18.03 Differential Equations — Fall 2011

COURSE INFORMATION

Lecturer:

Prof. Ben Brubaker, 2-267, ext. 3-4079, `brubaker@math.mit.edu`.
Office hours: MW 1-2:30 pm (to be confirmed), or by appointment.

Required texts:
Edwards and Penney, *Elementary Diffierential Equations*, 6th edition.
Arthur Mattuck, *18.03 Supplementary Notes and Exercises*, linked from the 18.03 course page (see below) or available by request for $13 at CopyTech in 11-004.

Course web page: `http://math.mit.edu/classes/18.03/fall11/`

Lectures: 54–100, MWF 3:00–4:00 p.m.

Recitations: Tuesday and Thursday, beginning Sept. 8; Meeting place and times posted on the course webpage. Section changes may be made online through Stellar.

Problem Sets: Roughly one each week, always due Fridays by NOON in 2-114 (see syllabus for precise PSET due dates); returned the following week in recitation. *NO late homework will be accepted due to the resulting logistical nightmares. So it is always better to at least turn in what you have managed to complete.*

Homework Rules: Collaboration on problem sets is encouraged, **but**

a) **Attempt each part of each problem yourself.** Read each portion of the problem before asking for help. If you don't understand what is being asked, ask for help interpreting the problem and then make an honest attempt to solve it.

b) **Write up each problem on your own.** On both Part A and B exercises, you may work together, so long as you **write the names of classmates with whom you consulted** and do the final write up of your solutions independently.

This course has a long tradition at MIT, with a large portion of the exercises repeated from year to year. You should not consult these materials in preparing your problem set solutions. Failure to follow these rules will be treated as an act of academic dishonesty and could lead to the invocation of administrative disciplinary action.

Tutoring: in 2-102, Mon-Tues-Wed-Thurs, 3-5 and 7:30-9:30 p.m.

Exams: 3 hour exams and one final exam. See Syllabus for dates and times. (Note that the third hour exam is an evening exam Tuesday night before Thanksgiving).

Make-up Exams: If you must miss an exam (e.g. for class conflicts or team sports), you may arrange to take a make-up exam in advance via the course administrator Jethro Van Ekeren by emailing `jethro@math.mit.edu`. Otherwise, make-up exams will only be granted after the fact with a valid medical excuse.

Grading: Approximate weighting: problem sets 25%, exams 45%, final 30%.

Questions: Concerns about homework, grading, exams: see your recitation instructor.

18.03 Syllabus – Fall 2011

I. First-Order Differential Equations

1.	W	Sept. 7	Outline of course, Direction fields and geometry of solutions (EP 1.1-1.4, Notes G)
2.	F	Sept. 9	Numerical methods (EP 6.1, 6.2, 6.3)
3.	M	Sept. 12	Autonomous equations and the phase line (EP 1.7, 7.1)
4.	W	Sept. 14	Linear equations – Integrating factors (EP 1.5)
5.	F	Sept. 16	Introduction to complex numbers (Notes C.1-3) **PSET 1 DUE**
6.	M	Sept. 19	Complex exponentials (Notes C.3-4)
	W	Sept. 21	No Class – Student Holiday
7.	F	Sept. 23	Input-Response models (Notes IR.1-6) **PSET 2 DUE**
8.	M	Sept. 26	Finish Input-Response, Review for Exam
	W	Sept. 28	**EXAM 1** (in class, covering lectures 1–8)

II. Higher-Order Linear Equations

9.	F	Sept. 30	Homogeneous equations, spring-mass-dashpot model (EP 2.1,2.3)
10.	M	Oct. 3	Complex and repeated roots, higher-order homog. equations (EP 2.2, 2.3)
11.	W	Oct. 5	Operators, Damping (Notes O.1-3, EP 2.4)
12.	F	Oct. 7	Exponential forcing and RLC circuits (EP 2.5, Notes O.4) **PSET 3 DUE**
	M	Oct. 10	No Class – Columbus Day
13.	W	Oct. 12	Undetermined coefficients (EP 2.5, Notes O.4-6)
14.	F	Oct. 14	Sinusoidal forcing, resonance (E.P 2.6) **(mini)-PSET 4 DUE**
15.	M	Oct. 17	Transients and Stability (EP 2.6, Notes S)
16.	W	Oct. 19	Electrical circuits and AM radios (EP 2.7)
17.	F	Oct. 21	Power series and regular singular points (EP 3.1-3.3) **PSET 5 DUE**
18.	M	Oct. 24	Review for Exam
	W	Oct. 26	**EXAM 2** (in class, covering lectures 9–17)

III. Fourier series and Laplace transforms

19.	F	Oct. 28	Fourier series (EP 8.1)
20.	M	Oct. 31	Convergence, Sine and Cosine decomposition (EP 8.2, 8.3)
21.	W	Nov. 2	Solving ODEs with Fourier series, Sound (EP 8.4)
22.	F	Nov. 4	Laplace transform (EP 4.1, Notes H) **PSET 6 DUE**
23.	M	Nov. 7	ODEs with the Laplace transform (EP 4.2, 4.3)
24.	W	Nov. 9	Convolution and the delta function (EP 4.4, Notes CG) **(mini)-PSET 7 DUE**
	F	Nov. 11	No Class - Veteran's Day
25.	M	Nov. 14	Step Functions (EP 4.5, Notes IR.4)
26.	W	Nov. 16	Weight function and transfer function (EP 4.6, Notes CG)
27.	F	Nov. 18	Laplace transform and convolution, pole diagrams **PSET 8 DUE**
28.	M	Nov. 21	Review for Exam
	T	Nov. 22	**EXAM 3** (7:30 pm, rooms on course webpage, covering lectures 19–27)
	W	Nov. 23	No Class

IV. First-order systems

29.	M	Nov. 28	Linear systems and matrices (EP 5.1-5.3, Notes LS.1)
30.	W	Nov. 30	Eigenvalues and eigenvectors (EP 5.4, Notes LS.2)
31.	F	Dec. 2	Complex and repeated eigenvalues (EP 5.4, 5.6, Notes LS.3) **PSET 9 DUE**
32.	M	Dec. 5	Decoupling and solution matrices (EP 5.7, Notes LS.4-6)
33.	W	Dec. 7	Inhomogeneous equations (EP 5.8)
34.	F	Dec. 9	Nonlinear systems (EP 7.2, 7.3, Notes GS) **PSET 10 DUE**
35.	M	Dec. 12	Examples of nonlinear systems (EP 7.4, 7.5, Notes GS)
36.	W	Dec. 14	Review for Final Exam
	TBA	Dec. 16–22	**FINAL EXAM**

EP = Edwards and Penney, Notes = Mattuck's Supplemental Notes and Exercises

18.03: Differential Equations (Fall 2011)

Lecture Notes | Homework | Announcements | Stellar (grades/sections) | mathlets

General information:

Lecturer	**Professor Benjamin Brubaker (brubaker@math.mit.edu**, 2-
Course Admin	**Jethro van Ekeren (jethro@math.mit.edu**, 2-587)
Lectures	Mon, Wed, Fri at 3:00 pm in room 54-100
Texts	(1) *Elementary Differential Equations with Boundary Value Proble* by Edwards and Penney, 6th edition. (2) **18.03 Notes and Exercises**, by Arthur Mattuck.
Course Info	An **information sheet** with course policies, grade breakdown and course.
Course Syllabus	**One page outline** of the course, including important dates (exams
Recitations	Tue. and Thu. For location, instructors, office hourse, and contact in
Grades/ Sections	To check your grades, visit the **18.03 Stellar page.** If you need to r you may also do this through Stellar. Use this page for all other activ
Extra Help	Free additional math tutoring for 18.03 and other undergraduate math classes is available. See **this page** for information.

Additional Course Materials

Handouts, announcements and course materials can be found using the following links:

- **Homework and Solutions**
- **Lecture Notes**
- The supplement **18.03 Notes and Exercises** by Arthur Mattuck is available at OpenCourseWare.
  Hard copies are also available on demand at Copy Tech (11-004) for $13.

Recitation Information

	Time	Room	Instructor	Office	Office Hours	Phone	E-Mail
Lec.1	MWF 3	54-100	B. Brubaker	2-267	?; ?	3-4079	brubaker@math.mit.edu
1	TR 10	2-143	Lyubov Chumakova	2-336	?; ?	3-6584	lyuba@math.mit.edu
2	TR 10	2-147	Ruben Rosales	2-337	?; ?	3-2784	rrr@math.mit.edu
3	TR 11	2-143	Lyubov Chumakova	2-336	?; ?	3-6584	lyuba@math.mit.edu
4	TR 11	2-147	Ruben Rosales	2-337	?; ?	3-2784	rrr@math.mit.edu
5							
6	TR 12	2-139	Neil Olver	2-332	?; ?	3-6770	olver@math.mit.edu
7	TR 1	2-139	Neil Olver	2-332	?; ?	3-6770	olver@math.mit.edu
8	TR 1	2-146	Vivek Shende	2-248	?; ?	3-4993	vivek@math.mit.edu
9	TR 2	4-145	Zhiwei Yun	2-173	?; ?	3-2685	zyun@math.mit.edu
10	TR 2	2-146	Vivek Shende	2-248	?; ?	3-4993	vivek@math.mit.edu
11	TR 3	4-145	Zhiwei Yun	2-173	?; ?	3-2685	zyun@math.mit.edu
12	TR 3	66-168	Jethro van Ekeren	2-587	?; ?	3-4129	jethro@math.mit.edu
	Course Administrator:		Jethro van Ekeren	2-587	?; ?	3-4129	jethro@math.mit.edu

Course Announcements

No announcements yet.

18.03 1st Class 9/7

Prof Brubaker again

Differential Equations

math.mit.edu/classes/18.03

P-Sets due Fri
- none this week
- 2-114 room

Math applets

Grades on stellar

No matlab here - ~~USE~~ 18.997

ODE : ordinary differential equations

only take derivites of single variable functions

Not a PDE

partial

1st order ODE - highest derivative appearing is 1st order derivitives

1st unit

(2) Later

Second-order ODE (all higher order ODEs)

Systems of first order ODEs

Exam 1 at end of month ⇒ 1st order ODEs

1st order ODEs

$$\frac{dx}{dt} = f(x,t)$$

ie $\frac{dx}{dt} = x^3 t + t^2 + \sin x$

Can't solve exactly

Need techniques

$$\frac{dx}{dt} = t^2 + 1$$

find anti-deriv / integrate w/ respect to t

$$x'(t) = \frac{dx}{dt} = t^2 + 1$$

$$\underbrace{\int x'(t)\,dt}_{x(t)} = \frac{1}{3}t^3 + t + c$$

③

$$\frac{dx}{dt} = g(x)\,h(t)$$

Not easy to do

So do algebra, move stuff to 1 side

$$\frac{1}{g(x)}\frac{dx}{dt} = h(t)$$

integrate both sides

$$\int \frac{1}{g(x)}\frac{dx}{dt}\,dt = \int h(t)\,dt$$

~~Can~~ Can clean up by changing variables

$u = x(t)$ ~~going to~~

$du = x'(t)\,dt$

going to be a bit gory

$x = x(t)$

$dx = x'(t)\,dt$

behaves (is <u>not</u>) like a fraction

$$\int \frac{1}{g(x)}dx = \int h(t)\,dt$$

4

Will ask comprehension qu w/ #s

$$\frac{dx}{dt} = 2xt + x$$

- seperable
- (totally forgot how to do this)

ans

$$Ce^{t^2+t}$$

is online

Factor out x

Integrate w/ respect to t

Watch abs value

Read book if confused

Why are we here?

Predict the future

1. Model a real world situation using diff eq
   ← hardest pt

(5)

2. Analyze the result

a) - exact solution - rare

- sep of variables
- etc

b) - numerical solutions

- Friday only

c) - qualative solutions

- slope fields
- etc

3. Make a prediction

- Usually about long term behavior of solution

(Was this what I was bad at in 14.02 ?)

Predict Future w/ Qualative Sols w/ Slope Field

$$\frac{dy}{dx} = y^2 - x$$

Find a sol., so its deriv $= y^2 - x$

Can draw a solution

Pt $(0,0) \rightarrow$ slope $= 0$

6

Repeat

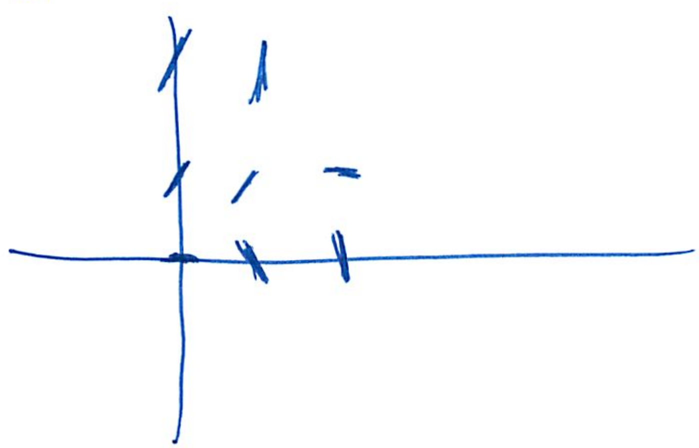
$(1,0) \to -1$

$(2,0) \to -2$

$(0,1) \to 1$

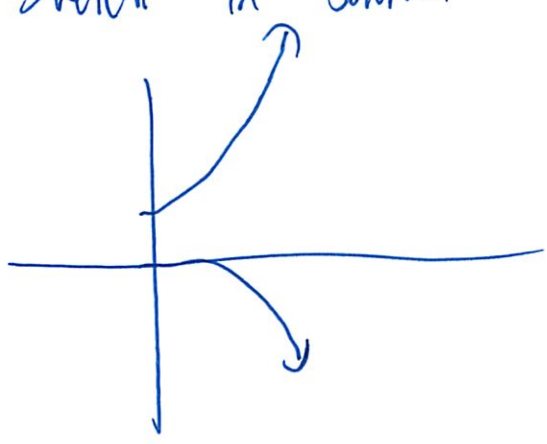
$(0,2) \to 4$

Plot



Etc

Sketch in solution



Tedious

Good for PCs. → Applet! "Isoclines"

⑦

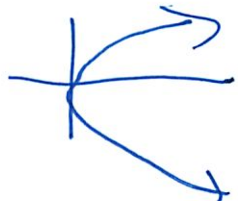
def **Isoclines** - fixed slope
"iso" "cline"

- curve w/ $f(x,y)$ = to a constant in $\frac{dy}{dx}=f(x,y)$

For us $y^2 - x = C$

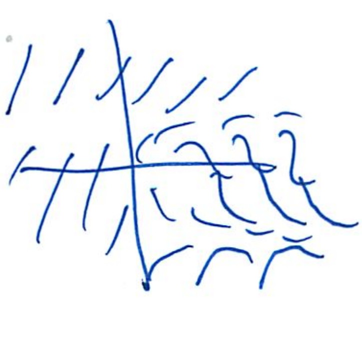
0 iso cline
$(C=0)$

$x = y^2$



Says where sol has deriv $= 0$
critical pt (max or min)

Draw in isocline
- where all curves have maxima



8

Inside curve sols have $\ominus$ derivitives
Outside $\oplus$ deriv
On 0

Once you get inside parabola
└ Ya never leave
└ 0-isocline is a fence for sols to diff eq

lower fence - curve such that sols never cross it from above

Upper fence - never crosses from below

Claim

$\sqrt{x}$ is an upper fence
$-\sqrt{x}$ is a lower fence

Need some property to prove

Prop A lower fence → ~~~~

$$L'(x) < f(x, L(x))$$

$$\frac{dy}{dx} = f(x,y)$$

9

Verify here $\frac{dy}{dx} = y^2 - x$

$L(x) = -\sqrt{x}$ here

$L'(x) = -\frac{1}{2\sqrt{x}} < 0 = f(x, L(x))$

always $\ominus$

Now plug $-\sqrt{x}$ for y

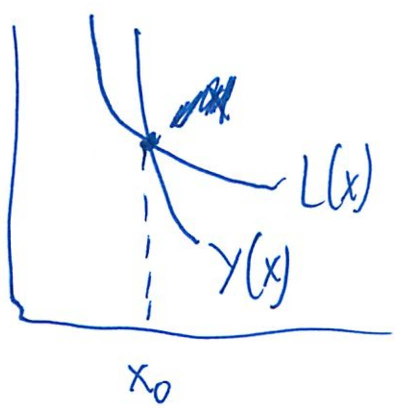
Value so deriv is always 0

<u>Proof</u> by contridiction

Suppose $L(x) \geq f(x, L(x))$ and sol'n $y(x)$

does cross it from above

Derive contridiction, then must this not be true



(10)

To cross $y'(x)$ must be more than $L'(x)$

$$y'(x_0) \leq L'(x_0)$$

can only prove weak inequality

$$y'(x_0) \leq L'(x_0) < f(x_0, L(x_0)) = f(x_0, y(x_0))$$

Said $= y'(x_0)$

$$y'(x_0) \leq L'(x_0) < y'(x_0)$$

Contradiction

So $y(x)$ can't cross from above

If 2 curves w/ same asy behavior then by squeeze theorm all sols inside have same asy behavior

a "funnel"

* Trying to make funnels out of fences *

9/7

Math 18.03 : Differential Equations

Lecture 1 Supplemental Notes

September 7, 2011

Quick Quiz

Which of the following functions is the general solution to the ordinary differential equation

$$\frac{dx}{dt} = 2xt + x \ ?$$

1. $t^2 + t + C$
2. $e^{t^2} + x + C$
3. Ce^{t^2+t}
4. Ce^{2t+1}

Quick Answer

Which of the following functions is the general solution to the ordinary differential equation

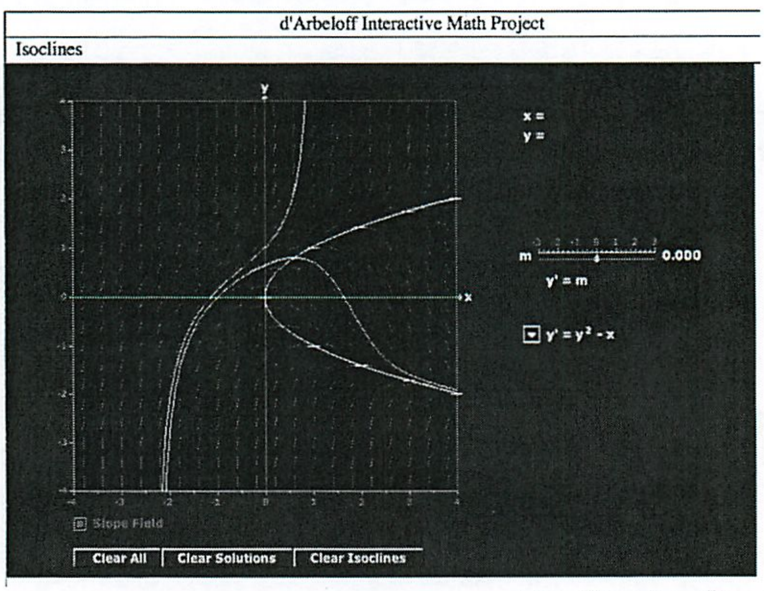
$$\frac{dx}{dt} = 2xt + x \; ?$$

1. $t^2 + t + C$
2. $e^{t^2} + x + C$
3. Ce^{t^2+t}
4. Ce^{2t+1}

Write $\frac{dx}{dt} = x(2t+1)$. That is, $\frac{1}{x}\frac{dx}{dt} = 2t+1$.
Now integrate both sides with respect to t:

$$\begin{aligned} \int \frac{1}{x}\,dx &= \int (2t+1)\,dt \\ \ln|x| &= (t^2+t) + c \\ |x| &= Ce^{t^2+t} \quad \text{where } C = e^c \\ x &= Ce^{t^2+t} \quad C \text{ arbitrary} \end{aligned}$$

Direction Fields: $y' = y^2 - x$



Lecture 1 9/7

math.mit.edu/classes/18.03 → notes pg. / hw pg. / applets

Reminders: course webpage / book / notes / Yossi's course: 18.997S (MATLAB, 6 units F: 1-3 + LAB T/W)

read carefully (w/ pencil & paper) before coming to class.

First unit is on first-order differential equations.

(Entire class focused on ODEs ← "ordinary" d.e.s: functions of one variable)

second course might discuss PDEs p: partial – functions of several vars.)

First-order ODE: $\frac{dx}{dt} = f(x,t)$ e.g. $\frac{dx}{dt} = x^3 t + t^2 + \sin x$

$\frac{dx}{dt} = t^2+1$

Later: ODE's with second derivs.

Systems of ODEs

Given $\frac{dx}{dt} = t^2+1$. Easy to solve. Integrate both sides w.r.t. t

$$\int \frac{dx}{dt}\, dt = \int (t^2+1)\, dt$$

$$x(t) = \frac{1}{3}t^3 + t + c \quad \text{any const. } c.$$

maybe prefer the notation $x'(t)$

similarly, can solve $\frac{dx}{dt} = f(t)$ for any function $f(t)$.

(Answer may not be expressible in terms of elementary functions)

Generalization — Separation of Vars.

$\frac{dx}{dt} = f(t)\cdot g(x)$. Ex: $\frac{dx}{dt} = x \cdot \cos t$

First some algebra: $\frac{1}{x}\frac{dx}{dt} = \cos t$ (valid if $x \neq 0$)

Now again integrate w.r.t. t

$$\int \frac{1}{x} \underbrace{x'(t)\, dt}_{\text{change of vars}} = \int \cos t\, dt$$

$u = x(t)$
$du = x'(t)\,dt$

$$\Rightarrow \quad \ln|x| = \sin t + c$$

$$\Rightarrow \quad |x| = e^{\sin t + c} = \underbrace{e^c}_{k\text{: const.} > 0} e^{\sin t} = k e^{\sin t}$$

$$\Rightarrow \quad x = k e^{\sin t} \qquad k\text{: any const.}$$

Same in general: $\dfrac{dx}{dt} = f(t)g(x) \xrightarrow[\text{in } t]{\text{int.}} \displaystyle\int \frac{1}{g(x)} \frac{dx}{dt}\, dt = \int f(t)\, dt$

(remark on fraction interp. of deriv.)

not always possible to solve for x:

e.g. $\dfrac{dx}{dt} = \dfrac{x}{1+x^2} \rightsquigarrow \ln|x| + \dfrac{x^2}{2} = t + c$

implicit solution $x(t)$.
perfectly usable for most purposes

Try an example using number cards 1-4.

Why do this? <u>Ans</u>. Want to predict the future.

(1) Model physical situation using differential equation

(2) Analyze diff. eqn:

(a) exact solution (e.g. sep. of vars.)

(b) numeric approx. (Friday's lecture)

(c) qualitative analysis (slope fields: rest of today)

(usually something like...)

(3) Draw conclusion: What is long-term behavior of solution?

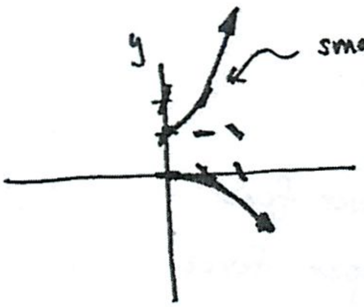
e.g. Is the population stable? Will business grow?
Do vibrations dampen or get amplified?

slope fields.

$$\frac{dy}{dx} = y^2 - x$$

then choose (x_0, y_0). Know slope of tangent line to solution curve at that point.

(x,y)	dy/dx
$(1,2)$	3



Keep going, filling in regularly spaced slope marks.

Can guess trajectory of solutions, given any initial value.

$(0,1)$	$dy/dx = 1$
$(0,0)$	$dy/dx = 0$
$(0,2)$	$dy/dx = 4$

Better : Use computer to draw them.

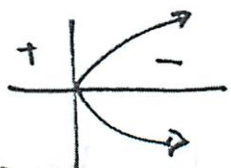
(See "Isoclines" Applet. Pick a few points and draw solutions.)

What is isocline? Curve with $\frac{dy}{dx} = c$ some fixed constant c.

"iso" - fixed
"cline" - slope

(In our example $y^2 - x = c$ so $y = \pm\sqrt{x+c}$)

0 - isocline : $x = y^2$



Inside parabola $x > y^2$ $dy/dx < 0$

Outside $dy/dx > 0$

Know that dy/dx has fixed sign <0 / >0 inside/outside of parabola.

Draw more curves, illustrate that sol'n max are acheived along parabola

Long term behavior? First, Fundamental theorem on Existence/Uniqueness

Fences and funnels :

In our example, solutions that enter the 0-isocline $x = y^2$ appear to be trapped inside it.

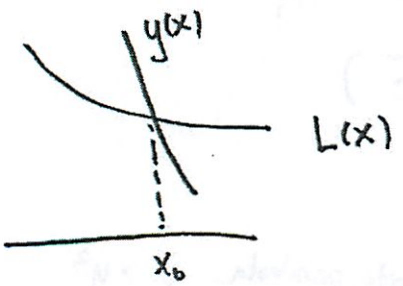
Say that curve $L(x)$ is a lower fence if solution can't cross it from above. $\mathcal{U}(x)$ is an upper fence if solution can't cross it from below.

In example $\frac{dy}{dx} = y^2 - x$, it seems $-\sqrt{x}$ is lower fence
$\sqrt{x}$ is upper fence.

claim $L(x)$ is lower fence if $L'(x) < f(x, L(x))$
(for $\frac{dy}{dx} = f(x,y)$)

proof : Suppose $y(x)$ is solution curve, crossing $L(x)$ from above at x_0 :

$$y'(x_0) \le L'(x_0) < f(x_0, L(x_0)) = f(x_0, y(x_0))$$
$$= y'(x_0)$$



Contradiction!

In our example, $L(x) = -\sqrt{x}$ has $L'(x) = -\frac{1}{2\sqrt{x}} < 0 = f(x, L(x))$

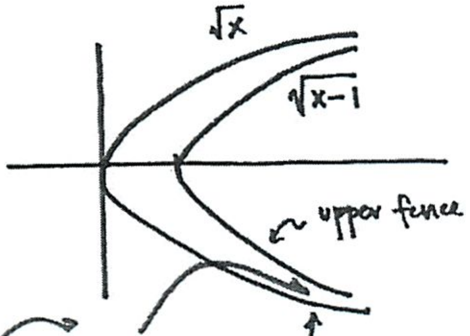
since L is 0-isocline.

Similarly upper fence $\mathcal{U}(x)$ satisfies :

$$\mathcal{U}'(x) > f(x, \mathcal{U}(x))$$

Funnels are regions of the plane trapped between fences having same asymptotic behavior.

Example : $\frac{dy}{dx} = y^2 - x$ then 0 and -1 isoclines look like:



Any solution entering "funnel" formed by two fences must asymptotically approach $-\sqrt{x}$ as $x \to \infty$.

Usually try to build fences out of isoclines and straight lines, but other choices are possible.

NEXT TIME : • Use slope fields to motivate method of numerical approximation. (Read Ch. 6 of EP part. 6.1, 6.2)

• state fundamental thm. about existence / uniqueness of ODEs.

IN SECTION : First look at modeling using population growth model.

Diff Eq Self Review

9/7

WP

- fn
- or or more unknown variables
- relates itself + derivitives

So ~~When~~ for constantly varrying quantities
- know rate of change
- predict long term future

Newton's Law in mechanics

~~Gravity~~ Ball falling = gravity - air resistance

constant   proportial to velocity

So acc (deriv velocity) <u>depends</u> on velocity

Need to solve diff eq

(would have liked worked example)

②

Closed form expression - if expressed analytically in terms of bounded well-known functions

$+ \ - \ \times \ \div$

$\sqrt[n]{\ }$ exp, log

Not ∞ series, $\int$ int, lim, ∞ contined fractions

Closed form solution if 1 sol can be closed for exp.

ODE - where dep variable function of 1 ind variable

order of higest deriv

Paul's Notes

differential eq - any eq w/ derivs

$F = ma$

$a = \frac{dv}{dt} = \frac{d^2u}{du^2}$ $\quad v = vel$, $u = pos$ at time t

3

So can write Newton's law as

$$m \frac{dv}{dt} = F(t, v)$$ ← just saying what function of time, actual velocity

← 1st order

$$\frac{m\, d^2 u}{du^2} = F\left(t, u, \frac{du}{dt}\right)$$

← 2nd order

? b/c 2 derivs down?

Not b/c ord or partial derivs

? ?

Linear any diff eq that can be writen in form ODE PDE

$$a_n(t)\, y^{(x)}(t) + a_{n-1}(t)\, y^{(n-x)} + \dots$$

$$+ a_1(t)\, y'(t) + a_0(t)\, y(t) = g(t)$$

No products of fns

it + its derivs only to 1st power

$a_0(t), \dots, a_n(t)$ are coefficents

- zero / non zero
- Constant / non constant
- linear / non / linear

Only $y(t)$ + its derivs ~~matter~~ effect if the diff eq is linear

(9)

Solution

A sol on interval $\alpha < t < \beta$ is any fn $y(t)$ that satisfies eq in interval

Often solutions come w/ intervals

example

$$4x^2 y'' + 12xy' + 3y = 0 \qquad x > 0$$

non linear ??? ...

So get 1st + 2nd derivs
- solve for y' y''

y''

$$4x^2 y'' = -12xy' - 3y$$

$$y'' = \frac{-12xy'}{4x^2} - \frac{3y}{4x^2}$$

$$= \frac{-3y'}{x} - \frac{3y}{4x^2}$$

X

oh they take derivs from sol trying to verify

(5)

Show $y(x) = x^{-\frac{3}{2}}$ is sol

(Need to get comfortable w/ vocab
and type of problems
today was a blur)

just recognize
its related

$y' = -\frac{3}{2} x^{-5/2}$ wrong way

$y'' = -\frac{5}{2} \cdot -\frac{3}{2} x^{-7/2}$ duh was thinking that

$+ \frac{15}{4} x^{-7/2}$

Plug in

$4x^2\left(\frac{15}{4}x^{-7/2}\right) + 12x\left(-\frac{3}{2}x^{-5/2}\right) + 3\left(x^{-3/2}\right) = 0$

$15x^{-3/2} - 18x^{-3/2} + 3x^{-3/2} = 0$

$0 = 0$

evens out

✓ satisfies eq

↳ solution ✓

Often need to restrict possible values of ind variable

(6)

Many possible sols sometimes

∞ #

like $x^{-1/2}$

$-9x^{-3/2}$

So look at inital conditions

$y(t_0) = y_0$ ← value ~~at certain~~ of sol at certain pts

$y^{(k)}(t_0) = y_k$ ← " of deriv " " "

you need depends on order of diff eq

~~Def~~ Inital Value Problem

- diff eq w/ proper # of ics.

Interval of Validity for IVP w/ inital conditions

is largest possible interval where sol valid

+ contains t_0

(7)

General Solution - no ics

ie $y(t) = \frac{3}{4} + \frac{c}{t^2}$ is general sol for $2t y' + 4y = 3$

(so now how to find that)

Actual sol - general sol w/ ics.

Explicit sol

$y = y(t)$

y is on this side

Implicit

Ys on both sides

or y^{power} on left

Direction Fields did in class

8

Gravity example

$F_g = ma$

$F_a = -\gamma v$

$$m \frac{dv}{dt} = F(t, v)$$

$$m \frac{dv}{dt} = mg - \gamma v$$

divide m from both sides

$$\frac{dv}{dt} = g - \frac{\gamma v}{m}$$

Plug some # in

$$\frac{dv}{dt} = 9.8 - \frac{.392}{2} v$$

9

Say $v = 30$ ¿"geometric view"???

¿it wont always be
but at a particular pt in time

then $\frac{dv}{dt} = 3.92 = acc$

¿velocity increasing at 3.92 $\frac{m}{s/s}$

¿¿right?

So try all these

Make pic 

See long term behavior

~~Draw~~ Start at a point + draw arrow

$\hookrightarrow$ = a solution curve

get family of sol curves / set integral curves
~~set~~

So here $v \to 50$ m/s as $t \uparrow$

Does not matter where you start

(10)

I think I will review lecture ~~date~~ problem now

$$\frac{dx}{dt} = 2xt + x$$

? That is the general sol for what ODE ?

? don't know - look at ans

$$\frac{dx}{dt} = x(2t+1)$$

Simplify

$$\frac{1}{x} \cdot \frac{dx}{dt} = 2t+1$$

integrate both sides

? what is goal ? x

$$\int \frac{1}{x}\,dx = \int (2t+1)\,dt$$

$$\ln|x| = (t^2+t) + c$$

$$|x| = Ce^{t^2+t}$$

where $C = e^c$

$$x = Ce^{t^2+t}$$

C arbitary

(? where more practice ?)

(1) Pauls

Whole bunch of ways to solve them

Many different cases

? But what was the 1st one?

~~Dirent~~

Seperable

18.03 Notes

$$\frac{dx}{dt} = t^2 + 1$$

? So want sol $x =$ w/r t

Integrate both sides by t

$$\int \frac{dx}{dt} dt = \int (t^2 + 1)\, dt$$

$$x = \frac{t^3}{3} + t + c$$

? don't forget

Notation

$$\frac{dx}{dt} = x'(t)$$

12

So what are we doing?

The change of x is t^2+1

So x is $\frac{t^3}{3}+t+C$

Is just a simple integration for ~~t~~ x

That is all we did today

"Needed to seperate variables 1st"

- Yeah, we put x back on its side

- all x on 1 side all t on one side

- Then knowing $\int \frac{1}{x}$ - still not natural at that

And handeling abs value

Generalization of Sep of Vales

$$\frac{dx}{dt} = f(t)\,g(x) \qquad \text{Ex} \quad \frac{dx}{dt} = x\cos t$$

(13)

Let me try

$$\frac{1}{x}\frac{dx}{dt} = \cos t$$

$$\int \frac{1}{x}\frac{\cancel{dx}}{\cancel{dt}}\cancel{dt}$$

$$\ln(x) = \int \cos(t) + c$$

TWA

abs

$$\ln|x| = \sin x + c$$ valid if $x \neq 0$

Take natural log?

No exponentiate

$$e^{\ln(x)} = x = e^{\sin x} + e^{c}$$

$$|x| = e^{\sin x} \cdot e^{c}$$ multiply!

abs value

$$|x| = e^{\sin x + c}$$

then what

$$= e^{c} e^{\sin t}$$

$$= k e^{\sin t}$$

abs value - how get rid does not make difference??

$$x = k e^{\sin(t)} \quad k = \text{const} > 0$$ just define

(14)

Not always possible to solve for x

$$\frac{dx}{dt} = \frac{x}{1+x^2}$$

$$\ln|x| + \frac{x^2}{2} = t + c$$

implicit solution for ~~x(t)~~ x(t)

Usable for most purposes
ys on both sides

? But example?

How do you use

Or ~~do~~ just say its a solution

(I never know what to do next

- need the why
- and outline of steps / procedures)

So we got exact sol today

Not always possible → see Fri

Good for today I think

Out of blue pen

18.03 Recitation

9/8

Vivek

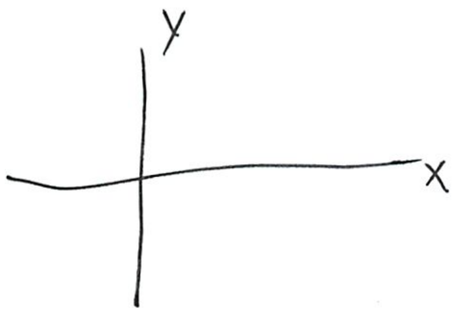
2-248

vivek@math.mit.edu

(Mostly sephomores - 80%)

Problem from lecture was seperate the variable

Slope fields



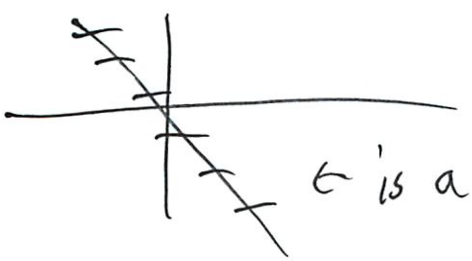
$$\frac{dy}{dx} = f(x,y)$$

at each pt, a diff slope

plot them!

isocline - line where curve is constant

when $\frac{dy}{dx} = C$



← is a lower fence

②

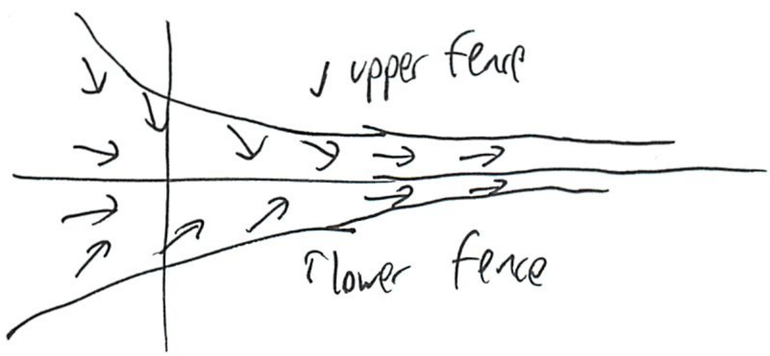
First lines are solution curves

Then draw in isoclines - kinda the limiting behavior

0-isoclines → slope 0

Can have lines that are both isoclines + solution curves

~~Fun~~ Squeeze



General Solution - encodes all the solutions

Problems

Population Model

1965 3.32 bil

For every 1000 people 50 born, 25 die

(3)

So $\dfrac{\Delta P}{\Delta T} = 50 - 25 \cdot \dfrac{\text{Pop(at certain time)}}{1000}$

Or in general

$$\frac{dP}{dt} = \underset{\substack{\uparrow \\ \text{constant}}}{a}\, P(t)$$

redefine a as .02

a) How long does it take for pop to double?

b) What is the pop in 2500?

c) How much space will each ~~people~~ person get?

$\boxed{\text{SA earth} = 5.5 \cdot 10^{15} \text{ sq. ft.}}$

Own So what do we want?

Value of P

$$\frac{1}{P}\frac{dP}{dt} = a$$

$$\int \frac{1}{P}\frac{dP}{dt}\,dt = \int a\,dt$$

$$\ln|P| = a \cdot t$$

(4)

Then

$e^{\wedge}$

$e^{\ln|P|} = |P| = e^{a \cdot t}$

say $P > 0$

$P = e^{at}$

at $t = 1965$

$3.32 \text{ billion} = e^{.02 \cdot 1965}$

? but no variable here

Can we verify

or define t here as 0?

or as something

$e^{.02 \cdot 1965} = 1.16 E17$

$\cancel{6.68 = e^{.02 \cdot 2?}}$

So solve for t

$3.32 \text{ billion} = e^{.02 t}$

$3.32 \text{ billion} = e^{.02} e^{t}$

$\ln\left(\frac{3.32 \text{ billion}}{e^{.02}}\right) = t$

(5)

Scrap

$$\frac{dP}{dt} = a\,P(t)$$

$$\frac{dP}{P(t)} = a\,dt$$

$$\int \frac{dP}{P(t)} = \int a\,dt$$

$$\ln P = at + C$$

I forgot

$$P = Ce^{at}$$

I need to figure out that algebra

Define 'inital' t

$$e^{.02t} = 2$$

$$.02t = \ln 2$$

$$t = \frac{\ln 2}{.02} = 50 \ln 2$$ ← where did they use this?

$C = 3.32 \cdot 10^9$ people ← I thought P was this → $\underbrace{Ce^{(.02 \cdot 0)}}_{\text{goes away}}$

over ←

So ↑ plugin

$Ce^{.02 \cdot 535}$ ← I guess they defined $t = 0$ 1965 for this

$2500 - 1965 = 535$

$P = 3.32 \cdot 10^9 \cdot 1.47 \cdot 10^{14} =$ ____ Then divide by space

(I think I got the idea - just stuck when my calc broke)

1. $6.64 \cdot 10^9 = 3.32 \cdot 10^9 e^{.02t}$

$2 = e^{.02t}$

$\ln 2 = .02t$

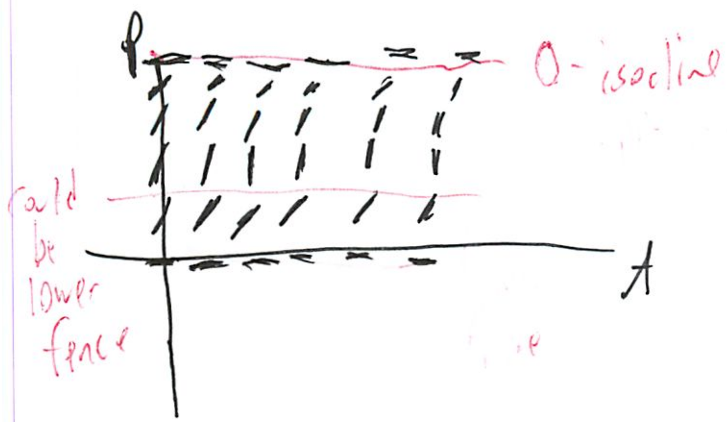
$t = \frac{\ln 2}{.02}$

≈ 34.7 years

$\approx 1965 + 34.7 = 2000$

⑥ dep → $\frac{dP}{dt}$ = $aP - bP^2$ $0 < b << a$
ind → ~~dt~~

1. Sketch slope field
2. Predict long term behavior



; so guess/define a a, b as constants

does not change left to right

Say b = 1
a = 5

So ~~...~~ P = 0 → 0
P = 1 → 4
P = 2 → 6
P = 3 → 6
P = 4 → 4
P = 5 → 0

Will go to P = 5
for a = 5, b = 1

Will go P = a ??
; would have to test more

(I should also do above/below stuff)
- but P > 0
and does not depend on t

0-isocline is that P level line
No fence - vector field tangent
not upper or lower tangents

More on fences and funnels: Lecture 2 (missed) 9/9 ①

In our last lecture, exploring slope fields (a.k.a. direction fields)

draw small chunks of lines having slope $f(x,y)$ for diff. eq'n

$\frac{dy}{dx} = f(x,y)$. Infer behavior of solution curves.

Today: Use slope fields in three ways:

(1) - more w/ fences & funnels to actually prove results about long-term behavior (not just guess them from picture)

(2) - Guess a result about existence/uniqueness of ODEs

(3) - Motivate method for numerical solutions (Euler's method)

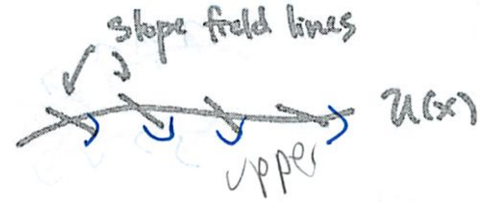
Recall lower fence $L(x)$ is curve s.t. $L'(x) < f(x, L(x))$

(in pictures

always above

think of slope field like a river current. It's pushing solutions above $L(x)$ away from lower fence.)

Similarly, upper fence $U(x)$ satisfies $U'(x) > f(x, U(x))$



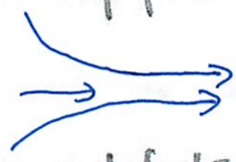
always goes under

More carefully $L(x)$ may be a lower fence only for x in interval e.g. (a,b) or (a,∞)

can limit

A (narrowing) funnel is a pair of fences $\mathcal{U}(x) > L(x)$ with

$$\lim_{x\to\infty} |\mathcal{U}(x) - L(x)| = 0$$ (same asymptotic behavior)



Example last time: $dy/dx = y^2 - x$

part of -1-isocline part of 0-isocline

funnel from $\mathcal{U}(x) = -\sqrt{x-1}$, $L(x) = -\sqrt{x}$.

Any solution landing in this funnel has asymptotic behavior like $-\sqrt{x}$. b/c goes to same pt

(Note: $\mathcal{U}'(x) = -\frac{1}{2}(x-1)^{-1/2}$ and $f(x, \mathcal{U}(x)) = -1$

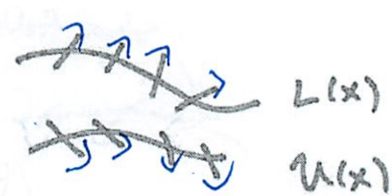
so get upper fence if $x > 5/4$.)

SOLUTIONS ALWAYS REMAIN INSIDE FUNNELS!

(look at applet to determine which initial conditions land in above funnel)

Even more exciting than funnels: ANTIFUNNELS

Antifunnel is pair of fences $L(x)$: lower, $\mathcal{U}(x)$: upper s.t. $L(x) > \mathcal{U}(x)$.



Solutions can't enter an antifunnel.

Surprising result about antifunnels: There always exists a solution inside antifunnel. (Hard to prove)

a point? depends where you start

Also: Given $\frac{dy}{dx} = f(x,y)$, and narrowing antifunnel,

-but can only start certain pts

if $\partial f/\partial y \geq 0$ in antifunnel, then there is a unique solution which stays in antifunnel.

?what

pf: $\partial f/\partial y$ measures dispersion rate of solutions (if initial conditions close, $\partial f/\partial y$ measures whether they remain close)

Given two solutions in anti-funnel, they must differ at some point x_0. ($y_1(x_0) \neq y_2(x_0)$ if y_1, y_2 sol'ns.)

But if $\partial f/\partial y \geq 0$, they can never get closer together, while narrowing antifunnel gets smaller and smaller. So one must exit the funnel.

In our example: $\frac{dy}{dx} = y^2 - x$ then $\underset{\mathcal{U}(x) =}{\sqrt{x}}$, $\underset{L(x) =}{\sqrt{x+1}}$ (y)

form antifunnel and $\partial f/\partial y = 2y \geq 0$ if $y \geq 0$. UNIQUE SOL'N $\to \sqrt{x}$ as $x \to \infty$

Last time, discussing slope fields. In playing around with applet "Isoclines" we see that picking any initial value $y(a) = b$ there seemed to be a unique soln $y(x)$ through the point (a,b)

- Search online for "dfield" — more flexible applet for drawing direction fields (i.e. slope fields)

Also, in our exact solutions to first-order ODEs, specifying a point (a,b) on the solution curve det'd the solution uniquely. So can find C

E.g.: $\frac{dx}{dt} = x \cos t \qquad x(0) = 5 \quad \rightsquigarrow$ gen. soln: $x(t) = ke^{\sin t}$

through $(0,5)$:

$x(t) = 5e^{\sin t}$

claim: Given $\frac{dx}{dt} = f(t,x)$, if f is nice near (t_0, x_0) and point (t_0, x_0)

then there exists a unique solution to this "initial value problem"

More precisely: BIG THEOREM: Given $dy/dx = f(x,y)$ and point (a,b)

if $f(x,y)$ and its partial $\frac{\partial}{\partial y} f(x,y)$ exist and are continuous on a rectangle containing (a,b), then the initial value problem with $y(a) = b$ has a unique solution on an interval I containing a.

Remarks: (1) I : interval can be specified in terms of a, b, and $f(x,y)$.

(2) The hypotheses are necessary. See EP p. 23-26 for discussion.

For example $\frac{dy}{dx} = 2\sqrt{y}$ with $y(0)=0$ has two solutions:

$y=0$, $y=x^2$. (partial in y not defined at $y=0$)

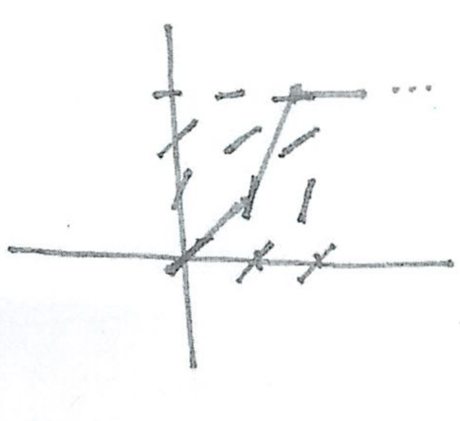
(3) Why does this matter? Helpful to know that solution exists when applying numerical / qualitative methods which can ~~[illegible]~~ otherwise lead to subtle mistakes, as we'll see later in lecture.

Numerical methods : Euler's method.

Idea : Given initial value problem $\frac{dy}{dx} = f(x,y)$ $y(x_0) = y_0$ then use linear approximation to estimate a solution as follows.

In direction field : start at $(0,0) = (x_0, y_0)$.

Flow along tangent line for one unit. ($\Delta x = 1$)



Get piece-wise linear curve that approximates a solution.

See applet "Euler's Method"

Math 18.03 : Differential Equations

Lecture 2 Supplemental Notes

Friday, September 9, 2011

Quick Quiz 2

True or False: The graph of any continuous function $C(x)$ is always made of upper fences and lower fences.

- TRUE
- FALSE

Quick Answer

True or False: The graph of any continuous function $C(x)$ is always made of upper fences and lower fences.

- TRUE-ish
- FALSE

The condition that $C(x)$ is a lower fence is

$$C'(x) < f(x, C(x))$$

and to be an upper fence

$$C'(x) > f(x, C(x))$$

and so the only time when one of these inequalities does not hold is if

$$C'(x) = f(x, C(x)).$$

The Logistical Model $dP/dt = aP - bP^2$

The 0-isoclines for this model are just the constant functions

$$P = 0 \quad \text{and} \quad P = a/b.$$

Experimentation shows that solutions do not cross them, but

$$P'(x) = 0 = f(x, P(x))$$

in both cases. Can we (or should we) call them fences?

YES. Hubbard and West call them "weak fences" when, for example for a lower fence $L(x)$,

$$L'(x) \leq f(x, L(x)). \qquad \text{NOTE it's } \leq \text{ here not } <.$$

The Logistical Model $dP/dt = aP - bP^2$

Under mild hypotheses, weak fences are also "non-porous" – don't let solutions leak through from one direction.

(The proof is much harder than Wednesday's one-line proof.)

Further discussion of issues arising in the Recitation 1 exercises are written in a handout on the course home page.

See "Announcements and handouts" section for the PDF file.

Common choices for fences

Any continuous function is made up of (at least weak) fences. But there are several choices of curves that are often used as fences:

- Constant functions or lines (Why? Their derivatives $C'(x)$ are constant, so easy to compare to $f(x, C(x))$)
- Isoclines (Why? The values $f(x, C(x))$ are constant, so easy to compare to $C'(x)$.)
- Solutions to easier, related differential equations (Why? Their behavior should be close to the behavior of the harder ODE.)

18.03 lecture 3

9/12

(Apparently I missed lecture 2 - did not know had lecture Fri)

PSet 1 Due Fri

Office hrs will be posted - can go any TA

Read books/notes in advance of lecture (Need to get book - had ordered it)

Last lecture - finished Euler's method for
- fences + funnels

$$\frac{dy}{dx} = f(x,y)$$

Pick inital pt (x_0, y_0) Flow along line w/ slope $f(x_0, y_0)$ for fixed Δx

Repeat

-

Use slope field to find numeric way of finding ways of solving diff eqs

Quiz $\frac{dy}{dx} = y^2 - 1$

$y(0) = 5$

(2)

Start $(0, 5)$

Compute slope 24

So go to $(1, 29)$

$\uparrow$
$5 + 24 \cdot 1$

$$(x_0, y_0) \to (x_1, y_1) = (x_0 + \Delta x,\ y_0 + \Delta x\, f(x_0, y_0))$$

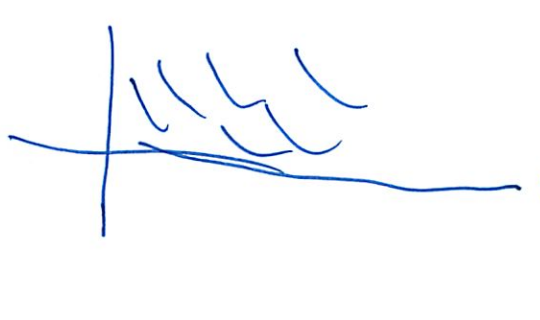
Not a great ans $y^2 - 1$ is curvy
we just used straight line

Then could

$$(x_1, y_1) \to (x_2, y_2) = (x_1 + \Delta x,\ y_1 + \Delta x\, f(x_1, y_1))$$

Have an applet for this "Euler's Method"

$$\frac{dy}{dx} = -xy$$



$\leftarrow$ so $y = 0$ is a sol

(3)

Is seperable

$$\int \frac{dy}{y} = \int -x \, dx$$

↳ inital value problem
w/ $y(0)=1$

$$y(x) = C \cdot e^{-\frac{1}{2}x^2}$$

constant could be 0
So 0 is a solution
(also $y=0$ is sol to $y(0)=0$)

Can see better if we reduce step size in applet
- then matches eqn / bell curve more

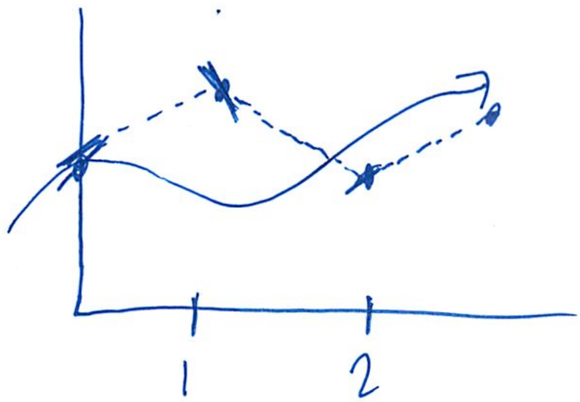
Table where $h =$ step size

When can solve eq exactly - can do it
but when can't - use numeric method
and better w/ small step sizes

(4) (I'm really annoyed I missed that lecture)

Easy for a computer to do

If sol looks like



doing Eulers method

Large possible room for errors

(1) - linear approximation for sol'n

(2) - and can start away from sol curve

- Compounding inital error
- error cumulative

Fix: make step size smaller

(3) - Rounding error from not enough digits

Fix: carry more digits

5

Two questions 1. How can we estimate error?

- so can choose step size

2. Does the solution actually exist?

- When can we guarentee a solution?

Assume if given an inital condition, we get unique sol. Is that always true?

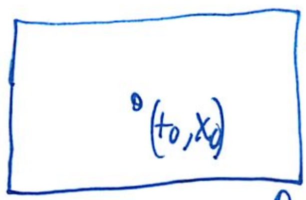
Some statement about existance + uniqueness

Rough $\frac{dx}{dt} = f(x,t)$ and inital pt (t_0, x_0)

Then if f is "nice" near (t_0, x_0) than a unique sol'n to the inital value problem exists.
└ diff eq and a point $x(t_0) = x_0$

"nice" = $f(t,x)$ and $\frac{\partial}{\partial x} f(t,x)$ exists* and are continuous
dep variable
on some rectangle containing the point of interest (t_0, x_0)

6



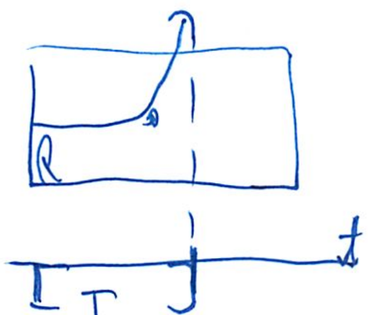
← rectangle

↑ within here it + deriv is "nice"

exists* = Sol'n exists on ~~th~~ some interval I ∋ t_0

I depends on t, x_0, $f(\cancel{x,w})$ t,x

So



↑ interval smaller than R

(See E.P 23-26 for discussion weird examples + reasons why we need the hypothesis)
↑ Edwards + Penny

Can we estimate error by choosing step size h

How find error in linear approx?

<u>Taylor Series w/ Remainer</u> from 18.01

error in estimation is controlled by next derivative

ie linear $|y''(x)|$ → find max over interval. →

~~NR~~ 7

→ result is C

<u>Then</u> Euler's ~~Err~~ Error

$$\left| \underbrace{y(x_n)}_{\text{actual}} - \underbrace{y_n}_{\text{estimate}} \right| \le C \cdot h$$

↑ step size

Table on slide 5

$$\frac{\text{Error}}{h} \approx C$$ ← about constant

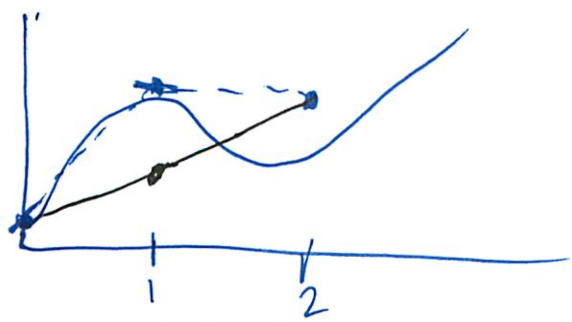
Use upper bound in Euler's Method

Normally just show is an error

- Don't really use in practice

Could find C

Improved Euler's Method



Could avg points

(not much improved here)

8

Then what is error b/w actual + improved

$$|\underbrace{y(x_n)}_{\text{actual}} - \underbrace{y_n}_{\text{improved est}}| \le ch^2$$

↑ Rounding error less likly

Extra work 3h calculations not h

Something like Simpson's rule

↳ Runge-kutta method - get h^4

Lecture 3

Euler's method: Initial value problem $\frac{dy}{dx} = f(x,y)$ with $y(x_0) = y_0$.

Then choose a step size $\Delta x = h$, value of solution $y(x_0+hn)$ is approximated as follows: (n: pos. int.)

$(x_1, y_1) = (x_0+h,\ y_0 + h\cdot f(x_0,y_0))$ ← means we flow along tangent direction w/ slope $\frac{dy}{dx} = f(x,y)$ for h units.

$(x_2, y_2) = (x_1+h,\ y_1 + h\, f(x_1,y_1))$

$\vdots$

Example: (with Euler's method applet)

$dy/dx = -xy$ (separable)

$y(0) = 1$

Can do exact: $\int \frac{dy}{y} = \int -x\, dx$

IVP: $y(x) = e^{-\frac{1}{2}x^2}$

step size $1 = \Delta x$.

$(0,0) \longrightarrow (0+1,\ 0 + 1\cdot f(0,0)) = (1,0)$

$(1,0) \longrightarrow (2,\ 0 + \underbrace{f(1,0)}_{-1}) = (2,0)$

$(2,0) \longrightarrow (3,0) \longrightarrow (4,0) \longrightarrow \cdots$

What is approx. value via Euler's method at $x = 3.0$

ACTUAL: $e^{-9/2} \approx .0111a.$

From picture, we see error comes from several sources:

(1) From (x_n, y_n) to (x_{n+1}, y_{n+1}) have error from following tangent line (not solution curve)

(2) Also (x_n, y_n) is result of earlier approx., and not generally on the solution curve $y(x)$.

(3) rounding errors from estimating resulting constants.

Small reason to be careful about # of steps we take.

Good reason to want to make step size smaller

- Show slide of table with smaller and smaller step sizes.

Two issues:

- How do we know that solution exists in first place? Back to existence/uniqueness in Lecture 2 Notes (skipped on Friday)

  (Euler's method can be insensitive to vertical asymptotes ...)

  Example: $dy/dx = x^2 + y^2$. Show table EP p. 439.

  In practice, see that approximations in table are approaching specific value.

- Can we give concrete error bounds on approx. in Euler's method?

Yes. If you remember Taylor's thm. with remainder, error in power series bounded by max. of next derivative.

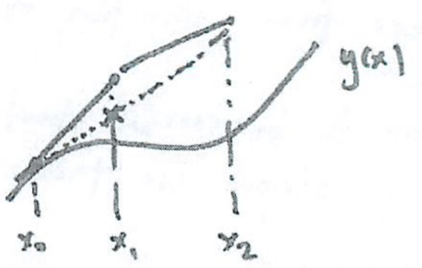
Error: if $\frac{dy}{dx} = f(x,y)$, $y(x_0) = y_0$ with f and f_x, f_y defined and continuous on rectangle containing (x_0, y_0), then

$$\underbrace{|\, y(x_n)}_{\text{actual value}} - \underbrace{y_n}_{\text{Euler approx}} | \leq C \cdot h$$

C: const. dep. on max of $|y''(x)|$. (hard to determine)

(Can use table of values to estimate C)
really just verifying thm. Not so useful in practice.

Refinements to Euler's method:
Two steps of Euler's method averaged
Error: $\leq C \cdot h^2$



Mention Runge-Kutta method.

Autonomous equations:

$$\frac{dy}{dx} = f(y).$$

(i.e. doesn't depend on x)

~~[illegible]~~ resulting slope field is constant along horizontal lines.

Example: Population model

$$\frac{dP}{dt} = aP - bP^2.$$

Example 2: $\frac{dy}{dx} = y(1-y)$.

Factor $f(y)$. Roots will be solutions

$y=0$, $y=1$ are called steady-state solutions.

(these are 0-isoclines)

In autonomous equation,

any shift of solution

$y(x)$ to $y(x-c)$: c constant

is also a solution.

(translation symmetry of solutions)

Only need to understand a single solution between steady solutions.

Summarize asymptotic behavior with a phase diagram.

For our example

if $y(0) > 1$ then $y(x) \to 1$ as $x \to \infty$

if $0 < y(0) < 1$ then $y(x) \to 1$ as $x \to \infty$

if $y(0) < 0$, then $y(x) \to -\infty$ as $x \to \infty$

Prove these properties using fences/funnels.

if neither sink nor source called "node"

Math 18.03 : Differential Equations

Lecture 3 Supplemental Notes

Monday, September 12, 2011

Quick Quiz 3

Given the differential equation

$$dy/dx = y^2 - 1$$

with initial condition $y(0) = 5$, then which of the following gives the first iteration in Euler's method with step size $\Delta x = 1$?

- (5, -1)
- (1, 4)
- (1, 24)
- None of the above

Quick Answer

Given the differential equation

$$dy/dx = y^2 - 1$$

with initial condition $y(0) = 5$, then which of the following gives the first iteration in Euler's method with step size $\Delta x = 1$?

1. (5, -1)
2. (1, 4)
3. (1, 24)
4. None of the above

We follow the line in the slope field with slope $\frac{dy}{dx}|_{(x,y)=(0,5)} = 24$ for $\Delta x = 1$ unit from the point $(0, 5)$. This gives $(1, 5 + 24 \cdot 1) = (1, 29)$.

Euler's method for $dy/dx = -xy$

x	$h = 1$	$h = 1/2$	$h = 1/4$	$h = 1/8$	$h = 1/16$	Actual
.5	–	1	.9375	.9089	.8954	.8825
1	1	.75	.6665	.6340	.6197	.6065
1.5	–	.375	.3436	.3331	.3287	.3247
2	0	.09375	.1208	.1290	.1324	.1353
2.5	–	0	.0264	.0359	.04008	.04394
3	0	0	.0031	.00697	.00903	.0111

Another approximation to $y(3)$ for smaller h:

$$h = 1/32 \ : .01007;$$

Error in Euler's method

We stated (without proof) that the error in Euler's method is roughly a constant times the step size h:

$$|y(x_n) - y_n| \leq Ch \quad \text{for some constant } C.$$

Here $y(x_n)$ is the value of the solution $y(x)$ at x_n and y_n is the approximation using n steps of Euler's method.

In our earlier example:

h	y_n	$y(3)$	$\lvert y(3) - y_n\rvert/h$
1/8	.00697	.0111	.03304
1/16	.00903	.0111	.03312
1.32	.01007	.0111	.03296

So it appears some constant C around .034 will do. This really just verifies the result on error. It isn't so useful in practice. Just decrease step size until the result appears to settle to a limit.

18.03 Recitation 2

9/13

- (Different TA) ~~[illegible]~~

You should be able to do everything on P-Set 1 except

- integrating factors in Part 1
- bifurcation in Part 2

Today Funnels + Fences, Euler's Method

$$\frac{dy}{dx} = 2y - y^2$$

Logistic equation of the form

$$\boxed{\frac{dy}{dx} = ay - by^2}$$

better model for pop growth than exp. model

$$\frac{dy}{dx} = ay$$

Sol'n = $y = Ae^x$

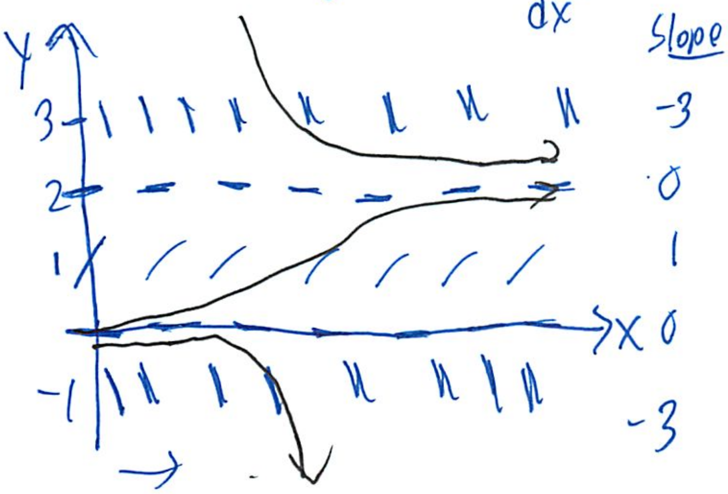
Since exp grows forever but growth may slow down

- Could solve w/ sep of variables
- but hard

(2)

1. Draw a slope field for this eq + solution curves

$$\frac{dy}{dx} = 2y - y^2$$



Same all accross

Draw some sol curves in black

So you can see b/w 0, 2 pop will grow till reaches limit at 2 asymptotically

~~How would be prove solution is 2~~

~~-Show~~ $\frac{dy}{dx}$ is

How would prove increasing?

If $0 < y < 2$ $\quad \frac{dy}{dx} = 2y - y^2$

$= y(2-y)$

(11) Could avg pts so error less likly

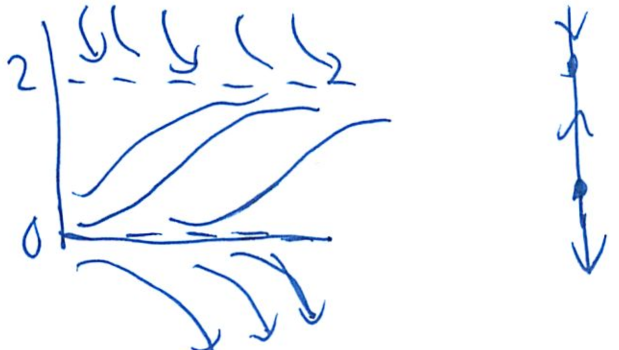
R2 **Logistic equation** - better model for pop growth than exponential

$$\frac{dy}{dx} = ay - by^2$$

Can solve w/ sep of variables, but hard

1. Draw slope field

example $= \frac{dy}{dx} = 2y - y^2$



How do we know sol $= 2$?

2. How prove is ↑ in range $0 < y < 2$

$$\frac{dy}{dx} = 2y - y^2$$
$$= y(2-y)$$

Note y must be (+) < 2

So ↑ in range

Could say fence $L(x) = 1$

↑ is an isocline, but not a 0-isocline

(3)

y must be positive and < 2

So

⊕ x ⊕ will be ⊕

So prove its increasing

(I get it now

- need to pay attention to what is going on)

A fence is at $L(x) = 1$ ← just a line

- look at slope along line
- its always +1 here
  - easy since same accross line

So sol curves when hit line are always going UP accross it

Nothing ever goes below it

(a)

Because we know slope is always (+)

can say we could draw fences (0, 2)

not including (actually don't know if includes)

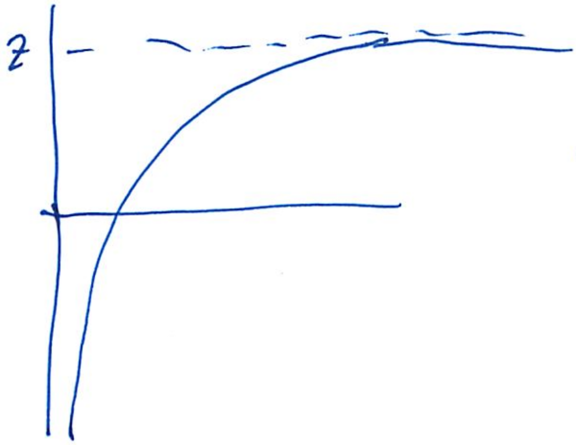
TA forgets too

Q. Show that the sol'n curve (a) tends towards $y=2$

How do we know it goes all the way up to 2 and asymptotes there

(I'm saying smart things - been helped by 6.042 thinking)
- other students prob reverse as most do 18.03 1st)

Could also have curve $L(x) = 2 - \frac{1}{x}$



Slope is $2y - y^2 = 2L(x) - L(x)^2$

$= 2\left(2-\frac{1}{x}\right) - \left(2-\frac{1}{x}\right)^2$

$= 4 - \frac{2}{x} - \left(4 - \frac{4}{x} + \frac{1}{x^2}\right)$

$= \frac{2}{x} - \frac{1}{x^2}$

(5)

Meanwhile Slope of $L(x)$ is

$$L'(x) = \frac{1}{x^2}$$

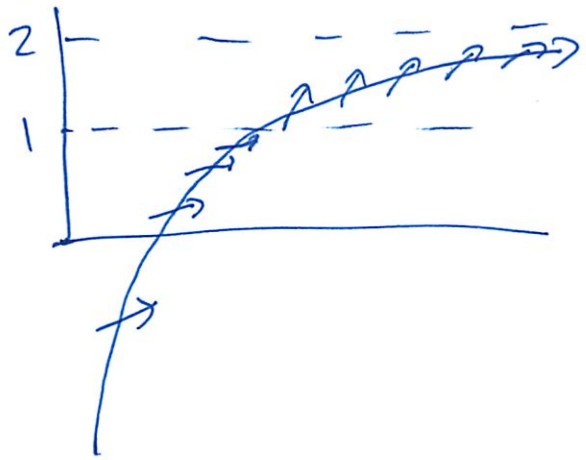
(Missed this - review math)

$$\frac{2}{x} > \frac{2}{x^2} \,?$$

$$x \overset{?}{<} x^2 \iff x > 1$$

Is a lower fence when $x > 1$

└ which happens to be $y > 1$



Solve analytically
- You can actually do it (lucky)
- Via seperation of variables
  - pretend is fraction
  - rearrange so all ys on one side, xs on other

6

$$\frac{dy}{2y-y^2} = dx$$

Integrate

$$\int \frac{dy}{2y-y^2} = \int dx$$

↓ easy

need to integrate using partial fractions $= x + C$

$\int \frac{1}{y}\, dy = \ln|y| + c$

$\int \frac{1}{a\cancel{x}+b}\, dy =$

$u = ay + b$

$du = a\,dy$

$= \int \frac{1}{u}\, dy$

Turn into du

$= \int \frac{1}{u} \frac{du}{a}$

$= \frac{1}{a} \ln|u| + c$

now sub that back in for y

$= \frac{1}{a} \ln|ay+b| + c$

(7)

But for fraction in denom $\rightarrow$ use partial fractions

$$\frac{1}{2y-y^2} = \frac{1}{y(2-y)} = \frac{A}{y} + \frac{B}{2-y}$$

Sum of things already done.

- Find constants

$$= \frac{A(2-y)+By}{y(2-y)}$$

$\uparrow$ denom here is same as original

So numerator should be same

$$A(2-y) + By = 1 + 0y$$

find A, B:

$$\boxed{\begin{matrix} 2A = 1 \\ -A+B=0 \end{matrix} \rightarrow \begin{matrix} A = \frac{1}{2} \\ B = \frac{1}{2} \end{matrix}}$$

$$= \frac{1}{2y} + \frac{1}{2(2-2y)}$$

So now we can integrate

- like we did earlier on right

$$\int \frac{dy}{2y-y^2} = \int \frac{dy}{2y} + \int \frac{dy}{2(2-y)}$$

(8)

$= \frac{1}{2}\ln|y| - \frac{1}{2}\ln|2(2-y)|$

$= \frac{1}{2}\ln|y| - \frac{1}{2}\ln|2(2-y)| = x + C$

Can combine

$= \frac{1}{2}\ln\left|\frac{y}{2(2-y)}\right| = x + C$

Still complicated

Use PC to plot

Can rearrange to get y as fn of x

numerator + denom would have exponentials

- or $x(y) = \frac{1}{2}\ln\left|\frac{y}{(2-y)}\right| + C$

now can shift left or right however you want

Retry Problem 1 from Textbook

9/13

Find general sol

$$\frac{dx}{dt} = 2xt + x$$

~~At~~

Move all x to dx side

$\int$ w/ respect dt

$$\frac{dx}{dt} = x(2t+1)$$

$$\frac{1}{x}\frac{dx}{dt} = 2t+1$$

$$\int \frac{1}{x}\frac{dx}{\cancel{dt}}\cancel{dt} = \int (2t+1)\,dt$$

this whole thing cancels, I believe

$$\ln|x| = \int (2t)\,dt + \int 1\,dt$$

$$= \frac{2t^2}{2} + t + C$$

$$= t^2 + t + C$$

now exponentate

$$e^{\ln|x|} = |x| = Ce^{t^2+t} = C\cancel{e^{t^2}\cdot e^{t}}$$

Then here

✓ got it right

don't need

Oh I get it now

C is e^c

but we don't care exactly what C is

9/13

lunar lander my try

$v(t_0) = 450 \, m/s$

$a(t) = 2.5 \, m/s/s$

At what height so $v = 0$ at touch down

$$\frac{dv}{dt} = \frac{2.5v}{t}$$

get all ts

$$\frac{t\,dv}{t\,dt} = 2.5v$$

$$\int \frac{1}{t}\,dt = \int 2.5v\,dt$$

↑ like normal $\frac{1}{\text{something}}$

$$\ln|t| = 2.5vt$$?? - integrating help

↑ then what

exponentiate both sides

$$t = C\,e^{2.5vt}$$ ↑ should not be!

↑ Constant

2

Can put L after #
- but when multiplication?

How do I find where it overflows?
Printing too slow!

How is 25 % 5 1? its 0
5 % 5 is 0 not 25

Spent hr on this - don't know

18.03 Lecture 4

9/14

PS1 due ~~tomorrow~~ Friday @ noon

Typo Part 2 # 3 $\frac{dy}{dx} = \cancel{f(x)} = f(y)$

+ Change labeling on axis

Today: 2 important classes of 1st order ODEs

- autonomous - "independent" $\frac{dy}{dx} = f(y)$ ↑ ind of x $\frac{dy}{dt} = f(y)$ ↑ ind of time
- linear

$$\frac{dy}{dx} + P(x)\, y = Q(x)$$

Rearrange $\frac{dy}{dx} = -P(x)\,y + Q(x)$ ← linear in y

For example $\frac{dy}{dx} = -(\sin x)\, y + e^{2x^2}$

Not example $\frac{dy}{dx} = \cos(y)\, x$ ↑ ind. linear

Autonomous eq

$$\frac{dy}{dt} = f(y) \cdot 1$$

$$\int \frac{dy}{f(y)} = t + C$$

↓ Can always solve
- integral may be nasty

Tricks for eval qual.

Examples

Radio active decay
- exponential

"rate of decay is proportional to the amt present at any time t"

②

$$\frac{dP}{dt} = -kP \qquad k > 0$$

Example 2 Hollywood

Tank has S_0 salt disolved in 200 gallons water.
At $t=0$, start pumping water w/ $\frac{1}{2}$ lb salt / gallon
at a rate 4 gallons/min
Then we instantenously stir liquid
And water leaves tank at 4 gallons/min
How much salt (in lbs) in tank at time t.

Write the diff eq!
Prof: Hard. Break into pieces

$$\frac{dS}{dt}$$

$$\frac{1}{2}\,\frac{\text{lbs salt}}{\text{gallon}} \cdot 4\,\frac{\text{gal}}{\text{min}} = 2\,\text{lbs}\,\frac{\text{salt}}{\text{min}} \quad \text{entering}$$

but also leaving

$$= \cancel{2\text{lbs}/\text{min}} \; 4\,\text{gal}/\text{min} \cdot \underbrace{S(t)}_{\#\text{ lbs salt}} \cdot \underbrace{\frac{1}{200\text{ gal}}\,\text{lbs}}_{\text{concentration}}$$

(3)

$$\frac{dS}{dt} = \underset{\text{entering}}{\text{salt}} - \underset{\text{leaving}}{\text{salt}}$$

$$= \frac{2\,\text{lb}}{\text{min}} - \frac{4\,\text{gal}}{200} \cdot S(t)$$

$$\frac{dS}{dt} = 2 - \frac{4}{200} S(t)$$

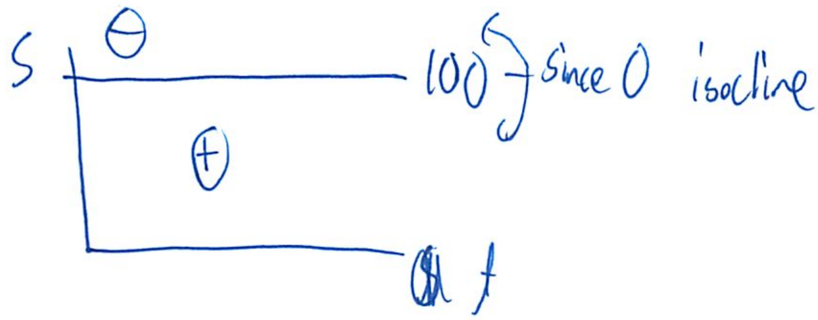
Now solve analytically

Could also look at qualitatively

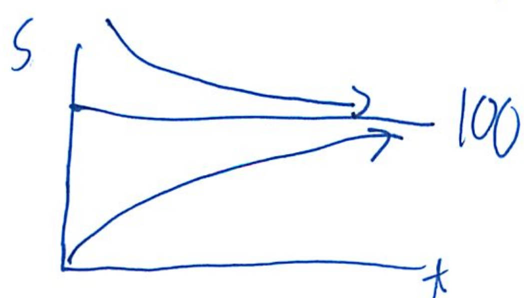
- draw 0 isocline

$$0 = 2 - \frac{S(t)}{50}$$

$$S(t) = 100$$



So assume will asy approach above



or be lazier

Phase line

(4)

Could do rigorous proof w/ fences + funnels

In general: the phase line has vertices which are roots of $f(y)$ in $\frac{dy}{dx} = f(y)$

Question 4

$$\frac{dy}{dx} = y^2 - 4y + 3$$

$$\int (-y^2 + 4)\,dy = \int 3\,dx$$

$$-\frac{y^3}{3} + 4y = 3x$$

"then what"

answer $f(y) = y^2 - 4y + 3$ oh factor ~~don't even do diffeq~~

$$(y-1)(y-3)$$

"didn't have to do integration"

$f(y) > 0$ if $y > 3$

$f(y) < 0$ if $3 > y > 1$

$f(y) > 0$ if $y < 1$

5

$$\frac{dy}{dx} = f(x)\ g(y)$$

$\uparrow$ $\uparrow$ integrate

auton

$$\frac{dy}{dx} = g(y) \qquad \frac{dy}{dy} = f(x)$$

$y=3$ "source"

otherwise "node"

$y=1$ "sink"

Important when measuring it source or sink

Parameters

- fun games we can play

Harvesting model - similar to logistical model

$$\frac{dP}{dt} = RP(M-P) - h$$

maximum; pos. const. "harvesting" constant

(6)

limited resources - like fish in a pond

$$\frac{dP}{dt} = P(4 - P) - h$$

hundreds of fish

Now vary h and see how affects long term behavior

Find roots of quadratic $P(4-P) - h$

Variables — Constant

$$P^2 - 4P + h = 0$$

roots $P = 2 \pm \sqrt{4-h}$

as change h, change real roots in quad. eqn

So when $h > 4$

- immaginary
- so no real roots
- no pts for phase line

When $h = 4$

- one real root

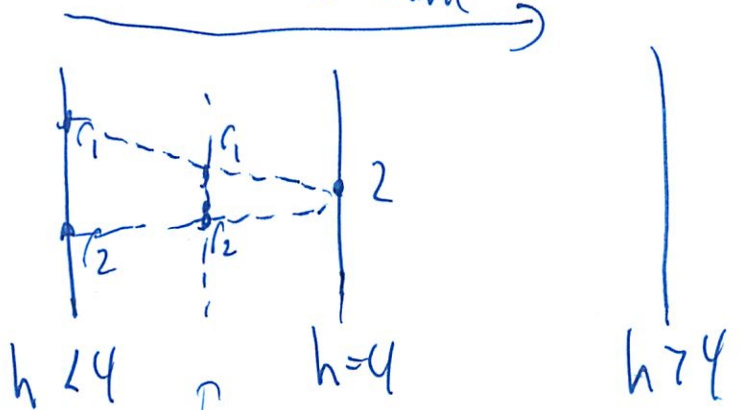
node

When $h < 4$

- two real roots

2

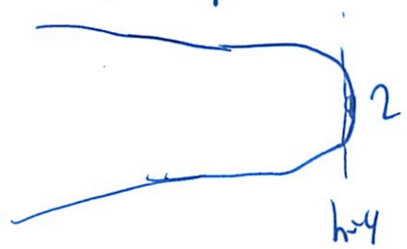
like a movie



bifurcation diagram

Then as get closer to 4, r_1 r_2 move together

like a ~~parapola~~ parabola



bifurcation pts - places on phase diagram where # of vertices on phase line increase / decrease

example $h=4$ bifurcation pt

So if fish warden don't set limit at bifurcation pt - too risky

8

Linear Equations

$$\frac{dy}{dx} + P(x) \cdot y = Q(x)$$

Only way to solve

Idea - ∫ both sides w/r to x

- but that is hard here

So slightly adjust

Better idea Multiply by $fn(x)$ on both sides

Then when ∫ w/r to x still fn x on other side.

The magic function. Multiply both sides by $e^{\int P(x)\,dx}$

LHS $\frac{dy}{dx} \cdot e^{\int P(x)\,dx} + P(x) e^{\int P(x)\,dx} \cdot y$

↑ is $\frac{d}{dx}$ of product of 2 fns

$$\frac{d}{dx}\left(y \cdot e^{\int P(x)\,dx}\right)$$

using prod rule

↓ int w/r to x

$$y \cdot e^{\int P(x)\,dx} + C$$

9

RHS $Q(x) \cdot e^{\int P(x)\, dx}$

int w/ r to x

$$\int Q(x)\, e^{\int P(x)\, dx}\, dx$$

must do inner $\int$ first!

Example $\dfrac{dy}{dx} - 2xy = x$

Identify P, Q

$P(x) = -2x$

$\int P(x)\, dx = -x^2$

any antideriv will do, so no +C

Now multiply by e^{-x^2}

$$\underbrace{\frac{dy}{dx} e^{-x^2} - 2x e^{-x^2} \cdot y}_{\frac{d}{dx}\left(y \cdot e^{-x^2}\right)} = x e^{-x^2}$$

$$y e^{-x^2} = \int x e^{-x^2}\, dx$$

do algebra

Lecture 4

9/14

Today: two important classes of ODEs: autonomous + linear

autonomous = "independent" as in independent of x:

$$\frac{dy}{dx} = f(y) \quad \text{or} \quad \frac{dP}{dt} = f(P)$$

linear = "linear in dep. variable y"

$$\frac{dy}{dx} + P(x)\,y = Q(x)$$ ← thinking $P(x)y - Q(x)$ as "linear" piece

Autonomous equations are special cases of separable eqns. Still may be difficult to solve:

$$\int \frac{dy}{f(y)} = x + C$$

But using qualitative analysis, find a few shortcuts.

Examples: radioactive decay $\frac{dN}{dt} = -kN$, $k > 0$. "rate of decay is proportional to the amount of substance present."

mixing problems: A tank contains S_0 lbs of salt dissolved in 200 gallons of water. At time $t = 0$, we start pumping water with $\frac{1}{2}$ lb salt / gallon at rate of 4 gal/min and then rapidly stirred mixture exits tank at same rate (4 gal/min)

How many lbs. of salt are in the tank at time t?

Ans: Use ODE because we can express $S'(t)$ easily.

$$S'(t) = \underset{\text{entering}}{\text{rate of salt}} - \underset{\text{leaving}}{\text{rate of salt}}$$

rate entering: $\frac{1}{2}$ lb/gal at 4 gal/min $=$ 2 lb/min.

rate leaving: 4 gal/min $\cdot$ $\Big($ $\underbrace{\text{lb/gallon in tank at time } t}_{S(t)/200}$ $\Big)$

Conclusion: $S'(t) = 2 - \dfrac{S(t)}{50}$

(autonomous)

solve by separation of variables $\rightsquigarrow$ $S(t) = S_0 e^{-t/50} + 100(1 - e^{-t/50})$

qualitative analysis: 0-isocline: $S(t) = 100$

if $S < 100$, then $S' > 0$

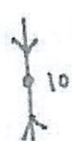
if $S > 100$ then $S' < 0$

How to illustrate solutions? Only need to draw 3: $S = 100$ + solns with $S(0) > 100$, $S(0) < 100$

Reason: solutions to autonomous equations are translation invariant. If $S(t)$ is sol'n, so is $S(t - t_0)$ for any t_0.

[One way to see this: slope field is constant along horizontal lines.]

Even easier: Phase line



(To argue solutions asymp. approach 100 requires a short argument with fences)

points on phase line: roots of $f(y)$ in $\dfrac{dy}{dx} = f(y)$

arrows: indicate whether $dy/dx > $ or < 0 on interval.

Slide question about phase line for logistic equation.

Variation of parameters. Example: Harvesting (logistical − const.)

$$\frac{dP}{dt} = \underbrace{kP(M-P)}_{\text{logistical}} - \underbrace{h}_{\text{constant harvesting rate}}$$

e.g. fish in pond.

$$\frac{dP}{dt} = P(4-P) - h \qquad 4: \text{max. pop. of fish (in hundreds)}$$

What are roots of this eqration? $-(P^2-4P+h)=0$

$\rightarrow (2 \pm \sqrt{4-h}) = P$

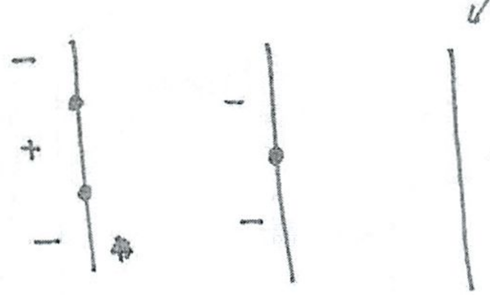
2 real roots if $h<4$

1 real root if $h=4$

0 real roots if $h>4$

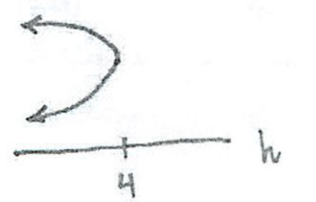
no points on phase line.

phase lines:



(Much cooler pictures later with systems of equations...)

bifurcation diagram (bifurcation point: value of parameter for which phase line changes behavior — inc. or dec # of roots)



NOTE: Not a solution curve! Just a curve that gives info about model as we vary parameter.

Why not set fishing limit to 400 fish per year? Too sensitive to small amount of overfishing.

linear equations: $\frac{dy}{dx} + P(x)y = Q(x)$ (Read EP §1.5)

if $Q(x)=0$ ("homogeneous" case), then separable:

$$\ln|y| = -\int P(x)\,dx + c'$$

Exponentiating: $|y(x)| = c\cdot e^{-\int P(x)dx}$

$$|y(x)\,e^{\int P(x)dx}| = c \Rightarrow y(x) = c\cdot e^{-\int P(x)dx}$$

In general linear equation, make $\frac{dy}{dx} + P(x)y$ look like $\frac{d}{dx}$(something)

then integrate both sides w.r.t. x. Have flexibility to mult. both sides by function in x

Idea: multiply both sides by $e^{\int P(x)dx}$

Why this factor?

Want $M(x)$ s.t.

$\frac{dy}{dx}M(x) + P(x)M(x)y$

$= \frac{d}{dx}(M(x)y)$ homog!

$\Rightarrow \frac{dM}{dx} - P(x)M = 0$

LHS: $\underbrace{\frac{dy}{dx}\cdot e^{\int P(x)dx} + P(x)\cdot e^{\int P(x)dx}y}_{\text{|| Product rule}}$

$$\frac{d}{dx}\left(e^{\int P(x)dx}\cdot y\right)$$

RHS: Just $e^{\int P(x)dx}Q(x)$. Looks messy, but at least it is a function of x alone which can be integrated.

Result: $y = e^{-\int P(x)dx}\cdot\int e^{\int P(x)dx}Q(x)\,dx$ (Don't memorize this formula)

Ex: $\frac{dy}{dx} - 2xy = x$ $\quad e^{\int 2x\,dx} = e^{-x^2}$ ...

Math 18.03 : Differential Equations

Lecture 4 Supplemental Notes

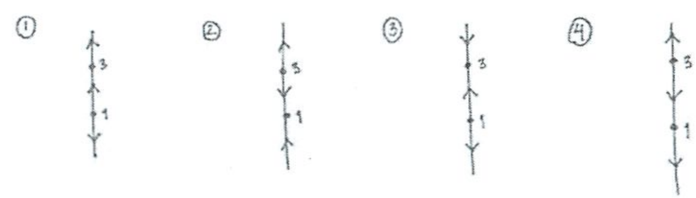
Wednesday, September 14, 2011

Quick Quiz 4

Given the differential equation

$$dy/dx = y^2 - 4y + 3$$

which of the following is a picture of the phase line for this autonomous equation?

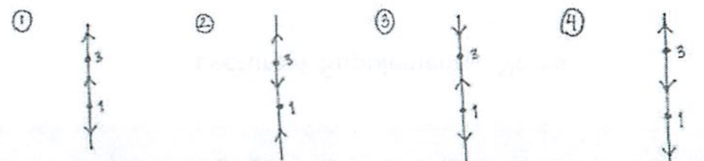


Quick Answer

Given the differential equation

$$dy/dx = y^2 - 4y + 3$$

which of the following is a picture of the phase line?



2! – $f(y) = y^2 - 4y + 3 = (y-1)(y-3)$ which satisfies:

$$f(y) > 0 \text{ if } y > 3, \quad f(y) < 0 \text{ if } 3 > y > 1, \quad f(y) > 0 \text{ if } y < 1$$

Practice 1.4 ex 1

$$\frac{dy}{dx} = -6xy \qquad y(0) = 7$$

Seperate variables

y on left (see the dy)

x on right ← like multiplying both sides by dx

$$\frac{1}{y}\frac{dy}{dx} = -6x$$

$$\int \frac{1}{y}\,dy = \int -6x\,dx$$

$$\ln|y| = -6\frac{x^2}{2} + C$$

$$= -3x^2$$

$$|y| = Ce^{-3x^2}$$

e^C

Now plug in IV ← never tried

$$7 = Ce^{-3\cdot 0^2}$$

well e^0 = WA says 1

2

Co

$C = 7$

Now answer q ✓

- just solve

$$|y| = 7e^{-3x^2}$$

Can see from inital condition that at $x=0$ its $+$ so can get rid of abs value symbol

But other than that correct :)

- getting the basics down
  - this is L1 only though :)

1.4 Ex 2

9/24

$$\frac{dy}{dx} = \frac{4-2x}{3y^2-5}$$

So y on left

$$\int (3y^2-5)\,dy = \int (4-2x)\,dx$$

: Can seperate
-well implicit

~~$\int 3y^2-5\,dx$~~

(know real math reasons - don't memorize short term shortcuts)

$$\frac{3y^3}{3} - 5y = 4x - \frac{2x^2}{2}$$

$$y^3 - 5y_{+C} = 4x - x^2 + C$$

: this is not 1st order ::

Thats all you can go

implicit solution (I should make vocab def sheet)

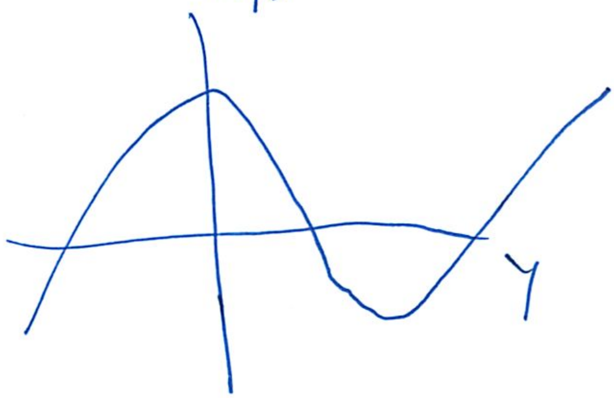
18.03 Recitation

9/15

(Do I have any qu
 - I think I've been working my way through P-set nicley so far)

Conceptual qu (about 50% people did not get)

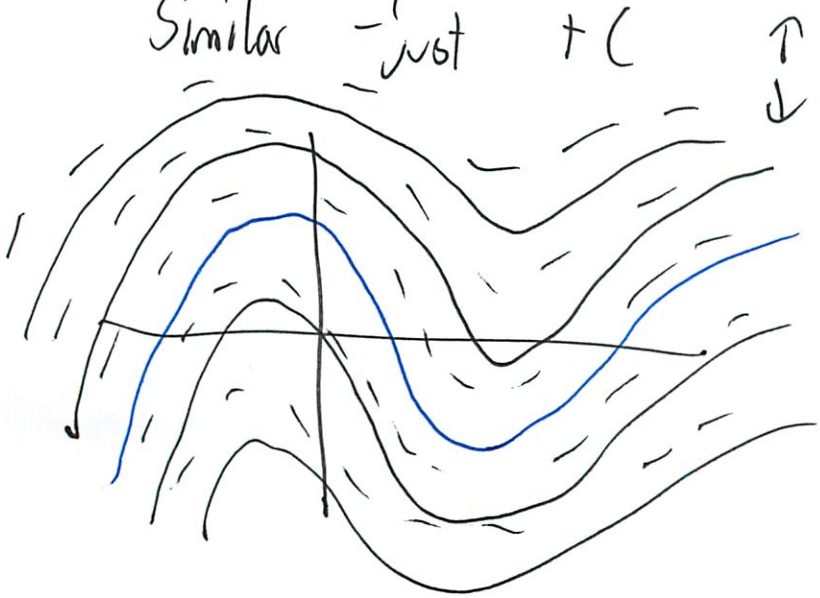
$\frac{dy}{dx} = f(y)$



Slope field
phase diagram
Sol curve
long term behavior

? Wait how can $\frac{dy}{dx} = f(y)$
That means some funct - from picture
Which was drawn

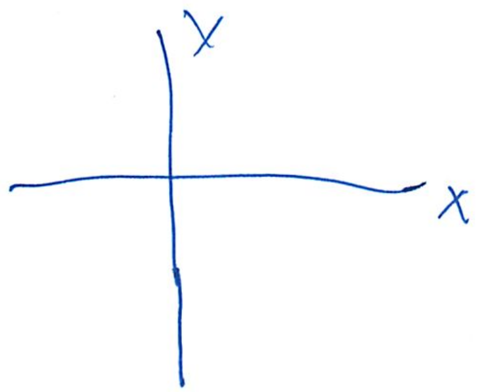
But how can we add extra slope field
Similar - just + C



long term behavior
to $+\infty$?
as y → ∞

(2)

Need some fn $y(x)$ so $\frac{dy}{dx}$ ~~dt~~ is that function of y



eval $f(y)$ at each point - That is your slope field

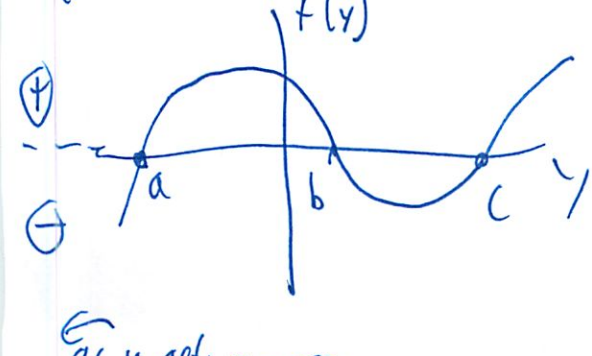
; so

$$\int dy = \int f(y)$$

$$\ln|y| = \cancel{e}\, f(y)$$

$$\cancel{e}\ y = e^{fy}$$

Have 3 points where $f(y)$ crosses y axis



← This is function of derivitive the slope - how it is (like in HS we drew this)

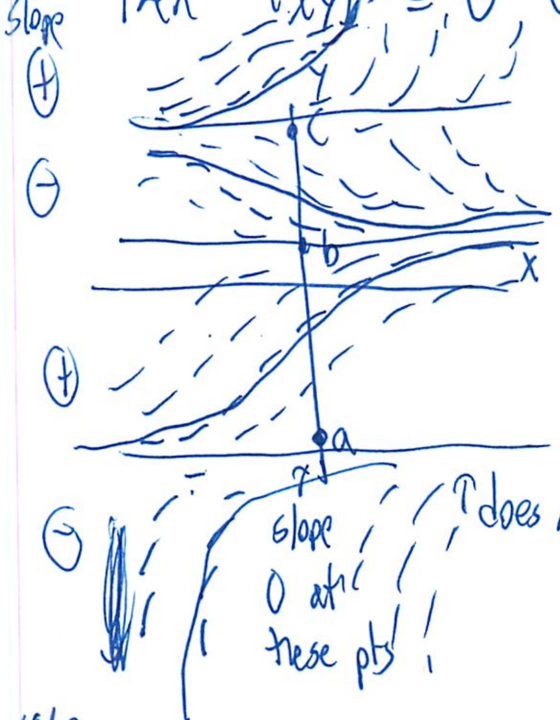
as y gets more ⊖
slope gets steeper ⊖

(3)

Looking for x, y

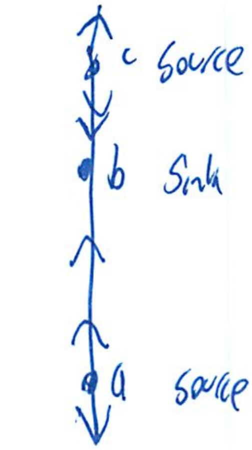
at y = a, b, c

Then f(x,y) = 0, 0, 0



does not matter what x is

Phase line



Then can draw in sol curves

Runge Kutta Method

Section 6.2 in book

TA does not know

Can we only draw phase diagram if f(x,y)

- not really since its now 2D.

(4)

First Order Linear Eq

$$\frac{dy}{dx} = y\,P(x) + Q(x)$$

eg $\frac{dy}{dx} = y + x$

Need to use integrating factor

$$\frac{dy}{dx} - y = x$$

$$IF = e^{\int -1\,dx} = e^{-x}$$

Or $P(x) = -1$ $\quad Q(x) = x$

in form $\frac{dy}{dx} + p(x)\cdot y = Q(x)$

to get into this form set P, Q to those values

~~Then integrate by parts~~

Multiply both sides by IF

$$\int e^{x}(y' - y) = \int x\,e^{-x}$$

Need product rule $\frac{d(A\cdot B)}{dx} = A\frac{dB}{dx} + B\frac{dA}{dx}$

(5)

Then integrate by parts

$u = x$

$du = dx$

$dv = e^{-x}$ $\quad v = e^{-x}$

etc

Finish integrating

Word problem people going into stadium

100 people arrive every hr

For every pair of people in statdium

Some prob p that 1 kills the other

So adding $\frac{dP}{dt} = \frac{100}{\text{hour}}$

killing $-(P/2 \cdot \text{Prob})$

did I mess up prob?

No all the possibility of pairs

6

So $\binom{P}{2}k$

Need to write out $\binom{P}{2}$ b/c can't integrate

$$\frac{P(P-1)}{2}$$

So

$$\frac{dP}{dt} = 100 - \frac{k\,P(P-1)}{2}$$

So what is the long term behavior

- Does not depend on t, can have a phase line
- Can solve for P - like beginning question
- Integrate - ugly

Find 0-isoclines

$$100 - \frac{kP^2}{2} + \frac{kP}{2} = 0$$

Solve w/ quadratic eq

(7)

$p^2 - p - \frac{200}{k} = 0$

Trying to find isocline not sol curve

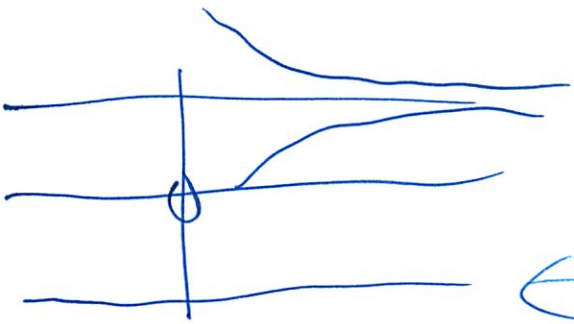
$\frac{dP}{dt} = 0$

Solve for $P(t)$

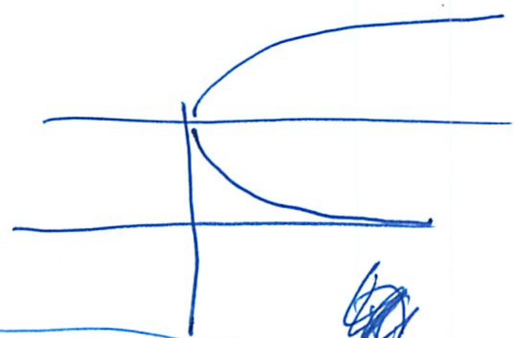
$P = \frac{1 \pm \sqrt{1 + \frac{800}{k}}}{2}$

2 isoclines for population - but note $P > 0$

Note $k \leq 1$



or



Need to make calculation

Say what happens if P is really big

- # will be huge + positive so

Isocline - Place where slope field is constant

18.03 FALL 2011 – Problem Set 1

Due Friday 9/16/11, high noon in 2-114

Part I consists of readings and exercises taken from Edwards and Penney and the Supplementary Notes (which are solved in the hand-written solutions at the end of the Notes). Of course, you should attempt to solve problems without referring to solutions in advance. These problems will be graded without many comments.

Part II consists of problems for which solutions are not given; it is worth more points. Some of these problems are longer multi-part exercises posed here because they do not fit conveniently into an exam or short-answer format. See the guidelines below (and also on the website) for which types of collaboration are acceptable, and follow them.

To encourage you to keep up with homework as it appears in lecture, both Part I and Part II problems are listed with the accompanying lecture in which the material will be covered.

Part I (20 points)

Lecture 1. Wed., Sept. 7: Separation of variables, direction fields.
Read: EP 1.1–1.4, Notes D, G HW: EP 1.4: 3, 9 (explaining the values taken by the undetermined constant in each case), 27; 1A-5c, 1C-1be

Lecture 2. Fri. Sept. 9: Numerical approximation
Read: EP 6.1–6.2, Notes G HW: 1C-3, 1C-6

Lecture 3. Mon. Sept. 12: Autonomous equations and the phase line.
Read: EP 1.7, 7.1 HW: EP 7.1: 9, 11, 23

Lecture 4. Wed. Sept. 14: Integrating factors.
Read: EP 1.5 HW: EP 1.5: 3, 16, 33

Lecture 5. Fri. Sept. 16 Introduction to complex numbers.
Read: 3.5, Notes C.1–3 HW: To be given on Problem Set 2.

Part II (29 points)

Directions and Rules: Collaboration on problem sets is strongly encouraged, **but**

a) Attempt each part of each problem yourself. Read each portion of the problem before asking for help. If you don't understand what is being asked, ask for help interpreting the problem and then make an honest attempt to solve it.

b) Write up each problem independently. On both Part I and II exercises you are expected to write the answer in your own words.

c) **Write on your problem set whom you consulted and the sources you used.** If you fail to do so, you may be charged with plagiarism and subject to serious penalties.

d) Do not consult materials from previous semesters.

0. (3 points) Write the names of all the people you consulted or with whom you collaborated and the resources you used, or say "none" or "no consultation". This includes visits outside recitation to your recitation instructor. If you don't know a name, you must nevertheless identify the person, as in, "tutor in Room 2-106," or "the student next to me in recitation." Optional: note which of these people or resources, if any, were particularly helpful to you.

This "Problem 0" will be assigned with every problem set. Its purpose is to make sure that you acknowledge (to yourself as well as others) what kind of help you require and to encourage you to pay attention to how you learn best (with a tutor, in a group, alone). It will also greatly help us to know what resources you find useful.

1. (Wed, 8 pts) This question concerns the differential equation

$$\frac{dy}{dx} = y^2 - x^2.$$

You may use the "Isoclines" mathlet (linked from the course homepage) for this exercise.

a) Draw the slope field for this differential equation and the 0, 2, and -2 isoclines.

b) Determine which pieces of the 0-isocline are upper fences and lower fences.

c) Suppose that the point (a, b) is a maximum of a solution curve $y(x)$. What can you say about the relationship between a and b? Explain.

d) Choose two nearby initial conditions for $y(0)$ whose corresponding solutions have radically different long-term behavior. (That is, pick points on the y-axis less than one unit apart whose solution curves have very different limiting behavior as $x \to \infty$.) Explain the reason for your answer using fences and funnels.

2. (Fri, 6 pts) Euler's method tends to give better and better approximations as we decrease the step size.

a) The initial value problem $dy/dx = y, y(0) = 1$ has exact solution $y = e^x$. Use Euler's method with step size $1/4$ and starting point $(0, 1)$ to obtain a rational approximation to e.

b) Use your method from part (a) to give an approximation to e using initial point $(0, 1)$ and step size $1/n$ for any integer n. Prove that as $n \to \infty$, the approximation by Euler's method converges to e.

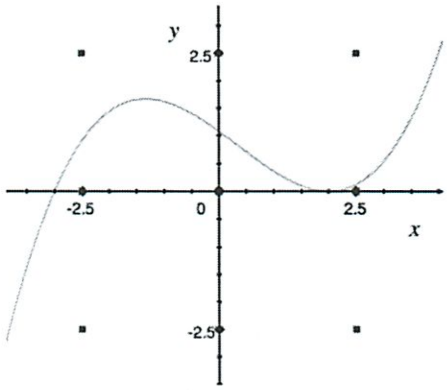
c) Euler's method with large step size can produce approximate solutions whose long term behavior is very different from the actual solution. Using the initial value problem

$$\frac{dy}{dt} = y^2 - t, \quad y(0) = .7,$$

show that Euler's method with step size 1 leads to inaccurate long-term behavior. (Explain your answer in terms of fences and funnels.)

3. (Mon, 6 pts)

a) Sketch the phase line (labeling nodes as sinks, sources, or nodes) for the differential equation $dy/dx = f(x)$ where $f(x)$ is the cubic equation pictured below. What changes in the phase line if the graph is shifted one unit downward?



b) By now, we are familiar with the logistical model for population growth. If we harvest a constant number C from this population per time step, then the resulting equation is

$$\frac{dP}{dt} = k\left(1 - \frac{P}{N}\right)P - C, \quad \text{where } k, C, N \text{ are constants.}$$

If we vary the parameter C on the right-hand side of the equation, for what value of C (in terms of k and N) does a bifurcation occur? What is the resulting long-term effect on the population for values of C larger than the bifurcation point? Can this effect be mitigated by later reducing the harvesting rate C to just below the bifurcation point? Why or why not?

4. (Wed, 6 pts) Our methods of differential equations allow us to solve (at least implicitly) all equations of the form

$$\frac{d}{dt}f(t,y) = 0. \tag{1}$$

Suppose that we are given a differential equation of the more common form

$$A(t,y) + B(t,y)\frac{dy}{dt} = 0.$$

It can be written in the form of equation (1) if and only if there is a function $f(t,y)$ with

$$A(t,y) = \partial f/\partial t \quad \text{and} \quad B(t,y) = \partial f/\partial y. \tag{2}$$

Prove that if A and B are functions with first partial derivatives in t and y that are continuous on a rectangle R, then there exists an $f(t,y)$ satisfying equation (2) if and only if

$$\partial A/\partial y = \partial B/\partial t$$

for all points (t,y) in R.

Michael Plasmeier

18.03 P-Set 1

9/16

Part 1

-15 = $\frac{34}{49}$

EP 1.4 #3

$$\frac{dy}{dx} = y \sin x$$

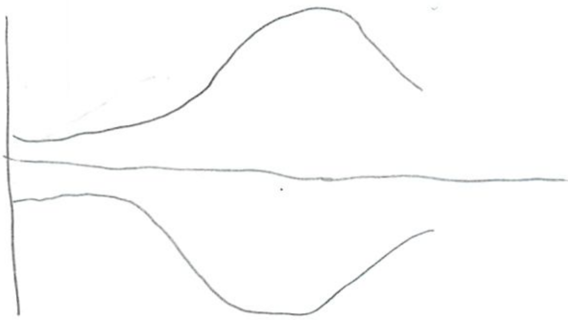
$$\frac{1}{y}\frac{dy}{dx} = \sin x$$

$$\int \frac{1}{y}\, dy = \int \sin x \, dx$$

$$\ln|y| = -\cos x + C$$

$$|y| = Ce^{-\cos x}$$

For y being + or −
look at $x=0$ $Ce^{-\cos 0}$
y will be 0
So its both + or −



Doesn't need to find c.

So what is C
- need initial value

②

9. $(1-x^2)\frac{dy}{dx} = 2y$

$$\frac{1}{2y}\frac{dy}{dx} = \frac{1}{1-x^2}$$

$$\int \frac{1}{2y}\,dy = \int \frac{1}{1-x^2}\,dx$$

$$\frac{\ln y}{2} = \frac{1}{2}\ln(x+1) - \ln(1-x) + c$$

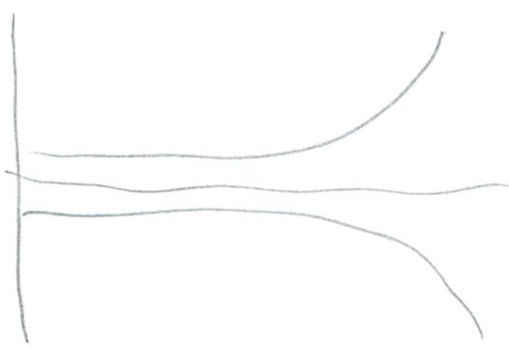
$$\ln y = \ln(x+1) - \ln(1-x) + c$$

$$\boxed{|y| = C\left(\frac{x+1}{1-x}\right)}$$ ✓

exponential rules

$$\boxed{x^{m-n} = \frac{x^m}{x^n}}$$

At $x=0$, $y=0$



(3)

27. $\frac{dy}{dx} = 6e^{2x-y}$ $\quad y(0)=0$

$\frac{dy}{dx} = 6\frac{e^{2x}}{e^{y}}$

$e^{y}\frac{dy}{dx} = 6e^{2x}$

$\int e^{y}\,dy = \int 6e^{2x}\,dx$

$e^{y} = \underline{6e^{2x} + C}$

ln ln ln

$y = (\ln 6)\cdot 2x + c$

x −1

Now IV

$0 = y(0) = \ln 6 \cdot 2(0) + C$

$0 = \ln 6 + c$

$C = -\ln 6$

So

$y(x) = 2x\ln 6 - \ln 6$

(4)

1A-5c $y' = \left(\frac{y-1}{x+1}\right)^2$

$$\frac{dy}{dx} = \left(\frac{y-1}{x+1}\right)^2$$

$$\frac{dy}{dx} = \frac{(y-1)^2}{(x+1)^2}$$

$$\frac{1}{(y-1)^2}\frac{dy}{dx} = \frac{1}{(x+1)^2}$$

$$\int \frac{1}{(y-1)^2}\,dy = \int \frac{1}{(x+1)^2}\,dx$$

$$\frac{1}{1-y} = -\frac{1}{x+1} + C$$

$$\wedge^{-1} \qquad \wedge^{-1}$$

$$1-y = -(x+1) + \frac{1}{C}$$

−.5

$$y = x+2+\frac{1}{C}$$; can simplify like this?

$$y = x+c$$

(5)

1C-1b Draw + verify analytically

$$\frac{dy}{dx} = 2x + y$$

$2x+y$ (y \ x)	-2	-1	0	1	2
-2	-6	-4	-2	0	2
-1	-5	-3	-1	1	3
0	-4	-2	0	1	4
1	-3	-1	1	3	5
2					
y	-2	0	2	4	6

flip in normal dimensions

ok

$$-y\frac{dy}{dx} = 2x$$

$$\int -y\,dy = 2x\,dx$$

$$-\frac{y^2}{2} = \frac{2x^2}{2} + C$$

(6)

$$-\frac{y^2}{2} = x^2 + c$$

$$-y^2 = 2x^2 + c$$

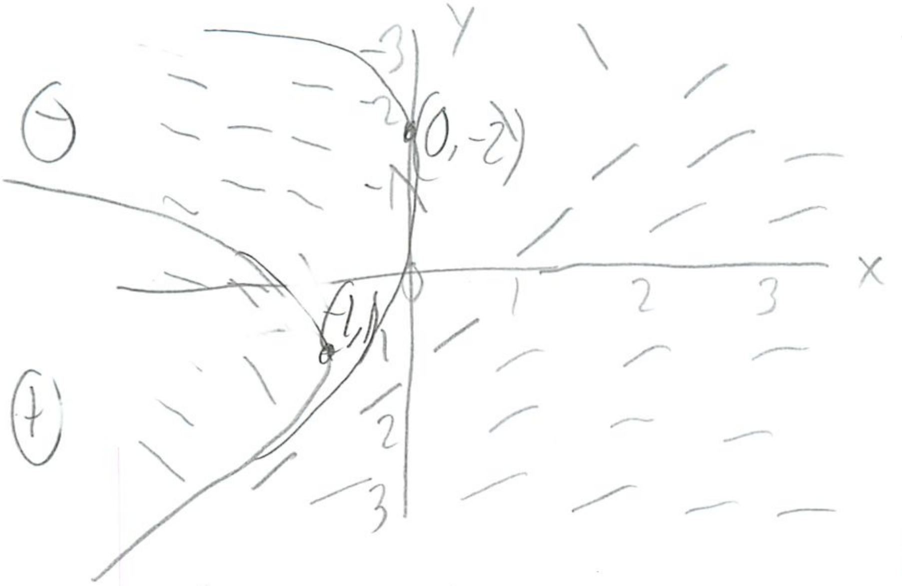
$$y^2 = -2x^2 + c$$

$$y = \sqrt{-2x^2} + c$$

1C-1e $\frac{dy}{dx} = \frac{1}{x+y}$

$\frac{1}{x+y}$	-3	-2	-1	0	1	2	3 x
-3	$-\frac{1}{6}$	$-\frac{1}{5}$	$-\frac{1}{4}$	$-\frac{1}{3}$	$-\frac{1}{2}$	-1	und
-2	$-\frac{1}{5}$	$-\frac{1}{4}$	$-\frac{1}{3}$	$-\frac{1}{2}$	-1	und	1
-1	$-\frac{1}{4}$	$-\frac{1}{3}$	$-\frac{1}{2}$	-1	und	1	$\frac{1}{2}$
0	$-\frac{1}{3}$	$-\frac{1}{2}$	-1	und	1	$\frac{1}{2}$	$\frac{1}{3}$
1	$-\frac{1}{2}$	-1	und	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$
2	-1	und.	1	$1/2$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$
3	Und.	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$
y							

17



flip in normal viewing

explanation?

−1

0, 0 undefined

1C-3 $y(x)$ is sol to IVP $y' = x - y$ $y(0) = 1$

a) Use Euler's method $h = .1$ step size

Find approx vale of $y(x)$ for $x = .1, .2, .3$

x	y	
0	1	← given
.1	.9	$y_0 + \Delta x f(x_0, y_0)$
.2	.82	$1 + .1(0-1)$
.3	.758 ✓	$.9 + .1(.1 - .9)$
		$.82 + .1(.2 - .82)$

Can check analytically

$$y \frac{dy}{dx} = x$$

$$\int y\,dy = \int x\,dx$$

(8)

$$\cancel{\frac{y^2}{2} = \frac{x^2}{2} + C}$$

$$\cancel{y^2 = x^2 + C}$$

$$\cancel{y = x + C}$$

Can't do sep variables

WA $y(x) = ce^{-x} + x - 1$

- What did I do wrong?

$$\cancel{1 = 0 + C}$$

$$\cancel{C = 1}$$

$$\cancel{y = .3 + 1}$$

$$\cancel{= 1.3}$$

see solution -1

wrong - I did something wrong

b) Modified Euler

- So this is make step smaller + carry more digits
- Oh no it is using mid points

So go to .2 then find half

$$1 + \frac{1 - .82}{2} = .89$$

x -1

mine was above that

(9)

11-6 $\frac{dy}{dx} = f(x)$ $y(0)$ y_0

We want $y(2nh)$

$\uparrow$ step size

n steps Runge-kutta

Exact value $= y_0 + \int_0^{2nh} f(x)\, dx$

Show this is same as Simpson Rule

(we didn't really learn this)

Prove w/ 1 step of each method

w/ Rungekutta method find one pt in each + take avg

$$y_{2h} = y_0 + 2h\left(\frac{① + 2② + 2③ + ④}{6}\right)$$

So

$$\left.\begin{aligned} ① &= f(0) \\ ② &= f(h) \\ ③ &= f(h) \\ ④ &= f(2h) \end{aligned}\right) \quad y' = f(x)$$

Then $y_{2h} = y_0 + \frac{2h}{6}\left(f(0) + 4(f(h)) + f(2h)\right)$ ✓

(10)

Then Simpson's Rule

$$y_{2h} = y_0 + \int_0^{2h} f(x)\,dx$$

$$= y_0 + \frac{2h}{6}\left(f(0) + 4f(h) + f(2h)\right)$$

Same!

✓

Lecture 3

Solve to find crit pts

Stable or unstable?

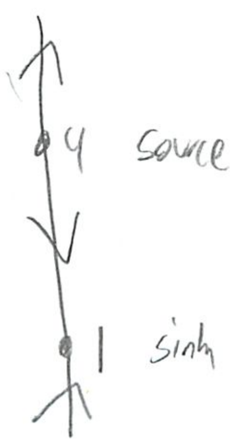
EP 7.1 #9

$$\frac{dx}{dt} = x^2 - 5x + 4$$

Find roots to find isoclines right?

$$(x-4)(x-1)$$

$$x = 1, 4$$



So test signs $f(5) = 5^2 - 5(5) + 4$ (+)

$f(3) = 3^2 - 5(3) + 4$ (−)

$f(0) = 0^2 - 5(0) + 4$ (+)

(11)

Solve explicity

$$\frac{d\cancel{x}}{dt} = x^2 - 5x + 4$$

$$\int \frac{dx}{x^2-5x+4} = \int dt$$

$$\int \frac{1}{x^2-5x+4}\,dx$$

$$\int x^{-2} - 5x^{-1} + \frac{1}{4}\,dx$$

$$\int \frac{x^{-1}}{-1} - 5\ln(x) + \frac{x}{4} = t + C$$

$$-\frac{1}{x} - 5\ln(x) + \frac{x}{4} = t + C$$

$$x(t) = \frac{e^{3C+3t} - 4}{e^{3C+3t} - 1}$$

So $x(5) = \dfrac{e^{3C+3(5)} - 4}{e^{3C+3(5)} - 1}$

Need C → know $x(4) = 0$

$$0 = \frac{e^{3C+3(4)} - 4}{e^{3C+3(4)} - 1}$$

(12)

$C = \frac{2}{3}\left(\log(2) - 6\right)$

So $x(5) = .96$

$x(3) = 4.7 \leftarrow$ wrong?

$x(0) = 4.8$

graph? $-.5$

#11 $\frac{dx}{dt} = (x-1)^3$

Find roots

$0 = (x-1)^3$

$x = 1$

Test pts $x(2) = (2-1)^3 = \oplus$

$x(0) = (0-1)^3 = \ominus$

Solve explictly

$\int \frac{1}{(x-1)^3}\,dx = \int dt$

graph $-.5$

$-\frac{1}{2(x-1)^2} = t + C$

(13)

$\circ -2 \qquad \circ -2$

$$\frac{1}{(x-1)^2} = -2t + C$$

$\circ\ 1-1$

$$(x-1)^2 = -\frac{1}{2t} + C$$

$$x-1 = \sqrt{-\frac{1}{2t}} + C$$

$$x = \sqrt{-\frac{1}{2t}} + C$$

So

at $t=2$

$$\sqrt{-\frac{1}{2(2)}} \Rightarrow \text{imaginary}$$

$t=0$

$$\sqrt{\frac{-1}{2(0)}} \rightarrow \text{division error}$$

Clearly wrong

WA

$$x(t) = \frac{-\sqrt{2}\sqrt{-c-t} + 2c + 2t}{2(c+1)}$$

(14)

23. Logistic eq $\frac{dx}{dt} = kx(M-x)$

$x(t)$ = pop of fish in lake after t months

Now fish are removed at hx/fish month

a) If $0 < h < kM$ show pop is still logistic.

Well the pop will steady state

at some value that is a sink,

So what is this pt

$\frac{dx}{dt} = 0 = (mk-h)-kx$

$\Rightarrow x = M - h/k$ ← new limit

$0 = kx(M-x) - hx$

$= kxM - kx^2 - hx$

Then need to plug in values
above and below

X

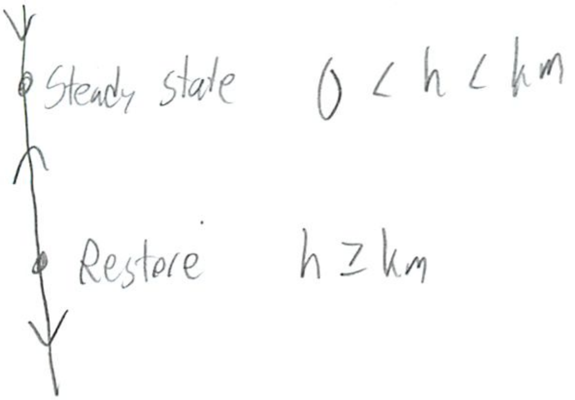
-2

$x(1) = +$

$x(kM-1) = -$

(15)

b) If $h \geq kM$ show that $x(t) \to 0$ as $t \to \infty$



? But how to show w/ variables?

Some how

$$kxM - kx^2 - hx = \ominus \text{ below }^{\text{some}} \text{ pt}$$

when $h > kM$

$$\frac{dx}{dt} = KMx - Kx^2$$

↓

analyze this term for $h \geq km$.

-1

16

Lecture 4 Integrating Factors

3 Find general solution

$$y' + 3y = 2x e^{-3x}$$

I think these are linear problems

General form

$$\frac{dy}{dx} + P(x)\, y = Q(x)$$

Integrating factor

$$\rho(x) = e^{\int P(x)\, dx}$$

So here

$$P(x) = 3$$
$$Q(x) = 2x e^{-3x}$$

derivitive of $y(x) e^{\int P(x)\, dx}$

$$y(x) e^{\int 3\, dx}$$

$$y(x) e^{3x}$$

Integrate right

17

$$\int \left(Q(x)\ e^{\int P(x)\,dx}\right) dx + C$$

$$\int \left(2x\ \cancel{e^{-3x}}\ \cancel{e^{3x}}\right) dx + C$$

$$\frac{x^4}{2e^6} + C$$

-.5

So

$$y e^{3x} = \frac{x^4}{2e^6} + C$$

$$y = \frac{x^4}{2e^6} \cdot \frac{1}{e^{3x}} + C$$

---

16. $y' = (1-y)\cos x \qquad y(\pi) = 2$

$$\frac{dy}{dx} = (1-y)\cos x$$

$$\frac{1}{1-y}\frac{dy}{dx} = \cos x$$

$$\int \frac{1}{1-y}\,dy = \int \cos x\,dx$$

$$-\log(1-y) = \sin x + C$$

$$e^{?} \qquad e^{?}$$

(18)

$\frac{1}{1-y} = Ce^{\sin x}$

$n-1$

$1-y = Ce^{-\sin x}$

$y = -Ce^{-\sin x}$?

Now inital

$2 = -Ce^{-\sin \pi}$

$C = -2$ -.5

So $y = -2e^{-\sin x}$

---

33. A tank has 1800 L of sol w/ 180 kg salt
Pure water pumped in 5 L/s
Stiring continously
Mixture at 5 L/s
How long till 18 kg salt?
(Did we do this in lecture?)
So first what is unit ; $\frac{ds}{dt}$

(19)

Entering $\frac{5 \text{ gallons}}{\text{sec}}$ with $\frac{0 \text{ lbs}}{L}$

Leaving $\frac{5 \text{ gallons}}{\text{sec}} \cdot \frac{S(t) \text{ lbs}}{1000 L}$

$$\frac{dS}{dt} = \text{Entering} - \text{Leaving}$$

$$= 0 - \frac{5S(t)}{1000}$$

we don't care - no add salt

$$\frac{dS}{dt} = -\frac{S(t)}{200}$$

Look at isocline

$$0 = -\frac{S(t)}{200}$$

$$0 = S(t)$$

So concentration will go to 0

But when? Solve analytically

$$-\frac{200}{S(t)} \frac{dS}{dt} = 1$$

(28)

$$\int -\frac{200}{s(t)}\, ds = \int 1\, dt$$

$$-200 \ln(s(t)) = t$$

$$\ln(s(t)) = -\frac{t}{200} + C$$

$e^{\wedge}$ $e^{\wedge}$

$$s(t) = Ce^{-t/200}$$

Do inital condion ← don't forget

$$1000 = C e^{-0/200}$$

$C = 100\cancel{0}$

Now when is 10 kg?

$$10 = 100\cancel{0}\, e^{-t/200}$$

Solve for t

$$t = 400 (\ln(2) + \ln(5))$$

$= 921.034$ sec 460. sec.

$-.5$

(I actually like word problems more)

(21)

Part 2

0. TBD ?

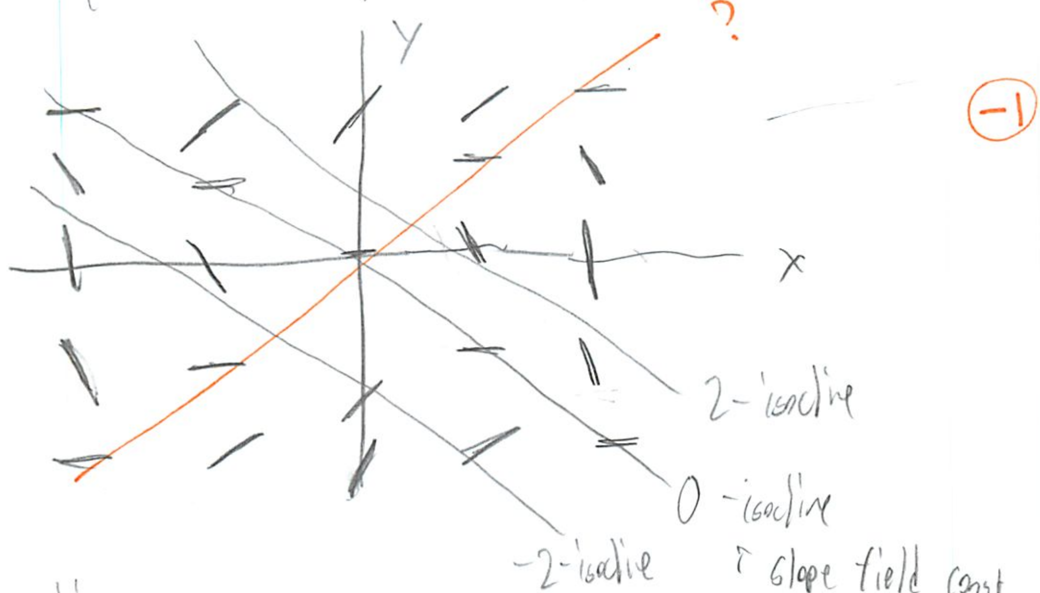
1. $\frac{dy}{dx} = y^2 - x^2$

a) Draw slope fields

y^2-x^2	-2	-1	0	1	2	x
2	0	3	4	3	0	
1	-3	0	1	0	-3	
0	-4	-1	0	-1	-4	
-1	-3	0	-1	0	-3	
-2	0	3	4	3	0	
y						

22

(So far not challenge problem)



0-isocline

$0 = y^2 - x^2$

$y^2 = x^2$

$y = \sqrt{x^2} = \pm x$

2-isocline

$2 = y^2 - x^2$

$y^2 = 2 - x^2$

$y = \sqrt{2} - x$

-2-isocline

$-2 = y^2 - x^2$

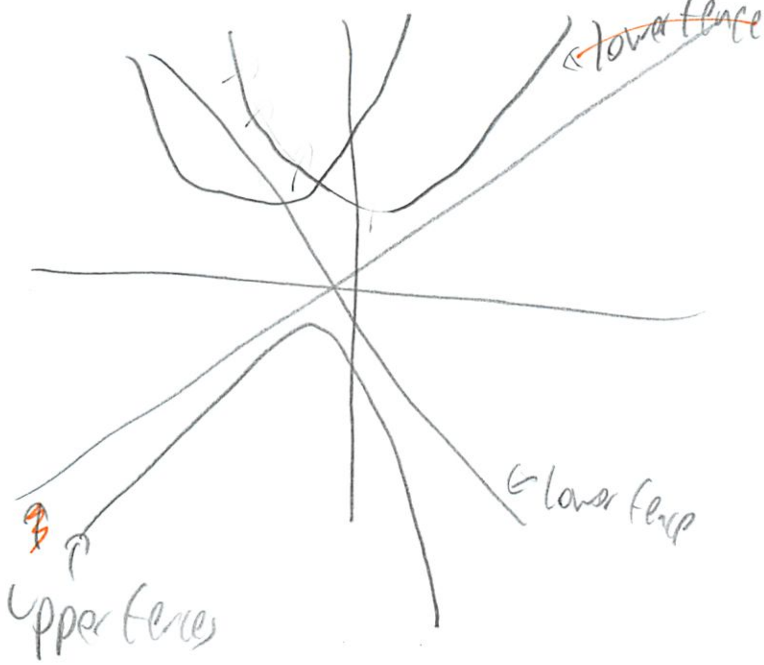
$y^2 = -x^2 - 2$

$y = -x - \sqrt{2}$

(23)

b) Determine which pieces of 0-isocline are upper/lower fences

So 0-isoclines include



use def. of up/low fences.

✓

c) Suppose (a,b) is max of sol curve $y(x)$

So one for example $x = .5$

$y = -.5$

$.5^2 - .5^2$

So $a = \pm b$

and will $= 0$

$-.5$

(24)

d) Choose two nearby inital conditions for $y(0)$ whose sol have very diff long term sols

So $y = 0.02$ goes to $-\infty$

but $y = .20$ goes to $+\infty$

This is because of the fences + funnels in place.

ok

25

2. Euler's Method

$$\frac{dy}{dx} = y \qquad y(0) = 1$$

has $y = e^x$

Now do Euler w/ step size $\frac{1}{4}$

X	Y	
0	1	$y_0 + h \cdot f(x_0, y_0)$
.25	1.25	← $1 + .25 \cdot 1$
.5	1.5625	← $1.25 + .25 \cdot 1.25$
.75	1.953125	
1	2.4414 ✓	

b) Now use smaller n

↳ formulaic

$$y_1 = y_0 + h \cdot y_0$$

but repeat w/ diff ys

$$y_2 = y_1 + h\, y_1$$

$$= (y_0 + h\, y_0) + h(y_0 + h\, y_1)$$

etc n times

(2b) Then this in lim approaches e (≈2.718)
? Not sure how I would represent mathmatically

show that $n \to \infty$, $y_n \to$ Taylor rep. of e. -2

c) Euler's method w/ large step sizes can be really bad

$$\frac{dy}{dt} = y^2 - t \qquad y(0) = .7$$

So using step 1

t	y
0	.7
1	.19 ← $.7 + 1 \cdot (.7^2 - 1)$
2	−1.77 ← $.19 + 1 \cdot (.19^2 - 2)$
3	−1.63
4	−2.97
5	.85

(27)

Now analytically!

$$\frac{dy}{dt} = y^2 - t$$

$$\int \frac{1}{y^2}\, dy = \int -t \, dt$$

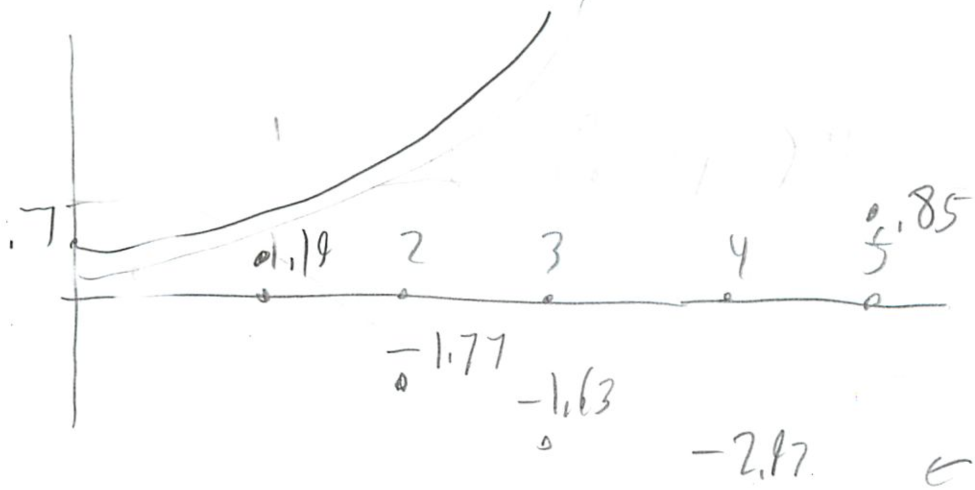
$$-\frac{1}{y} = -\frac{t^2}{2} + C$$

$$y = \frac{2}{t^2} + C$$

$$7 = \frac{2}{0^2} + C \quad \text{und.}$$

Actually can't

but WA can graph



$y(0) = 7$

← total wrong dir

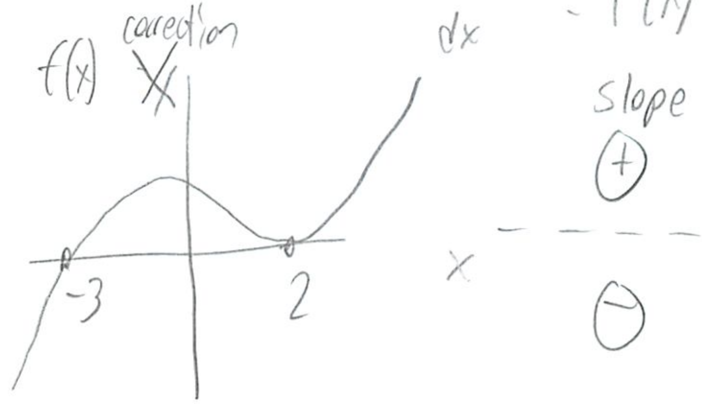
28

? Not sure how would argue w/ fences + funnels ?

Some how it jumped over the fense ✓ so it is now in a different funnel / anti-funnel

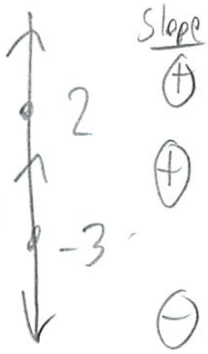
because first step is too big.

3. Sketch phase line for $\frac{dy}{dx} = f(x)$



(¿ Did we do this in recitation? There graph was f(x). Oh that was the typo. I fand it independly :))

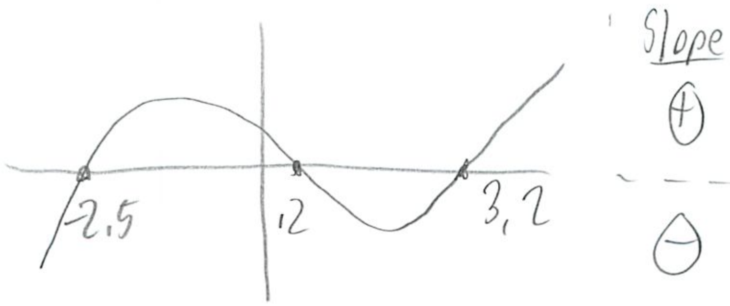
So the zeros are $-3, 2$



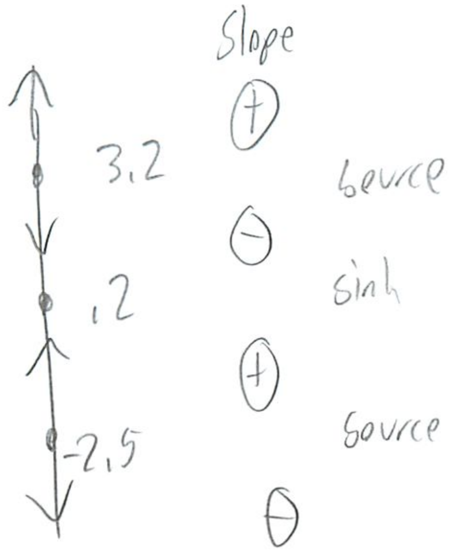
Try other pts
— well look at graph

So -3 source
2 is just node (since does not go under line)

What if shifted 1 unit down



(3)



b) Logistic model for pop growth

harvest C from pop per time step

$$\frac{dP}{dt} = k\left(1 - \frac{P}{N}\right)P - C \qquad k, C, N \text{ constants}$$

If we vary C on RHS, for what value of C does bifurcation occur?

- ~~do~~ missed this earlier
- but is it 0 isocline

$$0 = k\left(1 - \frac{P}{N}\right)P - C$$

Solve for P

$$C = kP - \frac{kP^2}{N}$$

$$P = \frac{kn - \sqrt{k^2n^2 - 4Ckn}}{2k}$$

~~set P = 0.~~

set $K^2 - 4KC/N = 0$.

(31)

But how to say what values of C and

What is C in terms of k, N, P?

$$C = kP - \frac{kP^2}{N}$$

Plug in P

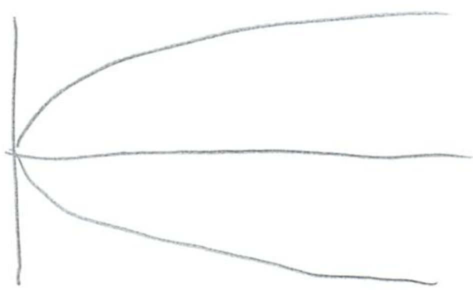
Then solve for C

(this seems messy)

But that gives us the point

(-3)

Can reduce C to below point, so pop grows



Bifurcation point = ?

37

#4 Our methods let us solve all eq (at least implicitly) of form $\frac{d}{dt} f(t,y) = 0$

If given

$$A(t,y) + B(t,y)\frac{dy}{dt} = 0$$

Write as

$$A(t,y) = \frac{\partial f}{\partial t} \quad B(t,y) = \frac{\partial f}{\partial y}$$

Prove that if A and B are fns w/ 1st partial derivs in t, y that are continuous on Rect R, then there exists a $f(t,y)$ satisfying eq if and only if $\frac{\partial A}{\partial y} = \frac{\partial B}{\partial t}$ for all pts (t,y) in R

If ~~(this)~~ $f(t,y)$ and its partials $\frac{\partial f}{\partial t}$ and $\frac{\partial f}{\partial y}$ exist and are continuous on a rectangle containing (a,b) than the function is continuous and $\frac{\partial A}{\partial y} = \frac{\partial B}{\partial t}$ ... and? You need to prove this! -3

Problem Set #1 answers.

Part I

The answers to this part can be found on the class notes. (including textbooks)

Part II

Problem #1. Equation $\frac{dy}{dx} = f(x,\, y) = y^2 - x^2$.

(a) See figure 1. The isoclines are given by: (A) $y = \pm x$ for $f = 0$, (B) $y = \pm\sqrt{2 + x^2}$ for $f = 2$, and (C) $x = \pm\sqrt{2 + y^2}$ for $f = -2$. (give equations)

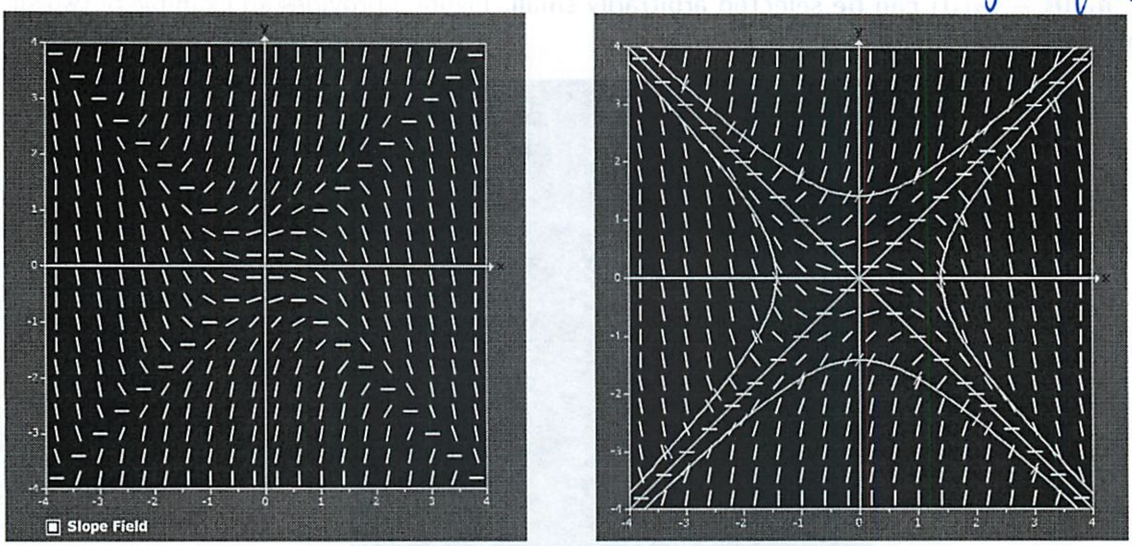


Figure 1: Left panel: slope field for the equation $\frac{dy}{dx} = f(x,\, y) = y^2 - x^2$. Right panel: same as the left panel, with the isoclines $f = 0$, $f = 2$, and $f = -2$ added.

(b) **Case $L = +x$.** Then $L' = +1 > 0 = f(x,\, L) = L^2 - x^2$. Hence **upper fence.**
Case $L = -x$. Then $L' = -1 < 0 = f(x,\, L) = L^2 - x^2$. Hence **lower fence.**

don't just look at symbols + ignore meaning/why

(c) $b = y(a)$ is a (local) maximum of the solution curve $y = y(x)$ if and only if $a = \pm b > 0$.

The proof of this is simple: At a (local) maximum of $y(x)$, $f(a,\, b) = y' = 0$. Hence there $a = \pm b$. Furthermore, using the equation to compute the second derivative at the maximum: $y'' = 2\, y\, y' - 2\, a = -2\, a$. Thus, it must be $a \geq 0$. Now, while $a > 0$ guarantees a (local) maximum, we need more information when $a = 0$. Going back to the equation, again, it is easy to see that $y''' = -2$ at $y = x = 0$, so this is an inflection point (not a maximum). Vice versa, it is easy to see that, if $a = \pm b > 0$, then $b = y(a)$ is a (local) maximum of the solution curve.

(d) Define $L = \sqrt{\delta + x^2}$ for $x \geq 0$, where $\delta > 1$ is a constant. Then **$L = L(x)$ is a lower fence,** since $L' = \frac{x}{L} < 1 < \delta = f(x,\, L)$.

From the answer to part (b), we see that the region **$0 \leq x < y < L(x)$ is an anti-funnel.** Since there $\frac{\partial f}{\partial y} \geq 0$, there is a unique solution $y = Y(x)$ that remains in the anti-funnel for all $x \geq 0$.

- Let $y = y_1(x)$ be the solution of the equation with initial data such that $Y(0) < y_1(0) < \sqrt{\delta}$. Then, for some $x = x_1 > 0$, $y = y_1(x)$ crosses the lower fence $y = L(x)$. Hence: **for $x > x_1$, $y_1(x) > L(x)$.** As a matter of fact, it can be shown that $y_1(x) \to \infty$ as $x \to x_*$, for some $x_* > x_1$ (but we do not need this here).
- Let $y = y_2(x)$ be the solution of the equation with initial data such that $0 < y_2(0) < Y(0)$. Then, for some $x = x_2 > 0$, $y = y_2(x)$ crosses the upper fence $y = x$. Hence for $x > x_2$, $y_2(x)$ is trapped[1] in the region $-x < y < x$ — where $y' < 0$. Thus: **$y_2(x) \leq y_2(x_2) = x_2$ for $x \geq x_2$.** As a matter of fact, it can be shown that $y_2(x)$ asymptotes $y = -x$ as $x \to \infty$ (but we do not need this here).

Any two solutions $y = y_1(x)$ and $y = y_2(x)$ provide an answer to this part. Note that $0 < y_1(0) - y_2(0)$ can be selected arbitrarily small. Figure 2 provides an example of two such solutions.

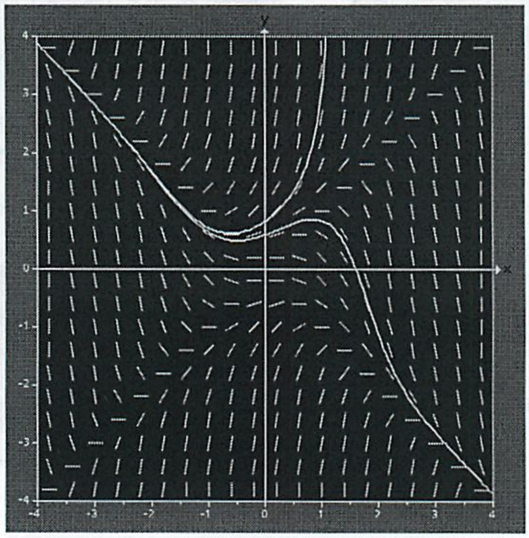


Figure 2: $\frac{dy}{dx} = f(x, y) = y^2 - x^2$. Two solutions with $0 < y_1(0) - y_2(0)$ "small", and radically different limiting behaviors as x grows — we cannot truly say: "as $x \to \infty$" because $y_1(x) \to \infty$ at some finite $x = x_*$, and it is not defined for $x \geq x_*$.

Problem #2. First we consider **Euler's method for $\frac{dy}{dx} = y$ with $y(0) = 1$.**

With step size $h = \Delta x$, $x_m = m\,h$ (where $m = 0, 1, 2, \ldots$) and y_m the approximation to $y(x_m)$, the method is

$$y_{m+1} = y_m + h\,y_m = (1 + h)\,y_m, \quad \text{with} \quad y_0 = 1. \tag{1}$$

Hence

$$y_m = (1 + h)^m. \tag{2}$$

(a) Since $y = e^x$, using $h = 1/4$ and $m = 4$, using (1 – 2) we obtain $e \approx \mathbf{\frac{625}{256} = 2.44\ldots}$

[1] Since $y = -x$ is a lower fence, see part (b).

(b) Take $h = 1/n$ and $m = n$ (where $n > 0$ is a natural number). We expect $y_m \approx y(1) = e$, with the approximation getting better and better as $n \to \infty$. This is correct, since (check your calculus notes/book)

$$y_m = \left(1 + \frac{1}{n}\right)^n \to e \quad \text{as} \quad n \to \infty. \tag{3}$$

To answer part (c), we must consider **Euler's method for $\frac{dy}{dx} = f(x, y) = y^2 - x$ with $y(0) = 0.7$,** which yields

$$y_{m+1} = y_m + h\,(y_m^2 - x_m), \quad \text{with} \quad y_0 = 0.7. \tag{4}$$

For this equation the 0-isocline is given by $x = y^2$. It is then easy to check that:
— $y = L_1(x) = +\sqrt{x}$ (for $x \geq 0$) is an upper fence.
— $y = L_2(x) = -\sqrt{x}$ (for $x \geq 0$) is a lower fence.
Hence **solutions that enter the region $\mathcal{I}$ given by $-\sqrt{x} \leq y \leq \sqrt{x}$, stay in it.** In fact, **these solutions are decreasing for $x > x_*$, and have their maximum at $x = x_*$ — where $y_* = y(x_*)$ is the point at which they enter the region** [Proof: $y' < 0$ for $-\sqrt{x} < y < \sqrt{x}$, and $y' > 0$ for $y^2 > x$].

We will now show that: **All the solutions in region $\mathcal{I}$ asymptote the curve $y = -\sqrt{x}$.**

(i) A solution in $\mathcal{I}$ has to cross into $y < 0$, where it stays ($y = 0$ is an upper fence for $x > 0$). Why? Because as long as $y > 0$, y^2 is decreasing, and thus so is $y' = y^2 - x$.

(ii) For $x \geq \gamma^{2/3}$, let $L_3 = -\sqrt{x} + \gamma/x \leq 0$, where $\gamma > 1/2$ is a constant. This is an upper fence, and forms a funnel together with L_2. The solutions in $\mathcal{I}$ always end up in one of these funnels (just take γ large enough, so that the solution crosses into $y < 0$ before $\gamma^{2/3}$).

The left panel in figure 3 shows an example of solutions of this type

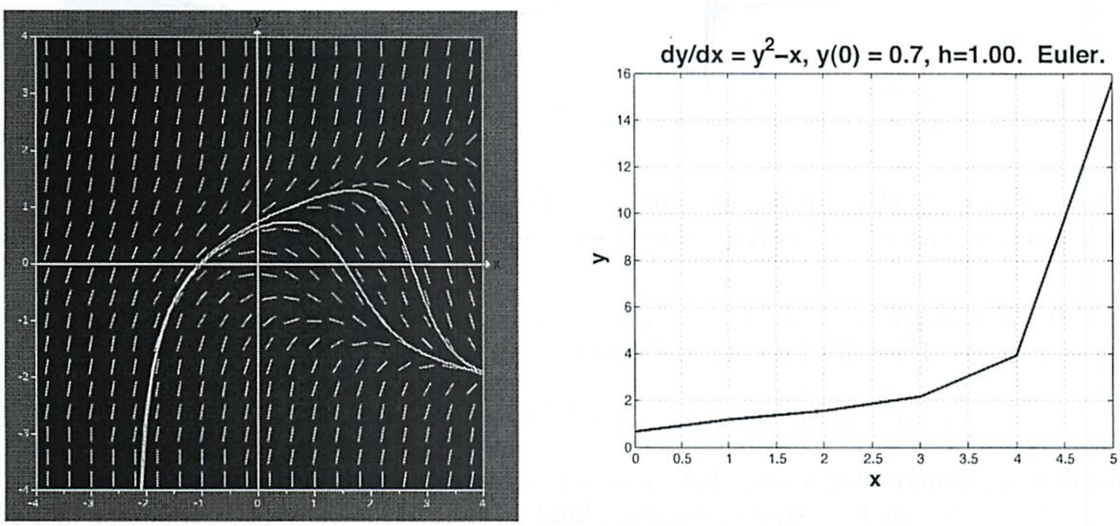


Figure 3: Equation $\frac{dy}{dx} = f(x, y) = y^2 - x$. Left panel: solutions with asymptote $y = -\sqrt{x}$. Right panel: Euler method for $y_0 = 0.7$ and $h = 1$.

Finally: Let $L_4 = \sqrt{x+\gamma}$, for $x \geq 0$, where $\gamma > 2^{-2/3} \approx 0.63$. Then **$L_4$ is a lower fence.**[2] It follows that **L_1 and L_4 form an anti-funnel, where $\frac{\partial f}{\partial y} \geq 0$.** Hence, there is a **unique solution $y = Y(x)$ that stays in the anti-funnel.** In particular, if a solution has $0 < y(x) < Y(0)$, it eventually must enter the region $\mathcal{I}$. It can be shown (more fences, but we will not do it here) that $Y(0) > 0.7$. Hence, we conclude that

$$\textbf{The solution with } y(0) = 0.7 \textbf{ asymptotes } y = -\sqrt{x} \textbf{ as } x \to \infty. \tag{5}$$

However, using Euler's method for this solution, with $h = 1$, yields an answer where the behavior is anything but the one in (5). The problem is that, with this h, the method misses the isocline $y = \sqrt{x}$, and goes into the region of constant growth that occurs above L_4.

Question: does taking a smaller value for h fix the problem? Figure 4 shows the result of using smaller values of h. For a while the numerical solution tracks the actual solution, but eventually oscillations appear. These oscillations grow in amplitude, till eventually the Euler solution latches on a regime where y_m grows rapidly with x_m (only the beginning of this is plotted, since y_n gets to be huge very quickly).

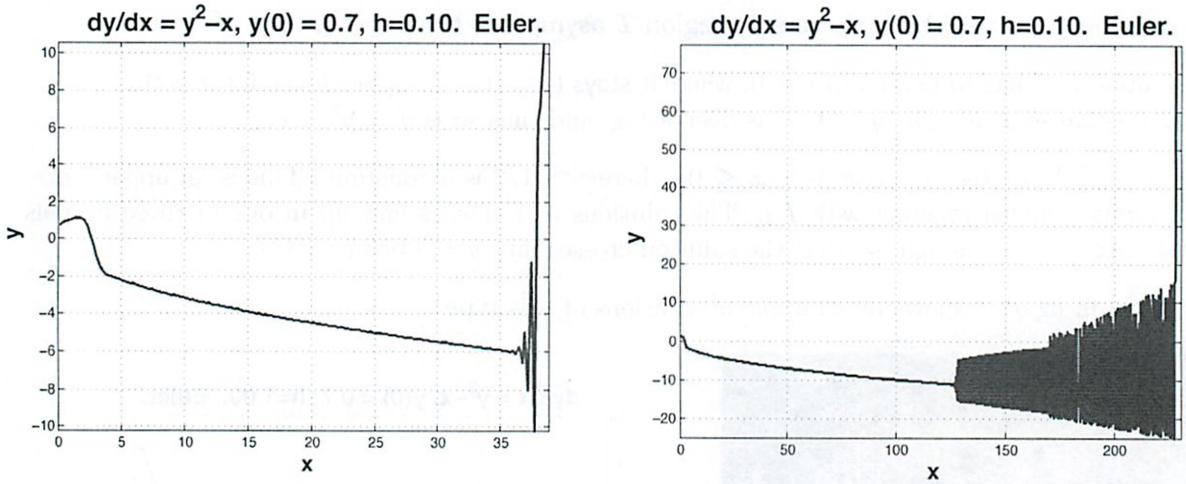


Figure 4: Euler's method for the equation $\frac{dy}{dx} = f(x, y) = y^2 - x$ and $y_0 = 0.7$. Left panel: solution with $h = 0.25$. Right panel: solution with $h = 0.10$.

To understand why this happens, write $y_m = -\sqrt{x_m} + z_m$ — note that, for the exact solution, z_m becomes small as x grows. Plug this into (4), and obtain

$$z_{m+1} = \sqrt{x_{m+1}} - \sqrt{x_m} + (1 + h\,(z_m - 2\,\sqrt{x_m}))\,z_m. \tag{6}$$

However, note that z_m cannot stay small! For, if so, for x_m large enough: $(\mathbf{1} + \boldsymbol{h}\,(\boldsymbol{z_m} - \mathbf{2}\,\sqrt{\boldsymbol{x_m}}))$ becomes large and negative, and z_m starts oscillating (and growing). This is exactly what the figure shows: for moderate values of x_m, z_m behaves as it should. But, at some critical value of x (larger the smaller h is), the mechanism here kicks in and triggers oscillations. Once the oscillations grow enough to

[2] Proof: $L_4' = \frac{1}{2\,L_4}$ and $L_4 \geq \sqrt{\gamma}$. Hence $L_4' \leq \dfrac{1}{2\sqrt{\gamma}} < \gamma = f(x,\, L_4)$.

launch y_m into the region $y_m > \sqrt{x_m + 1/2}$, the numerical solution enters a regime of rapid growth. To see why this last, look at (4): If $y_m > \sqrt{x_m + 1/2}$, then $y_{m+1} > y_m + \frac{1}{2}h$ so that $y_{m+1} > \sqrt{x_{m+1} + 1/2}$. Thus y_m grows. Since y_m grows at least linearly with m (since $y_{m+1} > y_m + \frac{1}{2}h$), $y_m^2 - x_m$ also grows — so that the growth rate of y_m accelerates, and so on ... this feedback loop leads to catastrophic growth of y_m with m.

Problem #3. Here we consider **autonomous equations of the form** $\frac{dy}{dx} = f(y)$.

(a) Figure 5 depicts phase lines (horizontal black lines) for the equation, with a plot of the function $f = f(y)$ super-imposed — so that the flow directions and critical points are easy to ascertain. The function $f = f(y)$ given in the problem statement corresponds to a "bifurcation at critical" state, where small perturbations to the function can trigger a **bifurcation.** Thus: (i) If the function is displaced slightly upwards, the semi-stable critical point on $y > 0$ disappears. (ii) If the function is displaced slightly downwards, the semi-stable critical point on $y > 0$ splits into a sink and a source.

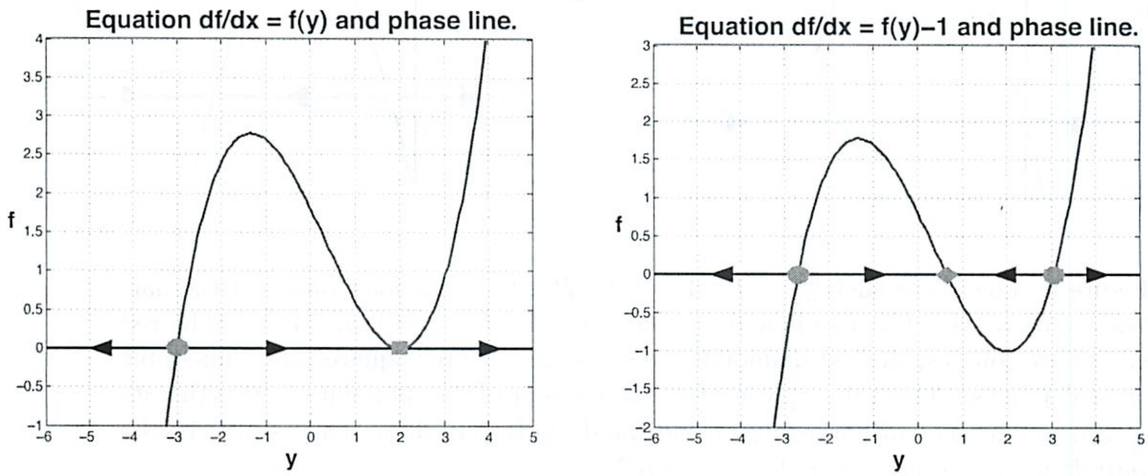


Figure 5: Phase lines for autonomous equations of the form $\frac{dy}{dx} = f(y)$. The flow directions are indicated by black arrows. The red circles are sources, the red diamonds are sinks, and the red squares are semi-stable critical points. **Left panel:** function $f = f(y)$ as given in the problem statement. **Right panel:** function $f = f(y)$ given in the problem statement, displaced downwards by one unit.

(a) Figures 6 and 7 illustrate the behavior of the logistic model $\frac{dP}{dt} = K\left(1 - \frac{1}{N}P\right)P - C$ for various values of the harvesting parameter C:

— For $0 < C < \frac{1}{4}KN$ two equilibrium states exist: a sink at $P = P_h$ (with a lower value than the steady state population $P = N$ without harvesting), and a source at $P = P_e$ (where $0 < P_e < P_h$). As C increases. P_h decreases and P_e increases.

This set up allows for a satisfactory steady state, with the net natural growth rate (birth minus death rate) balancing the harvesting at a steady **stable** equilibrium $P = P_h$.

However, note that:

> **If for whatever reasons the population falls below the source value (i.e. $P < P_e$) then extinction (i.e. $P = 0$) will occur in a finite time — unless C is modified.** (7)

Note that, in this model, the population is driven below zero $P < 0$ after the time of extinction. This is (of course) nonsense showing the serious limitations of the model: You have to take any conclusions from this model with due care.

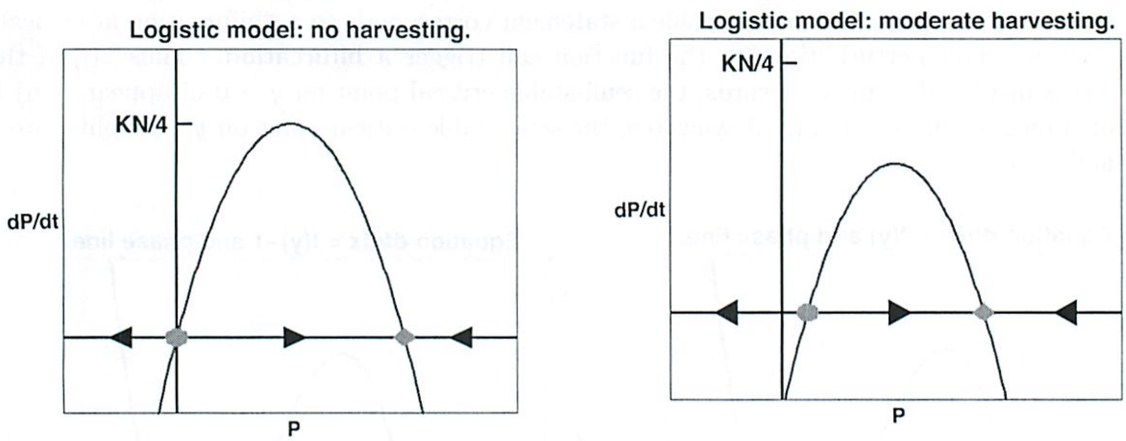


Figure 6: Logistic model $\frac{dP}{dt} = K\left(1 - \frac{1}{N}P\right)P - C$ for various values of the harvesting parameter C. The flow directions are indicated by black arrows. The red circles are sources, the red diamonds are sinks, and the red squares are semi-stable critical points. The value $\frac{1}{4}KN$ of the maximum of the population growth rate $\frac{dP}{dt}$, in the absence of harvesting, is indicated. **Left panel:** no harvesting. **Right panel:** moderate harvesting $0 < C < \frac{1}{4}KN$.

— At the critical value $C = \frac{1}{4}KN$, P_e and P_h coalesce in a single semi-stable equilibrium point. This is not a good regime to be at. In fact, even getting too close to it is not good: Then (see (7)) any small perturbation can push P below P_e.

— When the harvesting rate is above critical $C > \frac{1}{4}KN$, there are no equilibrium populations possible (neither stable, nor unstable), and extinction happens in a finite time.

Imagine now that harvesting has been happening at a rate $C > \frac{1}{4}KN$, but before catastrophe hits (i.e.: $P = 0$), C is lowered below critical.[3] Then

Question: *Will the population recover?*

Answer:[4] *Only as long as C is made low enough to make the population, at the time the change is implemented, satisfy $P > P_e$* — see (7).

[3] Notice that this need not happen because of any increase in wisdom. As P drops, harvesting becomes harder to do, commercial viability goes down, industries crumble, and C drops precipitously.

[4] At least if you believe in this model, though the answer seems quite reasonable.

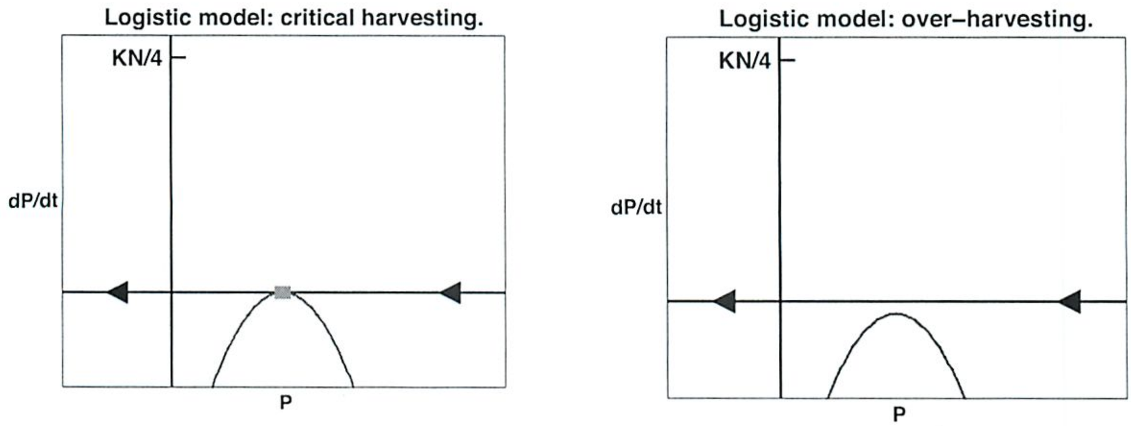


Figure 7: Same as figure 6, but with different harvesting parameters. **Left panel:** critical harvesting $C = \frac{1}{4}KN$. **Right panel:** over-harvesting $C > \frac{1}{4}KN$.

Problem #4. Here we will show that if $A = A(t, y)$ and $B = B(t, y)$ are functions with continuous first partial derivatives in a rectangle $\mathcal{R}$ given by $a \leq t \leq b$ and $y_1 \leq y \leq y_2$, then there exists a function $f = f(t, y)$ such that

$$A = f_t \quad \text{and} \quad B = f_y \tag{8}$$

if and only if

$$A_y = B_t. \tag{9}$$

Proof of the "only if". If (8) applies, then f is twice continuously differentiable, and

$$A_y = f_{ty} = f_{yt} = B_t,$$

so that (9) applies.

Proof of the "if". If (9) applies, let

$$f(t, y) = \int_{y_1}^{y} B(t, z)\, dz + \int_a^t A(s, y_1)\, ds.$$

It is easy to see that this satisfies (8).

THE END.

18.03 Lecture 5

next P-set due Fri

no Class Wed

So P-set shorter

Last class: linear first order ODEs

$$\frac{dy}{dt} + P(t)\,y = Q(t)$$

Multiply both sides by magic integrating factor $e^{\int P(t)\,dt}$

Integrate both sides with respect to t

Example $\frac{dy}{dt} + y = \frac{1}{1+t^2}$ $\quad y(2) = 3$

$P(t) = 1$

$Q(t) = \frac{1}{1+t^2}$

Int factor $e^{\int dt} = e^t$

(skipping step)

$$\frac{d}{dt}\left(y\, e^t\right) = \frac{e^t}{1+t^2}$$

product rule

20

WA ans $y(x) = c_1 e^x - x - 1$

What did I do wrong?

Or WA wrong?

Try one later

No now

$$\frac{dy}{dt} + y = \frac{1}{1+t^2}$$

$$P(x) = 1 \quad Q(y) = \frac{1}{1+t^2}$$

$$\int \frac{d}{dt}\left(y \cdot e^y\right) = \int \frac{1}{1+t^2} e^y \, dt$$

$y \cdot e^y =$

how $\int$ that?

Integrate by parts

but no parts

no dt

So do that

$(1+t^2)^{-1}$

$-1(1+t^2)^{-2} \cdot 2t \quad \underbrace{e^y}_{\text{like constant}}$

e^y cross out

$$y \cancel{e^y} = -\frac{2t}{(1+t^2)^2} + C$$

Oh never ended up solving - prof said that way too messy

(2)

$$y \cdot e^t = \int \frac{e^t}{1+t^2}\,dt + C$$

Integrated both sides w/r to t

integral is hard
integration by parts hard

Solve for c
etc (sloppy way)

<u>neat way</u>

Chane names so end up w/ t

$$\int_2^t \frac{d}{ds}\left(y e^s\right) ds = \int_2^t \frac{e^s}{1+s^2}\,ds$$

alt version of fund. theorm of calculus

$$y(t)\,e^t - y(2)\,e^2 =$$

$$y(t)\,e^t - 3e^2 = \int_2^t \frac{e^s}{1+s^2}\,ds$$

Can do algebra to solve for y

don't need to compute integral

③

First exam a week from Wed

└ last topic: input response models

Class

Input Response Models

$y(0) = y_0$

$$\frac{dy}{dt} + ky = Q(t)$$

k constant, > 0

Look familar: linear eq but $P(t) = k$

But some things to note

Integrating factor $\to e^{kt}$

So general solution to IR

$$y(t) = e^{-kt}\int_0^t Q(s)e^{ks}\,ds + y_0 e^{-kt}$$

(Read lecture notes for why IF not magic)

Since $k > 0$, $\underbrace{y_0 e^{-kt}}_{\text{"transient"}} \to 0$ as $t \to \infty$

Rest is ind. of inital condition y_0

(4)

So ~~e~~asy approach

$$\underbrace{e^{-kt}\int_0^t Q(s)\, e^{ks}}_{\substack{\text{"steady state" or}\\ \text{"long term" sol}}}\, ds$$

(class makes more sense after studying up)

Why IR model?

- Signal processing
- $Q(t)$ is input signal
- $y(t)$ is the response

Long digression to complex #

why $Q(t)$ sinusoid → solve $\int_0^t \sin(s)\, e^{ks}\, ds$

↑ can integrate by ~~the~~ parts twice

So change sign to complex exponential
So becomes pure exponential - easy to do

5

Sidebar

This is everything on exam

- ~~An~~ slope field
- fences / funnels / unit-funnel
- diff eq sep of variables
- linear ODE using IF
- modeling / word problem
  - mixing
  - harvesting
  - pop
  - exp growth/decay
- Complex #
- 4 or 5 questions
- no calculator

The notes has Complex #

- has story of how of imaginary # invented
  - $x^3 - 6x + 2$ what are its roots?
    - horrible formula w/ nested radicals
    - can express roots as combos of cubed root of -1 and $\pm$ root of -7

$$\sqrt[3]{-1 \pm \sqrt{7}}$$

8

Cartesan coords

$z = a + bi \quad a, b \in \mathbb{R}$

easy to add / subtract
- component wise

multiplying / dividing annoying
- must expand $(a+bi)(c+di)$

$\boxed{\text{remember } i^2 = -1}$ $= (ac - bd) + (bc + ad)i$

divison is worse $\frac{a+bi}{c+di}$
- rationalize denominator

$$\frac{a+bi}{c+di} \cdot \frac{c-di}{c-di} = \frac{(ac+bd) + (bc-ad)i}{c^2 + d^2}$$

Two useful functions

$Re(a+bi) = a$
$Im(a+bi) = b$
returns real #

(7)

<u>example</u>

$Re\left(\frac{1}{2+5i}\right) =$

need to rationalize denom

~~As~~ Multiply by $\frac{2-5i}{2-5i}$

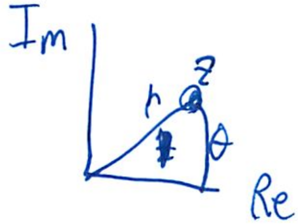
So

$= Re\left(\frac{2-5i}{29}\right)$

$= \frac{2}{29}$

Can also use a polar coord expansion

$z = r(\cos\theta + i\sin\theta) = re^{i\theta}$



$Re = r\cos\theta$

$Im = r\sin\theta$

$r = x^2 + y^2$ if $z = x + iy$

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$

So identity for $\cos\theta + i\sin\theta$

$e^{i\theta} = \cos\theta + i\sin\theta$

So if $z_1 = r_1 e^{i\theta_1}$

$z_2 = r_2 e^{i\theta_2}$ $\qquad z_1 \cdot z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$

⑧

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$$

So where does formula come from

$\theta = \pi$

$e^{i\pi} = -1 + 0$

$e^{i\pi} + 1 = 0$

↑ So many constants in 1 eq

π = ratio of circles

e = function whose deriv is itself

$i = \sqrt{-1}$

But why?

Power series at $t=0$

$$e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots$$

Converges everywhere

$$\sin t = t - \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots$$

$$\cos t = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots$$

These same defn's make sense in complex #s

Care about convergence

⑨

$|z| = r = \sqrt{x^2 + y^2}$
↑size

$= (z \cdot \bar{z})^{1/2}$
↑ $\bar{z}$ conjugate

$\bar{z} = x - iy$ if $z = x + iy$

Plug in $A = i\theta$

<u>Question</u> $e^{i\theta_1} \cdot e^{i\theta_2} = e^{i(\theta_1 + \theta_2)}$?

Use eq from earlier

$(\cos\theta_1 + i\sin\theta) \cdot$ etc... $= (\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))$
↑
trig
identity

<u>Example</u> $(2 + 5i)^{37}$ What is the qv here?

Pure misery w/ binomial expansion

So get polar coord form

$r = \sqrt{29}$

$2 + 5i = \sqrt{29} \cdot$ ~~tan⁻¹(5/2)~~ $e^{i\tan^{-1}(5/2)}$

So $(2 + 5i)^{37} = \sqrt{29}^{37} \cdot e^{37i\tan^{-1}(5/2)}$

(20)

$37 \tan^{-1}\left(\frac{5}{2}\right) = .0092995$

So the # is very close to real

.53 degrees

Lecture 5 4/16

On Wednesday, studying linear first-order ODEs.

No slides

$$\frac{dy}{dt} + P(t)\,y = Q(t)$$ ← solved by mult. by int. factor $e^{\int P(t)\,dt}$, then integrate both sides w.r.t. t.

Example: $\frac{dy}{dt} + y = \frac{1}{1+t^2}$, $y(2) = 3$.

Here $P(t) = 1$, so int. factor: $e^{\int 1\,dt} = e^t$

$$\frac{d}{dt}\left(e^t \cdot y\right) = \frac{e^t}{1+t^2}. \Rightarrow e^t \cdot y = \underbrace{\int \frac{e^t}{1+t^2}\,dt + c}$$

Sloppy. Haven't used initial cond.

Better: $$\underbrace{\int_2^t \frac{d}{ds}\left(e^s y\right) ds}_{e^t y - \underbrace{3e^2}_{= y(2)\cdot e^2}} = \int_2^t \frac{e^s}{1+s^2}\,ds$$

$$e^t y - 3e^2 = \int_2^t \frac{e^s}{1+s^2}\,ds.$$

Then solve for y to clean up a little.

In lecture notes, explain why integrating factor is motivated by solving simple equ:

$$\frac{dy}{dt} + P(t)\,y = 0$$

which is separable.

Last topic before exam: Input-Response models.

$$\frac{dy}{dt} + ky = Q(t)$$ with k: const > 0, $y(0) = y_0$.

(special case of linear equation with $P(t) = k$ ⇝ integ. factor is just e^{kt}

so $$y(t) = e^{-kt}\int_0^t Q(s)\,e^{ks}\,ds + y_0 e^{-kt}.$$

Since $k > 0$, then for long-term behavior: $y_0 e^{-kt} \to 0$ as $t \to \infty$

(other term is less certain as integral may grow rapidly in t.)

For this reason, $y_0 e^{-kt}$ called "transient" and main term

$e^{-kt}\int_0^t Q(s) e^{ks}\, ds$ is "steady state" or "long-term" solution.
(indep. of initial condition)

All IVPs have sol'ns asympt approaching steady state sol'n.

Called input-response because we think of $Q(t)$ as input signal and $y(t)$ as result or "response"

—

Study this model in detail for various inputs. Important class of inputs: sinusoids.

integrals of form $\int \sin(s)\, e^{ks}\, ds$.

Can do by parts, but slick method using complex numbers. Since cx. numbers are important part of the course, want to spend some time early on discussing their properties (which are beautiful)

—

Resummarize.

—

Exam topics:
- slope fields / fences / funnels
- numeric approx.
- separation of vars.
- linear ODEs / int. factor
- modeling (mixing, population models, exp. growth/decay)
- cx. numbers

Take practice exam on Thursday in recitation.
(no lecture Wednesday)

Complex numbers (mention history of cubic eqn.? $x^3 - 6x + 2$

with rts & given by combs of

$\sqrt[3]{-1 \pm \sqrt{-7}}$: real #)

problem in real numbers with real solution where we temporarily use cx. numbers.

(Do same for $\mathbb{R}$ models)

$a + bi$ with $a, b \in \mathbb{R}$: real numbers.

Simple real-valued functions "Re" : real part $\mathrm{Re}(a+bi) = a$

"Im" : imag. part $\mathrm{Im}(a+bi) = b$ ← NOT : bi

with $i^2 = -1$.

Multiplication : $(a+bi)\cdot(c+di) = (ac-bd) + i(bc+ad)$

Division : $\dfrac{a+bi}{c+di} \cdot \dfrac{c-di}{c-di} = \dfrac{(ac+bd) + i(bc-ad)}{c^2+d^2}$

Ex : $\mathrm{Re}\left(\dfrac{1}{2+5i}\right) = \mathrm{Re}\left(\dfrac{2-5i}{(2+5i)(2-5i)}\right) = \mathrm{Re}\left(\dfrac{2-5i}{29}\right) = \boxed{\dfrac{2}{29}}$

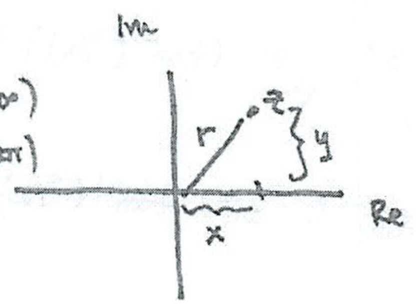
$z = x + iy$, then $\bar{z} = x - iy$: complex conjugate $z \cdot \bar{z} = \underbrace{|z|^2}_{\text{size of cx. number}}$

Alternate form (using polar coords)

$z = r\cdot(\cos\theta + i\sin\theta)$

$= x + iy$

unique if
$r \in [0, \infty)$
$\theta \in [0, 2\pi)$



$r^2 = x^2 + y^2$

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$

Even better polar expression when use Euler's formula : $e^{i\theta} = \cos\theta + i\sin\theta$

then $z = r\cdot e^{i\theta}$

Can regard $e^{i\theta} = \cos\theta + i\sin\theta$ as definition of cx. exponential.

But also define these via power series: (expanded about 0)

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Set $x = i\theta$ then compare.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

(notions of convergence make sense over cx. numbers using $|\cdot|$)

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

Can check: $e^{i\theta_1} \cdot e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$

How? $(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \rightsquigarrow$ expand and apply trig identities.

So to compute $z \cdot z' = r \cdot r' e^{i(\theta+\theta')}$

$$z/z' = \frac{r}{r'} \cdot e^{i(\theta-\theta')}$$

(addition / subtraction not so well behaved.)

Ex: $(2+5i)^{37}$ $\quad 2+5i \overset{\text{polar}}{\rightsquigarrow}$ $r = \sqrt{x^2+y^2} = \sqrt{29}$

$\theta = \tan^{-1}\left(\frac{5}{2}\right)$

$$= (\sqrt{29})^{37} \cdot e^{37i(\tan^{-1}(5/2))}$$

$.0092995 \approx 37i\tan^{-1}(5/2)$. $\Rightarrow$ almost a real #. (.53 deg.)

18.03 Lecture 6

9/14

Today: finish digression Complex #

Wed No Class

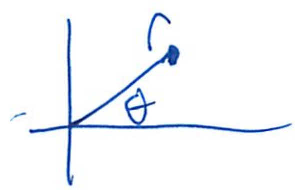
Thur Practice Test in Recitation

Fri: One more discussion on input/response models

$z = x + iy = re^{i\theta}$

$x, y \in \mathbb{R}$

polar coords only unique if require $r \in [0, \infty)$, $\theta \in [0, 2\pi)$



$e^{i\theta} = \cos\theta + i\sin\theta$

$e^{2\pi i \theta} = 1$

Complex # have n nth roots of 1.

$\left(e^{2\pi i/n}\right)^n = 1$

$z = e^{2\pi i/n}, e^{2\pi i 2/n}, e^{2\pi i(n-1)/n}, 1$

$= \{e^{2\pi i k/n}\}_{k=0,\dots,n-1}$

(2)

For example to solve $z^3 = 1$

$$(re^{i\theta})^3 = 1$$

$\Rightarrow$ if radii and θ are equal

$$r^3 = 1$$

$$3\theta = 0 + 2\pi k \text{ for some } k$$

$r = 1 \leftarrow$ one solution since real

$\theta : 2\pi/3,\ 4\pi/3,\ 0$

How many complex sols for i

$$e^z = 1 + i$$

Put everything in polar coords

$$e^z = 1 + i$$

$$z = x + iy$$

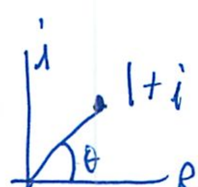
$$e^x \cdot e^{iy} = \sqrt{2}\, e^{i\pi/4}$$

↑ same only if r, θ same

$$re^{i\theta} = 1 + i$$

$$r = \sqrt{2}$$

$\theta = \frac{\pi}{4} \leftarrow \arctan(1)$



③

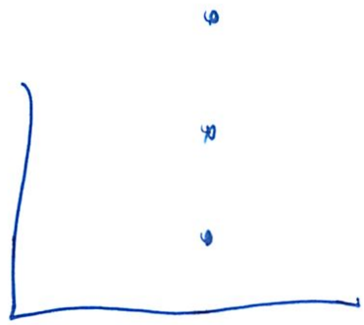
$x = \ln\sqrt{2} \to e^x = \sqrt{2}$

$y = \frac{\pi}{4}$

Can choose otherwise

$y = \frac{\pi}{4} + 2\pi k$

gives diff complex # sols



Can also take $-r$, πk

$x = \pm \ln\sqrt{2}$

$y = \frac{\pi}{4} + \pi k$

So ∞ many

Notice this question was subtly different from previous question

4

Find sol's to $z^5 = 3+3i$

↑5 solutions

Convert ___ to polar

$(re^{i\theta})^5 = \sqrt{18}\, e^{\pi i/4}$

$z_0 = \sqrt[10]{18}\, e^{i\pi/20}$

$5\theta = \frac{\pi}{4}$

↑ one solution

but we know there are 5 more

Can multiply any 5th root of unity

So all sol's are

$z_0,\ z_0 e^{2\pi i/5},\ z_0 e^{4\pi i/5},\ \ldots$

$(z_0 e^{2\pi i/5})^5 = z_0 \cdot e^{2\pi i} = z_0^5,$

Nice aplet ~~[illegible]~~ "complex roots"

(5) New calculus w/ complex #s

$$e^{z} = e^{x+iy} \overset{\text{def}}{=} e^{x}e^{iy} = e^{x}(\cos y + i \sin y)$$

Still true: $e^{z_1} e^{z_2} = e^{z_1 + z_2}$

Think e^{it} as complex valued fn
w/ real variable input

Any fn can be split so

$$f(t) = y(t) + iv(t)$$

$$\boxed{\begin{aligned} U(t) &= \mathrm{Re}(f(t)) \\ L(t) &= \mathrm{Im}(f(t)) \end{aligned}}$$ Remember

$$e^{it} = \cos t + i \sin t$$

$$\frac{d}{dt}(f(t)) \overset{\text{def}}{=} \frac{d}{dt} + i\frac{dv}{dt}$$

$$\int f(t) = \int u(t)\,dt + i\int v(t)\,dt$$

$$\frac{d}{dt}(e^{it}) \overset{?}{=} ie^{it}$$ hope it is nice

$$\frac{d}{dt}(\cos(t) + i\sin t)$$
$$\parallel$$
$$-\sin t + i\cos t$$

6

$= i(\cos t + i \sin t) = i e^{it}$

So it is just so nice

will have to check in HW the same way

$$\frac{d}{dt}\left(e^{(a+bi)t}\right) = (a+bi)e^{(a+bi)t}$$

<u>upshot</u> Can take deriv e^{rt} — does not matter if r is complex or real

$$\int e^{rt}\,dt = \frac{1}{r}e^{rt} + C$$

↑ Can be complex #

---

try: $\int e^x \sin 3x \, dx$

- would have done parts a few days ago
- but now think

$$\sin(3x) = \mathrm{Im}\left(e^{3xi}\right)$$

Instead

$$\int e^{(1+3i)x}\,dx$$

$$\mathrm{Im}\left(e^{(1+3i)x}\right) = \mathrm{Im}\left(e^x(\cos 3x + i \sin 3x)\right)$$

⑦

$= \cancel{\ } e^x \sin 3x$

$\int e^{(1+3i)x}\, dx = \frac{1}{1+3i} e^{(1+3i)x} + C$

Now find $\mathrm{Im}\left(\frac{1}{3i} e^{x(1+3i)}\right)$

$= \mathrm{Im}\left(\left(\frac{1-3i}{10}\right)\left(e^x \cdot \underbrace{e^{3ix}}_{\cos 3x + i\sin 3x}\right)\right)$

$= e^x\, \mathrm{Im}\left(\frac{1-3i}{10}\left(\cos 3x + i\sin 3x\right)\right)$

$= e^x\left(\frac{\sin 3x}{10} - \frac{3\cos 3x}{10}\right)$

Input-Response Model

Solve the diff eq

$$\frac{dy}{dt} + y = \sin 3t$$

a week ago: integrating factor

$e^{\int 1\, dt} = e^t$

$\frac{d}{dt} e^t y = e^t \sin 3t = \mathrm{Im}\left(e^{(1+3i)t}\right)$

integrate both sides w/r t t

Or ↓

8

Complexify

$\tilde{y}$ = complex variable

Solve $\frac{d}{dt}\left(e^{t}\tilde{y}\right) = e^{(1+3i)t}$

$\swarrow$ Im

$$\frac{d}{dt}\left(e^{t}y\right) = e^{t}\sin 3t$$

Or real (but diff diff eq !)

$$\frac{d}{dt}\left(e^{t}x\right) = e^{t}\cos 3t$$

2nd one is easy to ∞ w/r to t

$$e^{t}\tilde{y}$$

Last step, $\int$ both sides

$$e^{t}\tilde{y} = \frac{1}{1+3i}e^{(1+3i)t} + C$$

$$e^{t}y = \mathrm{Im}(\qquad) + \underbrace{\mathrm{Im}(C)}_{\text{now real}},$$

↑ Solve for C here if IV

Lecture 6

Complex numbers again:

$$z = x + iy = re^{i\theta}$$

r: size of z

θ: angle of rotation counter-clock. from real axis.

$x, y \in \mathbb{R}$

not unique unless we restrict $r \in [0, \infty)$ and $\theta \in [0, 2\pi)$

e.g. $e^{2\pi i \theta} = 1.$

complex numbers have n n^{th} roots of 1:

solutions to $z^n = 1$ are $z = \underbrace{e^{2\pi i/n}, e^{2\pi i/n \cdot 2}, \ldots, e^{2\pi i/n \cdot (n-1)}}_{}, 1$

points on complex unit circle. evenly spaced at $2\pi/n$ radians apart.

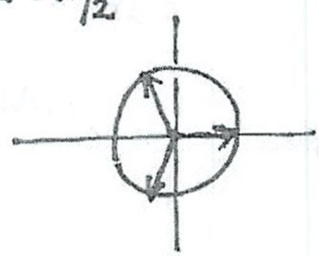
(mention issues w/ multi-valued functions?)

Ex: Find all solutions to

$$z^3 = 1 \quad : \quad 1, e^{2\pi i/3}, e^{4\pi i/3} = -1/2 - i\sqrt{3}/2$$

$$e^{2\pi i/3} = \cos 2\pi/3 + i \sin 2\pi/3 = -1/2 + i\sqrt{3}/2$$

(view this via polar forms: match r, θ)



Find all solutions to

$$z^5 = 3 + 3i = \sqrt{18}\, e^{i\pi/4}$$

one root: $\sqrt[10]{18} \cdot e^{i\pi/20}$. Given any root, can mult. by 5^{th} root of 1 (power of $e^{2\pi i/5}$)

Can also consider exponentials with both real and imaginary parts.

$$e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y)$$

still true that $e^{z_1} e^{z_2} = e^{z_1 + z_2}$ (apply rules to e^x, e^{iy} separately. Put back together again)

Complex functions + calculus:

The function $e^{i\theta}$ is complex-valued function of real variable θ.

Any such function $f(\theta) = u(\theta) + i\,v(\theta)$ $\quad u, v$: real val. functions

e.g. $e^{i\theta} = \underbrace{\cos\theta}_{u} + \underbrace{i\sin\theta}_{v}$ $\quad u = \mathrm{Re}(f(\theta))$, $v = \mathrm{Im}(f(\theta))$

To do calculus:

$$\frac{d}{d\theta}(f(\theta)) = \frac{du}{d\theta} + i\frac{dv}{d\theta}. \qquad \int f(\theta)\,d\theta = \int u\,d\theta + i\int v\,d\theta$$

What is $\frac{d}{d\theta}(e^{i\theta})$?

$$\frac{d}{d\theta}(\cos\theta + i\sin\theta) = -\sin\theta + i\cos\theta = i\cdot(\cos\theta + i\sin\theta) = i\,e^{i\theta}.$$

HW: check $\frac{d}{d\theta}\left(e^{(a+bi)\theta}\right) = (a+bi)\,e^{(a+bi)\theta}$.

Also implies:

$$\int e^{(a+bi)\theta}\,d\theta = \frac{1}{a+bi}e^{(a+bi)\theta} + \underbrace{c_1 + ic_2}_{\text{complex const.}}$$

Upshot: $\int e^{rt}\,dt = \frac{1}{r}e^{rt} + c$, $\quad \frac{d}{dt}(e^{rt}) = re^{rt}$

hold whether r is real or complex.

Application to integrals:

$$\int e^x \sin 3x\,dx \qquad \sin(3x) = \mathrm{Im}(e^{3xi})$$

$$e^x\sin(3x) = \mathrm{Im}\left(e^{x(1+3i)}\right)$$

$$\| $$

$$\mathrm{Im}\left(\int e^{x(1+3i)}dx\right) = \mathrm{Im}\left(\frac{1}{1+3i}e^{x(1+3i)} + c\right)$$

$$\frac{1}{1+3i} = \frac{1-3i}{10} \qquad e^{(1+3i)x} = e^x(\cos 3x + i\sin 3x)$$

...

Solve differential eqn : (Input-Response model with sinusoidal input.)

$$\frac{dy}{dt} + y = \sin 3t$$

method of Integrating factor : $\frac{d}{dt}(e^{t} \cdot y) = e^{t} \sin 3t$

$$\operatorname{Im}\left(\frac{d}{dt}(e^{t} \cdot z)\right) = \operatorname{Im}\left(e^{(3i+1)t}\right)$$

with $\operatorname{Im}(z) = y$.

Warning : Using y here is possibly misleading - may think we always take imaginary parts in cx. ODE to get dep. var.

solve cx. differential equation : $\frac{d}{dt}(e^{t} \cdot z) = e^{(3i+1)t}$

take imaginary parts. What would real parts give ?

Will use $y \to \tilde{y}$ for complexifying as in Mattuck's Notes.

Solving : $z = \frac{1}{1+3i} e^{3it} + \underset{\text{complex \#}}{c} e^{-t}$

$=$

$1+3i = \sqrt{10}\, e^{i\phi}$ $\phi = \tan^{-1}(3) \approx 5/4$ $(1+3i)^{-1} = \frac{1}{\sqrt{10}} e^{-i\phi}$

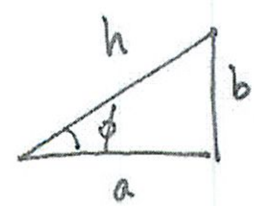
$$\Rightarrow z = \underbrace{\frac{1}{\sqrt{10}} e^{i(3t-\phi)}}_{\text{steady-state}} + \underbrace{c \cdot e^{-t}}_{\text{transient}}$$

Take Im part:

$$y = \frac{1}{\sqrt{10}} \sin(3t-\phi) + ce^{-t}$$

Same as before using trig ident: $a\cos\omega t + b\sin\omega t = h\cos(\omega t - \phi)$

$$\frac{1}{\sqrt{10}}\sin(3t-\phi) = \frac{-3\cos 3x + \sin 3x}{10}$$



Math 18.03 : Differential Equations

Lecture 6 Supplemental Notes

Monday, September 19, 2011

Quick Quiz 5

How many complex solutions z exist for the following equation:

$$e^z = 1 + i$$

1. 1
2. 2
3. 0
4. infinitely many

Quick Answer

How many complex solutions z exist for the following equation:

$$e^z = 1 + i$$

1. 1
2. 2
3. 0
4. infinitely many

To solve, we place the right-hand side in terms of polar coordinates:

$$1 + i = \sqrt{2}e^{\pi i/4}$$

Two complex numbers are equal if and only if they have the same real and complex parts. Write $z = x + iy$. Then

$$x = \pm \ln(\sqrt{2}), \; y = \pi/4 + \pi k, k \in \mathbb{Z}$$

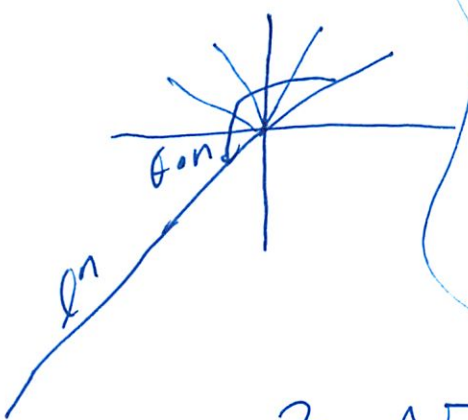
where we take $\pm$ depending on whether k is odd or even.

18.03 Recitation 9/20

Office Hrs tmo still on W-1-3

$z^n = a + bi$

Find root of ^



nth power in complex plane
means take angle
and go n times w/ it
take length write to nth power

$$z = \sqrt[n]{r}\, e^{i\left(\theta/n + \frac{2\pi k}{n}\right)} \qquad \overset{\shortparallel}{r e^{i\theta}}$$

1. How many sols to

$$x^2 + 3x + 5 = 0$$

How many real sols? complex sols?

2. Same $e^z = \frac{\pi^2}{17}$

3. Find real part of

$$(3+4i)^3$$

4. Write $3+4i$ in polar coords

5. " $\frac{1}{3+4i}$

(2)

(I have no clue - behind in studying this)

So 1, 3, 4, 5 easy (copy off boards)

1. $x = \dfrac{-3 \pm \sqrt{9-20}}{2}$ $\leftarrow$ 2 complex, 0 real

3. $\mathrm{Re}\left[(24i-7)(3+4i)\right] = -117$

4. $3+4i = 5e^{i\left(\tan^{-1}\left(\frac{4}{3}\right)\right)}$

5. $\dfrac{1}{3+4i} = \dfrac{3-4i}{9+16} = \dfrac{3-4i}{25}$

$= \dfrac{3}{25} \; \dfrac{-4}{25} i = \dfrac{1}{5} e^{i \tan\left(-\frac{4}{3}\right)}$

(coming back to me)

---

2. $e^{z} = 1 \quad z = a + bi$

$e^{a+bi} = e^{a}(\cos b + i \sin b) = 1$

$e^{a} \cos b = 1$

$\sin b = 0$

(3)

$b = 2k\pi$

$\cos b = 1$

$a = 0$

Poven sols are just

$2k\pi i$

k an integer

$e^z = \frac{\pi^2}{17}$

$\ln\left(\frac{\pi^2}{17}\right) + 2k\pi i$

divide both sides

$e^z \cdot e^{\ln \frac{\pi^2}{17}} = 1$

$e^{z - \ln \frac{\pi^2}{17}} = 1$

Now find sols for this

1 real $\ln\left(\frac{\pi^2}{17}\right)$

∞ complex $\ln\left(\frac{\pi^2}{17}\right) + 2k\pi i$

real # are complex # as well

often ∞ # complex #

but not always $e^z = 0$

(4)

How to solve

$$e^{z} = e^{\ln \frac{\pi^2}{17}}$$

divide by e^z

$$1 = e^{\ln \frac{\pi^2}{17}} \cdot e^{-z}$$

$$1 = e^{\underbrace{\ln \frac{\pi^2}{17} - z}_{y}}$$

Solved before so know the $e^{y} = 1$

$$\ln \frac{\pi^2}{17} - z = 2k\pi i$$

Differential Equation

$$\frac{dy}{dt} + y = \cos 4t$$

? how
- is in complex # chap
- so complex # method?

See notes 6 - end

$\tilde{y}$ = complex variable ← complexify

Solve $\frac{d}{dt}\left(e^{t}\,\tilde{y}\right) = e^{(1+4i)t}$

↑ ? how did we get

(5)

Find In( )

$\frac{d}{dy}(e^t y) = e^t \cos 4t$

$\int$ $\int$ both sides

$e^t y =$

(I lost what actually doing and what was explination

Is linear eq

$p(t) = 1$

~~$\int p(t) dt$~~

IF $e^{\int p(t)\, dt} = e^{\int 1\, dt} = e^t$

$\frac{dy}{dt} e^t + e^t y = e^t \cos 4t$

$\frac{d}{dt} y e^t = e^t \cos 4t$

$\int$ both sides $\int$

(6)

$$y e^t = \int \underbrace{e^t \cos 4t}_{\substack{\text{real part} \\ \text{of } \underbrace{e^{t+4ti}}_{e^t e^{4ti}}}} dt$$

$\leftarrow$ now trick

$= e^t (\cos 4t + i \sin 4t)$

$$\overset{Re}{=} \left[\int e^{t+4ti} \, dt \right]$$

$$\overset{Re}{=} \left[\frac{1}{1+4i} e^{t+4ti} \right]$$

But only want Real part
- multiple top + bottom by Conjugate

$$= \frac{1-4i}{1+16} e^t \cdot e^{4ti}$$

$$= \frac{1-4i}{17} e^t \left[\cos 4t + i \sin 4t \right]$$

multiply each part

18.03 Review

Get up to speed w/ what was taught in complex # unit

I've seen the basics of complex # before

- Multiplication $(a+bi)(c+di) = (ac-bd) + (bc+ad)i$

Division $\frac{a+bi}{c+di} \cdot \frac{c-di}{c-di} = \frac{(ac+bd)+(bc-ad)i}{c^2+d^2}$

So to get real part of something, break it up

$Re\left(\frac{1}{2+5i}\right)$ So multiply $\frac{(2-5i)}{(2-5i)}$

$$\frac{2-5i}{\cancel{4-25i}}$$

Look at the multiplication rules I just wrote!

$(4+25)+(10+-10)i$

$\frac{2-5i}{29}$ So $\frac{2}{29}$ ✓ will it always cancel?

Do we get a formula sheet? No

2

$z = r(\cos\theta + \sin\theta\, i)$

$= x + yi$

$r^2 = x^2 + y^2$

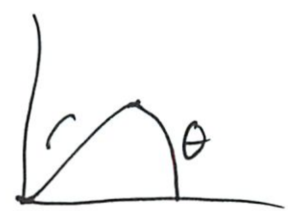
$\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$e^{i\theta} = \cos\theta + i\sin\theta$

Euler's Formula

"astanding"

Then if $z_1 = r_1 e^{i\theta_1}$



$z_2 = r_2 e^{i\theta_2}$

$z_1 \cdot z_2 = r_1 r_2\, e^{i(\theta_1 + \theta_2)}$

So multiplying two

$\theta = \pi$

$e^{i\pi} = -1 + 0$

$e^{i\pi} + 1 = 0$

Power Series at $A = 0$ converges everywhere

3

WP: Power Series ∞ series of form

$$\sum_{n=0}^{\infty} a_n (x-c)^n$$

$$= a_0 + a_1(x-c)^1 + a_2(x-c)^2 + \dots$$

$$|z| = r = \sqrt{x^2+y^2}$$

$$= (z \cdot \bar{z})^{1/2}$$

$$\uparrow \quad \uparrow$$

$$x+yi \quad x-yi$$

$$(2+5i)^{37}$$

$$r = \sqrt{29}$$

$$\underbrace{\sqrt{29}}_{\substack{r \\ \sqrt{x^2+y^2}}} e^{i \underbrace{\tan^{-1}\left(\frac{5}{2}\right)}_{\tan^{-1}\left(\frac{y}{x}\right)}}$$

Then apply exponential

$$\sqrt{29}^{37} e^{37 \tan^{-1}\left(\frac{5}{2}\right) i}$$

Can answer this

(4)

Oh next day

$e^{i\theta} = \cos\theta + \sin\theta\, i$ reminder

n nth roots of 1

$\left(e^{2\pi i/n}\right)^n = 1$

Root where crosses 1

which is 2π ?

$z = e^{2\pi i/n}$, $e^{2\pi i 2/n}$ etc

? So what does this look like?

Solve $z^3 = 1$

$\left(re^{i\theta}\right)^3 = 1$

$r^3 = 1 \rightarrow r = 1$ ← One sol since real

$3\theta = \underbrace{0}_{\text{more specific}} + \overbrace{2\pi k}^{\text{general}}$ for some k

$\theta = \frac{2\pi}{3}, \frac{4\pi}{3}, 0$

So how did I get this?

(5)

$3\theta = 2\pi k$

$\theta = \frac{2}{3}\pi k$

try multiple k from $\boxed{0 \to n-1}$

$\frac{2\cdot 0\pi}{3}, \frac{2\cdot 1\pi}{3}, \frac{2\cdot 2\pi}{3}$

So 1 real root

$\cancel{3}$ 2 complex roots?

? only 2 since real does not count

well actually a real # is also complex

So 3 complex roots of which one is real

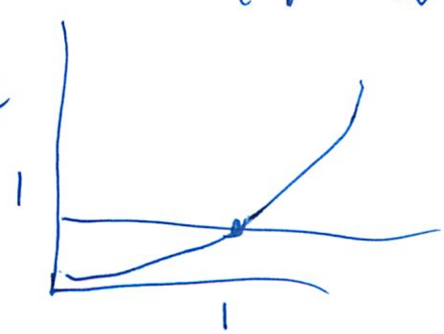
WA

$z = 1$

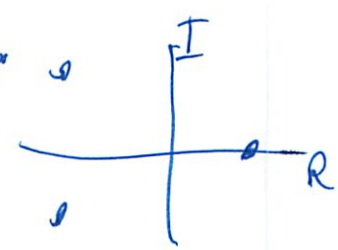
$z = -\sqrt[3]{-1} \approx -.5 - .866i$

$z = (-1)^{2/3} \approx -.5 + .866$

Real plot



Complex



Oh got this better

(6)

$e^z = 1 + i$

~~$z = x + iy$~~

Try myself

$e^{i\theta} = \cos\theta + i\sin\theta$

So $z = i\theta$

$\cos\theta = 1$

$\sin\theta = 1$

?

$z = x + iy$ ← How did we get that

$e^x \cdot e^{iy} = \sqrt{2}\, e^{i\pi/4}$

This was the hard qu

Place RHS in polar

$1 + i = \underbrace{\sqrt{2}}_{\text{So } \sqrt{x^2+y^2}}\, e^{\pi i/4}$

$\tan^{-1}\left(\frac{1}{1}\right)$

$\sqrt{1^2 + 1^2}$

are only equal if real, complex =

(7)

$z = x + yi$ ← where did we get? general format

Write $x = \pm \ln(\sqrt{2})$ ← then try to get rid of $e^{\wedge}$

$y = \frac{\pi}{4} + \underbrace{\pi k}_{\text{standard}}$ $\underbrace{k \in \mathbb{Z}}_{\text{so } \infty}$

$e^x = \sqrt{2}$

$z^5 = 3 + 3i$

Now do on own for real

$z = r e^{i\theta}$

$r = \sqrt{3^2 + 3^2} = 4.24$

$\theta = \tan^{-1}\left(\frac{3}{3}\right) = \frac{\pi}{4}$

$z^5 = 5r \, e^{i5\theta}$

$5 \cdot \sqrt{18} \, e^{\frac{5\pi}{4} i}$

So what next?

$r = 5\sqrt{18}$

$\theta = \frac{5\pi}{4} + k\,2\pi$ for $k \in \mathbb{Z}$

so ∞

Say $5\theta = \frac{\pi}{4}$

$0, \frac{7\pi}{4}, \frac{9\pi}{4}, \frac{11\pi}{4}, \frac{13\pi}{4}$

8 Here is defn

$$\boxed{\begin{array}{l} e^{z} = e^{x+iy} \underset{\sim}{\text{def}} e^{x} e^{iy} = e^{x}(\cos y + i\sin y) \\ e^{z_1} e^{z_2} = e^{z_1+z_2} \end{array}}$$

Any fn can be split

$$f(t) = y(t) + i\, v(v)$$

$$\frac{d}{d}(f(t)) \underset{=}{\text{def}} \frac{dv}{dt} + i\frac{dv}{dt}$$

$$\int f(t) = \int u(t)\,dt + i\int v(t)\,dt$$

↑ if want to differentiate

I think I have not resolved previous qu

One root $\sqrt[10]{18} \cdot e^{i\pi/20}$

then can multiply by 5th root of 1

So I missed w/ what to do when got x, y

so root $z = x + yi$

↑ this is the root I think

also $z = r \cdot e^{i\theta}$

Complex

Addition

Subtraction

Multiplication

$$(a+bi)(c+di) = (ac-bd)+(bc+ad)i$$

Division

$$\frac{a+bi}{c+di} \cdot \frac{c-di}{c-di} = \frac{(ac+bd)+(bc-ad)i}{c^2+d^2}$$

to get real part break up

$$\mathrm{Re}\left(\frac{1}{2+5i}\right) \cdot \frac{2-5i}{2-5i} = \frac{2-5i}{(4+25)+(10-10)i}$$

$$= \frac{2-5i}{29} \rightarrow \frac{2}{29}$$

$$z = r(\cos\theta + \sin\theta\, i)$$

$$= x+yi$$

$$r^2 = x^2+y^2 \quad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$ Euler

$$z_1 = r_1 e^{i\theta_1} \quad z_2 = r_2 e^{i\theta_2}$$

$$z_1 \cdot z_2 = r_1 r_2\, e^{i\theta_1 \theta_2}$$

$$\theta = \pi$$

$$e^{i\pi} = -1+0$$

$$e^{i\pi}+1 = 0$$

n nth roots of 1

$$\left(e^{2\pi i/n}\right)^n = 1$$

Break up into complex exponential

Get r, θ

Then put in z form

$$z = x+iy = re^{i\theta}$$

$$e^z = e^{x+iy} = e^x e^{iy}$$

$$= e^x(\cos y + i\sin y)$$

C. Complex Numbers

1. Complex arithmetic.

Most people think that complex numbers arose from attempts to solve quadratic equations, but actually it was in connection with cubic equations they first appeared. Everyone knew that certain quadratic equations, like

$$x^2 + 1 = 0, \qquad \text{or} \qquad x^2 + 2x + 5 = 0,$$

had no solutions. The problem was with certain cubic equations, for example

$$x^3 - 6x + 2 = 0.$$

This equation was known to have three real roots, given by simple combinations of the expressions

$$(1) \qquad A = \sqrt[3]{-1 + \sqrt{-7}}, \qquad B = \sqrt[3]{-1 - \sqrt{-7}};$$

one of the roots for instance is $A + B$: it may not look like a real number, but it turns out to be one.

What was to be made of the expressions A and B? They were viewed as some sort of "imaginary numbers" which had no meaning in themselves, but which were useful as intermediate steps in calculations that would ultimately lead to the real numbers you were looking for (such as $A + B$).

This point of view persisted for several hundred years. But as more and more applications for these "imaginary numbers" were found, they gradually began to be accepted as valid "numbers" in their own right, even though they did not measure the length of any line segment. Nowadays we are fairly generous in the use of the word "number": numbers of one sort or another don't have to measure anything, but to merit the name they must belong to a system in which some type of addition, subtraction, multiplication, and division is possible, and where these operations obey those laws of arithmetic one learns in elementary school and has usually forgotten by high school — the commutative, associative, and distributive laws.

To describe the complex numbers, we use a formal symbol i representing $\sqrt{-1}$; then a **complex number** is an expression of the form

$$(2) \qquad a + ib, \qquad a, b \quad \text{real numbers.}$$

If $a = 0$ or $b = 0$, they are omitted (unless both are 0); thus we write

$$a + i0 = a, \qquad 0 + ib = ib, \qquad 0 + i0 = 0\ .$$

The definition of *equality* between two complex numbers is

$$(3) \qquad a + ib = c + id \quad \Leftrightarrow \quad a = c,\ b = d\ .$$

This shows that the numbers a and b are uniquely determined once the complex number $a + ib$ is given; we call them respectively the **real** and **imaginary** parts of $a + ib$. (It would be more logical to call ib the imaginary part, but this would be less convenient.) In symbols,

$$(4) \qquad a = \mathrm{Re}\,(a + ib), \qquad b = \mathrm{Im}\,(a + ib)$$

Addition and multiplication of complex numbers are defined in the familiar way, making use of the fact that $i^2 = -1$:

(5a) **Addition** $(a+ib)+(c+id) = (a+c)+i(b+d)$

(5b) **Multiplication** $(a+ib)(c+id) = (ac-bd)+i(ad+bc)$

Division is a little more complicated; what is important is not so much the final formula but rather the procedure which produces it; assuming $c+id \neq 0$, it is:

(5c) **Division** $$\frac{a+ib}{c+id} = \frac{a+ib}{c+id} \cdot \frac{c-id}{c-id} = \frac{ac+bd}{c^2+d^2} + i\,\frac{bc-ad}{c^2+d^2}$$

This division procedure made use of *complex conjugation*: if $z = a+ib$, we define the **complex conjugate** of z to be the complex number

(6) $$\bar{z} = a - ib \qquad \text{(note that } z\bar{z} = a^2+b^2 \text{).}$$

The size of a complex number is measured by its **absolute value**, or *modulus*, defined by

(7) $$|z| = |a+ib| = \sqrt{a^2+b^2}; \qquad \text{(thus : } z\bar{z} = |z|^2 \text{).}$$

Remarks. One can legitimately object to defining complex numbers simply as formal expressions $a+ib$, on the grounds that "formal expression" is too vague a concept: even if people can handle it, computers cannot. For the latter's sake, we therefore define a complex number to be simply an ordered pair (a,b) of real numbers. With this definition, the arithmetic laws are then defined in terms of ordered pairs; in particular, multiplication is defined by

$$(a,b)(c,d) = (ac-bd,\ bc+ad)\ .$$

The disadvantage of this approach is that this definition of multiplication seems to make little sense. This doesn't bother computers, who do what they are told, but people do better at multiplication by being told to calculate as usual, but to use the relation $i^2 = -1$ to get rid of all the higher powers of i whenever they occur.

Of course, even if you start with the definition using ordered pairs, you can still introduce the special symbol i to represent the ordered pair $(0,1)$, agree to the abbreviation $(a,0) = a$, and thus write

$$(a,b) = (a,0)+(0,1)(b,0) = a+ib\ .$$

2. Polar representation.

Complex numbers are represented geometrically by points in the plane: the number $a+ib$ is represented by the point (a,b) in Cartesian coordinates. When the points of the plane are thought of as representing complex numbers in this way, the plane is called the **complex plane**.

By switching to polar coordinates, we can write any non-zero complex number in an alternative form. Letting as usual

$$x = r\cos\theta, \qquad y = r\sin\theta,$$

we get the **polar form** for a non-zero complex number: assuming $x+iy \neq 0$,

(8) $$x+iy = r(\cos\theta + i\sin\theta)\ .$$

When the complex number is written in polar form, we see from (7) that

$$r = |x+iy|, \qquad \textit{(absolute value, modulus)}$$

We call θ the *polar angle* or the *argument* of $x+iy$. In symbols, one sometimes sees

$$\theta = \arg\,(x+iy) \qquad \textit{(polar angle, argument)}\ .$$

The absolute value is uniquely determined by $x+iy$, but the polar angle is not, since it can be increased by any integer multiple of 2π. (The complex number 0 has no polar angle.) To make θ unique, one can specify

$$0 \le \theta < 2\pi \qquad \textit{principal value} \text{ of the polar angle.}$$

This so-called principal value of the angle is sometimes indicated by writing $\text{Arg}\,(x+iy)$. For example,

$$\text{Arg}\,(-1) = \pi, \qquad \arg\,(-1) = \pm\pi, \pm 3\pi, \pm 5\pi, \dots\ .$$

Changing between Cartesian and polar representation of a complex number is the same as changing between Cartesian and polar coordinates.

Example 1. Give the polar form for: $-i, \quad 1+i, \quad 1-i, \quad -1+i\sqrt{3}$.

Solution.

$$\begin{aligned} -i &= i\cos\tfrac{3\pi}{2} & \qquad 1+i &= \sqrt{2}\,(\cos\tfrac{\pi}{4} + i\sin\tfrac{\pi}{4}) \\ -1+i\sqrt{3} &= 2\,(\cos\tfrac{2\pi}{3} + i\sin\tfrac{2\pi}{3}) & \qquad 1-i &= \sqrt{2}\,(\cos\tfrac{-\pi}{4} + i\sin\tfrac{-\pi}{4}) \end{aligned}$$

The abbreviation $\text{cis}\,\theta$ is sometimes used for $\cos\theta + i\sin\theta$; for students of science and engineering, however, it is important to get used to the exponential form for this expression:

$$e^{i\theta} = \cos\theta + i\sin\theta \qquad \textbf{Euler's formula.} \tag{9}$$

Equation (9) should be regarded as the *definition* of the exponential of an imaginary power. A good justification for it however is found in the infinite series

$$e^t = 1 + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots .$$

If we substitute $i\theta$ for t in the series, and collect the real and imaginary parts of the sum (remembering that

$$i^2 = -1, \quad i^3 = -i, \quad i^4 = 1, \quad i^5 = i, \dots ,$$

and so on, we get

$$\begin{aligned} e^{i\theta} &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots\right) \\ &= \cos\theta \ + \ i\sin\theta\ , \end{aligned}$$

in view of the infinite series representations for $\cos\theta$ and $\sin\theta$.

Since we only know that the series expansion for e^t is valid when t is a real number, the above argument is only suggestive — it is not a proof of (9). What it shows is that Euler's formula (9) is formally compatible with the series expansions for the exponential, sine, and cosine functions.

Using the complex exponential, the polar representation (8) is written as

$$x + iy = r\,e^{i\theta} \tag{10}$$

The most important reason for polar representation is that multiplication of complex numbers is particularly simple when they are written in polar form. Indeed, by using (9) and the trigonometric addition formulas, it is not hard to show that

$$e^{i\theta}e^{i\theta'} = e^{i(\theta+\theta')}\ .$$

This gives another justification for definition (9) — it makes the complex exponential follow the same exponential addition law as the real exponential. Thus we can multiply two complex numbers in polar form by

(11) $$r\,e^{i\theta}\cdot r'e^{i\theta'} = r\,r'\,e^{i(\theta+\theta')}\ ;$$ **multiplication rule**

to multiply two complex numbers, you multiply the absolute values and add the angles.

By repeated application of this, we get the rule (sometimes called *DeMoivre's formula* for raising a complex number to a positive integer power: using the notation of (10),

(12) $$\left(r\,e^{i\theta}\right)^n = r^n e^{in\theta};\qquad \text{in particular,}\qquad (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

Example 2. Express $(1+i)^6$ in the form $a+bi$.

Solution. We change to the polar form, use (12), then change back to Cartesian form:

$$(1+i)^6 = (\sqrt{2}\,e^{i\pi/4})^6 = (\sqrt{2})^6 e^{i\,6\pi/4} = 8\,e^{i\,3\pi/2} = -8i\ .$$

The answer may be checked by applying the binomial theorem to $(1+i)^6$ and collecting the real and imaginary parts.

Division of complex numbers written in polar form is done by the rule (check it by crossmultiplying and using the multiplication rule):

$$\frac{r\,e^{i\theta}}{r'e^{i\theta'}} = \frac{r}{r'}\,e^{i\,(\theta-\theta')}\ ;$$ **division rule**

to divide by a complex number, divide by its absolute value and subtract its angle.

Combining pure oscillations of the same frequency. The equation which does this is widely used in physics and engineering; it can be expressed using complex numbers:

(13) $$A\cos\lambda t + B\sin\lambda t = C\cos(\lambda t+\phi),\qquad \text{where}\ \ A+Bi = Ce^{i\phi};$$

in other words, $C=\sqrt{A^2+B^2}$, $\phi = \tan^{-1} B/A$. To prove (13), we have

$$\begin{aligned} A\cos\lambda t + B\sin\lambda t &= \operatorname{Re}\big((A+Bi)\cdot(\cos\lambda t + i\sin\lambda t)\big) \\ &= \operatorname{Re}(C\,e^{i\phi}\cdot e^{i\lambda t}) \\ &= \operatorname{Re}(C\,e^{\lambda t+\phi}) = C\cos(\lambda t+\phi)\ . \end{aligned}$$

3. Complex exponentials

Because of the importance of complex exponentials in differential equations, and in science and engineering generally, we go a little further with them.

Euler's formula (9) defines the exponential to a pure imaginary power. The definition of an exponential to an arbitrary complex power is:

(14) $$e^{a+ib} = e^a e^{ib} = e^a(\cos b + i\sin b).$$

We stress that the equation (14) is a definition, not a self-evident truth, since up to now no meaning has been assigned to the left-hand side. From (14) we see that

(15) $$\operatorname{Re}(e^{a+ib}) = e^a\cos b,\qquad \operatorname{Im}(e^{a+ib}) = e^a\sin b\ .$$

The complex exponential obeys the usual law of exponents:

$$e^{z+z'} = e^z e^{z'}, \tag{16}$$

as is easily seen by combining (14) and (11).

The complex exponential is expressed in terms of the sine and cosine by Euler's formula (9). Conversely, the sin and cos functions can be expressed in terms of complex exponentials. There are two important ways of doing this, both of which you should learn:

$$\cos x = \operatorname{Re}(e^{ix}), \qquad \sin x = \operatorname{Im}(e^{ix}); \tag{17}$$

$$\cos x = \tfrac{1}{2}(e^{ix} + e^{-ix}), \qquad \sin x = \tfrac{1}{2i}(e^{ix} - e^{-ix}). \tag{18}$$

The equations in (18) follow easily from Euler's formula (9); their derivation is left for the exercises. Here are some examples of their use.

Example 3. Express $\cos^3 x$ in terms of the functions $\cos nx$, for suitable n.

Solution. We use (18) and the binomial theorem, then (18) again:

$$\begin{aligned}\cos^3 x &= \tfrac{1}{8}(e^{ix} + e^{-ix})^3 \\ &= \tfrac{1}{8}(e^{3ix} + 3e^{ix} + 3e^{-ix} + e^{-3ix}) \\ &= \tfrac{1}{4}\cos 3x + \tfrac{3}{4}\cos x. \qquad \square\end{aligned}$$

As a preliminary to the next example, we note that a function like

$$e^{ix} = \cos x + i \sin x$$

is a *complex-valued function of the real variable* x. Such a function may be written as

$$u(x) + i\,v(x), \qquad u,\ v \text{ real-valued}$$

and its derivative and integral with respect to x is defined to be

$$D(u+iv) = Du + iDv, \qquad \int (u+iv)\,dx = \int u\,dx + i\int v\,dx. \tag{19}$$

From this it follows by a calculation that

$$D(e^{(a+ib)x}) = (a+ib)e^{(a+ib)x}, \quad \text{and therefore} \quad \int e^{(a+ib)x}\,dx = \frac{1}{a+ib}e^{(a+ib)x}. \tag{20}$$

Example 4. Calculate $\int e^x \cos 2x\,dx$ by using complex exponentials.

Solution. The usual method is a tricky use of two successive integration by parts. Using complex exponentials instead, the calculation is straightforward. We have

$$\begin{aligned} e^x\cos 2x &= \operatorname{Re}\left(e^{(1+2i)x}\right), && \text{by (14) or (15); therefore} \\ \int e^x \cos 2x\,dx &= \operatorname{Re}\left(\int e^{(1+2i)x}\,dx\right), && \text{by (19).}\end{aligned}$$

Calculating the integral,

$$\begin{aligned}\int e^{(1+2i)x}\,dx &= \frac{1}{1+2i}\,e^{(1+2i)x} && \text{by (20);} \\ &= \left(\frac{1}{5} - \frac{2}{5}i\right)(e^x\cos 2x + i\,e^x \sin 2x),\end{aligned}$$

using (14) and complex division (5c). According to the second line above, we want the real part of this last expression. Multiply using (5b) and take the real part; you get

$$\tfrac{1}{5}e^x\cos 2x + \tfrac{2}{5}e^x \sin 2x. \qquad \square$$

In this differential equations course, we will make free use of complex exponentials in solving differential equations, and in doing formal calculations like the ones above. This is standard practice in science and engineering, and you need to get used to it.

4. Finding n-th roots.

To solve linear differential equations with constant coefficients, you need to be able find the real and complex roots of polynomial equations. Though a lot of this is done today with calculators and computers, one still has to know how to do an important special case by hand: finding the roots of

$$z^n = \alpha,$$

where α is a complex number, i.e., finding the n-th roots of α. Polar representation will be a big help in this.

Let's begin with a special case: the **n-th roots of unity**: the solutions to

$$z^n = 1\ .$$

To solve this equation, we use polar representation for both sides, setting $z = re^{i\theta}$ on the left, and using all possible polar angles on the right; using the exponential law to multiply, the above equation then becomes

apply

$$r^n e^{in\theta} \;=\; 1 \cdot e^{(2k\pi i)}, \qquad k = 0, \pm 1, \pm 2, \dots .$$

Equating the absolute values and the polar angles of the two sides gives

$$r^n = 1, \qquad n\theta = 2k\pi\ , \qquad k = 0, \pm 1, \pm 2, \dots ,$$

from which we conclude that

$$(*) \qquad r = 1, \qquad \theta = \frac{2k\pi}{n}\ , \qquad k = 0, 1, \dots, n-1\ .$$

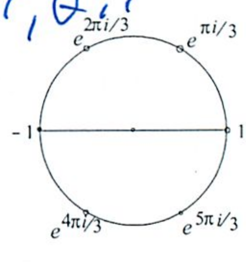
In the above, we get only the value $r = 1$, since r must be real and non-negative. We don't need any integer values of k other than $0, \dots, n-1$ since they would not produce a complex number different from the above n numbers. That is, if we add an, an integer multiple of n, to k, we get the same complex number:

$$\theta' \;=\; \frac{2(k+an)\pi}{n} \;=\; \theta + 2a\pi; \qquad \text{and} \qquad e^{i\theta'} = e^{i\theta}, \quad \text{since } e^{2a\pi i} = (e^{2\pi i})^a = 1.$$

We conclude from $(*)$ therefore that

So just put in r, θ ;;

(21) *the n-th roots of 1 are the numbers* $e^{2k\pi i/n}$, $\quad k = 0, \dots, n-1$.



This shows there are n complex n-th roots of unity. They all lie on the unit circle in the complex plane, since they have absolute value 1; they are evenly spaced around the unit circle, starting with 1; the angle between two consecutive ones is $2\pi/n$. These facts are illustrated on the right for the case $n = 6$.

after that starts repeating ;

From (21), we get another notation for the roots of unity (ζ is the Greek letter "zeta"):

(22) $\quad$ *the n-th roots of 1 are* $\quad 1, \zeta, \zeta^2, \ldots, \zeta^{n-1}, \quad$ where $\zeta = e^{2\pi i/n}$.

We now generalize the above to find the n-th roots of an arbitrary complex number w. We begin by writing w in polar form:

$$w = r\,e^{i\theta}\ ; \qquad \theta = \text{Arg}\ w,\ 0 \le \theta < 2\pi,$$

i.e., θ is the principal value of the polar angle of w. Then the same reasoning as we used above shows that if z is an n-th root of w, then

(23) $\quad z^n = w = r\,e^{i\theta}, \qquad \text{so} \qquad z = \sqrt[n]{r}\,e^{i(\theta+2k\pi)/n}, \qquad k = 0, 1, \ldots, n-1.$

Comparing this with (22), we see that these n roots can be written in the suggestive form

(24) $\quad \sqrt[n]{w} = z_0,\ z_0\zeta,\ z_0\zeta^2,\ \ldots,\ z_0\zeta^{n-1}, \qquad \text{where } z_0 = \sqrt[n]{r}\,e^{i\theta/n}\ .$

As a check, we see that all of the n complex numbers in (24) satisfy $z^n = w$:

$$\begin{aligned} (z_0\zeta^i)^n &= z_0^n\zeta^{ni} = z_0^n \cdot 1^i, && \text{since } \zeta^n = 1, \text{ by (22);} \\ &= w, && \text{by the definition (24) of } z_0 \text{ and (23).} \end{aligned}$$

Example 5. Find in Cartesian form all values of $\quad$ a) $\sqrt[3]{1}$ $\qquad$ b) $\sqrt[4]{i}$.

Solution. a) According to (22), the cube roots of 1 are $1, \omega$, and ω^2, where

So another rule $\qquad$ Simplify

$$\begin{aligned} \omega &= e^{2\pi i/3} &&= \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} &&= -\frac{1}{2} + i\frac{\sqrt{3}}{2} \\ \omega^2 &= e^{-2\pi i/3} &&= \cos\frac{-2\pi}{3} + i\sin\frac{-2\pi}{3} &&= -\frac{1}{2} - i\frac{\sqrt{3}}{2}\ . \end{aligned}$$

The greek letter ω ("omega") is traditionally used for this cube root. Note that for the polar angle of ω^2 we used $-2\pi/3$ rather than the equivalent angle $4\pi/3$, in order to take advantage of the identities

$$\cos(-x) = \cos x, \qquad \sin(-x) = -\sin x\ .$$

Note that $\omega^2 = \bar{\omega}$. Another way to do this problem would be to draw the position of ω^2 and ω on the unit circle, and use geometry to figure out their coordinates.

b) To find $\sqrt[4]{i}$, we can use (24). We know that $\sqrt[4]{1} = 1, i, -1, -i$ (either by drawing the unit circle picture, or by using (22)). Therefore by (24), we get

$$\begin{aligned} \sqrt[4]{i} &= z_0,\ z_0 i,\ -z_0,\ -z_0 i, && \text{where } z_0 = e^{\pi i/8} = \cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\ ; \\ &= a+ib,\ -b+ia,\ -a-ib,\ b-ia, && \text{where } z_0 = a+ib = \cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\ . \end{aligned}$$

Example 6. Solve the equation $x^6 - 2x^3 + 2 = 0$.

Solution. Treating this as a quadratic equation in x^3, we solve the quadratic by using the quadratic formula, the two roots are $1+i$ and $1-i$ (check this!), so the roots of the original equation satisfy either

$$x^3 = 1+i, \qquad \text{or} \qquad x^3 = 1-i\,.$$

This reduces the problem to finding the cube roots of the two complex numbers $1 \pm i$. We begin by writing them in polar form:

$$1+i = \sqrt{2}\,e^{\pi i/4}, \qquad 1-i = \sqrt{2}\,e^{-\pi i/4}\,.$$

(Once again, note the use of the negative polar angle for $1-i$, which is more convenient for calculations.) The three cube roots of the first of these are (by (23)),

← Then other form

$$\begin{aligned}
\sqrt[6]{2}\,e^{\pi i/12} &= \sqrt[6]{2}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)\\
\sqrt[6]{2}\,e^{3\pi i/4} &= \sqrt[6]{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right), \qquad \text{since } \frac{\pi}{12} + \frac{2\pi}{3} = \frac{3\pi}{4};\\
\sqrt[6]{2}\,e^{-7\pi i/12} &= \sqrt[6]{2}\left(\cos\frac{7\pi}{12} - i\sin\frac{7\pi}{12}\right), \qquad \text{since } \frac{\pi}{12} - \frac{2\pi}{3} = -\frac{7\pi}{12}.
\end{aligned}$$

Where is $\frac{2\pi}{3}$ from?

Some thing w/ l_h

The second cube root can also be written as $\sqrt[6]{2}\left(\frac{-1+i}{\sqrt{2}}\right) = \frac{-1+i}{\sqrt[3]{2}}$.

This gives three of the cube roots. The other three are the cube roots of $1-i$, which may be found by replacing i by $-i$ everywhere above (i.e., taking the complex conjugate).

The cube roots can also according to (24) be described as

$$z_1,\ z_1\omega,\ z_1\omega^2 \quad \text{and} \quad z_2,\ z_2\omega,\ z_2\omega^2, \qquad \text{where } z_1 = \sqrt[6]{2}\,e^{\pi i/12}, \quad z_2 = \sqrt[6]{2}\,e^{-\pi i/12}.$$

Exercises: Section 2E

OH Agenda

9/21

- Integrating factor example
  - what w/ ~~ddx~~ $\frac{d}{dx}(\quad)$
- Complex # example
  - ~~xx~~ $z = x + yi$
  - ending last problem

$\boxed{0+1}$ 9/21

4th root of 1

$(r,\theta)^4 = 1$

$r^4 e^{4\theta i} = 1$ ← do this since power expansion is hard

$e^{\theta i} = \cos\theta + i\sin\theta$

$e^{4\theta i} = \cos 4\theta + i\sin(4\theta)$

$r^4 e^{4\theta i} = r^4(\cos 4\theta + i\sin 4\theta)$

now solve $1 = \underbrace{r^4\cos 4\theta}_{\text{like } a} + \underbrace{r^4\sin(4\theta)}_{\text{like } b} i$

want $1 = a + bi$

$1 + 0i = a + bi$

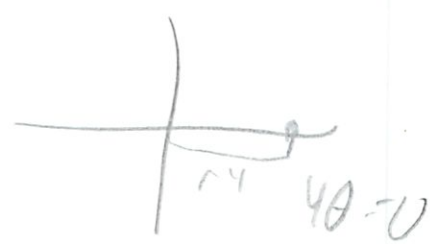
want $a = 1$

$b = 0$

$r^4 \cos 4\theta = 1$

$r^4 \sin 4\theta = 0$

Solve 2 eq 2 unknowns



$r^4 = 1$

$r = \pm 1$ but can only be $+$

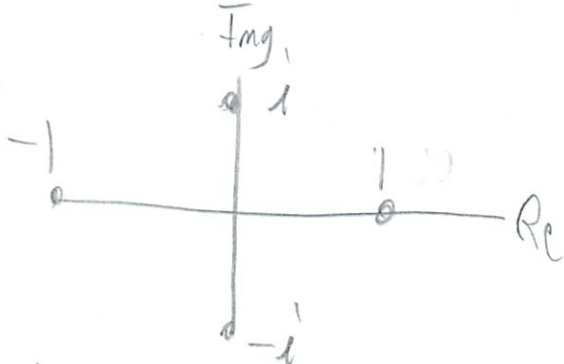
→

$$4\theta = 0 + 2\pi k \qquad k \in \mathbb{Z}$$

$$\theta = \frac{0}{4} + \frac{2\pi k}{4}$$

$$\theta = 0 + \frac{\pi}{2}$$

So



Write real img parts from pic

So these are our 4 roots of r, θ

OH

9/21

$$\frac{dy}{dt} + y = \sin t$$

$P(x) = 1 \quad Q(x) = \sin t$

$e^{\int 1\, dt} = e^t$

$$e^t \frac{dy}{dt} + ye^t = e^t \sin t$$

via product rule magic
- how this is done is magic
- just know can convert

$$\frac{d}{dt}(ye^t) = e^t \sin t$$

↑ if were to expand we would see was equal to via prod rule for derivs

$$\boxed{y \cdot e^t + e^t \cdot \frac{dy}{dt}}$$

Then do

$$ye^t = \int (e^t \sin t)\, dt$$

↑ Could $\int$ by parts
or could do complex exponential trick
↑ why we learned about them

$\sin t = Im(e^{it})$ ← we know since $e^{it} = \cos t + \sin t\, i$

So put in

$$= \int (Im(e^{it}) e^t)\, dt$$

Prod Rule

$$\frac{d}{dt} x \cdot y = x\frac{dy}{dt} + y\frac{dx}{dt}$$

$$\int \frac{d}{dt}(x \cdot y)\, dt = \int \left(x \frac{dx}{dt}\right) dt + \int \left(y \frac{dx}{dt}\right) dt$$

$= x \cdot y$ FTC

Chain rule

$$\frac{d}{dt}(f(g(t))) = g'(t) \cdot f'(g(t))$$

$f = e^x \quad g = 5t$

$5e^{5t} = \int \dots \frac{1}{5} e^{5t}$

↑ forget this

②

Now since e^t is a real # we need to apply it as a factor to <u>both</u> real + imag parts

since $a(b+ci) = (ab) + (ac)i$

$$= \int Im(e^t e^{it})\, dt$$

We can also move Im(part out of integral b/c we can $\int$ then find Im() or find Im() then integrate

$$= Im\left(\int (e^t e^{it}\right) dt$$

$$= Im\left(\int e^{t+it}\right) dt$$

Now actually integrate.

Note chain rule

$\frac{d}{dt} e^{ct} \rightarrow$ $f = e^x$ $f' = e^x$
$g = ct$ $g' = c$

$= c \cdot e^{ct}$

So by FTC

$\underbrace{\int \frac{d}{dt} e^{ct}}_{\text{did above}} = e^{ct}$

Now break it down

$\int c e^{ct}$

Take c outside

$c \int e^{ct}$ we know $= e^{ct}$

Divide by c

$\int e^{ct} = \frac{e^{ct}}{c}$ can also find Integration by parts

(3)

$= \mathrm{Im}\left(\underbrace{\frac{1}{1+i}} e^{t+it}\right)$

this was constant part of $t+it$.

Now need to find Im part, so distributed $\frac{1}{1+i}$

So multiply by $(1-i$

$$\frac{1}{1+i}\frac{(1-i)}{(1-i)} = \frac{1-i}{1^2+0-i^2} = \frac{1-i}{1--1} = \frac{1-i}{2}$$

Just want Im part $\frac{-i}{2}$

$$= \frac{-i}{2} e^{t+it}$$

He had to leave then Is this final ans? No

$$ye^t = \frac{-i}{2} e^{t+it}$$

what is the goal? solve diff eq - so get $\underline{y(t) = \text{something}}$

↑ yes

So $y = \dfrac{\frac{-i}{2} e^{t+it}}{e^t}$ ← think this is wrong

$$= \frac{-i e^{it}}{2}$$

Then expand e^{it}

$$= -\frac{i}{2}(\cos t + \sin(t) i)$$

$$= -\frac{i}{2}\cos t - \frac{(-1)}{2}\sin t$$

$$= \frac{\sin t}{2} - \frac{\cos t}{2} i$$

↑ not what got on WA - is -but signs switched ? ?

18.03 Fall 2011 – Practice Exam I

This exam is shorter than the one you'll take in class on Wednesday, in order to give you time at the end of recitation to discuss the solutions with your TA.

1. Solve the following differential equation exactly by any method:

$$(\cos x \tan y)\frac{dy}{dx} + (\sin x \cos y) = 0$$

2. Suppose we are given the differential equation with

$$\frac{dy}{dx} = f(x,y)$$

with $f(x,y)$ a continuous function with continuous first partial derivatives.

a) If the -1 isocline is $y = e^{-x}$ and the 1 isocline is $y = -e^{-x}$, for which initial values can you determine their long-term behavior? What is their long term behavior? Explain.

b) What if the situation was reversed so that the -1 isocline is $y = -e^{-x}$ and the 1 isocline is $y = e^{x}$. Can you say anything about the long term behavior of any solutions? Explain.

c) If the 2-isocline has equation $y = 2^x$, use Euler's method to approximate the value of the solution $y(3)$ passing through the point $(0,1)$.

3. Meadow flowers have been observed to satisfy the differential equation

$$\frac{dP}{dt} = -P(P-3)^2 + r$$

where $P(t)$ is the size of the flower population (in some appropriate units) and r is the replenishment rate – a constant rate at which we seed the meadow.

a) Draw a picture of the phase line for this differential equation when $r = 0$, indicating whether points are sinks, sources, or neither.

b) Find a value of $r > 0$ at which a bifurcation occurs.

c) What is the long term behavior of solutions for values of r greater than the bifurcation point you found in part (b)?

4. Find a complex differential equation which may be used to solve the ODE:

$$\frac{dA}{dt} + 3A = \sin(2t).$$

Use your answer to find the steady-state solution to this differential equation.

18.03 Practice Test

9/22

1. $(\cos x \tan y)\frac{dy}{dx} + (\sin x \cos y) = 0$

   ↳ complex exp

   ↳ sep of variables (is #1)

$$\frac{\cancel{\cos x}\ \tan y}{\cos y}\frac{dy}{dx} = -\frac{\cos x}{\sin x}$$

$\int$

1. Sepration variables

X on 1 side

Y on other

$\int$ and it works

Answers will be posted

- he does not want to do trig now

Don't need to write ans explicitly in terms of y

(I never ~~need~~ know when I am wrong

and when its not possible.

2. $\frac{dy}{dx} = f(x,y)$

{ Contineous Function, contineous 1st partial derivs

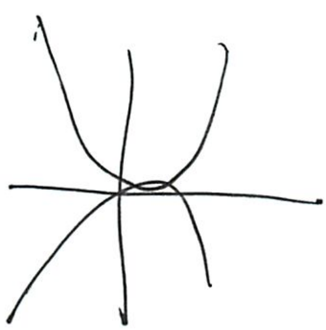
-1 iso is $y = e^{-x}$

1 $y = -e^{-x}$

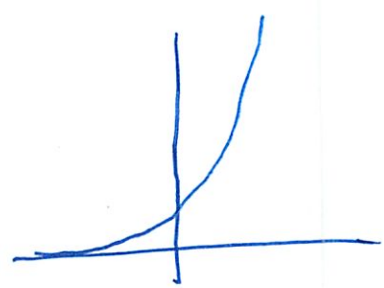
What vales can detemine long-term behavior

Well iso clines are lines where slope $= 1$

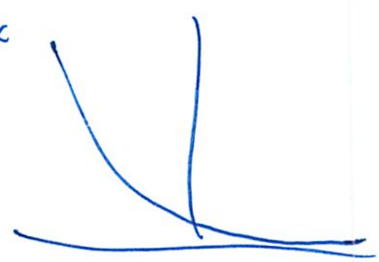
What is e^{-x} visually?



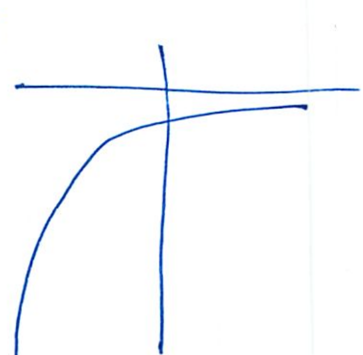
So $y = e^{x}$



$y = e^{-x}$

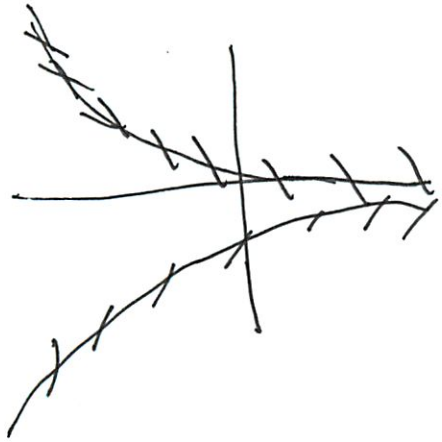


$y = -e^{-x}$



Learn the conventions

(2)



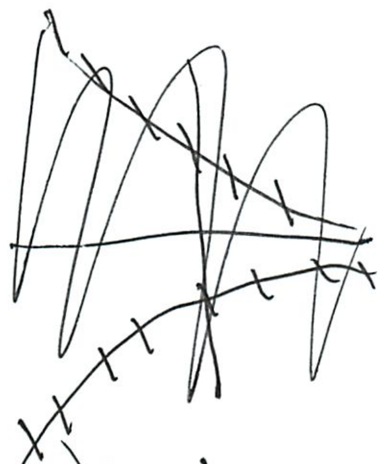
← yes Converges to 0

funnel (~~unit~~ lecture which I skipped)

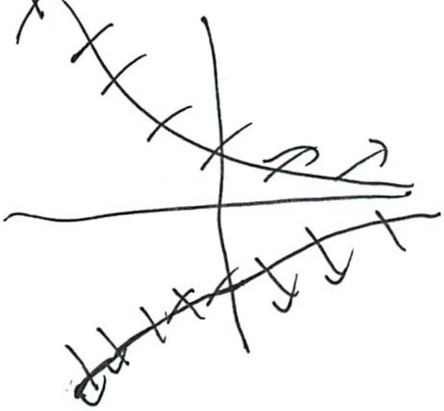
So all that was holding me back here was graph of e^—.

I guess could try to manually ~~plot~~ plot points.

b)



antifunnel



3

c) 2-isocline $y = 2^x$

Use Euler's method to find $y(3)$ through $(0,1)$

[Seperate problem - could try even if did not do above)

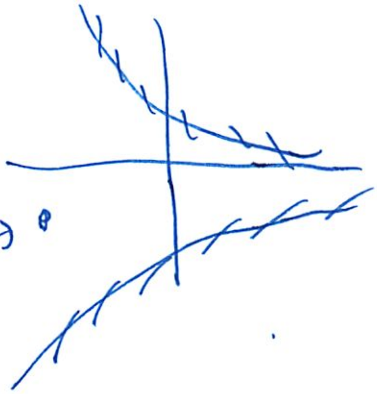
So

x	y
0	1
1	$1+2=3$
2	$3+4=7$
3	$7+8=15$

← slope along $y=2^x = 2$??

? actually can't
- only 1 step size where know slope

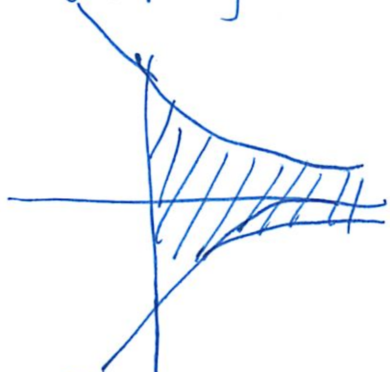
2. $\frac{dy}{dx} = f(x,y)$



Don't know this sol

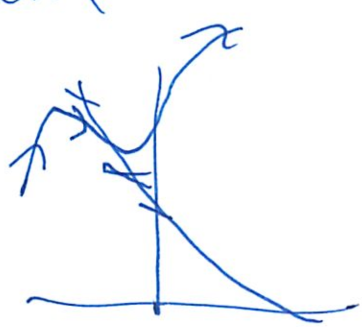
Not all solutions converge to 0

know everything with $x > 0$



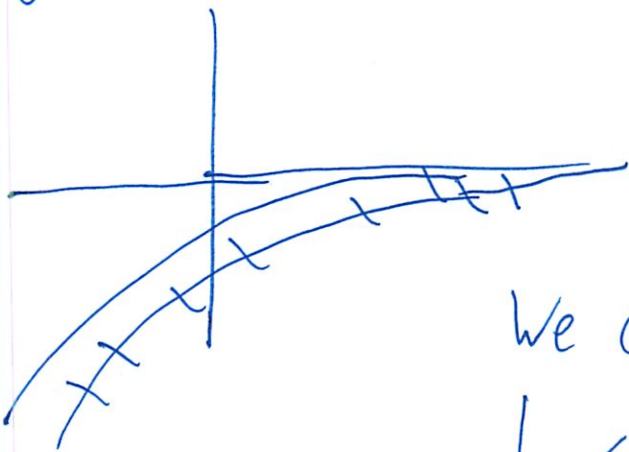
? here downward slope is greater than slope of line

Can't



b) Otherway don't have to go out

0

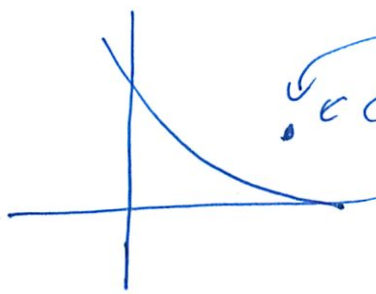


Don't know is antifunnel
Could be a funnel inside

We can't say anything

1 sol does converge to 0

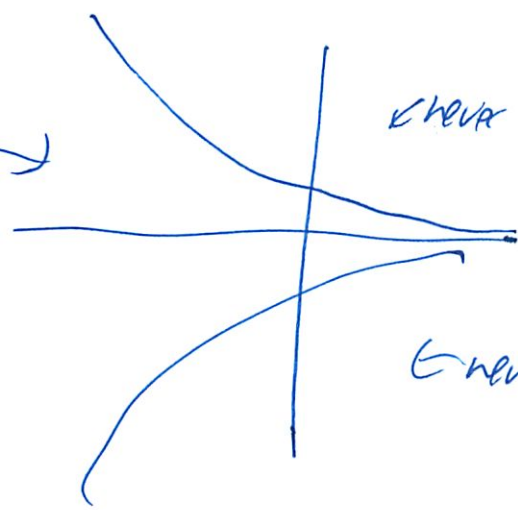
Could say lower fence for



← can't become negative
We can say that

(Much more complex than I thought)

Can't say anything →



← never 0

← never 0

3. Medow Flowers

$$\frac{dP}{dt} = -P\,(P-3)^2 + r$$

? flower pop ? replenishment rate

a phase line - What is this again ← this

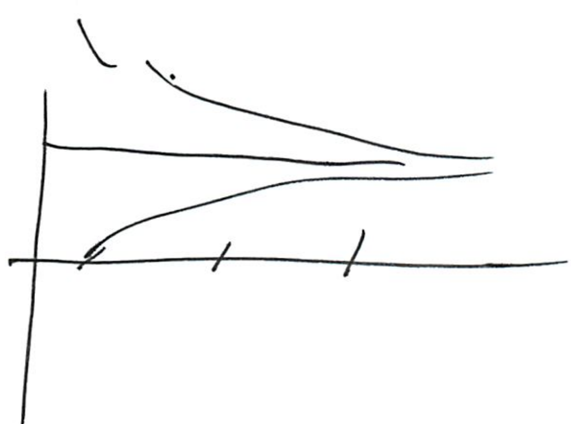
Can plot

$\frac{dP}{dt}$	0	1	2	3	4 P
0	0	-4	-2	0	4
1	1	-3	-1	0	5
2	2	-2	0	0	6
3	3	-1	1	0	7
~~$\frac{dP}{dt}$~~ r					

← ? but where time

Or put in r

· r is offset



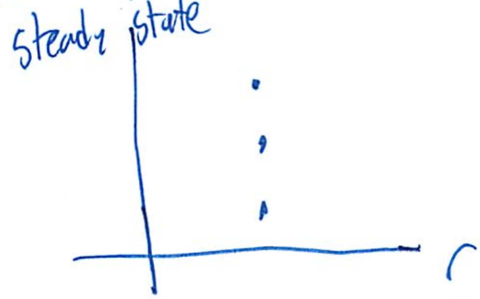
← long term behavior

b) So at r=3

c) To 3 (the bifurcation pt)

- ? not sure if did correctly
  - but think write track

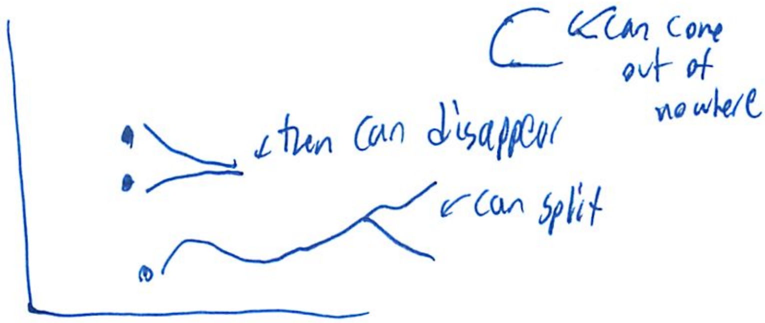
3. Bifurcation diagram

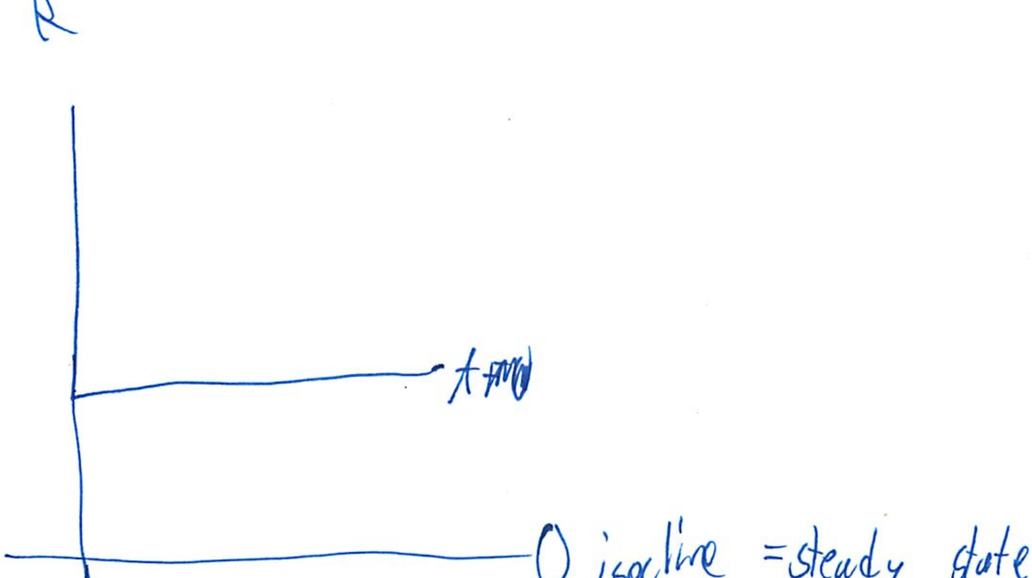


↑ parameter that varies

Only eq where t not on RHS

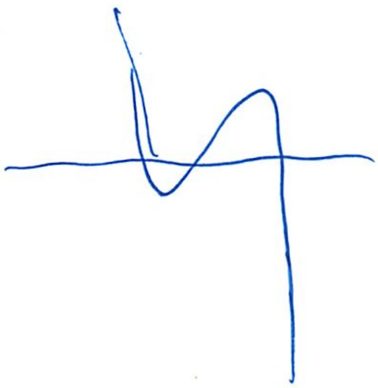
Can com together





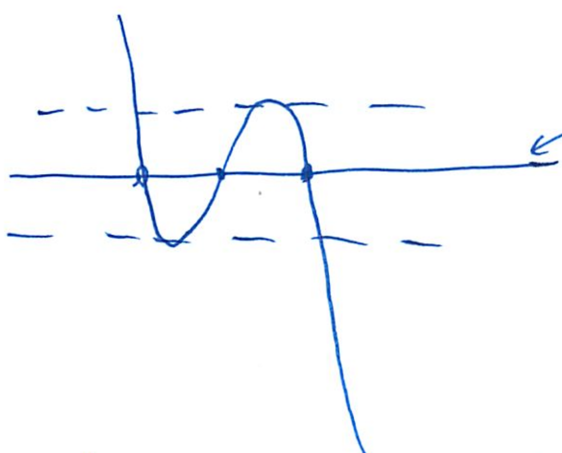
②

$$\frac{dP}{dt} = \underbrace{-p(p-3)^2 + 2}_{\text{want roots of polynomial of } p}$$



r shifts up + down
along x axis

So look at local mins / maxes



← roots of 0s
- the 0-isoclines

How to solve for r ?
- don't need to do

When does something happen to roots
- at local min / maxes

4. Find a complex diff eq
(why so many complex qv!

$$\frac{dA}{dt} + 3A = \sin(2t)$$

So this is that IF method
Going to look this up

$P(x) = 3 \quad Q = \sin 2t$

$e^{\int 3dt} = e^{3t}$

$$e^{3t}\frac{dA}{dt} + 3Ae^{3t} = e^{3t}\sin(2t)$$

$$\frac{dA}{dt}\left(3Ae^{3t}\right) = \int e^{3t}\sin 2t$$

$$= \int e^{3t}\, Im\left(e^{2t}\right)$$

$$3Ae^{3t} = \frac{Im\left(e^{5t}\right)}{5t}$$

$$A = \frac{Im\left(e^{5t}\right)}{5t \cdot 3e^{3t}}$$

(2) (Doing kinda fast + loose)

~~II~~ Need to distribut fraction

$$\frac{1}{15te^{5t}} \cdot \cos 3t - \sin 3t\, i$$

(Or am I confusing 2 diff problems?)

$$\frac{\cos 3t - \sin 3t\, i}{15t(\cos 5t - \sin 5t\, i)}$$

$$\frac{\cos 3t - \sin 3t}{15t\cos 5t} - \frac{\cancel{\cos 3t} \sin 3t\, i}{15t\sin 5t\, i}$$

$$\frac{-\sin 3t\cancel{i}}{15t\sin 5t\cancel{i}}$$

$$\frac{-\sin 3t}{15t\sin 5t}$$

? can reduce further ?? forget rules

4, $dA + 3A = \sin(2t)$

$If = e^{3t}$

$$e^{3t} \frac{dA}{dt} + e^{3t}\, 3t = \sin(2t)\, e^{3t}$$

$$\int \frac{d}{dt}\left(A e^{3t}\right) = \int \sin(2t)\, e^{3t}\, dt$$

↓ I forgot i

$$A e^{3t} = Im\left(e^{2ti + 3t}\, dt\right)$$

18.03 Fall 2011 – Practice Exam I With Solutions

This exam is shorter than the one you'll take in class on Wednesday, in order to give you time at the end of recitation to discuss the solutions with your TA.

1. Solve the following differential equation exactly by any method:

$$(\cos x \tan y)\frac{dy}{dx} + (\sin x \cos y) = 0$$

Solution: After a bit of algebra, we may rewrite the equation as

$$\frac{dy}{dx} = -\frac{\sin x \cos y}{\cos x \tan y}$$

and this is a separable equation which may be further rewritten as follows:

$$\frac{\sin y}{\cos^2 y}\frac{dy}{dx} = -\frac{\sin x}{\cos x}.$$

Of course, the right-hand side is just $-\tan x$ but since we're about to integrate, we leave it in terms of sine and cosine. Integrating both sides with respect to x:

$$\int \frac{\sin y}{\cos^2 y}\,dy = \int -\frac{\sin x}{\cos x}\,dx.$$

Then using a change of variables with cosine on both sides and simplifying:

$$y = \operatorname{arcsec}(\ln(\cos x) + c)$$

2. Suppose we are given the differential equation with

$$\frac{dy}{dx} = f(x, y)$$

with $f(x, y)$ a continuous function with continuous first partial derivatives.

a) If the -1 isocline is $y = e^{-x}$ and the 1 isocline is $y = -e^{-x}$, for which initial values can you determine their long-term behavior? What is their long term behavior? Explain.

b) What if the situation was reversed so that the -1 isocline is $y = -e^{-x}$ and the 1 isocline is $y = e^{-x}$. Can you say anything about the long term behavior of any solutions? Explain.

c) If the 3/2-isocline has equation $y = 2^x$, use Euler's method to approximate the value of the solution $y(4)$ passing through the point $(0, 1)$.

Solution: For part (a), the condition on a curve $y = C(x)$ being an upper fence is that $C'(x) > f(x, C(x))$. But $d/dx(e^{-x}) = -e^{-x} > -1$ if $x > 0$, so e^{-x} is an upper fence for these x values. Similarly $-e^{-x}$ is a lower fence since $e^{-x} < 1$ for $x > 0$. Together, they form a narrowing funnel. So for all initial values beginning in this funnel, the corresponding solutions are asymptotic to 0. (We can't say more because we don't have any more information about the function $f(x, y)$.)

For part (b), with the isoclines reversed, now e^{-x} has derivative $-e^{-x} < 1$ for all $x > 0$ so is a lower fence. Likewise, $-e^{-x}$ has derivative $e^{-x} > -1$ for $x > 0$, so forms an upper fence. Together, they form a narrowing antifunnel, so there exists some solution which remains inside it (and hence has long term behavior approaching 0). Note that we can't say that this solution is *unique* since we are told nothing about $\partial f/\partial y$ (which is required ≥ 0 for uniqueness).

For (c), we only know information about the 3/2 isocline. Since $(0, 1)$ is on the isocline $(2^0 = 1)$, we know the slope field at this point has value 3/2. Then flowing along the tangent line with step size 4, we get approximate y-value $y(4) \approx 1 + 3/2(4) = 7$.

3. Hothouse violets have been observed to satisfy the differential equation

$$\frac{dP}{dt} = -P(P-3)^2 + r$$

where $P(t)$ is the size of the flower population (in some appropriate units) and r is the replenishment rate – a constant rate at which we seed the hothouse beds.

a) Draw a picture of the phase line for this differential equation when $r = 0$, indicating whether points are sinks, sources, or neither.

b) Find a value of $r > 1$ at which a bifurcation occurs.

c) What is the long term behavior of solutions for values of r greater than the bifurcation point you found in part (b)?

Solution: When $r = 0$, the roots of the autonomous equation are just $P = 0$ and $P = 3$, so these are the vertices on our phase line. The value of dP/dt is negative if $P > 0$ and positive if $P < 0$ so $P = 0$ is a source while $P = 3$ is neither.

Our function $f(P)$ is cubic, and for polynomial equations, bifurcation points occur where the polynomial has one or more double roots. Thus $r = 0$ is a bifurcation point, but the question asks for a bifurcation point $r > 1$. Drawing a graph of the cubic, we see that a double root will occur if we shift the cubic up by r units, where $-r$ is equal to the value of f at the local minimum. A little calculus shows the local minimum of $f(P)$ is at $(t, P) = (1, -4)$ so setting $r = 4$ gives a bijection.

If $r > 4$, we can redraw the phase line to see that the only root is a number bigger than 3. It is a sink, so all solutions tend to this unique root, regardless of initial condition.

4. Find a complex differential equation which may be used to solve the ODE:

$$\frac{dA}{dt} + 3A = \sin(2t).$$

Use your answer to find the steady-state solution to this differential equation.

Solution: We complexify to

$$d\tilde{A}/dt + 3\tilde{A} = e^{(2t)i}$$

and plan to take the imaginary part of both sides, where $\text{Im}(\tilde{A}) = A$, since $e^{(2t)i} = \cos 2t + i \sin 2t$. Now the integration step is simple, using the integrating factor e^{3t}:

$$d/dt(\tilde{A}e^{3t}) = e^{(3+2i)t}$$

so that integrating with respect to t, we get

$$\tilde{A}e^{3t} = \frac{1}{3+2i}e^{(3+2i)t} + c$$

for a complex constant c, so

$$A = \mathrm{Im}\left(\frac{1}{3+2i}e^{(2i)t}\right) + ce^{-3t}$$

for a real constant c.

We put $3+2i$ in polar form $re^{i\theta}$ with $r = \sqrt{3^2+2^2} = \sqrt{13}$ and $\theta = \tan^{-1}(2/3)$ and then reciprocate. So we are left to compute:

$$\mathrm{Im}\left(\frac{1}{3+2i}e^{(2i)t}\right) = \mathrm{Im}\left(\frac{1}{\sqrt{13}}e^{-\tan^{-1}(2/3)}e^{(2i)t}\right) = \frac{1}{\sqrt{13}}\sin(2t - \tan^{-1}(2/3)).$$

Thus the general solution is:

$$A(t) = \frac{1}{\sqrt{13}}\sin(2t - \tan^{-1}(2/3)) + ce^{-3t}$$

and the steady-state solution is the one with $c = 0$. Recall that ce^{-3t} is called the transient solution as it tends to 0 as $t \to \infty$.

18.03 FALL 2011 – Problem Set 2

Due Friday 9/23/11, high noon in 2-114

To encourage you to keep up with homework as it appears in lecture, both Part I and Part II problems are listed with the accompanying lecture in which the material will be covered.

Part I (18 points)

Lecture 5. Fri. Sept. 16 Introduction to complex numbers.
Read: 3.5, Notes C.1–3, IR.1 HW: 2E 1, 2, 4, 14, 16. (Note: 2E-4b should read: $\overline{zw} = \overline{z}\,\overline{w}$.)

Lecture 6. Mon. Sept. 19: Complex Exponentials.
Read: Notes C.3–4 HW: 2E 9, 10, 15, 17

Wed. Sept 21 – No Class (Student Holiday)

Lecture 7. Fri., Sept. 23: Input-Response Models.
Read: EP 1.1–1.4, Notes IR HW: To be assigned on the next Pset

Part II (23 points)

0. (3 points) Write the names of all the people you consulted or with whom you collaborated and the resources you used, or say "none" or "no consultation". This includes visits outside recitation to your recitation instructor. If you don't know a name, you must nevertheless identify the person, as in, "tutor in Room 2-106," or "the student next to me in recitation." Optional: note which of these people or resources, if any, were particularly helpful to you.

1. (Fri, 6 pts)

a) Given a complex number $z = x+iy$ express the real and imaginary parts of the following complex numbers in terms of x and y:

$$z^4, \quad \frac{1}{z}, \quad \frac{z-1}{z+1}, \quad \frac{1}{z^2}$$

b) Show that

$$\left|\frac{a-b}{1-\bar{a}b}\right| = 1$$

if either $|a| = 1$ or $|b| = 1$. What exception, if any, must be made if $|a| = |b| = 1$?

c) Find an expression for $\sin 5t$ in terms of powers of $\sin t$ and $\cos t$ using Euler's formula and the binomial theorem.

2. (Mon, 6 pts) Determine the curves traced out by the following equations. Be sure to justify your answers by demonstrating the geometric properties of each curve. (E.g. if a horizontal line, show it has constant slope equal to 0.)

a) $f(t) = i + (4+i)t$ with t any real number.

b) $g(t) = \sin t + i\cos t$ with t any real number.

c) $h(t) = \dfrac{1+it}{1-it}$ with $t \in [0, \infty)$.

3. (Mon, 8 pts) This question uses the mathlet called "Complex Exponential." Use it, together with Euler's formula $e^{(a+bi)t} = e^{at}(\cos(bt) + i\sin(bt))$ to answer the following questions.

a) For what values of $a+bi$ is the curve $e^{(a+bi)t}$ a circle?

b) For what values of $a+bi$ is the curve $e^{(a+bi)t}$ a ray? Which rays are possible?

c) For what values of $a+bi$ does the curve $e^{(a+bi)t}$ converge to 0 as $t \to \infty$?

d) For what values of $a+bi$ is the curve $e^{(a+bi)t}$ a spiral moving counterclockwise away from the origin as t increases?

e) Given a non-zero complex number $a+bi$, for what values of z is the curve traced out by e^{zt} the same as the curve traced out by $e^{(a+bi)t}$? Explain.

Michael Plasmeier

18.03 P-Set 2

9/21

$\textcircled{-8} = 33/41$

Part 1

Lecture 5

2E #1. Change to Polar form

$$\underbrace{-1}_{x} \underbrace{+i}_{y}$$

$$r = \sqrt{x^2+y^2}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$r = \sqrt{(-1)^2+(1)^2} = \sqrt{2}$$

$$\theta = \tan^{-1}\left(-\frac{1}{1}\right) = -\frac{\pi}{4}$$

$$\left(\sqrt{2}, \frac{\pi}{4}\right)$$ ✓

write solution in $Ae^{i\theta}$ form.

#2. $\sqrt{3} - i$

$$r = \sqrt{\sqrt{3}^2 + (-1)^2} = \sqrt{4} = 2$$

$$\theta = \tan^{-1}\left(\frac{-1}{\sqrt{3}}\right) = -\frac{\pi}{6}$$

$$\left(2, \frac{\pi}{6}\right)$$ ✓

#4 Prove for any comp conj $\overline{z+w} = \bar{z} + \bar{w}$

(Not true w/ circuits!) ✓

Complex conjugate: 2 # w/ same real, imag have opposite signs

(2) So $z = a + ib$

$\bar{z} = a - ib$

i Are z, w complex conjugates together - no

Adding complex #

$(a+bi) + (c+di) = (a+c) + (b+d)i$

So let $z = a + ib$

$w = c + id$

Then $\bar{z} = a - ib$

$\bar{w} = c - id$

So $\bar{z} + \bar{w}$ is

$(a+c) + (-b + -d)i$

$z + w$ is

$(a+c) + (b+d)i$

Then — that

$(a+c) - (b+d)i$

Now show

$-(b+d) = (-b + -d)$

(3)

Since $-b-d = -(b+d)$

So proved. QED.

b) $\overline{zw} = \overline{z} \cdot \overline{w}$ ← note correction in notes

Multiplication rule

$$(a+bi)\cdot(c+di) = (ac-bd)+(ad+bc)i$$

Again $z = a+bi$

$w = c+di$

$\overline{z} = a-bi$

$\overline{w} = c-di$

So $\overline{z}\cdot\overline{w}$ is

$$(ac-(-b)(-d))+(a(-d)+(-b)c)i$$

Simplify

$$(ac-bd)+(-ad-bc)i$$

$$(ac-bd)-(ad+bc)i$$

$z\cdot w$ is

$$(ac-bd)+(ad+bc)i$$

$\overline{z\cdot w}$ is $(ac-bd)-(ad+bc)i$

QED

(4)

14. Express $\sin^4 x$ in terms of $\cos 4x$ and $\cos 2x$ using (18) and binomial theorm. Why would you not expect $\sin 4x$ or $\sin 2x$ in the answer?

(18) $\cos x = \frac{1}{2}(e^{ix} + e^{-ix})$

(BT) Expanding $(x+y)^n$ ie $(x+y)^4 = x^3 + x^2 y + x y^2 + y^3$

$\sin^4 x$

So $\sin x = \frac{1}{2}(e^{ix} - e^{-ix})$

$$\sin x^4 = \left(\frac{e^{ix} - e^{-ix}}{2i}\right)^4$$

Binomial theorm

$$= \frac{1}{16}\left(e^{4ix} - 4e^{3ix}e^{-ix} + 6e^{2ix}e^{-2ix} - 4e^{ix}e^{-3ix} + e^{-4ix}\right)$$

$$= \frac{1}{16}\left(e^{4ix} + e^{-4ix}\right) - \frac{4}{16}\left(e^{2ix} - 2ix\right) \cdot \frac{6}{16} \; 01$$

$$= \frac{1}{8}\cos 4x - \frac{1}{2}\cos 2x + \frac{3}{8}$$ ✓

So $\sin^4 x$ is even, than answer can not contain odd fns $\sin 4x$, $\sin 2x$

(5)

16.a) Prove (18) $\cos x = \frac{1}{2}(e^{ix} + e^{-ix})$

So (18) $\cos x = \frac{1}{2}(e^{ix} + e^{-ix})$

$e^{ix} = \cos x + i\sin x$

$e^{-ix} = \cos x - i\sin x$

Don't need

$\boxed{\begin{matrix}\cos(-x) = \sin x \\ \sin(-x) = -\sin(x)\end{matrix}}$

I need to put on formula sheet

Then add them

$e^{ix} + e^{-ix}$

$\cos x + i\sin x + \cos x - i\sin x$

$2\cos x$

So $\cos x = \frac{1}{2}(e^{ix} + e^{-ix})$ ✓

b) $\sin x = \frac{1}{2i}(e^{ix} - e^{-ix})$

So $e^{ix} - e^{ix}$ is

$\cos x + i\sin x - \cos x + i\sin x$

$2i\sin x = e^{ix} - e^{ix}$

So $\sin x = \frac{1}{2i}(e^{ix} - e^{ix})$ ✓

(6)

Lecture 6 2E #9. Express $a+bi$ the 6 sixth roots of 1

So $\left(e^{2\pi i/6}\right)^6$

with $k=0,\ldots,5$

? but where does k go in here?

So $r^5=1$

$\theta = \frac{2\pi}{6} \pm \frac{k2\pi}{6}$

So $\theta = \underset{0}{\frac{2\pi}{6}}, \underset{1}{\frac{4\pi}{6}}, \underset{2}{\frac{6\pi}{6}}, \underset{3}{\frac{8\pi}{6}}, \underset{4}{\frac{10\pi}{6}}, \underset{5}{\frac{12\pi}{6}}$

So together

$1e^{\frac{\pi}{3}i}, e^{\frac{2\pi}{3}i}, \underset{-1}{e^{\pi i}}, e^{\frac{4\pi}{3}}, e^{\frac{5\pi}{3}}, \underset{1}{e^{2\pi}}$

which matches ✓

✓

⑦

#10 Solve $x^4 + 16 = 0$

$$x^4 = -16$$

Vivek
OH

$x = \sqrt[4]{-16}$ ← don't do

Want 4th roots for -16

Before it was 1

Answer will be $(r, \theta)^4 = -16$

So just replace the 1 w/ -16 from OH problem

$$r^4 e^{4\theta i} = -16$$

$$e^{\theta i} = \cos\theta + i\sin\theta$$

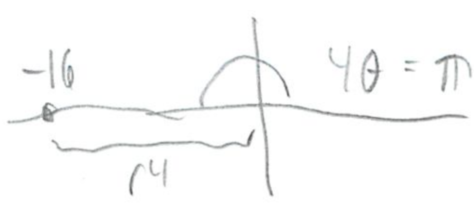
$$e^{4\theta i} = \cos 4\theta + i\sin 4\theta$$

$$r^4 e^{4\theta i} = r^4(\cos 4\theta + i\sin 4\theta)$$

$$-16 = \underbrace{r^4\cos 4\theta}_{a} + i\,\underbrace{\sin(4\theta)r^4}_{b}$$

want $a = r^4\cos 4\theta = -16$

$b = r^4\sin 4\theta = 0$



$4\theta = \pi$

8

$r^4 = -16$

$r = \sqrt[4]{-16}$

but 2 or -2 will be 16

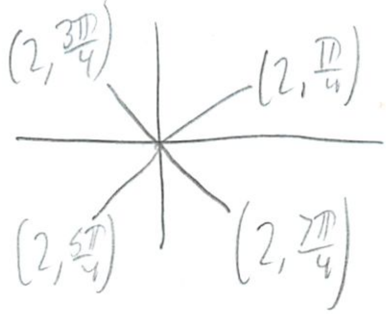
but r must be (+)

$r = 2$

$4\theta = \pi + 2\pi k$

$\theta = \frac{\pi}{4} + \frac{2\pi k}{4}$

$= \frac{\pi}{4} + \frac{\pi k}{2}$



✓ ok.

✓TA approved

#15) Find $\int e^{2x} \sin x \, dx$ using complex exponentials

- Did part of this in OH

- Could do integration by parts - but long + painful

- But $\sin x = \text{Im}(e^{ix})$ we can use

  └ since $e^{ix} = \cos x + \sin x \, i$

$= \int \left(e^{2x} \, \text{Im}(e^{ix})\right) dx$

Now since e^{2x} is real, need to distribute it to real + imag - but we only care img

$\int \text{Im}(e^{2x} e^{ix}) \, dx$

(9)

Also can move Img out as can Integrate first

$= \operatorname{Im}\left(\int\left(e^{2x} e^{ix}\right) dx\right)$

$= \operatorname{Im}\left(\int e^{2x+ix}\right) dx\Big)$

Now actually integrate, via chain rule

$f = e^x \qquad f' = e^x$

$g = 2x + ix \qquad g' = 2 + i$

$= \operatorname{Im}\left(\left[\frac{2}{5} - \frac{i}{5}\right] e^{2x+ix}\right)\Big)$

$= \frac{-i}{5} e^{2x+ix}$

is all we care about

-1

done ?

#17 | Derive formula (20) $D\left(e^{(a+ib)x}\right) = (a+ib)\, e^{(a+ib)x}$

from the def'n of a complex exponential and the derivitive formula (10) $D(u+iv) = Du + i\,Dv$

~~So~~ $\cancel{e^{(a+ib)x} = e^{ax} e^{ibx}}$

~~If we take derivitive of this we get (via prod rule)~~

$\cancel{e^{ibx} e^{ax} + a e^{ax} e^{ibx}}$

If we take deriv of LHS by chain rule

$f = e^x$

$g = (a+ib)x$

10

$(a+ib)x \quad e^{(a+ib)x}$

~~This is~~

$\cancel{(a+ib)x(2e^{ax}e^{ibx})}$

This is because $D(u+iv) = Du + iDv$

QED

(ii)

11

Part 2

0. Vivek OH for Part 1 - very helpful :)

Phil Reiser
Matt Fawk ✓

1. Given a complex # $z = x + iy$ express the real imag parts in terms of x and y

z^4)

$$z^4 = (x + iy)^4$$

$(x+iy)(x+iy)(x+iy)(x+iy)$

$(x^2 + 2iy + i^2y^2)(x^2 + 2iy + i^2y^2)$

$\cancel{x^4} + \cancel{x^2 2iy} + \cancel{x^2 i^2} y^2 + \cancel{2iy} x^2 + (2iy)^2 + y^4$

$x^4 + 4x^2 iy - x^2y^2 - 4y^2 + y^4$

Re $x^4 - x^2y^2 - 4y^2 + y^4$

Im $4x^2yi$

-.25

x Algebra!

$\frac{1}{z} = \frac{1}{x+yi}$

$$\frac{1}{x+yi}\frac{(x-yi)}{(x-yi)} = \frac{x-yi}{x^2 + x\cdot yi + xyi - y^2i^2}$$

$$= \frac{x-yi}{x^2+y^2}$$

Re $\frac{x}{x^2+y^2}$ ✓ Im $\frac{-yi}{x^2+y^2}$ ✓

(12)

$$\frac{z-1}{z+1} = \frac{x+yi-1}{x+yi+1} \frac{(x-yi+1)}{(x-yi+1)}$$

$$= \frac{x^2 - \cancel{xyi} + \cancel{x} + \cancel{yix} - y^2i^2 - 1 + \cancel{x} + yi - 1}{x^2 - \cancel{xyi} + \cancel{x} - xyi - y^2i^2 + yi + x + yi + 1}$$

$$= \frac{x^2 + 2x + y^2 - yi - 2}{x^2 - 2xyi + 2x + yi - y^2 i^2 + 1}$$

Real?
Im?

(-.5)

$$\frac{1}{z^2} = \frac{1}{(x+yi)^2} = \frac{1}{(x+yi)(x+yi)}$$

$$= \frac{1}{x^2 + 2xyi + y^2i^2}$$

$$= \frac{1}{x^2 + 2xyi - y^2} \frac{(x^2 - 2xyi + y^2)}{(x^2 - 2xyi + y^2)}$$

$$= \frac{x^2 - 2xyi + y^2}{x^4 - 2xyx^2 + x^2y^2 + x^2 2xyi - 4x^2y^2i^2 + y^2 2xyi - x^2y^2 - 2xy^2i - y^4}$$

R()?
Im()?

(-.5)

13

1b. Show that $\left| \dfrac{a-b}{1-\bar{a}b} \right| = 1$

if either $|a| = 1$ or $|b| = 1$

What exception must be made if $|a| = |b| = 1$

-2

(14)

1c. Find expression for $\sin 5t$ in terms of $\sin t$, $\cos t$ using Euler + B.T.

$\sin 5t$ is $\mathrm{Im}\left(e^{5\theta}\right)$

$\overset{\mathrm{Im}}{=}\left(e^{it}\right)^5$

$\overset{\mathrm{Im}}{=}\left(\cos t + i\sin t\right)^5$

$= \mathrm{Im}\Big[(\cos t)^5 + (\cos t)^4 \sin t \, i + (\cos t)^3(\sin t)^2 i^2$
$+ (\cos t)^2(\sin t)^3 i^3 + \cos t(\sin t)^4 i^4 + (\sin t)^5 i^5\Big]$

$= (\cos t)^4 \sin t + (\cos t)^2(\sin t)^3 + (\sin t)^5$

coefficient ?

(-.5)

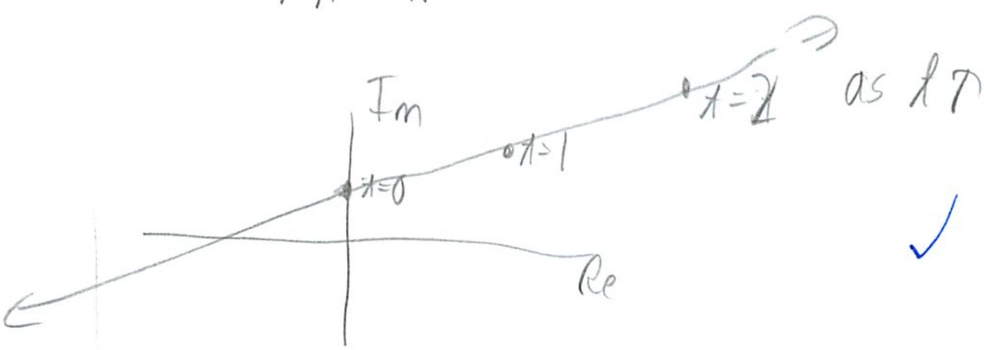
(15)

2. Determine curve traced out by equation.

a. $f(t) = i + (4+i)t$ w/ t any real #

Just manually plot - make table

$= 4t + it + i$



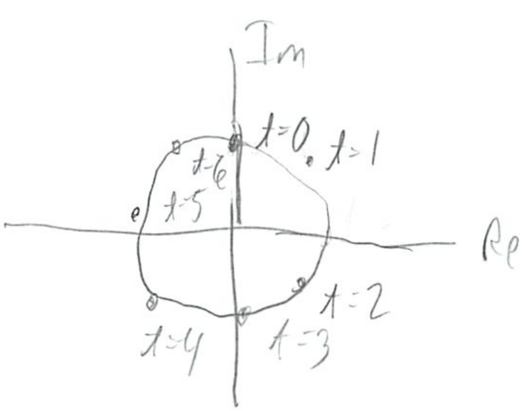
Can see linear

$4t$ = horiz change

$(t+1)i$ vertical change

b) $g(t) = \sin t + i\cos t$

degrees or radians



(1b)

Circle

Because $\sin t + i\cos t$ controls angle ✓

c) $h(t) = \dfrac{1+it}{1-it}$ w/ $t \in [0,\infty)$

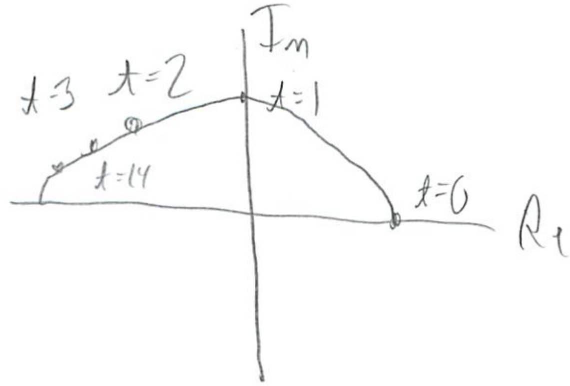
Split

$$= \frac{1+it}{1-it}\,\frac{(1+it)}{(1+it)}$$

$$= \frac{1^2+2it+i^2t^2}{1^2-it+it-i^2t^2}$$

$$= \frac{1+2it-t^2}{1+t^2}$$

$$= \frac{-t^2+1}{1+t^2} + \frac{2it}{1+t^2}.$$



t	Re	Im
0	1	0
1	0	1
2	$-\frac{3}{5}$	$\frac{4}{5}$
3	$-\frac{4}{5}$	$\frac{3}{5}$
4	$-\frac{15}{17}$	$\frac{8}{17}$
5	$-\frac{12}{13}$	$\frac{5}{13}$

✓

17

3. Mathlet "Complex Exponential"

Use with Euler $e^{(a+bi)t} = e^{at}(\cos(bt) + i\sin(bt))$

a) For what values of $a+bi$ is curve $e^{(a+bi)t}$ a circle?

So the default $a=0$

$b=2$ ← can be any value

$b \neq 0$,

b) Ray If $b=0$ and $a =$ anything

all rays w/ slope 0 $a \neq 0$

c) Converge as $t=0$

↓ negitive

So when it is a spiral (a b/w -1 and 0)

B between -8 and 0

Converges towards center

-1

d) when move away from origin

when a between 0 and 1

B between -8 and 0 still

-1

(18)

e) Given a non 0 complex # $a+bi$
for what values of z is curve traced out
by e^{zt} the same as traced out by $e^{(a+bi)t}$

Well when $z = (a+bi)$??

$z = r(\cos\theta + \sin\theta i)$

$e^{i\theta} = \cos\theta + \sin\theta\, i$

$z = r_1 e^{i\theta}$

ED.

So $a = r\cos\theta$
$b = r\sin\theta$

PSET 2. Solutions. 18.03. Fall 2011

Problem 1.a $z = x+iy$.

$$z^4 = (x+iy)^4 = x^4 + 4x^3(iy) + 6x^2(iy)^2 + 4x^1(iy)^3 + (iy)^4$$

$$\Rightarrow \begin{cases} Re(z^4) = x^4 - 6x^2y^2 + y^4 \\ Im(z^4) = 4x^3y - 4xy^3 \end{cases}$$

$$\frac{1}{z} = \frac{\bar{z}}{|z|^2} \Rightarrow \begin{array}{l} Re(1/z) = Re(\bar{z})/|z|^2 = x/(x^2+y^2) \\ Im(1/z) = Im(\bar{z})/|z|^2 = -y/(x^2+y^2) \end{array}$$

$$\frac{z-1}{z+1} = \frac{(z-1)\overline{(z+1)}}{(z+1)\overline{(z+1)}} = \frac{z\bar{z} + (z-\bar{z}) - 1}{|z+1|^2} = \frac{|z|^2 - 1 + i\,2\,Im(z)}{|z+1|^2}$$

$$\Rightarrow Re\left(\frac{z-1}{z+1}\right) = (x^2+y^2-1)/[(x+1)^2+y^2]$$

$$Im\left(\frac{z-1}{z+1}\right) = (2y)/[(x+1)^2+y^2]$$

$$\frac{1}{z^2} = \frac{(\bar{z})^2}{(z\bar{z})^2} = \frac{1}{|z|^4}(x-iy)(x-iy) = \frac{(x^2-y^2) - 2i(xy)}{|z|^4}$$

$$Re(1/z^2) = (x^2-y^2)/(x^2+y^2)^2$$

$$Im(1/z^2) = -2xy/(x^2+y^2)^2$$

Problem 1.b Show that $\left|\frac{a-b}{1-\bar{a}b}\right| = 1$ if either $|a|=1$ or $|b|=1$. What exception, if any, must be made if $|a|=|b|=1$?

Solution:

$$\left|\frac{a-b}{1-\bar{a}b}\right| = \frac{|a-b|}{|1-\bar{a}b|} = 1$$

$\Rightarrow$ if $1 \neq \bar{a}b \Rightarrow$

$$|a-b|^2 = |1-\bar{a}b|^2$$

$$(a-b)(\overline{a-b}) = (1-\bar{a}b)\overline{(1-\bar{a}b)} = (1-\bar{a}b)(1-a\bar{b})$$

$$a\bar{a} - \cancel{\bar{a}b} - \cancel{a\bar{b}} + b\bar{b} = 1 - \cancel{\bar{a}b} - \cancel{b\bar{a}} + a\bar{a}b\bar{b}$$

$$|a|^2 + |b|^2 - |a|^2|b|^2 - 1 = 0$$

$$(|a|^2 - 1)(|b|^2 - 1) = 0$$

$\Rightarrow$ either $|a| = 1$ or $|b| = 1$.

Should there be an exception if both $|a| = |b| = 1$?

If both $|a| = |b|$ then take $a = e^{i\theta_a}$, $b = e^{i\theta_b}$, and $1 - \bar{a}b = 1 - e^{i(-\theta_a + \theta_b)} = 0$.

$$\Rightarrow e^{i(\theta_b - \theta_a)} = e^{i(2\pi n)} \iff \theta_b = \theta_a + 2\pi n$$

$\Rightarrow$ When $|a| = |b| = 1$ we have to exclude cases when $Arg(a) = Arg(b) + 2\pi n$, $n \in \mathbb{Z}$.

Problem 1.c

$$\sin 5t = Im\left(e^{i5t}\right) = Im\left(\left(e^{it}\right)^5\right)$$

$$= Im\left(\left[\cos t + i\sin t\right]^5\right)$$

$$= Im\left(5\cos^4 t(i\sin t) + 10\cos^2 t(i\sin t)^3 + (i\sin t)^5\right)$$

$$= \boxed{5\cos^4 t\sin t - 10\cos^2 t(\sin t)^3 + \sin^5 t}$$

1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

$\boxed{\text{Problem 2}}$

(a) $f(t) = i + (4+i)t$ is a $\boxed{\text{straight line}}$ through the point
$= a + bt$ a with a tangent b. The line has a slope $1/4$.

(b) $g(t) = \sin(t) + i\cos(t)$ is a curve given by $\begin{cases} x = \sin(t) \\ y = \cos(t). \end{cases}$,

where $x^2+y^2 = 1$ => it's a $\boxed{\text{circle}}$ centered at $x=0, y=0$.

(c) $h(t) = \dfrac{1+it}{1-it} = \dfrac{(1+it)(1+it)}{1+t^2} = \dfrac{(1-t^2) + i(2t)}{1+t^2}$

is a curve given by $x = (1-t^2)/(1+t^2)$
$y = 2t/(1+t^2)$

Again $x^2+y^2 = \dfrac{(1-t^2)^2+(2t)^2}{(1+t^2)^2} = 1$, but
$t \in [0,\infty)$,
so the curve is not a circle, but a $\boxed{\text{semi-circle}}$ since $\boxed{y \geq 0}$

Problem 3 $\quad e^{(a+ib)t} = e^{at} e^{i(bt)}$, $\quad Arg = bt,\ R = e^{at}$.

ⓐ circle $\Leftrightarrow$ $R = const$, $Arg \propto t$ $\Leftrightarrow$ $a = 0$, $b \neq 0$

ⓑ a ray $\Leftrightarrow$ R changes, $Arg = const$ $\Leftrightarrow$ $a \neq 0$, $b = 0$

ⓒ converge to 0 as $t \to +\infty$ $\Leftrightarrow$ $R \to 0$ as $t \to \infty$ $\Leftrightarrow$ $a < 0$

NO RESTRICTIONS on b.

ⓓ Spiral moving counterclockwise away from the origin. $\Leftrightarrow$ R increases with t $\Leftrightarrow$ $a > 0$

Arg increases with t $\Leftrightarrow$ $b > 0$

ⓔ e^{zt} is the same curve as $e^{(a+ib)t}$ $\Leftrightarrow$ Both curves are at the same point for different times

$$e^{zt} = e^{(a+ib)T}$$

$$z = (a+ib)\frac{T}{t} \Leftrightarrow \text{when } \frac{T}{t} \text{ is not a function of time} \Leftrightarrow \frac{T}{t} = const = c$$

$$\Rightarrow \boxed{z = (a+ib)\cdot c \quad c \in \mathbb{R},\ c \neq 0.}$$

18.03 Lecture 7

9/23

Exam 1 Wed In Class

Sols to practice exams will be posted

Last qu was $y' + ky = f(t)$

- take linear eq and want to complexify and solve and return to reals

$\frac{dy}{dt} + 2y = \cos t$

Strategy 3

Only type of fs we solve are sin + cos

- never more complex than that

So strategy 1 ✓

$\cos t = Re(e^{it})$

If $\sin t = Im(e^{it})$ $\boxed{\text{← note!}}$

Need to take ~~Re~~ of left $Re(\tilde{y}) = y$

What did he say about the 2?

Keep at I think

Arrive at $\frac{d}{dt}\left(\tilde{y}\, e^{2it}\right) = e^{(2+i)t}$

(2)

Then Integrate

$$\tilde{y} e^{2t} = \int e^{(2+i)t} dt$$

$$= \frac{1}{2+i} e^{(2+i)t} + C$$

↑ complex constant
$a+bi$

$$\tilde{y} = \left(\frac{1}{2+i}\right) e^{ti} + C \cdot e^{-2t}$$

Solve complex problem

Then take Re( ) of each side

$$Re(\tilde{y}) = y = Re\left(\left(\frac{1}{2+i}\right) e^{ti} + C \cdot e^{-2t}\right)$$

$$= Re\left(\frac{1}{2+i} e^{ti} + \right) + a e^{-2t}$$ ← $Re(e^{-2t}) = e^{-2t}$

Two ways to find real? part of product

1) Can express both in rectangular coords

$(x+iy)(x'+iy')$

Expand + take real parts

Multiply top + bottom by conjugate

⋮

$$= \frac{2}{5}\cos t + \frac{1}{5}\sin t$$

③

Faster + better way

2) Convert $\frac{1}{2+i}$ into complex polar coords.

Do by finding polar form of $2+i$ and reciprocate answer

$$x + iy \to re^{i\theta}$$

$$r = \sqrt{x^2+y^2} = \sqrt{5}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{1}{2}\right)$$

$$\sqrt{5}\, e^{i\tan^{-1}(1/2)}$$ ↑ can just leave

So

$$\frac{1}{2+i} = \frac{1}{\sqrt{5}} e^{-1i\tan^{-1}(1/2)}$$

Then multiply by e^{it}

$$\frac{1}{\sqrt{5}} e^{i(t-\tan^{-1}(1/2))}$$

Take Re() – $\boxed{\text{remember } e^{i\theta} = \cos\theta + i\sin\theta}$

$$\frac{1}{\sqrt{5}} \cos\left(t - \tan^{-1}\left(\tfrac{1}{2}\right)\right)$$

Answer looks very different!

#2 is way better – hides wave info

amp = $\frac{1}{\sqrt{5}}$

(4)

Period $= 1$

Since $\cos(t - \tan^{-1}(\frac{1}{2}))$

phase shift $= \tan^{-1}(\frac{1}{2})$

aka phase lag

Can understand everything about wave

Moral Write sol'n in form

$$\underbrace{A}_{\substack{\text{amplitude} \\ \text{(max value)}}} \cdot \cos(\omega t - \phi)$$

period is $\frac{2\pi}{\omega}$

ϕ: phase shift (where 1st max occures)

If took sin - take Im() parts

Could sort of apply rules

ϕ would be first 0 instead of first max

- Could convert to sin by shifting by $\frac{\pi}{2}$

(5)

<u>Example</u> Newton's Law of Cooling

$$T' + kT = \underbrace{k\,T_e(t)}$$

can write as $f(t)$
to get what we started day w/

T: temp of object immersed in medium w/ ambient temp
$T_e(t)$

<u>example</u> Coffee in a room

T = temp of coffee
$T_e(t)$ = temp of room
— assume constant
t = time

Problem uses just int factor — NOT complex #

<u>example 2</u> Coffee on a sidewalk

T = temp of coffee

$T_e(t)$ = temp outside
— sinusoid by time of day

Then really have to do complexification
— would be evil

~~HW~~ 6

Linear ODE $k > 0$

Steady state ~~a~~ AND transient solutions

~~exist~~ $Ce^{-kt} \to 0$ as $t \to 0$

ind of initial condition

Did not combine answers earlier

Coffee will eventually match temp outside weither you brewed it or take out of freezer

So always remember a transient + steady state sol

Picture your ~~coffee as~~ diff eq as coffee on sidewalk

General linear ODE

$$y + p(x) y = Q(x)$$

not always constant

- Solve using IF

Special case Homogeneous Linear ODE

$$y' + P(x) y = 0$$

- Solve seperation of variables $\quad y = e^{-\int P(x)dx} \cdot y_0$

8

That is where IF comes from

Homogeneous eq: easy

<u>Theorm</u> If $y = \phi(x)$ is a sol'n to linear ODE then all sols are of form $y = \phi(x) + h(x)$ where $h(x)$ is a soln to homogenous eq

$y' + P(x)y = 0$

If find one eq for simpler - can solve then all and then tack it on

sols = special sol + homogeneous sol

<u>Superposition (Principle of)</u>: if ~~all~~ $y = \phi_1(x)$ is sol'n to $y' + P(x)y = Q_1(x)$ AND $y = \phi_2(x)$ is sol'n to $y' + P(x)y = Q_2(x)$ then $y = \phi_1(x) + \phi_2(x)$ is a sol'n to $y' + P(x)y = Q_1(x) + Q_2(x)$

- can check w/ 30 sec of algebra

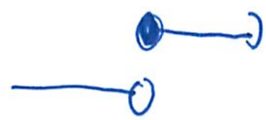
1st theorm is special case of principle of superposition

③ Could add diff eq of ~~[illegible]~~ each rock to get response to lots of rocks

---

Read in ~~notes~~ supplementary notes on IR on how to handle discontinuous inputs

<u>inputs</u> step function



Square wave

For example you could bring coffee in + out

Or flick switch on circuit

<u>IR mode</u> |

$$y' + P(x)\,y = 0$$ Zero inital condition

↑ nothing is happening

$y = 0$ is a sol

Then someone flips a switch

Its like saying at some pt in time fn changes

$$y' + P(x)\,y = Q(x)$$ ↖introduce new ODE $Q(x)$

(9)

Call the solution a response

How to model a store

- like 2nd order ODE
- instanteous effect

Come up w/ fn having instanous action

Take sequence of functions $\{h_n\}$

Such that

- $h_n \geq 0$ for all x
- $\int_{-\infty}^{\infty} h_n = 1$ aka $\int_{-\infty}^{\infty} h_n(x)\,dx = 1$

  - Improper integral

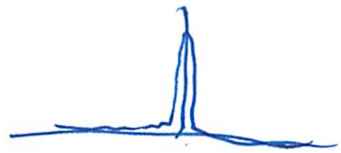
Most Important - Support $(h_n) \to 0$ as $h_n \to \infty$

the x values for which $h_n(x)$ is non 0 $(h_n(x) \neq 0)$

For example h_1

h_{5000}

$h_{5^{100}}$



$\lim_{n \to \infty} h_n = \delta \to$ Dirac delta "function"

- Mathmatican say this can't be a functions

But it would be a useful function to have

So can measure throwing stones into ponds

~~[illegible]~~

So redo analysis - whole idea of function needs to be redone

Not "put # in → get # out"

But "a generalized function; defined according integration againts other functions

Define δ so that $\int_{-\infty}^{\infty} \underbrace{\delta(x)}_{\text{only something you integrate againts}} g(x)\,dx = g(0)$

$=$

$\int \lim_{n \to \infty} \int_{-\infty}^{\infty} h_n(x)\, g(x)\, dx$

Lecture 7

On Monday, complexifying linear diff. equations in order to make integration step easy. $y' + ky = f(t)$ with f: sinusoidal

—

Begin with example on slide.

Arrive at: $\frac{d}{dt}(\tilde{y} \cdot e^{2t}) = \int e^{(2+i)t} dt$

$\tilde{y}$: complexified variable with $Re(\tilde{y}) = y$.

$$\tilde{y} e^{2t} = \frac{1}{(2+i)} e^{(2+i)t} + c'$$

$$\tilde{y} = \frac{1}{(2+i)} e^{it} + \underbrace{c' \cdot e^{-2t}}_{\text{transient}}$$

transient $\to 0$ as $t \to \infty$

Then take Re of both sides:

$$y = Re\left(\frac{1}{2+i} e^{it}\right) + c \cdot e^{-2t}$$

with $c = Re(c')$ const.

(think $c' = c + ki$ with c, k real.)

Two ways to find $Re\left(\frac{1}{2+i} \cdot e^{it}\right)$:

(1): Express $\frac{1}{2+i}$ and e^{it} in ~~rectangular coords~~ real and imag. parts, then expand:

$\frac{1}{2+i} = \frac{2-i}{5} = \frac{2}{5} - \frac{1}{5}i$. $e^{it} = \cos t + i \sin t$

Ans: $Re\left(\frac{2-i}{5} \cdot (\cos t + i \sin t)\right) = \frac{2}{5} \cos t + \frac{1}{5} \sin t$

(2) Express $\frac{1}{2+i}$ in polar form. Then multiply & convert:

$\frac{1}{2+i}$ written as $r \cdot e^{i\theta}$ with $r = \left|\frac{1}{2+i}\right| = \frac{1}{\sqrt{5}}$ $\quad \theta = \tan^{-1}\left(\frac{-1/5}{2/5}\right) = \tan^{-1}(-1/2)$

so $Re\left(\frac{1}{2+i} e^{it}\right) = Re\left(\frac{1}{\sqrt{5}} e^{i(t + \tan^{-1}(-1/2))}\right)$

Now take Re: $\frac{1}{\sqrt{5}} \cos\left(t + \tan^{-1}(-1/2)\right) = \frac{1}{\sqrt{5}} \cos\left(t - \tan^{-1}(1/2)\right)$

reciprocate

Better: convert $2+i$ to polar, then ~~reciprocate~~:

$$2+i = \sqrt{5}\, e^{i\tan^{-1}(1/2)} \rightarrow \frac{1}{2+i} = \frac{1}{\sqrt{5}} e^{-i\tan^{-1}(1/2)}.$$

The two solutions are equivalent according to trig. identity "the sinusoidal identity" on P. 8 of IR notes.

Often prefer polar method as form $A\cos(\omega t - \phi)$

allows us to read off important properties of response as a wave.

$|A|$: amplitude. (maximum)

ϕ : phase lag. (shifts graph right by ϕ, then location of first maximum)

ω : circular frequency

so $2\pi/\omega$ gives period.

These are harder to extract from the earlier form $a\cos\omega t + b\sin\omega t$.

What if we had $\sin\omega t$ instead? $\cos(\omega\{t - \pi/2\}) = \sin\omega t$.

Example. Newton's Law of Cooling (good to have physical metaphor...)

$$T' + kT = \underbrace{k\,T_e(t)}_{f(t)} \quad \text{where } T_e(t) : \text{ambient temperature}$$

T_e might be constant ← interior room with thermostat, sinusoidal ← varying according to day/night temperatures.

Reminder: general linear equation has

$$y' + P(x)y = Q(x)$$ with $P(x)$ not nec. constant.

Another special case: Homogeneous equation: $y' + P(x)y = 0$.

this is separable, with $\frac{y'}{y} = -P(x)$ solving

$$y = e^{-\int P(x)\,dx} \cdot y_0 .$$ (Motivates choice of integrating factor. Also useful in general linear equation.)

Theorem: If $y = \phi(x)$ is a solution to $y' + P(x)y = Q(x)$, then all solutions have the form $y = \phi(x) + h(x)$ where $h(x)$ is solution to the homogeneous equation.

(check they give solutions. Best pf: linear ops + linear alg. Skip.)

Principle of Superposition: If $\phi_1(x)$ is solution to $y' + P(x)y = Q_1(x)$ and $\phi_2(x)$ is solution to $y' + P(x)y = Q_2(x)$ (note P not varying) then $\phi_1 + \phi_2$ is a sol'n to $y' + P(x)y = Q_1(x) + Q_2(x)$.

Finally, discuss linear equations and IR with discontinuous inputs. Stone thrown into pond — instantaneous impulse.

Two stones in pond: use principle of superposition to handle separately.

In IR notes, can see how to handle other discontinuous inputs

(e.g. step functions  or "box functions": )

which might result from flipping a switch.

write t to emphasize time aspect

<u>Model</u>: $y' + P(t)y = 0$ with $y(0) = 0$ steady-state.

Then at some time t_0, introduce input $Q(t)$.

<u>Notice</u>: Solution is still $y = 0$ from time $t=0$ to $t = t_0$.

How to model instantaneous input? Delta function.

Sequence of functions $\{h_N\}$, non-neg., support $\to 0$ as $N \to \infty$

"Dirac sequence" with $\int h_N = 1$.

<u>Tempted to say</u>: limit of this sequence is a Function with integral 1 and supported only at the origin $x = 0$.

(and physicists do say)

But no such function exists.

Try to <u>prove</u>: $\lim_{N\to\infty} \int_{-\infty}^{\infty} h_N(x) g(x) = g(0)$ for any continuous function g.

so then defining property of δ:

$$\int_{-\infty}^{\infty} \delta(x) g(x) = g(0).$$

"generalized functions"

Math 18.03 : Differential Equations

Lecture 7 Supplemental Notes

Friday, September 23, 2011

Quick Quiz 6

When solving the differential equation

$$dy/dt + 2y = \cos t$$

one (successful) strategy would be to:

1. complexify to $d\tilde{y} + 2\tilde{y} = e^{it}$, solve, and take Re(.) of the result.
2. complexify to $d\tilde{y} + 2\tilde{y} = e^{it}$, solve, and take Im(.) of the result.
3. complexify to $d\tilde{y} + 2\tilde{y} = e^{2it}$, solve, and take Re(.) of the result.
4. complexify to $d\tilde{y} + 2\tilde{y} = e^{2it}$, solve, and take Im(.) of the result.

Quick Answer

When solving the differential equation

$$dy/dt + 2y = \cos t$$

one (successful) strategy would be to:

1

1. complexify to $d\tilde{y} + 2\tilde{y} = e^{it}$, solve, and take Re(.) of the result.

Indeed, $\mathrm{Re}(e^{it}) = \mathrm{Re}(\cos t + i \sin t) = \cos t$ and we choose complex variable $\tilde{y}$ such that $\mathrm{Re}(\tilde{y}) = y$.

Since taking Re and Im parts is compatible with doing calculus over the complex numbers (i.e., doesn't matter if you apply these functions before or after integrating), the strategy succeeds.

Real quick differentiation

$\frac{d}{dx}(a \cdot b)$

$= ab' + a'b$

?

↑ thats the product rule!

Chain rule

$\frac{d}{dt}(f(g(t)) = g'(t) \cdot f'(g(t))$

9/25

1 Using the Chain Rule in Reverse

Recall that the Chain Rule is used to differentiate composite functions such as $\cos(x^3+1)$, $e^{\frac{1}{2}x^2}$, $(2x^2+3)^{11}$, $\ln(3x+1)$. (The Chain Rule is sometimes called the Composite Functions Rule or Function of a Function Rule.)

If we observe carefully the answers we obtain when we use the chain rule, we can learn to recognise when a function has this form, and so discover how to integrate such functions.

Remember that, if $y = f(u)$ and $u = g(x)$

so that $y = f(g(x))$, (a composite function)

then $$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

oh that is intermediate

Using function notation, this can be written as

$$\frac{dy}{dx} = f'(g(x)) \cdot g'(x).$$

In this expression, $f'(g(x))$ is another way of writing $\frac{dy}{du}$ where $y = f(u)$ and $u = g(x)$ and $g'(x)$ is another way of writing $\frac{du}{dx}$ where $u = g(x)$.

This last form is the one you should learn to recognise.

Examples

By differentiating the following functions, write down the corresponding statement for integration.

i. $\sin 3x$ $3 \cdot \cos(3x)$

ii. $(2x+1)^7$

iii. e^{x^2}

Solution

But what is the reverse?

i $\frac{d}{dx}\sin 3x = \cos 3x \cdot 3,$ so $\int \cos 3x \cdot 3dx = \sin 3x + c.$

ii $\frac{d}{dx}(2x+1)^7 = 7(2x+1)^6 \cdot 2,$ so $\int 7(2x+1)^6 \cdot 2dx = (2x+1)^7 + c.$

iii $\frac{d}{dx}\left(e^{x^2}\right) = e^{x^2} \cdot 2x,$ so $\int e^{x^2} \cdot 2xdx = e^{x^2} + c.$

∫ oh they want us to recognize it

Exercises 1.1

Differentiate each of the following functions, and then rewrite each result in the form of a statement about integration.

i	$(2x-4)^{13}$	ii	$\sin \pi x$	iii	e^{3x-5}
iv	$\ln(2x-1)$	v	$\dfrac{1}{5x-3}$	vi	$\tan 5x$
vii	$(x^5-1)^4$	viii	$\sin(x^3)$	ix	$e^{\sqrt{x}}$
x	$\cos^5 x$	xi	$\tan(x^2+1)$	xii	$\ln(\sin x)$

The next step is to learn to *recognise* when a function has the forms $f'(g(x)) \cdot g'(x)$, that is, when it is the derivative of a composite function. Look back at each of the integration statements above. In every case, the function being integrated is the *product* of two functions: one is a composite function, and the other is the derivative of the "inner function" in the composite. You can think of it as "the derivative of what's inside the brackets". Note that in some cases, this derivative is a constant.

For example, consider

$$\int e^{3x} \cdot 3dx.$$

We can write e^{3x} as a composite function.
3 is the derivative of $3x$ i.e. the derivative of "what's inside the brackets" in $e^{(3x)}$.

This is in the form

$$\int f'(g(x)) \cdot g'(x)dx$$

with

$$u = g(x) = 3x, \text{ and } f'(u) = e^u.$$

Using the chain rule in reverse, since $\frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x)$ we have

$$\boxed{\int f'(g(x)) \cdot g'(x)dx = f(g(x)) + c.}$$

just basically recognize when Chain rule

In this case

$$\int e^{3x} \cdot 3dx = e^{3x} + c.$$

If you have any doubts about this, it is easy to check if you are right: *differentiate your answer!*

Now let's try another:

$$\int \cos(x^2+5) \cdot 2xdx.$$

try $= \sin(x^2+5) + C$

$\cos(x^2+5)$ is a composite function.
$2x$ is the derivative of x^2+5, i.e. the derivative of "what's inside the brackets".

So this is in the form

$$\int f'(g(x)) \cdot g'(x)dx \quad \text{with } u = g(x) = x^2 + 5 \text{ and } f'(u) = \cos u.$$

Recall that if $f'(u) = \cos u$, $f(u) = \sin u$.

So,

$$\int \cos(x^2 + 5) \cdot 2x dx = \sin(x^2 + 5) + c.$$ ✓

Again, *check that this is correct*, by differentiating.

People sometimes ask "Where did the $2x$ go?". The answer is, "Back where it came from."

haha

If we differentiate $\sin(x^2 + 5)$ we get $\cos(x^2 + 5) \cdot 2x$.

So when we integrate $\cos(x^2 + 5) \cdot 2x$ we get $\sin(x^2 + 5)$.

$\int x^2 = \frac{x^3}{3}$

Examples

Each of the following functions is in the form $f'(g(x)) \cdot g'(x)$. ← not banded

Identify $f'(u)$ and $u = g(x)$ and hence find an indefinite integral of the function.

$\frac{d}{dx} \frac{x^3}{3} =$

$\frac{3}{3} x^2$

i. $(3x^2 - 1)^4 \cdot 6x$ $\quad \frac{1}{5}(3x^2-1)^5 + C$ ✓

ii. $\sin(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}}$ $\quad \cos(\sqrt{x})$

Solutions

i. $(3x^2 - 1)^4 \cdot 6x$ is a product of $(3x^2 - 1)^4$ and $6x$.

Clearly $(3x^2 - 1)^4$ is the composite function $f'(g(x))$. So $g(x)$ should be $3x^2 - 1$.

$6x$ is the "other part". This should be the derivative of "what's inside the brackets" i.e. $3x^2 - 1$, and clearly, this is the case:

$\frac{d}{dx}(3x^2 - 1) = 6x$.

So, $u = g(x) = 3x^2 - 1$ and $f'(u) = u^4$ giving $f'(g(x)) \cdot g'(x) = (3x^2 - 1)^4 \cdot 6x$.

If $f'(u) = u^4$, $f(u) = \frac{1}{5}u^5$.

So, using the rule

$$\int f'(g(x)) \cdot g'(x)dx = f(g(x)) + c$$

we conclude

$$\int (3x^3 - 1)^4 \cdot 6x = \frac{1}{5}(3x^2 - 1)^5 + c.$$ ✓

You should differentiate this answer *immediately* and check that you get back the function you began with.

~~good~~ good way to ✓ don't be lazy

do for my #2

$\sin(\sqrt{x}) \cdot \sqrt{x}^{-2}$

$\sin(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}}$

ii. $\sin(\sqrt{x}) \cdot \dfrac{1}{2\sqrt{x}}$

This is a product of $\sin(\sqrt{x})$ and $\frac{1}{2\sqrt{x}}$.

Clearly $\sin(\sqrt{x})$ is a composite function.

The part "inside the brackets" is $\sqrt{x}$, so we would like this to be $g(x)$. The other factor $\frac{1}{2\sqrt{x}}$ ought to be $g'(x)$. Let's check if this is the case:

$g(x) = \sqrt{x} = x^{\frac{1}{2}}$, so $g'(x) = \dfrac{1}{2}x^{-\frac{1}{2}} = \dfrac{1}{2x^{\frac{1}{2}}} = \dfrac{1}{2\sqrt{x}}$.

So we're right! Thus $u = g(x) = \sqrt{x}$ and $f'(u) = \sin u$ giving

$$f'(g(x)) \cdot g'(x) = \sin(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}}.$$

Now, if $f'(u) = \sin u$, $f(u) = -\cos u$.

So using the rule

$$\int f'(g(x)) \cdot g'(x)dx = f(g(x)) + c$$

we conclude

$$\int \sin(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}}dx = -\cos(\sqrt{x}) + c.$$

Actually dxh no -2 or then ∫ would go away

Again, *check immediately* by differentiating the answer.

Note: The explanations given here are fairly lengthy, to help you to understand what we're doing. Once you have grasped the idea, you will be able to do these very quickly, without needing to write down any explanation.

Let me try $\frac{d}{dx}$ this

$\frac{d}{dx} - \cos(\sqrt{x})$

First $\frac{d}{dx}\sqrt{x} = x^{1/2}$

$\frac{1}{2}x^{-1/2}$

$+ \frac{1}{2\sqrt{x}} \sin(\sqrt{x}) + c$

$\int \sin x = -\cos x + c$

$\frac{d}{d} \cos x = -\sin x$

; So why couldn't I see earlier

was not comfortable w/ did not fully do proper differentiation

Example

Integrate $\displaystyle\int \sin^3 x \cdot \cos x dx$.

Solution

$\displaystyle\int \sin^3 x \cdot \cos x dx = \int (\sin x)^3 \cdot \cos x dx.$

So $u = g(x) = \sin x$ with $g'(x) = \cos x$.

And $f'(u) = u^3$ giving $f(u) = \frac{1}{4}u^4$.

Hence $\displaystyle\int \sin^3 x \cdot \cos x dx = \frac{1}{4}(\sin x)^4 + c = \frac{1}{4}\sin^4 x + c.$

Exercises 1.2

Each of the following functions is in the form $f'(g(x)) \cdot g'(x)$. Identify $f'(u)$ and $u = g(x)$ and hence find an indefinite integral of the function.

i	$\underbrace{\frac{1}{3x-1}}_{f'} \cdot \underbrace{3}_{u}$	ii	$\sqrt{2x+1} \cdot 2$	iii	$(\ln x)^2 \cdot \frac{1}{x}$
iv	$e^{2x+4} \cdot 2$ $e^{2x+4} + c$ ✓	v	$\sin(x^3) \cdot 3x^2$ $-\cos(x^3) + c$ ✓	vi	$\cos\left(\frac{\pi x}{2}\right) \cdot \frac{\pi}{2}$
vii	$(7x-8)^{12} \cdot 7$ $\frac{1}{13}(7x-8)^{13}$ ✓	viii	$\sin(\ln x) \cdot \frac{1}{x}$	ix	$\left(\frac{1}{\sin x}\right) \cdot \cos x$
x	$e^{\tan x} \cdot \sec^2 x$	xi	$e^{x^3} \cdot 3x^2$ $e^{x^3} + c$ ✓	xii	$\sec^2(5x-3) \cdot 5$
xiii	$(2x-1)^{\frac{1}{3}} \cdot 2$	xiv	$\sqrt{\sin x} \cdot \cos x$ $\frac{2}{3}(\sin x)^{3/2} + c$ ✓		

The final step in learning to use this process is to be able to recognise when a function is not quite in the correct form but *can be put into the correct form* by minor changes.

For example, we try to calculate $\int x^3\sqrt{x^4+1}dx$.

We notice that $\sqrt{x^4+1}$ is a composite function, so we would like to have $u = g(x) = x^4+1$. But this would mean $g'(x) = 4x^3$, and the integrand (i.e. the function we are trying to integrate) only has x^3. However, we can easily make it $4x^3$, as follows:

$$\int x^3\sqrt{x^4+1}dx = \frac{1}{4}\int \sqrt{x^4+1} \cdot 4x^3 dx.$$

Note: The $\frac{1}{4}$ and the 4 cancel with each other, so the expression is not changed.

So $u = g(x) = x^4 + 1, \qquad g'(x) = 4x^3$

And $f'(u) = u^{\frac{1}{2}} \qquad f(u) = \frac{2}{3}u^{\frac{3}{2}}$

So, $\displaystyle\int x^3\sqrt{x^4+1}dx = \frac{1}{4}\int \sqrt{x^4+1} \cdot 4x^2 dx = \frac{1}{4} \cdot \frac{2}{3}\left(x^4+1\right)^{\frac{3}{2}} + c.$

Note: We may only insert constants in this way, not variables.

We *cannot* for example evaluate $\int e^{x^2}dx$ by writing $\frac{1}{2x}\int e^{x^2} \cdot 2xdx$, because the $\frac{1}{x}$ in front of the integral sign does *not* cancel with the x which has been inserted in the integrand.

This integral cannot, in fact, be evaluated in terms of elementary functions.

The example above illustrates one of the difficulties with integration: many seemingly simple functions cannot be integrated without inventing new functions to express the integrals. There is no set of rules which we can apply which will tell us how to integrate any function. All we can do is give some techniques which will work for *some* functions.

Exercises 1.3

Write the following functions in the form $f'(g(x)) \cdot g'(x)$ and hence integrate them:

i	$\frac{1}{7}$ $\cos 7x$	ii	xe^{x^2}
iii	$\frac{x}{1-2x^2}$	iv	$x^2(4x^3+3)^9$
v	$\sin(1+3x)$	vi	$\frac{\sin\sqrt{x}}{\sqrt{x}}$
vii	$\frac{x}{\sqrt{(1-x^2)}}$	viii	e^{3x} $= \frac{1}{3}e^{3x} + C$
ix	$\tan 6x$		

Hint: Write $\tan 6x$ in terms of $\sin 6x$ and $\cos 6x$.

try

$\frac{1}{3} \cdot 3\, e^{3x}$

Careful to add right stuff

$= \frac{1}{7} \sin 7x + C$

try $\frac{d}{dx}$

$\frac{1}{7} \cdot 7 \cos 7x$

I think I get the picture

Just need to think

And rember identies

Which I had a hard time remembering even in HS

Should have studied

2 Solutions to exercises

Exercises 1.1

i $\quad \frac{d}{dx}(2x-4)^{13} = 13\cdot(2x-4)^{12}\cdot 2,\quad$ so $\int 13(2x-4)^{12}\cdot 2dx = (2x-4)^{13}+c.$

ii $\quad \frac{d}{dx}(\sin \pi x) = \cos \pi x \cdot \pi,\quad$ so $\int \cos\pi x\cdot \pi dx = \sin \pi x + c.$

iii $\quad \frac{d}{dx}(e^{3x-5}) = e^{3x-5}\cdot 3,\quad$ so $\int e^{3x-5}\cdot 3dx = e^{3x-5}+c.$

iv $\quad \frac{d}{dx}(\ln(2x-1)) = \frac{1}{2x-1}\cdot 2,\quad$ so $\int \frac{1}{2x-1}\cdot 2dx = \ln(2x-1)+c.$

v $\quad \frac{d}{dx}(\frac{1}{5x-3}) = -\frac{1}{(5x-3)^2}\cdot 5,\quad$ so $\int -\frac{1}{(5x-3)^2}\cdot 5dx = \frac{1}{5x-3}+c.$

vi $\quad \frac{d}{dx}(\tan 5x) = \sec^2 5x\cdot 5,\quad$ so $\int \sec^2 5x\cdot 5dx = \tan 5x + c.$

vii $\quad \frac{d}{dx}((x^5-1)^4) = 4(x^5-1)^3\cdot 5x^4,\quad$ so $\int 4(x^5-1)^3\cdot 5x^4dx = (x^5-1)^4+c.$

viii $\quad \frac{d}{dx}(\sin x^3) = \cos(x^3)\cdot 3x^2,\quad$ so $\int \cos(x^3)\cdot 3x^2dx = \sin(x^3)+c.$

ix $\quad \frac{d}{dx}(e^{\sqrt{x}}) = e^{\sqrt{x}}\cdot\frac{1}{2}x^{-\frac{1}{2}},\quad$ so $\int e^{\sqrt{x}}\cdot\frac{1}{2}x^{-\frac{1}{2}}dx = e^{\sqrt{x}}+c.$

x $\quad \frac{d}{dx}(\cos^5 x) = 5\cos^4 x\cdot(-\sin x),\quad$ so $\int 5\cos^4 x\cdot(-\sin x)dx = \cos^5 x + c.$

xi $\quad \frac{d}{dx}(\tan(x^2+1)) = \sec^2(x^2+1)\cdot 2x,\quad$ so $\int \sec^2(x^2+1)\cdot 2xdx = \tan(x^2+1)+c.$

xii $\quad \frac{d}{dx}(\ln(\sin x)) = \frac{1}{\sin x}\cdot\cos x,\quad$ so $\int \frac{1}{\sin x}\cdot \cos xdx = \ln(\sin x)+c.$

Exercises 1.2

(Before you read these solutions, *check* your work by differentiating your answer.)

i. $\int \frac{1}{3x-1}\cdot 3dx = \ln(3x-1)+c.$ I knew it looked familar but did not look long enough to realise $\int \frac{1}{x} = \ln x + c$

$$\left\{\begin{array}{llll} u = g(x) & = 3x-1 & \text{so } g'(x) & = 3 \\ f'(u) & = \frac{1}{u} & \text{so } f(u) & = \ln u \end{array}\right.$$

ii. $\int \sqrt{2x+1}\cdot 2dx = \frac{2}{3}(2x+1)^{\frac{3}{2}}+c.$

$$\begin{cases} u = g(x) & = 2x+1 & \text{so } g'(x) & = 2 \\ f'(u) & = \sqrt{u} & \text{so } f(u) & = \frac{2}{3}u^{\frac{3}{2}} \end{cases}$$

iii. $\displaystyle\int (\ln x)^2 \cdot \frac{1}{x} dx = \frac{1}{3}(\ln x)^3 + c.$

$$\begin{cases} u = g(x) & = \ln x & \text{so } g'(x) & = \frac{1}{x} \\ f'(u) & = u^2 & \text{so } f(u) & = \frac{1}{3}u^3 \end{cases}$$

iv. $\displaystyle\int e^{2x+4} \cdot 2dx = e^{2x+4} + c.$

$$\begin{cases} u = g(x) & = 2x+4 & \text{so } g'(x) & = 2 \\ f'(u) & = e^u & \text{so } f(u) & = e^u \end{cases}$$

v. $\displaystyle\int \sin(x^3) \cdot 3x^2 dx = -\cos(x^3) + c.$

$$\begin{cases} u = g(x) & = x^3 & \text{so } g'(x) & = 3x^2 \\ f'(u) & = \sin u & \text{so } f(u) & = -\cos u \end{cases}$$

vi. $\displaystyle\int \cos(\frac{\pi x}{2}) \cdot \frac{\pi}{2} dx = \sin(\frac{\pi x}{2}) + c.$

$$\begin{cases} u = g(x) & = \frac{\pi}{2}x & \text{so } g'(x) & = \frac{\pi}{2} \\ f'(u) & = \cos u & \text{so } f(u) & = \sin u \end{cases}$$

vii. $\displaystyle\int (7x-8)^{12} \cdot 7dx = \frac{1}{13}(7x-8)^{13} + c.$

$$\begin{cases} u = g(x) & = 7x-8 & \text{so } g'(x) & = 7 \\ f'(u) & = u^{12} & \text{so } f(u) & = \frac{1}{13}u^{13} \end{cases}$$

viii. $\displaystyle\int \sin(\ln x) \cdot \frac{1}{x} dx = -\cos(\ln x) + c.$

$$\begin{cases} u = g(x) & = \ln x & \text{so } g'(x) & = \frac{1}{x} \\ f'(u) & = \sin u & \text{so } f(u) & = -\cos u \end{cases}$$

ix. $\displaystyle\int \frac{1}{\sin x} \cdot \cos x dx = \ln(\sin x) + c.$

$$\begin{cases} u = g(x) & = \sin x & \text{so } g'(x) & = \cos x \\ f'(u) & = \frac{1}{u} & \text{so } f(u) & = \ln u \end{cases}$$

x. $\displaystyle\int e^{\tan x} \cdot \sec^2 x dx = e^{\tan x} + c.$

$$\begin{cases} u = g(x) & = \tan x & \text{so } g'(x) & = sec^2 x \\ f'(u) & = e^u & \text{so } f(u) & = e^u \end{cases}$$

xi. $\displaystyle\int e^{x^3}\cdot 3x^2dx = e^{x^3}+c.$

$$\left\{\begin{array}{llll} u=g(x) & =x^3 & \text{so } g'(x) & =3x^2 \\ f'(u) & =e^u & \text{so } f(u) & =e^u \end{array}\right.$$

xii. $\displaystyle\int \sec^2(5x-3)\cdot 5dx = \tan(5x-3)+c.$

$$\left\{\begin{array}{llll} u=g(x) & =5x-3 & \text{so } g'(x) & =5 \\ f'(u) & =\sec^2 u & \text{so } f(u) & =\tan u \end{array}\right.$$

xiii. $\displaystyle\int (2x-1)^{\frac{1}{3}}\cdot 2dx = \frac{3}{4}(2x-1)^{\frac{4}{3}}+c.$

$$\left\{\begin{array}{llll} u=g(x) & =2x-1 & \text{so } g'(x) & =2 \\ f'(u) & =u^{\frac{1}{3}} & \text{so } f(u) & =\frac{3}{4}u^{\frac{4}{3}} \end{array}\right.$$

xiv. $\displaystyle\int \sqrt{\sin x}\cdot\cos x dx = \frac{2}{3}(\sin x)^{\frac{3}{2}}+c.$

$$\left\{\begin{array}{llll} u=g(x) & =\sin x & \text{so } g'(x) & =\cos x \\ f'(u) & =\sqrt{u} & \text{so } f(u) & =\frac{2}{3}u^{\frac{3}{2}} \end{array}\right.$$

Exercises 1.3

(Before reading the solutions, check all your answers by differentiating!)

i. $\displaystyle\int \cos 7x dx = \frac{1}{7}\int \cos 7x\cdot 7dx = \frac{1}{7}\sin 7x + c.$

$$\left\{\begin{array}{rcl} u & = & g(x)=7x,\ \ g'(x)=7 \\ f'(u) & = & \cos u \textbf{ so } f(u)=\sin u \end{array}\right.$$

ii. $\displaystyle\int xe^{x^2}dx = \frac{1}{2}\int e^{x^2}\cdot 2xdx = \frac{1}{2}e^{x^2}+c.$

$$\left\{\begin{array}{rcl} u & = & g(x)=x^2,\ \ g'(x)=2x \\ f'(u) & = & e^u \textbf{ so } f(u)=e^u \end{array}\right.$$

iii. $\displaystyle\int \frac{x}{1-2x^2}dx = -\frac{1}{4}\int\frac{1}{1-2x^2}\cdot(-4x)dx = -\frac{1}{4}\ln(1-2x^2)+c.$

$$\left\{\begin{array}{rcl} u & = & g(x)=1-2x^2,\ \ g'(x)=-4x \\ f'(u) & = & \frac{1}{u} \textbf{ so } f(u)=\ln u \end{array}\right.$$

iv. $\displaystyle\int x^2(4x^3+3)^9dx = \frac{1}{12}\int(4x^3+3)^9\cdot 12x^2dx = \frac{1}{12}\cdot\frac{1}{10}(4x^3+3)^{10}+c = \frac{1}{120}(4x^3+3)^{10}+c.$

$$\left\{\begin{array}{rcl} u & = & g(x)=4x^3+3,\ \ g'(x)=12x^2 \\ f'(u) & = & u^9 \textbf{ so } f(u)=\frac{1}{10}u^{10} \end{array}\right.$$

v. $\displaystyle\int \sin(1+3x)dx = \frac{1}{3}\int \sin(1+3x)\cdot 3dx = -\frac{1}{3}\cos(1+3x) + c.$

$$\left\{\begin{array}{rcl} u &=& g(x) = 1+3x,\ \ g'(x) = 3 \\ f'(u) &=& \sin u \text{ so } f(u) = -\cos u \end{array}\right.$$

vi. $\displaystyle\int \frac{\sin\sqrt{x}}{\sqrt{x}}dx = 2\int \sin\sqrt{x}\cdot\frac{1}{2\sqrt{x}}dx = -2\cos\sqrt{x} + c.$

$$\left\{\begin{array}{rcl} u &=& g(x) = \sqrt{x},\ \ g'(x) = \frac{1}{2\sqrt{x}} \\ f'(u) &=& \sin u \text{ so } f(u) = -\cos u \end{array}\right.$$

vii. $\displaystyle\int \frac{x}{\sqrt{1-x^2}}dx = -\frac{1}{2}\int \frac{1}{\sqrt{1-x^2}}\cdot(-2x)dx = -\frac{1}{2}\cdot 2(1-x^2)^{\frac{1}{2}} + c = -(1-x^2)^{\frac{1}{2}} + c.$

$$\left\{\begin{array}{rcl} u &=& g(x) = 1-x^2,\ \ g'(x) = -2x \\ f'(u) &=& \frac{1}{\sqrt{u}} \text{ so } f(u) = 2u^{\frac{1}{2}} \end{array}\right.$$

viii. $\displaystyle\int e^{3x}dx = \frac{1}{3}\int e^{3x}\cdot 3dx = \frac{1}{3}e^{3x} + c.$

$$\left\{\begin{array}{rcl} u &=& g(x) = 3x,\ \ g'(x) = 3 \\ f'(u) &=& e^u \text{ so } f(u) = e^u \end{array}\right.$$

ix. $\displaystyle\int \tan 6x dx = \int \frac{\sin 6x}{\cos 6x}dx = -\frac{1}{6}\int \frac{1}{\cos 6x}\cdot -6\sin 6x = -\frac{1}{6}\ln(\cos 6x) + c.$

$$\left\{\begin{array}{rcl} u &=& g(x) = \cos 6x,\ \ g'(x) = -6\sin 6x \\ f'(u) &=& \frac{1}{u} \text{ so } f(u) = \ln u \end{array}\right.$$

9/25

Exponents Review

6.004/18.03

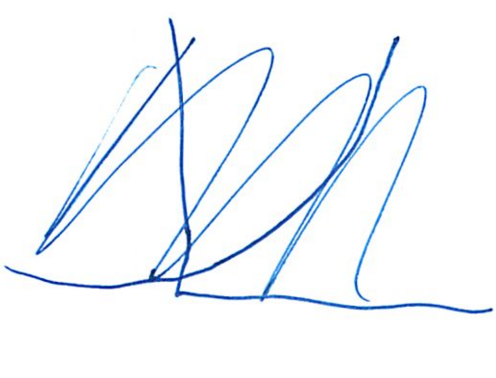
Logarithm The ~~#~~ log of a # is the exponent by which the base has to be raised to produce that #

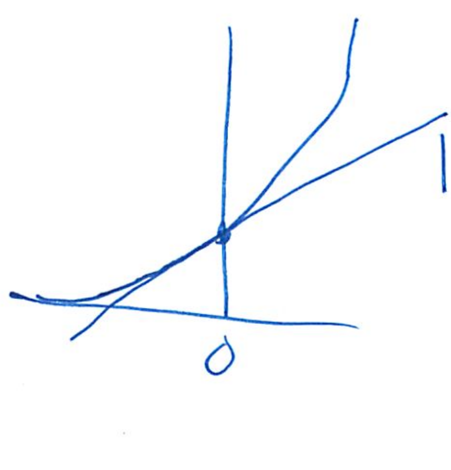
$10^3 = 1000$ $\qquad \log_{10}(1000) = 3$

$b^y = x$ $\qquad \log_b(x) = y$

- Invented to simplify Calculus

~~Note~~ **Euler's # e** value of derivative ~~(slope ...)~~ of $f(x) = e^x$ at point $x = 0$ $= 1$





2.71828 .... etc

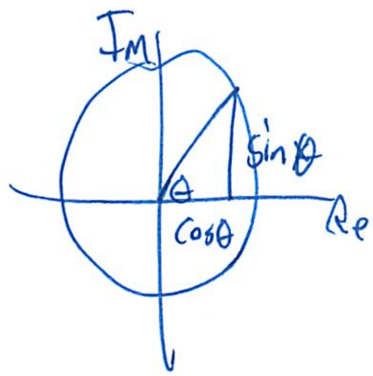
2

In <u>Euler's Identity</u>

$$e^{i\pi} + 1 = 0$$

$$e^{ix} = \cos x + i \sin x$$

$$= -1 + 0i$$



<u>Natural Log</u> Is log to e

$\ln(x)$

$\log_e(x)$

Power $e^y = x$

e raised to to equal x

$y = \ln(x)$

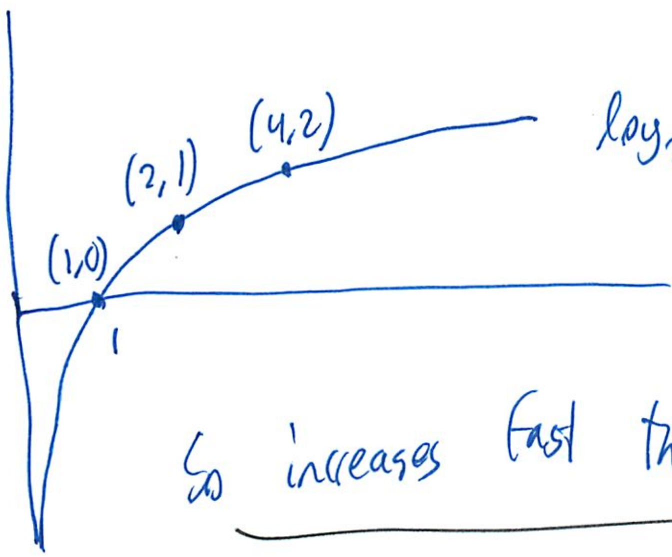
$\ln(e) = 1$ since $e^1 = e$

③ $\ln(1) = 0$ since $e^0 = 1$

I feel so dumb not knowing this junior year

"natural" since appears so much in math

<u>Log</u>



So increases fast than slow

Like inserting items into sorted list

Like our circuit w/ $2^{\log_2 N}$ levels

No printed notes

18.03 Lecture 8

9/26

Exam Review

Exam Wed
- in class, 54-100

P-Set 3 due next friday Oct 7

Practice exams posted online

Autonomas Eq / Bifurcations

↳ equations

$$\frac{dP}{dt} = F(P)$$

↑fn ind of t

like where F is a polynomial

logistical model ←

$$\frac{dP}{dt} = aP - bP^2$$

a, b constants

What is role of a?

then $\frac{dP}{dt} = aP$ exponential growth

↑growth rate

Constant of proportionality to pop

a = growth rate w/o limiting influences

2

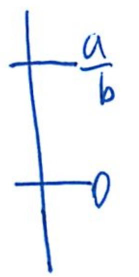
Then realize this is weird

So add limiting term "$-bP^2$"

What is leveling out value

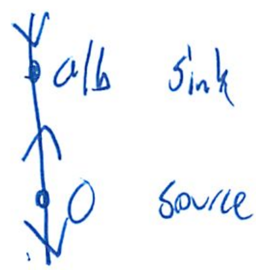
$$\frac{dP}{dt} = aP\left(1 - \frac{b}{a}P\right)$$

If drawing phase line,



(ohhh - so helpful vs what I was studying last night)

Since Pop model, we know sighs



So if initial condition $0 < IC < \frac{a}{b}$ then will end up a/b

"Limiting size of population" $= \frac{a}{b}$

(3)

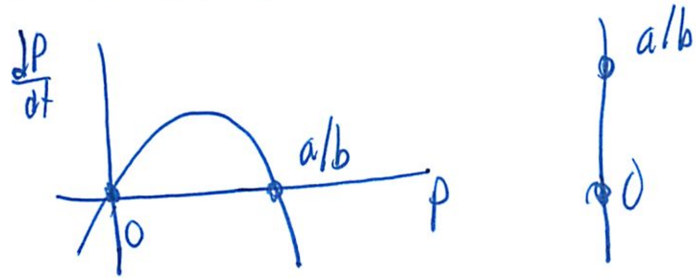
Logistical Model w/ harvesting

- remove some at constant rate
- makes more complicated

$$\frac{dP}{dt} = aP\left(1 - \frac{b}{a}P\right) - r$$

- So varies w/ choice of r
- when $r=0$, its like before

$r=0$



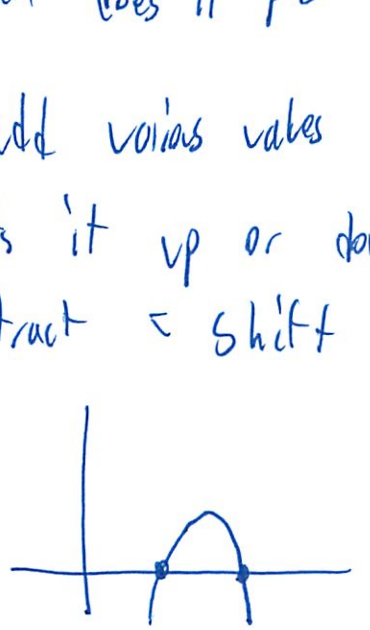
???

When does a bifurcation occur?

- when does # pts on phase line change?

Now add various values of r

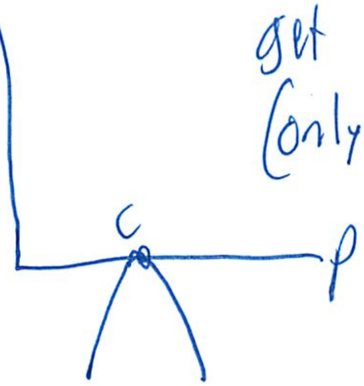
- shifts it up or down
- subtract = shift ↓

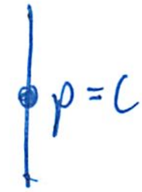


(4)

When r = max value of parabola
get a bifurcation
(only 1 ~~intersection~~ phase line pt)

$F(P) = \frac{dP}{dt}$

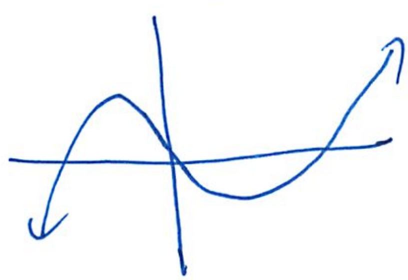




To find max - take deriv (of deriv, but does not matter)

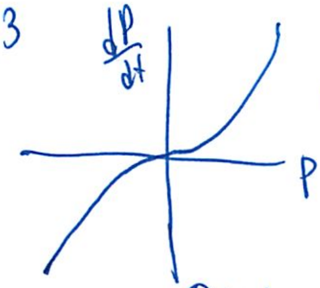
$$\boxed{\begin{gathered} aP - bp^2 \\ \text{then set} = \text{to } 0 \\ 0 = a - 2bp \end{gathered}}$$

Cubic - same idea



need to know relative min / max
are 2 bifurcation pts here

p^3 $\frac{dP}{dt}$



Only has 1 real 0 - where $\frac{dP}{dt} = 0$
Only has 0 bifurcation pt

↑Not for drawing solutions!!!

(5)

After get phase line, then can draw in solution lines on

W/ $\frac{dP}{dt} = \cos P + r$ when $r = 1/-1$

it has crazy biforcation pts

Have ∞ many

Then at $r = .999$, have half as many!

$\frac{1}{2}\infty$

Parameter can be anywhere

Can be - # of biforcation pts

Lets' talk Money

Savings acts make 5%

Inital contribution \$100,000

Withdraw \$8,000/year "contineauly"

Write diff eq modeling

Find amt of time before run out of \$, if we do

(6) Can quickly see at year 1 $\$100,000 + 5,000 - 8,000$
= losing \$
So would run out

Now need to write diff eq

$$\frac{d\$}{dt} = .05\$ - 8,000$$

this is
amt changed

amt after 1 year = 1.05

think about
$\Delta S = .05S\,\Delta t - 8000\,\Delta t$
divide by Δt
make continuous

Can solve multiple ways
- linear w/ IF
- sep of variables

Can try OP modeling qu

7. Fences + Funnels

$$\frac{dy}{dx} = 1 - xy$$

Not seperable

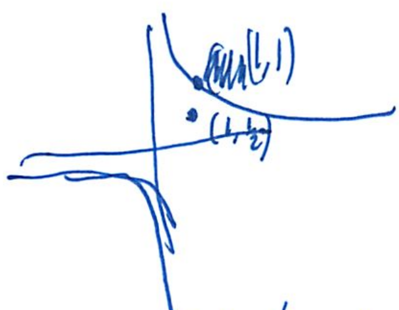
But what is long-term behavior of sol'n passing through $(1, \frac{1}{2})$

Can do slope field

- don't spend too much time on it
- 1st try for fence = 0-isocline

$$1 - xy = 0$$

$$y = \frac{1}{x}$$



- Then try origin/rough points - origin $1-0=1$
- Then try ± 1-isocline

• 1-isocline

$$1 - xy = 1$$

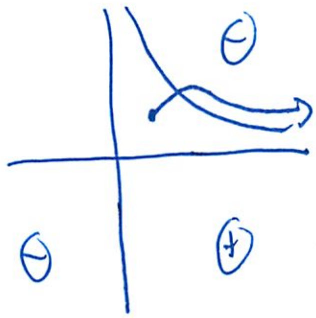
$$xy = 0$$

$$x = 0 \text{ or } y = 0$$

- So axis are slope lines

8

- Test our pt $1 - 1 \cdot \frac{1}{2} = \frac{1}{2}$



So know it will be this rough pattern

- Only care what happens going forward
- Now try -1 isoline

$$1 - xy = -1$$

$$y = \frac{2}{x} = c(x)$$

call $c(x)$

- Check for fences

Then take $c'(x)$

Compare $c'(x)$ vs $f(x, c(x))$

$$-\frac{2}{x^2} \quad \text{vs} \quad -1$$

Care when x is $+$

$$-2/x^2 > -1$$

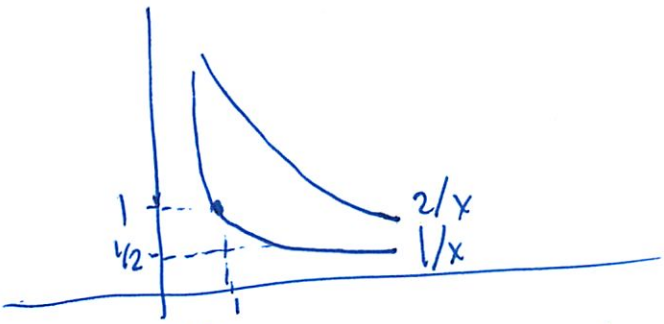
So upper fence

- x must be $> \sqrt{2}$

9

Leave to us to check $\frac{1}{x}$ is lower fence if $x \geq 1$

Bring 2nd color on Wed



Point starting at not in funnel

Need to argue will go in funnel

Could argue time runs forward - so can't drop below slope $\frac{1}{2}$

put vertical line

must go in funnel

So we win!

Easist qui start in funnel - fence above, below

Harder " ; Argue will enter into funnel

Antifunnel

- always a sol
- Unique sol if if + partial deriv w/r to dependent variable

(10)

Complex

— Tue in recitation

Input - Response (IR) model

- usually for linear ODEs

- $\frac{dy}{dt} + ky = f(t)$

For $f(t) = \sin t$ or $\cos t$ use complex methods

But if $f(t) = \sin 2t + \cos 2t$

- could use superposition (1)
- do each part independently.
- left side staying same, so can add solutions

Can't complexify right away

- Not immediately Im, Re anything

(2) could use identity $a\cos\theta + b\sin\theta = A\cos(\theta - \varphi)$

$A = \sqrt{a^2+b^2}$ $\tan\varphi = \frac{b}{a}$

Recitation 18.03

9/27

Exam 1 tomorrow

$e^z = c$

$\uparrow a + bi$

$\hookrightarrow re^{i\theta} = \underbrace{r\cos\theta}_{a} + \underbrace{r\sin\theta}_{b} i$

Solve system of linear eq for r, θ

$e^z = re^{i\theta}$

$\uparrow$ pos real #

$z = \ln r + i(\theta + 2k\pi)$

↑ haven't seen that - should remember

Antifunnel

there is 1 value that is always a solution, is unique --- (well always true)

(2)

$$\frac{dy}{dx} = f(x,y)$$

If function + anti derivitive contineous and start at initial pt
then there is a unique soltion

$y(x)$ with $y(x_0) = y_0$

Calculating Harvest Constant

Mushroom farmer

$y(t)$ = amt of mushrooms in tons

Logistic equation $\qquad$ t = months

$$\frac{dy}{dt} = 8y - y^2$$

1. What is max sustainable rate of harvest?

8-isocline

$$8y - y^2 = 8$$

$$8y = y^2$$

$$y = 8$$

So will always have 8 tons

?, so then how to do sustainable harvest

(3)

But then there is always raise/lower $\pm r$

$\updownarrow$

¿Can harvest 8 tons/month?

2. Suppose harvest 12 tons/month. How many tons long term

¿So $8y - y^2 - 12$?

$8y - y^2 = 12$

Can quadratic eq (roots!)

$y = $ ~~[illegible]~~ ~~[illegible]~~ G+V

7

56 - 49

6 42 - 36

$y = 6$ = long term ~~[illegible]~~ mushrooms / month

But that does not completly make sense

Perhaps I have isocline result in wrong result

1, With harvesting

¿I half did this - did in 1 qu
- didn't in other

$$\frac{dm}{dt} = 8m - m^2 - H$$

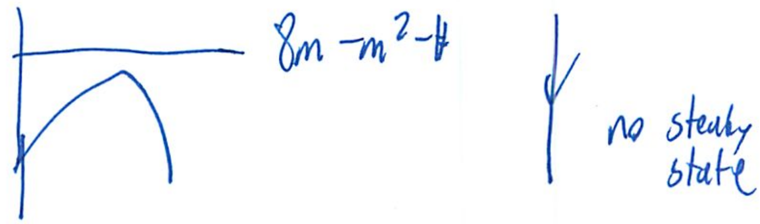
$8m - m^2 - \underbrace{H}$

Shifts up + down

(4)

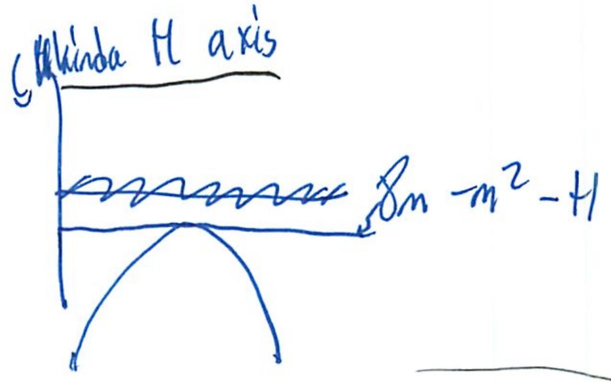
When $H = 0$

Then as H ↑ So if



So care when $8m - m^2 - H = 0$

So find the max



So $\frac{d}{dm}(8m - m^2 - H) = 0$

$m = 4$

$H = 16$

→ any point on curve is steady state for diff value of Hs

<u>Goal Get max H</u>

- vary H to find max Steady state $8m - m^2 - H$

(5)

Check previous result

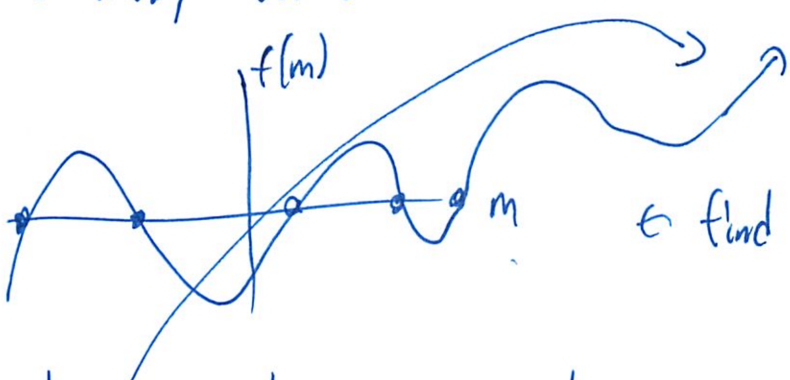
2. $\frac{dm}{dt} = 8m - m^2 - 12$

$= -(m-2)(m-6)$

← 2 possible steady states

Use the stable one, or it will fall off

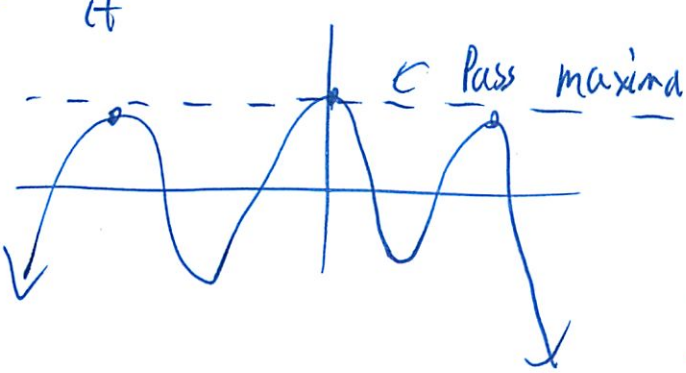
If had crazy curve



← find 0-isocline

there can harvest as much as you want

But if



← find the highest one

By taking deriv + set = 0

Its like moving it down that many units

(6)

Mixing Problem

30 gal of water w/ 10 gals water
10 oz Salt

Salt water flows in at 4 gallon/min
— 3 oz of salt / gallon

Immediate mixing

Water flows out 2 gallons/min

What is the concentration of salt when tank is $\frac{2}{3}$ full?

$\frac{dS}{dt}$ = or $\frac{dW}{dt}$? or both???

? I think

$$\frac{dS}{dt} \text{ in} = \frac{4 \text{ gallons}}{\text{min}} \cdot \frac{3 \text{ oz salt}}{\text{gallon}} = 12 \text{ oz/min}$$

$$\frac{dS}{dt} \text{ out} = 2 \cdot S(t) \frac{\text{oz}}{\text{gallon}}$$

$$\frac{dS}{dt} = 12 - 2S(t)$$

?or is it logistic

~~But~~ then IV $S(0) = \frac{10 \text{ oz}}{10 \text{ gal}} = \frac{1 \text{ oz}}{\text{gal}}$

(7)

But then how do we know when tank $\frac{2}{3}$ full

Oh easy

$$\frac{dW}{dt} = 4 - 2 = 2$$

So when 20 gal → in 5 min

So want $S(5) = ?$

Now need to solve (optional)

$$2S \cdot \frac{dS}{dt} = 12$$

$$\int 2S\, dS = \int 12\, dt$$

$$\frac{2S^2}{2} = 12t + C$$

$$S = \sqrt{12t} + C$$

Find C

$$1 = \sqrt{12 \cdot 0} + C$$

$$C = 1$$

$$S(5) = \sqrt{12(5)} + 1$$

I think did pretty well - 70% confident

only part is word problem / setup

⑧

Ans I Figure out when $\frac{2}{3}$ full

in 5 min ✓

Then how salt changing

define $S =$ amt salt

$$\frac{dS}{dt} = \frac{4\,gal}{min} \cdot \frac{3\,oz}{gal} - \underbrace{\frac{S}{10+2t}}_{\text{do need this}} \cdot \frac{2\,gal}{min}$$

Current concentration
- how much salt is in this water

Plus It asks for concentration

$$= \frac{S(5)}{20}$$

(I think I've paid much more attention to my math ability - stop + Think)
in last week

18.03 Exam 1 Review

9/25

I never really read book
- in part since spread b/w book + notes ....

Perhaps I should to review

ODE = single variable
1st order = only 1 deriv

$$\frac{dx}{dt} = f(x,t)$$

ie $\frac{dx}{dt} = x^3 t + t^2 + \sin x$

Many can't be solved ~~an~~ exactly
- Some can ; seperation of variables

$$\frac{dx}{dt} = g(x)\, h(t)$$

$$\frac{1}{g(x)} \frac{dx}{dt} = h(t)$$

$$\int \frac{1}{g(x)}\, dx = \int h(t)\, dt$$

- ? $\boxed{\frac{dx}{dt}\, dt}$

(2)

Try Day 1 problem

$$\frac{dx}{dt} = 2xt + x$$

all x all t

but first factor out x

$$\frac{dx}{dt} = x(2t+1)$$

$$\frac{1}{x}\frac{dx}{dt} = 2t+1$$

$$\int \frac{1}{x}\frac{dx}{\cancel{dt}}\cancel{dt} = \int 2t+1\,dt$$

$$\ln(x) = \frac{2t^2}{2} + t + c$$

Raise to e since want just x =

$$e^{\wedge \ln(x)} = e^{t^2+t+c}$$

$$x = e^{t^2} e^{t} \underbrace{e^{c}}_{\text{call } A}$$

$$x = A e^{t^2} e^{t}$$

$$= A e^{t^2+t}$$

C √bingo

3

Slope fields

Plot

$\frac{dy}{dx} = y^2 - x$

y \ x	-2	-1	0	1	2
-2	6				
-1					
0					
1					
2					

etc

then plot ↑ these are slopes

Pick a point + follow it

↳ solution line

Line where all slopes same = isocline

; Solutions avoid ;

Like our example

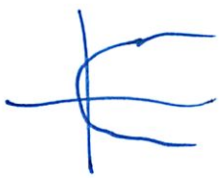
doesn't look like isocline

?

(9) But can find algebraically. Like 0-isocline

$0 = y^2 - x$

$x = y^2$



Somehow on curve 0 deriv

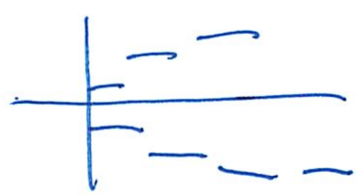
Because I'm thinking 0 deriv is —

So everywhere were curve is flat

Used <u>isoclines</u> app

I was right in both case

Are actually slopes



but hard to see

esp in my drawling

<u>Lower fence</u> = sol never crosses from above

<u>Upper fence</u> = below

'Pieces of 0-isocline

? must it always be an isocline

(5)

L'(x) will always be ⊖ ; for upper, lower, or both ?

↑ this is a lower

If it were to be ⊕ would be contradiction

Squeeze Theorm → 2 curves w/ same asy. behavior all curves inside must have same behavior

---

Lecture 2

- Was this fenses + funnels like prof said
- Or numeric methods like on Calendar ?

But 1st self review

Closed form - bounded/expressed by well known fns ⊕ ⊖ ⊗ ⊘ no ∞ series $\int$, lim, ∞

linear - not product of functions

Solution - t that satisfies y(t) on given interval

- if you were to plug it in, would satisfy eq
- Sometimes ≥ 1 possible sol (move up + down)
  - So look at inital condition

general solution - ~~know~~ no inital value, leave as variable

(6)

explicit sol y = something

implicit y is on both sides

—or—

y^{power} =

---

8)

isoclines are limiting behavior

Do a population model example

1965 = 3.32 billion people

For every 1000 people 50 born, 25 die each year

? So how would you even write this so units are correct?

$$\frac{\Delta P}{\Delta t} = 50 - 25 \cdot \frac{pop(t)}{1000}$$

? then have to multiply later

? divide by thousand

No this is change

don't think will have to

General model $\frac{dP}{dt} = a\,P(t)$

makes sense like a 2% increase each year

~~[illegible]~~ $a = .02$

(7)

a) How long for pop to double?

$$\frac{dP}{dt} = a\,P(t)$$

all P all t

$$\frac{1}{P(t)}\frac{dP}{dt} = a$$

$$\int \frac{1}{P(t)}\,dP = \int a\,dt$$

$$\ln P = at + c$$

$$e^{\ln P} = e^{at}e^{c}$$

$$P = Ae^{at}$$

Now set base - inital condition

$$3.32\text{b} = A\underbrace{e^{.02\,0}}_{e^0=1} \leftarrow \text{year } 0$$

$$A = 3.32 \text{ billion}$$

Now av - when 6.64 b

$$6.64\text{b} = 3.32\text{bill}\,e^{.02t}$$

[solve for t

too hard w/o calc

(a) Or is it:

$$2 = e^{.02t}$$

$$\ln \quad \ln$$

$$\ln 2 = .02t$$

$$t = \frac{\ln 2}{.02}$$ ← that can't do

good - did not get originally

b) What is pop in 2500?

$t = 2500 - 1968$

$$P = 3.32b\, e^{.02(2500-1968)}$$

etc.

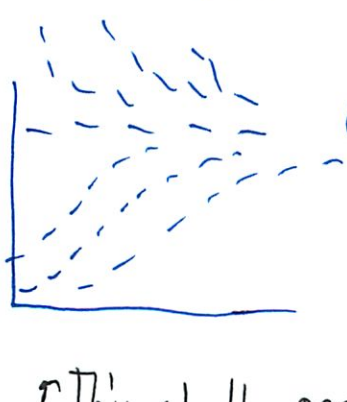
c) How much space

- jus do question

$$\frac{\text{total space (given)}}{\text{answer to b}}$$

~~Sketch slope field~~

Now more complex $\frac{dP}{dt} = aP - bP^2$ $\quad 0 < b << a$



0-isocline

↑ where stable

↑ This shall practice on - not 100% clear

9

Lecture 2 (missed)

1. Prove F+F long term behavior
2. Guess existance/uniqueness ODEs
3. Numerical Sols - Euler's method

↑ with the steps

Lower $L'(x) < f(x, L(x))$

Upper $U'(x) > f(x, U(x))$

Funnel (narrowing) $U(x) > L(x)$

$$\lim_{x \to \infty} |U(x) - L(x)| = 0$$

Sols always remain inside

Antifunnels $L(x) > U(x)$

- is always a sol inside
- if $\frac{df}{dy} \geq 0$ then unique sol stays inside

Theorm about IVP and drawing a box

I never really got this whole box business

Use theorm before trying numeric methods

(10)

Euler's method Pick initial pt (x_0, y_0). Flow along line. Repeat

$$(x_0, y_0)$$
$$\downarrow$$
$$(x_1, y_1) = \left(x_0 + \Delta x,\ y_0 + \Delta x\, f(x_0, y_0)\right)$$

Can be very wrong
Compounds errors, so errors cumulative
So (1) small step size
(2) Carry lots of digits

. Does solution exist + is unique?
- if f is "nice" near initial pt
  - function exists
  - derivative exists

  } in rectangle

- Solution exists on some interval

(Get back that reasoning about math from 6.042)

Estimate error w/ Taylor Series + remainder

$$\left| \underbrace{y(x_n)}_{\text{actual}} - \underbrace{y_n}_{\text{estimate}} \right| \le C \cdot \underbrace{h}_{\text{step size}}$$

$$C = \frac{\text{error}}{h}$$

(12)

You can solve this one via sep of variables

$$\frac{dy}{dx} = 2y - y^2$$

$$\frac{1}{2y-y^2}\frac{dy}{dx} = 1$$

$$\int \frac{1}{2y-y^2}\,dy = \int 1\,dx$$

$$\ln(2y-y^2) = x+C$$

$$e^{\ln(2y-y^2)} = e^x \cdot e^C$$

$$2y-y^2 = Ae^x$$

Now solve for y

No integrated wrong
need to use partial fractions

$$= \int \frac{1}{2y} + \frac{1}{2(2-2y)}\,dy$$

$$= \frac{1}{2}\ln|y| - \frac{1}{2}\ln|2(2-y)|$$

Still complicated

13

But I never answered how we find 2

I guess we could see $\downarrow$ ~~pts~~ slopes > 2

$\uparrow$ ~~pts~~ slopes < 2

Or ~~I'll~~ I think this will come up later when we do line plots

---

No! We can solve for 0-isocline like for previous ← think!!!

$$\frac{dy}{dy} = \cancel{\quad} 2y - y^2$$

$$0 = 2y - y^2$$

Solve for y

$$y^2 = 2y$$

How do you solve that?

Can G+V see $y=2$

But is there better way

Could plot both + find visual intersections

Think textbook way was look at TI tables

EJ: could use quadratic formula

$a=1$ $b=2$ $c=0$

(14)

9/27

~~At~~ Autonomas ODEs

~~At~~ - independent

- $\frac{dy}{dx} = f(y)$

ind of x

So like $\frac{dy}{dx} = f(y) \cdot 1$

Can $\int$ both sides, but nasty

The salt qu are ~~not~~ autonomas

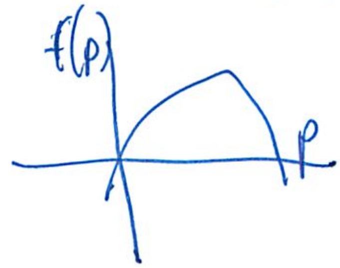
So concentration is in this too!

$S(t)$ = lbs salt

- not concentration!

In recitation today did what is harvest w/ largest

Steady state



~~Bal~~ So question $\frac{ds}{dt} = 2 ~~\#~~ - \frac{s}{400} + l$

it would be

(15)

$$\frac{d}{ds}\left(2 - \frac{s}{400} + l\right)$$

d constant

$\uparrow s$

$$0 - \frac{1}{400} + l$$

First do ~~class~~ recitation one

$$\frac{d}{dm}\left(8m - m^2 - H\right) = 0$$

$$8 - 2m - H = 0$$

$$8 = 2m + H$$

But still 2 variables

Yeah - no forget H - just get m

$$8 = 2m$$

$$m = 4$$

Then plug in for 0 iso

$$8 \cdot 4 - (4)^2 + H = 0$$

$$32 - 16 + H = 0$$

$$H = -16$$

↑ harvesting

So at $s = -\frac{1}{400}$

Then plug in to find H ..

~~then to get~~ If have H, can solve (find roots) to see steady states

(16)

Harvesting

$$\frac{dP}{dt} = R\,P(M-P) - h$$

oh man subtract
So is positive

Varry h to find behavior

the phase line pts change

(Wow this is coming back - reviewed in class a lot)

When # phase line pts change (as h changes)
= biforcation point

Linear Eq

$$\frac{dy}{dx} + P(x) \cdot y = Q(x)$$

So multiply both sides by $e^{\int P(x)\,dx}$

Then LHS is

$$\frac{d}{dx}\left(y \cdot e^{\int P(x)\,dx}\right)$$

RHS

$$Q(x)\,e^{\int P(x)\,dx}$$

+ integrate

(17) Practice men

$e^{\int p(x)\,dx}$

$$\underbrace{\frac{d}{dy}\left(y\, e^{\int p(x)\,dx}\right)}_{\text{integrating this is simple}} = \underbrace{Q(x)\, e^{\int p(x)\,dx}}_{\text{integrating much harder}}$$ ✓

$y\, e^{\int p(x)\,dx}$

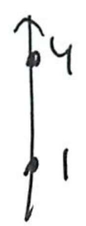
Quick Quiz 4

$$\frac{dy}{dx} = y^2 \overset{+}{-} 4y \overset{\times}{+} 3$$

~~Solve~~ Find phase line

Find roots

$(x-1)\cancel{(x-4)}$



Now test pts

$y=5$ $25-20+3$ (+)

$y=3$ $9-12+3$ 0

Opps $(x-3)$!

$y=2$ $4-8+3$ (−) ✓

$y=6$ 3 (+)

(18)

Runge Kutta - skipping

Practice a simple linear

$$\frac{dy}{dx} = \cancel{x}\, y + x$$

So want

$$\frac{dy}{dx} + P(x) = Q(x)$$

$\uparrow$ -1 $\quad$ y

✓ Is dy

$$e^{\int -1\, dx} = e^{-x}$$

$$\int \frac{d}{dy}\left(y e^{-y}\right) = y e^{-y}$$

I messed up setup

$$\frac{dy}{dx} + P(x) \cdot y = Q(x)$$

So $P(x) = -1$ $\quad$ $Q(x) = x$ ✓

$$y e^{-y} = \int x\, dx = \frac{x^2}{2} + C$$

Need to x this as well

$\int e^x (y' - y)$

Not explicit
- can't solve further?

integrate by parts (ug)

(19)

No actually try

$$\boxed{\begin{array}{l} u = x \\ du = dx \\ dv = e^{-x} \qquad v = e^{-x} \end{array}}$$

So try to find $u' f'(u)$

No thats not right

~~Didn't~~ No dx is in there

$dx\ e^{-x}$

So

$-y e^{-x}$??

WA no response!

So I am concerned about this
Feel like skipping but shouldn't

Oh I wrote earlier (in black)...
~~Prob~~ TA never explains

On other side forgot It

$\int x e^{-y} dx$ ~~Prod rule~~

$\frac{x^2}{2} e^{-y} + x e^{-y}$

Now divide out e^{-y}

$y = \frac{x^2}{2} + x + C$

No! Its integration

$= \frac{1}{2} x^2 e^{-y}$

still cross out

So $y = \frac{1}{2} x^2 + C$

(21)

Neat way

$$\int_2^t \frac{d}{ds}\left(y e^s\right) ds$$

$$= \int_2^t \frac{e^s}{1+s^2}\, ds$$

alt. fund theorem calculus

$y(t) e^t - y(2) e^2 =$

I totally forget this

Use algebra to solve for y

$$y(t) e^t - 3e^2 = \int_2^t \frac{e^s}{1+s^2}\, ds$$

I can't imagine that being on there...

Input Response

Do want to look over

Not sure what is

$$\frac{dy}{dt} + ky = Q(t) \qquad y(0) = 0, \quad k \text{ const}, > 0$$

like a reduced version of linear

(22)

Since $k > 0$, $\underbrace{Y_0 e^{-kt}}_{\text{transient}} \to 0$ as $t \to \infty$

- Signal prossing

So asy appraoch

$$e^{-kt} \int_0^t Q(s) e^{ks}$$

¿do we need to remember anything special?

Complex #

Remembel formulas!

$z = a + bi$

to add - add reals, add Img

to divide - rationalize denom so only reals

$z = r(\cos\theta + i\sin\theta) = re^{i\theta}$

$z_1 \cdot z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$

$e^{i\pi} + 1 = 0$

Think w/ OTL remember better

And if get lost, think about it!

$e^{2\pi i} = 1$

(23)

This nth root of unity thing I never really got

$\left(e^{2\pi i/n}\right)^n = 1$

$z = e^{2\pi i/n}, e^{2\pi 2i/n}, \dots e^{2\pi i(n-1)/n}, 1$

Did one in OH

Try One

$z^3 = 1$

$(re^{i\theta})^3 = 1$

$r^3 = 1$

$3\theta = 0 + 2\pi k$ for some k

$r = 1$

$\theta = \frac{2\pi}{3}, \frac{4\pi}{3}, 0$

Evaluate complex sols in polar

$e^z = 1 + i$

; do OH one

(24) OH Problem

4th root of 1

$$(r,\theta)^4 = 1$$

$$r^4 e^{4\theta i} = 1$$

$$e^{\theta i} = \cos\theta + i\sin\theta$$

$$e^{4\theta i} = \cos 4\theta + i\sin 4\theta$$

$$r^4 e^{4\theta i} = \underbrace{r^4\cos 4\theta}_{a} + \underbrace{r^4\sin 4\theta}_{b}\, i$$

Want $1 = a + bi$

+ ~~since~~ want.

since want 1 i

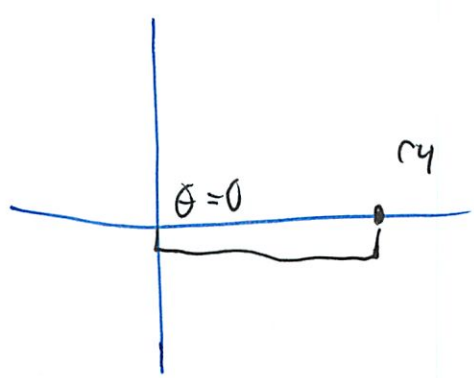
So want $1 + 0i = a + bi$

$a = 1$ $\quad b = 0$

$r^4\cos 4\theta = 1$

$r^4\sin 4\theta = 0$

Solve sim.



So $r^4 = 1$

$r = 1$

$\theta = 0$

(25)

But only 1 sol

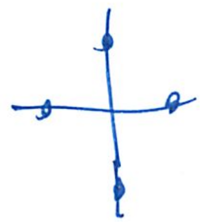
$4\theta = 0 + 2\pi k \quad k \in \mathbb{Z}$

Divide

$\theta = \frac{0}{4} + \frac{2\pi}{4}$

$\theta = 0 + \frac{\pi}{2} k$

So



$\int$ Complex $= \int Re + i \int Im$

- can $\int$ before or after
pulling out Re / Im part

Big trick $\int \sin 3x$

is $\int Im(e^{\cancel{\quad} \to 3xi}))$

Just want im / sin part

e^{x+3xi} logic easier

(26)

What is ~~A~~ $\tilde{y}$ again

- the complex variable
- but what exactly?

Seems to be just y but when other side

written as $e^{(1+4t)}$

kinda skipping practice on this

· should try one at one point

Thats pretty much it

- rest of notes are review
- no one more

Lecture 7

Oh $Re(\tilde{y}) = y$

(I don't get this pg)

$$\frac{d}{dt}\tilde{y}e^{2t} = \int e^{(2+i)t}\,dt$$

$$\tilde{y}e^{2t} = \frac{1}{2+i}e^{(2+i)t} + c'$$

$$\tilde{y} = \frac{1}{2+i}e^{it} + \underbrace{c \cdot e^{-2t}}_{\text{transient}}$$

27

Take Re of both sides

$$y = \mathrm{Re}\left(\frac{1}{2+i} e^{it}\right) + c e^{-2t}$$

Two ways to find Re

1. Express as real + imag parts + expand

   traditional way

2. Express in polar r, θ form

$$\underset{\text{amp}}{A} \cos(\underbrace{\omega}_{\substack{\text{circular} \\ \text{freq}}} t - \underset{\text{phase lag}}{\phi})$$

$$\frac{2\pi}{\omega} = \text{period}$$

example Newton's Law of Cooling

Superposition

etc

Step / Box functions

Now do practice!

Retry Practice Exam

9/27

1. Damn

$\cancel{\cos x}$ $\cos x \tan y \frac{dy}{dx} + \sin x \cos y = 0$

Ys Xs

$\overline{\cos x}$ $\overline{\cos y}$

$$\frac{\tan y}{\cos y}\frac{dy}{dx} = -\frac{\sin x}{\cos x}$$

trig identites ??

$$\int \frac{\tan y}{\cos y}\, dy = -\int \frac{\sin x}{\cos x}\, dx$$

can do $e^{\wedge}$ ← dont think makes sence

Cheat ~~on~~ $\frac{\sin x}{\cos x} = \tan x$ $\frac{\tan y}{\cos y} = \tan y \sec y$

Hopefully are written - this is not a calc class

$\int \tan x = -\ln(\cos x)$ $\int \tan y \sec y = \sec y$

$\sec y = -\ln \cos x + C$

No explicit sol

②

$$\frac{dy}{dx} = \frac{-\sin x \cos y}{-\cos x \tan y}$$

$$\frac{\sin y}{\cos^2 y} \frac{dy}{dx} = \frac{-\sin x}{\cos x}$$

Oh expand tan to be $\frac{\sin}{\cos}$ - duh!

$\int \qquad \int$

$y = \arccos(\ln(\cos x) + c)$

So basically what I had
plus last step take arcsec() both sides
But I would not know trig to do

---

2. $\frac{dy}{dx} = f(x, y)$ continuous + deriv continuous

a) -1 isocline $\quad y = e^{-x}$

$\quad 1 \quad y = -e^{-x}$

for which I.V know

(3)

So Fence + Funnel qu

What was test for it again

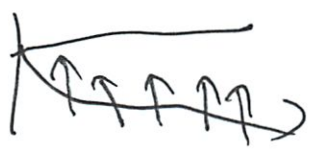
deriv > 0

deriv < 0

So draw

e^x First is

e^{-x}



$-e^{-x}$

So antifunnel

for $x > 0$ to $y \to \infty$

(I think I drew e^x wrong)

$C'(x) > f(x, C(x))$ upper

$\frac{d}{dx}(e^{-x}) = -e^{-x} > -1$ if $x > 0$ so

e^{-x} upper fence

Funnel - all values $\to 0$, But can't say else - don't know $f(x,y)$

4

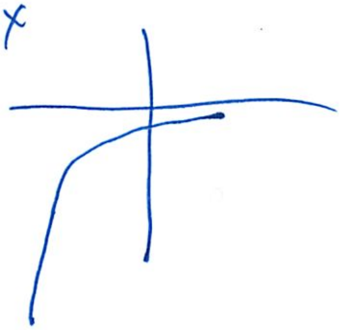
e^x

damn

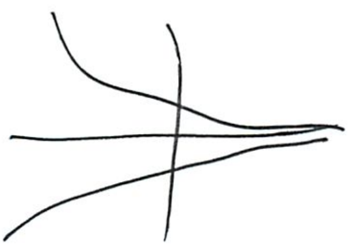
e^{-x}

wrong way

$-e^{-x}$



So



all sol $\to 0$

Look more at fence rule

$$\underbrace{C'(x)}_{\text{take deriv}} > f(x, C(x))$$

since looking $y = C(x)$

b) Reversed

Guessing is now anti-funnel

So can't say anything - except not go to 0

(5)

No there is a sol which is inside

Can't say if unique since don't know $\frac{\partial f}{\partial y}$

($\frac{\partial f}{\partial t} \geq 0$ for uniqueness)

C. If 2-isocline has eq $y = 2^x$ use Eulers

- So I remember needed to be done in 1 step for some reason
- Had to eval 2^x or something

0	1
3	$(3, 1+2^3)$

$2^0 = 1$ is on isocline

Only know ~~0~~ 2-isocline

Flow along

$y(3) = 1 + 2(3) \approx 7$

So what did I mess up?

$y(t_0 + h) \approx y(t_0) + h f(t_0, y(t_0))$

1 2

← Oh the result

but isocline tells us about slope

(c)

: Or we only know when slope $= 2$

$$y = 2^x$$

So need slope $= 2$

$2^0 = 1$ is on 2-isocline

So follow it

- only pt we know

Don't forget h.

And use old values

---

3. $$\frac{dP}{dt} = -P(P-3)^2 + r$$

Ok test at recitation

Now $r=0$

Find 0-isoclines

$$0 = -P(P-3)^2$$

$$\cancel{-P^2 - 3P}\ (P-3)(P-3)$$

$$-P(P^2 - 6P + 9)$$

(7)

How am I supposed to find cubic root?

~~#~~ 0 + v Integers?

0 ✓

Just look at other side

already have 3

Now test

$p=4 \quad -4(16-24+9)$ ⊖

$p=2 \quad -2(4-12+9)$ ⊖ ??

$p=-1 \quad \cdot 1(1+6+9)$ ⊕

Re do 1st one

$-4(16-24+9)$ ⊖

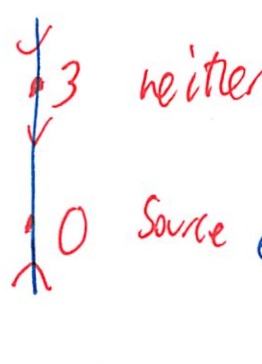
$p=5 \;|\; -5(25-30+9)$ ⊖

$p=2$? $\quad -2(4-12+9)$ ⊖

$p=1 \quad -1(1-6+9)$ ⊖

So 3 is not a root?

0 is stable



but that should be a sink ????

↓ was error in solutions before

(8)

b) Find biforcation pt

$$\frac{d}{dp}\left(-p(p-3)^2\right)$$

chain rule
prod rule

$$-p \cdot 2(p-3) \cdot 1 + (p-3)^2 \cdot -1$$

$$-2p^2 + 6p + -p^2 + 6p - 9$$

$$-3p^2 + 12p - 9 = 0$$

? How solve this
– divide by 3

$$-p^2 + \overset{+}{4}p - \overset{\times}{3} = 0$$

? sign errors

$$p^2 - \overset{+}{4}p + \overset{\times}{3} = 0$$

$$(x-3)(x-1)$$

which is it ? – try 3
– should only be one ??

(9)

Try $x = 3$

~~BZA~~

$$-3(3-3)^2 + H = 0$$

$H = 0$ ← don't need

So at $x = 3$

Oh no at what value c (or H)

- So I did wrong
- But did what did in recitation today
  - or what I thought we did

$c = 0$ is bifurcation

But need $c > 1$

Can draw

See double root when shift ↑ c units where

- c is eq to f at missing

Local min of $f(P)$ is $(1, -4)$ at (t, P)

i so $df(P)$

(10)

Let me try deriv of whole thing

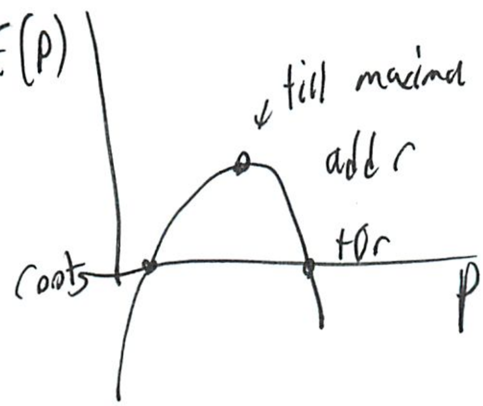
$$\frac{d}{dp}\left(-p\,(p-3)^2 - r\right) = 0$$

$$-3p^2 + 12p - 9\,\cancel{r} = 0$$

; but what solve this for

what is p where this is max

Think in pic



So r falls away

Thought max at $P=0$

$f(P)$ is 0

Then r will be that $f(p)$ value

— but that is 0!

What is $f(p)$ exactly?

$$\frac{dP}{dt}$$

10

Lecture: take deriv $aP - bP^2$, set $= 0$

$$0 = a - 2bp$$

what I did

But then does not say what

Take P plug in

No dP will be 0 ~~when P~~

~~Need~~ No only when r ~~[illegible]~~ $= 0$

~~So find~~ but then my deriv is shotty

~~Mis~~ P is -4

So set at 4

So likely

WA: No roots 1, 3

Oh well move on

3c) If r really high, will bleed meadow dry

Oh this happens with all # anyway

(12)

4. Complex ODE + Find steady state

$$\frac{dA}{dt} + 3A = \sin(2t)$$

$P(t) = 3 \quad Q(t) = \sin 2t$

$$\int \frac{d}{dt}\left(A e^{3A}\right) = \int \sin 2t \, dt$$

$$A e^{3A} = \int \left(\text{Im}\left(e^{2ti}\right)\right) dt$$

; what next

$$\int e^{2ti} = 2 e^{2ti}$$

$$A e^{3A} = \text{Im}\left(2 e^{2ti}\right) + c$$

; so now convert back

$$A e^{3A} = 2 \sin 2t + c$$

; can't find explicit

~~at A=0~~ will always ossilate

Complexify to $\dfrac{d\tilde{A}}{dt} + 3\bar{A} = e^{(2t)i}$

Take Im() both sides

$\text{Im}(\bar{A}) = A$ since $e^{(2t)i} = \cos 2t + i \sin 2t$

(13)

$$\frac{d}{dt}\left(\tilde{A} e^{3t}\right) = \underbrace{e^{(3+2i)t}}$$

I forgot to multiply this side by the IF!!

e^{3t}

$$\tilde{A} e^{3t} = \frac{1}{3+2i} e^{(3+2i)t} + c$$

also the integration gives $\frac{1}{\text{value}}$

$$A = I_m\left(\frac{1}{3+2i} e^{(2i)t}\right) + c e^{-3t}$$

* IF $\int$ is same as main problem

$e^{\int 3\, dt}$

- no did not do wrong earlier
- remembered wrong

So put polar form

$$3 + 2i \rightarrow r e^{i\theta}$$

$$r = \sqrt{3^2 + 2^2} = \sqrt{13}$$

$$\theta = \tan^{-1}\left(\frac{2}{3}\right)$$

(14)

So compute

$$\mathrm{Im}\left(\frac{1}{3+2i} e^{(2i)t}\right)$$

$$= \mathrm{Im}\left(\frac{1}{\sqrt{13}} e^{-\tan^{-1}(2/3)} e^{(2i)t}\right)$$

$$= \frac{1}{\sqrt{13}} \sin\left(2t - \tan^{-1}(2/3)\right)$$

So $A(t) = \frac{1}{\sqrt{13}} \sin\left(2t - \tan^{-1}\left(\frac{2}{3}\right)\right) + c\, e^{-3t}$

Steady state sol is when $c=0$

?? really?

~~Ut~~ ce^{-3t} is transient sol as tends to 0 as $t \to \infty$

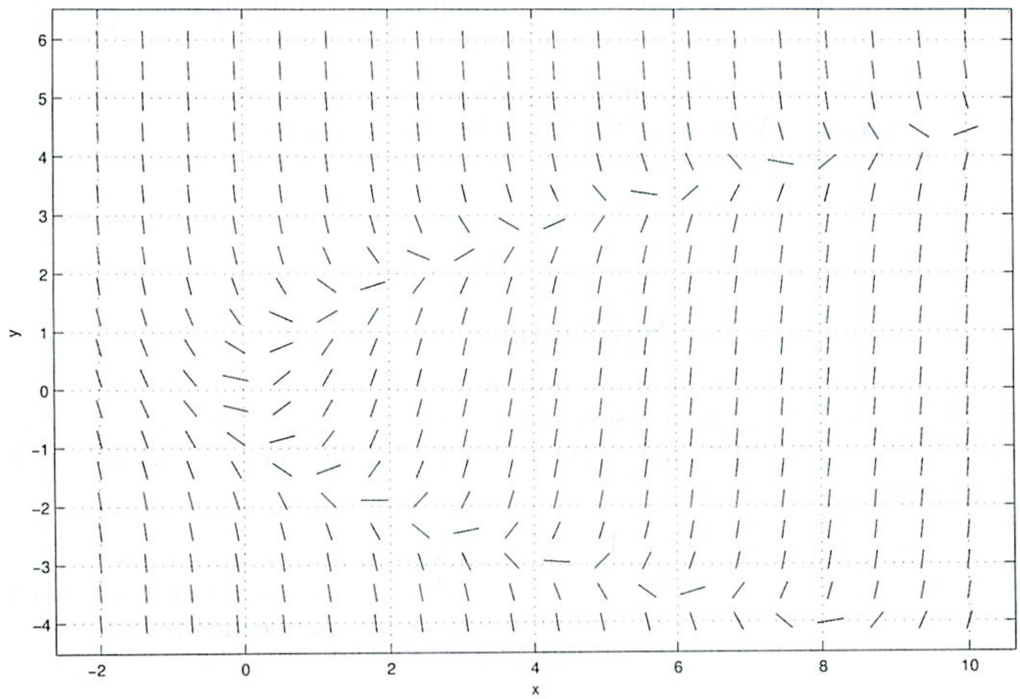
I think that was that last lecture day
I did not understand.

18.03 Practice Hour Exam I, February, 2008

1. (a) A pot of water cools at a rate proportional to the difference T between its temperature and room temperature. It is observed that over an interval of $10\ln 2 \simeq 6.93$ minutes the value of T is halved. On the stove, though, there is a second process at work: heat is added to the water at a constant rate, increasing temperature at the rate of 8 degrees per minute. Write down a differential equation for T which controls it while the pot is on the stove.

(b) Estimate $y(2)$ where y is the solution of the differential equation $y' = x + y$ with $y(0) = 0$, using Euler's method with step size 1. Do you expect your estimate to be too high or too low? Why?

2. The direction field of a differential equation $y' = F(x, y)$ is illustrated.



(a) On the direction field, sketch the graph of the solution $y(x)$ with $y(0) = 0$. Continue it in both directions till it leaves the direction field box.

(b) Some solutions of this differential equation grow when x is large and some do not. On the graph, sketch the curve representing the boundary between these two behaviors. Continue it in both directions till it leaves the direction field box.

(c) The null-cline is given by the equation $2x = y^2$ Estimate the value $y(50)$ of the solution you graphed in (a).

Is your estimate high or low?

3. Find a solution of $\dot{x} = x + 2te^t$, by any method.

4. Find a sinusoidal solution to the differential equation $\dot{x} - 2x = 4\cos(3t)$. Express your answer as a sum of sines and cosines. You may use any method to find this solution.

5. (a) Compute explicitly in the form $a + bi$ all the cube roots of i.

(b)–(e) relate to the sinusoidal function $2\sin(\pi t) - 2\cos(\pi t)$. Determine

(b) its period P

(c) its amplitude A

(d) its phase lag ϕ

(e) its time lag t_0

6. A grower of mushrooms wants to maximize his harvest rate a (in tons per week). The number of tons of mushrooms in his farm obeys the logistic equation $\dot{y} = -y^2 + 4y$ in the absence of harvesting.

(a) Sketch the phase portrait of this differential equation as it is, in the absence of harvesting. Sketch some solutions, including all equilibrium solutions.

(b) What is the largest harvesting rate a that will allow for a constant mass of mushrooms?

(c) The farmer maintains a 3 ton per week harvest rate over a long period of time, with a constant mushroom mass. What must that mass be, approximately?

Solutions

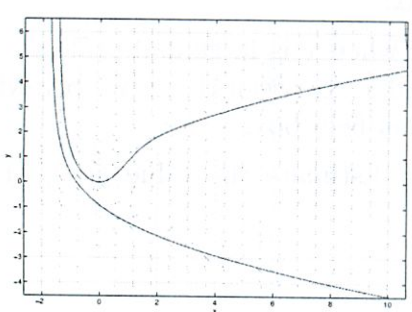
1. (a) $\dot{T} = -kT$ off the stove; this has solution $T = Ce^{-kt}$. Thus $C/2 = (1/2)T(0) = T(10\ln 2) = Ce^{-k\cdot 10\ln 2} = C(1/2)^{10k}$ and so $10k = 1$ or $k = 1/10$. On the stove the ODE is $\dot{T} = -(1/10)T + 8$.

(b)

k	x_k	y_k	$A_k = x_k + y_k$	$hA_k = A_k$
0	0	0	0	0
1	1	0	1	1
2	2	1		

So $y(2)$ is approximately 1. Since $y_1 = 0 < A_1 = 1$, the vector field has been rising under that segment, and the estimate is too low.

2.



(a) The top curve is the solution through $(0, 0)$.

(b) The lower curve is the separatrix: solutions above it grow for x large, solutions below it do not. [In fact, solutions below it reach $-\infty$ in finite time. The separatrix is a solution itself, the only solution which is always falling and which is defined for all large x.]

(c) The graphed solution is trapped by the funnel having the nullcline as its upper fence, so $y(50)$ is very near to $\sqrt{100} = 10$. Since it's approaching from below, the estimate is (very slightly) high.

3. The standard form is $\dot{x} - x = 2te^t$. The homogeneous solution is e^t, so we substitute $x = e^t u$: $\dot{x} = e^t u + e^t \dot{u}$, so $2te^t = \dot{x} - x = e^t \dot{u}$ or $\dot{u} = 2t$. This integrates to $u = t^2 + c$, so $x = t^2 e^t + ce^t$. Since only one solution was asked for we can take $c = 0$ or anything else.

Alternatively, e^{-t} is an integrating factor, and $2t = e^{-t} \cdot 2te^t = e^{-t}(\dot{x} - x) = \dfrac{d}{dt} e^{-t} x$ so $e^{-t} x = \int 2t\, dt = t^2 + c$ or $x = (t^2 + c)e^t$. Again c can be anything.

4. First solve the complex-valued equation $\dot{z} - 2z = 4e^{3it}$. This can be done using integrating factors or variation of parameters, or by trying $z_p = Ae^{3it}$ and solving for A: $A3ie^{3it} - 2Ae^{3it} = 4e^{3it}$ implies $A = 4/(3i - 2)$.

Thus $z_p = \dfrac{4}{-2+3i} e^{3it} = \dfrac{4(-2-3i)}{13} e^{3it}$, whose real part is $x_p = (4/13)(-2\cos(3t) + 3\sin(3t))$.

5. (a) The magnitude of i is 1, so the magnitude of each of its cube roots is 1. The argument of i is $\pi/2$, so the argument of one cube root is $\pi/6$. The others differ by $2\pi/3$ and $4\pi/3$ and so are $5\pi/6$ and $9\pi/6 = 3\pi/2$. The last gives $-i$, whose cube is indeed i. The others give $(\sqrt{3} + i)/2$ and $(-\sqrt{3} + i)/2$.

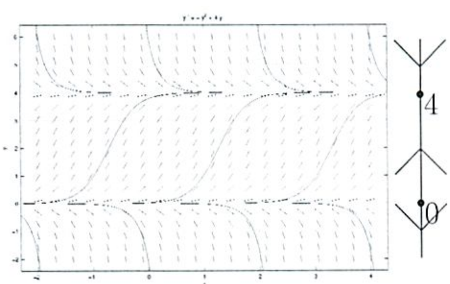
(b) $P = 2\pi/\omega = 2\pi/\pi = 2$.

(c) A is the length of the segment joining $(0, 0)$ to $(-2, 2)$: $2\sqrt{2}$.

(d) ϕ is the polar angle of $(-2, 2)$, which is $3\pi/4$.

(e) $t_0 = (P/2\pi)\phi = (1/\pi)(3\pi/4) = 3/4$.

6. (a)



(b) The maximum value of $g(y) = -y^2 + 4y$ is 4 (at $y = 2$), so $\dot{y} = -y^2 + 4y - 4$ has a semi-stable critical point at $y = 2$ and no larger harvest rate leads to any critical points. The largest sustainable harvest rate is 4 tons per week.

(c) $\dot{y} = -y^2 + 4y - 3$ has critical points at the roots of $y^2 - 4y + 3$, namely at $y = 1$ and $y = 3$. The critical point at $y = 1$ is unstable and can't be maintained over a long period of time; so the farmer must have $y = 3$.

Pre-exam Do Sin/Cos Question 9/28

from OH

$\frac{dy}{dt} + y = \sin t$

$P(x) = 1 \quad Q(x) = \sin t$

$e^{\int 1 \, dt} = e^{t}$ ← I did this wrong again!

$\frac{d}{dy}(y e^{t}) = \int \sin t \, e^{t}$

$= \int Im(e^{ti}) e^{t}$ ✓

$Im(\int e^{ti+t} \, dt)$ ✓

$\cancel{1 \cdot e^{ti+y} \quad dy}$

← think doing something wrong here w/ Im and ∫

It has dt

and $\frac{1}{blah}$ ✓ $\frac{1}{i+1} e^{ti+t}$

$y e^{t} =$

distribute

(2)

$$y e^t = \frac{1}{i+1} e^{ti} + \frac{1}{i+1} e^{it}$$

? and Were did I n part go?

First want just i part since Im( )

$$\frac{1}{i+1} \frac{(i-1)}{(i-1)} \quad \frac{i-1}{i^2+i-i-1} = \frac{i-1}{-1-1} = \frac{i-1}{2}$$

So $\frac{i}{2} e^{t+ti}$

$$y e^t = \frac{i}{2} e^{t+it}$$

Now solve for y

$$y = \frac{\frac{i}{2} e^{t+it}}{e^t}$$

Leave it at that I guess

- get most credit

18.03 Debrief

9/28

Majorly F-ed up logistic eq qu
I did not know what it was

$$\frac{dP}{dt} = P(t)\left(1 - P(t)\right)$$

The rest I prob did Ok on
My usual partial credit

18.03 EXAM I

Wednesday, September 28, 2011

Name: Michael Plasmeier

Recitation Instructor (Circle One): L. Chumakova / N. Olver / R. Rosales / V. Shende / J. van Ekeren / Z. Yun

Recitation Hour: #10

Instructions: You may not use calculators, notes, textbooks, or personal electronic devices. As a courtesy to other students, please turn off all cell phones. Read each question carefully. Whenever possible, include justification for your reasoning and show your work. Answers without any work or explanation will receive little or no credit. There are 5 questions and a 55 minute time limit on this exam. Good luck.

Question	Score	Maximum
1	3	4
2	6	8
3	2	9
4	8	14
5	0	6
Total	19	41

 Name: Plasmeier

1. Solve the following initial value problem exactly by any method:

$$(1+x)^2\frac{dy}{dx} - y^2 = 0, \quad y(0) = 1$$

ys xs

$$(1+x)^2 \frac{dy}{dx} = y^2$$

$$\frac{1}{y^2}\frac{dy}{dx} = \frac{1}{(1+x)^2}$$

$$\int y^{-2}\,dy = \int (1+x)^{-2}\,dx$$

$y^2 = \frac{y^3}{3}$

$$\frac{y^{-1}}{-1} = \frac{(1+x)^{-1}}{-1} + C$$

$$-\frac{1}{y} = -\frac{1}{(1+x)} + C \checkmark$$

$\cdot -1^{-1}$

$$\underline{y = (1+x) + C} \ X$$

$$1 = 1 + 0 + C$$

$$C = 0$$

$$\underline{y = 1 + x} \checkmark$$

2. a) Use complex methods to find the general solution to the differential equation

$$\frac{dy}{dt} + 4y = \sin 3t.$$

3

$e^{\int 4dt} = e^{4t}$

$P(x) = 4 \quad Q(x) = \sin 3t$

$\frac{d}{dt}\left(ye^{4t}\right) = \sin 3t \, e^{4t}$

$= \mathrm{Im}\left(e^{3ti}\right) e^{4t}$

$\int \frac{d}{dt}\left(ye^{4t}\right) = \int \mathrm{Im}\left(e^{3ti+4t}\right) dt$

$ye^{4t} = \mathrm{Im}\left(\frac{1}{3i+4} e^{3ti+4t}\right)$

$\longrightarrow$

Over

b) Thinking of this as an Input-Response model, what is the amplitude of the resulting response wave?

0

board $a\cos\theta + b\sin\theta = A\cos(\theta - \phi)$ w/ $A = \sqrt{a^2+b^2}$

So we had $a = 0$

$b = -\frac{1}{8}$ so $A = -\frac{1}{8}$

Must be positive

$-\frac{1}{8}\sin 3t = -\frac{1}{8}\cos(3t - \phi)$

from sin $\to$ cos $\frac{\pi}{2}$ offset

may be getting these

$$ye^{4t} = Im\left(\frac{1}{3i+4} e^{3ti+4t}\right)$$

$$\frac{1}{3i+4} \frac{(3i-4)}{(3i-4)}$$

$$\frac{3i-4}{9i^2-12i+12i-16}$$

Will always be +ve with conj. transpose

$$\frac{3i-4}{-9-16} = \frac{3i-4}{\cancel{24}\,25}$$

Only want Im

$$ye^{4t} = -\frac{1}{8} Im\left(e^{3ti+4t}\right)$$

$$ye^{4t} = -\frac{1}{8} Im\left(e^{3ti} e^{4t}\right)$$

prob messed up

$$y = -\frac{1}{8} Im\left(e^{3ti}\right)$$

$$= -\frac{1}{8} \sin 3t \quad + Ce^{-4t}$$

gr forgot on!

 Name: ______________________

2

c) Find the general solution to the differential equation (using any method or earlier work in (a) and (b))

$$\frac{dy}{dt} + 4y = 1 + \sin 3t.$$

This is prob easy - if knew trick

$$ye^{4t} = \int 1 e^{4t}\,dt + \int \sin 3t\, e^{4t}\,dt$$

same

$$= \frac{1}{4}e^{4t} - \frac{1}{8}\sin 3t\, e^{4t}$$

$$y = \frac{1}{4} - \frac{1}{8}\sin 3t + Ce^{-4t}$$

I took off a mark for (a) and (c) together for this

 Name: ______________________

$\frac{dP}{dt} = aP + bP^2$

3. A population of island turtles obeys a logistical growth model. Their limiting population is 400 turtles. But far away from this limit, the population grows at a rate of 75% of the current population each year.

a) Write a differential equation which models the population growth rate dP/dt.

gain $\frac{dP}{dt} = .75P(1)$ Seems too simple

$\frac{1}{.75P}\frac{dP}{dt} = 1$

$\int \frac{1}{.75P}\, dP = \int 1\, dt$

$(.75P)^{-1}$

$P = .375P^2$?

0

messy

b) Draw a phase line for this equation, labeling equilibrium solutions as sinks, sources, or neither.

0-isocline

$0 = .75P$

$P = 0$

1

$P = 10 \quad .375(10)^2 = +$

$P = -1 \quad +$

 Name: ____________________

c) Suppose that we allow harvesting of turtles at a rate of r turtles per year, harvested continuously. Write a differential equation that models the new growth rate with harvesting.

$$\frac{dP}{dt} = .75P - r$$

1

d) Which value of harvesting r is a bifurcation point for this model? What is the significance of this point for the turtle population?

$$\frac{d}{dP}(.75P) = 0$$

$$.75 = 0$$

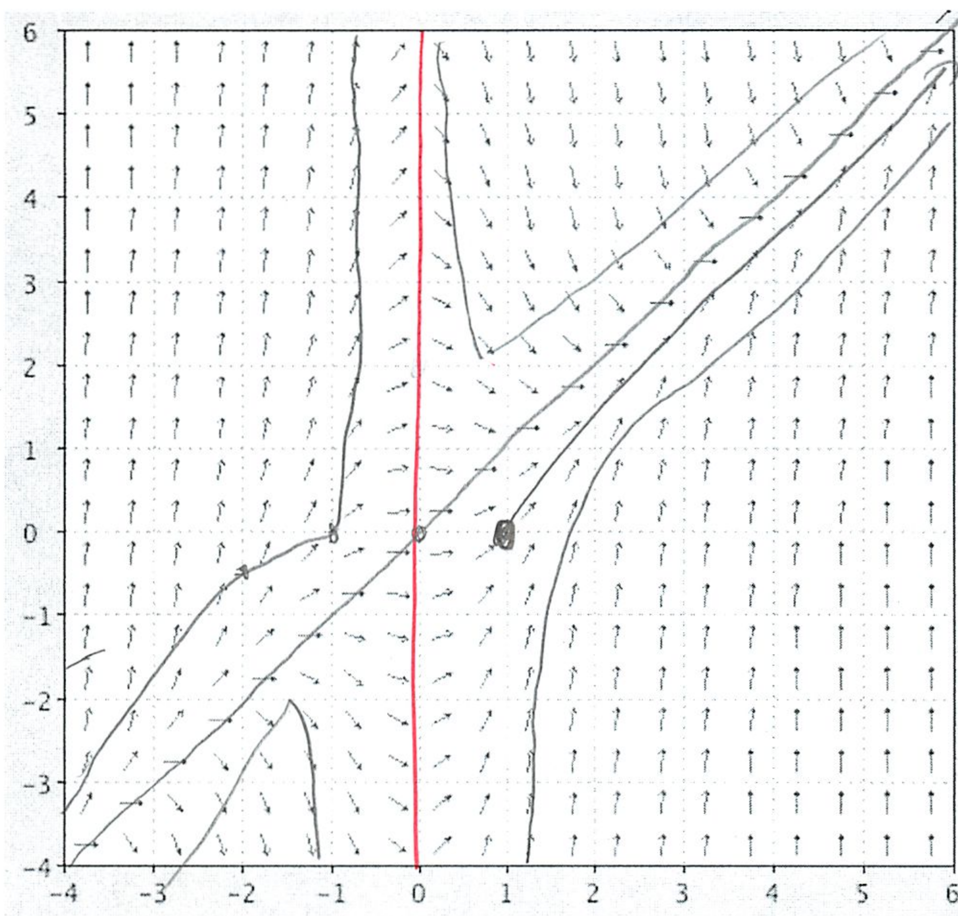
Take deriv $\frac{dP}{dt}$ + set = 0

Solve for max P

Then r is offset that is dP

I messed up earlier somewhere so this is not working

4. The differential equation $dy/dx = x^2 - xy$ has the following direction field:



(a) Write the equations for the 0, 2 and −2 isoclines. Then sketch them on the direction field.

2

$0 = x^2 - xy$

$xy = x^2$

$y = \frac{x^2}{x}$

$= x$

$\boxed{y = x \text{ and } x = 0}$

$2 = x^2 - xy$

$xy = x^2 - 2$

$y = x - \frac{2}{x}$

Start at $x = -4$

$y(-2) = -1.5$

$-2 = x^2 - xy$

$xy = x^2 + 2$

$y = x + \frac{2}{x}$

(b) Sketch the solution passing through the point $(1, 0)$ on the direction field.

3

$y(0) =$ und $y(2) = 1$

$y(-1) = 0$

$y(-4) = -4.5$

 Name: ____________________

2

(c) What is the long-term behavior of the solution through $(1,0)$? Be sure to justify your answer.

It approachey $y=x$

I guess you can see visually,

But also make funnel argument

$y=x$ upper fence $U' > f(x,y)$ ✓

$y=x-\frac{2}{x}$ lower fence $L' < f(x,y)$ ← show this

$U>L$ means fence

(d) Use Euler's method with step size 1/2 to approximate the value of ~~x~~ y(1) where ~~$x(t)$~~ y(x) is the solution passing through $(0,2)$.

y Correction

Correction y(x)

0	2
1	$2 = 2+\frac{1}{2}f(0,2)$
	$= 2+\frac{1}{2}\cdot 0$

1

I get it now

(e) Do you expect your answer in (d) is more or less than the actual value of ~~x~~(1)? Explain.

Correction y

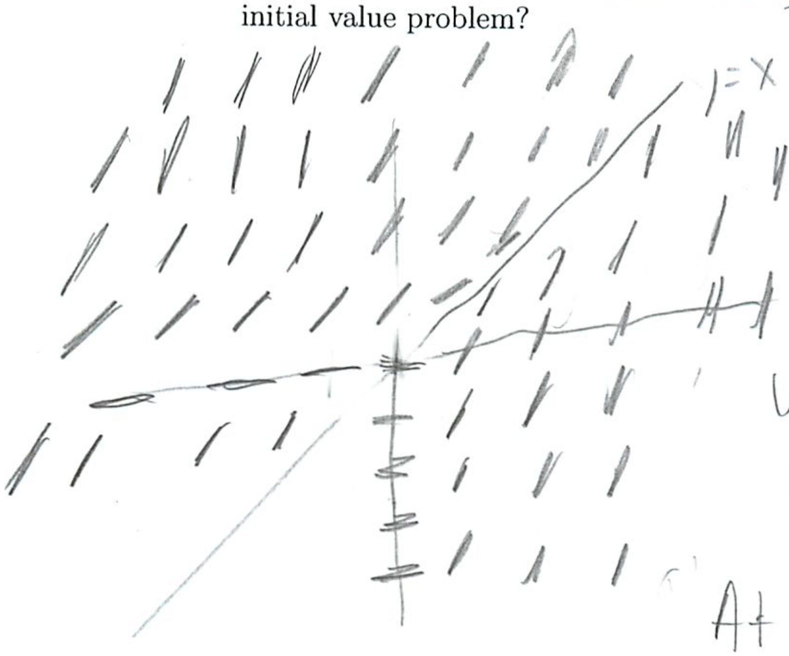
Less than actual. At $(0,2)$ the slope is 0 so the y does not change. But at $y(1)$ the slope is actually $y=2$ from the isocline we were told

0

5. Consider the initial value problem

$$\frac{dy}{dx} = \begin{cases} y^2 & \text{if } y \geq x \\ x^2 & \text{if } x \geq y \end{cases}, \quad y(0) = 0.$$

For what value of c is the line $x = c$ a vertical asymptote for the solution to the initial value problem?



Where is this a vertical asymptote

At $c = \cancel{\infty}$

or all values of c

Answers for

18.03 EXAM I

Wednesday, September 28, 2011

Name: Jethro

Recitation Instructor (Circle One): L. Chumakova / N. Olver / R. Rosales / V. Shende / J. van Ekeren / Z. Yun

Recitation Hour: ________________

Instructions: You may not use calculators, notes, textbooks, or personal electronic devices. As a courtesy to other students, please turn off all cell phones. Read each question carefully. Whenever possible, include justification for your reasoning and show your work. Answers without any work or explanation will receive little or no credit. There are 5 questions and a 55 minute time limit on this exam. Good luck.

Question	Score	Maximum
1		4
2		8
3		9
4		14
5		6
Total		41

 Name: ____________

1. Solve the following initial value problem exactly by any method:

$$(1+x)^2\frac{dy}{dx} - y^2 = 0, \quad y(0) = 1$$

Separate: $\dfrac{dy}{y^2} = \dfrac{dx}{(1+x)^2}$.

$\displaystyle\int \frac{dy}{y^2} = -\frac{1}{y}$, $\displaystyle\int \frac{dx}{(1+x)^2} = -\frac{1}{1+x}$.

So $-\dfrac{1}{y} = -\dfrac{1}{1+x} + C$

$y(0) = 1 \Rightarrow -\dfrac{1}{1} = -\dfrac{1}{1} + C \Rightarrow C = 0$

So $-\dfrac{1}{y} = -\dfrac{1}{1+x}$

$\Rightarrow \boxed{y = 1 + x}$

 Name: ______________________

2. a) Use complex methods to find the general solution to the differential equation

$$\frac{dy}{dt} + 4y = \sin 3t.$$

Integ. factor $e^{4t} \rightsquigarrow (e^{4t}y)' = e^{4t}\sin 3t.$

$= \mathrm{Imag}\left(e^{(4+3i)t}\right).$

$$\int e^{(4+3i)t}\,dt = \frac{1}{4+3i}e^{(4+3i)t} + C$$

$$\frac{1}{4+3i}e^{(4+3i)t} = \frac{4-3i}{25}(\cos 3t + i\sin 3t)e^{4t}$$

$$\text{Imag part} = \frac{e^{4t}}{25}(4\cos 3t - 3\sin 3t).$$

$$e^{4t}y(t) = \quad + C \quad \therefore \quad y(t) = \frac{1}{25}(4\cos 3t - 3\sin 3t) + Ce^{-4t}.$$

b) Thinking of this as an Input-Response model, what is the amplitude of the resulting response wave?

Amplitude of $A\cos\theta + B\sin\theta$ is $\sqrt{A^2+B^2}$.

So ampl. of $y(t)$ is

$$\frac{1}{25}\cdot\sqrt{4^2+(-3)^2} = \frac{5}{25} = \frac{1}{5}.$$

c) Find the general solution to the differential equation (using any method or earlier work in (a) and (b))

$$\frac{dy}{dt} + 4y = 1 + \sin 3t.$$

By linearity solution is solution to part (a) + solution to $\left(\frac{dy}{dt} + 4y = 1.\right)$.

But the solution to this is $y = \frac{1}{4} + Ce^{-4t}$.

So

$$\boxed{y = \frac{1}{25}(4\cos 3t - 3\sin 3t) + \frac{1}{4} + Ce^{-4t}.}$$

 Name: ______________________

3. A population of island turtles obeys a logistical growth model. Their limiting population is 400 turtles. But far away from this limit, the population grows at a rate of 75% of the current population each year.

a) Write a differential equation which models the population growth rate dP/dt.

~~Far from~~ Logistic $\Rightarrow \frac{dP}{dt} = aP - bP^2$.

"Far from limit" means P small so P^2 can be ignored.

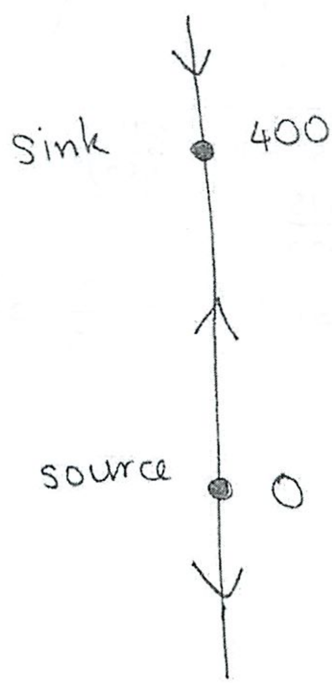
$\frac{dP}{dt} = aP$ & 75% growth per year suggests $a = 0.75$

Remember $\frac{a}{b} = \text{limit} = 400$,

so

$$\boxed{\frac{dP}{dt} = 0.75\left(P - \frac{1}{400}P^2\right).}$$

b) Draw a phase line for this equation, labeling equilibrium solutions as sinks, sources, or neither.



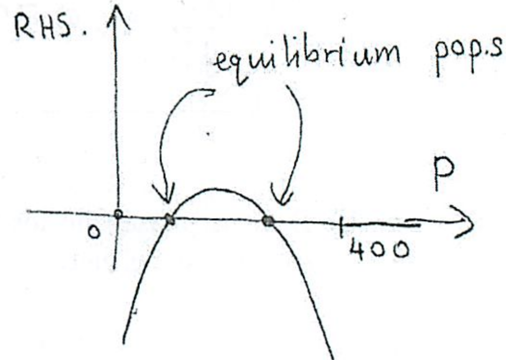
c) Suppose that we allow harvesting of turtles at a rate of r turtles per year, harvested continuously. Write a differential equation that models the new growth rate with harvesting.

$$\boxed{\frac{dP}{dt} = 0.75\left(P - \frac{1}{400}P^2\right) - r}$$

d) Which value of harvesting r is a bifurcation point for this model? What is the significance of this point for the turtle population?

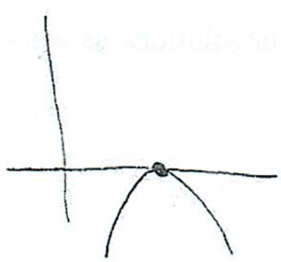
$$\frac{dP}{dt} = 0.75\left(P - \frac{1}{400}P^2\right) - r$$

Plot of RHS:



The bifurc'. pt is the 'r' for which there is only 1 equilib'. pop.

i.e.,



Solve: $0.75\left(P - \frac{1}{400}P^2\right) - r = 0$

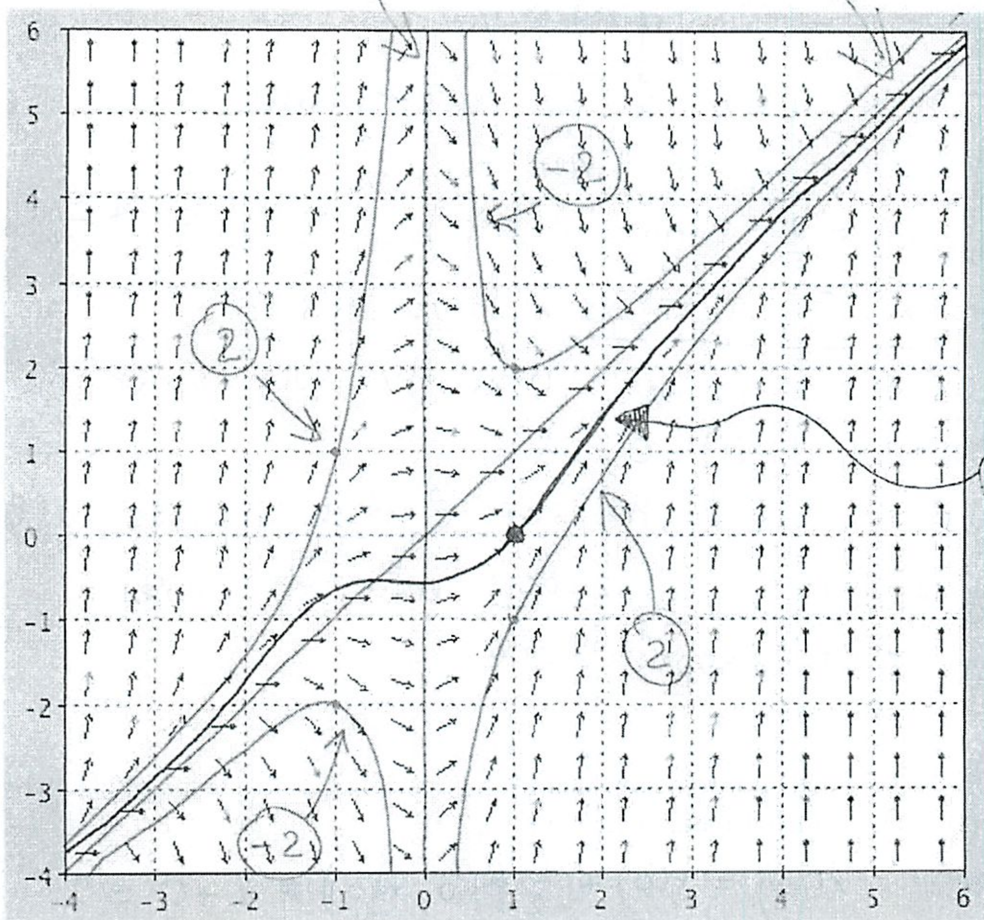
$$\rightsquigarrow P = \frac{-0.75 \pm \sqrt{0.75^2 - 4r\left(\frac{0.75}{400}\right)}}{-\frac{0.75}{400}}$$

To get 1 sol'n, need $\left(0.75^2 = 4r\,\frac{0.75}{400}\right) \Rightarrow 0.75 = \frac{r}{100} \Rightarrow \boxed{r=75}$

Higher $r \Rightarrow$ Extinction.

0 iso

4. The differential equation $dy/dx = x^2 - xy$ has the following direction field:



(a) Write the equations for the 0, 2 and −2 isoclines. Then sketch them on the direction field.

0 :	2 :	−2 :
$x(x-y)=0$	$x^2 - xy = 2$	$x^2 - xy = -2$.
$\boxed{x=0 \ \& \ x=y.}$	$\boxed{y = x - \frac{2}{x}}$	$\boxed{y = x + \frac{2}{x}.}$

(b) Sketch the solution passing through the point (1, 0) on the direction field.

 Name: ______________________

(c) What is the long-term behavior of the solution through $(1, 0)$? Be sure to justify your answer.

$\boxed{y \sim x.}$

0-cline $y = x$ is an upper fence.

$L(x) = x - \frac{2}{x}$ is a lower fence?

$f(x, L(x)) = 2$ (from part a), $L'(x) = 1 + \frac{2}{x^2}$.

For $x > \sqrt{2}$, $L(x)$ is a lower fence.

So we have funnel. Does the sol'n enter the funnel?

Yes. b/c $L(\sqrt{2}) = 0$, &

$y(1) = 0$, but the slope field is +ve, so $y(x)$ is increasing.

So $y(\sqrt{2}) \geq 0$ $\therefore$ $y(x)$ does enter the funnel.

Then it is trapped there.

(d) Use Euler's method with step size 1/2 to approximate the value of $x(1)$ where $x(t)$ is the solution passing through $(0, 2)$.

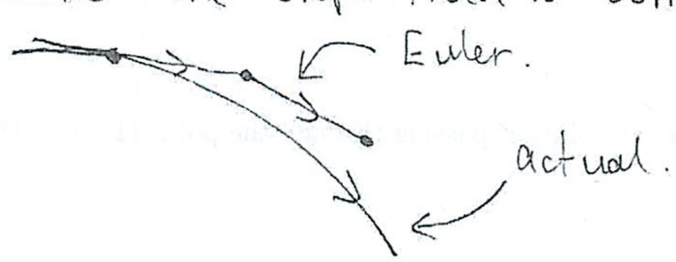
Euler: $x(\frac{1}{2}) = x(0) + \frac{1}{2} f(0, x(0)) = 2 + 0 = 2$.

$x(1) = x(\frac{1}{2}) + \frac{1}{2} f(\frac{1}{2}, x(\frac{1}{2})) = 2 + \frac{1}{2}(\frac{1}{2})(\frac{1}{2} - 2)$

$= 13/8$.

(e) Do you expect your answer in (d) is more or less than the actual value of $x(1)$? Explain.

More b/c the slope field is concave down.

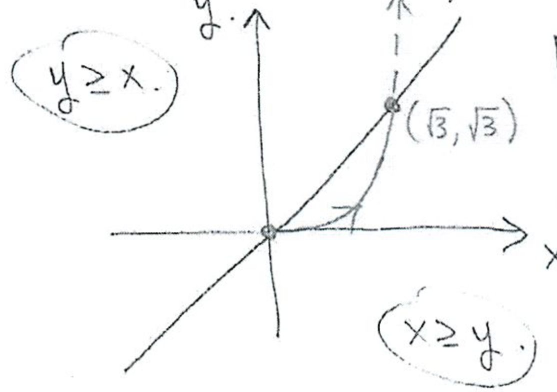


5. Consider the initial value problem

$$\frac{dy}{dx} = \begin{cases} y^2 & \text{if } y \geq x \\ x^2 & \text{if } x \geq y \end{cases}, \quad y(0) = 0.$$

For what value of c is the line $x = c$ a vertical asymptote for the solution to the initial value problem?

$y(0)=0$, slope field $=0$ at that point.
So sol'n starts moving due east into $x \geq y$ territory.



So we solve

$$\frac{dy}{dx} = x^2.$$

$$\Rightarrow y = \frac{x^3}{3} + C.$$

$y(0)=0$ means $C=0$.

But then $y = \frac{x^3}{3}$ hits the line $y=x$ when

$x = x^3/3 \Rightarrow x = \sqrt{3}$

The eqn changes to $\frac{dy}{dx} = y^2$. $\Rightarrow \int \frac{dy}{y^2} = \int dx$

$\Rightarrow -\frac{1}{y} = x + D \rightsquigarrow y = \frac{-1}{x+D}$. Now $y(\sqrt{3}) = \sqrt{3}$

So $\sqrt{3} = \frac{-1}{\sqrt{3}+D} \rightsquigarrow \sqrt{3}+D = \frac{-1}{\sqrt{3}} \rightsquigarrow D = -\frac{1}{\sqrt{3}} - \sqrt{3}$.

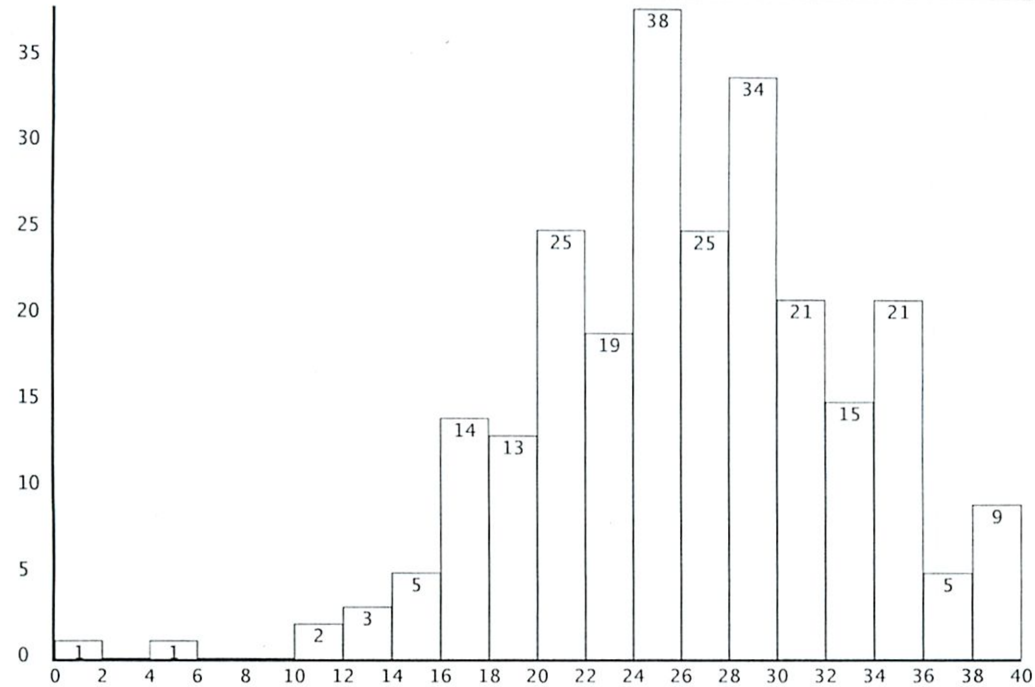
$\therefore y = \frac{-1}{x - (\sqrt{3} + \frac{1}{\sqrt{3}})}$. This has a vertical asymptote @

$$\boxed{x = \sqrt{3} + \frac{1}{\sqrt{3}}}.$$

18.03 Differential Equations

Dashboard | Students | Assignments

Grading Summary for Exam 1



Number of Scores: 251

Average: 26.15

Standard Deviation: 6.53

Review Answers Exam 1 OH 9/30

$\operatorname{Im}\left(\frac{1}{3i+4} e^{3ti+4t}\right)$

I tried $\operatorname{Im}(\ ) \cdot \operatorname{Im}(\ )$

$\underset{\text{1st part}}{\wedge}$ $\underset{\text{2nd part}}{\wedge}$

Not correct - need to first multiple

e^{4t} = real piece

So

$e^{4t} \frac{1}{25} \operatorname{Im}\left((3i-4) \cdot e^{3it}\right)$

$(3i-4)\ (\cos 3t + i \sin 3t)$

$\cancel{e^{4t} \frac{1}{25}}$ $\left(\begin{array}{l} -3 \sin t - 4 \cos 3t \\ + \cancel{BB}\ 3i \cos 3t - 4i \sin 3t \end{array} \right)$ two pieces

So Im part $= (3 \cos t - 4 \sin 3t)\, e^{4t} \cdot \frac{1}{25} + C e^{-4t}$

forgot to add C

then $\overline{e^{4t}}$

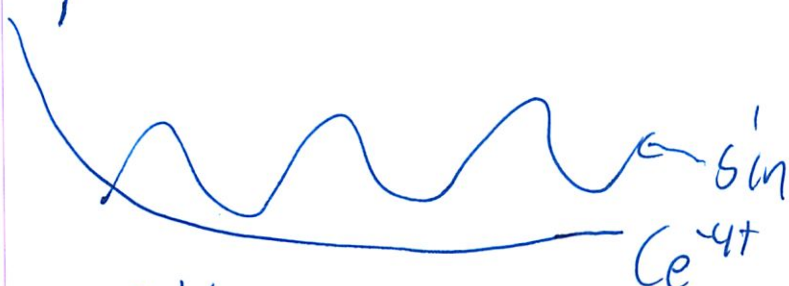
when you do it

2

Input Response

- boost input by sin function
- it behaves by sin equation

Only care about sin, cos terms
Constant terms fall to 0



add so

(3)

Now find amplitude

$$A\cos(\theta-\phi) = \overset{A}{\cos\theta}\cos\phi + \overset{A_1}{\sin\theta}\sin\phi$$

Want to write what we got in other for
So eaiser to see amp
So take ans but remove irrelevant parts

$$\underbrace{\frac{3}{25}}\cos 3t - \frac{4}{25}\sin 3t$$

$A\cos\theta = \frac{3}{25}$ $\quad A\sin\theta = -\frac{4}{25}$ $\quad \phi = 3t$

Solve for A

know $\cos^2\theta + \sin^2\theta = 1$

So $A^2 = (A\cos\theta)^2 + (A\sin\theta)^2$

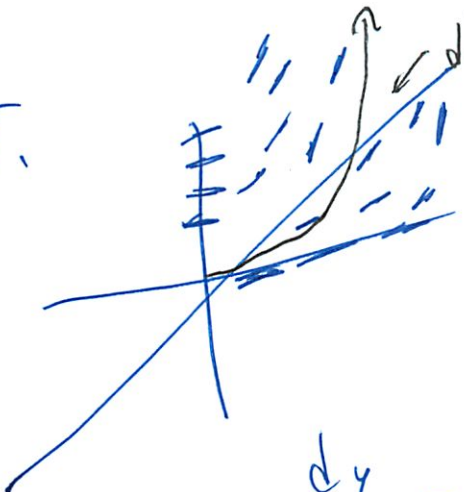
$$= \left(\frac{3}{25}\right)^2 + \left(-\frac{4}{25}\right)^2$$

$$A = \frac{1}{5}$$

If wanted θ $\frac{\cos\theta}{\sin\theta}$ get $\tan\theta$

4

#5.



different when crosses line
-need to use different equation
Find point

$$\frac{dy}{dx} = x^2$$

$$y = \frac{1}{3}x^3$$

want to know when $y = x$

$$x = \frac{1}{3}x^3$$

Hits 3 places $(\sqrt{3}, \sqrt{3})$
One we care about

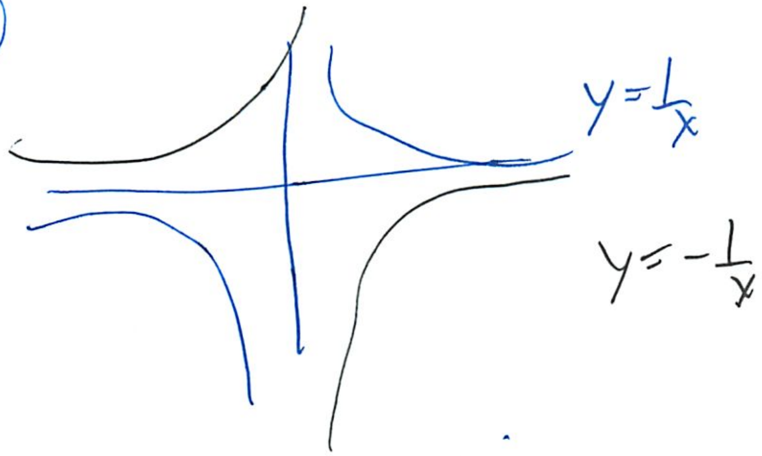
$$\frac{dy}{dx} = \cancel{x}\, y^2$$

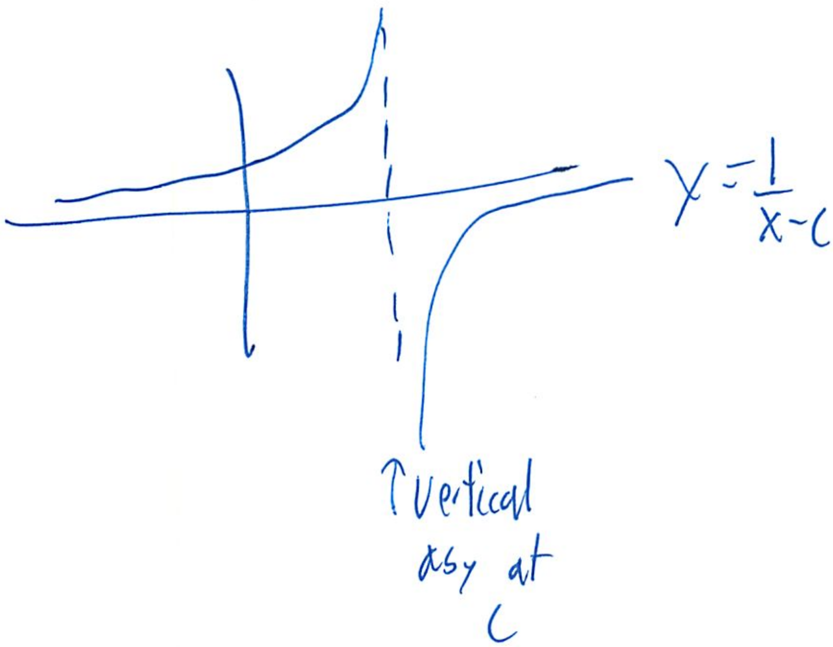
$$\int \frac{dy}{y^2} = -\int dx$$

$$-\frac{1}{y} = x - c$$

$$y = \frac{-1}{x - c}$$

(5)





Want to find c – thats the problem

$$\sqrt{3} = \frac{-1}{\sqrt{3}-c}$$

O. Linear Differential Operators

1. Linear differential equations. The general linear ODE of order n is

(1) $$y^{(n)} + p_1(x)y^{(n-1)} + \ldots + p_n(x)y \;=\; q(x).$$

If $q(x) \neq 0$, the equation is **inhomogeneous**. We then call

(2) $$y^{(n)} + p_1(x)y^{(n-1)} + \ldots + p_n(x)y \;=\; 0.$$

the **associated homogeneous** equation or the *reduced* equation.

The theory of the n-th order linear ODE runs parallel to that of the second order equation. In particular, the general solution to the associated homogeneous equation (2) is called the **complementary** function or solution, and it has the form

(3) $$y_c \;=\; c_1y_1 + \ldots + c_ny_n \;, \qquad c_i \;\text{ constants},$$

where the y_i are n solutions to (2) which are *linearly independent*, meaning that none of them can be expressed as a linear combination of the others, i.e., by a relation of the form (the left side could also be any of the other y_i):

$$y_n \;=\; a_1y_1 + \ldots + a_{n-1}y_{n-1} \;, \qquad a_i \;\text{ constants}.$$

Once the associated homogeneous equation (2) has been solved by finding n independent solutions, the solution to the original ODE (1) can be expressed as

(4) $$y \;=\; y_p + y_c,$$

where y_p is a particular solution to (1), and y_c is as in (3).

2. Linear differential operators with constant coefficients

From now on we will consider only the case where (1) has constant coefficients. This type of ODE can be written as

(5) $$y^{(n)} + a_1y^{(n-1)} + \ldots + a_ny \;=\; q(x) \;;$$

using the differentiation operator D, we can write (5) in the form

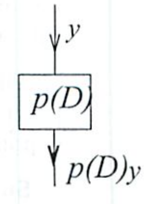
(6) $$(D^n + a_1D^{n-1} + \ldots + a_n)\,y \;=\; q(x)$$

or more simply,

$$p(D)\,y \;=\; q(x) \;,$$

where

(7) $$p(D) = D^n + a_1D^{n-1} + \ldots + a_n \;.$$



We call $p(D)$ a **polynomial differential operator with constant coefficients**. We think of the formal polynomial $p(D)$ as operating on a function $y(x)$, converting it into another function; it is like a black box, in which the function $y(x)$ goes in, and $p(D)y$ (i.e., the left side of (5)) comes out.

Our main goal in this section of the Notes is to develop methods for finding particular solutions to the ODE (5) when $q(x)$ has a special form: an exponential, sine or cosine, x^k, or a product of these. (The function $q(x)$ can also be a sum of such special functions.) These are the most important functions for the standard applications.

The reason for introducing the polynomial operator $p(D)$ is that this allows us to use polynomial algebra to help find the particular solutions. The rest of this chapter of the Notes will illustrate this. Throughout, we let

$$p(D) = D^n + a_1 D^{n-1} + \ldots + a_n\,, \qquad a_i \text{ constants.} \tag{7}$$

3. Operator rules.

Our work with these differential operators will be based on several rules they satisfy. In stating these rules, we will always assume that the functions involved are sufficiently differentiable, so that the operators can be applied to them.

Sum rule. If $p(D)$ and $q(D)$ are polynomial operators, then for any (sufficiently differentiable) function u,

$$[p(D) + q(D)]u = p(D)u + q(D)u\,. \tag{8}$$

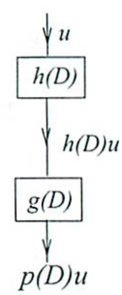
Linearity rule. If u_1 and u_2 are functions, and c_i constants,

$$p(D)\,(c_1 u_1 + c_2 u_2) \;=\; c_1 p(D)\,u_1 + c_2 p(D)\,u_2\,. \tag{9}$$

The linearity rule is a familiar property of the operator $a\,D^k$; it extends to sums of these operators, using the sum rule above, thus it is true for operators which are polynomials in D. (It is still true if the coefficients a_i in (7) are not constant, but functions of x.)

Multiplication rule. If $p(D) = g(D)\,h(D)$, as polynomials in D, then

$$p(D)\,u \;=\; g(D)\big(h(D)\,u\big)\,. \tag{10}$$



The picture illustrates the meaning of the right side of (10). The property is true when $h(D)$ is the simple operator $a\,D^k$, essentially because

$$D^m(a\,D^k u) \;=\; a\,D^{m+k}u;$$

it extends to general polynomial operators $h(D)$ by linearity. Note that a must be a constant; it's false otherwise.

An important corollary of the multiplication property is that *polynomial operators with constant coefficients commute*; i.e., for every function $u(x)$,

$$g(D)\big(h(D)\,u\big) \;=\; h(D)\big(g(D)\,u\big)\,. \tag{11}$$

For as polynomials, $g(D)h(D) \;=\; h(D)g(D) \;=\; p(D)$, say; therefore by the multiplication rule, both sides of (11) are equal to $p(D)\,u$, and therefore equal to each other.

The remaining two rules are of a different type, and more concrete: they tell us how polynomial operators behave when applied to exponential functions and products involving exponential functions.

Substitution rule.

(12) $$p(D)e^{ax} = p(a)e^{ax}$$

Proof. We have, by repeated differentiation,

$$De^{ax} = ae^{ax}, \quad D^2e^{ax} = a^2e^{ax}, \quad \ldots, \quad D^ke^{ax} = a^ke^{ax};$$

therefore,

$$(D^n + c_1D^{n-1} + \ldots + c_n)\, e^{ax} = (a^n + c_1a^{n-1} + \ldots + c_n)\, e^{ax},$$

which is the substitution rule (12). □

The exponential-shift rule This handles expressions such as x^ke^{ax} and $x^k \sin ax$.

(13) $$p(D)\, e^{ax}u = e^{ax}p(D+a)\, u\ .$$

Proof. We prove it in successive stages. First, it is true when $p(D) = D$, since by the product rule for differentiation,

(14) $$De^{ax}u(x) = e^{ax}Du(x) + ae^{ax}u(x) = e^{ax}(D+a)u(x)\ .$$

To show the rule is true for D^k, we apply (14) to D repeatedly:

$$\begin{aligned} D^2e^{ax}u &= D(De^{ax}u) = D(e^{ax}(D+a)u) && \text{by (14);} \\ &= e^{ax}(D+a)\big((D+a)u\big), && \text{by (14);} \\ &= e^{ax}(D+a)^2u\ , && \text{by (10).} \end{aligned}$$

In the same way,

$$\begin{aligned} D^3e^{ax}u &= D(D^2e^{ax}u) = D(e^{ax}(D+a)^2u) && \text{by the above;} \\ &= e^{ax}(D+a)\big((D+a)^2u\big), && \text{by (14);} \\ &= e^{ax}(D+a)^3u\ , && \text{by (10),} \end{aligned}$$

and so on. This shows that (13) is true for an operator of the form D^k. To show it is true for a general operator

$$p(D) = D^n + a_1D^{n-1} + \ldots + a_n\ ,$$

we write (13) for each $D^k(e^{ax}u)$, multiply both sides by the coefficient a_k, and add up the resulting equations for the different values of k. □

Remark on complex numbers. By Notes C. (20), the formula

(*) $$D\,(c\,e^{ax}) = c\,a\,e^{ax}$$

remains true even when c and a are complex numbers; therefore the rules and arguments above remain valid even when the exponents and coefficients are complex. We illustrate.

Example 1. Find $D^3e^{-x}\sin x$.

Solution using the exponential-shift rule. Using (13) and the binomial theorem,

$$\begin{aligned} D^3e^{-x}\sin x &= e^{-x}(D-1)^3\sin x = e^{-x}(D^3 - 3D^2 + 3D - 1)\sin x \\ &= e^{-x}(2\cos x + 2\sin x), \end{aligned}$$

since $D^2\sin x = -\sin x$, and $D^3\sin x = -\cos x$.

Solution using the substitution rule. Write $e^{-x}\sin x = \operatorname{Im} e^{(-1+i)x}$. We have

$$\begin{aligned} D^3e^{(-1+i)x} &= (-1+i)^3e^{(-1+i)x}, && \text{by (12) and (*);} \\ &= (2+2i)\, e^{-x}(\cos x + i\sin x), \end{aligned}$$

by the binomial theorem and Euler's formula. To get the answer we take the imaginary part: $e^{-x}(2\cos x + 2\sin x)$.

4. Finding particular solutions to inhomogeneous equations.

We begin by using the previous operator rules to find particular solutions to inhomogeneous polynomial ODE's with constant coefficients, where the right hand side is a real or complex exponential; this includes also the case where it is a sine or cosine function.

Exponential-input Theorem. *Let $p(D)$ be a polynomial operator with onstant coefficients, and $p^{(s)}$ its s-th derivative. Then*

$$p(D)y = e^{ax}, \qquad \text{where } a \text{ is real or complex} \tag{15}$$

has the particular solution

$$y_p = \frac{e^{ax}}{p(a)}, \qquad \text{if } p(a) \neq 0; \tag{16}$$

$$y_p = \frac{x^s e^{ax}}{p^{(s)}(a)}, \qquad \text{if } a \text{ is an } s\text{-fold zero}^1 \text{of } p. \tag{17}$$

Note that (16) is just the special case of (17) when $s = 0$. Before proving the theorem, we give two examples; the first illustrates again the usefulness of complex exponentials.

Example 2. Find a particular solution to $(D^2 - D + 1)\; y = e^{2x}\cos x$.

Solution. We write $e^{2x}\cos x = \text{Re}\,(e^{(2+i)x})$, so the corresponding complex equation is

$$(D^2 - D + 1)\;\tilde{y} = e^{(2+i)x},$$

and our desired y_p will then be $Re(\tilde{y}_p)$. Using (16), we calculate

$$p(2+i) = (2+i)^2 - (2+i) + 1 = 2 + 3i\,, \quad \text{from which}$$

$$\begin{aligned}
\tilde{y}_p &= \frac{1}{2+3i}\, e^{(2+i)x}, && \text{by (16);} \\
&= \frac{2-3i}{13}\, e^{2x}(\cos x + i \sin x)\;; && \text{thus} \\
Re(\tilde{y}_p) &= \frac{2}{13}\, e^{2x}\cos x + \frac{3}{13}\, e^{2x}\sin x\,, && \text{our desired particular solution.}
\end{aligned}$$

Example 3. Find a particular solution to $y'' + 4y' + 4y \;=\; e^{-2t}$.

Solution. Here $p(D) = D^2 + 4D + 4 = (D+2)^2$, which has -2 as a double root; using (17), we have $p''(-2) = 2$, so that

$$y_p = \frac{t^2 e^{-2t}}{2}.$$

Proof of the Exponential-input Theorem.

That (16) is a particular solution to (15) follows immediately by using the linearity rule (9) and the substitution rule (12):

$$p(D)y_p \;=\; p(D)\frac{e^{ax}}{p(a)} \;=\; \frac{1}{p(a)}p(D)e^{ax} \;=\; \frac{p(a)e^{ax}}{p(a)} \;=\; e^{ax}.$$

---

[1] John Lewis communicated this useful formula.

For the more general case (17), we begin by noting that to say the polynomial $p(D)$ has the number a as an s-fold zero is the same as saying $p(D)$ has a factorization

(18) $$p(D) = q(D)(D-a)^s, \qquad q(a) \neq 0.$$

We will first prove that (18) implies

(19) $$p^{(s)}(a) = q(a)\, s!\ .$$

To prove this, let k be the degree of $q(D)$, and write it in powers of $(D-a)$:

(20) $$q(D) = q(a) + c_1(D-a) + \ldots + c_k(D-a)^k; \quad \text{then}$$
$$p(D) = q(a)(D-a)^s + c_1(D-a)^{s+1} + \ldots + c_k(D-a)^{s+k};$$
$$p^{(s)}(D) = q(a)\, s! + \text{positive powers of } D-a;$$

substituting a for D on both sides proves (19). □

Using (19), we can now prove (17) easily using the exponential-shift rule (13). We have

$$\begin{aligned}
p(D)\frac{e^{ax}x^s}{p^{(s)}(a)} &= \frac{e^{ax}}{p^{(s)}(a)}\, p(D+a)x^s, && \text{by linearity and (13);}\\
&= \frac{e^{ax}}{p^{(s)}(a)}\, q(D+a)\, D^s x^s, && \text{by (18);}\\
&= \frac{e^{ax}}{q(a)s!}\, q(D+a)\, s!, && \text{by (19);}\\
&= \frac{e^{ax}}{q(a)s!}\, q(a)\, s! \;=\; e^{ax},
\end{aligned}$$

where the last line follows from (20), since $s!$ is a constant:

$$q(D+a)s! = (q(a) + c_1 D + \ldots + c_k D^k)\, s! = q(a)s!\ .$$

Polynomial Input: The Method of Undetermined Coefficients.

Let $r(x)$ be a polynomial of degree k; we assume the ODE $p(D)y = q(x)$ has as input

(21) $$q(x) = r(x), \quad p(0) \neq 0; \qquad \text{or more generally,} \quad q(x) = e^{ax}\, r(x), \quad p(a) \neq 0.$$

(Here a can be complex; when $a = 0$ in (21), we get the pure polynomial case on the left.)

The method is to assume a particular solution of the form $y_p = e^{ax}h(x)$, where $h(x)$ is a polynomial of degree k with unknown ("undetermined") coefficients, and then to find the coefficients by substituting y_p into the ODE. It's important to do the work systematically; follow the format given in the following example, and in the solutions to the exercises.

Example 5. Find a particular solution y_p to $y'' + 3y' + 4y = 4x^2 - 2x$.

Solution. Our trial solution is $y_p = Ax^2 + Bx + C$; we format the work as follows. The lines show the successive derivatives; multiply each line by the factor given in the ODE, and add the equations, collecting like powers of x as you go. The fourth line shows the result; the sum on the left takes into account that y_p is supposed to be a particular solution to the given ODE.

$$\begin{aligned}
\times 4 \quad y_p &= Ax^2 + Bx + C\\
\times 3 \quad y_p' &= \qquad 2Ax + B\\
y_p'' &= \qquad\qquad 2A\\
4x^2 - 2x &= (4A)x^2 + (4B + 6A)x + (4C + 3B + 2A).
\end{aligned}$$

Equating like powers of x in the last line gives the three equations

$$4A = 4, \qquad 4B + 6A = -2, \qquad 4C + 3B + 2A = 0;$$

solving them in order gives $A = 1$, $B = -2$, $C = 1$, so that $y_p = x^2 - 2x + 1$.

Example 6. Find a particular solution y_p to $y'' + y' - 4y = e^{-x}(1 - 8x^2)$.

Solution. Here the trial solution is $y_p = e^{-x}u_p$, where $u_p = Ax^2 + Bx + C$.

The polynomial operator in the ODE is $p(D) = D^2 + D - 4$; note that $p(-1) \neq 0$, so our choice of trial solution is justified. Substituting y_p into the ODE and using the exponential-shift rule enables us to get rid of the e^{-x} factor:

$$p(D)y_p = p(D)e^{-x}u_p = e^{-x}p(D-1)u_p = e^{-x}(1 - 8x^2),$$

so that after canceling the e^{-x} on both sides, we get the ODE satisfied by u_p:

$$(22) \qquad p(D-1)u_p = 1 - 8x^2; \qquad \text{or} \qquad (D^2 - D - 4)u_p = 1 - 8x^2,$$

since $p(D-1) = (D-1)^2 + (D-1) - 4 = D^2 - D - 4$.

From this point on, finding u_p as a polynomial solution to the ODE on the right of (22) is done just as in Example 5 using the method of undetermined coefficients; the answer is

$$u_p = 2x^2 - x + 1, \qquad \text{so that} \qquad y_p = e^{-x}(2x^2 - x + 1).$$

In the previous examples, $p(a) \neq 0$; if $p(a) = 0$, then the trial solution must be altered by multiplying each term in it by a suitable power of x. The book gives the details; briefly, the terms in the trial solution should all be multiplied by the smallest power x^r for which none of the resulting products occur in the complementary solution y_c, i.e., are solutions of the associated homogeneous ODE. Your book gives examples; we won't take this up here.

5. Higher order homogeneous linear ODE's with constant coefficients.

As before, we write the equation in operator form:

$$(23) \qquad (D^n + a_1 D^{n-1} + \ldots + a_n)\, y \;=\; 0,$$

and define its **characteristic equation** or *auxiliary equation* to be

$$(24) \qquad p(r) = r^n + a_1 r^{n-1} + \ldots + a_n = 0.$$

We investigate to see if e^{rx} is a solution to (23), for some real or complex r. According to the substitution rule (12),

$$p(D)\, e^{rx} = 0 \quad \Leftrightarrow \quad p(r)\, e^{rx} = 0 \quad \Leftrightarrow \quad p(r) = 0\ .$$

Therefore

(25) e^{rx} *is a solution to (7)* $\Leftrightarrow$ r *is a root of its characteristic equation (16).*

Thus, to the real root r_i of (16) corresponds the solution $e^{r_i x}$.

Since the coefficients of $p(r) = 0$ are real, its complex roots occur in pairs which are conjugate complex numbers. Just as for the second-order equation, to the pair of complex conjugate roots $a \pm ib$ correspond the complex solution (we use the root $a + ib$)

$$e^{(a+ib)x} = e^{ax}(\cos bx + i \sin bx),$$

whose real and imaginary parts

$$e^{ax} \cos bx \qquad \text{and} \qquad e^{ax} \sin bx \tag{26}$$

are solutions to the ODE (23).

If there are n distinct roots to the characteristic equation $p(r) = 0$, (there cannot be more since it is an equation of degree n), we get according to the above analysis n real solutions $y_1, y_2, \ldots, y_n$ to the ODE (23), and they can be shown to be linearly independent. Thus the the complete solution y_h to the ODE can be written down immediately, in the form:

$$y = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n \ .$$

Suppose now a real root r_1 of the characteristic equation (24) is a k-fold root, i.e., the characteristic polynomial $p(r)$ can be factored as

$$p(r) = (r - r_1)^k g(r), \qquad \text{where } g(r_1) \neq 0 \ . \tag{27}$$

We shall prove in the theorem below that corresponding to this k-fold root there are k linearly independent solutions to the ODE (23), namely:

$$e^{r_1 x}, \ xe^{r_1 x}, \ x^2 e^{r_1 x}, \ \ldots, \ x^{k-1} e^{r_1 x} \ . \tag{28}$$

(Note the analogy with the second order case that you have studied already.)

Theorem. *If a is a k-fold root of the characteristic equation $p(r) = 0$, then the k functions in (28) are solutions to the differential equation $p(D)\, y = 0$.*

Proof. According to our hypothesis about the characteristic equation, $p(r)$ has $(r-a)^k$ as a factor; denoting by $g(x)$ the other factor, we can write

$$p(r) = g(r)(r - a)^k ,$$

which implies that

$$p(D) = g(D)(D - a)^k \ . \tag{29}$$

Therefore, for $i = 0, 1, \ldots, k - 1$, we have

$$\begin{aligned}
p(D)x^i e^{ax} &= g(D)(D - a)^k x^i e^{ax} \\
&= g(D)\big((D - a)^k x^i e^{ax}\big), && \text{by the multiplication rule,} \\
&= g(D)\big(e^{ax} D^k x^i\big) , && \text{by the exponential-shift rule,} \\
&= g(D)\big(e^{ax} \cdot 0\big), && \text{since } D^k x^i = 0 \text{ if } k > i; \\
&= 0,
\end{aligned}$$

which shows that all the functions of (20) solve the equation. □

If r_1 is real, the solutions (28) give k linearly independent real solutions to the ODE (23).

In the same way, if $a+ib$ and $a-ib$ are k-fold conjugate complex roots of the characteristic equation, then (28) gives k complex solutions, the real and imaginary parts of which give $2k$ linearly independent solutions to (23):

$$e^{ax}\cos bx,\ e^{ax}\sin bx,\ \ xe^{ax}\cos bx,\ xe^{ax}\sin bx,\ \ \ldots,\ \ x^{k-1}e^{ax}\cos bx,\ x^{k-1}e^{ax}\sin bx\ .$$

Example 6. Write the general solution to $(D+1)(D-2)^2(D^2+2D+2)^2y=0$.

Solution. The characteristic equation is

$$p(r)=(r+1)(r-2)^2(r^2+2r+2)^2\ =\ 0\ .$$

By the quadratic formula, the roots of $r^2+2r+2=0$ are $r=-1\pm i$, so we get

$$y=c_1e^{-x}+c_2e^{2x}+c_3xe^{2x}+e^{-x}(c_4\cos x+c_5\sin x+c_6x\cos x+c_7x\sin x)$$

as the general solution to the differential equation. □

As you can see, if the linear homogeneous ODE has constant coefficients, then the work of solving $p(D)y=0$ is reduced to finding the roots of the characteristic equation. This is "just" a problem in algebra, but a far from trivial one. There are formulas for the roots if the degree $n\le 4$, but of them only the quadratic formula ($n=2$) is practical. Beyond that are various methods for special equations and general techniques for approximating the roots. Calculation of roots is mostly done by computer algebra programs nowadays.

This being said, you should still be able to do the sort of root-finding described in Notes C, as illustrated by the next example.

Example 7. Solve: a) $y^{(4)}+8y''+16y=0$ b) $y^{(4)}-8y''+16y=0$

Solution. The factorizations of the respective characteristic equations are

$$(r^2+4)^2=0 \quad\text{and}\quad (r^2-4)^2=(r-2)^2(r+2)^2=0\ .$$

Thus the first equation has the double complex root $2i$, whereas the second has the double real roots 2 and -2. This leads to the respective general solutions

$$y=(c_1+c_2x)\cos 2x+(c_3+c_4x)\sin 2x \qquad\text{and}\qquad y=(c_1+c_2x)e^{2x}+(c_3+c_4x)e^{-2x}.$$

6. Justification of the method of undetermined coefficients.

As a last example of the use of these operator methods, we use operators to show where the method of undetermined coefficients comes from. This is the method which assumes the trial particular solution will be a linear combination of certain functions, and finds what the correct coefficients are. It only works when the inhomogeneous term in the ODE (23) (i.e., the term on the right-hand side) is a sum of terms having a special form: each must be the product of an exponential, sin or cos, and a power of x (some of these factors can be missing).

Question: What's so special about these functions?

Answer: They are the sort of functions which appear as solutions to some linear homogeneous ODE with constant coefficients.

With this general principle in mind, it will be easiest to understand why the method of undetermined coefficients works by looking at a typical example.

Example 8. Show that

$$(D-1)(D-2)y \;=\; \sin 2x. \tag{30}$$

has a particular solution of the form

$$y_p \;=\; c_1 \cos 2x + c_2 \sin 2x\ .$$

Solution. Since $\sin 2x$, the right-hand side of (30), is a solution to $(D^2+4)y=0$, i.e.,

$$(D^2+4)\,\sin 2x \;=\; 0.$$

we operate on both sides of (30) by the operator D^2+4 ; using the multiplication rule for operators with constant coefficients, we get (using y_p instead of y)

$$(D^2+4)(D-1)(D-2)y_p \;=\; 0. \tag{31}$$

This means that y_p is one of the solutions to the *homogeneous* equation (31). But we know its general solution: y_p must be a function of the form

$$y_p \;=\; c_1 \cos 2x + c_2 \sin 2x + c_3 e^x + c_4 e^{2x}\ . \tag{32}$$

Now, in (32) we can drop the last two terms, since they contribute nothing: they are part of the complementary solution to (30), i.e., the solution to the associated homogeneous equation. Therefore they need not be included in the particular solution. Put another way, when the operator $(D-1)(D-2)$ is applied to (32), the last two terms give zero, and therefore don't help in finding a particular solution to (30).

Our conclusion therefore is that there is a particular solution to (30) of the form

$$y_p \;=\; c_1 \cos 2x + c_2 \sin 2x\ .$$

Here is another example, where one of the inhomogeneous terms is a solution to the associated homogeneous equation, i.e., is part of the complementary function.

Example 9. Find the form of a particular solution to

$$(D-1)^2 y_p \;=\; e^x. \tag{33}$$

Solution. Since the right-hand side is a solution to $(D-1)y=0$, we just apply the operator $D-1$ to both sides of (33), getting

$$(D-1)^3 y_p \;=\; 0.$$

Thus y_p must be of the form

$$y_p \;=\; e^x(c_1 + c_2 x + c_3 x^2).$$

But the first two terms can be dropped, since they are already part of the complementary solution to (33); we conclude there must be a particular solution of the form

$$y_p \;=\; c_3 x^2 e^x\ .$$

Exercises: Section 2F

The Exponential Shift

Occasionally, a differential equation with constant coefficients appears that would be simpler to solve if the forcing function did not have an exponential. In these cases, the exponential shift can transform the equation into one that does not have that exponential.

First, let's prove the exponential shift theorem:

$$e^{ax} f(D)y = f(D-a)[e^{ax} y]$$

To start, let's evaluate $(D-a)[e^{ax} y]$.

$$\begin{aligned}
(D-a)[e^{ax} y] &= D[e^{ax} y] - a[e^{ax} y] \\
&= e^{ax} D\, y + y\, De^{ax} - ae^{ax}\, y \\
&= e^{ax} D\, y + y\, ae^{ax} - ae^{ax}\, y \\
&= e^{ax} D\, y
\end{aligned}$$

Next, let's evaluate $(D-a)^2[e^{ax} y]$.

$$\begin{aligned}
(D-a)^2[e^{ax} y] &= (D-a)\,(D-a)\,[e^{ax} y] \\
&= (D-a)\,[e^{ax}\, Dy] \\
&= D\,[e^{ax}\, Dy] - a[e^{ax}\, Dy] \\
&= e^{ax}\, D^2 y + ae^{ax}\, Dy - ae^{ax}\, Dy \\
&= e^{ax}\, D^2 y
\end{aligned}$$

From here, induction will show that $(D-a)^n[e^{ax} y] = e^{ax} D^n y$ is true for all n greater than one.

If $(D-a)^k[e^{ax} y] = e^{ax} D^k y$, then:

$$\begin{aligned}
(D-a)^{k+1}[e^{ax} y] &= (D-a)\,(D-a)^k\,[e^{ax} y] \\
&= (D-a)\,[e^{ax}\, D^k y] \\
&= D\,[e^{ax}\, D^k y] - a[e^{ax}\, D^k y] \\
&= e^{ax}\, D^{k+1} y + ae^{ax}\, D^k y - ae^{ax}\, D^k y \\
&= e^{ax}\, D^{k+1} y
\end{aligned}$$

John Baliga
Triton College
MAT 341

If f is a polynomial, then $f(D)$ is just a linear combination of D^n terms. The fact that $(D-a)^n[e^{ax}y] = e^{ax}D^n y$ would hold for each term in the equation to be proved:

$$e^{ax}f(D)y = f(D-a)[e^{ax}y]$$

Let's use this in some examples. We will only use the exponential shift to help find the particular solution.

Example 1

$$[D^2-2D+5]\,y = 16x^3e^{3x} \tag{1}$$

We will treat the homogeneous solution, which is $y_h = e^x(A\cos 2x + B\sin 2x)$, separately.

$$[D^2-2D+5]\,y_p = 16x^3e^{3x} \tag{2}$$

$$e^{-3x}[D^2-2D+5]\,y_p = 16x^3 \tag{3}$$

$$[(D+3)^2-2(D+3)+5]\,(e^{-3x}y_p) = 16x^3 \tag{4}$$

$$[D^2+4D+8]\,(e^{\ 3x}y_p) = 16x^3 \tag{5}$$

From here, we can say that $(e^{\ 3x}y_p) = Cx^3+Ex^2+Fx+G$. We can find y_p in a familiar way:

$$(e^{\ 3x}y_p) = Cx^3+Ex^2+Fx+G \tag{6}$$

$$D(e^{-3x}y_p) = 3Cx^2+2Ex+F \tag{7}$$

$$D^2(e^{\ 3x}y_p) = 6Cx+2E \tag{8}$$

Substitution gives the following:

$$[D^2+4D+8]\,(e^{-3x}y_p) = (6Cx+2E)+4(3Cx^2+2Ex+F)+8(Cx^3+Ex^2+Fx+G) \tag{9}$$

$$= 8Cx^3+(12C+8E)x^2+(6C+8E+8F)x+2E+4F+8G \tag{10}$$

From which we find that: $C=2$, $E=-3$, $F=3/2$, $G=0$, and:

$$e^{\ 3x}y_p = 2x^3-3x^2+\tfrac{3}{2}x \tag{11}$$

$$y_p = \left(2x^3-3x^2+\tfrac{3}{2}x\right)e^{3x} \tag{12}$$

The general solution is then:

$$y = y_h + y_p = e^x\left(A\cos 2x + B\sin 2x\right)+\left(2x^3-3x^2+\tfrac{3}{2}x\right)e^{3x} \tag{13}$$

Example 2

$$[D^2 - 2D + 1]\,y \quad = \quad xe^x + 7x - 2 \tag{14}$$

In this case, the homogeneous solution is: $y_h = e^x(A + Bx)$. We would expect that the particular solution would have this form: $y_p = e^x(Cx^2 + Ex^3) + Fx + G$. Here, we will use superposition and split the particular solution into two pieces, the first of which we will find using the exponential shift.

$$y_p \quad = \quad y_{p1} + y_{p2} \tag{15}$$

$$[D^2 - 2D + 1]\,y_{p1} \quad = \quad xe^x \tag{16}$$

$$[D^2 - 2D + 1]\,y_{p2} \quad = \quad 7x - 2 \tag{17}$$

Solving (16):

$$[D^2 - 2D + 1]\,y_{p1} \quad = \quad xe^x \tag{18}$$

$$e^{-x}\,[D^2 - 2D + 1]\,y_{p1} \quad = \quad x \tag{19}$$

$$[(D + 1)^2 - 2(D + 1) + 1]\,(e^{-x}y_{p1}) \quad = \quad x \tag{20}$$

$$D^2\,(e^{-x}y_{p1}) \quad = \quad x \tag{21}$$

We can think of this as being $D^2v = x$, where $v = e^{-x}y$. From this, we could say that $v_h = C_1x + C_2$, and that $v_p = C_3x^3 + C_4x^2$. Note that the two statements, $v_h = C_1x + C_2$ and $y_h = e^x(A + Bx)$, are equivalent. It is a straightforward thing to show that $v_p = \frac{1}{6}x^3$. So:

$$e^{-x}y_{p1} \quad = \quad \tfrac{1}{6}x^3 \tag{22}$$

$$y_{p1} \quad = \quad \tfrac{1}{6}x^3e^x \tag{23}$$

For (17), it is clear that $y_{p2} = C_5x + C_6$, and:

$$[D^2 - 2D + 1]\,y_{p2} \quad = \quad 7x - 2 \tag{24}$$

$$0 - 2(C_5) + C_5x + C_6 \quad = \quad 7x - 2 \tag{25}$$

$$C_5x + (C_6 - 2C_5) \quad = \quad 7x - 2 \tag{26}$$

From this, we find that $C_5 = 7$ and $C_6 = 12$. Putting this all together, we have:

$$y = y_h + y_p \quad = \quad e^x\left(A + Bx\right) + \tfrac{1}{6}x^3e^x + 7x + 12 \tag{27}$$

This was time consuming, but it might have been more so if y_{p1} were found without using the exponential shift.

Reference: Elementary Differential Equations, 5th edition, Earl D. Rainville and Phillip E. Bedient

John Baliga
Triton College
MAT 341