1}{5}$

Then $\emptyset = \tan^{-1}\left(\frac{B}{A}\right)$ I believe ✓

But pre form must be $A\cos\theta + B\sin\theta$

Can also be Amp $\cos(\theta)\cos(\emptyset)$ + Amp $\sin(\theta)\sin(\emptyset)$

I can see it get confusing

(9)

c) Find the general sol

$$\frac{dy}{dt} + 4y = \underbrace{1}_{\text{new}} + \sin 3t$$

Trick: by linearity

Sol to part (a) + Sol to $\frac{dy}{dt} + 4y = 1$

$y = \frac{1}{4} + Ce^{-4t}$

$$Y = \frac{1}{25}(4\cos 3t - 3\sin 3t) + \frac{1}{4} + Ce^{-4t}$$

Cool trick. I never realized that...

10

3. Logistical

400 turticles
75 growth

So $\frac{dP}{dt} = aP - bP^2$

75 $\quad \frac{a}{b} = 400$

$\frac{75}{b} = 400$

$75 = b400$

$b = \frac{75}{400}$

75% each year

$\frac{dP}{dt} = .75p - \frac{.75}{400}p^2 - r$

don't forget harvesting!

Remember the variables

b) What is biforcation pt r

$\frac{dP}{dt} = .75p - \frac{.75}{400}p^2 - r$

So I know what this is
but how to find?

where $dP/dt = 0$ - does that make sense - no changes...

(11)

✓ yes

$0 = .75p - \frac{.75}{400}p^2 - r$

Solve for p interms of r

$r = .75p - \frac{.75}{400}p^2$

$r = \left(.75 - \frac{.75}{400}p\right)p$

? how to get rid of ps ?

Quadratic?

$\cancel{\frac{.75}{400}}$

$-\frac{.75}{400}p^2 + .75p - r$

$$\frac{-.75 \pm \sqrt{.75^2 - 4\cdot\left(-\frac{.75}{400}\right)\cdot -1}}{2\cdot -\frac{.75}{400}}$$

algebra hell

✓ this is what they use

(12)

Solve for r

L basically algebra

$.75^2 = 4r \quad \frac{.75}{400}$

$.75 = \frac{r}{100}$

$r = 75$

Above this r = extiction

b) Sources / sinks

I forgot this qu
Did it earlier
basically at that P is decir ⊕/⊖ I belie

(13)

4. $\frac{dy}{dx} = x^2 - xy$

Pictured direction field

a) eqations for 0, 2, -2 isoclines

$$0 = x^2 - xy$$

? draw or put in terms of y

$$xy = x^2$$

$$\cancel{y = \frac{x^2}{x} = x}$$; that simple

Same extra mistake on exam

$$x(x-y) = 0$$

$$x = 0$$

$$x = y$$

Ask when is this true

Don't rearrange for some reason...

(14)

$2 = x^2 - xy$

↑ when is this true?

why can't I solve for ?

$2 = x(x - y)$

Oh you can

$y = x - \frac{2}{x}$

Silly - why can't ya before...

Sketch sol - done

c) Long term behavior

Oh that fences + funnels business :(

0-isocline $y = x$ is an upper fence

↳ guess could prove

$L(x) = x - \frac{2}{x}$

$f(x, L(x)) = 2$ ↙ just the #

$L'(x)\left(1 + \frac{2}{x^2}\right)$

(15)

then for $x > \sqrt{2}$ $L(x)$ is a lower fence

Then argue if sol enters funnel

Yes b/c $L(\sqrt{2}) = 0$ + $y(1) = 0$

$y(x)$ increasing

$y(\sqrt{2}) \geq 0$ so $y(x)$ does enter funnel

d) Euler $h = \frac{1}{2}$ for $(0, 2)$

0	2		
$\frac{1}{2}$	$2 + \frac{1}{2} \cdot f(0, 2)$ ← old? ←	$x^2 - xy$, $0^2 - 0\,2$	$= 2$ ✓
1	$2 + \frac{1}{2} \cdot f(\frac{1}{2}, 2)$	$(\frac{1}{2})^2 - \frac{1}{2} \cdot 2$	$= 2\frac{1}{8}$

↑ want up to 1

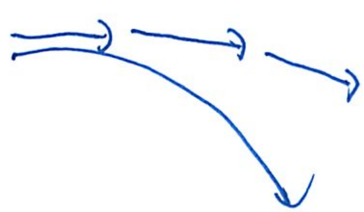
?i did they get formula wrong

did totally wrong wht I did

e) More or less

More since slope field is concave down

↑clever



16

5. For IVP given

What value of c is $x=c$ a vertical asy?

Challenge problem I missed

Skip for time

Test 2

1a) $y''-3y+2y=0$

(s^2-3s+2)

$(s-1)(s-2)$

$Ae^{t}+Be^{2t}$ ✓

b) ~~IW~~ IVP ~~at~~ $y(0)=1$ $y'(0)=-3$

I wanted to do this problem

$1=Ae^{0}+Be^{20}$

$1=A+B$

(17)

$$y'(t) = Ae^{t} + 2Be^{2t}$$

$$-3 = Ae^{0} + 2Be^{20}$$

$$-3 = A + 2B$$

$$A = \cancel{B}\ 1 - B$$

$$-3 = (1 - B) + 2B$$

$$-3 = 1 + B$$

$$B = -4$$

A) $1 = A + (-4)$

$$A = 5$$

$$y = \cancel{a}\ 5e^{t} + -4e^{2t}$$

(18)

c) Find particular sol

$$y'' - 3y' + 2y = x^2$$

Had $y = Ae^{t} + Be^{2t}$

Eliminate x^2 w/ D^2

$$(D^2 + 0) = 0$$

↓ do long way to check

$$\frac{-\cancel{B}0 \pm \sqrt{0^2 - 4 \cdot 1 \cdot 0}}{2}$$

$$\frac{-0 \pm \sqrt{0}}{2}$$ umm - no

Oh need D^3

$D = 2x$

$D^2 = 2$

$D^3 = 0$

They seem to do it a very diff way

↓

(19)

$$y = ax^2 + bx + c$$
$$y' = 2ax + b$$
$$y'' = 2a$$

$$(2a) - 3(2ax+b) + 2(ax^2+bx+c)$$
$$= (2a)x^2 + (2b-6a)x + (2c-3b+2a)$$

$2a = 1$ $\qquad$ $a = 1/2$

$2b - 6a = 0$ $\qquad$ $b = 3/2$

$2c - 3b + 2a = 0$ $\qquad$ $c = 7/4$

$$y(x) = \frac{1}{2}x^2 + \frac{3}{2}x + \frac{7}{4}$$

---

homogeneous sol has nothing to do with it

So whats wrong w/ my orginal method?

L They said Completly wrong form of guess...

~~Don't have~~ Don't have time to research full ans - but it seems

(20)

$D^3(D-1)(D-2)$

↑ method does not cut it here

Since not like $(D-2)^2$

I had a similar issue on #2

↳ solved differently than book did it

So I guess it do it that way when D^3

Confused now why it is that way...

Ah flipped right to pg

When D^3

the guess is

$C_0 + C_1 x + C_2 x^2 + C_3 x^3$

Matches this

(21)

2a) Find general sol to

$$(D-1)^2(D^2+2D+4)y=0$$

$$\frac{-2 \pm \sqrt{4-4\cdot1\cdot4}}{2\cdot1}$$

$$\frac{-2 \pm \sqrt{12}\,i}{2}$$

$$\frac{-2 \pm 2\sqrt{3}\,i}{2}$$

$$-1 \pm \sqrt{3}\,i$$

Convert to sin + cos

$$Ae^{t} + Bte^{t} + \left[\begin{array}{l} e^{(-1+\sqrt{3}i)t} \\ e^{-t}e^{\sqrt{3}it} \\ e^{-t}\cos\sqrt{3}t + i\sin\sqrt{3}t \end{array}\right.$$

$$Ce^{-t}\cos\sqrt{3}t + De^{-t}\sin\sqrt{3}t$$

(22)

8) <u>State the form</u> of the inhomogeneous ODE

a) $= \sin 3x$

$A\cos 3x + B\sin 3x$ ✓ Really got wrong originally

b) $= e^{-x}\cos\sqrt{3}x$

Not ~~[illegible]~~ addition, so can't split

Hmm - no clue on top of mind

$xe^{-x}(A\cos\sqrt{3}x + B\sin\sqrt{3}x)$

So what rule is this?

Well on exam I tried takeing deriv of it

?Are repeated roots involved

└ ~~We~~ see above - yes !!

c)

x^2e^x

$Ax^4e^x + Bx^3e^x + Cx^2e^x$

At a loss on these harder ones

└ skip for now...

(24) p28 skipped

3. Find a recurrance relationship

$$y'' - 2xy' + 8y = 0$$

Darn not good at these

Need to practice

$$y = \sum_{n=0}^{\infty} a_n x_n$$

$$y' = \sum_{n=1}^{\infty} a_n n x^{n-1} = \sum_{n=0}^{\infty} a_{n+1} (n+1) x^n$$

$$y'' = \sum_{n=2}^{\infty} a_n (n)(n-1) x^{n-2} = \sum_{n=0}^{\infty} a_{n+2} (n+1)(n) x^{n}$$

Ok write that 2-3 times to memorize

$$Y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=1}^{\infty} a_n n x^{n-1} = \sum_{n=0}^{\infty} a_{n+1} (n+1) x^n$$

$$y'' = \sum_{n=2}^{\infty} a_n (n)(n+1) x^{n-2} = \sum_{n=0}^{\infty} a_{n+2} (n+1)(n) x^n$$

(25)

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=1}^{\infty} a_n n x^{n-1} = \sum_{n=0}^{\infty} a_{n+1} (n+1) x^n \checkmark$$

$$y'' = \sum_{n=2}^{\infty} a_n (n)(n+1) x^{n-2} = \sum_{n=0}^{\infty} a_{n+2} (n)(n+1) x^n \checkmark$$

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=1}^{\infty} a_n n x^{n-1} = \sum_{n=0}^{\infty} a_{n+1}(n+1) x^n \checkmark$$

$$y'' = \sum_{n=2}^{\infty} a_n (n)(n+1) x^n = \sum_{n=0}^{\infty} a_{n+2} (n)(n+1) x^n \checkmark$$

Oh after that Plug in (Have not done one of these...)

$$\sum_{n=0}^{\infty} a_{n+2} (n+1)(n) x^n - 2x \left(\sum_{n=0}^{\infty} a_{n+1} (n+1)\right) x^n$$

$$+ 8 \sum_{n=0}^{\infty} a_n x^n = 0$$

(26)

Urg did I memorize wrong values?

Textbook) $f'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1}$

Had slighty wrong vales

Textbook)

$$y' = \sum_{n=1}^{\infty} n c_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) c_{n+1} x^n$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n$$

That sounds better

Why did I just memorize wrong values?

$$y' = \sum_{n=1}^{\infty} n c_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) c_{n+1} x^n$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n$$

27

$$y' = \sum_{n=1}^{\infty} n c_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) c_{n+1} x^n$$

$$y'' = \sum_{n=2}^{\infty} (n-1) n c_n x^{n-2} = \sum_{n=0}^{\infty} (n+1)(n+2) c_{n+2} x^n$$

Ok try plugging in again

$$\sum_{n=0}^{\infty} (n+1)(n+2) c_{n+2} x^n - 2x \sum_{n=0}^{\infty} (n+1) c_{n+1} x^n$$

but should not conver

$$+ 8 \sum_{n=0}^{\infty} c_n x^n$$

$$\sum_{n=0}^{\infty} (n+1)(n+2) c_{n+2} x^n - 2 n c_n x^n + 8 c_n x^n = 0$$

Now put terms of highest
drop summation

$$c_{n+2} = \frac{2 n c_n x^n + 8 c_n x^n}{(n+1)(n+2) x^n}$$

→ drop x^n as well

(28)

$$C_{n+2} = \frac{2n\,c_n \mp 8c_{n}}{(n+1)(n+2)}$$

(✓) Thats it for part a
That is the recurrence relationship

b) find the first 3 non-0 terms

So base pattern

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

Crank that

$n=0$ $\quad C_2 = \dfrac{2(0)\,c_0 \mp 8c_0}{(1)\,(2)}$ ← do we have C_0? leave in terms of

$= \dfrac{8c_0}{2} = -4c_0$ had earlier sign error (✓) otherwise

↑ leave in larger terms if have to make a pattern

$h=1$ $\quad C_3 = \dfrac{2(1)\,C_1 \mp 8c_1}{(2)\,(3)}$

$= \dfrac{2c_1 \mp 8c_1}{6} = \dfrac{\cancel{10}^{-6}c_1}{6} = \dfrac{5}{3}c_1 \;\; -1c_1$ (✓)

(29)

$n=2$ $$c_4 = \frac{2(2)c_2 + 8c_2}{(3)(4)}$$

$$= \frac{4c_2 + 8c_2}{12}$$

$$= \cancel{\frac{12c_2}{12} = c_2 = -4c_0} \quad -\frac{1}{3}c_2 = \frac{4}{3}c_0$$

$n=3$ $$c_5 = \frac{2(3)c_3 + 8c_3}{(4)(5)}$$

$$= \frac{6c_3 + 8c_3}{20}$$

$$= \cancel{\frac{14c_3}{20} = \frac{7c_3}{10} = \left(\frac{7}{10}\right)\left(\frac{5}{3}c_1\right)} = \frac{35}{30}c_1$$

$$-\frac{1}{10}c_3 = \frac{1}{10}c_1$$ prob just some algebra thing

Put it together

$$y(t) = c_0\left(1 - 4x^2 + \frac{4}{3}x^4 + \ldots\right) + c_1\left(x - x^3 + \frac{1}{10}x^5 + \ldots\right)$$

can have even/odd

two linearly ind sols

(30)

Ok - hope I can get that one

4. Spring - Mass - Dashpot

$m = 1$
$c = 4$
$k = 20$
$F(t) = 120 \cos 2t$

$$m\ddot{x} + c\dot{x} + k = F(t)$$
$$\ddot{x} + 4\dot{x} + 20 = 120 \cos 2t$$

Oh question is amplitute

Oh this is the question w/ that formula

$$A(\omega) = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}$$

← which is just 120!

In recitation we solved the whole thing 1st

Guess $A \cos 2t + B \sin t$

Solve system like before

(31)

Then finding amp is easy $\sqrt{A^2+B^2}$

Exam sols much more complicated

b) Maximize amplitude

So basically keep ω as a variable

$$A = \frac{F_0}{\sqrt{(40-\omega^2)^2+(4\omega)^2}}$$

Max whole thing

So min inside

expand at

$$\omega^4 - 24\omega^2 + 4\omega$$

Take deriv

$$4\omega^3 - 48\omega$$

Set $= 0$

$$4\omega^3 - 48\omega = 0$$

$$\omega = 0 \pm \sqrt{12}$$

(32)

#5 ~~LC~~ LC circuit

¿just plug in like always?

Solved it before

Skip now

<u>Key</u> $\omega = \sqrt{\frac{k}{m}}$

Make sure to remember other formula

$$A(\omega) = \frac{F_0}{\sqrt{(k-m\omega^2)^2 + (c\omega)^2}}$$

$$= \frac{F_0}{\sqrt{(k-m\omega^2)^2 + (c\omega)^2}}$$

$$= \frac{F_0}{\sqrt{(k-m\omega^2)^2 + (c\omega)^2}}$$

$$= \frac{F_0}{\sqrt{(k-m\omega^2)^2 + (c\omega)^2}}$$

(33)

$$\omega = \sqrt{\frac{k}{m}}$$

$$= \sqrt{\frac{k}{m}}$$

$$= \sqrt{\frac{k}{m}}$$

$$A = \frac{F_0}{\sqrt{(k-m\omega^2)^2 + (c\omega)^2}}$$

Test 3

1. LaPlace Transform

$$y'' + 6y' + 9y = 0 \qquad y(0) = 1$$
$$y'(0) = 3$$

Forgot how to do these...

$$(s^2 + 6s + 9)\mathcal{L}(y) - s(1) - 3 - 6 = 0$$

↑ dont forget

↓ multiplied not of

$$\mathcal{L}(y) = \frac{s(1) + 9}{s^2 + 6s + 9}$$

Now Heavysides

34)

$$\frac{s+9}{s^2+6s+9}$$

$$\frac{-6 \pm \sqrt{36-4\cdot 1\cdot 9}}{2}$$

$$-3 \pm 0$$

$$\frac{s+9}{(s+3)^2} = \frac{A}{s+3} + \frac{B}{(s+3)^2}$$

Can only get 1 w/

$$\frac{(s+9)\cancel{(s+3)}}{(s+3)^2} = \frac{A\cancel{(s+3)}}{\cancel{(s+3)}} + \frac{B\cancel{(s+3)}}{(s+3)^2}$$

i can't find

$$\frac{s+9\cancel{(s+3)^2}}{\cancel{(s+3)^2}} = \frac{A(s+3)^2}{\cancel{(s+3)}} + \frac{B\cancel{(s+3)^2}}{\cancel{(s+3)^2}}$$

$$s+9 = A(s+3) + B$$

$s=-3$

$$6 = B$$

35

Now this way

$$\frac{s+9}{(s+3)^2} = \frac{A}{s+3} + \frac{-3}{(s+3)^2}$$

$s=0$ ← something boring

$$\frac{9}{9} = \frac{A}{3} + -\frac{3}{9}$$

$1\frac{1}{3} = A/3$

$$\frac{12}{9} = \frac{4}{3} = \frac{A}{3}$$

$A=4$ ← they got
hmm ?

$$= \frac{4}{s+3} - \frac{3}{(s+3)^2}$$

Oh plugged wrong thing in for B

$$\frac{(s+9)}{(s+3)^2} = \frac{A}{s+3} + \frac{6}{(s+3)^2}$$

$s=0$

$$\frac{9}{9} = \frac{A}{3} + \frac{6}{9}$$

$\frac{3}{9} = \frac{A}{3} \to \frac{1}{3} = \frac{A}{3} \to A=1$ ✓ Better

36

Now back to the problem

$$\mathcal{L}^{-1}\left(\frac{1}{s+3}\right) = e^{-3t}$$ check tables on test

$$6\,\mathcal{L}^{-1}\left(\frac{1}{(s+3)^2}\right) = 6\,t\,e^{-3t}$$

quota fixes for you

$$y = e^{-3t} + 6\,t\,e^{-3t}$$

---

b) Now a non homogeneous

$$\mathcal{L}(RHS) = \frac{1}{s+3}$$

So $$y = \frac{1}{(s+3)^3} + \underbrace{\frac{6}{(s+3)^2} + \frac{1}{s+3}}_{\text{from a}}$$

how is it added; well is repeated root

$$\mathcal{L}^{-1}\left(\frac{1}{(s+3)^3}\right) = \frac{1}{2}\,t^2 e^{-3t}$$

I guess note is a double deriv

I don't think I know this well enough to get it right

(37)

I didn't look at all the patterns that much

So supposedly

$\frac{1}{(s+3)^2}$ is $-F'(s)$

$\mathcal{L}(e^{3t}) = \frac{1}{s-3}$

$\mathcal{L}(t e^{3t}) = -\left(\frac{1}{s-3}\right)'$

$= +\left(\frac{1}{s+3}\right)^2$

Yeah I guess

WA: $u = s-3$

$\frac{d}{du}\frac{1}{u} = -\frac{1}{u^2}$

$= \frac{-\frac{d}{ds}(s-3)}{(s-3)^2}$

$= \frac{\frac{d}{ds}(s) + \frac{d}{ds}(-3)}{(s-3)^2}$

38

$$= -\frac{1}{(s-3)^2}$$

Think about it next time...

c) $y'' + 6y' + 9y = \delta(t-3)$

Now one of these QU to practice

Recitation keeps it like this!

$$\mathcal{L}^{-1}\left(\frac{\mathcal{L}(\delta(t-3))}{(s+3)^2}\right)$$

Can convert

$$= \mathcal{L}^{-1}\left(\frac{e^{-3s}}{(s+3)^2}\right)$$

$$= u(t-3)\, e^{-3(t-3)}\, (t-3)$$

I should have studied LaPlace more..

2. Weight function

no good at

To Study Sticky Note

- Linear eqns
  - Would like more practice
  - This is integrating factor
- Word problems
  - Reviewed
- Complexification
  - Think I got it
- Solving eqn systems
  - Not part 4
  - But the homogonneous thing
  - Got it
- Radius of convergence
  - Did not pratice more
- Power series
  - Not really
  - Actually I did a problem
- Fourier series
  - No
- Fourier transform
  - Should pratice more
- LaPlace
  - Did not review all of the possible combos much
  - Not feeling all that confident
- Weight done

Final Formula Sheet

USEFUL FORMULAS:

The convolution of functions $f(t)$ and $g(t)$ is defined to be

$$[f * g](t) = \int_0^t f(u)g(t-u)du.$$

Convolution is commutative: $f * g = g * f$.

The Laplace transform of a function $f(t)$ is defined to be

$$F(s) = \mathcal{L}[f(t)] = \int_{0^-}^{\infty} f(t)e^{-st}dt \quad \text{(defined for } s \gg 0\text{)}.$$

It satisfies the following properties:

Linearity: $\mathcal{L}[af(t) + bg(t)] = aF(s) + bG(s)$.

t-Shift: $\mathcal{L}[u(t-a)f(t-a)] = e^{-as}F(s)$.

s-Shift: $\mathcal{L}[e^{at}f(t)] = F(s-a)$.

t-Derivatives: $\mathcal{L}[f'(t)] = sF(s) - f(0)$.

$\mathcal{L}[f''(t)] = s^2F(s) - sf(0) - f'(0)$.

s-Derivative: $\mathcal{L}[tf(t)] = -F'(s)$.

Convolution: $\mathcal{L}[f(t) * g(t)] = F(s)G(s)$.

Laplace transforms of common functions:

$$\mathcal{L}[1] = \frac{1}{s} \qquad \mathcal{L}[e^{at}] = \frac{1}{s-a} \qquad \mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}[\cos(at)] = \frac{s}{s^2+a^2} \qquad \mathcal{L}[\sin(at)] = \frac{a}{s^2+a^2}$$

$$\mathcal{L}[\delta(t-a)] = e^{-as} \qquad \mathcal{L}[u(t-a)] = \frac{e^{-as}}{s}.$$

– CONTINUED ON NEXT PAGE –

If the Fourier series of a function $f(x)$ with period $2L$ is

$y = c,$

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right],$$

then

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right) \quad \text{and} \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right).$$

The differential equation for current $I(t)$ in an RLC circuit with voltage $V(t)$ is given by:

$$LI'' + RI' + \frac{1}{C}I = V'(t)$$

The differential equation for displacement $x(t)$ of a spring of mass m with constant k, damping constant c, and external force $F(t)$ is:

$$mx'' + cx' + kx = F(t)$$

Sinusoidal identity:

$$a\cos\omega t + b\sin\omega t = A\cos(\omega t - \phi)$$

where

$$A^2 = a^2 + b^2, \qquad \phi = \tan^{-1}\left(\frac{b}{a}\right)$$

Exponential Response Formula for $P(D)y = e^{at}$:

$$y_p(t) = \frac{t^k e^{at}}{P^{(k)}(a)}$$

where a is a root with multiplicity k in the characteristic polynomial P.

The square wave function $Sq(t)$ of period $P = 2L$ and height 1, defined by

$$Sq(t) = \begin{cases} 1 & t \in (0, L) \\ 0 & t = 0 \text{ or } L \\ -1 & t \in (-L, 0) \end{cases}$$

$\omega_{res} = \sqrt{\frac{k}{m}}$

has Fourier series:

$$Sq(t) = \frac{4}{\pi}\sum_{n:\,\text{odd}} \frac{1}{n}\sin\left(\frac{n\pi t}{L}\right).$$

$A = \dfrac{F_0}{\sqrt{(k - m\omega^2)^2 - (\omega)^2}}$

18.03 Debreif

12/20

Ouch

Prob got half wrong

↳ but isn't that usual for me on the final

Should have done more pratice exams

↳ less "static review"

↳ at least for 18.03

Particulary pratice La Place + Fourier

which had lot of trouble on

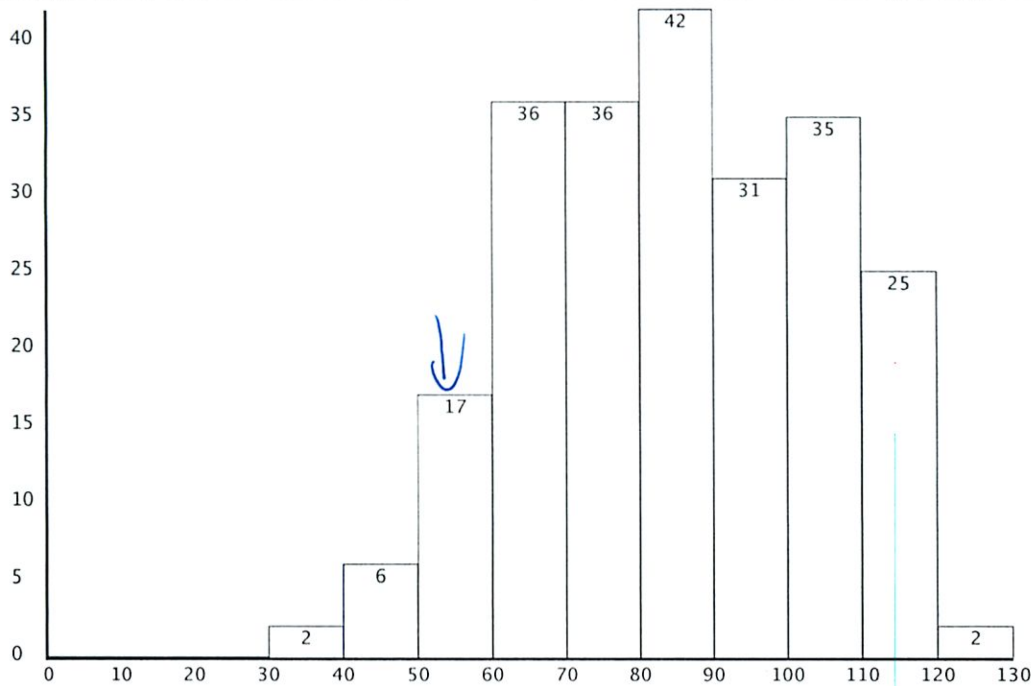
And advanced concepts at the end

↳ kinda predicted what I would mess up...

18.03 Differential Equations

Dashboard | Students | Assignments

Grading Summary for final



Number of Scores:	232
Average:	83.91
Standard Deviation:	19.35

18.03 Differential Equations

Grade Report

Grade Report for Michael E. Plasmeier

Assignment/Exam Name	Graph	Due Date	Points	Max Pts
Homework 1		09.16.2011	34.00	49.00
Homework 2		09.23.2011	33.00	41.00
Exam 1		09.28.2011	19.00	41.00
Homework 3		10.07.2011	53.00	66.00
Homework 4		10.14.2011	8.00	30.00
Homework 5		10.21.2011	37.50	55.00
Exam 2		10.26.2011	20.00	36.00
Homework 6		11.07.2011	40.50	59.00
Homework 7		11.13.2011	20.50	23.00
Homework 8		11.18.2011	44.00	58.00
Exam 3		11.28.2011	20.00	41.00
Homework 9		12.02.2011	25.00	33.00
Homework 10		12.09.2011	38.00	49.00
final		12.20.2011	57.50	122.00

CUMULATIVE SCORE **450.00**

Instructor's Comments