18.03 Lecture 19

10/28

Replacement Prof today

Fourier Series for Periodic Functions

Start w/ periodic function

def $f(x)$ satisfies $f(x+P) = f(x)$ for all x

shift over by period) same thing!

repeats twice P, etc

If P is a period for f Than so is $2P, 3P, -P, -2P$, etc

We ~~understand~~ use these.

- ~~periodic~~ Fourier
- music
- astronomy: planet position
- etc

---

example

$\sin(x), \cos(x) \to$ period 2π

Most functions we work w/ have a smallest period

eg $\sin(mx)$, $\cos(mx)$

also periodic w/ period $\frac{2\pi}{m}$

②

Constant function is also periodic

└ no smallest period

└ doesn't exist

Can add functions to get periodic function

- if periods ≠ sum <u>might</u> not be periodic

<u>eg</u> if $f(x)$, $g(x)$ both are periodic by P

└ then so is $f(x)+g(x)$

$f(x)\,g(x)$

eg $f(x) = \underbrace{\frac{a_0}{2}}_{\text{constant}} + a_1\cos(x) + a_2\cos(2x) + \dots + a_n\cos(nx)$

$+ b_1\sin(x) + b_2\sin(2x) + \dots + b_n\sin(nx)$

so it is periodic by 2π

└ if change x to $x+2\pi$ still same

Describe values for 1 period - a "window"

so describe on $[-\pi,\pi]$

Notice

a_i are infront of even functions

b_i odd

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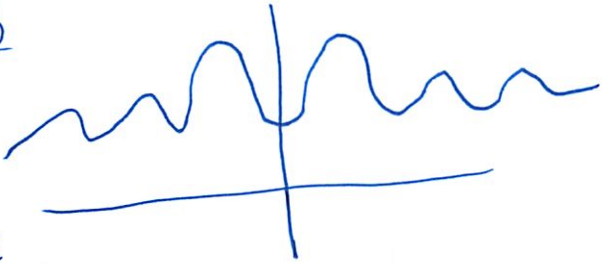
Even functions

$f(x) = f(-x)$
for all x



Symmetric
y-axis

Constants
$\cos(x)$
$\cos(2x)$
$\vdots$

Odd functions

$f(x) = -f(-x)$

reflected

Symmetric
across
origin

$\sin(x)$
$\sin(2x)$
$\vdots$

Fact (exercise)

Any function $f(x)$ is a sum of an even fn
and an odd fn

~~even * even = odd~~
~~odd~~
odd + even - arbitrary
even + even = even
odd + odd = odd
even · even = even   like XOR
even · odd = odd
odd · odd = even

④

Main thing: calculation

Our $f(x) = \frac{a_0}{2} + a_1 \cos(x) + a_2 \cos(2x) + \dots + a_N \cos(Nx)$
$+ b_1 \sin(x) + b_2 \sin(2x) + \dots + b_N \sin(Nx)$

If we know the values of function $f(x)$ how do we determine the coeffs $a_0, a_1, \dots a_n, b_1, \dots b_n$

Analogous example: Power Series

$g(x) = a_0 + a_1 x + a_2 x^2 + \dots$

L converges

Don't know coeffs a_i's, b_i's

Need to figure them out

We do know all derivs

L as many as we want

~~1st strike~~

So plug in 0 for x

$a_0 = g(0)$

1st deriv

$g'(x) = a_1 + 2a_2 x + 3a_3 x^2$

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If plug in 0, get a_1 term

$$a_1 = g'(0)$$

Etc

$$a_2 = \frac{g''(0)}{2}$$

So general terms

$$g^{(n)}(0) = n!\, a_n$$ ← n differentiations

$$a_n = \frac{g^{(n)}(0)}{n!}$$

Same for Fourier

~~all~~ but w/ integrating, not differentiating

For example

$$f(x) = \frac{a_0}{2} + a_1\cos(x) + \cdots + a_n \sin(Nx)$$
$$+ b_1 \sin(x) + \ldots b_n \sin(Nx)$$

We'll figure out a_is, b_is by integration

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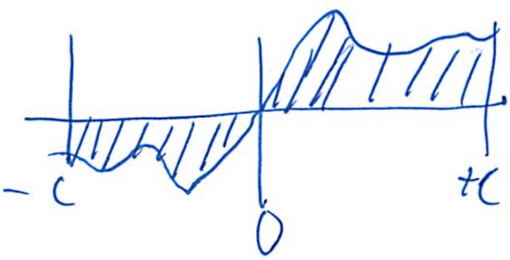
Preliminaries

1. $\int_{-\pi}^{\pi} \sin(mx)\,dx = -\frac{\cos(mx)}{m}\Big|_{-\pi}^{\pi} = 0$

because $\cos()$ is an even function

or $\int$ odd fn

$\int_{-c}^{c}$ odd fn $= 0 =$



2. $\int_{-\pi}^{\pi} \cos(mx)\,dx = \frac{\sin(mx)}{m}\Big|_{\pi}^{\pi} = 0$

0 because $\sin(m\pi)$ is 0 for all integers m

integer multiples of π

3. $\int_{-\pi}^{\pi} \cos(mx)\cos(nx)\,dx \begin{cases} = 2\pi & \text{if } m=n \\ = 0 & \text{if } m \neq n \end{cases}$

4. $\int_{-\pi}^{\pi} \sin(mx)\sin(nx)\,dx = \begin{cases} = \pi & \text{if } m=n \\ 0 & \text{if } m \neq n \end{cases}$

5. $\int_{-\pi}^{\pi} \sin(mx)\cos(nx)\,dx = 0$

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Now we can go back to original puzzle

$$\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \int_{-\pi}^{\pi} \Big(\frac{a_0}{2} + a_1 \cos x + \dots + a_n \cos(Nx) +$$
$$+ b_1 \sin x + \dots b_n \sin(Nx) \,dx$$

Integrate each sepertly

$$= \int_{-\pi}^{\pi} \frac{a_0}{2}\,dx + \int_{-\pi}^{\pi} a_1 \cos x\,dx + \dots + \int_{-\pi}^{\pi} a_N \cos(Nx)\,dx$$
$$+ \int_{-\pi}^{\pi} b_1 \sin(x)\,dx + \dots + \int_{-\pi}^{\pi} b_N \sin(Nx)$$

$$= \frac{a_0}{2} \cdot 2\pi + 0 + \dots + 0 + 0 + \dots + 0$$

$$= \pi a_0$$

So $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,dx$

$$\int_{-\pi}^{\pi} f(x) \cos(mx)\,dx = \int_{-\pi}^{\pi} \Big(\frac{a_0}{2} + a_1 \cos x + \dots + a_n \cos(Nx)$$
$$+ b_1 \sin x + \dots + b_N \sin Nx \Big)$$

only stuff that survives

key part of calc

$$\cdot \cos(mx)\,dx$$

$$= a_m \int_{-\pi}^{\pi} \cos(mx) \cos(mx)\,dx + 0$$

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This we had before but here $n = m$

So

$$a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(mx)\, dx$$

$$b_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(mx)\, dx$$

} orthonormal functions
- lets us pick out coefficients

~~"Leap of faith"~~

Now lets compute for arbitrary function $f(x)$
periodic by 2π

- can be written by sum of sins + coss
- can be written as ∞ sum
- sum of constant, $\sin(mx)$, $\cos(mx)$
- not all fns
  must be reasonably nice

Given $f(x)$ define

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(mx)\, dx$$

by 2π

$$b_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(mx)\, dx$$

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Key Fact / Theorm For a periodic $\downarrow 2\pi$ function

$$f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + \dots \to \infty$$
$$+ b_1 \sin x + \dots \to \infty$$

Remember

As long as f is nice

ie F is contineos

Fourier Series of f(x)

$\llcorner$ transformed it to #s

Helps you do math

10/28 1

Today : $\boxed{\text{Fourier series for periodic functions}}$.

We want to understand periodic functions, i.e. functions which repeat. That is, there is a P such that

$f(t+P) = f(t)$ for all t.

Of course, 2P, 3P, −P etc. are all periods for f. For most nice periodic functions, they'll have a smallest period, we'll take it to be positive.

Many examples in the physical world : electrical signals, heartbeats, positions of planets around sun, sound waves etc.

Examples (mathematical)

1) $\cos(x)$, $\sin(x)$ are periodic (period 2π)

2) more generally, $\cos(nx)$ and $\sin(nx)$ are periodic (their smallest period is $\frac{2\pi}{n}$, but 2π is also a period)

A constant function is also periodic (it doesn't have a smallest period).

So a function of the form

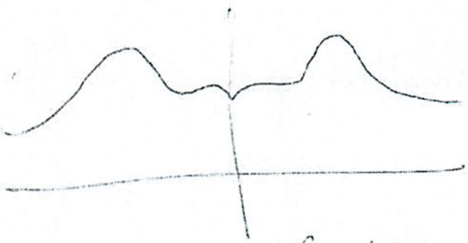
$$f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + \dots + a_N \cos Nx + b_1 \sin x + b_2 \sin 2x + \dots + b_N \sin Nx \quad \Big\} \; a_0, a_1, \dots, a_N, \; b_1, \dots, b_N \text{ constants}$$

is periodic. Enough to describe values on the interval $[-\pi, \pi]$.

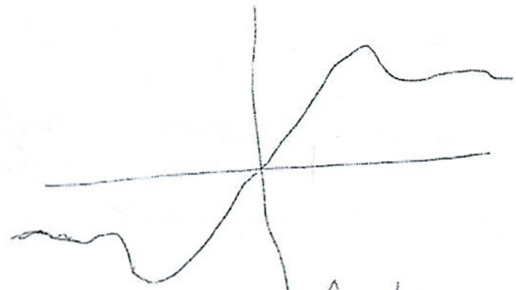
Notice that I clumped together the constant function with the cosines. Reason: these are even functions

Even function: $f(x) = f(-x)$ e.g. $\cos(mx) = \cos(m(-x))$

Odd function: $f(x) = -f(-x)$ e.g. $\sin(mx) = -\sin(m(-x))$



Even function
(Symmetrical abt. y-axis)



Odd function
(Symmetrical abt origin)

Any function $f(x)$ is the sum of an even function and an odd function (Exercise! $f(x)$ doesn't have to be periodic).

Now suppose we ~~take~~ have a function of the form

$$f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + \cdots + a_N \cos Nx$$
$$+ b_1 \sin x + \cdots + b_N \sin Nx$$

I don't tell you what $a_0, a_1, \ldots, a_N, b_1, \ldots, b_N$ are. But I'll tell you $f(x)$ at any point x if you ask.

How do we deduce $a_0, \ldots, a_N, b_1, \ldots, b_N$?

Let's see an analogous example: a power series

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_N x^N + \cdots$$

How do we deduce $a_0, a_1, \ldots$, from values of f and its derivatives?

$$a_0 = f(0), \quad a_1 = f'(0), \quad a_2 = \frac{f''(0)}{2!}, \quad a_3 = \frac{f'''(0)}{3!}, \text{ etc.}$$

Similarly for the sines and cosines example, we can deduce a_i, b_i not by differentiation but by integration!

Preliminaries:

$$\int_{-\pi}^{\pi} \sin(mx)\,dx = \left.\frac{-\cos(mx)}{m}\right|_{-\pi}^{\pi} = 0 \quad \text{for any nonzero integer } m$$

(since $\cos(\ )$ is even)

$$\int_{-\pi}^{\pi} \cos(mx)\,dx = \left.\frac{\sin(mx)}{m}\right|_{-\pi}^{\pi} = 0 \quad \text{for any nonzero integer } m$$

(since $\sin(\ )$ is 0 at integer multiples of π)

Similarly, you can show, by using sum and difference formulas for angles, that for positive integers m and n,

$$\int_{-\pi}^{\pi} \cos(mx)\cos(nx)\,dx = \begin{cases} \pi, & \text{if } m = n \\ 0, & \text{if } m \neq n \end{cases}$$

$$\int_{-\pi}^{\pi} \sin(mx)\sin(nx)\,dx = \begin{cases} \pi & \text{if } m = n \\ 0, & \text{if } m \neq n \end{cases}$$

Finally. $\int_{-\pi}^{\pi} \cos(mx)\sin(nx)\,dx = 0$ for any positive integers m and n

(one can see the last one directly by noticing the integrand is an odd function).

So if $f(x) = \frac{a_0}{2} + a_1 \cos x + \cdots + a_N \cos Nx + b_1 \sin x + \cdots + b_N \sin Nx$

then $\int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{\pi} \frac{a_0}{2}\,dx = a_0 \pi$ (all other integrals vanish)

$$\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = \int_{-\pi}^{\pi} a_n \cos nx \cos nx \,dx$$

$$= a_n \pi$$

$\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = b_n \pi$. So we can get $a_0, a_1, \ldots, b_1, \ldots$ this way

Now here's the <u>key idea of Fourier series</u>

For an arbitrary periodic function $f(x)$ (periodic by 2π)

<u>define</u>

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

Then if $f(x)$ is nice enough, we will have

$$f(x) = \frac{a_0}{2} + a_1 \cos x + \cdots \quad + b_1 \sin x + b_2 \sin 2x + \cdots$$

$$= \frac{a_0}{2} + \sum_{n \geq 1} (a_i \cos nx + b_i \sin nx)$$

The reason this works: sines and cosines are in some sense complete: they are all the periodic functions for $[-\pi, \pi]$.

If $f(x)$ is nice enough, the series on the right will converge, and to $f(x)$ "almost everywhere". More on this next time.

Chap 8

Fourier Reading

10/29

$$\frac{d^2x}{dt^2} + \omega_0^2 x = f(t)$$

Models mass + spring

L natural freq ω

L under influence of external force of magnitude $f(t)$ per unit mass

If $f(t)$ sin, cos can find w/ guess method

Can be linear combo of sins, coss

$$\sum_{n=1}^{N} A_n \cos \omega_n t$$

Can represent any periodic fn as linear combo of trig terms

Periodic $f(t+p) = f(t)$

period

$$\cos n\left(t + \frac{2\pi}{n}\right) = \cos(nt + 2\pi) = \cos nt$$

Any linear combo of sin + cos has period 2π

$$f(t) = 3 + \cos t - \sin t + 5\cos 2t + 17\sin 3t$$

②

We can write any $f(t)$ w/ $p = 2\pi$ as

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)$$

called the <u>Forier Series</u>

Integral stuff we saw in class

<u>Orthagonal</u> $\underset{=}{def}$ $\int_a^b u(t)\, v(t)\, dt = 0$

Forier series converges to $f(t)$ for every t

Can $\int$ by terms

$$\int_{-\pi}^{\pi} f(t)\, dt = \pi a_0$$

all trig terms vanish

So $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)\, dt$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos mt$$

(3)

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dt$$

Sometimes F.S. might not converge
└see 8.2

Some are piecewise coefficent

$\underline{\text{def}}$ $\sum_{n \text{ odd}} = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty}$

Can overshoot near discontinuity = Gibb's phenomenon

The more terms you have the better it looks

(I kinda remember this from 6.02)

Some helpful integration formulas

$\underline{8.2}$ $p = 2L$

half period

$g(u) \stackrel{\text{def}}{=} f\left(\frac{Lu}{\pi}\right)$

is periodic w/ period 2π

if $t = \frac{Lu}{\pi}$

$u = \frac{\pi t}{L}$ then $f(t) = g(u)$

(4)

So $f(t) = g\left(\frac{\pi t}{L}\right)$.

$$\sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi t}{L} + b_n \frac{n\pi t}{L}\right)$$

So $$a_n = \frac{1}{L}\int_{-L}^{L} g\left(\frac{\pi t}{L}\right) \cos \frac{n\pi t}{L}\, dt$$

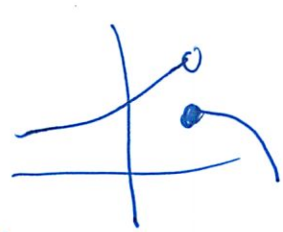
$$= \frac{1}{L}\int_{-L}^{L} f(t) \cos \frac{n\pi t}{L}\, dt$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(t) \sin \frac{n\pi t}{L}\, dt$$

(whats the point of this?)

Example finding Fourier series of ~~place~~ square wave

piecewise continuous - only isolated finite jumps



piecewise smooth - if deriv f' is piecewise continuous

(5)

Theorem Convergence of Fourier Series

Suppose periodic fn $f(t)$ is piecewise smooth

Then converges

a) to the value $f(t)$ at each pt where f is continuous

b) to the value $\frac{1}{2}[f(t+) + f(t-)]$ at each pt f is discontinuous

avg of left and right hand limits

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi t}{L} + b_n \sin \frac{n\pi t}{L} \right)$$

(i don't get importance of)

(8)

8.3 Sin + Cos

even $f(-x) = f(x)$

odd $f(-x) = -f(x)$

Fourier cosine series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}$$

w/ $$a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} \, dx$$

Fourier Sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx$$

Term wise Differentation

- Contineus
- peiordic $2L$
- f' is piecewise smooth

$$f'(x) = \sum_{n=1}^{\infty} \left(-\frac{\pi n}{L} a_n \sin \frac{n\pi x}{L} + \frac{n\pi}{L} b_n \cos \frac{n\pi x}{L} \right)$$

(6)

Fourier Series Sol of Diff Eq

$$ax'' + bx' + cx = f(t) \quad 0 < t < L \quad (1)$$

$$x(0) = x(L) = 0 \quad (2)$$

Use techniques of last unit

Or
$$f(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left(A_n \cos \frac{n\pi t}{L} + B_n \sin \frac{n\pi t}{L}\right) \quad (3)$$

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi t}{L} + b_n \sin \frac{n\pi t}{L}\right) \quad (4)$$

can differentate twice termwise

Plug (4) into (3) and (4) into (1)

Equate coeffs of like terms

like in the lucky guess method

If resulting series in (4) also satisfies endpoints in (2) then we have a formal Fourier series sol

see example

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(a lot of stuff confusing will be less so after lecture goal to see twice)

Termwise Integration

Can always $\int$ term by term

$$\int_0^t f(s)\,ds = \frac{a_0 t}{2} + \sum_{n=1}^{\infty} \frac{L}{n\pi}\left[a_n \sin\frac{n\pi t}{L} - b_n\left(\cos\frac{n\pi t}{L} - 1\right)\right]$$

$\boxed{\text{see example}}$

~~both~~

also RHS always converges for all t

a_0 must $= 0$

Fourier series

From Wikipedia, the free encyclopedia

how many you have depends on quality of match

In mathematics, a **Fourier series** decomposes periodic functions or periodic signals into the sum of a (possibly infinite) set of simple oscillating functions, namely sines and cosines (or complex exponentials). The study of Fourier series is a branch of Fourier analysis. Fourier series were introduced by Joseph Fourier (1768–1830) for the purpose of solving the heat equation in a metal plate.

Fourier transforms

Continuous Fourier transform

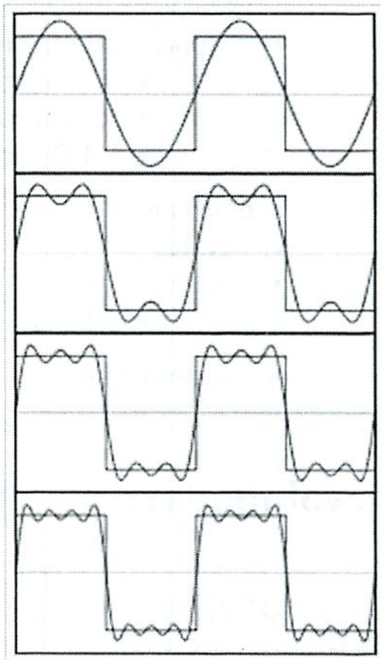
Fourier series

Discrete Fourier transform

Discrete-time Fourier transform

Related transforms

The heat equation is a partial differential equation. Prior to Fourier's work, no solution to the heat equation was known in the general case, although particular solutions were known if the heat source behaved in a simple way, in particular, if the heat source was a sine or cosine wave. These simple solutions are now sometimes called eigensolutions. Fourier's idea was to model a complicated heat source as a superposition (or linear combination) of simple sine and cosine waves, and to write the solution as a superposition of the corresponding eigensolutions. This superposition or linear combination is called the Fourier series.



The first four Fourier series approximations for a square wave.

Although the original motivation was to solve the heat equation, it later became obvious that the same techniques could be applied to a wide array of mathematical and physical problems, and especially those involving linear differential equations with constant coefficients, for which the eigensolutions are sinusoids. The Fourier series has many such applications in electrical engineering, vibration analysis, acoustics, optics, signal processing, image processing, quantum mechanics, econometrics,[1] thin-walled shell theory,[2] etc.

The Fourier series is named in honour of Joseph Fourier (1768–1830), who made important contributions to the study of trigonometric series, after preliminary investigations by Leonhard Euler, Jean le Rond d'Alembert, and Daniel Bernoulli. He applied this technique to find the solution of the heat equation, publishing his initial results in his 1807 *Mémoire sur la propagation de la chaleur dans les corps solides* (trans. Treatise on the propagation of heat in solid bodies), and publishing his *Théorie analytique de la chaleur* in 1822.

From a modern point of view, Fourier's results are somewhat informal, due to the lack of a precise notion of function and integral in the early nineteenth century. Later, Dirichlet and Riemann expressed Fourier's results with greater precision and formality.

Contents

Revolutionary article

> “ $$\varphi(y) = a\cos\frac{\pi y}{2} + a'\cos 3\frac{\pi y}{2} + a''\cos 5\frac{\pi y}{2} + \cdots.$$
>
> Multiplying both sides by $\cos(2k+1)\frac{\pi y}{2}$, and then integrating from $y = -1$ to $y = +1$ yields:
>
> $$a_k = \int_{-1}^{1} \varphi(y)\cos(2k+1)\frac{\pi y}{2}\,dy.$$ ”
>
> —Joseph Fourier, Mémoire sur la propagation de la chaleur dans les corps solides. (1807)[3][nb 1]

This immediately gives any coefficient a_k of the trigonometrical series for $\varphi(y)$ for any function which has such an expansion. It works because if φ has such an expansion, then (under suitable convergence assumptions) the integral

$$a_k = \int_{-1}^{1} \varphi(y)\cos(2k+1)\frac{\pi y}{2}\,dy$$
$$= \int_{-1}^{1} \left(a\cos\frac{\pi y}{2}\cos(2k+1)\frac{\pi y}{2} + a'\cos 3\frac{\pi y}{2}\cos(2k+1)\frac{\pi y}{2} + \cdots\right)dy$$

can be carried out term-by-term. But all terms involving $\cos(2j+1)\frac{\pi y}{2}\cos(2k+1)\frac{\pi y}{2}$ for $j \neq k$ vanish when integrated from -1 to 1, leaving only the kth term.

In these few lines, which are close to the modern formalism used in Fourier series, Fourier revolutionized both mathematics and physics. Although similar trigonometric series were previously used by Euler, d'Alembert, Daniel Bernoulli and Gauss, Fourier believed that such trigonometric series could represent arbitrary functions. In what sense that is actually true is a somewhat subtle issue and the attempts over many years to clarify this idea have led to important discoveries in the theories of convergence, function spaces, and harmonic analysis.

When Fourier submitted a later competition essay in 1811, the committee (which included Lagrange, Laplace, Malus and Legendre, among others) concluded: *...the manner in which the author arrives at these equations is not exempt of difficulties and...his analysis to integrate them still leaves something to be desired on the score of generality and even rigour.*

Birth of harmonic analysis

Since Fourier's time, many different approaches to defining and understanding the concept of Fourier series have been discovered, all of which are consistent with one another, but each of which emphasizes different aspects of the topic. Some of the more powerful and elegant approaches are based on mathematical ideas and tools that were not available at the time Fourier completed his original work. Fourier originally defined the Fourier series for real-valued functions of real arguments, and using the sine and cosine functions as the basis set for the decomposition.

Many other Fourier-related transforms have since been defined, extending the initial idea to other applications. This general area of inquiry is now sometimes called harmonic analysis. A Fourier series, however, can be used only for periodic functions, or for functions on a bounded (compact) interval.

Definition

Must period $= 2\pi$

In this section, $f(x)$ denotes a function of the real variable x. This function is usually taken to be periodic, of period 2π, which is to say that $f(x + 2\pi) = f(x)$, for all real numbers x. We will attempt to write such a function as an infinite sum, or series of simpler 2π–periodic functions. We will start by using an infinite sum of sine and cosine functions on the interval $[-\pi, \pi]$, as Fourier did (see the quote above), and we will then discuss different formulations and generalizations.

Fourier's formula for 2π-periodic functions using sines and cosines

For a periodic function $f(x)$ that is integrable on $[-\pi, \pi]$, the numbers

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx, \quad n \geq 0$$

and

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx, \quad n \geq 1$$

are called the Fourier coefficients of f. One introduces the *partial sums of the Fourier series* for f, often denoted by

$$(S_N f)(x) = \frac{a_0}{2} + \sum_{n=1}^{N} [a_n \cos(nx) + b_n \sin(nx)], \quad N \geq 0.$$

The partial sums for f are trigonometric polynomials. One expects that the functions $S_N f$ approximate the function f, and that the approximation improves as N tends to infinity. The infinite sum

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)]$$

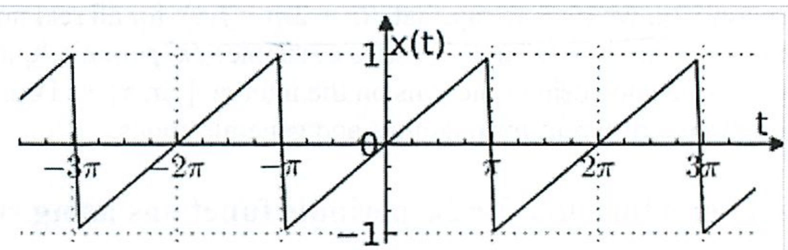
is called the **Fourier series** of f. These trigonometric functions can themselves be expanded, using multiple angle formulae.

The Fourier series does not always converge, and even when it does converge for a specific value x_0 of x, the sum of the series at x_0 may differ from the value $f(x_0)$ of the function. It is one of the main questions in harmonic analysis to decide when Fourier series converge, and when the sum is equal to the original function. If a function is square-integrable on the interval $[-\pi, \pi]$, then the Fourier series converges to the function at *almost every* point. In engineering applications, the Fourier series is generally presumed to converge everywhere except at discontinuities, since the functions encountered in engineering are more well behaved than the ones that mathematicians can provide as counter-examples to this presumption. In particular, the Fourier series converges absolutely and uniformly to $f(x)$ whenever the derivative of $f(x)$ (which may not exist everywhere) is square integrable.[4] See Convergence of Fourier series.

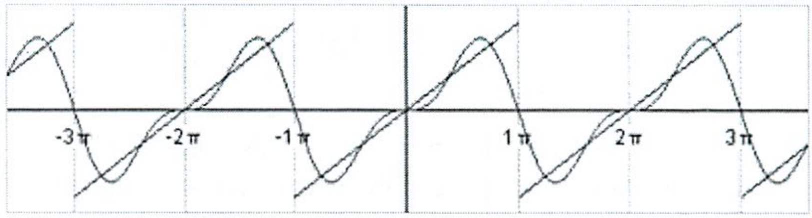
It is possible to define Fourier coefficients for more general functions or distributions, in such cases convergence in norm or weak convergence is usually of interest.

Example 1: a simple Fourier series

We now use the formula above to give a Fourier series expansion of a very simple function. Consider a sawtooth wave



Plot of a periodic identity function—a sawtooth wave.



Animated plot of the first five successive partial Fourier series.

$$f(x) = x, \quad \text{for} - \pi < x < \pi,$$
$$f(x + 2\pi) = f(x), \quad \text{for} - \infty < x < \infty.$$

In this case, the Fourier coefficients are given by

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x\,dx = 0.$$
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos(nx)\,dx = 0, \quad n \geq 0.$$
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(nx)\,dx = -\frac{2}{n}\cos(n\pi) + \frac{2}{\pi n^2}\sin(n\pi) = 2\,\frac{(-1)^{n+1}}{n}, \quad n \geq 1.$$

It can be proven that the Fourier series converges to $f(x)$ at every point x where f is differentiable, and therefore:

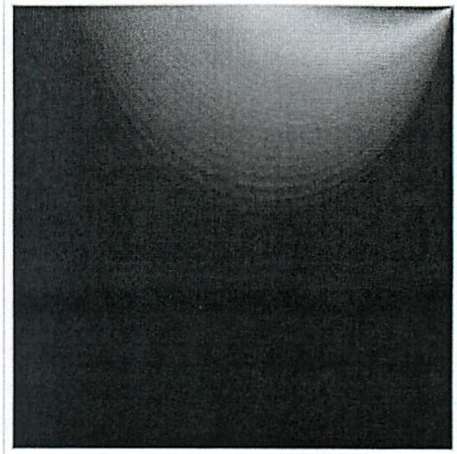
$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos(nx) + b_n \sin(nx)\right] \\ &= 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(nx), \quad \text{for} \quad x - \pi \notin 2\pi\mathbf{Z}. \end{aligned} \qquad \textbf{(Eq.1)}$$

When $x = \pi$, the Fourier series converges to 0, which is the half-sum of the left- and right-limit of f at $x = \pi$. This is a particular instance of the Dirichlet theorem for Fourier series.

Example 2: Fourier's motivation

One notices that the Fourier series expansion of our function in example 1 looks much less simple than the formula $f(x) = x$, and so it is not immediately apparent why one would need this Fourier series. While there are many applications, we cite Fourier's motivation of solving the heat equation. For example, consider a metal plate in the shape of a square whose side measures π meters, with coordinates $(x, y) \in [0, \pi] \times [0, \pi]$. If there is no heat source within the plate, and if three of the four sides are held at 0 degrees Celsius, while the fourth side, given by $y = \pi$, is maintained at the temperature gradient $T(x, \pi) = x$ degrees Celsius, for x in $(0, \pi)$, then one can show that the stationary heat distribution (or the heat distribution after a long period of time has elapsed) is given by

$$T(x,y) = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nx)\frac{\sinh(ny)}{\sinh(n\pi)}.$$


Heat distribution in a metal plate, using Fourier's method

Here, sinh is the hyperbolic sine function. This solution of the heat equation is obtained by multiplying each term of **Eq.1** by sinh(*ny*)/sinh(*nπ*). While our example function *f*(*x*) seems to have a needlessly complicated Fourier series, the heat distribution *T*(*x*, *y*) is nontrivial. The function *T* cannot be written as a closed-form expression. This method of solving the heat problem was made possible by Fourier's work.

Other applications

Another application of this Fourier series is to solve the Basel problem by using Parseval's theorem. The example generalizes and one may compute ζ(2*n*), for any positive integer *n*.

Exponential Fourier series

We can use Euler's formula,

$$e^{inx} = \cos(nx) + i\sin(nx),$$

where *i* is the imaginary unit, to give a more concise formula:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}.$$

The Fourier coefficients are then given by:

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx.$$

The Fourier coefficients a_n, b_n, c_n are related via

$$\begin{aligned} a_n &= c_n + c_{-n} \quad &&\text{for } n = 0,1,2,\dots \\ b_n &= i(c_n - c_{-n}) \quad &&\text{for } n = 1,2,\dots \end{aligned}$$

and

$$c_n = \begin{cases} \frac{1}{2}(a_n - ib_n) & n > 0 \\ \quad\frac{1}{2}a_0 & n = 0 \\ \frac{1}{2}(a_{-n} + ib_{-n}) & n < 0 \end{cases}$$

The notation c_n is inadequate for discussing the Fourier coefficients of several different functions. Therefore it is customarily replaced by a modified form of f (in this case), such as F or $\hat{f}$, and functional notation often replaces subscripting. Thus:

$$\begin{aligned} f(x) &= \sum_{n=-\infty}^{\infty} \hat{f}(n) \cdot e^{inx} \\ &= \sum_{n=-\infty}^{\infty} F[n] \cdot e^{inx} \quad \text{(engineering).} \end{aligned}$$

In engineering, particularly when the variable x represents time, the coefficient sequence is called a frequency domain representation. Square brackets are often used to emphasize that the domain of this function is a discrete set of frequencies.

Fourier series on a general interval $[a, a + \tau]$

The following formula, with appropriate complex-valued coefficients $G[n]$, is a periodic function with period τ on all of **R**:

$$g(x) = \sum_{n=-\infty}^{\infty} G[n] \cdot e^{i2\pi \frac{n}{\tau} x}.$$

If a function is square-integrable in the interval $[a, a + \tau]$, it can be represented in that interval by the formula above. I.e., when the coefficients are derived from a function, $h(x)$, as follows:

$$G[n] = \frac{1}{\tau} \int_a^{a+\tau} h(x) \cdot e^{-i2\pi \frac{n}{\tau} x}\, dx,$$

then $g(x)$ will equal $h(x)$ in the interval $[a,a+\tau]$. It follows that if $h(x)$ is τ-periodic, then:

- $g(x)$ and $h(x)$ are equal everywhere, except possibly at discontinuities, and
- a is an arbitrary choice. Two popular choices are $a = 0$, and $a = -\tau/2$.

Another commonly used frequency domain representation uses the Fourier series coefficients to modulate a Dirac comb:

$$G(f) \stackrel{\text{def}}{=} \sum_{n=-\infty}^{\infty} G[n] \cdot \delta\left(f - \frac{n}{\tau}\right),$$

where variable f represents a **continuous** frequency domain. When variable x has units of seconds, f has units of hertz. The "teeth" of the comb are spaced at multiples (i.e. harmonics) of $1/\tau$, which is called the fundamental frequency. $g(x)$ can be recovered from this representation by an inverse Fourier transform:

$$\begin{aligned}\mathcal{F}^{-1}\{G(f)\} &= \int_{-\infty}^{\infty}\left(\sum_{n=-\infty}^{\infty} G[n]\cdot\delta\left(f-\frac{n}{\tau}\right)\right)e^{i2\pi fx}\,df,\\ &= \sum_{n=-\infty}^{\infty} G[n]\cdot\int_{-\infty}^{\infty}\delta\left(f-\frac{n}{\tau}\right)e^{i2\pi fx}\,df,\\ &= \sum_{n=-\infty}^{\infty} G[n]\cdot e^{i2\pi\frac{n}{\tau}x} \stackrel{\text{def}}{=} g(x).\end{aligned}$$

The function $G(f)$ is therefore commonly referred to as a **Fourier transform**, even though the Fourier integral of a periodic function is not convergent at the harmonic frequencies.[5]

Fourier series on a square

We can also define the Fourier series for functions of two variables x and y in the square $[-\pi, \pi]\times[-\pi, \pi]$:

$$\begin{aligned}f(x,y) &= \sum_{j,k\in\mathbb{Z}\text{ (integers)}} c_{j,k}e^{ijx}e^{iky},\\ c_{j,k} &= \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x,y)e^{-ijx}e^{-iky}\,dx\,dy.\end{aligned}$$

Aside from being useful for solving partial differential equations such as the heat equation, one notable application of Fourier series on the square is in image compression. In particular, the jpeg image compression standard uses the two-dimensional discrete cosine transform, which is a Fourier transform using the cosine basis functions.

Hilbert space interpretation

Main article: Hilbert space

In the language of Hilbert spaces, the set of functions $\{e_n = e^{inx}, n \in \mathbb{Z}\}$ is an orthonormal basis for the space $L^2([-\pi,\pi])$ of square-integrable functions of $[-\pi,\pi]$. This space is actually a Hilbert space with an inner product given for any two elements f and g by:

$$\langle f,\, g\rangle \stackrel{\text{def}}{=} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{g(x)}\,dx.$$

The basic Fourier series result for Hilbert spaces can be written as

$$f = \sum_{n=-\infty}^{\infty}\langle f, e_n\rangle\, e_n.$$

This corresponds exactly to the complex exponential formulation given above. The version with sines and cosines is also justified with the Hilbert space interpretation. Indeed, the sines and cosines form an orthogonal set:

$$\int_{-\pi}^{\pi} \cos(mx)\, \cos(nx)\, dx = \pi\delta_{mn}, \quad m, n \geq 1,$$
$$\int_{-\pi}^{\pi} \sin(mx)\, \sin(nx)\, dx = \pi\delta_{mn}, \quad m, n \geq 1$$

(where δ_{mn} is the Kronecker delta), and

$$\int_{-\pi}^{\pi} \cos(mx)\, \sin(nx)\, dx = 0\,;$$

furthermore, the sines and cosines are orthogonal to the constant function **1**. An *orthonormal basis* for $L^2([-\pi, \pi])$ consisting of real functions is formed by the functions **1**, and $\sqrt{2}\cos(n\ x)$, $\sqrt{2}\sin(n\,x)$ for n = 1, 2,... The density of their span is a consequence of the Stone–Weierstrass theorem, but follows also from the properties of classical kernels like the Fejér kernel.

Properties

We say that f belongs to $C^k(\mathbb{T})$ if f is a 2π-periodic function on **R** which is k times differentiable, and its kth derivative is continuous.

- If f is a 2π-periodic odd function, then $a_n = 0$ for all n.
- If f is a 2π-periodic even function, then $b_n = 0$ for all n.
- If f is integrable, $\lim_{|n|\to\infty} \hat{f}(n) = 0$, $\lim_{n\to+\infty} a_n = 0$ and $\lim_{n\to+\infty} b_n = 0$. This result is known as the Riemann–Lebesgue lemma.
- A doubly infinite sequence $\{a_n\}$ in $c_0(\mathbb{Z})$ is the sequence of Fourier coefficients of a function in $L^1[0,2\pi]$ if and only if it is a convolution of two sequences in $\ell^2(\mathbb{Z})$. See [1] (http://mathoverflow.net/questions/46626/characterizations-of-a-linear-subspace-associated-with-fourier-series)
- If $f \in C^1(\mathbb{T})$, then the Fourier coefficients $\widehat{f'}(n)$ of the derivative f' can be expressed in terms of the Fourier coefficients $\hat{f}(n)$ of the function f, via the formula $\widehat{f'}(n) = in\hat{f}(n)$.
- If $f \in C^k(\mathbb{T})$, then $\widehat{f^{(k)}}(n) = (in)^k\hat{f}(n)$. In particular, since $\widehat{f^{(k)}}(n)$ tends to zero, we have that $|n|^k\hat{f}(n)$ tends to zero, which means that the Fourier coefficients converge to zero faster than the kth power of n.
- Parseval's theorem. If $f \in L^2([-\pi, \pi])$, then $\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2\, dx.$
- Plancherel's theorem. If $c_0,\ c_{\pm 1},\ c_{\pm 2}, \ldots$ are coefficients and $\sum_{n=-\infty}^{\infty} |c_n|^2 < \infty$ then there is a unique function $f \in L^2([-\pi, \pi])$ such that $\hat{f}(n) = c_n$ for every n.

- The first convolution theorem states that if f and g are in $L^1([-\pi, \pi])$, then $\widehat{f * g}(n) = 2\pi \hat{f}(n)\hat{g}(n)$, where $f * g$ denotes the 2π-periodic convolution of f and g. (The factor 2π is not necessary for 1-periodic functions.)

- The second convolution theorem states that $\widehat{f \cdot g} = \hat{f} * \hat{g}$.

- The Poisson summation formula states that the periodic summation of a function, f, has a Fourier series representation whose coefficients are proportional to discrete samples of the continuous Fourier transform of f:

$$\sum_{n=-\infty}^{\infty} f(t+nT) = \frac{1}{T} \sum_{k=-\infty}^{\infty} \hat{f}\left(\frac{k}{T}\right) \exp\left(i2\pi \frac{k}{T} t\right).$$

Similarly, the periodic summation of $\hat{f}$ has a Fourier series representation whose coefficients are proportional to discrete samples of f, a fact which provides a pictorial understanding of aliasing and the famous sampling theorem.

- Also see Relations between Fourier transforms and Fourier series.

Compact groups

Main articles: Compact group, Lie group, and Peter–Weyl theorem

One of the interesting properties of the Fourier transform which we have mentioned, is that it carries convolutions to pointwise products. If that is the property which we seek to preserve, one can produce Fourier series on any compact group. Typical examples include those classical groups that are compact. This generalizes the Fourier transform to all spaces of the form $L^2(G)$, where G is a compact group, in such a way that the Fourier transform carries convolutions to pointwise products. The Fourier series exists and converges in similar ways to the $[-\pi, \pi]$ case.

An alternative extension to compact groups is the Peter–Weyl theorem, which proves results about representations of compact groups analogous to those about finite groups.

Riemannian manifolds

Main articles: Laplace operator and Riemannian manifold

The atomic orbitals of chemistry are spherical harmonics and can be used to produce Fourier series on the sphere.

If the domain is not a group, then there is no intrinsically defined convolution. However, if X is a compact Riemannian manifold, it has a Laplace-Beltrami operator. The Laplace-Beltrami operator is the differential operator that corresponds to Laplace operator for the Riemannian manifold X. Then, by analogy, one can consider heat equations on X. Since Fourier arrived at his basis by attempting to solve the heat equation, the natural generalization is to use the eigensolutions of the Laplace-Beltrami operator as a basis. This generalizes Fourier series to spaces of the type $L^2(X)$, where X is a Riemannian manifold. The Fourier series converges in ways similar to the $[-\pi, \pi]$ case. A typical example is to take X to be the sphere with the usual metric, in which case the Fourier basis consists of spherical harmonics.

Locally compact Abelian groups

Main article: Pontryagin duality

The generalization to compact groups discussed above does not generalize to noncompact, nonabelian groups. However, there is a straightfoward generalization to Locally Compact Abelian (LCA) groups.

This generalizes the Fourier transform to $L^1(G)$ or $L^2(G)$, where G is an LCA group. If G is compact, one also obtains a Fourier series, which converges similarly to the $[-\pi, \pi]$ case, but if G is noncompact, one obtains instead a Fourier integral. This generalization yields the usual Fourier transform when the underlying locally compact Abelian group is $\mathbb{R}$.

Approximation and convergence of Fourier series

An important question for the theory as well as applications is that of convergence. In particular, it is often necessary in applications to replace the infinite series $\sum_{-\infty}^{\infty}$ by a finite one,

$$(S_N f)(x) = \sum_{n=-N}^{N} \hat{f}(n) e^{inx}.$$

This is called a *partial sum*. We would like to know, in which sense does $(S_N f)(x)$ converge to $f(x)$ as N tends to infinity.

Least squares property

We say that p is a trigonometric polynomial of degree N when it is of the form

$$p(x) = \sum_{n=-N}^{N} p_n e^{inx}.$$

Note that $S_N f$ is a trigonometric polynomial of degree N. Parseval's theorem implies that

Theorem. The trigonometric polynomial $S_N f$ is the unique best trigonometric polynomial of degree N approximating $f(x)$, in the sense that, for any trigonometric polynomial $p \neq S_N f$ of degree N, we have $\|S_N f - f\|_2 < \|p - f\|_2$.

Here, the Hilbert space norm is

$$\|g\|_2 = \sqrt{\frac{1}{2\pi} \int_{-\pi}^{\pi} |g(x)|^2 \, dx}.$$

Convergence

Main article: Convergence of Fourier series

See also: Gibbs phenomenon

Because of the least squares property, and because of the completeness of the Fourier basis, we obtain an elementary convergence result.

Theorem. If f belongs to $L^2([-\pi, \pi])$, then the Fourier series converges to f in $L^2([-\pi, \pi])$, that is, $\|S_N f - f\|_2$ converges to 0 as N goes to infinity.

We have already mentioned that if f is continuously differentiable, then $in\hat{f}(n)$ is the nth Fourier coefficient of the derivative f'. It follows, essentially from the Cauchy–Schwarz inequality, that the Fourier series of f is absolutely summable. The sum of this series is a continuous function, equal to f, since the Fourier series converges in the mean to f:

Theorem. If $f \in C^1(\mathbb{T})$, then the Fourier series converges to f uniformly (and hence also pointwise.)

This result can be proven easily if f is further assumed to be C^2, since in that case $n^2\hat{f}(n)$ tends to zero as $n \to \infty$. More generally, the Fourier series is absolutely summable, thus converges uniformly to f, provided that f satisfies a Hölder condition of order $\alpha > ½$. In the absolutely summable case, the inequality $\sup_x |f(x) - (S_N f)(x)| \le \sum_{|n|>N} |\hat{f}(n)|$ proves uniform convergence.

Many other results concerning the convergence of Fourier series are known, ranging from the moderately simple result that the series converges at x if f is differentiable at x, to Lennart Carleson's much more sophisticated result that the Fourier series of an L^2 function actually converges almost everywhere.

These theorems, and informal variations of them that don't specify the convergence conditions, are sometimes referred to generically as "Fourier's theorem" or "the Fourier theorem".[6][7][8][9]

Divergence

Since Fourier series have such good convergence properties, many are often surprised by some of the negative results. For example, the Fourier series of a continuous T-periodic function need not converge pointwise. The uniform boundedness principle yields a simple non-constructive proof of this fact.

In 1922, Andrey Kolmogorov published an article entitled "Une série de Fourier-Lebesgue divergente presque partout" in which he gave an example of a Lebesgue-integrable function whose Fourier series diverges almost everywhere. He later constructed an example of an integrable function whose Fourier series diverges everywhere (Katznelson 1976).

See also

- Gibbs phenomenon
- Laurent series — the substitution $q = e^{ix}$ transforms a Fourier series into a Laurent series, or conversely. This is used in the q-series expansion of the j-invariant.
- Sturm–Liouville theory
- ATS theorem

Integrating Practice

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) \sin nt \, dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} (-\sin nt)\, dt + \frac{1}{\pi}\int_{0}^{\pi} \sin nt \, dt$$

$$= \frac{1}{\pi}\left[\frac{1}{n} \cos nt\right]_{-\pi}^{0} + \frac{1}{\pi}\left[-\frac{1}{n} \cos nt\right]_{0}^{\pi}$$

$$= \frac{2}{n\pi}(1 - \cos n\pi) -$$

$$= \frac{2}{n\pi}\left(1 - (-1)^n\right)$$

$$= \begin{cases} \frac{4}{n\pi} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

$$\cos(-n\pi) = \cos(n\pi) = (-1)^n$$

Why ??

WA gives - actually no value

Can use table

n	$\cos(\pi n)$
1	-1
2	1
3	-1
4	1
5	-1

2

Ok so that is where it came from
I was not making the jump

~~KK~~ How about $\sin(n\pi)$
always 0 for integer values n

$\sin(-n\pi)$
Same

Integrating by Parts

$$\int f(x)\, g'(x)\, dx$$ ← deriv of

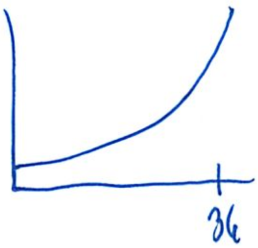
$$= f(x)\, g(x) - \int f'(x)\, g(x)\, dx$$

18.03 Lecture 20

10/31

Exam 2

Histogram



$\frac{1}{3}$ got perfect or 1 or 2 off

Exam 3 will have to be more challenging

Goal for new unit: develop new methods (Fourier Series)

to handle more general inputs

piecewise ~~continuous~~

smooth

└ no kinks

eg: ~~swan~~ square wave

– most electrical circuits w/ on off switch

$= S_q(x)$

↑ period 2π

(theory ↑ clearest)

②

On finite intervals - no discontinuities

$p(D)y = Sq(x)$

Could try power series methods

└ But fail for square wave fn!

Idea Think of sq wave ~~fn~~ as superposition (aka sum) of sinusoidal wave
↙ any piecewise smooth

See slide 2

- Sum 2 curves
- add y values at every x

Slide 3 adds square wave to compare

Can add even more terms till it is closer (slide 4)

- 1st ~~squiggle~~ maximum is closer to y axis
- as ~~added~~ adding terms

* So straight line is a wave ossilating ∞ often *

On Fri: learned how to make these functions

(3)

Assume piecewise nice function has expression in terms of sinusoids

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx$$

only term that contributes

Integrating against $\cos nx$ kills all terms in Fourier series

stick something in to integral w/ $\cos x$

integrals are 0

$$\int_{-\pi}^{\pi} \cos mx \cos nx \, dx = 0$$

unless $m=n$

$$\int_{-\pi}^{\pi} \sin mx \cos nx \, dx = 0$$

for any m, n

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dy$$

<u>Example</u> Square wave function

$$Sq(x) = \begin{cases} 1 & \text{if } x \in [0, \pi] \\ -1 & \text{if } x \in (-\pi, 0) \end{cases}$$

Repeat fn to get 2π fn

did slightly different on last pg

pts $0, \pi$ don't matter

(4)

I always forget this too!

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} Sq(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} (-1)\,dx + \frac{1}{\pi}\int_{0}^{\pi} (1)\,dx$$

$$= 0$$

would know since odd fn

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{Sq(x)}_{\text{odd}} \cdot \underbrace{\cos nx}_{\text{even}}\,dx = 0$$

~~same odd 2 fn~~

odd · even = odd

$$\boxed{\int_{-\pi}^{\pi} \text{odd} = 0}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} Sq(x)\sin nx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} (-1)\sin nx\,dx + \int_{0}^{\pi} 1\,\sin nx\,dx\right]$$

can do - easy - need $\ominus$ signs correct

$$= \frac{1}{\pi}\underbrace{\left[\frac{1}{n}\cos nx\Big|_{x=-\pi}^{x=0}\right]}_{\frac{1}{n}(1-\cos(-\pi n))} + \frac{1}{\pi}\left[-\frac{1}{n}\cos nx\Big|_{x=0}^{x=\pi}\right]$$

depends if n is odd or even

$$\begin{cases} 2/n & \text{if } n \text{ odd} \\ 0 & \text{if } n \text{ even} \end{cases}$$

$$= \begin{cases} \frac{4}{n\pi} & \text{if } n \text{ odd} \\ 0 & \text{if } n \text{ even} \end{cases}$$

(5)

$Sq(x)$ has Fourier Series

$$\frac{4}{\pi}\sin x + \frac{4}{\pi 3}\sin 3x + \cdots$$

$$= \sum_{n=1}^{\infty} \frac{4}{\pi} \frac{\sin(2n-1)x}{2n-1}$$

¿ Does this function $= Sq(x)$

└ if you have enough terms and squint] see also

<u>Convergence</u>

When is a function = to its fourier series?

If $f()$ is nice - then are = ~~[illegible]~~ wherever f is continuous

<u>nice</u> = <u>piecewise smooth</u>

└ means f' is <u>piecewise ~~smooth~~ continuous</u>

~~piecewise cont~~

└ has finitely many jump discontinuities on a bounded interval

? finite

<u>example</u> Square wave $Sq(x)$

<u>Claim</u>: $Sq(x)$ is piece wise smooth

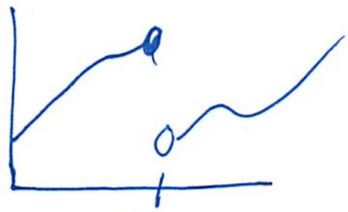
$Sq'(x) = 0$ unless x is a multiple of π

└ not defined here

zero fn w/ holes

(6) On finite intervals, only finite # of holes
↳ So satisfies test. $Sq(x)$ is piecewise smooth

In general <u>jump discontinuity</u>



x_0 ↙ here a jump discontinuity at x_0

Also

limits to left and right are finite

Proof does not matter - only the statement

A <u>counter example</u> $g(x) = \begin{cases} 1 & \text{if } x = \text{rational} \\ 0 & x = \text{irrational} \end{cases}$

↳ ∞ly many discontinuities

<u>Theorm</u> If f is piecewise smooth - than the fourier series ~~converg~~ of f converges to f at every pt that f is contineous

At the jump discontinuity, the Fourier series converges to the avg of the left and right limits

7

$$\frac{1}{2}\left[\lim_{x \to x_0^+} f(x) + \lim_{x \to x_0^-} f(x)\right]$$

Conclusion: Fourier Series for $Sq(x)$ converges everywhere
- even at discontinuities
  - $\sin(\pi) = 0$

~~Also~~ can see this directly for the sq wave fn since the Fourier series looked like

$$Sq(x) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{2n-1}$$

if $x =$ ~~odd~~ multiple of π then the Fourier series identically 0 as predicted by theorem

Will take Sq wave and accept it from now

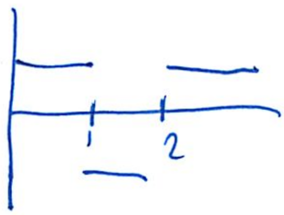
Other periods: if fn is 2π periodic then use $\cos(nx)$, $\sin(nx)$
- have 2π as period

if function is p periodic
- other than 2π

- use $\sin\left(\frac{2\pi x}{p}\right)$, $\cos\left(\frac{2\pi x}{p}\right)$ or (in sums) $\sin\left(\frac{2\pi n x}{p}\right)$, $\cos\left(\frac{2\pi n x}{p}\right)$

8

ex Sq wave w/ period 2



Fourier expansion w/ a lot more stuff

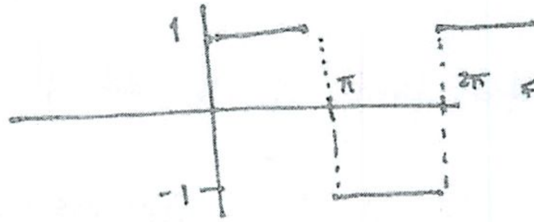
$$\sum_{n=1}^{\infty} \frac{4}{\pi} \frac{\sin((2n-1)\pi x)}{2n-1}$$

$\llcorner$ since $\frac{2\pi}{p} = \frac{2\pi}{2} = \pi$

In general $x \to \frac{2\pi}{p} x$

Lecture 20 10/31

<u>Goal</u>: Develop new methods (Fourier series) that can handle "piecewise continuous" inputs – e.g. square wave function. $Sq(x)$



← later we'll have a way of altering period using change of vars.

(think of a switch.)

Even power series methods fail here. ← Always produces a constant function as Taylor series at a point.

<u>Idea</u>: Think of square wave function as a superposition of sinusoidal waves.

(show slides of how sinusoids approximate square wave)

To find the sinusoids, assume $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$

notice that $\frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx \, dx = a_n$

integrating against $\cos nx$ kills every term in Fourier expansion except a_n.

Think of this integral calculation as "picking off" the coeff. a_n in the expansion.

i.e. $\int_{-\pi}^{\pi} \cos mx \cos nx = 0$ unless $n=m$

$\int_{-\pi}^{\pi} \sin mx \cos nx = 0$ always.

Similarly, $\frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin mx \, dx = b_m$. ← picks off coeff. of $\sin mx$ in expansion of f.

Example: Square wave function.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0}(-1)\,dx + \frac{1}{\pi}\int_{0}^{\pi}(1)\,dx = \boxed{0}$$

(could notice that f is odd.)

similarly $$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{\underbrace{f(x)}_{\text{odd}}\ \underbrace{\cos nx}_{\text{even}}}_{\text{odd}}\,dx = 0$$ since integrating odd function on interval symmetric about origin: $[-\pi,\pi]$.

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx =$$

$$= \frac{1}{\pi}\int_{-\pi}^{0}(-1)\cdot\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi}(+1)\cdot\sin nx\,dx$$

$$= \underbrace{\frac{1}{\pi}\left[+\frac{1}{n}\cos nx\Big|_{x=-\pi}^{x=0}\right]}_{\text{value at } -\pi \text{ depends on } n \text{ odd / } n \text{ even.}} + \frac{1}{\pi}\left[-\frac{1}{n}\cos nx\Big|_{x=0}^{x=\pi}\right]$$

$$\left(\frac{1}{n}\left(1 - \cos(-\pi n)\right)\right) = \begin{cases} 0 & \text{if } n \text{ even} \\ \frac{2}{n} & \text{if } n \text{ odd} \end{cases}$$

same for second integral. ...

$$b_n = \begin{cases} 4/n\pi & \text{if } n \text{ odd} \\ 0 & \text{if } n \text{ even.} \end{cases}$$

Conclusion: Square wave function $Sq(x) \sim \frac{4}{\pi}\sin x + \frac{4}{\pi\cdot 3}\sin 3x + \cdots$

or in summation notation: $$Sq(x) \sim \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{4}{\pi(2n-1)}\cdot\sin(2n-1)x$$

"seems possibly equal to" — need to address convergence!

Other periods: 2π - periodic $\longrightarrow$ used $\sin nx$, $\cos nx$

for arbitrary period P $\longrightarrow$ use $\sin\left(\frac{2\pi}{P} nx\right)$, $\cos\left(\frac{2\pi}{P} nx\right)$

(Book uses $L = \frac{1}{2}P$, so that $\sin\left(\frac{\pi}{L} nx\right)$, $\cos\left(\frac{\pi n x}{L}\right)$ are used)

Always compare apples to apples $\leftarrow$ P must be period for sines/cosines used in Fourier expansion.

Two ways to compute coefficients:

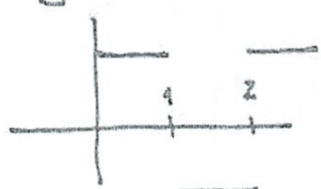
① Analogous formulas — if f has period P, then

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{2\pi n x}{P} + b_n \sin \frac{2\pi n x}{P}$$

with $a_n = \frac{2}{P} \int_{-P/2}^{P/2} f(x) \cos \frac{2\pi n x}{P}\, dx.$ (or with $\frac{P}{2} = L$: $\frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{\pi n x}{L}\, dx$ etc.)

$b_n = \frac{2}{P} \int_{-P/2}^{P/2} f(x) \sin \frac{2\pi n x}{P}\, dx.$

② Use formulas for 2π-period + change of vars.

Ex: Square wave w/ period 2. :  $:= g(x)$

Since square wave w/ period 2π $f(x)$ is

$$f(x) \sim \sum_{n=1}^{\infty} \frac{4}{\pi} \cdot \frac{\sin(2n-1)x}{2n-1} \longrightarrow g(x) \sim \sum_{n=1}^{\infty} \frac{4}{\pi} \frac{\sin((2n-1)\pi x)}{2n-1}$$

since $x \mapsto \pi x$ changes period from 2π to 2.
($x \mapsto \frac{\pi}{L} x$ in general)

See p. 565 for ex. with period 4 using method ①

Convergence : Roughly, if f nice, then Fourier series of f converges to f at each point where f continuous.

nice : piece-wise smooth (i.e. f' is piecewise continuous)

A piece-wise continuous function has finitely many jump discontinuities on any bounded interval (otherwise continuous.)

Ex : Square wave function f. w/ period 2π. $f'(x) = 0$ unless x is multiple of π (then f' undefined)

this is a piecewise continuous function. (∞-ly many discontinuities, but only finitely many on bounded interval.)

Thm : If f is piecewise smooth then

(I) Fourier series of f converges to f where f continuous

(II) converges to $\frac{1}{2}\left[\lim_{x \to x_0^+} f(x) + \lim_{x \to x_0^-} f(x)\right]$ at any (jump) discontinuity x_0.

Ex : If we define square wave $f(x)$ by $f(x) = \begin{cases} 1 & \text{if } x \in (0,\pi) \\ 0 & \text{if } x = \pi, 2\pi \\ -1 & \text{if } x \in (\pi, 2\pi) \end{cases}$ on $(0, 2\pi]$, then repeat.

then Fourier series of $f(x)$ converges everywhere!

In particular if $x = \pi$. then $\sin (2n-1)\pi = 0 \ \forall n$, so Fourier series is identically 0 as expected from theorem.

Math 18.03 : Differential Equations

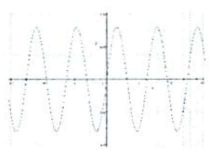
Lecture 20 Supplemental Notes

Monday, October 31, 2011

Superposition of waves

Add the two waves (pictured below)

$$\frac{4}{\pi}\sin x \qquad \text{and} \qquad \frac{4}{3\pi}\sin(3x)$$



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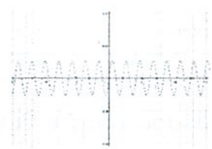
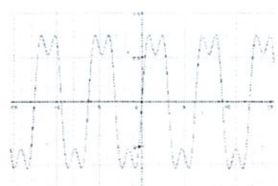
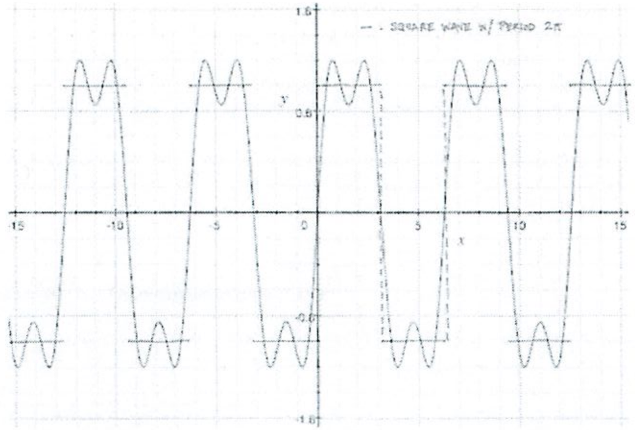


$=$

$\frac{4}{\pi}\sin x + \frac{4}{3\pi}\sin(3x)$:

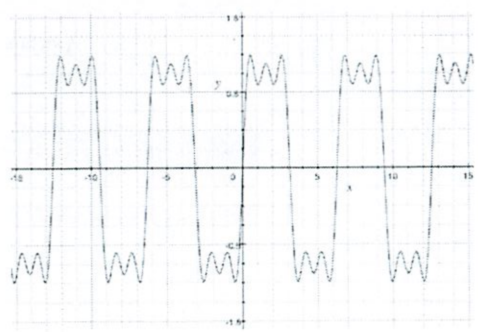


Comparison with square wave (in red)



Adding a third non-zero term to approx. square wave

$$\frac{4}{\pi}\sin x + \frac{4}{3\pi}\sin(3x) + \frac{4}{5\pi}\sin(5x)$$

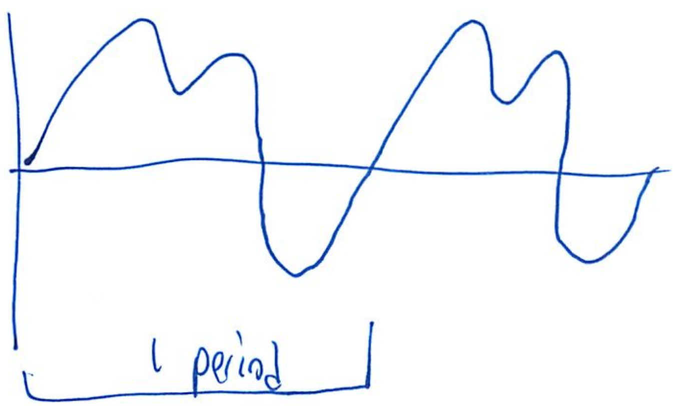


18.03 Recitation

11/1

Periodic functions

- do something
- then do it again + again



$1P = 2L$

- $\sin(x)$, $\cos(x)$ have period 2π
- $f(x) = f(x+p)$
- for period
- $\sin(2x)$, 2

period π

- care about smallest period
could say $2P$ = period
we don't care
Constant has no smallest period

- $\sin(kx) \to \frac{2\pi}{k}$ = period

$\hookrightarrow \sin(kx) = \sin(kx + 2\pi) = \sin\left(k\left(x + \frac{2\pi}{k}\right)\right)$

(2)

Fourier: these are all periodic functions

If f is a "nice" periodic function of period $p = \frac{2\pi}{k}$
then can write (uniquely)

$$f = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(knx) + b_n \sin(knx)$$

To get coefficents a_n, b_n

$$a_0 = \int_{-p/2}^{p/2} f(x)\,dx$$

period

$$a_n = \int_{-p/2}^{p/2} f(x) \cos nkx \, dx$$

$$b_n = \int_{-p/2}^{p/2} f(x) \sin nkx \, dx$$

Note

$$\int_{-p/2}^{p/2} \underbrace{\underbrace{\cos nkx}\ \underbrace{\sin mkx}}_{\text{odd}} = 0$$

$\int$ to even function

(3)

$\int \cos(mkx)\cos(nkx)$

$\downarrow$

$\cos(mkx + nkx) = \cos(mkx)\cos(nkx) - \sin(mkx)\sin(nkx)$

$\cos(mkx - nkx)$ " " + " "

$$\int_{-P/2}^{P/2} \cos(Tkx)\,dx$$

$T = m+n,\ m-n$

$$= \frac{2}{Tk}\left(\sin(T\pi x)\right) = 0$$

except if $T=0$ $\int \cos \neq \sin$

$$P = \int_{-P/2}^{P/2} \cos(2nkx) + \int_{-P/2}^{P/2} 1$$

double angle formula

$$= 2\int \cos(nkx)\cos(nkx) = \boxed{\int_{-P/2}^{P/2} \cos(nkx)^2\,dx = \frac{P}{2}}$$

So can use these shortcuts when $\int$

- know what goes to 0
- What coefficients stay around

L like when $\int \underbrace{\cos nkx}_{f(x)} \cos nkx$

c. only if $n=m$

4

So expanded at $L = \frac{p}{2}$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi}{L} x\right) dx$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi}{L} x\right) dx$$

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(x)\, dx$$

$$p = \frac{2\pi}{k}$$

$$L = \frac{\pi}{k}$$

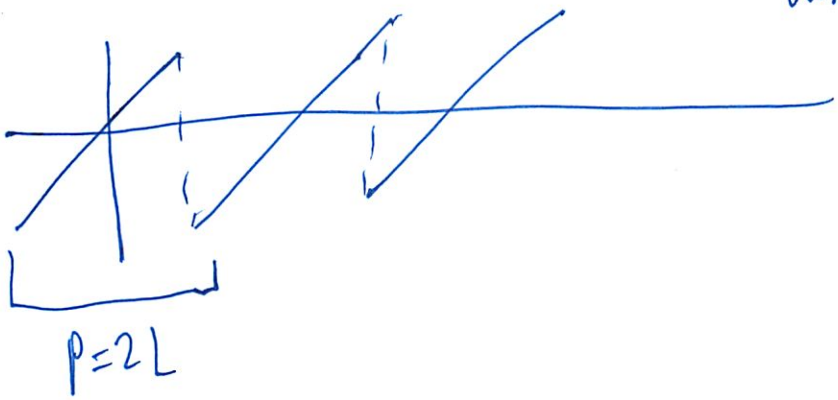
$$k = \frac{\pi}{L}$$

Not covering complex Fourier series

↳ coefficents should be complex conjugates

(5)

Example



which looks like $f(x)=x$
for $x \in [-L, L)$

So we use our formulas

$$a_n = \frac{1}{L}\int_{-L}^{L} \underbrace{x}_{\text{odd}} \underbrace{\cos\left(\frac{n\pi x}{L}\right)}_{\text{even}} dx$$

keep as variable

odd

So $\int$ odd = even so 0

more generally for odd f the $a_n = 0$

$$b_n = \frac{1}{L}\int_{-L}^{L} x \sin\left(\frac{nx\pi}{L}\right) dx$$

integration by parts

$$= \frac{1}{L}\left[\frac{L}{n\pi} x \cos\left(\frac{n\pi k}{L}\right)\Big|_{-L}^{L} - \int_{-L}^{L} \frac{-L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)dx \right.$$

$$= \frac{1}{L}\left| \frac{L}{n\pi} x \cos\left(\frac{n\pi k}{L}\right)\right|_{-L}^{L} - \left|\left(\frac{L}{n\pi}\right)^2 \sin\left(\frac{n\pi x}{L}\right)\right|_{-L}^{L}$$

$$= \frac{1}{L}\left(\frac{-L^2}{n\pi}\cos(n\pi) - (-L)^2\cos(-n\pi)\right) + 0$$

6

$$= \frac{1}{L} \cdot -\frac{2L^2}{n\pi}(-1)^n$$

$\cos(n\pi) = 0$

$\cos(-n\pi) = (-1)^n$

~~So all together~~

$$= \frac{2L}{n\pi}(-1)^{n+1}$$ ← got rid of neg sign

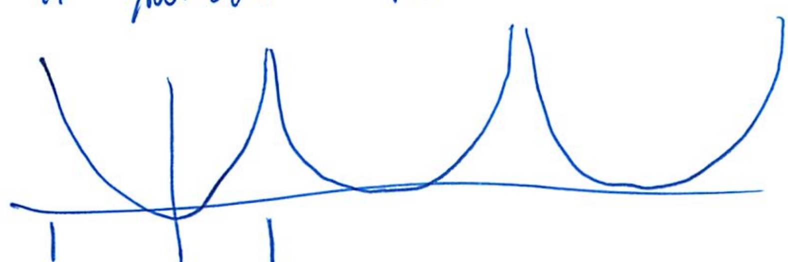
So all together

$$f(x) = \frac{2L}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\left(\frac{n\pi}{L}x\right)$$

So that is complete process from begining to end

Do it yourself for

a)



$p = 2L$ ← still a variable

$\phi(x) = x^2$ for $x \in (-L, L)$

b)

$h = x^n$ for $x \in (-L, L)$

OH

11/2

Conceptual exam qu

- find some way to use fact

18.03 Lecture 21

11/2

P-Set scores are dropping

Fri: Ruben Rosales - Sub Prof
Laplace Transforms

Next Friday - Holiday
PS due Wed - mini pset
6-7 problems

Exam 3 Tue Nov 22
7:30 PM
(I don't have any other night exams)
4 rooms across campus

$\frac{1}{3}$ Fourier Series
$\frac{2}{3}$ Laplace Transforms

Today Fourier Series
2 applications
- Square Wave ODE
- Boundary Value Problem

will be on exam

Fourier Series

Like power series, but w/ sin, cos

Quick Quiz 1

$$f(t) = \begin{cases} 1+t & t \in (-1,0] \\ 1-t & t \in (0,1) \end{cases}$$

2

~~Answer 2 → $\frac{1}{2}$~~

It's the average $\frac{1}{L}\int_{-L}^{L} f(t)\,dt$

weighted average

$L = \frac{1}{2}P = 1$

So prof got it wrong

Answer

$\frac{1}{1}\cdot 1 = \boxed{1}$ #3

Now Quick Quiz 2

what is coeff b_1

answer #1 $\boxed{0}$

$$b_1 = \frac{1}{1}\int_{-1}^{1} \underbrace{\underbrace{f(t)}_{\text{even}}\ \underbrace{\sin \pi t}_{\text{odd}}}_{\text{odd}} = 0$$

$\int \text{odd} = 0$

Won't make $\int$ hard on exam

(3)

Applications Undamped spring w/ square wave input

$mx'' + kx = Sq(t)$

undamped

where $Sq(t) = \begin{cases} 1 & \text{if } 0 < t < 1 \\ -1 & \text{if } 1 < t < 2 \end{cases}$

repeat

so period $= 2$

discontinuites every 1

So must do Fourier Series

Like power Series - same game

$$Sq(t) = \sum_{n=1}^{\infty} \frac{4}{\pi} \frac{\sin(2n-1)\pi t}{(2n-1)}$$

this means odd

$$= \sum_{n \text{ odd}} \frac{\sin(n\pi t)}{n}$$

(I never realized was same)

Guess a particular sol'n

$$x_p(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi t) + b_n \sin(n\pi t)$$

(4)

So should guess fns of ~~same~~ same flavor

$\cos(n\pi t)$ $\sin(n\pi t)$ etc

Here diff eq ~~has~~ only even derivs

aka So ODE has symmetry (Study!)

input is odd

So can throw out the cosines!

$$x_p(t) = \cancel{\frac{a_0}{2}} + \sum_{n=1}^{\infty} \cancel{a_n\cos(n\pi t)} + b_n \sin(n\pi t)$$

Will only see odd

$$x_p(t) = \sum_{\cancel{n=1}\ \text{odd}}^{\infty} b_n \sin(n\pi t)$$

Guess ~~∞~~

$$x_p(t) = \sum_{n \text{ odd}} b_n \sin(n\pi t)$$

Plug guess in

Calc 2nd deriv

Plug that in too

(5)

$mx'' + kx =$

$$x_p''(t) = \sum_{n \text{ odd}} -n^2 \pi^2 b_n \sin(n\pi t)$$

Coeff of $\sin n\pi t$ (LHS):

$$\underbrace{-mn^2\pi^2 b_n}_{\text{2nd deriv}} + \underbrace{k \cdot b_n}_{\text{reg}}$$

= coeffs on RHS

$$= \cancel{\quad} \frac{4}{\pi} \cdot \frac{1}{n}$$

So

$$-mn^2\pi^2 b_n + k \cdot b_n = \frac{4}{\pi}\frac{1}{n}$$

Like before, except equating coeffs of $\sin \underset{n}{3} \pi t$

So solve for b_n

$$b_n = \frac{4}{\pi} \cdot \frac{1}{n(k - mn^2\pi^2)}$$

Have found particular sol'n

$$x_p(t) = \sum_{n \text{ odd}} \frac{4}{\pi} \frac{1}{n(k - mn^2\pi^2)} \sin n\pi t$$

Don't need to visualize

(6)

Should it be continous or piecewise

- Just you applying force to spring
- $x_p(t)$ is position
- Can't be discontinuous! $\xrightarrow{\text{would be}}$ teleportation!

- So expecting a wave/osillation

Amplitude Gain

- before had to use trig identity
- fn should converge
- so smallest end should have most
- so can graph 1st 10 terms
- take max
- can find critical pt in orginal
  - See where $\cos = 0$

Can check $t = \frac{1}{2}$ is a critical pt & fn is concave down at $t = \frac{1}{2}$

max of $x_p(t)$ is at $t = \frac{1}{2}$

$$x_p\left(\tfrac{1}{2}\right) = \sum_{n\ odd} \frac{4}{\pi}\left(\frac{1}{n(k-mn^2\pi^2)}\right)\sin\frac{n\pi}{2}$$

then take like 100 terms

⑦

It is possible a guess on LHS was included in homogeneous sol → ~~oscillation~~ resonance

roots $\pm\sqrt{\frac{k}{m}}$ ⤳ $\cos\sqrt{\frac{k}{m}}\,t$

$\sin\sqrt{\frac{k}{m}}\,t$

Fails if any of the fns $= \sin\sqrt{\frac{k}{m}}$

* Fails if $n\pi = \sqrt{\frac{k}{m}}$ for any n

Example

$Sq(t)$ w/ period 2π

$$Sq(t) = \frac{4}{\pi}\left(\sin t + \frac{1}{3}\sin 3t + \frac{1}{5}\sin 5t + \dots\right)$$

So in!

$mx'' + kx = Sq(t)$ has resonance

if $m = 1 \quad k = 9$

then $\sqrt{\frac{k}{m}} = \sqrt{\frac{9}{1}} = \sqrt{9} = 3$

So our particular sol'n involves $\sin nt$ — n is odd but $\neq 3$

~~Then~~ Here must put t in front of $\sin$ and $\cos$

$$\underbrace{\underbrace{t}_{\text{odd}}\ \underbrace{\cos 3t}_{\text{even}}}_{\text{odd}}, \dots$$

8

Damping problem in notes

2nd application <u>Boundry Value Problems</u>

eg. $P(D)x = f(t)$ on a finite interval $t \in (0,L)$

But specify boundry conditions for sol

w/ ~~xxx~~ $x(0) = x(L) = 0$

ex: $x'' + 9x = t$

↳ using easy example

On $t \in (0,1)$ w/ $x(0) = 0 = x(1)$

↑ On unit interval

↑ Starts and ends on 0

Where is Fourier series?

Draw:

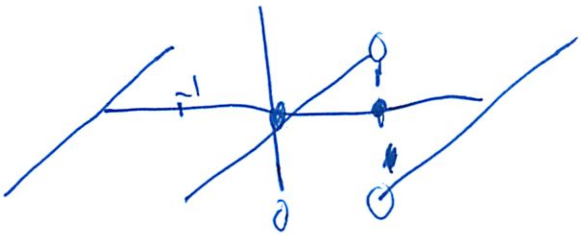
<u>Sneaky idea</u> extend $f(t)$ to periodic fn ~~xxx~~ of Period $2L$ w/ boundry conditions $f(0) = f(1) = 0$

↳ fn on right

↑ matching above

⑨

Chose an extension to make period



So can do same as before
Replace w/ Fourier Series + solve

$f(t)$ has Fourier Series

$$= \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin n\pi t$$

Make guess for undet. coeff

$$x_p(t) = \sum_{n=1}^{\infty} b_n \sin n\pi t$$

2 coeffs for every n
not just odd n

Plug into LHS and solve for b_n

$$x_p(t) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sin n\pi t}{n(9 - n^2\pi^2)}$$

⋮ skipping last few steps

(10)

Could have done the same solve sys of eqns

But this works w/ non linear and piecewise

This is called Harmonic analysis on finite interval

Fri: Harmonic analysis on real line (∞ interval)

Methods that are opposite of operator method

Lecture 21 11/2

Differential equation with square wave input:

Ex: Undamped spring $mx'' + kx = F(t)$ where

Exam 3
Nov. 22 (T) Evening
7:30 pm.

$$F(t) = \begin{cases} 1 & \text{if } 0 < t < 1 \\ -1 & \text{if } 1 < t < 2 \end{cases} \quad (\text{period } 2)$$

Find general sol'n to this ODE. Last time $F(t) = \sum_{n=1}^{\infty} \frac{4}{\pi} \frac{\sin (2n-1)\pi t}{(2n-1)}$

$$= \sum_{n \text{ odd}} \frac{4}{\pi} \frac{\sin n\pi t}{n}.$$

Book's notation

(What kind of response are we expecting? continuous?)

Homogeneous solution: $c_1 \cos \omega_0 t + c_2 \sin \omega_0 t$ with $\omega_0 = \sqrt{\frac{k}{m}}$.

Guess particular sol'n: ODE preserves even/odd-ness of functions

so guess $x_p(t) = \sum_{n \text{ odd}} b_n \sin n\pi t$ (just like power series method, only now we're using Fourier series)

$\Rightarrow x_p''(t) = \sum_{n \text{ odd}} -n^2\pi^2 b_n \sin n\pi t$ (using fact about term-wise diff. of Fourier series)

(This guess ok provided none of $\sin n\pi t$ have $n\pi = \sqrt{\frac{k}{m}}$)

This will always be the case if k, m don't involve π.

Solving, we get: $-mn^2\pi^2 b_n + k b_n = \frac{4}{\pi n}$ equating coeffs. of $\sin n\pi t$ on both sides.

$$b_n = \frac{4}{\pi n (k - mn^2\pi^2)}$$

That is $x_p(t) = \frac{4}{\pi} \sum_{n\cdot\text{odd}} \frac{\sin n\pi t}{n(k - mn^2\pi^2)}$

What is amplitude of this steady-state response? Try to maximize $x_p(t)$.

$t = 1/2$ is critical point. Gives $x_p(\frac{1}{2}) = \frac{4}{\pi} \sum_{n\cdot\text{odd}} \frac{1}{n(k-mn^2\pi^2)} \sin(\frac{n\pi}{2})$

concave down there,

so local maximum.

$= \frac{4}{\pi} \sum_{n\cdot\text{odd}} \frac{1}{n\cdot(k-mn^2\pi^2)} \cdot (-1)^{\frac{n+3}{2}}$

Kind of like alternating harmonic series.

converges.

Approximate by summing up finite number of terms.

Say 100 or 50.

When would we have resonance?

Say square wave of period 2π

$F(t) = \frac{4}{\pi}(\sin t + \frac{1}{3}\sin 3t + \cdots)$

so if $\sqrt{\frac{k}{m}}$ is odd integer, then resonance.

E.g. if $k = 9$, $m = 1$, then $\sqrt{\frac{k}{m}} = 3$

so solution will involve $t \cos 3t$ (odd function since t: odd, $\cos 3t$: even)

(don't have to guess $t \sin 3t$ which is even)

What if there is damping?

$mx'' + cx' + kx = F(t)$ with

$F(t)$ odd function of period $p = 2L$.

Then $F(t)$ has Fourier series made of $\sin \frac{n\pi t}{L}$, n: integer.

Write $F(t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi t}{L}$ some coeffs b_n.

Remember that we calculate b_n via the formula:

$$\frac{1}{L}\int_{-L}^{L} f(t)\ \sin\frac{n\pi t}{L}\,dt = b_n.$$

Could try Fourier series sol'n again by substituting.

Messy. ODE doesn't preserve odd-ness, so guess is ugly. <u>Better</u>: Infinite sum of superpositions (limit of finite sum of superpositions)

Know that for any single sine function

$F(t) = F_0 \sin \omega t$, answer is

$$x_p(t) = \frac{F_0}{\sqrt{(k-m\omega^2)^2+(c\omega)^2}} \sin(\omega t - \alpha) \qquad \alpha = \tan^{-1}\left(\frac{c\omega}{k-m\omega^2}\right)$$

(follows from ERF, for example: $\tilde{x}_p(t) = \dfrac{e^{i\omega t}}{P(i\omega)}$)

with $P(r) = mr^2+cr+k$

So by superposition, if

$$F(t) = \sum b_n \sin\frac{n\pi t}{L}$$

then $F_{0,n} = b_n$, $\omega_n = \dfrac{n\pi}{L}$, $\alpha_n = \cdots$ for each n gives solution.

so

$$x_p(t) = \sum_{n=1}^{\infty} \frac{b_n}{\sqrt{(k-m\omega_n^2)^2+(c\omega_n)^2}} \sin(\omega_n t - \alpha_n)$$

Final application: Boundary value problems.

E.g. $\quad p(D)x = f(t)$ on interval $t \in [0, L]$

with $x(0) = x(L) = 0$.

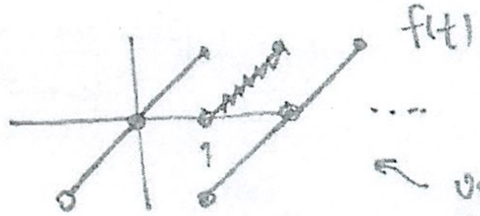
(or more generally, could make $x(0)$, $x(L)$ anything we want. These are the "boundary values")

Example: $\quad x'' + 9x = t$

with $\quad x(0) = x(1) = 0$.

How do Fourier series help?

Extend $f(t) = t$ on $t \in (0,1)$ to a periodic function, satisfying endpoint conditions.



Use odd period 2 extn!

Then substitute the Fourier series for $f(t)$, solve using series methods.

$f(t)$ has Fourier expansion $\quad f(t) = \dfrac{2}{\pi} \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin n\pi t$

Again ODE preserves odd-ness, so guess $\quad x_p(t) = \displaystyle\sum_{n=1}^{\infty} b_n \sin n\pi t$.

Substitute and solve!

$$x_p(t) = \frac{2}{\pi} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin n\pi t}{n(9 - n^2\pi^2)}$$

Note 1: We could have solved this exactly via old methods for linear ODEs, but this works much more generally.

Note 2: Subject in mathematics - Harmonic analysis on finite interval (period of our function)

Next, explore techniques of Harmonic analysis on infinite interval: real line.

Math 18.03 : Differential Equations

Lecture 21 Supplemental Notes

Wednesday, November 2, 2011

Quick Quiz

The periodic function $f(t)$ having period 2, and defined for $t \in (-1, 1)$ by

$$f(t) = \begin{cases} 1+t & \text{if } t \in (-1,0) \\ 1-t & \text{if } t \in (0,1) \end{cases}$$

has Fourier expansion $a_0/2 + \cdots$ with constant term a_0 equal to

1. 0
2. 1/2
3. 1
4. This function doesn't have a Fourier expansion.

Quick Answer

The periodic function $f(t)$ having period 2, and defined for $t \in (-1, 1)$ by

$$f(t) = \begin{cases} 1+t & \text{if } t \in (-1,0) \\ 1-t & \text{if } t \in (0,1) \end{cases}$$

has Fourier expansion $a_0/2 + \cdots$ with constant term a_0 equal to

1. 0
2. 1/2
3. 1
4. This function doesn't have a Fourier expansion.

The constant term a_0 is computed by:

$$\frac{1}{1}\int_{-1}^{1} f(t)\,dt$$

which is just the average of the function $f(t)$ over a complete period. The integral is the area of two right triangles of area 1/2, so the total integral is 1 and the average value is also 1.

Quick Quiz 2

The periodic function $f(t)$ having period 2, and defined for $t \in (-1, 1)$ by

$$f(t) = \begin{cases} 1+t & \text{if } t \in (-1,0) \\ 1-t & \text{if } t \in (0,1) \end{cases}$$

has Fourier expansion $a_0/2 + a_1 \cos \pi t + b_1 \sin \pi t + \cdots$ with coefficient b_1 equal to

1. 0
2. 1/2
3. 1
4. This function still doesn't have a Fourier expansion.

Quick Answer 2

The periodic function $f(t)$ having period 2, and defined for $t \in (-1, 1)$ by

$$f(t) = \begin{cases} 1+t & \text{if } t \in (-1,0) \\ 1-t & \text{if } t \in (0,1) \end{cases}$$

has Fourier expansion $a_0/2 + a_1 \cos \pi t + b_1 \sin \pi t + \cdots$ with coefficient b_1 equal to

1. 0
2. 1/2
3. 1
4. This function still doesn't have a Fourier expansion.

The coefficient b_1 is computed by:

$$\int_{-1}^{1} f(t) \sin \pi t \, dt$$

which is an odd function, as $\sin \pi t$ is odd and $f(t)$ is even. Hence the integral symmetric about the origin is 0.

18.03 Recitation

Fourier Series

11/3

No more difficult than power series

Warmup What is smallest period

1. $\sin \frac{7}{3}t + 4 \sin \frac{8}{15}t$
2. $\sin t + 2 \sin \pi t$
3. $\sin \frac{1}{t}$

"How do you do this since we just learned $\frac{2\pi}{n}$ trick - how does that work w/ sums

Or when you have $\frac{t}{T}$

(No one else got it)

$\sin kt = \sin(kt + 2\pi) = \sin\left(\left(k\left(t + \boxed{\frac{2\pi}{k}}\right)\right)\right)$

1. $P = \frac{2\pi}{7/3}$ $\quad P = \frac{2\pi}{8/15}$

$= \frac{6\pi}{7}$ $\quad = \frac{15\pi}{4}$

What is smallest multiple of both?

- no denom common factor

②

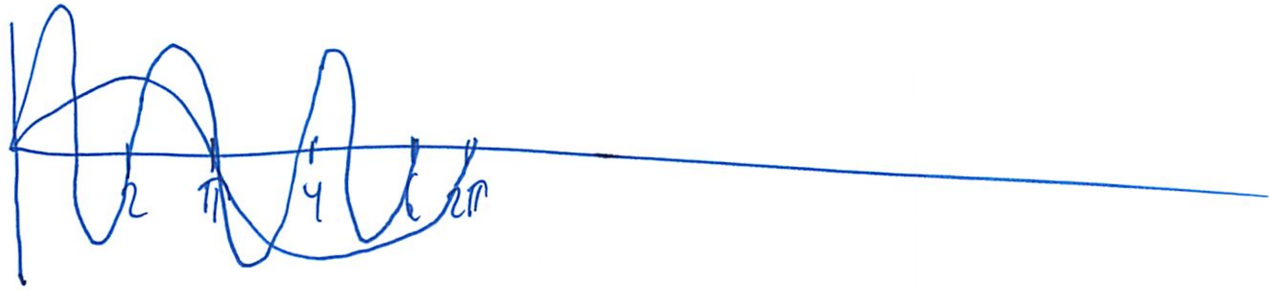
Take 7 of $\frac{6\pi}{7}$ and 4 $\frac{15\pi}{4}$

So 6π, 15π

Least common multiple 6, 15 is 30

So 30π

2. $P = 2\pi$ + $P = 2$



Add them

If say 2π then

~~So at 2π~~ - should have same value at 0

$f(2\pi) = \underbrace{\cancel{\sin 2\pi}}_{0} + \underbrace{2\sin 2\pi^2}_{\text{not } 0}$

~~Not~~

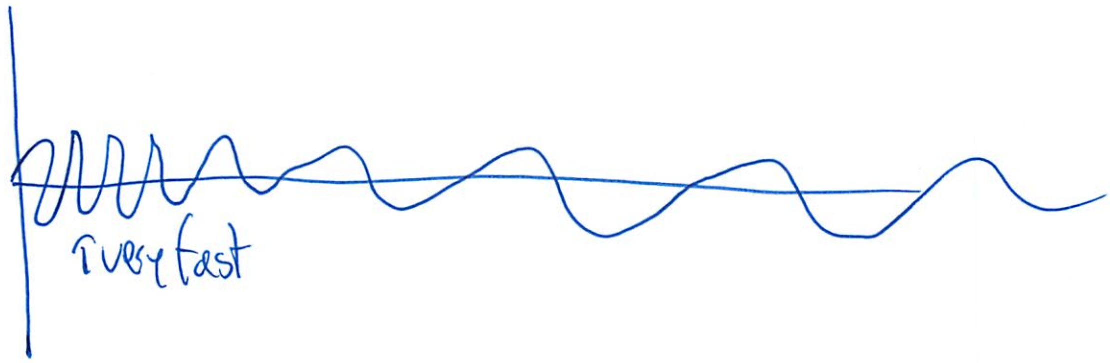
So no period

Lmust be multiple of each of their periods

So fn not periodic

(3)

3. $\sin \frac{1}{t}$



Obviously not periodic

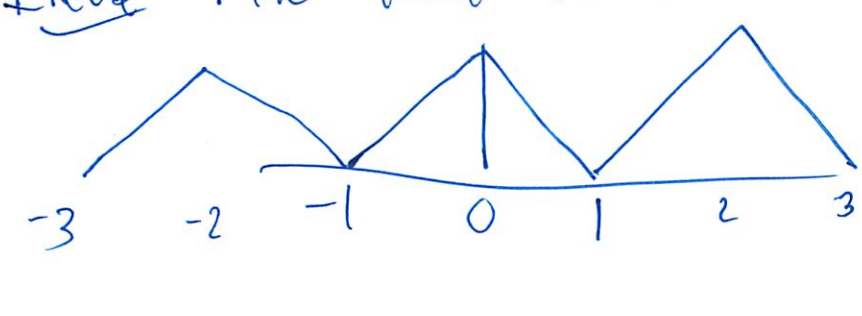
Fourier Series

$$f = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi t}{2} + b_n \sin \frac{n\pi t}{2}$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(t) \cos \frac{n\pi t}{2}$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(t) \sin \frac{n\pi t}{2}$$

Exercise Find fourier fn for



(4)

Grrr, done so many times

But never did w/ even/odd practice

~~cos n~~

period = 2

$$a_n = \frac{1}{1}\left[\int_{-1}^{0} (x+1)\cos\frac{n\pi x}{1} + \int_{0}^{1}(-x+1)\cos n\pi x\right]$$

whatever this is?

L is $\frac{1}{2}$ period

L = 1

I need to study odd even tricks

- what multiplied by what = what
- $\int$ what = what

$\int$ odd = even $\int$ even = odd + c	$\int_{-x}^{x}$ odd = 0 $\int_{-x}^{x}$ even = some value

Other people are actually integrating

Must actually integrate like did several times on hw

$\int$ cos is what

+ sin

(5)

$a_0 = \cancel{}\, 1 \int_{-1}^{1} f(t)\,dt$

$= 1\int_{-1}^{0} (1+t)\,dt + 1\int_{0}^{1} 1-t\,dt$

$= t+\frac{t^2}{2}\Big|_{-1}^{0} + t-\frac{t^2}{2}\Big|_{0}^{1}$

$= -1-\frac{1}{2} + 1-\frac{1}{2} = -1$

$a_n = 1\int f(t)\cos(n\pi t)\,dt$

$= \int_{-1}^{0} (1+t)\cos(n\pi t)\,dt + \int_{0}^{1} (1-t)\cos(n\pi t)\,dt$

Need to $\int$ by parts
which you should remember

⋮

$b_n = 1\int_{-1}^{1} \underbrace{\underbrace{f(t)}_{\text{even}}\ \underbrace{\sin(n\pi t)}_{\text{odd}}}_{\text{odd}} = 0$

~~odd even~~ odd

$\int_{-x}^{x}$ ~~even~~ odd $= 0$

Then $f(x) = \frac{1}{2} + \sum_{n\ \text{odd}} \frac{-4}{(n\pi)^2}\cos(n\pi t)$

$\boxed{\frac{a_0}{2} \text{ should be avg of fn}}$

6

<u>Example</u> Spring Mass Dashpot

$$mx'' + cx' + kx = f(t) = \sum a_n \cos nt$$

Solve in 2 parts

Remember for

$$mx'' + cx' + kx = a_n \cos nt$$

$\hookrightarrow x = X_n \cos(nt + \theta_n)$

solved this many times

get X_n, θ_n

$\hookrightarrow$ functions of

ERF(a_n, n, m, c, k)

So to solve this

Put $\sum$ sign on both sides

$x_n = X_n \cos(nt + \theta)$

$$m x_n'' + c x_n' + k x_n = a_n \cos nt$$

$\hookrightarrow$ know $x_n = X_n \cos(nt - \theta_n)$

⑦

$x = \sum x_n$ solves $mx'' + cx' + kx = F$

basically using superposition principle an ∞ amt of times

Boundary Value Problem w/ Fourier way to Solve

$P(D)\, x = q(t)$

$x(0) = n_0$

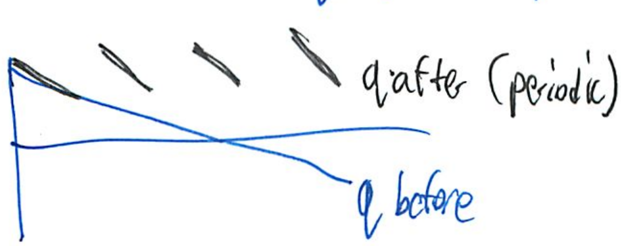
$x_1(t) = n_1$

Want to know which sols take this form

For the purpose of solving - only care about

$\underbrace{x(t)}_{\text{sol'n}},\ q(t)$ for $0 \le t \le A$

We instead solve $q(t) \to$ replace $\to q^{\text{periodic}}(t)$



(8)

Discuss when they are =

What ever $x(t)$ was call n_0

say $n_0 = \frac{1}{2}$

Starts and ends @ $\frac{1}{2}$

can do anything in Middle

So instead of looking for $x(t)$

look for $x^{\text{periodic}}(t)$

$$x(t) = \frac{a_0}{2} + \sum a_n \cos \frac{n\pi t}{1/2} + b_n \sin \frac{n\pi t}{1/2}$$

The thing about sin, cos

Sins all 0) at boundries

Cos all 1

Enforcing boundries means what

Sums a_n, b_n

(TA confused)

(9)

Diff example

$x(0) = 0$ ~~and~~ $x_1(\ell) = 0$

$q(x)$ even

$P(D)$ has only even powers

$D^4 + D^2 + 1$ ~~etc~~

know sol is just sum of sines

Sin is 0 at bondry

So know it will satisfy bondry condition

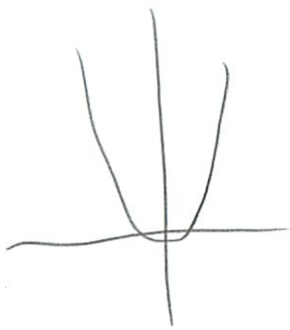
Can patch together general sols w/ sins and coses

- discuss next week

Even/odd fact sheet

11/3

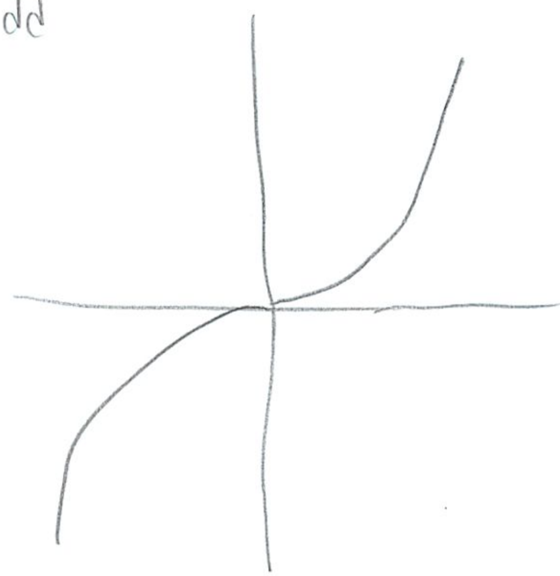
even



x^2 highest polynomial

$f(x) = f(-x)$

odd



x^3

$-f(x) = f(-x)$

even · even = even

odd · odd = even

even · odd = odd

$\int$ even = odd

$\int$ odd = even

$\int_{-x}^{x}$ even $= 2\int_{0}^{x}$ even

$\int_{-x}^{x}$ odd $= 0$

BVP

11/3

What is this exactly?

a formula?

type of problem?

name of section, but also brought up in applications section

Specify boundry conditions for solution

$t \in (0,1)$ w/ $x(0) = 0 = x(1)$

recitation

? So these are just the IVs?

WP: Diff eq w/ extra constraints → boundry conditions

Include IV problems

└ gives specific solutions

(So nothing special?)

18.03 FALL 2011 – Problem Set 6A

Due Friday 11/04/11, high noon in 2-106 with Problem Set 6B

To encourage you to keep up with homework as it appears in lecture, both Part I and Part II problems are listed with the accompanying lecture in which the material will be covered.

Part I (14 points)

Lecture 17. Fri. Oct. 21: Power Series Methods
READ: EP 3.2, 3.3, HW: EP 3.1: 21, EP 3.2: 1, 5, 11, 16, 19, 26

Lecture 18. Mon. Oct. 24: Review for Exam II
Sample Exams posted in the Handouts section of the course webpage.

Wed. Oct. 26: EXAM II - WALKER GYM (during class hour 3-4 pm.)

Lecture 19. Fri. Oct. 28: Introduction to Fourier Series
READ: EP 8.1, HW: To be assigned on the next problem set.

Part II (9 points)

0. (3 points) Write the names of all the people you consulted or with whom you collaborated and the resources you used, or say "none" or "no consultation". This includes visits outside recitation to your recitation instructor. If you don't know a name, you must nevertheless identify the person, as in, "tutor in Room 2-102," or "the student next to me in recitation." Optional: note which of these people or resources, if any, were particularly helpful to you.

1. (Friday, 6 pts) EP 3.2, Problem 32 (Hermite Polynomials)

18.03 FALL 2011 – Problem Set 6B

Due Friday 11/04/11, high noon in 2-106 with Problem Set 6A

To encourage you to keep up with homework as it appears in lecture, both Part I and Part II problems are listed with the accompanying lecture in which the material will be covered.

Part I (14 points)

Lecture 19. Fri. Oct. 28: Introduction to Fourier Series
READ: EP 8.1, HW: Notes 7A-1, 2, 3, 4

Lecture 20. Mon. Oct. 30: Convergence, Sines and Cosines
READ: EP 8.2. 8.3, HW: Notes 7B-1

Lecture 21. Wed. Nov. 1: Solving ODEs with Fourier Series
READ: EP 8.4, HW: Notes 7C-1, 2

Lecture 22. Fri. Oct. 24: Laplace Transform
READ: EP 4.1, Notes H HW: To be assigned on the next problem set.

Part II (22 points)

0. (3 points – already tallied in part a) Write the names of all the people you consulted or with whom you collaborated and the resources you used, or say "none" or "no consultation". This includes visits outside recitation to your recitation instructor. If you don't know a name, you must nevertheless identify the person, as in, "tutor in Room 2-102," or "the student next to me in recitation." Optional: note which of these people or resources, if any, were particularly helpful to you.

1. (Friday, 12 pts) This problem will use the Mathlet "Fourier Coefficients." When the applet opens you are presented with a series of sliders labeled b_n. By clicking the [Formula] radio button, you can see that they are coefficients of sines in a Fourier series made up entirely of sine functions. If you press the [Cosine] radio button you'll see a_n's instead, coefficients of cosines. Move one of the slider handles: a cosine or sine curve appears and changes amplitude. Release it at some value and move another one. The white curve shows the new sinusoid until you release the slider, and the yellow curve shows the sum of the two. By moving more sliders you can build up more complicated sums and more complicated functions.

Now select the Target [B]. Is it an even function or an odd function? Based on this, decide whether to approximate it using sines or cosines. Select one or the other appropriately (using [All terms]) and do the best you can by eyeballing the result to get the best approximation you can to the green target curve.

a) Does it appear that only even terms are needed? Only odd term? or both? The [Odd terms] and [Even terms] buttons allow you to choose just the even terms or the odd terms, and gives you more of the one you select. If it seems that just the even or odd terms will be useful, explain (in words) why.

b) Write down these values of the coefficients.

c) The target function [B] is the periodic function with period 2π which is given by

$$f(t) = \pi/4 \quad \text{for } -\pi/2 < t < \pi/2 \quad \textit{and} \quad f(t) = -\pi/4 \quad \text{for } \pi/2 < t < 3\pi/2$$

and $f(\pm\pi/2) = 0$. Compute the Fourier coefficients for this function, using the integral formulas for them, and compare with your answers from (a). Reset the sliders to the computed values and see if it looks like a better fit.

d) Now set the sliders to some random set of values. Still with target function [B] displayed, select the [Distance] button. A number appears at the upper right corner of the screen. This is the root mean square distance from the target function to the selected Fourier sum – a measure of how good a fit the Fourier sum gives. Instead of eyeballing the t as before, start from the bottom and successively adjust the sliders to minimize the distance. Write down the optimal values of the coefficients. Compare with the computed values.

2. (Monday, 5 pts) In class, we discussed the "real form" of the Fourier series:

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + \sum_{n=1}^{\infty} b_n \sin nt$$

where the coefficients a_n, b_n are real. But we can also use the complex form:

$$\sum_{n=-\infty}^{\infty} \alpha_n e^{int}$$

where the α_n are complex and satisfy $\alpha_{-n} = \overline{\alpha}_n$. Show that the real form can be uniquely expressed in terms of the complex form and vice versa. That is, give an expression for the α_n in terms of a_n and b_n and vice versa.

3. (Wednesday, 5 pts)

a) Using your answer in $7A - 2b$, find the cosine series of $f(t) = t$ on $0 < t < 1$.

b) Find the formal Fourier series solution on $0 < t < 1$ to the initial value problem

$$x'' + 10x = t \qquad x'(0) = 0, \quad x'(1) = 0.$$

(Use cosines that satisfy these boundary conditions.)

c) Write down the three most significant terms of the solution $x(t)$ expressed with numerical approximations to the amplitudes. Use this to describe the graph of $x(t)$.

Michael Plasmeier 18.03 P-Set 6 10/25

-18.5 $\frac{40.5}{59}$

Part A

Part 1

Lecture 17 Power Series Methods

EP 3.1 #21 First derive a recurrence relationship given C_n for $n \geq 2$ in terms of C_0, C_1

Then amplify given IC to find C_0, C_1. Next find C_n

Then identify elementy closed form sol.

$$y'' - 2y' + y = 0 \qquad \begin{aligned} y(0) &= 0 = C_0 \\ y'(0) &= 1 = C_1 \end{aligned}$$

1. Plug in generic stuff

$$\sum_{n=2}^{\infty} n(n-1) C_n x^{n-2} - 2 \sum_{n=1}^{\infty} n C_n x^{n-1} + \sum_{n=0}^{\infty} C_n x^n = 0$$

2. Shift index

$$\sum_{n=0}^{\infty} (n+2)(n+1) C_{n+2} x^n - 2(n+1) C_{n+1} x^n + C_n x^n = 0$$

3. Pull out x^n coefficents

$$(n+2)(n+1) C_{n+2} - 2(n+1) C_{n+1} + C_n$$

(2)

4. Set = to C_{n+2}

↓ what I forgot on exam

$$C_{n+2} = \frac{2(n+1)\,C_{n+1} - C_n}{(n+2)\,(n+1)}$$

C_{n+2} for $n \geq 0$

which is

C_n for $n \geq 2$

C_0, C_1 not pre determined. Can find w/ given IV

$y(0) = C_0 = 0$

$y'(0) = C_1 = 1$

So write

$$y(x) = 0 + 1 + \sum_{n=0}^{\infty} \frac{2(n+1)\,C_{n+1} - C_n}{(n+2)\,(n+1)}$$

No want terms written out

$n=0$ $\quad C_2 = \dfrac{2(1)C_1 - C_0}{(2)\,(1)} = \dfrac{2C_1 - C_0}{2}$

$n=1$ $\quad C_3 = \dfrac{2(2)\,C_2 - C_1}{(3)(2)} = \dfrac{4C_2 - C_1}{6}$

$n=2$ $\quad C_4 = \dfrac{2(3)\,C_3 - C_2}{(4)(3)} = \dfrac{6C_3 - C_2}{12}$

(3)

Convert terms back to c_1, c_0

$$c_3 = \frac{4\left(\frac{2c_1 - c_0}{2}\right) - c_1}{6}$$

$$= \frac{4\left(c_1 - \frac{c_0}{2}\right) - c_1}{6}$$

$$= \frac{4c_1 - 2c_0 - c_1}{6}$$

$$= \frac{3c_1 - 2c_0}{6}$$

$$c_4 = \frac{\overset{}{\underset{2}{\cancel{6}}}\left(\frac{3c_1 - 2c_0}{\cancel{6}}\right) - \left(\frac{2c_1 - c_0}{\cancel{2}}\right)}{12} \quad 2$$

$$= \frac{3c_1 - 2c_0 - c_1 \pm \frac{c_0}{2}}{12}$$

$$= \frac{2c_1 - \frac{3}{2}c_0}{12} \rightsquigarrow \frac{4c_1 - 3c_0}{24}$$

So now need pattern for this
- difficult
- preferably with c_1, c_0 on outside
- this one does not seem set up for that

(4)

OH Plug in values given C_1, C_0

$$C_0 = 0$$
$$C_1 = 1$$

$$C_2 = \frac{2(1)-0}{2} = 1$$

$$C_3 = \frac{4(1) - 1}{6} = \frac{1}{2}$$

$$C_4 = \frac{6\left(\frac{1}{2}\right) - 1}{12} = \frac{1}{6}$$

So now write values out

$$y(x) = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \dots$$
$$= 0 + 1x + 1x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4$$
$$= x + \frac{x^3}{2!} + \frac{x^4}{3!} + \frac{x^5}{4!} + \dots \quad 3!$$

know

$$= xe^x$$

✓ Correct

(5)

EP 3.2 #1 Find the general solution

$$(x^2-1)y'' + 4xy' + 2y = 0$$

$$P = \frac{4x}{(x^2-1)} \quad Q = \frac{2}{(x^2-1)}$$

$$y'' + P(x)y' + Q(x)y = 0$$

¿I totally don't get this chap

Where are these points singular?

¿So where is denom 0?

So where $x = \pm 1$

So radius of convergence is at least 1?

Then solve normally

$$(x^2-1)\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + 4x\sum_{n=1}^{\infty} n c_n x^{n-1} + 2\sum_{n=0}^{\infty} c_n x^n = 0$$

distribute

$$\sum_{n=2}^{\infty} n(n-1)c_n x^n - \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + 4\sum_{n=1}^{\infty} n c_n x^n$$

$$+ 2\sum_{n=0}^{\infty} c_n x^n = 0$$

6

Shift

$$\sum_{n=0}^{\infty} n(n-1) c_n x^n - \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n + 4 \sum_{n=0}^{\infty} n c_n x^n + 2 \sum_{n=0}^{\infty} c_n x^n = 0$$

Can replace w/ $n=0$ w/ no effects on sum

Pull out x_n terms

$$n(n-1) c_n - (n+2)(n+1) c_{n+2} + 4 n c_n + 2 c_n$$

$$c_{n+2} = \frac{-n(n-1) c_n - 4 n c_n - 2 c_n}{(n+2)(n+1)}$$

$n=0$ $\quad c_2 = \dfrac{-0(-1) c_0 - 4(0) c_n - 2 c_0}{(2)(1)} = \dfrac{-2c_0}{2} = -c_0$

$n=1$ $\quad c_3 = \dfrac{-1(0) c_1 - 4(1) c_1 - 2 c_1}{(3)(2)} = \dfrac{-6c_1}{6} = -c_1$

$n=2$ $\quad c_4 = \dfrac{-2(1) c_2 - 4(2) c_2 - 2 c_2}{(4)(3)} = \dfrac{-12 c_2}{12} = -c_2$

Easy!

$c_{n+2} = -c_n$ $\quad$ X $\quad$ $\boxed{c_n = c_{n+2}}$ $\quad$ $-.5$

7

So no IV given

$c_0 + c_1 x - c_0 x^2 - c_1 x^3 - c_2 x^4 - c_3 x^5$

Want in terms of c_0, c_1

$$y = -c_0 x^2 - c_1 x^3 + c_2 x^4 + c_3 x^5 - c_0 x^6 - c_1 x^7 + \dots$$

I think did something wrong w/ sign

-1

Book answer

$$y(x) = \sum_{n=0}^{\infty} c_n x^n + \sum_{n=0}^{\infty} c_{n+1} x^{n+1}$$

$$y(x) = c_0 \sum_{n=0}^{\infty} x^{2n} + c_1 \sum_{n=0}^{\infty} x^{2n+1}$$

$$= \frac{c_0 + c_1 x}{1 - x^2}$$

8

5. $(2-x^2)y'' - xy' + 16y = 0$

$$P = \frac{-x}{(2-x^2)} \qquad Q = \left(\frac{16}{2-x^2}\right)$$

Sigularity at $x = \sqrt{2}$ ✓ so $\rho = \sqrt{2}$

$$(2-x^2)\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - x\sum_{n=1}^{\infty} n c_n x^{n-1} + 16\sum_{n=0}^{\infty} c_n x^n = 0$$

$$2\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=0}^{\infty} n(n-1)c_n x^n - \sum_{n=0}^{\infty} n c_n x^n + 16$$

↑ change — ↑ more — ↑ more

$$2\sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n \qquad \text{"}$$

So $2(n+2)(n+1)c_{n+2} - n(n-1)c_n - nc_n + 16c_n$

$$c_{n+2} = \frac{n(n-1)c_n + nc_n - 16c_n}{2(n+2)(n+1)}$$

$n=0$ $\quad C_2 = \dfrac{0(-1)c_0 + 0c_n - 16c_0}{2(2)(+1)} = \dfrac{-16c_0}{4} = -4c_0$

$n=1$ $\quad C_3 = \dfrac{1(0)c_1 + 1c_n - 16c_1}{2(3)(2)} = \dfrac{-15c_1}{12}$

4

No IV either

where is your recurrence relation?

$y(x)\ c_0 + c_1 x - 4c_0 - \frac{15c_1}{12} + \dots$

-2

Book

$3(n+2)\ c_{n+2} = n c_n$

is this a reduction of what I have

$$c_{n+2} = \frac{n^2 c_n - n c_n + n c_n - 16c_n}{2(n+2)(n+1)}$$

$$c_{n+2} = \frac{n^2 c_n - 16c_n}{2(n+2)(n+1)}$$

$$2(n+2)(n+1)\ c_{n+2} = n^2 c_n - 16c_n$$

much more complex then given answer?
did I do the inital expansion wrong?
(but then would do whole series wrong

Book $y(x) = c_0 + c_1 \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)3^n}$

(10)

11. $5y'' - 2xy + 10y = 0$

$P = \frac{-2x}{5}$ $Q = \frac{10}{5}$

; so no singular points

$5\sum_{n=2}^{\infty} n(n-1)C_n x^{n-2} - 2\sum_{n=0}^{\infty} nC_n x^n + 10\sum_{n=0}^{\infty} C_n x^n = 0$

↓move ↑move

$5\sum_{n=0}^{\infty} (n+2)(n+1)C_{n+2} x^n \ldots$ " etc

$5(n+2)(n+1)C_{n+2} - 2nC_n + 10C_n$

$C_{n+2} = \frac{2nc_n - 10c_n}{5(n+2)(n+1)}$

$n=0$ $C_2 = \frac{2(0)C_0 - 10c_0}{5(2)(1)} = \frac{-10c_0}{10} = -C_0$

$n=1$ $C_3 = \frac{2(1)C_1 - 10c_1}{5(3)(2)} = \frac{-8c_1}{30}$

$n=2$ $C_4 = \frac{2(2)C_2 - 10c_2}{5(4)(3)} = \frac{-6c_2}{60} = \frac{-C_2}{10} = \frac{C_0}{10}$

11

$$y(x) = c_0 + x c_1 - c_0 - \frac{8c_1}{30} + \frac{c_0}{10}$$

; So I am never good at finding pattern
Which often we are not required to do in class
But I should be able to use what I had before

$$y(x) = c_0 + x c_1 + \sum \frac{2n c_n - 10 c_n}{5(n+2)(n+1)}$$

but should be in terms
of c_0, c_1
this is the generic part
I am bad at

Did not put out enough steps to do X -2

Book

$$5(n+1)(n+2)c_{n+2} = 2(n-5)c_n$$

$$y(x) = c_0\left(1 + \frac{2x^2}{3} + \frac{x^4}{27}\right) +$$

$$c_1\left(1 - x^2 + \frac{x^4}{10} + \frac{x^6}{750} + 15\sum_{n=4}^{\infty} \frac{(2n-7)!!\,2^n x^{2n}}{(2n)!\,5^n}\right)$$

? I would
have never
gotten this

(12)

16. Use power series to solve IV problem

$$(1+x^2)y'' + 2xy' - 2y = 0 \qquad y(0) = 0 = c_0 \quad y'(0) = 1 = c_1$$

$$(1+x^2)\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + 2x\sum_{n=1}^{\infty} n c_n x^{n-1} - 2\sum_{n=0}^{\infty} c_n x^n = 0$$

shift

$$\sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n + \sum_{\substack{n=0 \\ \text{move}}}^{\infty} n(n-1)c_n x^n + 2\sum_{\substack{n=0 \\ \text{move}}}^{\infty} n c_n x^n - 2$$

$$(n+2)(n+1)c_{n+2} + n(n-1)c_n + 2nc_n - 2c_n$$

$$c_{n+2} = \frac{-n(n-1)c_n - 2nc_n + 2c_n}{(n+2)(n+1)}$$

$$n=0 \quad c_2 = \frac{-0(-1)c_0 - 2(0)c_0 + 2c_0}{(2)(1)} = \frac{2c_0}{2} = c_0$$

$$n=1 \quad c_3 = \frac{-1(0)c_1 - 2(1)c_1 + 2c_1}{(3)(2)} = \frac{0c_1}{6} = 0$$

$$n=2 \quad c_4 = \frac{-2(1)c_2 - 2(2)c_2 + 2c_2}{(4)(3)} = \frac{-4c_2}{12} = \frac{c_2}{4}$$

13

$$y(x) = c_0 + x c_1 + c_0 + \frac{c_0}{4}$$

So this is eaiser to see

$$= c_0 + x c_1 + \sum_{n=0}^{\infty} \frac{c_0}{n!}$$

so not quite $n!$

$n!$ is $1 \cdot 2 = 2$

6

24

120

Oh is IV problem

$$y(x) = 0 + x \cdot 1 + 0 + \frac{0}{4}$$

$$= x$$ ✓ corect

(only good when IV - no #s)

✓

(14)

19. Solve IV. Make sub of $t = x - a$

Find a sol $\sum c_n t^n$ of transformed diff eq

$(2x - x^2)y'' - 6(x-1)y' - 4y = 0$ $\quad y(1) = 0$, $y'(1) = 1$

$(2x-x^2)\sum_{n=2}^{\infty} n(n-1)C_n x^{n-2} - 6(x-1)\sum_{n=1}^{\infty} n C_n x^{n-1} - 4\sum_{n=0}^{\infty} C_n x^n = 0$

Distribute

$\sum_{n=1}^{\infty} n(n-1)C_n x^{n-1} - \sum_{n=0}^{\infty} n(n-1)C_n x^n - 6\sum_{n=0}^{\infty} nC_n x^n - \sum_{n=1}^{\infty} nC_n x^{n-1} - 4\sum_{n=0}^{\infty} C_n x^n = 0$

move, move, move

shift

$\sum_{n=0}^{\infty}(n+1)(n)C_{n+1}x^n - \sum n(n-1)C_n x^n - 6\sum nC_n x^n - (n+1)C_{n+1}x^n - 4C_n x^n$

$(n+1)(n)C_{n+1} - n(n-1)C_n - 6nC_n + (n+1)C_{n+1} - 4C_n$

$$C_{n+1} = \frac{n(n-1)C_n + 6nC_n + 4C_n}{(n+1)^2(n)}$$

$n=0$ $\quad C_1 = \dfrac{0(-1)C_0 + 6(0)C_0 + 4C_0}{(0)(0+1)^2}$ = invalid

$n=1$ $\quad C_2 = \dfrac{1(0)C_1 + 6(1)C_1 + 4C_1}{1(2)^2} = \dfrac{10C_1}{4} = \dfrac{5C_1}{2}$ invalid

(15)

$n=2 \quad C_3 = \dfrac{2(1)C_2 + 6(2)C_2 + 4C_2}{2(3)^2} = \dfrac{12C_2}{18} = \dfrac{6C_2}{9} = \dfrac{2C_2}{3}$

$= \dfrac{2}{3}\left(\dfrac{5}{2}\right)C_1 = \dfrac{5C_1}{3}$

$$y(x) = C_0 + C_1 x + \frac{5C_1}{2}x^2 + \frac{5C_1}{3}x^3$$

$$= C_0 + C_1 x + \sum_{n=2}^{\infty} \frac{5C_1}{n} x^n$$

X (-1).

Book

$$y(x) = \frac{1}{3}\sum_{n=0}^{\infty} (2n+3)(x-1)^{2n+1}$$ Converges if $0 < x < 2$

So I didn't keep variable
x seperate
Must have done it very wrong
with x

(16)

26. Find a 3-term recurrence relationship for sols of the form $y = \sum c_n x^n$. Then find first 3 non-zero terms in each of 2 lin. ind sols.

$$(1+x^3)y'' + x^4 y = 0$$

$$(1+x^3)\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + x^4 \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=0}^{\infty} n(n-1)c_n x^{n+1} + \sum_{n=0}^{\infty} c_n x^{n+4}$$

n, make??

$$\sum_{n=0}^{\infty}(n+2)(n+1)c_n x^n + \sum_{n=1}^{\infty}(n-1)(n-2)c_{n-1}x^n + \sum_{n-4}^{\infty} c_{n-4}x^n$$

$$(n+2)(n+1)c_n + (n-1)(n-2)c_{n-1} + c_{n-4}$$

$$c_n = \frac{-(n-1)(n-2)c_{n-1} - c_{n-4}}{(n+2)(n+1)}$$

i did other way here

$$n=4 \quad c_4 = \frac{(4-1)(4-2)c_3 - c_0}{6 \cdot 5} = \frac{6c_3 - c_0}{30}$$

$$n=5 \quad c_5 = \frac{(5-1)(5-2)c_4 - c_1}{7 \cdot 6} = \frac{12c_4 - c_1}{42}$$

(17)

Need 3 terms

$$c_4 = \frac{(6-1)(6-2)c_5 - c_2}{8\cdot 7} = \frac{20c_5 - c_2}{56}$$

Need first 3 non recurring terms

c_0, c_1, c_2, c_3

But how get this?

How are there 2 sols ???

└ was on exam

we never learned

Notes one sol $c_1 = c_3 = 0$

other sol $c_0 = 0$ $c_1 = 1 \to c_2 = 0$ $c_3 = \frac{1}{6}$

So they had

$$c_4 = \frac{2c_2 + c_0}{12}$$

$$c_5 = \frac{3c_3 + c_1}{20}$$

} different $c_{n+2} = \frac{nc_n + c_{n-2}}{(n+2)(n+1)}$

But I did not have this here × (-1).

Book

$$y(x) = c_0\left(1 - \frac{x^6}{30} + \frac{x^9}{72} + \cdots\right) + c_1\left(x - \frac{x^7}{42} + \frac{x^{10}}{90} + \cdots\right) \cdots$$

(18)

Part 2

0. No one yet
OH Vivek

1. EP 3.2 # 32. Hermite Polynomial

Do they mean # 33 That is Hermite Polynomial...

OH Find Power Series sol
When are polynomials - See 0 at top
Evaluate
Show formula gives polynomial of degree n
Fact: Can rewrite

$$(e^{x^2})\frac{d^n}{dx^n}(e^{-x^2}) = \underbrace{\left(e^{x^2}\frac{d}{dx}e^{-x^2}\right)^n}_{\text{ones between } \frac{da}{dx} \text{ cancel}}$$

Hermite Polynomial of order α is

$$y'' - 2xy' + 2\alpha y = 0$$

a) Derive the 2 power series sols

$$y_1 = 1 + \sum_{m=1}^{\infty} (-1)^m \frac{2^m \alpha(\alpha-2)\cdots(\alpha-2m+2)}{(2m)!} x^{2m}$$

$$y_2 = x + \sum_{m=1}^{\infty} (-1)^m \frac{2^m(\alpha-1)(\alpha-3)\cdots(\alpha-2m+1)}{(2m+1)!} x^{2m+1}$$

(19)

Do it normally

$$\sum_{n=2}^{\infty} n(n-1) C_n x^{n-2} - 2x \sum_{n=1}^{\infty} n C_n x^{n-1} + 2\alpha \sum_{n=0}^{\infty} C_n x^n$$

Shift

& adjust

$$\sum_{n=0}^{\infty} (n+2)(n+1) C_{n+2} x^n - 2 \sum_{n=0}^{\infty} n C_n x^n + 2\alpha \sum C_n x^n$$

[illegible]

$$(n+2)(n+1) C_{n+2} - 2 n C_n + 2\alpha C_n$$

$$C_{n+2} = \frac{(2n - 2\alpha) C_n}{(n+2)(n+1)}$$ ✓

$n=0$ $$C_2 = \frac{-2\alpha C_0}{(2)(1)} = -\alpha C_0$$

$n=1$ $$C_3 = \frac{(2-2\alpha) C_1}{(3)(2)} = \frac{2C_1 - 2\alpha C_1}{6} = \frac{C_1 - \alpha C_1}{3}$$

$n=2$ $$C_4 = \frac{(4-2\alpha) C_2}{(4)(3)} = \frac{(4-2\alpha)(-\alpha C_0)}{12}$$

$$= \frac{-4\alpha C_0 + 2\alpha^2 C_0}{12}$$

$$= \frac{2\alpha C_0 + \alpha^2 C_0}{6}$$

(20)

Put it together

$$Y(x) = c_0 + c_1 x + -\alpha c_0 x^2 + \left(\frac{c_1 - \alpha c_1}{3}\right) x^3 + \ldots$$

But this is far cry from the formulas

? So some start c_0, some c_1
then do evens, odds

So I see how given sol works

I guess I would have that if expand out

✓

(2)

Show that y_1 is a polynomial if α is an even integer, whereas y_2 is a polynomial if α is odd

What do they mean by "is polynomial"?

I guess y_1 "engages" on even
y_2 odd

So when α odd in y_1 goes to 0?
say $\alpha = 1$

$$y_1 = 1 + \sum_{m=1}^{\infty} (-1)^m \frac{2^m(-1) \cdots (-2m+2)}{(2m)!} x^{2m}$$

; does not show anything?

But when $\alpha = 2$ above then one term goes to 0

So whole multiplication goes 0

So $y_1 = 1 \leftarrow$ polynomial?

Likewise $\alpha = 3$ then

$y_2 = x$

(2)

b) The <u>Hermite Polynomial</u> of degree n is denoted by $H_n(x)$. It is the nth degree polynomial sol of Hermite's eqn multiplied by a suitable constant so that the coefficent of x^n is 2^n

Show 1st 6 Hermite polynomials are

$H_0(x) = 1$

$H_1(x) = 2x$

$H_2(x) = 4x^2 - 2$

$H_3(x) = 8x^3 - 12x$

$H_4(x) = 16x^4 - 48x^2 + 12$

$H_5(x) = 32x^5 - 160x^3 + 120x$

General formula for Hermite Polynomials is

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}\left(e^{-x^2}\right)$$

<u>Verify</u> formula does give an n-th degree polynomial

So, n is the nth term

but want coeff x^n to be 2^n

But what is α?

Look for $H_1(x)$ $\sum_{m=1}^{1} \cdots = (-1)^1 \frac{2^1}{2} \alpha(\alpha-2) \cdots x^2$

23

$= -1 \, \alpha(\alpha-2)\cdots(\alpha-2m+2)\, x^2$

$= -x^2$

$\otimes$ Not what had before $\times \, (-1)$.

(24)

General Formula

Can use our hint

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} \left(e^{-x^2}\right)$$

$$= (-1)^n \left(e^{x^2} \frac{d}{dx} e^{-x^2}\right)^n$$

So can try values

n=1

$$= (-1)\ e^{x^2}\ 2e^{x^2}\ x$$

$$= -2x\ e^{2x^2}$$

that is not $2x$!

n=2

$$= (-1)^2 \left(e^{x^2}\ 2e^{x^2} x\right)^2$$

$$= 1x^2 e^{4x^2}$$

Should be $4x^2 - 2$

Nope does not work

(25)

Part B Part 1

Lecture 19 Introduction to Fourier Series

7A-1 Find the smallest period for each function

a) $\sin \frac{\pi t}{3}$

Period of $\sin nt$ is $\frac{2\pi}{n}$ ~~positive integer~~

Wolfram Alpha: period = 2

Must n be an integer. Here could say $n = \frac{\pi}{3}$

Then

period $= \frac{2\pi}{\frac{\pi}{3}} = 2\pi \cdot \frac{3}{\pi} = 6$ (J) OH

not what got w/ WA

How do we compute period again?

We could plot

OH plotted wrong

$\frac{\pi}{3} \sim 1$

So period $\sim 2\pi \sim 6.5$

(26)

b) $|\sin t|$

So normally this would be 2π
Which ↳away from origin
is what we care about, right?

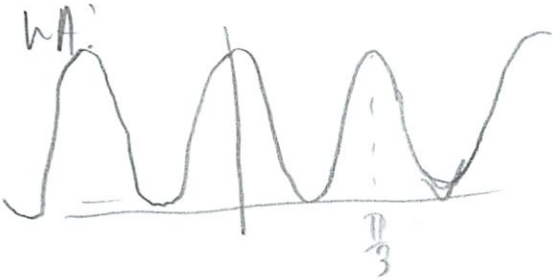
WA:

Actually it does not change near origin
Period is 2π ✗ π $-\frac{1}{2}$
Again method to find not in book

c) $\cos^2 3t$

$\cos(3t)\cos(3t)$
?what does function multiplied together do to period

WA:



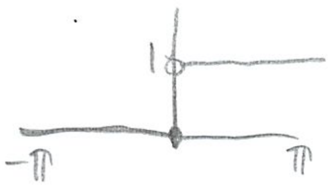
$\frac{\pi}{3}$ = period

(27)

#2. Find the Fourier series of $f(t)$ of period 2π which is given over $-\pi < t \leq \pi$

a) $f(t) = \begin{cases} 0 & -\pi < t \leq 0 \\ 1 & 0 < t \leq \pi \end{cases}$

So



Do piecewise

But a_0 is same

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)\, dt \quad \leftarrow \text{general form}$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^{0} 0 \, dt + \int_{0}^{\pi} 1 \, dt \right]$$

$$= \frac{1}{\pi} \left[C + t \Big|_0^{\pi} \right]$$

$$= \frac{1}{\pi} \left[\underset{\text{ignore?}}{C} + \pi \right]$$

$$= 1$$

(23)

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) \cos nt \, dt \quad \leftarrow \text{general form}$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0 \cos nt \, dt + \int_{0}^{\pi} 1 \cos nt \, dt\right]$$

$$= \frac{1}{\pi}\left[\underset{\text{ignore?}}{C} + \frac{\sin(nt)}{n}\Big|_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[C + \frac{\overset{\swarrow \text{ any integer mutiple}}{\sin(n\pi)}}{n} - \frac{\sin(n0)}{n}\right] \qquad \begin{array}{l} \sin(\pi)=0 \\ \sin(0)=0 \end{array}$$

$$= 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) \sin nt \, dt \quad \leftarrow \text{general form}$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0 \sin nt + \int_{0}^{\pi} 1 \sin nt\right]$$

$$= \frac{1}{\pi}\left[\underset{\text{ignore?}}{C} + \frac{-\cos(nt)}{n}\Big|_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[C + \frac{-1}{n} - \frac{1}{n}\right] \qquad \begin{array}{l} \cos(\pi) = -1 \\ \cos(0) = 1 \end{array}$$

$$= \frac{1}{\pi}\left[\frac{-2}{n}\right]$$

So how do we bring it together?

$$f(t) \sim \frac{1}{2} + \sum_{n=0}^{\infty} \frac{1}{\pi}\left[\frac{-2}{n}\right] \sin nt \quad \checkmark$$

(29)

So book had

$$b_n = -\frac{\cos nx}{n\pi}\Big|_0^{\pi}$$

But got $$-\frac{(-1)^n - (-1)}{n\pi}$$???

missing part b. (G)

$$= \frac{1-(-1)^n}{n\pi}$$

$$= \begin{cases} 0 & n \text{ even} \\ \frac{2}{n\pi} & n \text{ odd} \end{cases}$$

How did they get that? I did integral on WA as well

I see. I missed a relationship in the book

$$\sin(n\pi) = \sin(-n\pi) = 0$$

$$\cos(n\pi) = \cos(-n\pi) = (-1)^n$$

$$\sin(0 \cdot n) = \sin(0) = 0$$

$$\cos(0 \cdot n) = \cos(0) = 1$$

So redo b_n

$$b_n = \frac{1}{\pi}\left[\frac{-\cos(nx)}{n}\Big|_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{-(-1)^n}{n} - -\frac{1}{n}\right]$$

(30)

$$= \frac{1}{n\pi} - \frac{(-1)^n}{n\pi}$$

= So now even, odd

So say $n=1$ (odd)

$$= \frac{1}{n\pi} - \frac{(-1)^1}{n\pi}$$

$$= \frac{2}{n\pi} \quad n \text{ odd}$$

Say $n=2$ (even)

$$= \frac{1}{n\pi} - \frac{(1)}{n\pi}$$

$$= 0 \quad n \text{ even}$$

So verified book answer

Put all together

$$f(x) \sim \frac{1}{2} + \sum_{n \text{ odd}} \frac{2}{n\pi}$$

(31)

b $f(x) = \begin{cases} -x & -\pi < x < 0 \\ x & 0 \le x \le \pi \end{cases}$

$$a_0 = \frac{1}{\pi}\left[\int_{-\pi}^{\pi} f(x)\,dx\right] = \frac{1}{\pi}\left[\int_{-\pi}^{0} -x\,dx + \int_{0}^{\pi} x\,dx\right]$$

$$= \frac{1}{\pi}\left[-\frac{x^2}{2}\Big|_{-\pi}^{0} + \frac{x^2}{2}\Big|_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[-\frac{0^2}{2} - \frac{(-\pi)^2}{2} + \frac{\pi^2}{2} - \frac{0^2}{2}\right]$$

$$= \frac{1}{\pi}\left[0 + \frac{\pi^2}{2} + \frac{\pi^2}{2} - 0\right]$$

$$= \frac{1}{\pi} \cdot 2 \cdot \frac{\pi^2}{2} = \pi$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -x\cos nx\,dx + \int_{0}^{\pi} x\cos nx\,dx\right]$$

integrate by parts

$\int f\,dg = fg - \int g\,df$

$f = x \quad dg = \cos(nx)\,dx$

$df = dx \quad g = \frac{\sin(nx)}{n}$

$$= \frac{1}{n}\int \sin(nx)\,dx - \frac{x\sin(nx)}{n}$$

$u = nx$

$du = n\,dx$

$$= \frac{1}{n^2}\int \sin(u)\,du - \frac{x\sin(nx)}{n}$$

(32)

Integral $\sin(v) = -\cos(v)$

$$= \frac{-\cos(v)}{n^2} - \frac{x \sin(nx)}{n}$$

sub back

$$= \frac{-nx\sin(nx) + \cos(nx)}{n^2}$$

$$= \frac{1}{\pi}\left[\frac{-nx\sin(nx) + \cos(nx)}{n^2} + \frac{nx\sin(nx) + \cos(nx)}{n^2}\right]$$

$= 0$ (?)

→ Book does integration differently

$$\frac{2}{\pi}\int_0^{\pi} x \cos nx$$

$f\uparrow$ $dg\uparrow$

$f\downarrow$ $g\downarrow$ $\int dg$ $\downarrow df$

$$= \frac{2}{\pi}\left(x\frac{\sin nx}{n} - \int \frac{\sin nx}{n}\, dx\right)\Bigg|_0^{\pi}$$

$$= \frac{2}{\pi}\ 0 + \frac{\cos nx}{n^2}\Bigg|_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{(-1)^n - 1}{n^2}\right]$$

So what did I do wrong?

Oh I did not do definite parts

$$\frac{-nx\sin(nx) + \cos(nx)}{n^2}\Bigg|_{-\pi}^{0}$$

$$= 0 - \left[\frac{-\pi x\sin(-\pi x) + \cos(-\pi x)}{\pi^2}\right]$$

(33)

$$-\left(+\frac{\pi x \cdot 0}{n^2}+\frac{(-1)^n}{n^2}\right)$$

$$-\left(\frac{(-1)^n}{\pi^2}\right)$$

$$\left.\frac{nx\sin(nx)+\cos(nx)}{n^2}\right|_0^{\pi}$$

$$\frac{\pi x\sin(\pi x)+\cos(\pi x)}{\pi^2}-0$$

$$\pi x\cdot 0+\frac{(-1)^n}{\pi^2}$$

So $\frac{1}{\pi}\left[-\frac{(-1)^n}{\pi^2}+\frac{(-1)^n}{\pi^2}\right]$

Still $=0$

I think I converted it wrong since WA definite integral is different

Ok lets take

$a_n=\frac{2}{\pi}\left[\frac{(-1)^n-1}{n^2}\right]$ and more on

(34)

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\ \sin nt\ dt$$

$$= \frac{1}{\pi}\left(\int_{-\pi}^{0} -t \sin nt \quad + \int_{0}^{\pi} t \sin nt\right)$$

Use trick book does

$$= \frac{2}{\pi}\int_{0}^{\pi} |t| \sin nt\ dt$$

Now try integrating on my own

$f = t \qquad dg = \sin nt\ dt$

$df = dt \qquad g = -\dfrac{\cos(nt)}{n} \qquad \int f g' = fg - \int f' g\ dx$

$$= \frac{2}{\pi}\left(-\frac{t\cos(nt)}{n} - \int \frac{-\cos(nt)}{n}\,dt\right)\Bigg|_0^{\pi}$$

apply each part

$$= \frac{2}{\pi}\left(-\frac{\pi(-1)^n}{n} + \left[\frac{\sin(nt)}{n^2}\right]_0^{\pi}\right)$$

$$= \frac{2}{\pi}\left[-\frac{\pi(-1)^n}{n} + 0\right]$$

$$= \frac{-2\pi(-1)^n}{n}$$

×

-1

ⓧ No should be 0

- ; can we see immediatly since odd?

WA gives Non-zero value

35

3. Give another proof of orthogonality relationship

$$\int_{-\pi}^{\pi} \cos mx \cos nx \, dx = \begin{cases} 0 & m \neq n \\ \pi & m = n \end{cases}$$

? So this eq is given in book and lecture

Orthangonal: $u \cdot v = 0$

So lets actually integrate (using WA)

1. trig identity $\cos(\alpha)\cos(\beta) = \frac{1}{2}\left(\cos(\alpha - \beta) + \cos(\alpha + \beta)\right)$

$\alpha = mx$
$\beta = nx$

$$= \frac{1}{2}\int_{-\pi}^{\pi} \cos(mx - nx) + \cos(mx + nx) \, dx$$

2. Integrate each term

$$= \frac{1}{2}\int_{-\pi}^{\pi} \cos(mx - nx) + \frac{1}{2}\int_{-\pi}^{\pi} \cos(mx + nx) \, dx$$

3. Simplify

$$= \frac{1}{2}\int_{-\pi}^{\pi} \cos(x(m-n))dx + \frac{1}{2}\int_{-\pi}^{\pi} \cos(x(m+n)) \, dx$$

4. By parts $u = x(m-n)$
$du = m - n \, dx$

$$= \frac{1}{2(m-n)}\int_{-\pi}^{\pi} \cos(u) \, du + \frac{1}{2(m+n)}\int_{-\pi}^{\pi} \cos(u) \, du$$

(36)

5. Integrate

$$= \frac{1}{2(m-n)} \cdot \sin(u) \Big|_{-\pi}^{\pi} + \frac{1}{2(m+n)} \cdot \sin(u) \Big|_{-\pi}^{\pi}$$

6. Simplify

$$= \frac{-\sin(u)}{2(m-n)} + \frac{\sin(u)}{2(m+n)} \Big|_{-\pi}^{\pi}$$

7. Sub back

$$= \frac{\sin(x(\cancel{m+n}))}{2(m-n)} + \frac{\sin(x(m+n))}{2(m+n)} \Big|_{-\pi}^{\pi}$$

8. Get common factors

$$= \frac{m \sin(mx)\cos(nx) - n\cos(mx)\sin(nx)}{m^2 - n^2} \Big|_{-\pi}^{\pi}$$

9. Now definite terms

$$= \frac{2m\sin(\pi m)\cos(\pi n) - 2n\cos(\pi m)\sin(\pi n)}{m^2 - n^2}$$

Now test $m = n$

$$= \frac{2m\sin(\pi m)\cos(\pi m) - 2m\cos(\pi n)\sin(\pi m)}{m^2 - m^2}$$

$$= \frac{0}{0} \to \text{undefined}$$

x -2

hmmm.

37

4. Suppose that $f(x)$ has period P

Show that $\int_I f(x)\,dx$ has the same value over any interval I of length P

a) Show that for any a, we have

$$\int_P^{a+P} f(x)\,dx = \int_0^a f(x)\,dx$$

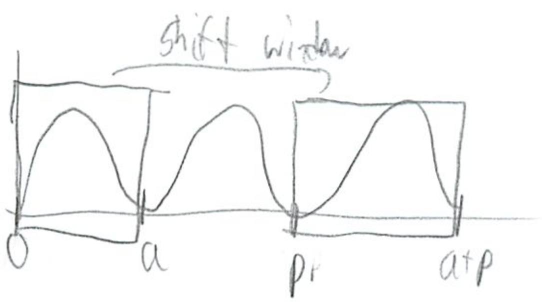
So this goes back to periodicity.

$$f(x+P) = f(x)$$

So lets say

$$\int_0^a f(x)\,dx = \int_0^a f(x+P)\,dx$$

? but how to integrate this ?



Or subtract P from interval

$$\int_P^{a+P} f(x) = \int f(a+P) - \int f(P)$$

$$= \int f(a) - \int f(0)$$

(38)

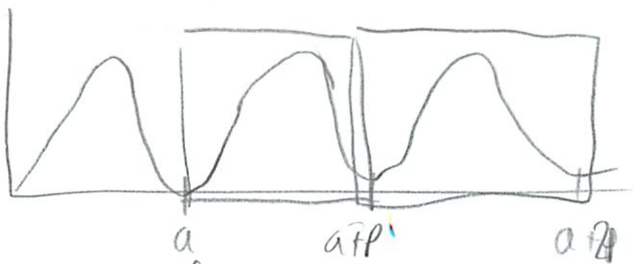
b. Show that $\int_a^{a+P} f(t)\,dt = \int_0^P f(t)\,dt$

Did that above - kinda

Here $0 \to P$, before $0 \to a$, $P \to a+P$

from $a \to a+P$

$$\int_a^{a+P} f(t)\,dt = \int_a^{a+P} f(t+P)\,dt$$



↑ a, P are same thing

$= \int f(a+P+P) - \int f(a+P)$

$= \int f(a+P)$

(39) 11/3

Lecture 20 Convergence, Sines, Cosines

7B-1 a) Find the Fourier cosine series of the function $1-x$ over the interval $0 < x < 1$ and then draw over the integral $[-2, 2]$ the graph of the function $f(x)$ which is the sum of this Fourier cosine series

So find Fourier

$$a_0 = \frac{1}{.5} \int_0^1 1-x \, dx$$

period = 1
$L = \frac{1}{2}$

$$= 2 \left(x - \frac{x^2}{2} \right) \Big|_0^1$$

$$= 2\left(1 - \frac{1}{2}\right)$$

$$= 1$$

$$a_n = \frac{1}{.5} \int_0^1 (1-x) \cos n\pi x \, dx$$

neither even nor odd

$$= \frac{1}{.5} \int_0^1 \cos n\pi x - x\cos n\pi x \, dx$$

(40)

$$= \frac{1}{.5}\left(\int_0^1 \cos n\pi x\, dx - \int_0^1 x \cos n\pi x)\, dx\right)$$

$f = x$ $\quad dg = \cos n\pi x$

$df = dx$ $\quad g = \frac{\sin(n\pi x)}{n\pi}$

Really need to learn to S manually

$$= \frac{1}{.5}\left(-\frac{\cos(n\pi x)}{\pi^2 n^2} - \frac{x\sin(n\pi x)}{\pi n} + \frac{\sin(n\pi x)}{\pi n}\right)\Bigg|_0^1$$

$$= 2\left(\frac{1-\cos(\pi n)}{\pi^2 n^2}\right)$$

✓

$$b_n = \frac{1}{.5}\int_0^1 (1-x)\sin \pi n x\, dx$$

$$= \frac{1}{.5}\left(\frac{\pi n(1-x)\cos(\pi n x) - \sin(\pi n x)}{\pi^2 n^2}\right)\Bigg|_0^1$$

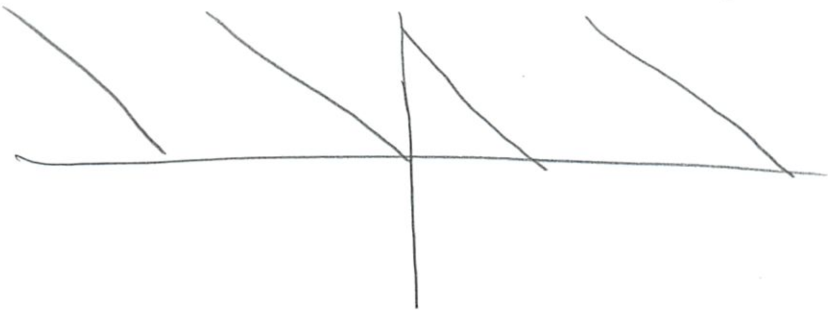
$$= 2\left(\frac{\pi n - \sin(\pi n)}{\pi^2 n^2}\right)$$

✓

$$Y(x) = \frac{1}{2} + \sum_{n=1}^{\infty} 2\left(\frac{1-\cos(\pi n)}{\pi^2 n^2}\right)\cos(n\pi) + 2\left(\frac{\pi n - \sin(\pi n)}{\pi^2 n^2}\right)\sin(n\pi)$$

(4)

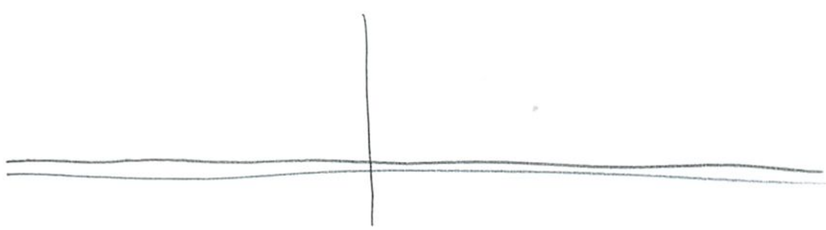
Now the new thing - boundry value problems



Now supposed to draw this $y(x)$ X (-1)

Plug into WA

Oh just = 0



b) What is Fourier Sine series?

Oh did above - we never learned about in class

just one part

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}$$

$$a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} \, dx$$

(12)

So a_n is slightly different

$\leftarrow$ is L $\frac{1}{2}$ here - this fn has no real period

oh here given interval $[0, L]$ so $L = 1$

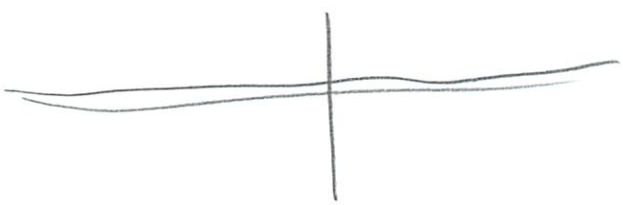
$$a_n = \frac{2}{1} \int_0^1 (1-x) \cos \frac{n\pi x}{1}$$

pretty much same

- is the exact same

$$y(x) = \frac{1}{2} + \sum_{n=1}^{\infty} 2 \left(\frac{1-\cos(\pi n)}{n^2 \pi^2} \right) \cos(n\pi)$$

Graph



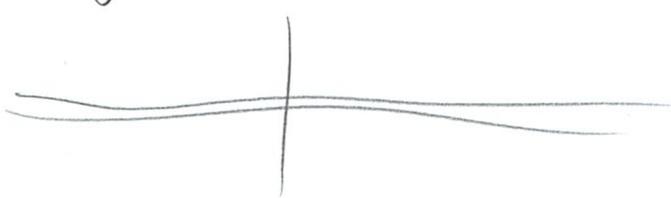
still $= 0$

X

b) Sine series

$$y(x) = \sum_{n=1}^{\infty} 2 \left(\frac{\pi n - \sin(\pi n)}{n^2 \pi^2} \right) \sin(n\pi)$$

$= 0$



x

(-1)

(42)b

Lecture 21 Solving ODEs w/ Fourier Series

7C-1 Applications to Resonate freqs

For each spring-mass system find weither pure resonance occurs w/o actually calculating

Book

So if non zero term $B_N \sin(N\pi t / L)$ for which $\frac{N\pi}{L} = \omega_0$ then resonance

$$x(t) = \frac{-B_N}{2m\omega_0} t \cos \omega_0 t$$

Lecture Notes

Lecture Notes differently

if $\sqrt{\frac{k}{m}}$ odd integer then resonance

ie $\sqrt{\frac{9}{1}} = 3$ = odd integer

a) $\underset{m}{2}x'' + \underset{k}{10}x = F(t)$ $F(t) = 1$ on $(0,1)$ "integral of"

$F(t)$ is odd and period $= 2$

$\sqrt{\frac{10}{2}} = \sqrt{5} = \times$ Not integer

No

$\frac{n\pi}{1}$

(43)

b) $x'' + 4\pi^2 x = F(t)$ $F(t) = 2t$ on $(0,1)$
m k

$F(t)$ is odd and of period 2

$\sqrt{\frac{4\pi^2}{1}} = \sqrt{4}\sqrt{\pi^2} = 2\pi$

is that a solution like $A\cos \underbrace{2\pi}\, t$

Well period is 2 - not 2π

Does $2\pi = \frac{n\pi}{?}$ for an integer n? ↳ so ~~no~~?

Yes $n=2$ yes ✓

c) $x'' + 9x = F(t)$ $F(t) = 1$ on $(0,\pi)$

$F(t)$ is odd, and of period 2π

$\sqrt{\frac{9}{1}} = 3$ but periods are of 2π

↳ in class example period was of 2π - yes ✓

$3 = \frac{n\pi}{2\pi}$ yeah $n=9$

Sols: typical term expansion $\cos \frac{n\pi}{L} t$ $\sin \frac{n\pi}{L} t$

pure resonance if $\frac{n\pi}{L} = \omega_0$

Ah makes more sense - fixed

(44)

7C-2 Find a periodic solution as a Fourier series to $x'' + 3x = F(t)$ where $F(t) = 2t$ on $(0, \pi)$ $F(t)$ odd and has period 2π

Steps 1. Find general sol $x_c = C_1 x_1 + C_2 x_2$ of homogeneous

$$(D^2 + 3)x = 0$$

$$x = i\sqrt{3}$$
$$-i\sqrt{3}$$

$$x_c = Ae^{i\sqrt{3}x} + Be^{-i\sqrt{3}x}$$

?wrong/unneeded?

Choose periodic extension of $f(t)$ so solve endpoint

So $p = 2\pi$

$$f(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left(A_n \cos \frac{n\pi t}{L} + B_n \sin \frac{n\pi t}{L} \right)$$

compute like before

then

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi t}{L} + b_n \sin \frac{\pi n t}{L} \right)$$

Now have ∞ly many terms to determine

Want to also solve endpoint conditions

(45)

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} 2t \cos n\pi t \, dt$$

$$= \frac{1}{2\pi}\left[\frac{2\pi n t \sin(\pi n t) + \cos(\pi n t)}{\cancel{n}^2 \pi^2}\right]_0^{2\pi}$$

$$= \frac{1}{\cancel{2}\pi} \frac{\cancel{2}(2\pi^2 n \sin(2\pi^2 n) + \cos(2\pi^2 n) - 1)}{\pi^2 n^2}$$

$$= \frac{2\pi^2 n \sin(2\pi^2 n) + \cos(2\pi^2 n) - 1}{\pi^3 n^2}$$

$$b_n = \frac{1}{2\pi}\int_0^{2\pi} 2t \sin n\pi t \, dt$$

$$= \frac{1}{2\pi}\left[\frac{2\sin(\pi n t) - 2\pi n t \cos(\pi n t)}{\pi^2 n^2}\right]_0^{2\pi}$$

$$= \frac{\pi\left(\frac{\sin(2\pi^2 n)}{n^2} - 2n\cos(2\pi n)\right)}{n^2}$$

So x is $\frac{a_0}{2} + \sum\left(a_n \cos\frac{n\pi t}{L} + b_n \sin\frac{\pi n t}{L}\right)$

Then double deriv for x''

(4b)

Looking at solutions - this is way wrong

only sine series?

$F(x) = 4\left(\sin x - \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x \right) \ldots$ $L = \pi$

$X' = \sum B_n \sin nt$

$X'' = \sum - B_n n^2 \sin nt$

$F(t) = \sum B_n (3 - n^2) \sin nt$

$\therefore B_n = (-1)^{n+1} \frac{4}{n} \frac{1}{(3-n^2)} = \frac{(-1)^n 4}{n(n^2-3)}$

(47)

Part B

0. On previous section
1. Mathlet: Fourier Coefficents.

   (Playing with it is cool!)

   Target B is a step wave - even fn

   └ so use sins?

   Oh fun to build

   a_0 moves ↕ leave alone

   a_{evens} is wrong - moving it off 0 is worse

   a_{odd} - move in opposite directions?

   $a_1 \to 2$

   $a_3 \to -1$

   $a_5 \to .5$ ← oh factors getting smaller

   Oh you can see more odd terms - if click that

   But my swap pattern did not work

(48)

a) So I have only odd terms ✓

b) $a_1 = .866$

$a_3 = -.240$

$a_5 = 2.5$

$a_7 = -.32$

$a_9 = -.14$ ✓

$a_{11} = 0$

c) Target is $f(t) = \begin{cases} \pi/4 & -\frac{\pi}{2} < t < \frac{\pi}{2} \\ -\frac{\pi}{4} & \frac{\pi}{2} < t < \frac{3\pi}{2} \\ 0 & \text{for } \pm\frac{\pi}{2} \end{cases}$

Compute w/ math

$a_0 \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt$

$= \frac{1}{\pi}\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\pi}{4}\,dt + \int_{\pi/2}^{3\pi/2} -\frac{\pi}{4}\,dt\right)$

$\frac{1}{\pi}\left(\frac{t\pi}{4}\Big|_{-\pi/2}^{\pi/2} + \frac{-t\pi}{4}\Big|_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\right)$

$\frac{1}{\pi}\left(\frac{\pi^2}{8} + \frac{\pi^2}{8} + -\frac{3\pi^2}{8} + \frac{\pi^2}{8}\right)$ ✓

$? \int_a^b = F(b) - F(a)$!

(49)

$$= \frac{1}{\pi}(0)$$

$$= 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt$$

$$= \frac{1}{\pi}\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\pi}{4}\cos nt\,dt + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} -\frac{\pi}{4}\cos nt\,dt\right)$$

move out constants

$$= \frac{1}{\pi}\left(\frac{\pi}{4}\left.\frac{\sin nt}{n}\right|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + -\frac{\pi}{4}\left.\frac{\sin nt}{n}\right|_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\right)$$

$$= \frac{1}{\pi}\left(\frac{\pi}{4}\left(\frac{\sin\frac{\pi}{2}n}{n} - \frac{\sin(-\frac{\pi}{2})n}{n}\right) - \frac{\pi}{4}\frac{\sin\frac{3\pi}{2}n}{n} - \frac{\sin\frac{\pi}{2}n}{n}\right)$$

not 0! its

$$-\frac{\sin\frac{\pi}{2}n}{n}$$

n	$\sin\frac{n\pi}{2}$
1	1
2	0
3	-1
4	0
5	1
6	0
7	-1

0 n even

$(-1)^{\lfloor N/2 \rfloor}$ odd

n	$\sin\frac{3\pi n}{2}$
1	-1
2	0
3	1
4	0
5	-1
6	0
7	+1

reverse

0 n even

$(-1)^{\lceil N/2 \rceil}$ odd

✓

(50)

$$= \frac{1}{4} \cdot 2 \frac{\sin \frac{\pi}{2} n}{n} - \frac{1}{4} \frac{\sin \frac{\pi}{2} n}{n} - \frac{1}{4} \frac{\sin \frac{3\pi}{2} n}{n}$$

$$= \frac{\sin \frac{\pi}{2} n}{4n} - \frac{\sin \frac{3\pi}{2} n}{4n}$$

So 0 n even
but for n odd

n	$\frac{\sin \frac{\pi}{2} n}{4n} - \frac{\sin \frac{3\pi}{2} n}{4n}$
1	$\frac{1}{4n} - -\frac{1}{4n} = \frac{1}{2n}$
3	$\frac{-1}{4n} - \frac{1}{4n} = -\frac{1}{2n}$
5	$\frac{1}{4n} - -\frac{1}{4n} = \frac{1}{2n}$

So $\frac{1}{2n}$ every 4 starting at 1

$$b_n = \frac{1}{\pi} \int f(x) \cos nx \, dx$$

$$= \frac{1}{\pi} \left(\int_{-\pi/2}^{\pi/2} \frac{\pi}{4} \sin nx \, dx + \int_{\pi/2}^{3\pi/2} -\frac{\pi}{4} \sin nx \, dx \right)$$

$$= \frac{1}{\pi} \left(\frac{\pi}{4} \frac{\cos nx}{n} \Big|_{-\pi/2}^{\pi/2} + -\frac{\pi}{4} \cdot \frac{\cos nx}{n} \Big|_{\pi/2}^{3\pi/2} \right)$$

$$= \frac{1}{\pi} \left(\frac{\pi}{4} \left(\frac{\cos \frac{\pi}{2} n}{n} - \frac{\cos -\frac{\pi}{2} n}{n} \right) + \frac{\pi}{4} \left(\frac{\cos \frac{3\pi}{2} n}{n} - \frac{\cos \frac{\pi}{2} n}{n} \right) \right)$$

n	$\cos \frac{\pi}{2} n$
1	0
2	-1
3	0
4	1
5	0
6	-1

0 n odd
$(-1)^{\lceil n/2 \rceil}$

Same
↳ here subtracts to be 0

n	$\cos \frac{3\pi}{2} n$
1	0
2	-1
3	0
4	1
5	0
6	-1

← same

↑ so subtracts out

$= 0$

(51)

So

$$f(t) \sim \sum_{\substack{n=1 \\ \text{odd}}}^{\infty} (-1)^{\lfloor N/2 \rfloor} \frac{1}{2N}$$

Now try that out

$a_1 = \frac{1}{2} = .5$

$a_3 = -\frac{1}{6} \sim -.1666$

$a_5 = \frac{1}{10} = .1$

$a_7 = -\frac{1}{14} \sim -.0714$

$a_9 = \frac{1}{18} \approx .0555$

$a_{11} = -\frac{1}{22} \approx -.0454$

X off by 2x.

see part b).

(-2)

52

d) Use distance function

Eyeballed value: .2966

Computed value: .4132

; So do we want more or less?

It looks worse

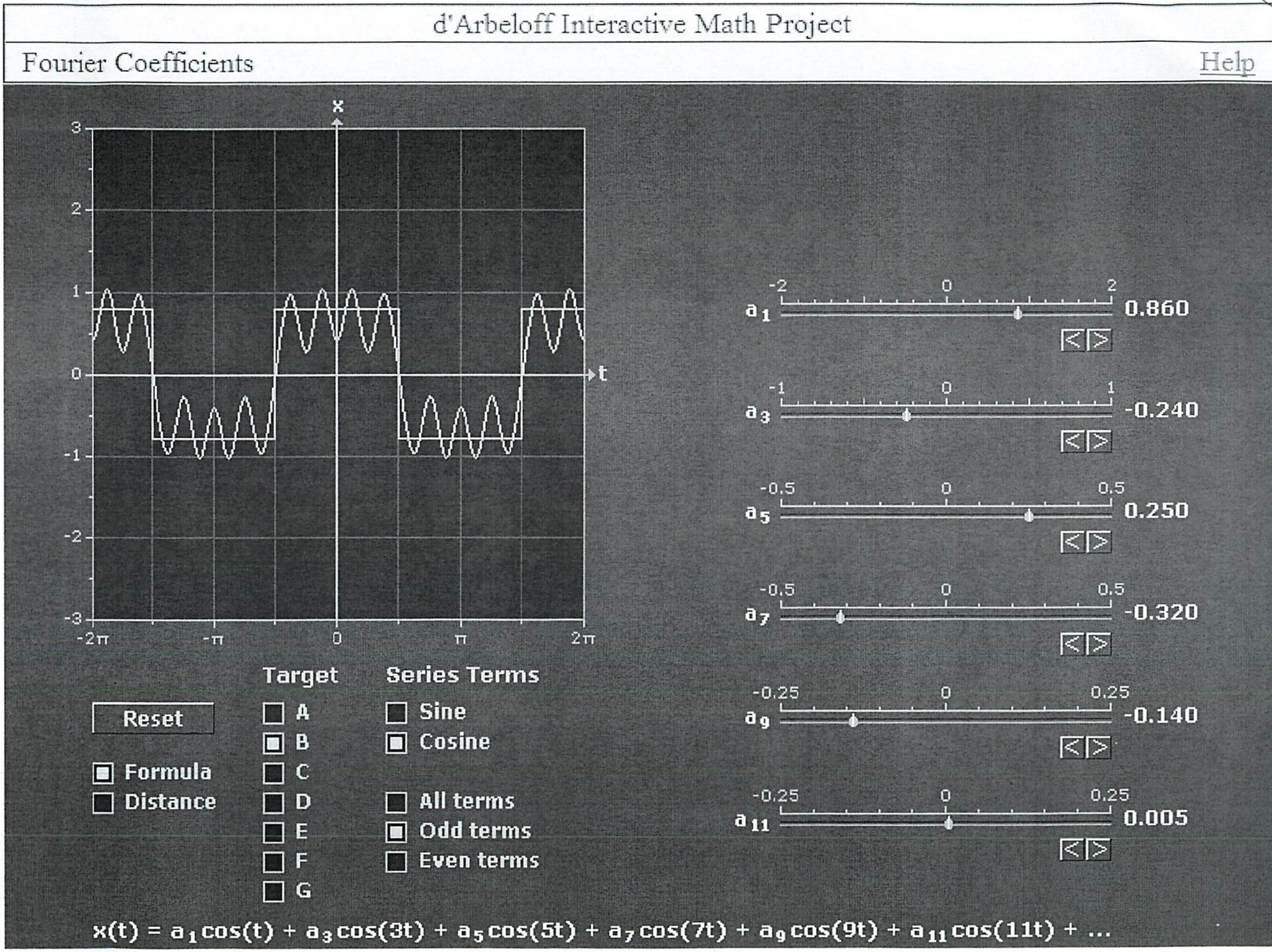
- just need gain?
- its not a_0 or a_1

moves up or down

adds large peak

53

b)

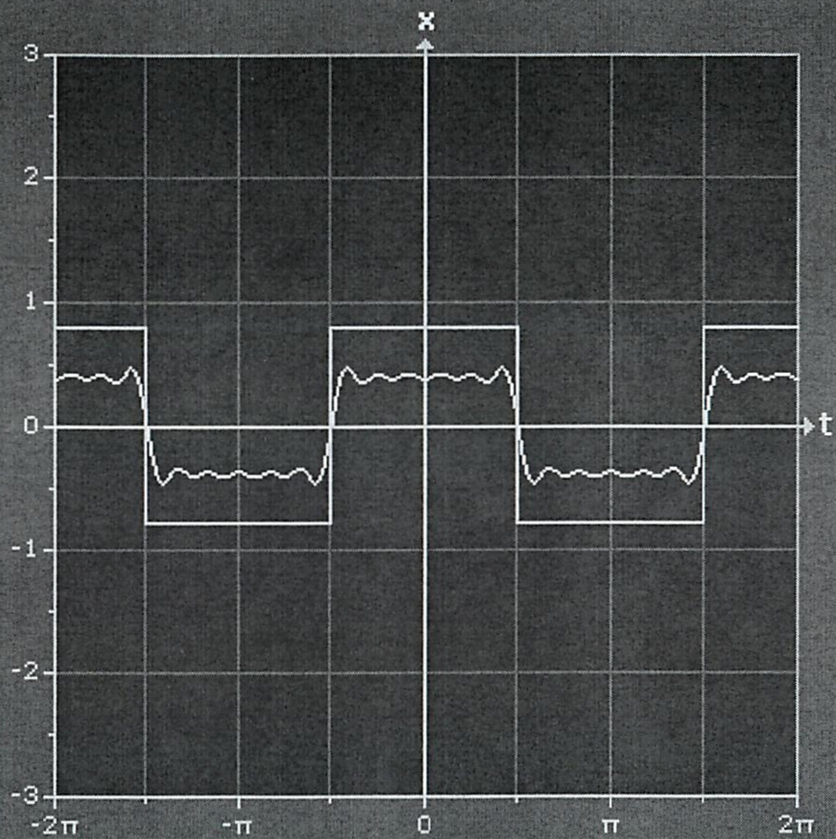


d'Arbeloff Interactive Math Project
Fourier Coefficients
Help
x
t
3
2
1
0
-1
-2
-3
-2π
-π
0
π
2π
Target
A
B
C
D
E
F
G
Series Terms
Sine
Cosine
All terms
Odd terms
Even terms
Reset
Formula
Distance
a1 -2 0 2 0.860
a3 -1 0 1 -0.240
a5 -0.5 0 0.5 0.250
a7 -0.5 0 0.5 -0.320
a9 -0.25 0 0.25 -0.140
a11 -0.25 0 0.25 0.005
x(t) = a1 cos(t) + a3 cos(3t) + a5 cos(5t) + a7 cos(7t) + a9 cos(9t) + a11 cos(11t) + ...

54

d'Arbeloff Interactive Math Project

Fourier Coefficients | Help



0 . 4 1 3 2 0

a_1	-2 … 0 … 2	0.500
a_3	-1 … 0 … 1	-0.160
a_5	-0.5 … 0 … 0.5	0.100
a_7	-0.5 … 0 … 0.5	-0.075
a_9	-0.25 … 0 … 0.25	0.055
a_{11}	-0.25 … 0 … 0.25	-0.045

Reset

Formula
Distance

Target
A
B
C
D
E
F
G

Series Terms
Sine
Cosine

All terms
Odd terms
Even terms

$x(t) = a_1\cos(t) + a_3\cos(3t) + a_5\cos(5t) + a_7\cos(7t) + a_9\cos(9t) + a_{11}\cos(11t) + \dots$

Used calculated values

Scale all up

55

2. In class we discussed the "real form" Fourier Series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + \sum_{n=1}^{\infty} b_n \sin nt$$

where the coefficents a_n, b_n are real

We can also use the complex form

$$\sum_{n=-\infty}^{\infty} \alpha_n e^{int}$$

α_n are complex

$\alpha_{-n} = \overline{\alpha_n}$ ← what does the bar mean.

Show that the real form can be uniquely expressed in terms of the complex form and vice versa. Hint α_n in terms of a_n, b_n

So $\alpha_n = a_n + b_n i$

$$e^{int} = \cos nt + i \sin nt$$

So

$$\alpha_n e^{int} = a_n \cos nt + i\, b_n \sin nt$$

OK

?, seems too simple

(56)

3. Using answer in 7A-2b find cosine series of $f(x)$ on x for $0 < x < 1$

Cosine Series
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}$$

$$\cancel{= \frac{\pi}{2} + \sum_{n\,odd} \frac{-4}{\pi n^2} \cos \frac{n\pi x}{\pi}}$$

No need to do for $0 < x < 1$

$$a_n = \frac{2}{\pi}\left[0 + \left[\frac{\cos nx}{n^2} \right|_0^1 \right.$$

$$= \frac{2}{\pi} \frac{\cos n}{n^2} - \frac{2}{\pi} \frac{\cos 0}{n^2}$$

$$= \frac{2}{\pi} \frac{\cos n}{n^2} \cancel{- \frac{2}{\pi n^2}}$$

$$a_0 = \frac{2}{\pi} \frac{\cos 0}{0^2} - \frac{2}{\pi 0^2}$$

divide by 0 error

Prob should do long way

$$= \frac{1}{\pi} \int_0^1 |x| - \frac{1}{\pi} \frac{x^2}{2} \Big|_0^1 = \frac{1}{\pi} \frac{1}{2}$$

(57)

$$f(x) = \frac{1}{\pi 4} + \sum_{n=1}^{\infty} \frac{2}{\pi} \frac{\cos n}{n^2} - \frac{2}{\pi \pi^2}$$

b) Find the formal Fourier series sol on $0 < t < 1$ to the inital Value problem

$$X'' + 10x = t \qquad x'(0) = 0$$
$$x'(1) = 0$$

(how is this connected to the above??)

t = calced in book 8.3 (16)

$$= \frac{2L}{\pi}\left(\sin \frac{\pi t}{L} - \frac{1}{2}\sin\frac{2\pi t}{L} + \frac{1}{3}\sin\frac{3\pi t}{L} - \dots\right)$$
for $0 < t < L$

So anticipate a sine sol

$$x(t) = \sum_{n=1}^{\infty} b_n \sin n\pi t$$

Sub in series

~~don't see it~~

~~$b_n =$? how do we calculate this?~~

(58) Take x', x''

$$x' = \sum b_n n\pi \cos n\pi t$$

$$x'' = \sum b_n n^2\pi^2 - \sin(n\pi t)$$

Plug in

$$-\sum b_n n^2\pi^2 \sin(n\pi t) + 10 \sum b_n \sin n\pi t = t$$

Coeffs on LHS + RHS must match
(wish we did > 1 example)

$$b_n n^2\pi^2 + b_n = -\frac{2L}{\pi}\left(\frac{1}{n}\right)$$

$$b_n(n^2\pi^2 + 1) = -\frac{2L}{n\pi}$$

$$b_n = \frac{-2L}{(n\pi)(n^2\pi^2+1)}$$

Similar to before

Found particular sol

$$x_p(t) = \sum_{n=1}^{\infty} \frac{-2L}{(n\pi)(n^2\pi^2+1)} \sin n\pi t$$

$L=1$

(59)

We didn't use the IVs

What do we do here again?

8.3 Ex 2 sucks

Want something that satisfies $X'(0)=0$

$X'(1)=0$

$\underline{x'(0)=0}$

$$X'_p(t) = \sum_{n=1}^{\infty} \frac{-2L\cancel{n\pi}}{\cancel{n\pi}(n^2\pi^2+1)} \cos n\pi t$$

$$0 = \sum_{n=1}^{\infty} \frac{-2L}{(n^2\pi^2+1)} \cos 0$$

$$= \sum_{n=1}^{\infty} \frac{-2L}{n^2\pi^2+1}$$

? what are we trying to set?? n

- no examples of this

$\underline{x'(1)=0}$

$$0 = \sum_{n=1}^{\infty} \frac{-2L}{n^2\pi^2+1} \cos n\pi$$

Oh L is 1

? Or just check IV to see right

(60)

c) Write down the 3 most significant terms of sol $x(t)$
expressed w/ numerical approx to amplitudes.
Describe graph $x(t)$

$$x_p(t) = \frac{-2}{\pi^3+1} \sin \pi t + \frac{-2}{(2\pi)(4\pi^2+1)} \sin 2\pi t$$

$$- \frac{2}{(3\pi)(9\pi^2+1)} \sin 3\pi t$$

X See solution. (-2) 3/5

; numerical approx to amplitudes
like $\pi = 3.14$;
$2\pi = 6.28$
$3\pi = 9.42$

Solutions to Pset 6 (part II)

6a Hermite equation

$$y'' - 2xy' + 2\alpha y = 0$$

(a) Let $y = \sum_{n=0}^{\infty} a_n x^n$. Then

$$\sum n(n-1) a_n x^{n-2} - 2x \sum n a_n x^{n-1} + 2\alpha \sum a_n x^n = 0$$

$$\rightsquigarrow \sum (n+1)(n+2) a_{n+2} x^n - 2 \sum n a_n x^n + 2\alpha \sum a_n x^n = 0.$$

$$(n+1)(n+2) a_{n+2} = (2n - 2\alpha) a_n.$$

$$\boxed{a_{n+2} = \frac{2(n-\alpha)}{(n+1)(n+2)} a_n.}$$

Therefore

$$y = a_0 + a_1 x + \frac{2(-\alpha)}{1\cdot 2} a_0 x^2 + \frac{2(1-\alpha)}{2\cdot 3} a_1 x^3$$

$$+ \frac{2(-\alpha)}{1\cdot 2} \cdot \frac{2(2-\alpha)}{3\cdot 4} a_0 x^4 + \frac{2(1-\alpha)}{2\cdot 3} \cdot \frac{2(3-\alpha)}{4\cdot 5} a_1 x^5 + \cdots$$

$$= a_0 \left[1 + \sum_{m=1}^{\infty} (-1)^m \frac{2^m \alpha(\alpha-2)\cdots(\alpha-2m+2)}{(2m)!} x^{2m} \right]$$

$$+ a_1 \left[x + \sum_{m=1}^{\infty} (-1)^m \frac{2^m (\alpha-1)\cdots(\alpha-2m+1)}{(2m+1)!} x^{2m+1} \right]$$

If α is an even non negative integer then One of α, $\alpha-2$, $\alpha-4$, ... is zero. All coefficients a_n beyond some finite n, are zero.

This $\Rightarrow$ $y_1(x)$ is a polynomial in x.

The same goes for $y_2(x)$ if α is an odd positive integer.

(b) By formula for $y_1(x)$:

$\alpha = 0 \rightsquigarrow y_1 = 1$ $\boxed{= H_0(x)}$

$$\alpha = 2 \rightsquigarrow 1 + (-1)\frac{2(2)}{2}x^2 = 1 - 2x^2$$

mult. by -2 to get $H_2(x)$.

$$\alpha = 4 \rightsquigarrow 1 + (-1)\frac{2(4)}{2}x^2 + (-1)^2\frac{4(4)(2)}{24}x^4 = 1 - 4x^2 + \frac{4}{3}x^4$$

mult. by 12 to get $H_4(x)$.

Now $y_2(x)$:

$\alpha = 1 \rightarrow y_1 = x$ mult. by 2 to get $H_1(x)$.

$$\alpha = 3 \rightsquigarrow x + (-1)\frac{2(2)}{6}x^3 = x - \frac{2}{3}x^3$$

mult. by -12 to get $H_3(x)$.

$$\alpha = 5 \rightsquigarrow x + (-1)\frac{2(4)}{6}x^3 + (-1)^2\frac{4(4)(2)}{120}x^5$$

$$= x - \frac{4}{3}x^3 + \frac{4}{15}x^5$$

mult. by 120 to get $H_5(x)$.

$(-1)^n e^{x^2} \dfrac{d^n}{dx^n}(e^{-x^2})$ equals an n^{th} degree polynomial?

Well, $\dfrac{d}{dx}(e^{-x^2}) = -2xe^{-x^2}$ & in general

$\dfrac{d}{dx}(x^k e^{-x^2}) = kx^{k-1}e^{-x^2} - 2x^{k+1}e^{-x^2}$. Therefore if $p(x)$ is a k^{th} degree polynomial then $\dfrac{d}{dx}(p(x)e^{-x^2})$ is $q(x)e^{-x^2}$ for some $(k+1)^{th}$ degree polynomial q.

Hence $\dfrac{d^n}{dx^n}(e^{-x^2}) = q(x)e^{-x^2}$ for some n^{th} degree polynomial $q(x)$ & so

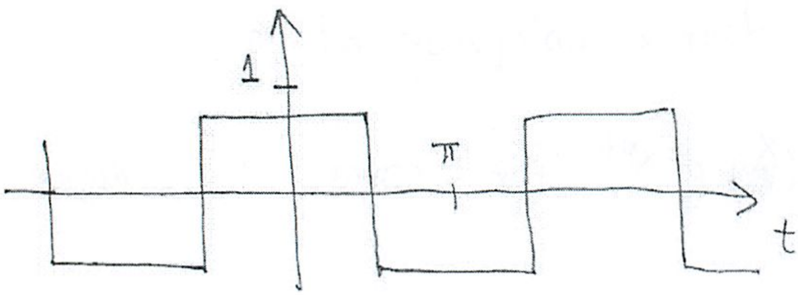
$$(-1)^n e^{x^2} \frac{d^n}{dx^n}(e^{-x^2}) = (-1)^n q(x)$$ is

an n^{th} degree polynomial.

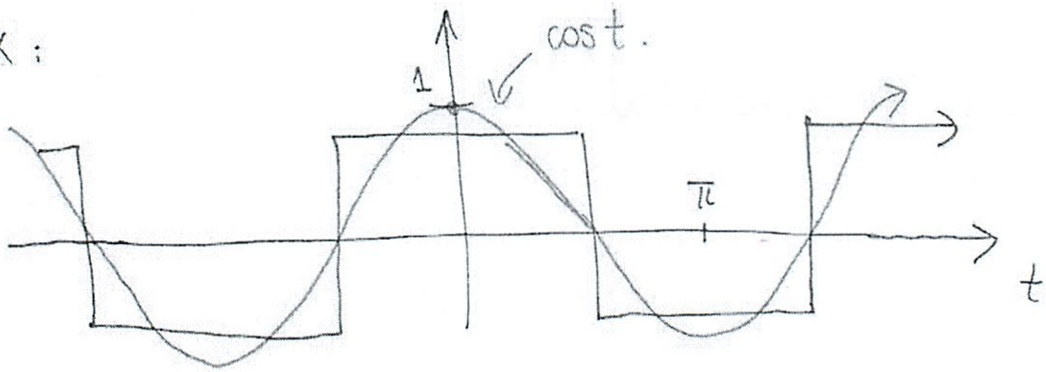
Solutions to 6b Part II

(1) Target B is even so we should approximate with cosines.

(a) Certainly no a_0 term is needed b/c the function has t-average $= 0$

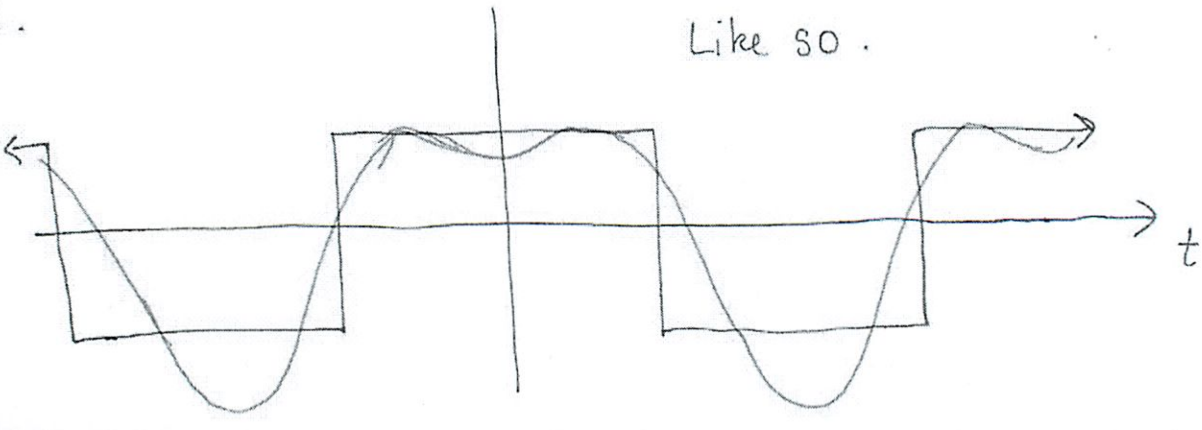


A pure cosine wave is the 1st approx:



Adding an $a_2 \cos 2t$ term spoils the symmetry of the approx.

Like so.



This is not a proof that $a_2 = 0$, but it does suggest that probably $a_2 = 0$.
Similarly with $a_4, a_6 \ldots$. So let's just use Odd Terms.

So

$$\text{Target B} = a_1 \cos t + a_3 \cos 3t + a_5 \cos 5t + \ldots$$

After some eyeballing I got

(b) $a_1 = 1.000$,

$a_3 = -0.330$,

$a_5 = 0.180$,

$a_7 = -0.100$,

$a_9 = 0.06$,

& $a_{11} = -0.047$.

(c) $f(t)$ is odd, so we need only cosines, Period $= 2\pi$.

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt \quad , \quad \text{where } a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt \, dt.$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt = \frac{1}{\pi}\int_{-\pi/2}^{3\pi/2} f(t)\,dt = \frac{1}{\pi}\left(\int_{-\pi/2}^{\pi/2} f(t)\,dt + \int_{\pi/2}^{3\pi/2} f(t)\,dt\right)$$

$$= \frac{1}{\pi}\left(\int_{-\pi/2}^{\pi/2} \frac{\pi}{4}\,dt + \int_{\pi/2}^{3\pi/2} \left(-\frac{\pi}{4}\right) dt\right) = \frac{1}{4}\left(\left(\frac{\pi}{2} - \frac{-\pi}{2}\right) - \left(\frac{3\pi}{2} - \frac{\pi}{2}\right)\right)$$

$$= 0.$$

Similarly

$\underline{n \geq 1}$

$$a_n = \frac{1}{\pi}\left(\int_{-\pi/2}^{\pi/2} f(t)\cos nt\,dt + \int_{\pi/2}^{3\pi/2} f(t)\cos nt\,dt\right)$$

$$= \frac{1}{\pi}\left(\int_{-\pi/2}^{\pi/2}\left(\frac{\pi}{4}\right)\cos nt\,dt - \int_{\pi/2}^{3\pi/2}\left(\frac{\pi}{4}\right)\cos nt\,dt\right).$$

$\int \cos nt\,dt = \frac{1}{n}\sin nt + C$

$$= \frac{1}{4}\left(\left[\frac{1}{n}\sin nt\right]_{-\pi/2}^{\pi/2} - \left[\frac{1}{n}\sin nt\right]_{\pi/2}^{3\pi/2}\right) \quad (*)$$

If n is even, $\sin nt$ is 0 @ $t = \pm\frac{\pi}{2}$ & $\frac{3\pi}{2}$ since $\sin(n\pi) = 0$

So only odd n terms appear.

	$n=1$	$n=3$	$n=5$	$n=7$	$n=9$
$\sin(n\frac{\pi}{2})$	1	-1	1	-1	1
$\sin(n\cdot\frac{-\pi}{2})$	-1	1	-1	1	-1
$\sin(n\frac{3\pi}{2})$	-1	1	-1	1	-1
	$\boxed{m=1}$	$\boxed{m=2}$	$\boxed{m=3}$	$\boxed{m=4}$	$\boxed{m=5}$

OK, so the pattern is $\sin\left((2m-1)\frac{\pi}{2}\right) = (-1)^{m+1}$

$$\sin\left((2m-1)\frac{-\pi}{2}\right) = (-1)^m$$

$$\sin\left((2m-1)\frac{3\pi}{2}\right) = (-1)^m.$$

We've changed to m where $n = 2m-1$.

In terms of m, $(*)$ now becomes

$$\frac{1}{4}\left(\frac{1}{2m-1}\left((-1)^{m+1}-(-1)^{m}\right)-\frac{1}{2m-1}\left((-1)^{m}-(-1)^{m+1}\right)\right)$$

$$=\frac{(-1)^m}{4(2m-1)}\left(\left((-1)-1\right)-\left(1-(-1)\right)\right)$$

$$=\frac{(-1)^m}{4(2m-1)}\cdot(-4)=\frac{(-1)^{m+1}}{2m-1}.$$

The first few terms are :

$m=1$: $a_1 = 1 \sim 1.000$

$m=2$: $a_3 = -1/3 \sim -0.333$

$m=3$: $a_5 = 1/5 \sim 0.200$

$m=4$: $a_7 = -1/7 \sim -0.143$

$m=5$: $a_9 = 1/9 \sim 0.111$

$m=6$: $a_{11} = -1/11 \sim 0.091$

Not too far from the experimental values.

---

d) That was fun. I got

$a_1 = 1.000$

$a_3 = -0.330$

$a_5 = 0.200$

$a_7 = -0.145$

$a_9 = 0.110$

$a_{11} = -0.090$

This is very close to the theoretical values.

(2) $$\sum_{n=-\infty}^{\infty} \alpha_n e^{int} = \sum_{n=-\infty}^{\infty} \alpha_n (\cos nt + i \sin nt)$$

$$= \sum_{n=1}^{\infty} \left[\alpha_n (\cos nt + i \sin nt) + \alpha_{-n} (\cos nt - i \sin nt) \right]$$
$$+ \alpha_0$$

using
$\cos(-x) = \cos x$
$\sin(-x) = \sin x$.

$$= \boxed{\alpha_0} + \sum_{n=1}^{\infty} \left[\boxed{(\alpha_n + \alpha_{-n})} \cos nt + \boxed{i(\alpha_n - \alpha_{-n})} \sin nt \right]$$

This $$= \boxed{\frac{a_0}{2}} + \sum_{n=1}^{\infty} \left[\boxed{a_n} \cos nt + \boxed{b_n} \sin nt \right]$$

where
$$\boxed{\begin{aligned} \alpha_0 &= \frac{a_0}{2} \\ a_n &= \alpha_n + \alpha_{-n} \\ b_n &= i(\alpha_n - \alpha_{-n}). \end{aligned}}$$

$\alpha_n = \overline{\alpha_{-n}} \Rightarrow \alpha_0$ real.
$\Rightarrow a_0$ real.

Also a_n real
& b_n real. ✓

Conversely, given a_0, α_0 is determined uniquely. And given a_n, b_n $(n \geq 1)$, α_n, α_{-n} are determined by solving equations:

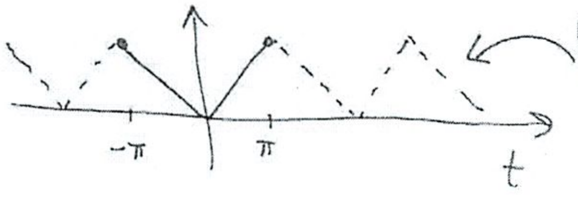
$$\alpha_n = \frac{1}{2}(a_n - i b_n)$$
$$\alpha_{-n} = \frac{1}{2}(a_n + i b_n)$$

a_n, b_n real $\Rightarrow$ $\alpha_n = \overline{\alpha_{-n}}$ as required.

(3)

(a) The answer to 7A-2b is

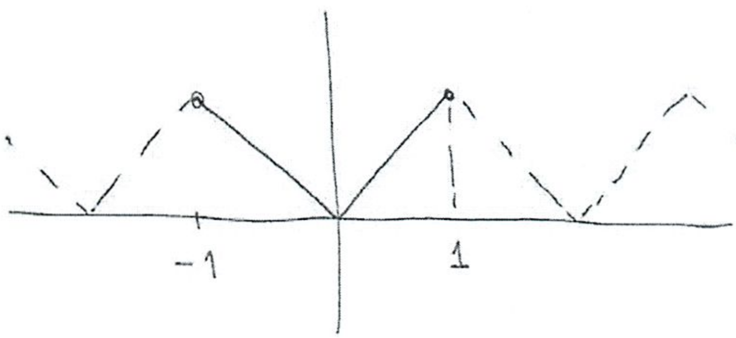
$$F(t) \sim \frac{\pi}{2} - \frac{4}{\pi}\left(\cos t + \frac{1}{3^2}\cos 3t + \frac{1}{5^2}\cos 5t + \dots\right)$$



where F is this.

But now we want the cosine series for

$f(t) =$



Note the period has changed.

Indeed $f(t) = \frac{1}{\pi} F(\pi t)$

$$= \frac{1}{2} - \frac{4}{\pi^2}\left(\cos \pi t + \frac{1}{3^2}\cos 3\pi t + \frac{1}{5^2}\cos 5\pi t + \dots\right)$$

(b) The homogeneous eqn $x'' + 10x = 0$ has solutions $A\sin\sqrt{10}\,t + B\cos\sqrt{10}\,t$, so we need not worry about resonance.

$f(t) =$ const. + ($\cos \pi t$ term) + ($\cos 3\pi t$ term) + ...

We find a solution to $x'' + 10x = f(t)$ term-by-term.

$$x'' + 10x = \frac{1}{2} \Rightarrow x = \frac{1}{20}.$$

$$x'' + 10x = \cos n\pi t \Rightarrow \text{Guess } A\cos n\pi t + B\sin n\pi t.$$

(Sin $n\pi t$ unnecessary.)

Subst. & get

$$A\left(-(n\pi)^2 + 10\right)\cos n\pi t = \cos n\pi t.$$

So $$A = \frac{1}{10 - \pi^2 n^2}.$$

∴ The solution (gotten by adding up all the contributions) is

$$x(t) = \frac{1}{20} - \frac{4}{\pi^2}\left(\frac{1}{10-\pi^2}\cos \pi t + \frac{1}{9(10-9\pi^2)}\cos 3\pi t + \frac{1}{25(10-25\pi^2)}\cos 5\pi t + \cdots\right).$$

(or in Σ notation):

$$x(t) = \frac{1}{20} - \frac{4}{\pi^2}\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{1}{n^2(10-n^2\pi^2)}\cos n\pi t.$$

(c) First few terms:

-0.000571

$$x(t) \cong \boxed{\frac{1}{20}} - \boxed{\frac{4}{\pi^2(10-\pi^2)}} \cos \pi t - \boxed{\frac{4}{9\pi^2(10-9\pi^2)}} \cos 3\pi t$$

$$- \boxed{\frac{4}{25\pi^2(10-25\pi^2)}} \cos 5t \; \sim\sim$$

0.05

3.11

- 0.0000685

So the terms $\frac{1}{20} - \frac{4}{\pi^2(10-\pi^2)} \cos \pi t.$

really dominate the solution.

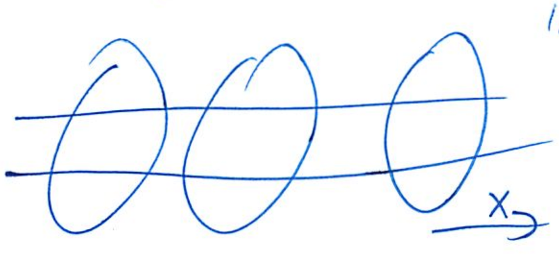
18.03 Lecture 22

11/4

Heat Eqn

- Substitute prof
- History: First Application of Fourier Series
- Have a wire

insulated



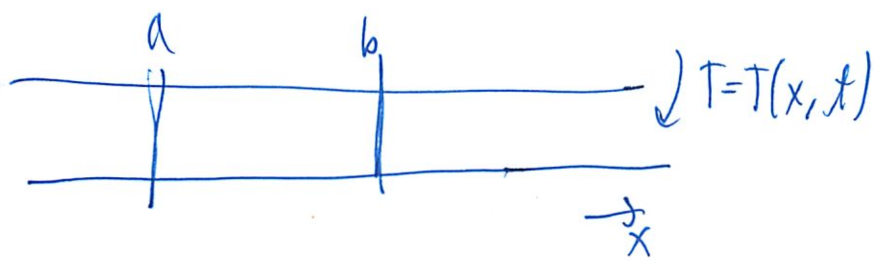
- care about temp on wire

  $T = T(x,t)$

- Thin wire - no cross section

- c = heat capacity

cT is amt of heat per unit value



pull at a section of wire $a \to b$

- F = heat flux

2

Rate of change

$$\frac{d}{dt}\int_a^b cT\,dx\,A =$$

Must be balanced

can only insert heat at right or left

$$\frac{d}{dt}\int_a^b cT\,dx\,A = F_A A - F_B - B$$

$$= -A\int_a^b F_x\,dx$$

$$\int_a^b \left(c\frac{\partial T}{\partial t} + \frac{\partial F}{\partial x}\right)dx = 0 \quad \text{true for any } a,b$$

$$c\frac{\partial T}{\partial t} + \frac{\partial F}{\partial x} = 0$$

What is F?

heat flux

- heat flows from hot → cold

~~Fick's law~~

Fick's law

$$\boxed{F = -\mu T_x}$$

c = const

μ = cont = heat conductivity

3

$$cT_t - \mu T_{xx} = 0$$

$$T_t = \nu T_{xx}$$

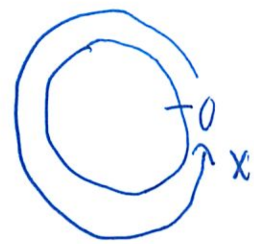
Symbol?

ν = heat diff

Example

$\mu = 1$ to make easy

$T_t = T_{xx}$



$0 \le x \le 2L$

Need to solve eq for T periodic of period 2L in x

Want to get into ODE

- You have current temp
- want to know how heat changes over time

Same idea as ODEs to solve

Can transform to ______ (missed)

(4)

will look like characteristic eq'n

But now ∞ly many solution

$$T = \varphi(x)\, e^{rt}$$

Plug into eqn

Simplify

$$r\varphi = \varphi''$$ ✓ partial diff eq became ODE

find

should be periodic w/ period 2L

is like our charistic eq'n

easy to solve since φ is constant

want periodic - must be sin, cos

~~[illegible]~~

$$r = -\lambda^2 \qquad \varphi = b_n \sin \lambda x + a_n \cos \lambda x$$

are periodic

Does it have the right period

$$\lambda_n = \frac{n\pi}{L}$$ ← only allowed λ

$$r_n = -\lambda_n^2 = -\frac{n^2\pi^2}{L^2} \qquad n = 1, 2, 3, \ldots$$

(5)

Also need $\psi = \frac{1}{2}a_0 \quad r_0 = 0$

r_s are sols of periodic eq

So now have tons of solutions to heat solution

$$T = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{L}x\right)e^{r_n t} + \left(b_n \sin \frac{n\pi}{L}x\right)e^{r_n t}$$

? is a fourier series

Fourier was trying to solve this problem

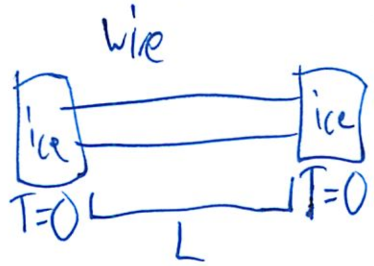
when set t=0 get this

$$T(x,0) = T_0(x) = \frac{1}{2}a_0 + \sum\left(a_n \cos \frac{n\pi}{t}x + b_n \sin \frac{n\pi}{L}x\right)$$

Can solve any inital problem

Then can extend to any periodic function

Example 2 Solve Hill's eqn



$T_t = T_{xx} \quad 0 < x < L$

$T = 0$ at $x = 0, L$ ← the ice (boundry conditions)

⑥

$$r\varphi = \varphi'' \qquad \varphi(0)=0$$
$$\varphi(L)=0$$

? r costant

~~so~~ φ is sin, cos etcc

must be a sin

$r = -\lambda^2$

? trick since it will be negitive

$\varphi = a \sin(\lambda x)$ ← satifies one condition

$\sin(\lambda L) = 0$

$\lambda L = n\pi$

$$\lambda = \frac{n\pi}{L} \qquad r = -\lambda^2 = -\frac{n^2\pi^2}{L^2} \qquad n = 1, 2, 3, \ldots$$

Now get

$$T = \sum_{1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{n^2\pi^2}{L^2}t} \quad \text{Sine F. S.}$$

⟶ got it w/ these boundry conditions

Can change boundries and get other one

This converges very fast ~~converges~~ exponential

– get good approximation

7

Example 3   insulate all sides of wire

$T_t = T_{xx} \quad 0 < x < L$

No heat flows at end points

heat flux $= T_x$

$T_x = 0$ at $x = 0, L$

Do it

Will get cosine fourier series

L since want deriv $= 0$

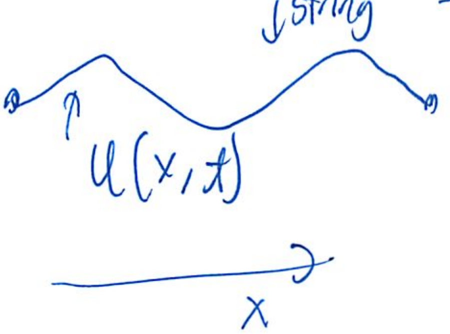
Could cool other end of wire

Boundry Value Problem - given values at 2 points

will see them a lot

(8)

Another Equation: The Wave Eq'n



- under tension T

has mass per unit length $\rightarrow \rho$ (rho)

string vibrating

$U = 0$ at $x = 0, L$

Look at conservation of angular motion

Do it

take 2 derivs

$$u_{tt} - c^2 u_{xx} = 0$$

$$c^2 = \frac{T}{\rho}$$

must not be real

$$u = \mathrm{Re}\left[e^{i\omega t} \varphi(x)\right]$$

Now equation becomes

$$-\omega_0^2 \varphi = c^2 \varphi''$$

similar to before

$$\varphi(0) = \varphi(L) = 0$$

(9)

Will be sins since it must be u at 0

Solution

$$u = \sin\left(\frac{\omega}{c} x\right)$$

$$\cancel{\sin\left(\frac{\omega}{c}\right) = 0} \quad \sin\left(\frac{\omega}{c} L\right) = 0$$

So

$$\frac{L\omega}{c} = n\pi$$

So $\omega = \dfrac{n\pi c}{L}$

Now can write solution

$$U = \sum_{n=1}^{\infty} \mathrm{Re}\left(A_n e^{i\omega_n t} \sin\left(\frac{n\pi x}{L}\right)\right)$$

get sine series again

When you pluck string - it will oscilate at this freq

~~Ch~~ Like a guitar : change ~~freq~~ tension to change freq

(10) Similar eqn w/ air vibrating in pipe organ

- when pipe closed, 0 velocity
- " " open - need to do some things
  will get some variation of this

Note freq

$$\frac{L\omega}{c} = n\pi$$

↑ discretized

Quantum Mechanics

- discretized energy levels

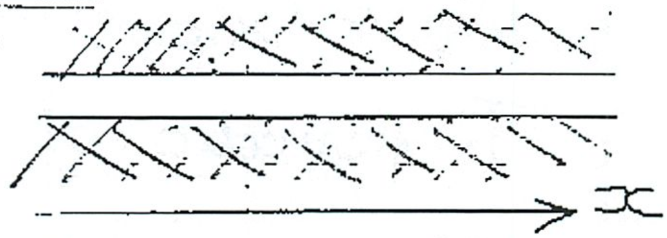
Lect. 22

18.03 Friday Nov. 4 (1)

Applications of Fourier Series to pde

Heat Equation 1-D

1) Derivation



Thin, insulated wire

$T = T(x,t)$ temprature

c = heat capacity

A = crossectional area

$F = -\mu T_x$ heat flow (Fick's Law)

μ = thermal conductivity

Conservation of energy:

$$\frac{d}{dt}\int_a^b cT\,dx = F\Big|_a - F\Big|_b = -\int_a^b F_x\,dx$$

Valid for any $a<b \rightarrow cT_t = -F_x$

$\therefore \quad T_t = \nu T_{xx}, \; \nu = \mu/C$

2) Example: Circular wire of length $2L$

$T_t = T_{xx} \quad (\nu = 1)$

T periodic of period $2L$

$T(x,0) = T_0(x)$ Initial Cond.

As for ode's look for an exponential time dependence
$\Rightarrow$ time derivatives into multiplication

$T = y(x)\, e^{rt} \Rightarrow$

$r y = y''$ Need $2L$ periodic solution

$y = \cos \frac{n\pi}{L} x, \quad r = -\frac{n^2 \pi^2}{L^2}$

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or $y = \sin\frac{n\pi}{L}x$, $r = -\frac{n^2\pi^2}{L^2}$

n natural #

Lots of solutions! Linear combination also a solution

$$T = \frac{1}{2}a_0 + \sum_1^\infty a_n e^{-\frac{n^2\pi^2}{L^2}t}\cos\frac{n\pi}{L}x$$

$$+ \sum_1^\infty b_n e^{-\frac{n^2\pi^2}{L^2}t}\sin\frac{n\pi}{L}x$$

Fourier Series! Get coeff. from data

$$T_0(x) = \frac{1}{2}a_0 + \sum_1^\infty a_n\cos\frac{n\pi}{L}x$$

$$+ \sum_1^\infty b_n\sin\frac{n\pi}{L}x$$

3) <u>Example</u> Wire of length L with temperature prescribed at ends

$$T_t = T_{xx} \quad \text{and} \quad T = 0 \text{ at } x = 0, L$$

Again try

$$T = y(x)e^{rt} \Rightarrow$$

$$ry = y'' \quad 0 < x < L$$

with $y(0) = y(L) = 0$

"Boundary Value problem for ode"

Solutions $y = \sin \frac{n\pi}{L} x$

$$r = n^2\pi^2/L^2$$

So

$$T = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x \, e^{-\frac{n^2\pi^2}{L^2} t}$$

4) **Example** Wire of length L with insulated ends

$$T_t = T_{xx} \quad 0<x<L$$

$$T_x = 0 \quad \text{at } x=0, L$$

You do it $\Rightarrow$ Cosine Fourier Ser.

Wave Equation 1-D

String under tension

$u = u(x,t)$ deviation from straight line

$$u_{tt} - c^2 u_{xx} = 0$$

$$c^2 = T/\rho \qquad T = \text{tension}$$

ρ = mass per unit length

See book for derivation

5) <u>Example</u> String of Length L tied at ends

$$u_{tt} - c^2 u_{xx} = 0 \quad 0 < x < L$$

$$u = 0 \text{ at } x = 0, L$$

$$u = e^{rt} y(x) \Rightarrow$$

$$r^2 y = c^2 y'' \quad y = 0 \text{ at } x = 0, L$$

$$y = \sin\left(\frac{n\pi x}{L}\right)$$

$$r = \pm i \frac{n\pi}{L} c = \pm i\omega$$

Modes of vibration
ω = natural frequencies

Again: full sln by Sine Fourier Series

<u>NEXT</u>: Laplace Transform

Let $f(t)$ be a function known to be a combination of exponentials multiplied by powers of t. Exaple

$$f(t) = x e^{t} + e^{\pi t} + (2x^3 + 7x^2) e^{3t}$$

How do you find the exponents, coefficients, etc from knowledge of the values of f only

Note

$$\int_0^\infty e^{rt} e^{-st}\, dt = \frac{-1}{r-s}$$

$$\int_0^\infty (t e^{rt}) e^{-st}\, dt = \frac{1}{(r-s)^2}$$

$$\int_0^\infty (t^2 e^{rt}) e^{-st}\, dt = \frac{-2}{(r-s)^3}$$

So, let

$$F(s) = \int_0^\infty f(t)e^{-st}\,dt$$

Laplace Transform of f

By looking at the "singularities" of F (places where $|F|$ goes to ∞) we can find the exponents.

Looking at the details of how $|F|$ blows up, the polynomial part of f [powers of t] follow.

The L.T. has many interesting properties ... next week.

18.03 Lecture 23

Laplace Transform

11/[illegible]

Brubaker is back

Laplace Transform

Admin Two techniques in this unit

1. Fourier Series
   - finished
2. LaPlace transform
   - start today

Fourier Series applications

1. IR model w/ square wave (or other piecewise smooth) fn
   - gives answer
   - but converges slowly
2. IR model w/ smooth input
   - converges fast
     - matters in applications
   - Could use Power series - but converges much more slowly
3. Partial differential eq's

   1-D heat eq'n

   Lwill see again in recitation

   $$\frac{\partial T}{\partial t} = V \frac{\partial^2 T}{\partial^2 x}$$ v const ← don't need to memorize

(2)

Since this has $\downarrow$ 2 factors solution $T(x,t)$ and Initial condition

$T(x,0)$ = some init. fn of x $f(x)$

- We guessed solution
- if $t=0 \to$ The ∞ly many sols looked like Fourier Series
- didn't use series method - used guess method
- but in special case $(t=0)$ could use Fourier series
- We guessed $e^{(\cdots t)} \sin(\cdots)x$
- When set $t=0$ then just have $N \sin x$ ↑horrible mess
  but lots of other stuff dropped off

Laplace Transform

Somewhat reminiscent of operator method

$D = \frac{d}{dx}$

Now use integration as operator

$$\mathcal{L}(f) = \int_0^\infty f(t)\, e^{-st}\, dt$$

↑improper integral

s is a real variable

$s >> 0$ so that integral converges

③

$$\mathcal{L}(f) = \int_0^\infty f(t)\, e^{-st}\, dt$$

$\quad\;\; \|\qquad\qquad\qquad\; \|$

$\mathcal{L}(f)(s) \qquad\quad F(s)$ two notations

<u>Ex</u> $\mathcal{L}(1)$

- do you remember improper integrals?

$$\int_0^\infty 1\, e^{-st}\, dt = \lim_{A\to\infty} \int_0^A e^{-st}\, dt$$

fund theorem of calculus

$$= \lim_{A\to\infty} \left(-\frac{1}{s} e^{-st}\right)\Bigg|_{0=t}^{A=t}$$

since

$e^{-sA} \to 0$ as $A\to\infty$ $\qquad = \dfrac{1}{s}$

if $s > 0$

<u>Solving ODEs using Laplace Transform</u>

1. Start w/ ODE
2. Laplace transform both sides
   - converts deriv's to algebra
3. Do algebra to simplify problem

(4)

4. Take Inverse of LaPlace Transform

Ex 2

$$\mathcal{L}(e^{at}) = \int_0^\infty e^{at} e^{-st}\,dt$$

$$= \int_0^\infty e^{(a-s)t}\,dt$$

$\vdots$ like before

$$= \frac{1}{s-a}$$

Start building a basic library of functions and ways to combine them

---

Ex 3+4

$\mathcal{L}(\cos bt)$ ← could do int by parts twice

$\mathcal{L}(\sin bt)$ or complexify via ex 2 then restore

$$\mathcal{L}(\cos bt) = \frac{s}{s^2+b^2}$$

$$\mathcal{L}(\sin bt) = \frac{b}{s^2+b^2}$$

(5) Properties

$\mathcal{L}(f) = F(s)$ then

(diff capital and lower F

oh $F(s) = \int_0^\infty f(t) e^{-st} dt$

$f(t)$ = original function

(A) $\mathcal{L}(e^{at} \cdot f(t)) = F(s-a)$

— can confirm if we think about $f(t) = 1$

(B) $\mathcal{L}(t \cdot f(t)) = -F'(s)$

(C) $\mathcal{L}(f'(t)) = sF(s) - f(0)$

Look strange - but these are the parts we use

Example

$\mathcal{L}(e^{at} \sin bt) \overset{(A)}{=} F(s-a)$ where F is the Laplace transform of $\sin bt$.

$$= \frac{b}{(s-a)^2 + b^2}$$

Proving (A) easy - just change of variables

Proof (B)

$$F'(s) = \frac{d}{ds}\left(\int_0^\infty f(t) e^{-st} dt\right)$$

Want to bring $\frac{d}{ds}$ inside

(difference quotient

But integration in t, differentiation in s

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$$= \int_0^\infty f(t) \underbrace{\frac{d}{ds}\left(e^{-st}\right)}_{-t e^{-st}} dt$$

*Pay close attention to which variable manipulating

$$= -\mathcal{L}(t \cdot f(t))$$

Getting Polynomials

- is a linear operator

Ex $\mathcal{L}(t^n)$ — if can do for some polynomial, can do for all

do by recursion

$\mathcal{L}(1) = \frac{1}{s}$

$\mathcal{L}(t) = \mathcal{L}(t \cdot 1)$

$\overset{B}{=} -\frac{d}{ds}\left(\frac{1}{s}\right)$

$= \frac{1}{s^2}$

repeat

$\mathcal{L}(t^2) = \mathcal{L}(t) \cdot \mathcal{L}(t)$

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$$= -\frac{d}{ds}\left(\frac{1}{s^2}\right)$$

$$= \frac{2}{s^3}$$

$\vdots$

$$\boxed{\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}}$$

Notice Laplace has turned everything into rational fn
- which are easy to handle

Now inverse Laplace transform of arbitrary rational fn
- break it up into easy pieces
- identify pieces of Laplace transforms of interesting fns

ex $\mathcal{L}^{-1}\left(\frac{1}{s^2+5s+4}\right)$

we want fn $f(t)$ so that

$$\mathcal{L}(f(t)) = \frac{1}{s^2+5s+4}$$

Can make it simpler w/ partial fractions

Factor $\frac{1}{s^2+5s+4} = \frac{1}{(s+4)(s+1)} = \frac{A}{s+4} + \frac{B}{s+1}$

common denom for both

cross multiply for common denom

(8)

$A(s+1) + B(s+4) = 1$ & want coefficents to be =

$(A+B)s + A + 4B = 1$

$\downarrow$

$A + B = 0$

$A + 4B = 1$

$3B = 1$ Similar to having repeated roots

$B = 1/3$

$A = -1/3$

If $\frac{1}{(s+4)^2} = \cancel{\frac{A}{s+4} + \frac{B}{s+4}}$ no unique denom

$= \cancel{\frac{A}{(s+4)} + \frac{B}{(s+4)^2}}$ nope

$= \frac{A}{(s+4)} + \frac{Bs+C}{(s+4)^2}$

Shortcut to partial fractions: Hevaisides' Cover-Up Method

$\llcorner$ see notes section H from 18.01

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$$\mathcal{L}^{-1}\left(\frac{+1/3}{s+1} + \frac{-1/3}{s+4}\right)$$

$$= +\frac{1}{3}\mathcal{L}^{-1}\left(\frac{1}{s+1}\right) - \frac{1}{3}\mathcal{L}^{-1}\left(\frac{1}{s+4}\right)$$

$$= \frac{1}{3}e^{-t} - \frac{1}{3}e^{-4t}$$

since $\mathcal{L}^{-1}\left(\frac{1}{s-a}\right) = e^{at}$

So how can we actually use to solve a diff eq

Application: Spring Model

$$x'' + w_0^2 x = f(t)$$

$$\mathcal{L}(x'') + w_0^2 \mathcal{L}(x) = \mathcal{L}(f)$$

key: Find $\mathcal{L}(x'')$

First show Laplacian $\mathcal{L}(x') = sX(s) - x(0)$

where $X(s) = \mathcal{L}(x)$

$$L(x') = \int_0^\infty \underbrace{x'(t)}_{dv}\underbrace{e^{-st}}_{u}\,dt$$

Integration by parts

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$$= -\int_0^\infty x(t)(-s)e^{-st}\,dt + \underbrace{x(t)}_{v}\,\underbrace{e^{-st}}_{u}\Big|_0^\infty$$

$$= s\,X(s) \cancel{} - x(0)$$

if do again

$$\mathcal{L}(x'') = s^2 X(s) - s\cdot x(0) - x'(0)$$

If trying to solve diff eq

$$X(s) = \frac{F(s)}{s^2+\omega_0^2} \quad \text{if} \quad x(0) = x'(0) = 0$$

terms drop out
divide out

$$x(t) = \mathcal{L}^{-1}\left(\frac{F(s)}{s^2+\omega_0^2}\right)$$

Need $F(s)$ to play around and
find ans

General method for finding linear ODEs

no slides

Lecture 23

11/7

For Laplace transform, we choose to include impulsive forces at $t=0$.

Sometimes write $\mathcal{L}(f) = \int_{0^-}^{\infty} f(t)\, e^{-st}\, dt$ to emphasize this.

Note $-$ sign

Then $\mathcal{L}(\delta) = \int_{0^-}^{\infty} \delta(t)\, e^{-st}\, dt = e^{-st}\Big|_{t=0} = 1.$

Remembering that differentiation becomes multiplication by s in Laplace transform,

might guess: $\mathcal{L}(\delta') = s$ (except $\delta(0)$ not defined ...)

Better

$$\mathcal{L}(\delta') = \int_{0^-}^{\infty} \delta'(t)\, e^{-st}\, dt$$

$$\overset{\text{int by parts}}{=} -\int_{0^-}^{\infty} \delta(t) \frac{d}{dt}(e^{-st})\, dt$$

since e^{-st} vanishes for t suff. large

$$= s\, e^{-st}\Big|_{t=0} = s. \checkmark$$

Try to solve:

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = e^{-t} + 3\delta(t-1)$$ with

$y(0) = y'(0) = 0.$

Two techniques for solving ODEs in this unit:

(1) Fourier series ; (2) Laplace transform (start today)

Uses of Fourier series

(a) Square-wave inputs (or other piecewise smooth inputs) in IR models

(b) Any smooth function in Input-Response models. Here, we could use power series as well, but Fourier series tend to converge to the solution more rapidly.

(c) In PDEs like 1-dim'l heat equation:

Solve $\frac{\partial T}{\partial t} = \nu \frac{\partial^2 T}{\partial x^2}$ ν: const. for $T(x,t)$ with

initial condition $T(x,0) = f(x)$

some interesting function in x

Here, guess infinitely many solutions, whose linear combination remains a solution.

At $t=0$, this linear comb. becomes Fourier series. Solve for coeffs. in linear comb. by computing Fourier coeffs of $f(x)$.

Laplace transform : $\mathcal{L}(f(t)) = \int_0^\infty f(t)\, e^{-st}\, dt$

where s is a variable (with $s \gg 0$ to guarantee convergence of improper integral)

Often convenient to use several notations for Laplace transform.

E-P uses capital letters (reminder about antiderivatives in def.):

$$\mathcal{L}(f(t)) = F(s).$$

Example: $\mathcal{L}(1) = \int_0^\infty 1 \cdot e^{-st}\, dt = \lim_{A \to \infty} \int_0^A e^{-st}\, dt$

$$= \lim_{A \to \infty} \left[-\frac{1}{s} e^{-st} \right]_0^A = \lim_{A \to \infty} -\frac{1}{s} e^{-sA} + \frac{1}{s} = 0 + \frac{1}{s} = \boxed{\frac{1}{s}}$$

since $e^{-sA} \to 0$ as $A \to \infty$ provided $\underline{s > 0}$. ← these are values for which improper integral converges.

Rough idea of how to solve ODE w/ Laplace:

ODE → take Laplace transform of both sides → Do some algebraic manipulations → Take inverse Laplace transforms.

↑ this converts an ODE to an algebra problem

↑ gives solution to ODE.

For the moment, we just want to investigate properties of Laplace transform:

(1) $\mathcal{L}(e^{at}) = \dfrac{1}{s-a}$

(2) $\mathcal{L}(\cos bt) = \dfrac{s}{s^2+b^2}$

(3) $\mathcal{L}(\sin bt) = \dfrac{b}{s^2+b^2}$

<u>pf</u>: $\mathcal{L}(e^{at}) = \int_0^\infty e^{at} e^{-st}\, dt = \int_0^\infty e^{(a-s)t}\, dt$

$$= \lim_{A \to \infty} -\frac{e^{-(s-a)t}}{s-a}\Bigg|_0^A = \boxed{\frac{1}{s-a}}$$

To prove (2) + (3) repeat this calculation with e^{ibt}, taking real and imaginary parts at end.

<u>Properties of Laplace Transforms</u>: Throughout, use $F(s) = \mathcal{L}(f(t))$

(A) $\mathcal{L}(e^{at} f(t)) = F(s-a)$ (Exponential Shift)

(B) $\mathcal{L}(t f(t)) = -F'(s)$

(C) $\mathcal{L}(f') = sF(s) - f(0)$

(D) $\mathcal{L}(c_1 f_1 + c_2 f_2) = c_1 \mathcal{L}(f_1) + c_2 \mathcal{L}(f_2)$ c_1, c_2 constants

(Linearity)

(A) + (D) are simple, so we leave them to you ((A) is just like pf. above)

pf. of (B): $F'(s) = \frac{d}{ds}\int_0^\infty f(t)\, e^{-st}\, dt \overset{*}{=} \int_0^\infty f(t)\, \frac{d}{ds}(e^{-st})\, dt$

$= \int_0^\infty f(t)\,(-t e^{-st})\, dt = -\mathcal{L}(t f(t))$.

* : can write out $\frac{d}{ds}$ as difference quotient to formally justify pulling derivative inside. Note it is diff. in $\underline{\underline{s}}$ not int. var. t.

Using (B) we can show (4): $\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$.

pf. of (4): $\mathcal{L}(t) = \mathcal{L}(t\cdot 1) \overset{(B)}{=} -F'(s)$ where $F(s) = \mathcal{L}(1) = \frac{1}{s}$

$= +\frac{1}{s^2}$

$\mathcal{L}(t^2) = \mathcal{L}(t\cdot t) = -F'(s)$ where now $F(s) = \mathcal{L}(t) = \frac{-1}{s^2}$

$= +\frac{2}{s^3}$

... can now prove easily by induction.

Inverse Fourier transforms (we'll state existence/uniqueness results for $\mathcal{L}$ next time, which guarantee well-defined.)

Notice all of our basic functions had Laplace transforms which were rational functions. How to find inverse transform of arbitrary rational function?

Ex. $\mathcal{L}^{-1}\left(\frac{1}{s^2+5s+4}\right)$. Want to break this into simpler pieces. partial Fractions!

$$s^2+5s+4 = (s+1)(s+4)$$

So write $\frac{1}{s^2+5s+4} = \frac{A}{s+1} + \frac{B}{s+4}$ and then solve for A, B by making common denom.

$$A(s+4) + B(s+1) = 1$$

$$\Rightarrow \left.\begin{array}{ll} A+B=0 & \text{(coeff. of } s) \\ 4A+B=1 & \text{(const. coeffs. on both sides)} \end{array}\right\} \quad A = 1/3, \; B = -1/3$$

$$\Rightarrow \frac{1}{s^2+5s+4} = \frac{1/3}{s+1} + \frac{-1/3}{s+4}.$$

Now wanted $f(t)$ s.t. $\mathcal{L}(f(t)) = \frac{1}{s^2+5s+4} = \frac{1/3}{s+1} - \frac{1/3}{s+4}$

But $\mathcal{L}(e^{at}) = \frac{1}{s-a}$ so choose $f(t) = \frac{1}{3}e^{-t} - \frac{1}{3}e^{-4t}$.

Solve an ODE with Laplace:

$$x'' + \omega_0^2 x = f(t) \xrightarrow[\text{Take Laplace}]{} \mathcal{L}(x'') + \omega_0^2 \mathcal{L}(x) = \mathcal{L}(f(t))$$

Let $X(s) = \mathcal{L}(x)$, $F(s) = \mathcal{L}(f)$.

First, want to relate $\mathcal{L}(x'')$ to $\mathcal{L}(x) = X(s)$. Use property (c) twice

((c) is proven using integration by parts)

Get $\mathcal{L}(x'') = s \cdot \mathcal{L}(x') - x'(0)$

$$= s\left(s\,\mathcal{L}(x) - x(0)\right) - x'(0)$$

$$= s^2\,\mathcal{L}(x) - s \cdot (x(0)) - x'(0)$$

Let's assume, for simplicity, that $x(0) = x'(0) = 0$.

Then $\mathcal{L}(x'') = s^2\,\mathcal{L}(x). = s^2\,X(s)$

Substituting back into original ODE:

$$s^2\,\mathcal{L}(x) + \omega_0^2\,\mathcal{L}(x) = \mathcal{L}(f(t))$$

i.e. $$X(s) = \frac{F(s)}{s^2+\omega_0^2}.$$

To finish, take inverse Laplace transform:

$$x(t) = \mathcal{L}^{-1}\left(\frac{F(s)}{s^2+\omega_0^2}\right).$$

Try this with your favorite choice of $f(t)$. See how it matches known answer by other methods.

Partial Fn Decomposition 11/7

WP Example

$$f(x) = \frac{1}{x^2+2x-3}$$

↳ split denom
$(x+3)(x-1)$

So

$$f(x) = \frac{A}{x+3} + \frac{B}{x-1}$$

Multiply through by x^2+2x-3

$$1 = \frac{A\cancel{(x+3)}(x-1)}{\cancel{(x+3)}} + \frac{B(x+3)\cancel{(x-1)}}{\cancel{(x-1)}}$$

since $\boxed{\frac{1(x+3)(x-1)}{x^2-2x-3}}$

$$1 = A(x-1) + B(x+3)$$

? then what

Oh is this called Heavisides ?

"Mentally cover up one side"
Lok set that factor to not matter
Lbe multiplied by x something

(2)

So $x = -3$ makes B term disappear

$$1 = A(x-1)$$
$$1 = A(-3-1)$$
$$A = -\frac{1}{4}$$

And unrelatedly on other side

$$x = 1$$
$$1 = B(1+3)$$
$$B = \frac{1}{4}$$

Then put together in original

$$\frac{-\frac{1}{4}}{x+3} + \frac{\frac{1}{4}}{x-1}$$

Simplify

$$\frac{1}{4}\left(\frac{-1}{x+3} + \frac{1}{x-1}\right)$$

✓ Done

From 18.01

Read 11/7

F. HEAVISIDE'S COVER-UP METHOD

Rational fns

$f(x) = \frac{P(x)}{Q(x)}$

? two polynomial fns

? define terms

The eponymous method was introduced by Oliver Heaviside ~~as~~ a fast way to do a decomposition into partial fractions. In 18.01 we need the partial fractions decomposition in order to integrate rational functions (i.e., quotients of polynomials). In 18.03, it will be needed as an essential step in using the Laplace transform to solve differential equations, and in fact this was more or less Heaviside's original motivation.

here are using

The cover-up method can be used to make a partial fractions decomposition of a rational function $\frac{p(x)}{q(x)}$ *whenever the denominator can be factored into distinct linear factors.*

We illustrate with an example; though simple, it should convince you that the method is worth learning.

Example 1. Decompose $\frac{x-7}{(x-1)(x+2)}$ into partial fractions.

Solution. We know the answer will have the form

jumped fast vs wikipedia - would not seen

$$\frac{x-7}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}. \tag{1}$$

Partial Fns

technique to reduce the degree of either numerator or denominator of rational fns

Convert $\frac{f(x)}{g(x)}$ into $\sum_j \frac{f_j(x)}{g_j(x)}$ ↳ factors of $g(x)$ generally lower degree

To determine A by the cover-up method, on the left-hand side we mentally remove (or cover up with a finger) the factor $x-1$ associated with A, and substitute $x = 1$ into what's left; this gives A:

$$\left.\frac{x-7}{(x+2)}\right|_{x=1} = \frac{1-7}{1+2} = -2 = A. \tag{2}$$

Similarly, B is found by covering up the factor $x+2$ on the left, and substituting $x = -2$ into what's left. This gives

$$\left.\frac{x-7}{(x-1)}\right|_{x=-2} = \frac{-2-7}{-2-1} = 3 = B.$$

Thus, our answer is

$$\frac{x-7}{(x-1)(x+2)} = \frac{-2}{x-1} + \frac{3}{x+2}. \tag{3}$$

yeah - why?

Why does the method work? The reason is simple. The "right" way to determine A from equation (1) would be to multiply both sides by $(x-1)$; this would give

$$\frac{x-7}{(x+2)} = A + \frac{B}{x+2}(x-1). \tag{4}$$

Now if we substitute $x = 1$, what we get is exactly equation (2), since the term on the right disappears. The cover-up method therefore is just any easy way of doing the calculation without going to the fuss of writing (4) — it's unnecessary to write the term containing B since it will become 0.

from 18.01

In general, if the denominator of the rational function factors into the product of distinct linear factors:

$$\frac{p(x)}{(x-a_1)(x-a_2)\cdots(x-a_r)} = \frac{A_1}{x-a_1} + \ldots + \frac{A_r}{x-a_r}\,, \quad a_i \neq a_j\,,$$

you sub its term

then A_i is found by covering up the factor $x - a_i$ on the left, and setting $x = a_i$ in the rest of the expression.

Example 2. Decompose $\frac{1}{x^3 - x}$ into partial fractions.

Solution. Factoring, $x^3 - x = x(x^2-1) = x(x-1)(x+1)$. By the cover-up method,

$$\frac{1}{x(x-1)(x+1)} = \frac{-1}{x} + \frac{1/2}{x-1} + \frac{1/2}{x+1}\,.$$

To be honest, the real difficulty in all of the partial fractions methods (the cover-up method being no exception) is in factoring the denominator. Even the programs which do symbolic integration, like Macsyma, or Maple, can only factor polynomials whose factors have integer coefficients, or "easy coefficients" like $\sqrt{2}$. and therefore they can only integrate rational functions with "easily-factored" denominators. (Of course, these are the only kind you'll see in 18.01 or 18.03.)

Heaviside's cover-up method also can be used even when the denominator doesn't factor into distinct linear factors. To be sure, it gives only partial results, but these can often be a big help. We illustrate.

Example 3. Decompose $\frac{5x+6}{(x^2+4)(x-2)}$.

Solution. We write

oh this

(5) $$\frac{5x+6}{(x^2+4)(x-2)} = \frac{Ax+B}{x^2+4} + \frac{C}{x-2}\,.$$

We first determine C by the cover-up method, getting $C = 2$. Then A and B can be found by the method of undetermined coefficients; the work is greatly reduced since we need to solve only two simultaneous equations to find A and B, not three.

"lucky guess method"

Following this plan, using $C = 2$, we combine terms on the right of (5) so that both sides have the same denominator. The numerators must then also be equal, which gives us

(6) $$5x + 6 = (Ax+B)(x-2) + 2(x^2+4)\,.$$

Comparing the coefficients say of x^2 and of the constant terms on both sides of (6) then gives respectively the two equations

x^2 Constant

$$0 = A + 2 \qquad \text{and} \qquad 6 = -2B + 8,$$

so match coefficients

from which $A = -2$ and $B = 1$.

X

In using (6), one could have instead compared the coefficients of x, getting $5 = -2A + B$; this provides a valuable check on the correctness of our values for A and B.

In Example 3, an alternative to undetermined coefficients would be to substitute two numerical values for x into the original equation (5), say $x = 0$ and $x = 1$ (any values other than $x = 2$ are usable). Again one gets two simultaneous equations for A and B. This method requires addition of fractions, and is usually better when only one coefficient remains to be determined (as in Example 4 below).

Still another method would be to factor the denominator completely into linear factors, using complex coefficients, and then use the cover-up method, but with complex numbers. At the end, conjugate complex terms have to be combined in pairs to produce real summands. The calculations are sometimes longer, and require skill with complex numbers.

The cover-up method can also be used if a linear factor is repeated, but there too it gives just partial results. It applies only to the *highest power of the linear factor*. Once again, we illustrate. use other method to Complete

Example 4. Decompose $\dfrac{1}{(x-1)^2(x+2)}$.

Solution. We write Only partial

$$\frac{1}{(x-1)^2(x+2)} = \frac{A}{(x-1)^2} + \frac{B}{x-1} + \frac{C}{x+2}. \tag{7}$$

To find A cover up $(x-1)^2$ and set $x = 1$; you get $A = 1/3$. To find C, cover up $x+2$, and set $x = -2$; you get $C = 1/9$.

This leaves B which cannot be found by the cover-up method. But since A and C are already known in (7), B can be found by substituting any numerical value (other than 1 or -2) for x in equation (7). For instance, if we put $x = 0$ and remember that $A = 1/3$ and $C = 1/9$, we get any value, it will balance out

$$\frac{1}{2} = \frac{1/3}{1} + \frac{B}{-1} + \frac{1/9}{2},$$

from which we see that $B = -1/9$.

B could also be found by applying the method of undetermined coefficients to the equation (7); note that since A and C are known, it is enough to get a single linear equation in order to determine B — simultaneous equations are no longer needed. — I should practice

The fact that the cover-up method works for just the *highest power* of the repeated linear factor can be seen just as before. In the above example for instance, the cover-up method for finding A is just a short way of multiplying equation (5) through by $(x-1)^2$ and then substituting $x = 1$ into the resulting equation.

Exercises 5E

From 18.01

Read 11/7

INT. IMPROPER INTEGRALS

In deciding whether an improper integral converges or diverges, it is often awkward or impossible to try to decide this by actually carrying out the integration, i.e., finding an antiderivative explicitly. For example both of these two improper integrals converge:

$$\int_1^\infty \frac{dx}{x^6+3x^3+2x^2+1} \quad \text{and} \quad \int_1^\infty \frac{dx}{\sqrt{x^3+1}},$$

but there is no explicit antiderivative for the second integral, and finding one for the first would be a hairy exercise in partial fractions, even if one were able to factor the denominator. factoring can be impossible

Instead of explicit integration, therefore, we show they converge by using estimation instead, comparing them with simpler integrals which are known to converge. Thus, for the first,

$$\frac{1}{x^6+3x^3+2x^2+1} \le \frac{1}{x^6}, \qquad x>0,$$

so that

$$\int_1^\infty \frac{dx}{x^6+3x^3+2x^2+1} \le \int_1^\infty \frac{dx}{x^6} = \frac{1}{5}.$$

Since the right hand integral converges, so does the left, which is smaller (but still positive). In a similar way, for the second integral, we estimate

$$\frac{1}{\sqrt{x^3+1}} \le \frac{1}{x^{3/2}}, \qquad x>0,$$

so that

$$\int_1^\infty \frac{dx}{\sqrt{x^3+1}} \le \int_1^\infty \frac{dx}{x^{3/2}} = 2.$$

In the same way we can show the divergence say of $\displaystyle\int_1^\infty \frac{x\,dx}{\sqrt{x^3+1}}$:

$$\frac{x}{\sqrt{x^3+1}} \ge \frac{1}{2\sqrt{x}}, \qquad x>1,$$

this last being equivalent to $2x^{3/2} \ge \sqrt{x^3+1}$, i.e., to $8x^3 \ge x^3+1$; thus we get

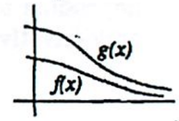
$$\int_1^\infty \frac{x\,dx}{\sqrt{x^3+1}} \ge \int_1^\infty \frac{dx}{2\sqrt{x}} = \infty.$$

We call the general principle we are using here the

Comparison Test for Improper Integrals. If $0 \le f(x) \le g(x)$ for $a \le x < \infty$,

(1) $$\int_a^\infty g(x)\,dx \text{ converges } \Rightarrow \int_a^\infty f(x)\,dx \text{ converges;}$$

(1′) $$\int_a^\infty f(x)\,dx \text{ diverges } \Rightarrow \int_a^\infty g(x)\,dx \text{ diverges.}$$



In other words, if the area under $g(x)$ is finite and $f(x)$ lies below $g(x)$ (but still over the x-axis), then the area under $f(x)$ will be finite also. Or equivalently, if the area under $f(x)$ is infinite, so is the area under $g(x)$.

In using the test, the lower limit of integration is of no importance, since if $a < b$,

$$\int_a^\infty f(x)\,dx = \int_a^b f(x)\,dx + \int_b^\infty f(x)\,dx, \quad \text{so that}$$

(3) $$\int_a^\infty f(x)\,dx \text{ converges } \Leftrightarrow \int_b^\infty f(x)\,dx \text{ converges.}$$

As another example of the use of the comparison test, we recall one used to illustrate the Second Fundamental Theorem (cf. p. FT.3):

$$e^{-x^2} \le e^{-x} \quad \text{for } x \ge 1, \text{ and therefore}$$

$$\int_0^\infty e^{-x^2}\,dx \text{ converges, since } \int_0^\infty e^{-x}dx \text{ converges.}$$

(Here we use (3) above to shift the lower limit from 1 to 0; also the comparison test (1).)

In using the comparison test, we have to have some standard integrals that we know converge or diverge, to use for comparison purposes. The most useful are:

(4) $$\int_1^\infty \frac{dx}{x^p} \quad \text{converges if } p > 1, \text{ diverges if } p \le 1;$$

(5) $$\int_0^\infty e^{-ax} \quad \text{converges if } a > 0.$$

It is important to notice that in using the test, you must make the inequality go in the right direction, which means you must *guess in advance* whether the integral will converge or diverge. For example, to test $\displaystyle\int_2^\infty \frac{x\,dx}{x^3-2}$, we see that as $x \to \infty$,

$$\frac{x}{x^3-2} \sim \frac{x}{x^3} = \frac{1}{x^2},$$

so from (4) we guess it will converge. Unfortunately, $\dfrac{x}{x^3-2} > \dfrac{1}{x^2}$, $x \ge 2$, so we can't use $\displaystyle\int_1^\infty \frac{dx}{x^2}$ as the comparison integral; however, only a slight change is needed:

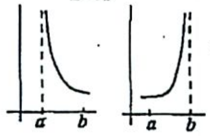
$$\frac{x}{x^3-2} < \frac{2}{x^2}, \quad x \ge 2,$$

as we see by cross-multiplying: $x^3 < 2x^3 - 4$; thus our integral converges, using (1). □

Improper integrals of the second kind

The comparison test also works for improper integrals of the form

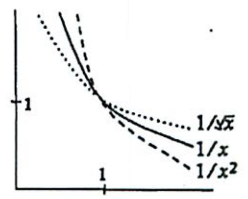
$$\int_{a^+}^{b} f(x)\,dx, \quad \text{where} \quad \lim_{x\to a^+} f(x) = \infty, \qquad \text{and} \qquad \int_{a}^{b^-} f(x)\,dx, \quad \text{where} \quad \lim_{x\to b^-} f(x) = \infty.$$



Sometimes these are called improper integrals of the **second kind** — the first kind being the previous type of improper integral, where one of the limits of integration is ∞ or $-\infty$.

For improper integrals of the second kind, useful standard comparison integrals are

$$\text{(6)} \qquad \int_{a}^{b^-} \frac{dx}{(b-x)^p} \quad \text{and} \quad \int_{a^+}^{b} \frac{dx}{(x-a)^p},$$



which *converge if* $p < 1$, *diverge if* $p \ge 1$. Note that this is just the opposite of (4). However, it's easy to remember which is which if you think of the picture at the right, which compares the graphs for $p < 1$ and $p > 1$.

Example. Test $\displaystyle\int_0^1 \frac{dx}{\sqrt{2x - x^2}}$ for convergence.

Solution. The integrand becomes infinite when $x = 0$ and when $x = 2$, but 2 is of no importance, since it's not in the interval over which we are integrating. In making our guess as to convergence or divergence, we note that

$$\frac{1}{\sqrt{2x - x^2}} \approx \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{x}}, \qquad x \approx 0^+.$$

Thus, according to (6) above, we expect convergence, and make the comparison

$$\text{(7)} \qquad \frac{1}{\sqrt{2x - x^2}} \le \frac{1}{\sqrt{x}}.$$

Where is (7) valid (if at all)? Crossmultiplying, it claims that

$$\sqrt{x} \le \sqrt{2x - x^2}, \quad \text{or squaring,} \quad x \le 2x - x^2, \text{ or } x^2 \le x\,.$$

This last inequality is true if $0 \le x \le 1$, so we're safe, since this is the region of integration. So we see by the comparison test above that our integral converges because according to (6), the integral $\displaystyle\int_{0+}^1 \frac{dx}{\sqrt{x}}$ converges.

Remark. It isn't customary to include the $+$ and $-$ symbols in a^+ and b^- when one writes integrals of the second kind. We only did it here to point out where they are improper, and on which side. It doesn't appear in the solutions to the exercises on improper integrals.

Exercises: Section 6B

Laplace transform

From Wikipedia, the free encyclopedia

In mathematics, the **Laplace transform** is a widely used integral transform. Denoted $\mathcal{L}\left\{f(t)\right\}$, it is a linear operator of a function $f(t)$ with a real argument t ($t \geq 0$) that transforms it to a function $F(s)$ with a complex argument s. This transformation is essentially bijective for the majority of practical uses; the respective pairs of $f(t)$ and $F(s)$ are matched in tables. The Laplace transform has the useful property that many relationships and operations over the originals $f(t)$ correspond to simpler relationships and operations over the images $F(s)$.[1] The Laplace transform has many important applications throughout the sciences. It is named for Pierre-Simon Laplace who introduced the transform in his work on probability theory.

The Laplace transform is related to the Fourier transform, but whereas the Fourier transform resolves a function or signal into its modes of vibration, the Laplace transform resolves a function into its moments. Like the Fourier transform, the Laplace transform is used for solving differential and integral equations. In physics and engineering, it is used for analysis of linear time-invariant systems such as electrical circuits, harmonic oscillators, optical devices, and mechanical systems. In this analysis, the Laplace transform is often interpreted as a transformation from the *time-domain*, in which inputs and outputs are functions of time, to the *frequency-domain*, where the same inputs and outputs are functions of complex angular frequency, in radians per unit time. Given a simple mathematical or functional description of an input or output to a system, the Laplace transform provides an alternative functional description that often simplifies the process of analyzing the behavior of the system, or in synthesizing a new system based on a set of specifications.

LTI
through
time
linear
$x(t)=\delta(t-t_1)$

Contents

History

The Laplace transform is named in honor of mathematician and astronomer Pierre-Simon Laplace, who used the transform in his work on probability theory. From 1744, Leonhard Euler investigated integrals of the form

$$z = \int X(x)e^{ax}\, dx \quad \text{and} \quad z = \int X(x)x^{A}\, dx$$

as solutions of differential equations but did not pursue the matter very far.[2] Joseph Louis Lagrange was an admirer of Euler and, in his work on integrating probability density functions, investigated expressions of the form

$$\int X(x)e^{-ax}a^{x}\, dx,$$

which some modern historians have interpreted within modern Laplace transform theory.[3][4]

These types of integrals seem first to have attracted Laplace's attention in 1782 where he was following in the spirit of Euler in using the integrals themselves as solutions of equations.[5] However, in 1785, Laplace took the critical step forward when, rather than just looking for a solution in the form of an integral, he started to apply the transforms in the sense that was later to become popular. He used an integral of the form:

$$\int x^s \phi(x)\,dx,$$

akin to a Mellin transform, to transform the whole of a difference equation, in order to look for solutions of the transformed equation. He then went on to apply the Laplace transform in the same way and started to derive some of its properties, beginning to appreciate its potential power.[6]

Laplace also recognised that Joseph Fourier's method of Fourier series for solving the diffusion equation could only apply to a limited region of space as the solutions were periodic. In 1809, Laplace applied his transform to find solutions that diffused indefinitely in space.[7]

Formal definition

The Laplace transform of a function $f(t)$, defined for all real numbers $t \geq 0$, is the function $F(s)$, defined by:

$$\boxed{F(s) = \mathcal{L}\left\{f(t)\right\} = \int_0^\infty e^{-st} f(t)\,dt.}$$

The parameter s is a complex number:

$\boxed{s = \sigma + i\omega,}$ with real numbers σ and ω. A + iB

The meaning of the integral depends on types of functions of interest. A necessary condition for existence of the integral is that f must be locally integrable on $[0,\infty)$. For locally integrable functions that decay at infinity or are of exponential type, the integral can be understood as a (proper) Lebesgue integral. However, for many applications it is necessary to regard it as a conditionally convergent improper integral at ∞. Still more generally, the integral can be understood in a weak sense, and this is dealt with below.

One can define the Laplace transform of a finite Borel measure μ by the Lebesgue integral[8]

$$(\mathcal{L}\mu)(s) = \int_{[0,\infty)} e^{-st}\,d\mu(t).$$

An important special case is where μ is a probability measure or, even more specifically, the Dirac delta function. In operational calculus, the Laplace transform of a measure is often treated as though the measure came from a distribution function f. In that case, to avoid potential confusion, one often writes

$$(\mathcal{L}f)(s) = \int_{0^-}^\infty e^{-st} f(t)\,dt$$

where the lower limit of 0^- is short notation to mean

$$\boxed{\lim_{\varepsilon \to 0^+} \int_{-\varepsilon}^\infty .}$$

This limit emphasizes that any point mass located at 0 is entirely captured by the Laplace transform. Although with the Lebesgue integral, it is not necessary to take such a limit, it does appear more naturally in connection with the Laplace–Stieltjes transform.

Probability theory

In pure and applied probability, the Laplace transform is defined by means of an expectation value. If X is a random variable with probability density function f, then the Laplace transform of f is given by the expectation

$$(\mathcal{L}f)(s) = E\left[e^{-sX}\right].$$

By abuse of language, this is referred to as the Laplace transform of the random variable X itself. Replacing s by $-t$ gives the moment generating function of X. The Laplace transform has applications throughout probability theory, including first passage times of stochastic processes such as Markov chains, and renewal theory.

Bilateral Laplace transform

Main article: Two-sided Laplace transform

When one says "the Laplace transform" without qualification, the unilateral or one-sided transform is normally intended. The Laplace transform can be alternatively defined as the *bilateral Laplace transform* or two-sided Laplace transform by extending the limits of integration to be the entire real axis. If that is done the common unilateral transform simply becomes a special case of the bilateral transform where the definition of the function being transformed is multiplied by the Heaviside step function.

The bilateral Laplace transform is defined as follows:

$$F(s) = \mathcal{L}\{f(t)\} = \int_{-\infty}^{\infty} e^{-st} f(t)\, dt.$$

Inverse Laplace transform

For more details on this topic, see Inverse Laplace transform.

The inverse Laplace transform is given by the following complex integral, which is known by various names (the **Bromwich integral**, the **Fourier-Mellin integral**, and **Mellin's inverse formula**):

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi i} \lim_{T\to\infty} \int_{\gamma-iT}^{\gamma+iT} e^{st} F(s)\, ds,$$

where γ is a real number so that the contour path of integration is in the *region of convergence* of $F(s)$. An alternative formula for the inverse Laplace transform is given by Post's inversion formula.

I thought this was just go backwards

Region of convergence

If f is a locally integrable function (or more generally a Borel measure locally of bounded variation), then the Laplace transform $F(s)$ of f converges provided that the limit

$$\lim_{R\to\infty} \int_0^R f(t) e^{-ts}\, dt$$

Convergence approaches some value in the limit

exists. The Laplace transform converges absolutely if the integral

$$\int_0^\infty |f(t)e^{-ts}|\, dt$$

Not or

exists (as a proper Lebesgue integral). The Laplace transform is usually understood as conditionally convergent, meaning that it converges in the former instead of the latter sense.

The set of values for which $F(s)$ converges absolutely is either of the form $\mathrm{Re}\{s\} > a$ or else $\mathrm{Re}\{s\} \geq a$, where a is an extended real constant, $-\infty \leq a \leq \infty$. (This follows from the dominated convergence theorem.) The constant a is known as the abscissa of absolute convergence, and depends on the growth behavior of $f(t)$.[9] Analogously, the two-sided transform converges absolutely in a strip of the form $a < \mathrm{Re}\{s\} < b$, and possibly including the lines $\mathrm{Re}\{s\} = a$ or $\mathrm{Re}\{s\} = b$.[10] The subset of values of s for which the Laplace transform converges absolutely is called the region of absolute convergence or the domain of absolute convergence. In the two-sided case, it is sometimes called the strip of absolute convergence. The Laplace transform is analytic in the region of absolute convergence.

Similarly, the set of values for which $F(s)$ converges (conditionally or absolutely) is known as the region of conditional convergence, or simply the **region of convergence** (ROC). If the Laplace transform converges (conditionally) at $s = s_0$, then it automatically converges for all s with $\mathrm{Re}\{s\} > \mathrm{Re}\{s_0\}$. Therefore the region of convergence is a half-plane of the form $\mathrm{Re}\{s\} > a$, possibly including some points of the boundary line $\mathrm{Re}\{s\} = a$. In the region of convergence $\mathrm{Re}\{s\} > \mathrm{Re}\{s_0\}$, the Laplace transform of f can be expressed by integrating by parts as the integral

$$F(s) = (s - s_0) \int_0^\infty e^{-(s-s_0)t} \beta(t)\, dt, \quad \beta(u) = \int_0^u e^{-s_0 t} f(t)\, dt.$$

That is, in the region of convergence $F(s)$ can effectively be expressed as the absolutely convergent Laplace transform of some other function. In particular, it is analytic.

A variety of theorems, in the form of Paley–Wiener theorems, exist concerning the relationship between the decay properties of f and the properties of the Laplace transform within the region of convergence.

In engineering applications, a function corresponding to a linear time-invariant (LTI) system is *stable* if every bounded input produces a bounded output. This is equivalent to the absolute convergence of the Laplace transform of the impulse response function in the region $\mathrm{Re}\{s\} \geq 0$. As a result, LTI systems are stable provided the poles of the Laplace transform of the impulse response function have negative real part.

Properties and theorems

ah nice

The Laplace transform has a number of properties that make it useful for analyzing linear dynamical systems. The most significant advantage is that differentiation and integration become multiplication and division, respectively, by s (similarly to logarithms changing multiplication of numbers to addition of their logarithms). Because of this property, the Laplace variable s is also known as *operator variable* in the L domain: either *derivative operator* or (for s^{-1}) *integration operator*. The transform turns integral equations and differential equations to polynomial equations, which are much easier to solve. Once solved, use of the inverse Laplace transform reverts back to the time domain.

Given the functions $f(t)$ and $g(t)$, and their respective Laplace transforms $F(s)$ and $G(s)$:

$$f(t) = \mathcal{L}^{-1}\{F(s)\}$$
$$g(t) = \mathcal{L}^{-1}\{G(s)\}$$

the following table is a list of properties of unilateral Laplace transform:[11]

Properties of the unilateral Laplace transform

	Time domain	's' domain	Comment
Linearity	$af(t) + bg(t)$	$aF(s) + bG(s)$	Can be proved using basic rules of integration.
Frequency differentiation	$tf(t)$ Just that multiplication	$-F'(s)$	F' is the first derivative of F.
Frequency differentiation	$t^n f(t)$	$(-1)^n F^{(n)}(s)$	More general form, n^{th} derivative of F(s).
Differentiation	$f'(t)$	$sF(s) - f(0)$	f is assumed to be a differentiable function, and its derivative is assumed to be of exponential type. This can then be obtained by integration by parts
Second Differentiation	$f''(t)$	$s^2F(s) - sf(0) - f'(0)$	f is assumed twice differentiable and the second derivative to be of exponential type. Follows by applying the Differentiation property to $f'(t)$.
General Differentiation	$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$	f is assumed to be n-times differentiable, with n^{th} derivative of exponential type. Follow by mathematical induction.
Frequency integration	$\dfrac{f(t)}{t}$	$\int_s^\infty F(\sigma)\, d\sigma$	
Integration	$\int_0^t f(\tau)\, d\tau = (u * f)(t)$	$\dfrac{1}{s}F(s)$	$u(t)$ is the Heaviside step function. Note $(u * f)(t)$ is the convolution of $u(t)$ and $f(t)$.
Time scaling	$f(at)$	$\dfrac{1}{\|a\|}F\left(\dfrac{s}{a}\right)$	
Frequency shifting	$e^{at}f(t)$	$F(s-a)$	
Time shifting	$f(t-a)u(t-a)$	$e^{-as}F(s)$	$u(t)$ is the Heaviside step function
Multiplication	$f(t)g(t)$	$\dfrac{1}{2\pi i}\lim_{T\to\infty}\int_{c-iT}^{c+iT} F(\sigma)G(s-\sigma)\, d\sigma$	the integration is done along the vertical line $Re(\sigma) = c$ that lies entirely within the region of convergence of F.[12]
Convolution	$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)\, d\tau$	$F(s) \cdot G(s)$	$f(t)$ and $g(t)$ are extended by zero for $t < 0$ in the definition of the convolution.
Complex conjugation	$f^*(t)$	$F^*(s^*)$	
Cross-correlation	$f(t) \star g(t)$	$F^*(-s^*) \cdot G(s)$	
Periodic Function	$f(t)$	$\dfrac{1}{1-e^{-Ts}}\int_0^T e^{-st}f(t)\, dt$	$f(t)$ is a periodic function of period T so that $f(t) = f(t+T),\ \forall t \geq 0$. This is the result of the time shifting property and the geometric series.

- **Initial value theorem**:

$$f(0^+) = \lim_{s\to\infty} sF(s).$$

- **Final value theorem**:

$f(\infty) = \lim_{s\to 0} sF(s)$, if all poles of $sF(s)$ are in the left half-plane.

The final value theorem is useful because it gives the long-term behaviour without having to perform partial fraction decompositions or other difficult algebra. If a function's poles are in the right-hand plane (e.g. e^t or $\sin(t)$) the behaviour of this formula is undefined.

Wikipedia 11/7

Table of selected Laplace transforms

The following table provides Laplace transforms for many common functions of a single variable. For definitions and explanations, see the *Explanatory Notes* at the end of the table.

Because the Laplace transform is a linear operator: ~ linear $f(x+y) = f(x) + f(y)$
$f(ax) = af(x)$ for all a

- The Laplace transform of a sum is the sum of Laplace transforms of each term.

$$\mathcal{L}\{f(t) + g(t)\} = \mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\}$$

- The Laplace transform of a multiple of a function is that multiple times the Laplace transformation of that function.

$$\mathcal{L}\{af(t)\} = a\mathcal{L}\{f(t)\}$$

These two rules

The unilateral Laplace transform takes as input a function whose time domain is the non-negative reals, which is why all of the time domain functions in the table below are multiples of the Heaviside step function, u(t). The entries of the table that involve a time delay τ are required to be causal (meaning that $\tau > 0$). A causal system is a system where the impulse response $h(t)$ is zero for all time t prior to $t = 0$. In general, the region of convergence for causal systems is not the same as that of anticausal systems.

$f(t) = \mathcal{L}^{-1}(F(s))$

zero except for one point (I remember from 6.02)

Functio n	Time domain $f(t) = \mathcal{L}^{-1}\{F($	Laplace s-domain $F(s) = \mathcal{L}\{f(t)\}$	Region of convergence	Refere nce
unit impulse	$\delta(t)$	1	all s	inspect ion
delayed impulse	$\delta(t-\tau)$	$e^{-\tau s}$		time shift of unit impuls e
unit step	$u(t)$	$\frac{1}{s}$	$\mathrm{Re}\{s\} > 0$	integra te unit impuls e
delayed unit step	$u(t-\tau)$	$\frac{e^{-\tau s}}{s}$	$\mathrm{Re}\{s\} > 0$	time shift of unit step

1 afterwards

$u(t) = \int_{-\infty}^{x} \delta(t)\,dt$

ramp	$t \cdot u(t)$	$\frac{1}{s^2}$	$\mathrm{Re}\{s\} > 0$	integrate unit impulse twice
delayed nth power with frequency shift	$\frac{(t-\tau)^n}{n!} e^{-\alpha(t-\tau)}$	$\frac{e^{-\tau s}}{(s+\alpha)^{n+1}}$	$\mathrm{Re}\{s\} > -\alpha$	Integrate unit step, apply frequency shift, apply time shift
nth power (for integer n)	$\frac{t^n}{n!} \cdot u(t)$	$\frac{1}{s^{n+1}}$	$\mathrm{Re}\{s\} > 0$ $(n > -1)$	Integrate unit step n times
qth power (for complex q)	$\frac{t^q}{\Gamma(q+1)} \cdot u(t)$	$\frac{1}{s^{q+1}}$	$\mathrm{Re}\{s\} > 0$ $(\mathrm{Re}\{q\} > -1)$	ref?
nth power with frequency shift	$\frac{t^n}{n!} e^{-\alpha t} \cdot u(t)$	$\frac{1}{(s+\alpha)^{n+1}}$	$\mathrm{Re}\{s\} > -\alpha$	Integrate unit step, apply frequency shift
exponential decay	$e^{-\alpha t} \cdot u(t)$	$\frac{1}{s+\alpha}$	$\mathrm{Re}\{s\} > -\alpha$	Frequency shift of unit step
exponential approach	$(1-e^{-\alpha t}) \cdot u(t)$	$\frac{\alpha}{s(s+\alpha)}$	$\mathrm{Re}\{s\} > 0$	Unit step minus exponential decay

~~Who~~ what does unit input do here?

sine	$\sin(\omega t) \cdot u(t)$	$\dfrac{\omega}{s^2 + \omega^2}$	$\operatorname{Re}\{s\} > 0$	ref?
cosine	$\cos(\omega t) \cdot u(t)$	$\dfrac{s}{s^2 + \omega^2}$	$\operatorname{Re}\{s\} > 0$	ref?
hyperbolic sine	$\sinh(\alpha t) \cdot u(t)$	$\dfrac{\alpha}{s^2 - \alpha^2}$	$\operatorname{Re}\{s\} > \|\alpha\|$	ref?
hyperbolic cosine	$\cosh(\alpha t) \cdot u(t)$	$\dfrac{s}{s^2 - \alpha^2}$	$\operatorname{Re}\{s\} > \|\alpha\|$	ref?
Exponentially-decaying sine wave	$e^{-\alpha t} \sin(\omega t) \cdot u($	$\dfrac{\omega}{(s+\alpha)^2 + \omega^2}$	$\operatorname{Re}\{s\} > -\alpha$	ref?
Exponentially-decaying cosine wave	$e^{-\alpha t} \cos(\omega t) \cdot u($	$\dfrac{s+\alpha}{(s+\alpha)^2 + \omega^2}$	$\operatorname{Re}\{s\} > -\alpha$	ref?
*n*th root	$\sqrt[n]{t} \cdot u(t)$	$s^{-(n+1)/n} \cdot \Gamma\left(1 + \frac{1}{n}\right)$	$\operatorname{Re}\{s\} > 0$	ref?
natural logarithm	$\ln\left(\frac{t}{t_0}\right) \cdot u(t)$	$-\frac{t_0}{s}\left[\ln(t_0 s) + \gamma\right]$	$\operatorname{Re}\{s\} > 0$	ref?
Bessel function of the first kind, of order *n*	$J_n(\omega t) \cdot u(t)$	$\dfrac{\omega^n \left(s + \sqrt{s^2 + \omega^2}\right)^{-n}}{\sqrt{s^2 + \omega^2}}$	$\operatorname{Re}\{s\} > 0$ $(n > -1)$	ref?
Modified Bessel function of the first kind, of order *n*	$I_n(\omega t) \cdot u(t)$	$\dfrac{\omega^n \left(s + \sqrt{s^2 - \omega^2}\right)^{-n}}{\sqrt{s^2 - \omega^2}}$	$\operatorname{Re}\{s\} > \|\omega\|$	ref?
Bessel function of the second kind,	$Y_0(\alpha t) \cdot u(t)$	$-\dfrac{2 \sinh^{-1}(s/\alpha)}{\pi \sqrt{s^2 + \alpha^2}}$	$\operatorname{Re}\{s\} > 0$	ref?

of order 0				
Modified Bessel function of the second kind, of order 0	$K_0(\alpha t) \cdot u(t)$			ref?
Confluent hypergeometric function	$e^{-t^2} t^{2c-2} M(a, c,$	$2^{1-2c}\Gamma(2c-1) U \left(c - \frac{1}{2}, a + \frac{1}{2}\right.$	$\mathrm{Re}\, c > \frac{1}{2}$, Re	ref?
Incomplete Gamma function	$e^{-t^2} \frac{\gamma(\alpha, -t^2)}{(-1)^\alpha}$	$\frac{\Gamma(2\alpha)}{4^\alpha} e^{\frac{s^2}{4}} \Gamma\left(\frac{1}{2} - \alpha, \frac{s^2}{4}\right)$	$\mathrm{Re}\, c > \frac{1}{2}$, Re	ref?
Error function	$\mathrm{erf}(t) \cdot u(t)$	$\frac{e^{s^2/4}\left(1 - \mathrm{erf}\left(s/2\right)\right)}{s}$	$\mathrm{Re}\{s\} > 0$	ref?
Rational function in s	$\sum_{i=1}^{l} \sum_{j=1}^{k_i} \frac{a_{ij}}{(j-1)!} t$	$F(s) = \frac{N(s)}{(s-\lambda_1)^{k_1} \cdots (s-\lambda_l)^{k}}$ $a_{ij} = \frac{1}{(k_i - j)!} \lim_{s \to \lambda_i} \frac{d^{k_i-j}}{ds^{k_i-j}} \left((s\right.$	$\mathrm{Re}\{s\} > max$	Partial fraction expansion and nth power with frequency shift

Explanatory notes:

- $u(t)$ represents the Heaviside step function.
- $\delta(t)$ represents the Dirac delta function.
- $\Gamma(z)$ represents the Gamma function.
- γ is the Euler–Mascheroni constant.
- t, a real number, typically represents *time*, although it can represent *any* independent dimension.
- s is the complex angular frequency, and $\mathrm{Re}\{s\}$ is its real part. a+bi
- α, β, τ, and ω are real numbers.
- *n* is an integer.

WTF is it flat?

LT. Laplace Transform

1. Translation formula. The usual L.T. formula for translation on the t-axis is

(1) $$\mathcal{L}\big(u(t-a)f(t-a)\big) \;=\; e^{-as}F(s), \qquad \text{where } F(s)=\mathcal{L}\big(f(t)\big)\,, \quad a>0.$$

This formula is useful for computing the inverse Laplace transform of $e^{-as}F(s)$, for example. On the other hand, as written above it is not immediately applicable to computing the L.T. of functions having the form $u(t-a)f(t)$. For this you should use instead this form of (1):

recognize steps

(2) $$\mathcal{L}\big(u(t-a)f(t)\big) \;=\; e^{-as}\mathcal{L}\big(f(t+a)\big), \quad a>0.$$

Example 1. Calculate the Laplace transform of $u(t-1)(t^2+2t)$.

ii $(t+1)^2+2(t+1)$

Solution. Here $f(t)=t^2+2t$, so (check this!) $f(t+1)=t^2+4t+3$. So by (2),

$t^2+2t+1+2t+2$

$$\mathcal{L}\big(u(t-1)(t^2+2t)\big) \;=\; e^{-s}\mathcal{L}(t^2+4t+3) \;=\; e^{-s}\left(\frac{2}{s^3}+\frac{4}{s^2}+\frac{3}{s}\right).$$

t^2+4t+3

Oh works

Example 2. Find $\mathcal{L}\big(u(t-\frac{\pi}{2})\sin t\big)$.

Solution.
$$\mathcal{L}\big(u(t-\tfrac{\pi}{2})\sin t\big) \;=\; e^{-\pi s/2}\mathcal{L}\big(\sin(t+\tfrac{\pi}{2})\big)$$
$$= e^{-\pi s/2}\mathcal{L}(\cos t) \;=\; e^{-\pi s/2}\frac{s}{s^2+1}.$$

Proof of formula (2). According to (1), for any $g(t)$ we have
$$\mathcal{L}\big(u(t-a)g(t-a)\big) \;=\; e^{-as}\mathcal{L}\big(g(t)\big);$$
this says that to get the factor on the right side involving g, we should replace $t-a$ by t in the function $g(t-a)$ on the left, and then take its Laplace transform.

Apply this procedure to the function $f(t)$, written in the form $f(t)=f((t-a)+a)$; we get ("replacing $t-a$ by t and then taking the Laplace Transform")
$$\mathcal{L}\big(u(t-a)f((t-a)+a)\big) \;=\; e^{-as}\mathcal{L}\big(f(t+a)\big),$$
exactly the formula (2) that we wanted to prove. □

Exercises. Find: a) $\mathcal{L}\big(u(t-a)e^t\big)$ b) $\mathcal{L}\big(u(t-\pi)\cos t\big)$ c) $\mathcal{L}\big(u(t-2)te^{-t}\big)$

Solutions. a) $e^{-as}\dfrac{e^a}{s-1}$ b) $-e^{-\pi s}\dfrac{s}{s^2+1}$ c) $e^{-2s}\dfrac{e^{-2}(2s+3)}{(s+1)^2}$

2. The transfer function and Green's function.

If we use the Laplace transform to solve the IVP

$$y'' + ay' + by \;=\; r(t), \qquad y(0) = 0,\; y'(0) = 0,$$

the transform of the IVP, with the usual notation, is

$$s^2Y + asY + bY \;=\; R(s)\;;$$

whose solution for $Y = \mathcal{L}^{-1}(y)$ is

$$Y \;=\; R(s)\frac{1}{s^2 + as + b}\;;$$

using the convolution operator to take the inverse transform, we get as the solution (further down the function $w(t)$ is defined):

$$y \;=\; r(t) * w(t) \;=\; \int_0^t r(u)w(t-u)\,du\;. \tag{3}$$

In this form of the solution, the following terminology is often used. Let $p(D) = D^2 + aD + b$ be the differential operator; then we write

$$\begin{aligned} W(s) &= \frac{1}{s^2 + as + b} && \text{the } \textbf{transfer function} \text{ for } p(D),\\ w(t) &= \mathcal{L}^{-1}(W(s)) && \text{the } \textbf{weight function} \text{ for } p(D),\\ G(t,u) &= w(t-u) && \text{the } \textbf{Green's function} \text{ for } p(D). \end{aligned}$$

The important thing to note is that each of these depends only on the operator, not on the forcing function $r(t)$; once they are calculated, the solution (3) to the IVP can be written down immediately as the definite integral there, and used for a variety of different $r(t)$.

The weight function $w(t)$ can be thought of as the unique solution to the IVP

$$y'' + ay' + by \;=\; 0; \qquad y(0) = 0,\;\; y'(0) = 1; \tag{4}$$

or as the solution to the IVP

$$y'' + ay' + by \;=\; \delta(t);\;\; y(0) = 0,\;\; y'(0^-) = 0; \tag{5}$$

in the second equation, $\delta(t)$ is the Dirac delta function. Both IVP's model (for $a, b > 0$) a damped spring-mass system which is initially at rest, but whose mass is given a unit impulse at time zero, say by a sharp blow.

It is an easy exercise to show that $w(t)$ is the solution to both IVP's. As an example of Green's functions, see the last few Laplace Transform exercises (in Section 3D).

18.03 Recitation

11/8

$$\mathcal{L}(f(t)) = \int_0^\infty f(t) e^{-st}\, dt$$

Warmup

$$\mathcal{L}(0) = \int_0^\infty 0 \cdot e^{-st}\, dt$$

$$= \lim_{A\to\infty} \int_0^A 0\, dt$$

$$= t \Big|_0^A$$

$$= A$$

$$A \to \infty \quad = \cancel{\infty} \quad 0$$

$\mathcal{L}(1)$ = did in class

$$\mathcal{L}(t) = \int_0^\infty t\, e^{-st}\, dt$$

$$= \lim_{A\to\infty} \int_0^A t e^{-st}\, dt$$

$$-\frac{t}{s} e^{-st} \Big|_0^A$$

$e^{-sA} \to 0$ as $A \to \infty$ $\quad = \cancel{-\frac{t}{s}} \quad \frac{1}{s^2}$

③

$$\mathcal{L}(e^t) = \int_0^\infty e^t e^{-st}\, dt$$

$$= \int_0^\infty e^{(1-s)t}\, dt$$

like before

$$\frac{1}{s-1} \checkmark \quad s > 1$$

s is a formal variable
(not a missing constant

$$\mathcal{L}(t^n e^{rt}) = \text{use (A)} \quad \mathcal{L}(e^{at} \cdot f(t)) = F(s-a)$$

$$= \frac{1}{(s-a)-a}$$

(what do we convert again ?
(?) - what does F refer to ?
F of t^n ?

$$= \int_0^\infty t^n e^{-(s-a)t} \quad \leftarrow \text{I believe}$$

(so for $t^n \Rightarrow \frac{1}{s^2}$

$$= \cancel{\frac{1}{(s-a)^2}} \quad \text{instead I believe}$$

$$\frac{n!}{(s-r)^{n+1}}$$

(3)

$\mathcal{L}(e^{-t^2})$ = does not exist

does not converge

$$\mathcal{L}\left(\sum_{n=0}^{\infty} \alpha_n t^n\right) = \quad \mathcal{L} \text{ is linear}$$

α_n ← Constants

$$= \sum \alpha_n \mathcal{L}(t^n)$$

$$= \sum_{n=0}^{\infty} \alpha_n \frac{n!}{s^{n+1}}$$

Also

$$\mathcal{L}(e^t) = \mathcal{L}\left(\sum_{n=0}^{\infty} \frac{t^n}{n!}\right)$$

$$= \sum_{n=0}^{\infty} \frac{1}{\cancel{n!}} \frac{\cancel{n!}}{s^{n+1}}$$

$$= \sum_{n=0}^{\infty} \frac{1}{s^{n+1}}$$

Sum geometric series

$$= \frac{1}{s}\left(\frac{1}{1-\frac{1}{s}}\right)$$

$$= \frac{1}{s-1}$$

Can also expand Power Series + compute Laplace from that

④

$F(s) = \mathcal{L}(f(t))$

$\mathcal{L}(e^{at} f(t)) = F(s-a)$

$\mathcal{L}(t\, f(t)) = F'(s)$

$\mathcal{L}(f') = sF(s) - f(0)$

$\mathcal{L}^{-1}(F(s))$ means go and find fn $f(t)$ that $\mathcal{L}(f(t)) = F(s)$

- may not be any
- or may be a lot
- we don't care when or if unique solution exists

in this class - only working w/ nice values

Now do this

$$\frac{t^n}{n!} e^{rt} = \mathcal{L}^{-1}\left(\frac{1}{(s-r)^{n+1}}\right)$$

you may not be wanting to take inverse of this

<u>Any</u> function of the form

$$\frac{\text{poly}(s)}{\text{poly}(s)}$$

~~can~~ be

5

Can be expanded as a sum of terms of the form $\frac{C}{(s-r)^4}$ using partial fractions method

---

Now to use this to sole diff eq

- We already know how to solve these problems

want to solve

$x'' + x' + x = f(t)$ ~~want~~ $\underline{IV}$ $x(0)$

$x'(0)$

$\mathcal{L}(x'' + x' + x) = \mathcal{L}(f(t))$

$\mathcal{L}(x'') + \mathcal{L}(x') + \mathcal{L}(x) = \mathcal{L}(f(t))$

$\mathcal{L}(x') = s\,\mathcal{L}(x)(s) - x(0)$

$\mathcal{L}(x'') = s\,\mathcal{L}(x')(s) - x'(0)$

plug in

$= s^2\,\mathcal{L}(x)(s) - s\,x(0) - x'(0)$

(6)

LaPlace transform of x - which is a function of S

$$s^2 \mathcal{L}(x)(s) - s\,x(0) - x'(0)$$
$$+ s\,\mathcal{L}(x)(s) - x(0)$$
$$+ \mathcal{L}(x)(s) = \mathcal{L}(f)(s)$$

Put everything besides $\mathcal{L}(x)$ on RHS

$$s^2 \mathcal{L}(x)(s) + s\,\mathcal{L}(x)(s) + \mathcal{L}(x)(s) = \mathcal{L}(f)(s) + x'(0) + (1+s)\,x(0)$$

factor

$$\mathcal{L}(x)(s) = \frac{\mathcal{L}(f)(s) + x'(0) + (1+s)\,x(0)}{1+s+s^2}$$

Do inverse notation

$$x(t) = \mathcal{L}^{-1}\left(\frac{\mathcal{L}(f)(s) + x'(0) + (1+s)\,x(0)}{1+s+s^2}\right)$$

7

Now if f is any function whose Laplace transform is $\frac{\text{poly}(s)}{\text{poly}(s)}$ then can solve by Laplace Transforms the IVP $P(D)x = f(t)$

? we already knew how to solve

Exercise Finish calcs for

1. $f(t) = x(0) = x'(0) = 0$

2. $f(t) = 0 \quad x(0) = 2 \quad x'(0) = 3$

3. $f(t) = e^t \quad x(0) = 0 \quad x'(0) = 0$

Confused what to do here - don't remember seeing

Fill in IVs

what is this? $f(t) = 0$

$$\mathcal{L}^{-1}\left(\frac{\mathcal{L}(f)(s) + 0 + (1+s)0}{1+s+s^2}\right)$$

? how to factor?

$$= \mathcal{L}^{-1}\left(\frac{\cancel{\mathcal{L}(f)(s)}}{} 0\right)$$

$$= 0$$

8

$$X(t) = \mathcal{L}^{-1}\left(\frac{0 + 3 + (1+s)2}{1+s+s^2}\right)$$ ← flipped

do quadratic forma

¿ factor denom? — roots $\frac{-1 \pm \sqrt{-3}}{2}$

then try to match pattern

$$= \mathcal{L}^{-1}\left(\frac{A}{s-\left(\frac{-1+\sqrt{-3}}{2}\right)} + \frac{B}{s-\left(\frac{-1-\sqrt{-3}}{2}\right)}\right)$$

Cross multiplication

$$= \mathcal{L}^{-1}\left(s(A+B) - A\left(\frac{-1-\sqrt{-3}}{2}\right) - B\left(\frac{-1+\sqrt{-3}}{2}\right)\right)$$

$$= \mathcal{L}^{-1}\left(s(A+B) + \frac{A+B}{2} + \frac{(A-B)\sqrt{-3}}{2}\right)$$

$$= 5 + 2s$$

$$2 = A+B$$

$$5 = \frac{A+B}{2} + \frac{(A-B)\sqrt{-3}}{2}$$

A, B can be complex #s

⋮

Then inverse LaPlace transform

$$= Ae^{\frac{-1+\sqrt{-3}}{2}} + Be^{\left(\frac{-1-\sqrt{-3}}{2}\right)t}$$

want A, B's imag part so that they cancle over whole thing

(4)

#3

$$X(t) = \mathcal{L}^{-1}\left(\frac{\frac{1}{s-1}}{1+s+s^2}\right)$$

$$= \mathcal{L}^{-1}\left(\frac{1}{s^3-1}\right)$$

⋮ figure out on own

$$= \mathcal{L}^{-1}\left(\frac{A}{s-1} + \frac{B}{s-e^{\frac{2\pi i}{3}}} + \frac{C}{s-e^{\frac{2\pi i}{3}}}\right)$$

roots of unity

Heat Eq
Reading

11/8

partial diff eq

dep variable function of ≥ 2 independent variables

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$$

$u(x,t)$

k = constant

$A\phi(x,t)$  $A\phi(x+\Delta x, t)$

x $\qquad$ $x+\Delta x$

$\phi(x,t)$ = heat flux $= -k \frac{\partial u}{\partial x}$

= rate of flow of heat

u_t = time rate of change of temp

ρ = density

c = specific heat = amt of heat (in calories) to $\uparrow$ 1°C in 1 g of material

(2)

So $c\delta v$ calories needed to raise $1\,cm^3$ from 0 to v

Slice has rod length dx

Volume $A\,dx$

heat content $Q(t) = \int_x^{x+\Delta x} c\delta A u(x,t)\,dx$

↑ to raise $[x, x+\Delta x]$ section of rod to temp $u(x,t)$

Heat can only leave/enter at 1 end

$$Q'(t) = kA\left(u_x(x+\Delta x, t) - u_x(x,t)\right)$$

because $R = \frac{dQ}{dt}$

↑ rate of flow of heat

So differentiate + apply mean v t

$$Q'(t) = c\delta A u_t(\bar{x}, t)\,\Delta x$$

So set =

$$c\delta A u_t(\bar{x},t)\,\Delta x = kA\left[u_x(x+\Delta x, t) - u_x(x,t)\right]$$

(3)

So

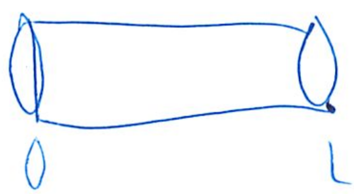
$$u_t(\bar{x},t) = k\,\frac{u_x(x+\Delta x,t) - u_x(x,t)}{\Delta x}$$

where $k = \frac{k}{c\sigma}$ = thermal diffusivity of material

Take limit
$$\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2}$$

So temp must satisfy this diff eq

Boundry Value



apply $u(x,t)$

before w/ ~~ordinary~~ ODE eq had partial constants

Now w/ PDE have arbitrary functions 'partial

~~ff~~ ~~Both~~ 2 ends

Other end $u(x,0) = f(x)$

(4)

$u(0,t) = u(L,t) = 0$

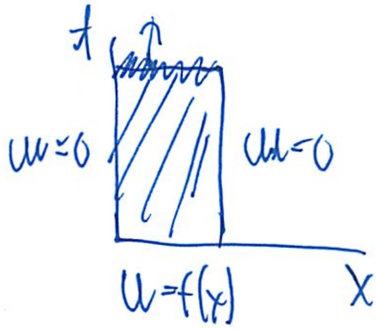
if both ends held againts ice

So BV problem

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$$

$$u(0,t) = u(L,t) = 0$$
$$u(x,0) = f(x)$$

So can draw



So what is u?

Values on region

So can do "heatmap" for u

Must satisfy diff eq inside

And Boundry values on the edge

5

Heat eq is linear

So linear combo

$$U = C_1 U_1 + C_2 U_2$$

homogeneous

So if 3 U fns

add them

18.03 FALL 2011 – Problem Set 7

Due WEDNESDAY 11/09/11, high noon in 2-106

To encourage you to keep up with homework as it appears in lecture, both Part I and Part II problems are listed with the accompanying lecture in which the material will be covered.

Part I (16 points)

Lecture 22. Fri. Nov. 2: The Heat Equation
READ: EP 8.5 HW: EP 8.4: 9, 10

Lecture 23. Mon. Nov. 5: Intro to the Laplace Transform
READ: EP 4.1, Notes H HW: Notes 3A-2, 3bc, EP 4.1: 5, 7, 8, 9.

Lecture 24. Wed. Nov. 7: Solving ODEs with Laplace Transforms
READ: EP 4.2, 4.3 HW: To be assigned on the next pset

Friday, Nov. 9 – No Class (Veteran's Day)

Part II (7 points)

0. (3 points – ~~already tallied in part~~ a) Write the names of all the people you consulted or with whom you collaborated and the resources you used, or say "none" or "no consultation". This includes visits outside recitation to your recitation instructor. If you don't know a name, you must nevertheless identify the person, as in, "tutor in Room 2-102," or "the student next to me in recitation." Optional: note which of these people or resources, if any, were particularly helpful to you.

1. (Monday, 4 pts)

a) Find a formula expressing $\mathcal{L}(f(at))$ (where a is any constant) in terms of $\mathcal{L}(f)$.

b) Confirm your answer in part (a) is correct by checking it for $f(t) = \cos t$ and $f(t) = t^n$ – that is, find the answer a second way.

Michael Plasmeier 18.03 P-Set 7 11/7

-2.5 $\frac{20.5}{23}$

Part 1

Lecture 22 Heat Equation

EP. 8.4 #9 Will pure Resonance occur?

$m=3 \quad k=12$

$F(t)$ is odd, period 2

$F(t)=1 \quad 0<t<1$

(not even heat eq problems!)

For $mx''+kx=B_n \sin \omega_0 t$

$$x(t) = \frac{-B_n}{2m\omega_0} t \cos \omega_0 t$$

If $\omega_0 = \sqrt{\frac{k}{m}}$

$$x(t) = \frac{-B_n}{2m\omega_0} t \cos \omega_0 t$$

So $\omega = \sqrt{\frac{12}{3}} = \sqrt{4} = 2$

$$F(t) = \frac{4}{\pi} \sum_{n \text{ odd}} \frac{\sin n\pi t}{n}$$

? no $\sin 2t$ terms

So no resonance ✓

②

10. $m = 1 \quad k = 4\pi^2$

$F(t)$ odd, period 2

$F(t) = 2t$ for $0 < t < 1$

$$\omega_0 = \sqrt{\frac{4\pi^2}{1}} = \sqrt{4\pi^2} = \sqrt{4}\sqrt{\pi^2} = 2\pi$$

$$\cancel{F(t) = x'' + 4\pi^2 x = N\omega \sin \omega_0 t}$$

$$\cancel{a_0 = \frac{1}{1}\int_{-1}^{1}}$$

$$\cancel{L = \frac{2}{2}}$$

Have formula

$F(t)$ = external force

$\cancel{x(t) = -}$

$m x'' + kx = F(t)$

Basically care if sin term in Fourier series of term

$$a_n = \frac{1}{1}\int_0^2 2t \cos \frac{n\pi t}{1}$$

$$= \frac{2(\pi n t \sin(\pi n t) + \cos(\pi n t))}{\pi^2 n^2}\Bigg|_0^2$$

$$= \frac{2(2\pi n \sin(2\pi n) + \cos(2\pi n) - 1}{\pi^2 n^2}$$

still think integrating wrong

(3)

$$b_n = \frac{1}{1}\int_0^2 2t \sin n\pi t \, dt$$

$$= \frac{2\sin 2\pi n - 2\pi n \cos 2\pi n}{\pi^2 n^2}$$

---

apparently (since $\omega\omega_0 = 2\pi$) there are 2π terms (when $n=1$?) So resonance occurs ✓

(4)

Lecture 23 Laplace Transform

3A-2 Derive the formulas for $\mathcal{L}(e^{\alpha t}\cos bt)$ and $\mathcal{L}(e^{\alpha t}\sin bt)$

by assuming formula $\mathcal{L}(e^{\alpha t}) = \frac{1}{s-\alpha}$

Also $\mathcal{L}(u + iv) = \mathcal{L}(u) + i\,\mathcal{L}(v)$

So $\mathcal{L}(f) = \int_0^\infty f(t)\, e^{-st}\, dt$

for $f(t) = 1$

$$= \lim_{A\to\infty} \int_0^A e^{-st}\, dt$$

$$= \lim_{A\to\infty} \left(-\frac{1}{s} e^{-st}\right)\Big|_{0=t}^{A=t}$$

$$= \frac{1}{s}$$

$e^{-sA} \to 0$ as $A \to \infty$

We have property (A) $\mathcal{L}(e^{at}\cdot f(t)) = F(s-a)$

So $F(s-a)$ means get Laplace transform value and plug in $s-a$ instead?

Know from lecture

$$\mathcal{L}(\cos bt) = \frac{s}{s^2+b^2}$$

5

But we need to confirm that:

$$= \int_0^\infty \cos bt \, e^{-st} \, dt$$

is $Re(e^{bt})$? No

$$\boxed{\begin{aligned} &e^{ix} = \cos x + i \sin x \\ &\text{So } \cos x = Re(e^{ix}) = \frac{e^{ix} + e^{-ix}}{2} \end{aligned}}$$

Can use something like this

$$\cos bt = \frac{e^{ibt} + e^{-ibt}}{2}$$

We know e^{ibt} from class is $\frac{1}{s-ib}$

So $\cos bt = \dfrac{\frac{1}{s-ib} + \frac{1}{s+ib}}{2}$ ← get common denom

$$= \frac{1}{2}\left(\frac{(s+ib)}{(s-ib)(s+ib)} + \frac{(s-ib)}{(s-ib)(s+ib)} \right)$$

$$= \frac{1}{2}\left(\frac{(s+ib) + (s-ib)}{s^2 + ibs - ibs - i^2 b^2} \right)$$

$$= \frac{1}{2} \frac{2s}{s^2+b^2} = \frac{s}{s^2+b^2}$$ ✓ Nice I actually did something right for a change

6

So now use that rule A from class

$$\mathcal{L}(e^{at} f(t)) = F(s-a)$$

laplace fn $\frac{s}{s^2+b^2}$

$$= \frac{(s-a)}{(s-\alpha)^2+b^2}$$

✓ Woot got one right! -did differently then book

$$\mathcal{L}(\sin bt) = \frac{b}{s^2+b^2}$$ ← from textbook $\frac{e^{ix}-e^{-ix}}{2}$

So $F(s-a)$ is

$$= \frac{b}{(s-\alpha)^2+b^2}$$ ✓

Note $\alpha = a$ here

7

3) Find $\mathcal{L}^{-1}(F(s))$ for each

(we're not doing an original?)

b) $\frac{3}{s^2+4}$

So lets look at table

$t \to \frac{1}{s^2}$

$e^{at} \to \frac{1}{s-a}$

$\cos kt \quad \frac{s}{s^2+k^2}$

$\sin kt \quad \frac{k}{s^2+k^2}$

Hmm non are good match

Ahh we can factor out $\frac{3}{2}$

$$\frac{\frac{3}{2} \; 2}{s^2+4}$$

$\frac{3}{2} \mathcal{L}^{-1}\left(\frac{2}{s^2+4^2}\right)$

? matches $\sin 2t$

$\frac{3}{2} \sin 2t$ ✓ tricky

✓

(8)

c) $\frac{1}{s^2-4}$ ← partial fraction decomposition

$1 = \frac{A}{s+2} + \frac{B}{s-2}$

$1 = A(s-2) + B(s+2)$

$s=-2$ $s=2$

$1 = A(-2-2)$ $1 = B(2+2)$

$A = -\frac{1}{4}$ $B = \frac{1}{4}$

$1 = \frac{-\frac{1}{4}}{s+2} + \frac{\frac{1}{4}}{s-2}$

$\frac{1}{4}\left(\frac{-1}{s+2} + \frac{1}{s-2}\right)$

$\frac{1}{4}\left(-e^{-2t} + e^{2t}\right)$ ✓

(1)

EP. 4.1 #5 $f(t) = \sinh kt$ Find LaPlace transform

$$\sinh x = \frac{e^{x} - e^{-x}}{2}$$

$$= \frac{1}{2}\left(\frac{1}{s-1} - \frac{1}{s--1}\right)$$

$$= \frac{1}{2}\left(\frac{1}{s-1} - \frac{1}{s+1}\right)$$

common denom

$$= \frac{1}{2}\left(\frac{(s+1) - (s-1)}{s^2 - s + s - 1}\right)$$

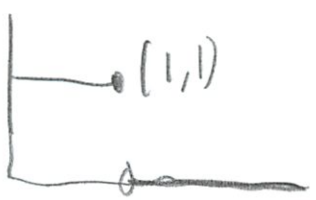
$$= \frac{1}{2}\,\frac{2}{s^2-1}$$

$$= \frac{1}{s^2-1}$$ ✓

✓

Book Don't forget the disclaimer $\boxed{s > 1}$

#7



¿ how do piece wise ?
↳ see book

$1 \to \frac{1}{s}$ for $0 < t < 1$

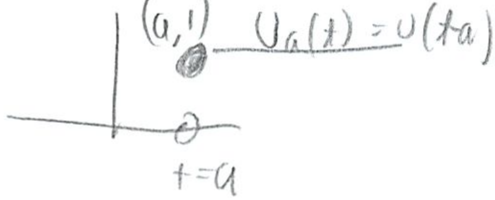
So $\frac{1}{s}$ for $0 < s < 1$?

(11)

Sol $\dfrac{1-e^{-s}}{s}$ $\quad s>0$ ✓

? what pattern does that match?

Book Also other unit step example



$$U_a(t) = u(t-a) = \begin{cases} 0 & \text{for } t<a \\ 1 & \text{for } t>a \end{cases}$$

Transform $\dfrac{e^{-as}}{s}$

? So here $a=0$

So $\dfrac{e^{-0s}}{s}$ so $\dfrac{1}{s}$ – what I had said

Or $\dfrac{1}{s}$ – going back down?
at $a=1$
but turn at 0

Ok I see 2 just subtract

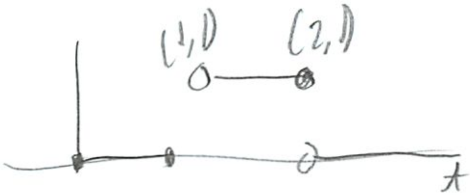
$$\frac{1}{s} - \frac{e^{-1s}}{s}$$

initial up — minus raised part

Makes sense

(11)

8.



Ok this should be similar to last one

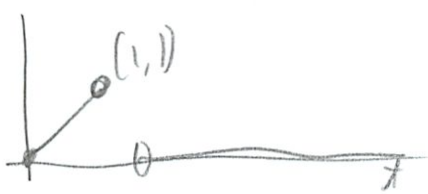
$$= \frac{e^{-1s}}{s} - \frac{e^{-2s}}{s}$$

$$= \frac{e^{-s} - e^{-2s}}{s} \qquad s > 0$$

don't forget ✓

This is kinda fun ...

9.



$$t \to \frac{1}{s^2}$$

$$\frac{1}{s^2} - \frac{e^{-s}}{s} \qquad s > 0 \qquad \leftarrow \quad -.5$$

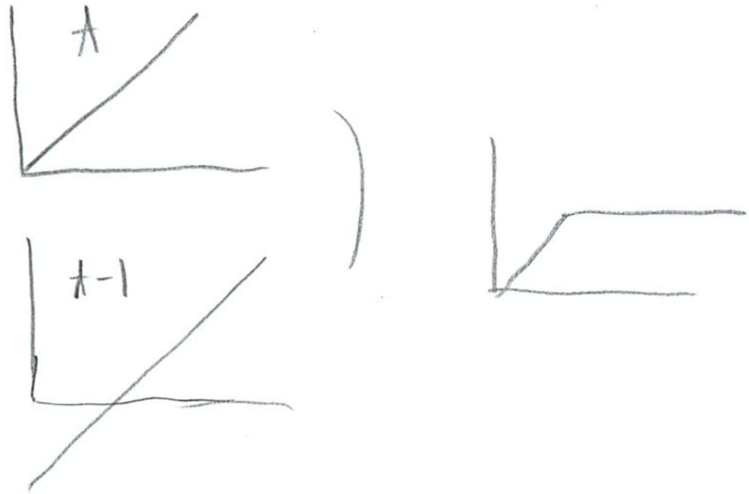
Sols Nope

$$\frac{1 - e^{-s} - se^{-s}}{s^2} \qquad s > 0$$

what is this part $\frac{e^{-s}}{s^2}$

(12)

Oh need to take off slope — righhh tttt...



Then need

(13)

Part 2

0. No one yet

Note there is no part a here

1.a. Find a formula expressing $\mathcal{L}(f(at))$ any constant in terms of $\mathcal{L}(f)$

So
$$\mathcal{L}(f) = \int_0^\infty f(t)\, e^{-st}\, dt$$

function

$$\mathcal{L}(f(at)) = \int_0^\infty f(at)\, e^{-st}\, dt$$

$$= a \int_0^\infty f(t)\, e^{-st}\, dt$$

can bring constant out (linear)

$$= a\, \mathcal{L}(f(t))$$

(I think I got it - actually understand this chap!)

X −2

(14)

b) Confirm a) by Checking it for

$$f(t) = \cos t$$
$$f(t) = t^n$$

i So

$$\mathcal{L}(\cos at) = \int_0^\infty \cos at \; e^{-st} \, dt$$

but split

$$\cos at = \frac{1}{2}\left(e^{iat} + e^{-iat}\right)$$

$$= \frac{1}{2}\left(\frac{1}{s-ia} + \frac{1}{s+ia}\right)$$

$$= \frac{1}{2}\left(\frac{(s+ia)+(s-ia)}{s^2+ias-ias-i^2a^2}\right)$$

$$= \frac{1}{2}\frac{2s}{s^2+a^2}$$

$$= \frac{s}{s^2+a^2}$$

is same as

$a \, \mathcal{L}(\cos 1t)$

(15)

So

$$a \cdot \frac{s^2}{s^2+1^2} \stackrel{?}{=} \frac{s}{s^2+a^2}$$

I don't think thats right ...

Could only take a out of f(t) if constant

$\therefore f(a)\,\mathcal{L}(f(t))$

can't always split $f(at) \rightarrow f(a)f(t)$

Online is "time scaling" property

$$f(at) \rightarrow \frac{1}{|a|} F\left(\frac{s}{a}\right)$$

Try to verify w/ cos

using property original

$$\frac{1}{|a|} \frac{\left(\frac{s}{a}\right)}{\left(\frac{s}{a}\right)^2+a^2} \stackrel{?}{=} \frac{s}{s^2+a^2}$$

$$= \frac{1}{|a|} \frac{\frac{s}{a}}{\frac{s^2}{a^2}+a^2}$$

$$= \frac{1}{|a|} \frac{s}{a\left(\frac{s^2}{a^2}+a^2\right)}$$

can only be positive well say

16

$$= \frac{s}{a^2\left(\frac{s^2}{a^2} + a^2\right)}$$

$$= \frac{s}{a^4 + s^2}$$

hmm not quite ....

Oh proof of time scaling

$$= \int_0^\infty f(at)\, e^{-st}\, dt$$

Online site

Multiply $-st$ in exponential by $\frac{a}{a} = 1$

Rewrite $(st)\frac{a}{a}$ as $\left(\frac{s}{a}\right)at$

Rewrite $\frac{a}{a}dt = \frac{1}{a}\, adt$

a is constant $\rightarrow$ take $\frac{1}{a}$ outside

└ write $a\,dt = d(at)$

$$= \frac{1}{a} \int_0^\infty e^{-\frac{s}{a}(at)}\, f(at)\, d(at)$$

Let $t' = at$

$dt' = d(at)$

When $t = 0$ $v = t'$ as $t \rightarrow \infty$

t' goes to ∞

So $t' \rightarrow f(at) = f(t')$

$$= \frac{1}{a} \int_0^\infty e^{(s/a)t'}\, f(t)\, dt'$$ So $\frac{1}{a} F\left(\frac{s}{a}\right)$

(17)

Try t^n

So t^n normally is $\frac{n!}{s^{n+1}}$ $n \geq 0$, $s > 0$

Modified w/ "time scale"

$$= \frac{1}{a} \frac{n!}{\left(\frac{s}{a}\right)^{n+1}} \stackrel{?}{=} \frac{n!}{s^{n+1}}$$

? is this the right question?

$$= \frac{1}{a} \frac{n!}{\frac{s^{n+1}}{a^{n+1}}}$$

$$= \frac{n!}{\frac{s^{n+1}}{a^n}}$$

$$= \frac{n!\,a^n}{s^{n+1}}$$ ← not quite

← but again is this the right qu?

Pset 7 Solutions (Part II)

1

(a) Let $F(s) = \mathcal{L}(f(t)) = \int_0^\infty f(t)e^{-st}\,dt$

by definition.

Let a be a constant.

$$\mathcal{L}(f(at)) = \int_{t=0}^{t=\infty} f(at)e^{-st}\,dt$$

Let $u = at$
So $du = a\,dt$.

$$= \int_{u=0}^{u=\infty} f(u)e^{-su/a}\,\frac{1}{a}\,du = \frac{1}{a}F\left(\frac{s}{a}\right).$$

(Note that this only works for $a > 0$)

Full marks for writing the computation above.

(b) $\mathcal{L}(t^n) = \dfrac{1}{s^{n+1}}$. So $\mathcal{L}((at)^n) = a^n \mathcal{L}(t^n)$

$$= a^n \frac{1}{s^{n+1}}.$$

Formula from (a) says

$$\mathcal{L}((at)^n) = \frac{1}{a} \cdot \frac{1}{\left(\frac{s}{a}\right)^{n+1}} = \frac{a^{n+1}}{a} \frac{1}{s^{n+1}} = a^n \cdot \frac{1}{s^{n+1}}$$

$\checkmark$

$$\mathcal{L}(\cos at) = \mathrm{Re}\, \mathcal{L}(e^{ait}) = \mathrm{Re} \int_0^\infty e^{-(s-ia)t}\, dt$$

$$= \mathrm{Re}\left[\frac{1}{-(s-ia)} e^{-(s-ia)t}\right]_0^\infty = \mathrm{Re}\, \frac{1}{(s-ia)} = \frac{s}{s^2+a^2}$$

$$\mathcal{L}(\cos t) = \frac{s}{s^2+1}.$$

$\checkmark$

Formula from (a) says

$$\mathcal{L}(\cos at) = \frac{1}{a} \cdot \frac{\frac{s}{a}}{\left(\frac{s}{a}\right)^2 + 1} = \frac{\frac{s}{a}}{\frac{s^2}{a} + a} = \frac{s}{s^2+a^2}$$

18.03 OH

11/4

8.4 10

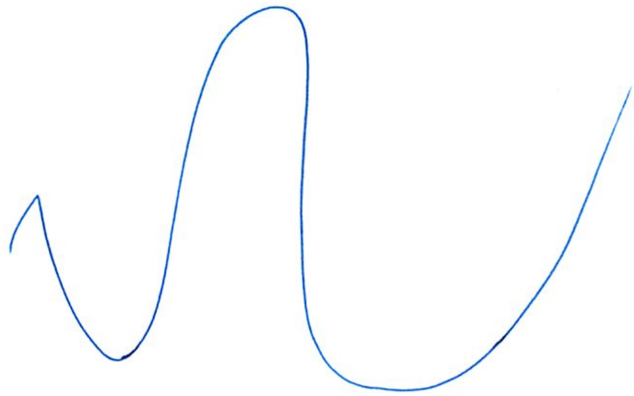
$m\ddot{x} + kx = F(t)$

$P\cos t\sqrt{\frac{k}{m}} + Q\sin t\sqrt{\frac{k}{m}}$

when repeated roots

$\mathrm{Re}\left(\underbrace{Ae^{i\sqrt{k/m}\,t} + Be^{-i\sqrt{k/m}\,t}}_{\text{can be anything}} + C(t)e^{i\sqrt{k/m}\,t} + D(t)e^{-i\sqrt{k/m}\,t}\right.$

specific C, D



resonance b/c added t

②

$$m\ddot{x} + kx = F(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}t\right) + b_n \sin\left(\frac{n\pi}{L}t\right)$$

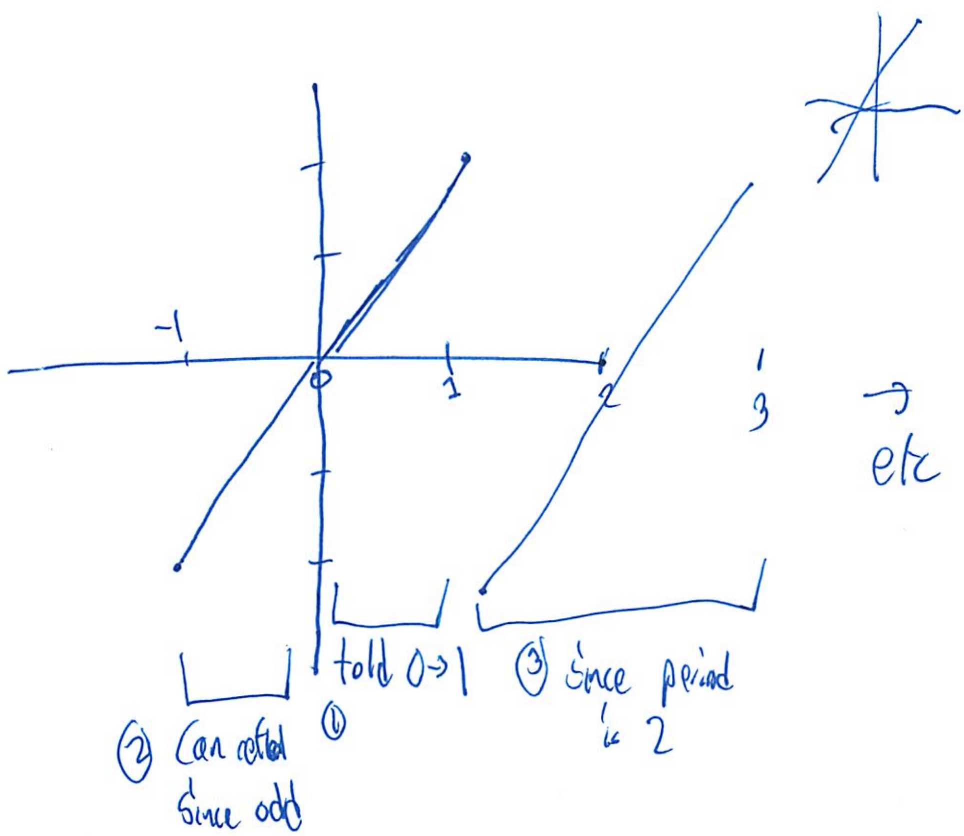
resonance when

① $\frac{n\pi}{L} = \sqrt{k/m}$ for some n

② & a_n or $b_n \neq 0$.

$m=1,\ k=4\pi^2 \rightsquigarrow \sqrt{k/m} = 2\pi$.

$F(t)$ odd fn period 2 $F(t)=2t$ $0<t<1$



(3) $L = 1$

$$\cancel{a_n = \frac{1}{L}\int_{-1}^{1} \cos\left(\frac{n\pi}{L}t\right)\cdot 2t \, dt}$$

Since even · odd = odd

$\int_{-1}^{1} \text{odd} = 0$

$$b_n = \int_{-1}^{1} \sin\left(\frac{n\pi}{L}t\right) \underbrace{2t}_{u} \, dt$$

also Fourier ~~Transf~~ Expansion (odd) only has b_n terms

Fourier Expansion (even) only a_n terms

Can also have neither odd nor even
↳ must do both $\int$

$$\int u\,dv = uv - \int v\,du$$

$d(uv) = u\,dv + v\,du$

$\int d(uv) = \int u\,dv + \int v\,du$

$= uv$

Want to get rid of $2t$
since can $\int \sin$

$u = 2t \qquad dv = \sin\left(\frac{n\pi t}{L}\right)$

$du = 2 \qquad v = \dfrac{-\cos\left(\frac{n\pi t}{L}\right)}{\frac{n\pi}{L}} = -L\dfrac{\cos\left(\frac{n\pi t}{L}\right)}{n\pi}$

(4)

$$b_n = 2t \cdot \frac{L\cos\left(\frac{n\pi t}{L}\right)}{n\pi}\Bigg]_{-1}^{1} + \int_{-1}^{1} \frac{L\cos\left(\frac{n\pi x}{L}\right)}{n\pi} \cdot 2$$

Can just do
No ∫ by parts

$$= 2t \cdot \frac{L\cos\left(\frac{n\pi t}{L}\right)}{n\pi}\Bigg]_{-1}^{1} + \frac{2L^2}{n^2\pi^2}\sin\left(\frac{n\pi t}{L}\right)\Bigg|_{-1}^{1}$$

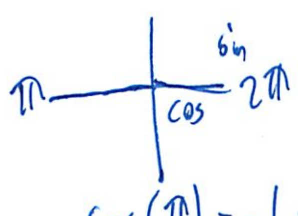
factors comp
again so 2

$$= \frac{2(1)\cdot 1\cos\left(\frac{n\pi(1)}{1}\right)}{n\pi} - \left(\frac{2(-1)\cdot(1)\cos\left(\frac{n\pi(-1)}{1}\right)}{n\pi}\right)$$

$$+ \frac{2(1)^2}{n^2\pi^2}\sin\left(\frac{n\pi 1}{1}\right) - \frac{2(1)^2}{n^2\pi^2}\sin\left(\frac{n\pi -1}{1}\right)$$

$$= \frac{2\cos(n\pi)}{n\pi} + \frac{2\cos(-n\pi)}{n\pi} + \frac{2\sin(n\pi)}{n^2\pi^2} - \frac{2\sin(-n\pi)}{n^2\pi^2}$$

Can simplify further
n must be (+) integer



$\cos(\pi) = -1 \Big\} (-1)^n$ say $\quad \sin(\pi) = 0$

$\cos(2\pi) = 1 \qquad \sin(2\pi) = 0$

← what WA was not doing

(5)

$$= \frac{4(-1)^n}{n\pi} + \sigma\, r()$$

As simple as can go

$$F(t) = \sum_{n=1}^{\infty} \frac{4}{n\pi}(-1)^n \sin\left(\frac{n\pi t}{L}\right)$$

$\|$

$x'' + 4\pi^2 x$

is this ever $\sqrt{k/m} = 2\pi$

yes $n=2$.

and this coefficent is <u>not</u> 0 when $n=2$

Resonance does occur (✓) Done

So basically

1. Memorize the shortcuts
2. Integrate by parts correctly
3. know $\sin(0)$ $\sin(\pi)$ etc
   Lor figure them out
4. Not make mistakes

and was good at Laplace on hw — so think I am set up well

6

Γ = gamma

means the factoral

└ but for fraction things - not just integers

don't worry about

$\Gamma(a+1)$ when a is integer is ~~(a+1)!~~ $a!$

└ +1 disappears - how gamma function works

Not caring it

18.03 Lecture 24

11/9

Solving ODEs w/ Laplace
Delta function

Table

f	$\mathcal{L}(f)$
e^{at}	$1/{s-a}$
$\cos bt$	$\frac{s}{s^2+b^2}$
$\sin bt$	$\frac{b}{s^2+b^2}$
t^n	$\frac{n!}{s^{n+1}}$
$e^{at}f$	$F(s-a)$
tf	$-F'(s)$
f'	$sF(s)-f(0)$
1	$\frac{1}{s}$

$te^{at} = \frac{1}{(s-a)^2}$

$\frac{t^2}{2}e^{at} = \frac{1}{(s-a)^3}$

$x = \frac{1}{s^2}$

where $F(s) = \mathcal{L}(f)$

[illegible]

$$\mathcal{L}(f'') = \mathcal{L}((f')') = s(\mathcal{L}(f')) - f'(0)$$

use property twice
keep track of order

$$= s(sF(s) - f(0)) - f'(0)$$

$$= s^2F(s) - sf(0) - f'(0)$$

nth derivative

$$\mathcal{L}(f^{(n)}) = s^nF(s) - s^{n-1}f(0) - \dots - f^{(n-1)}(0)$$

original fn

(2)

notice terms always smaller

LaPlace transform turns differentiation into multiplication by s,

---

Now apply LaPlace to ODEs we can already solve

$y'' - 4y' + 3y = e^{3t}$ $y(0)=1$ $y'(0)=2$

expected (using previous method) $r^2 - 4r + 3 = (r-3)(r-1)$

$C_1 e^{3t} + C_2 e^{t} \Rightarrow$ homogenous sol

Guess for $y_p(t)$: would do e^{3t}

but already in homogenous

So bump up

$= C_3 t e^{3t}$

Solve C_3

Then get C_2, C_1

Lots of solving of systems of eq way

(3)

LaPlace way

LaPlace transform both sides

$$\mathcal{L}(y''-4y'+3y) = \mathcal{L}(y'')-4\mathcal{L}(y')+3\mathcal{L}(y)$$

$$\mathcal{L}(y'') = s^2\mathcal{L}(y) - s\,y(0) - y'(0)$$

need to include inital conditions

$$= s^2\mathcal{L}(y) - \underset{y(0)}{s} - \underset{y'(0)}{2}$$

$$\mathcal{L}(y') = s\mathcal{L}(y) - \underset{y(0)}{1}$$

where get that last lecture

LHS ← group togeter everything

$$(s^2-4s+3)\mathcal{L}(y) \quad -s-2+4$$

characteristic polynomial pops out in front

polynomial evaluated at s

RHS ← Laplace trasform (e^{3t})

$$\mathcal{L}(e^{3t}) = \frac{1}{s-3}$$

So $$\mathcal{L}(y) = \frac{\frac{1}{s-3}+s-2}{s^2-4s+3}$$

④

Not a proper fraction

$$= \frac{1 + (s-2)(s-3)}{(s-3)^2(s-1)}$$

Now partial fractions

expand

$$= \frac{s^2 - 5s + 7}{(s-3)^2(s-1)}$$

$$Y = \mathcal{L}^{-1}\left(\frac{s^2-5s+7}{(s-3)^2(s-1)}\right)$$

Heavysides (Quick Quiz 1)

$$\frac{A}{(s-3)^2} + \frac{B}{s-3} + \frac{C}{s-1}$$

A: Clear $(s-3)^2$ denominar

w/ $s=3$

~~Heaviside~~

A drops at

evaluate left w/ $s=3$

$A = \frac{1}{2}$

B: Can't be found w/ coverup method

(5)

Can find A, C w/ heavysides

$A = 1/2$

$C = \frac{3}{4}$

Can't find B w/ coverup

Pick any $s \neq 3, 1$ to find B

Can plug in A, C

└ some choices easier than others

$\underline{s=2}$ $\quad 1 = \frac{1}{2} + \frac{B}{-1} + \frac{3}{4}$

$B = \frac{1}{4}$

So answer

$$y = \mathcal{L}^{-1}\left(\frac{1/2}{(s-3)^2} + \frac{1/4}{s-3} + \frac{3/4}{s-1} \right)$$

Choose fn that is sum of 3 things

$\underset{\text{need}}{\underline{\qquad\qquad}} + \frac{1}{4}e^{3t} + \frac{3}{4}e^{t}$

[illegible]

(6)

know $(s-2)^2$ is derivitive

known $\mathcal{L}(e^{3t}) = \frac{1}{s-3}$

So $\mathcal{L}(t e^{3t}) = -\left(\frac{1}{s-3}\right)'$

$= + \left(\frac{1}{s-3}\right)^2$

So first term can be
$\frac{1}{2} t e^{3t}$

↑ using like rule (B) or (C) ?

So solution

$y = \frac{1}{2} t e^{3t} + \frac{1}{4} e^{3t} + \frac{3}{4} e^{t}$

Can realize can just move charastic eq up front
don't need to do all of the algebra

General formula

$p(D) y = f(t)$

$\mathcal{L}[p(D) y] = \mathcal{L}(f)$

$p(s) \mathcal{L}(y) - \cancel{}\, r(s)$ ← junk note $\deg r(s) < \deg(p)$

↑ it was prof that called it junk

↑ charastic eq - eval at s

7

So general solution

$$\mathcal{L}(y) = \frac{\mathcal{L}(f) + r(s)}{p(s)}$$

$$y = \mathcal{L}^{-1}\left(\frac{\mathcal{L}(f) + r(s)}{p(s)}\right)$$

? So just need to do some partial fractions

So now can do every ODE we already know

See notes about statements of uniqueness

Reverse LaPlace transform might not work

Will consider them unique here

Why LaPlace transform?

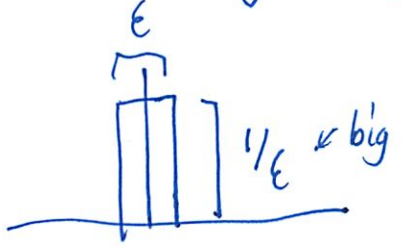
1 - beautiful algebra associated with
- a multiplication in LaPlace world = convolution (?correct)

2 - we can handle impulse functions
- idealize them using a delta function

(8)

Impulse function

- very large near $t=0$
- very small away from $t=0$
- total integral 1



ϵ is very small #

└ like a skyscraper

delta function is idealized, generalized function

does not actually exist

$\delta(t)$ = Delta function

$$\int_a^b f(t)\,\delta(t)\,dt = f(0)$$

provided $a<0<b$

(if f continous at 0)

(9)

Often convention

$$\int_{0^-}^{1} \delta(t)\, dt = 1$$

? include impulse fn

$$\int_{0^+}^{1} \delta(t)\, dt = 0$$

? don't include impulse fn

We will want to include delta function

So rethink LaPlace transform

$$\mathcal{L}(f) = \int_{0^-}^{\infty} f(t)\, e^{-st}\, dt$$

approaching from left

$$\mathcal{L}(\delta) = \int_{0^-}^{\infty} \delta(t)\, e^{-st}\, dt$$

include 0 — So value of fn at 0

$$= e^{-st}\Big|_{0=t}$$

$$= 1$$ very important LaPlace

(10)

Multiplicative identity in Convolution World

Take deriv of δ fn
$\downarrow$

Guess: $\mathcal{L}(\delta') = s$

How are we going to differentiate a fn that does not exist!

Integration by parts

↳ allows us to define δ'

$$\mathcal{L}(\delta') = \int_{0^-}^{\infty} \delta'(t) e^{-st}\, dt$$

$$\stackrel{\text{define}}{=} \int_{0^-}^{\infty} \delta(t) \frac{d}{dt}\left(e^{-st}\right) dt + \text{another term goes to } 0$$

$\uparrow$ antideriv $\uparrow$ deriv

$\downarrow \int_{-\infty}^{0} (?!)$

another term goes to 0 since $e^{-st} \to 0$ as $t \to \infty$

$$= s \int_{0^-}^{\infty} \delta(t) e^{-st}\, dt$$

$$= s \cdot 1$$

$$= s$$

(11)

Can check

$$\int_a^b \delta'(t)\, f(t)\, dt = f'(0)$$ ← magical functional picks off first deriv at 0

if $a < 0 < b$

Can you solve diff eq w/ delta fn?

$$\frac{d^2 y}{dt^2} + 2\frac{dy}{dt} + y = e^{-t} + 3\delta(t-1)$$

Math 18.03 : Differential Equations

Lecture 24 Supplemental Notes

Wednesday, November 9, 2011

Quick Quiz

Using the Heaviside cover-up method, we find that the coefficient A in the partial fractions expansion:

$$\frac{s^2 - 5s + 7}{(s-3)^2(s-1)} = \frac{A}{(s-3)^2} + \frac{B}{s-3} + \frac{C}{s-1}$$

is

1. 0
2. 1/2
3. 3/4
4. 3/2

11/9

Quick Answer

Using the Heaviside cover-up method, we find that the coefficient A in the partial fractions expansion:

$$\frac{s^2-5s+7}{(s-3)^2(s-1)} = \frac{A}{(s-3)^2} + \frac{B}{s-3} + \frac{C}{s-1}$$

is

1. 0
2. 1/2
3. 3/4
4. 3/2

The value of C is computed by clearing $(s-3)^2$ on both sides:

$$\frac{s^2-5s+7}{(s-1)} = A + B(s-3) + \frac{C(s-3)^2}{s-1}$$

and evaluating both sides at $s=3$ gives $C = 1/2$.

Lecture 29

11/9

Specific functions

f	$\mathcal{L}(f)$
e^{at}	$1/s-a$
$\cos bt$	s/s^2+b^2
$\sin bt$	b/s^2+b^2
t^n	$n!/s^{n+1}$

General formulas

f	$\mathcal{L}(f)$
f	$F(s)$
$e^{at} f$	$F(s-a)$
tf	$-F'(s)$
f'	$s\,F(s)-f(0)$
f''	$s^2 F(s) - sf(0) - f'(0)$

Formula for f' follows by integration by parts.

Using it twice: $\mathcal{L}(f') = s\,F(s) - f(0)$

$$\mathcal{L}((f')') = s(s\,F(s) - f(0)) - f'(0)$$
$$= s^2 F(s) - sf(0) - f'(0).$$

← Notice that Laplace transform turns differentiation into mult. by s.

Repeating this, $\mathcal{L}(f^{(n)}) = s^n F(s) - s^{n-1} f(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$.

where $f^{(n)}$, as usual, means taking the n^{th} derivative.

Ex: $y'' - 4y' + 3y = e^{3t}$ $\quad y(0) = 1,\ y'(0) = 2$

$$\mathcal{L}(y'') = s^2 \mathcal{L}(y) - s - 2$$
$$\mathcal{L}(y') = s\,\mathcal{L}(y) - 1$$

LHS: $(s^2 - 4s + 3)\,\mathcal{L}(y) - s + 2$ $\quad$ RHS: $\mathcal{L}(e^{3t}) = \dfrac{1}{s-3}$

After algebra: $(s^2-4s+3)\,\mathcal{L}(y) = \dfrac{1}{s-3} + s - 2$

or $\mathcal{L}(y) = \dfrac{1}{(s-3)(s^2-4s+3)} + \dfrac{s-2}{(s^2-4s+3)}$

$= \dfrac{1 + (s-2)(s-3)}{(s-3)(s-3)(s-1)} = \dfrac{s^2-5s+7}{(s-3)^2(s-1)}$

Do slide on partial fraction expansion.

$= \dfrac{A}{(s-3)^2} + \dfrac{B}{(s-3)} + \dfrac{C}{(s-1)}$ $A = \dfrac{1}{2}$, $C = \dfrac{3}{4}$ then solve for B.

Set $\underline{s=2}$: $1 = 1/2 - B + 3/4 \Rightarrow B = 1/4$.

Remark: Heaviside cover-up method can find the dominant term in solution. (as $t \to \infty$)

Take inverse Laplace transform to finish:

What is $\mathcal{L}^{-1}\left(\dfrac{1}{(s-3)^2}\right)$?   Ans. $\mathcal{L}(e^{3t}) = \dfrac{1}{s-3}$

To differentiate $F(s)$, mult. $f(t)$ by t, take Laplace.

so $\mathcal{L}^{-1}\left(\dfrac{1}{(s-3)^2}\right) = te^{3t}$

$\mathcal{L}(te^{3t}) = -\left(\dfrac{1}{s-3}\right)' = \dfrac{1}{(s-3)^2}$ ✓.

Get (after taking inverse Laplace of everything):

$A\,te^{3t} + Be^{3t} + Ce^{t}$

$= \underbrace{\tfrac{1}{2}\,te^{3t}} + \tfrac{1}{4}e^{3t} + \tfrac{3}{4}e^{t}$

this is dominant term in soln as $t \to \infty$. Easily found by cover up.

In general, given $p(D)y = f(t)$ then

taking Laplacian: $\mathcal{L}(p(D)y) = p(s)\mathcal{L}(y) + \underbrace{r(s)}$

polynomial depending on initial conditions

$\deg(r) < \deg(p)$.

so $\mathcal{L}(y) = \dfrac{\mathcal{L}(f) - r(s)}{p(s)}$

$\Rightarrow y = \mathcal{L}^{-1}\left(\dfrac{\mathcal{L}(f) - r(s)}{p(s)}\right)$.

Notice $\mathcal{L}$ is an integral from 0 to ∞. Doesn't record behavior of f at negative values. ($\mathcal{L}(f) = \int_0^\infty f(t)e^{-st}\,dt$)

So if we want to have unique Laplace transforms (so that inverse is defined!) we regard them as functions on $[0,\infty)$.

Sometimes use step function $u(t) = \begin{cases} 1 & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$

then write $\mathcal{L}^{-1}(1/s) = \underbrace{u(t)}$ rather than $\mathcal{L}^{-1}(1/s) = 1$.

unique function with support on $[0,\infty)$.

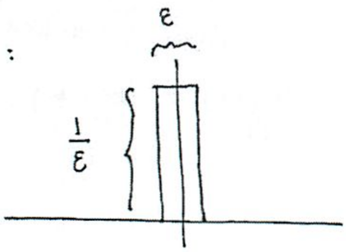
So why use Laplace transforms? Seems harder than just guessing / D-operators.

A: Handle impulsive functions easily.

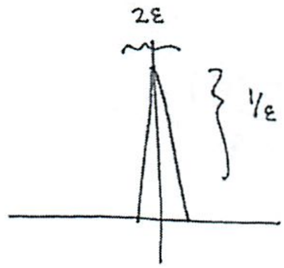
Delta function is idealized version of impulsive function

that is
- very large near $t = 0$
- very small away from $t = 0$
- has total integral $= 1$.

Pictures :



or



ε very small

Formal definition :

$$\int_a^b f(t)\,\delta(t)\,dt = f(0) \quad \text{if } f \text{ continuous at } 0.$$

provided $a < 0 < b$.

Often use convention

$$\int_{0^-}^{1} \delta(t)\,dt = 1 \quad , \quad \int_{0^+}^{1} \delta(t)\,dt = 0$$

to indicate whether we include impulsive function at 0.

Physical interpretation : impulsive force $f(t)$ acting for short period $\leftarrow$ strike of hammer. at time $t=1$

then change in momentum $= \int_{.999}^{1.001} f(t)\,dt = I \nwarrow$ mechanical impulse.

$m\,v(1.001) - m\,v(.999)$

can be modeled as $I\,\delta(t-1)$.

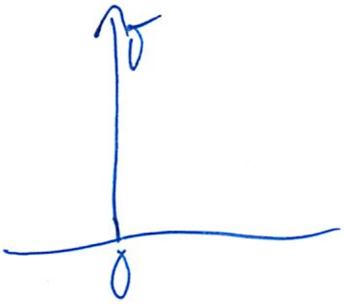
18.03 Recitation

11/10

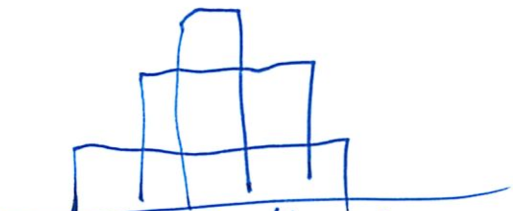
Delta "Function"

δ

lots of force all at once



very tall and very thin
to limit of



tall of area 1

Makes sense when look at $\int$ of

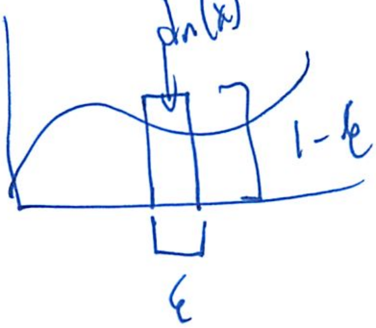
$$\int \delta(x)\, f(x)\, dx$$

|| by def

$$f(0)$$

$$= \lim_{n \to \infty} \int f(x)\, d_n(x)\, dx$$

2



$$\int_{-\frac{\epsilon}{2}}^{\frac{\epsilon}{2}} \frac{1}{\epsilon} f(x)\, dx$$

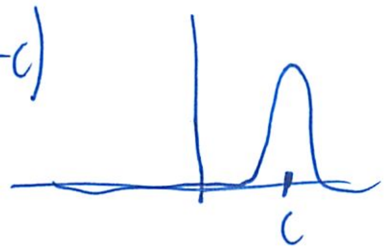
avg ~~val~~ value
of $f(x)$ over $-\frac{\epsilon}{2}, \frac{\epsilon}{2}$

Converges onto value

$\mathcal{L}(\delta(t-c))$

shifts over to c

$\delta(x-c)$



$\int \delta(x-c)\, f(x)\, dx = f(c)$

Some interval containing c

③

$\int \delta(x-c) f(x)\, dx = 0$

interval not containing c

$\mathcal{L}(\delta(t-c)) = \int_0^\infty e^{-st} \delta(t-c) = e^{-sc}$

if $c \geq 0$

makes sense to take Laplace transform of δ - function

$x'' + x = \delta(t-2)$

$x'(0) = x(0) = 0$

Exercise: try to solve

Take LaPlace transform of both sides

$(s^2+1)\,\mathcal{L}(x) = e^{-2s}$

└ dropped 0 terms

$\mathcal{L}(x) = \dfrac{e^{-2s}}{s^2+1}$

(4)

~~Ddn~~

More verbose

$$\mathcal{L}(x') = s\,\mathcal{L}(x) - x(0)$$

$$\mathcal{L}(x'') = s\,\mathcal{L}(x'') - x'(0)$$

$$= s^2\,\mathcal{L}(x) - s\,x(0) - x'(0)$$

$$\mathcal{L}(x'' + x) = s^2 + 1\ \mathcal{L}(x)$$

Otherside : compute

$$x = \mathcal{L}^{-1}\left(\frac{e^{-2s}}{s^2+1}\right)$$

Then take this

Need to do heavysides

- Partial fractions when $\frac{P(s)}{Q(s)}$ ← polynomials
- lets you expand into

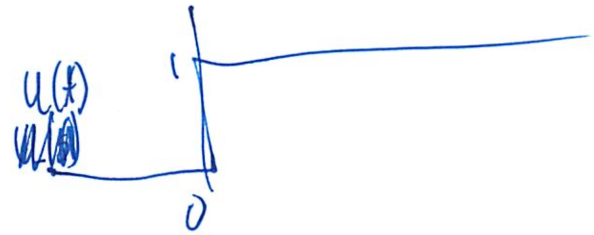
$$\sum \frac{c}{(s-r)^k}$$ ← constant ← constant

can take laplace inverse of

(5)

But we don't have a polynomial

Step Function



Exercise

① Find $\mathcal{L}(u(t-c))$ ← constant $c > 0$

② Find $\mathcal{L}(u(t-c)\,f(t-c))$

① $\dfrac{e^{-cs}}{s}$

since $\displaystyle\int_0^\infty u(t-c)\,e^{-st}\,dt$

$$= \int_c^\infty e^{-st}\,dt$$

$$= \frac{e^{-cs}}{s}$$

(1)

(2) $\mathcal{L}(u(t-c)\ f(t-c))$

$$= \int_c^\infty f(t-c)\, e^{-st}\, dt$$

$$= \mathcal{L}(f)\, e^{-cs}$$

Why is this useful?

$$\mathcal{L}^{-1}(\mathcal{L}(f)\, e^{-cs}) = u(t-c)\, f(t-c)$$

So we can

$$X = \mathcal{L}^{-1}\left(\frac{e^{-2s}}{s^2+1}\right)$$

Note $\mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) = \sin(t)$

$$= \mathcal{L}^{-1}\left(e^{-2s}\, \mathcal{L}(\sin t)\right)$$

$$= u(t-2)\, \sin(t-2)$$

(7)

Solve

$$x'' + 2x' + x = e^{3x} + \delta(x-1$$

$$x'(0) = x(0) = 0$$

$$(s^2 + 2s + 1)\,\mathcal{L}(x) = \frac{1}{s-3} + e^{-s}$$

$$x = \mathcal{L}^{-1}\left(\frac{1}{(s-3)(s+1)^2} + \frac{e^{-s}}{(s+1)^2}\right)$$

$$x = \mathcal{L}^{-1}\left(\frac{1}{(s-3)(s+1)^2}\right) + \mathcal{L}^{-1}\left(\frac{e^{-s}}{(s+1)^2}\right)$$

1st part - use partial fractions

$$\frac{1}{(s-3)(s-1)^2} = \frac{A}{(s+1)} + \frac{B}{(s+1)^2} + \frac{C}{(s-3)}$$

$$A(s-1)(s-3) + B(s-3) + C(s+1)^2$$

$$(A+C)(s^2) + (-2A + B + 2C)s + (3A - 3B + C) = 1$$

$$A = -C$$

$$B + 4C = 0$$

$$B = -4C$$

(8)

$4C + 12C = 1$

$C = \frac{1}{16}$

Now plug in

$$\frac{1}{(s-3)(s+1)^2} = \frac{-1/16}{s+1} + \frac{-1/4}{(s+1)^2} + \frac{1/16}{s-3}$$

No inverse Laplace of this

$$\mathcal{L}^{-1}\left(\frac{1}{(s-3)(s+1)^2}\right) = -\frac{1}{16}e^{-s} + -\frac{1}{4}se^{-s} + \frac{1}{16}e^{3s}$$

$$\mathcal{L}^{-1}\left(e^{-s}\mathcal{L}(te^{-t})\right) = u(t-1)(t-1)e^{-(t-1)}$$

$$x = -\frac{1}{16}e^{-s} - \frac{1}{4}se^{-s} + \frac{1}{16}e^{3s} + u(t-1)(t-1)e^{-(t-1)}$$

18.03 Lecture 25

11/14

Convolution

Next Mon review
Next Tue recitation optional : Test review
Next Tue evening : Test

Plan: Solve ODEs via Laplacean
eg spring model w/ impulse force

Example $x'' + 9x = 0$ using Laplace method
? w/o impulse force

RHS
$\mathcal{L}(0) = 0$

LHS
linear - so split

$\mathcal{L}(x'' + 9x) = \mathcal{L}(x'') + 9\mathcal{L}(x)$

$= \underbrace{(s^2+9)}_{\text{Charastic polynomial}} \mathcal{L}(x) - ~~\ldots~~ sx(0) - x'(0)$

Since $\mathcal{L}(x'') = s^2 \mathcal{L}(x) - s\,x(0) - x'(0)$
? might be ~~on~~ table on exam
well maybe just $\mathcal{L}(x')$

②

$$\mathcal{L}(x) = \frac{s\,x(0) + x'(0)}{s^2+9}$$

↓

$$x = \mathcal{L}^{-1}\left(\frac{s\,x(0)}{s^2+9} + \frac{x'(0)}{s^2+9}\right)$$

↑ can't reduce further

So what is the Laplace transform?

$$x(t) = x(0) \cdot \cos 3t + \frac{1}{3}\, x'(0) \sin 3t$$

(get table on exam).

$$\mathcal{L}(\sin 3t) = \frac{3}{s^2+9}$$

need to add factor

Same we would have got w/ guessing $\pm 3i$

So $A\cos 3t + B\sin 3t$

But no system of eqs to solve

So ~~this~~ ~~Mea~~ the models are roughly the same
↳ with enough practice.

(3)

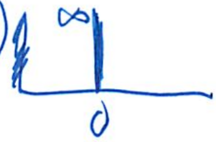
example 2 add impulsive force

Smack mass w/ hammer at time 2

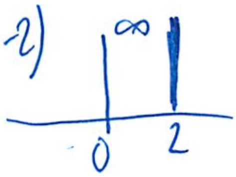
$$x'' + 9x = 4\underbrace{\delta_2(t)}_{\text{means } \delta(t-2)}$$

↑ impulsive force

$\delta(t)$



$\delta(t-2)$



LHS) same

$$\mathcal{L}(x''+9x) = (s^2+9)\,\mathcal{L}(x) - s\,x(0) - x'(0)$$

RHS)

$$= 4\,\mathcal{L}(\delta_2 t)$$

$$= 4\int_0^\infty \delta(t-2)\,e^{-st}\,dt$$

↑ picks off s at 0
shift by 2

$$= 4\,\underbrace{e^{-2s}}_{\text{a bit sneaky}}$$

at $\mathcal{L}(\delta(t)) = 1$ but here $t=2$

(4)

$$\mathcal{L}(x) = \frac{e^{-2s}}{s^2+9} + \frac{s\,x(0) + x'(0)}{s^2+9}$$

What is $\mathcal{L}^{-1}\left(\frac{e^{-2s}}{s^2+9}\right)$?

Rough guess: translation by 2 of fn whose Laplace transform is $\frac{1}{s^2+9}$

$$\boxed{\mathcal{L}^{-1}\left(\frac{1}{s^2+9}\right) = \frac{1}{3}\sin 3t}$$

$\mathcal{L}(f) = F(s)$ what is $e^{-as}F(s)$

LaPlace transform of some related fn?

$$e^{-as}F(s) = e^{-as}\int_0^\infty e^{-s\tau} f(\tau)\,d\tau$$

So bring inside

$$= \int_0^\infty e^{-s(\tau+a)} f(\tau)\,d\tau$$

$$t = \tau + a$$
$$dt = d\tau$$

$\infty + a = \infty$

$$= \int_a^{\infty+a} e^{-st} f(t-a)\,dt$$

(5)

$= ? \cancel{\mathcal{L}\{f(t-a)\}}$ No too quick

↓ cheat

$$\int_0^\infty e^{-st} f(t-a) \cdot \underbrace{u(t-a)}_{\text{The cheap fn defined to make this true}} \, dt$$

$u(t-a)$

$\|$

$u_a(t)$

$$= \begin{cases} 1 & \text{if } t \geq a \\ 0 & \text{if } t < a \end{cases}$$

unit step fn at a

aka Heaviside function

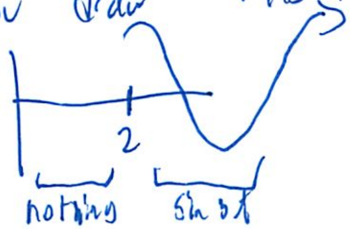
$$\mathcal{L}^{-1}\left(\frac{e^{-2s}}{s^2+9}\right) = \frac{1}{3} \sin(3(t-2)) \cdot u(t-2)$$

except now need ↗

So in general

$$\mathcal{L}^{-1}\left(e^{-as} F(s)\right) = f(t-a) \cdot u(t-a)$$

Could you draw this?



Thats when hammer hits!

⑥

Can Read Section 4.5 for lots more examples

Notice we didn't care about duplication of roots

<u>No</u> don't need to worry about it

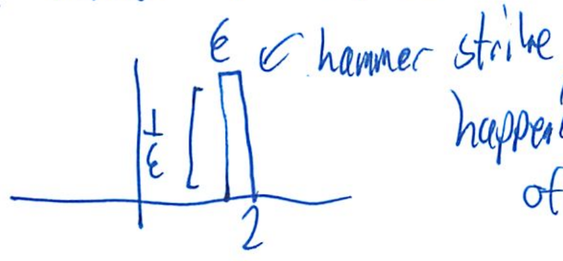
It comes out automatically when you do inverse LaPlace

But what is this impulse fn?

- not even a fn
- then why is it in our diff eq?

<u>Problem</u> $\delta_2(t) = \delta(t-2)$ is not a function!

idealized model for impulse



happening over short period of time (ϵ)

means small

$\delta_{2,\epsilon}(t)$ is really what it is

Claim replace $\delta_{2,\epsilon}(t)$ w/ $\delta_2(t)$ then get almost same answer - just a few ~~least~~ less significant ~~digits~~ decimal digits

Q —

Its Ok to write $\delta_2(t)$ in ODE since we immediately put into LaPlace transform

But taking $\int \delta(t)$ is doable

So ok to do it

If actually take $\delta_{2,\epsilon}(t) = \frac{1}{\epsilon}\left[\underbrace{u_2(t)}_{\text{go up}} - \underbrace{u_{2+\epsilon}(t)}_{\text{then quickly go back down}}\right]$

to get skyscraper

Can use to find

$$\mathcal{L}\left(\delta_{2,\epsilon}(t)\right) = e^{-2s}\underbrace{\frac{1}{s\epsilon}\left(1-e^{-s\epsilon}\right)}_{\text{junk}}$$

goes to 1 as $\lim_{\epsilon \to 0}$ using Lehopital's rule

Since $\mathcal{L}(u(t)) = \frac{e^{-2s}}{s}$ ← Laplace of 1

$\mathcal{L}^{-1}(1) = \frac{1}{s}$

But why did we multiply $\delta(t-2)\cdot \underbrace{4}$?

thats like $\infty \cdot 4$?

But when we $\int$ it matters

Since $\int \delta = 1$

⑧

So in a word problem (pt 2 of hw)
told that impulse force has

$$\int_2^{2+\epsilon} f(t)\,dt = 4 \quad (\text{"impulse"})$$

Model with $4\,\delta_2(t)$

since $\int_{2^-}^{2+\epsilon} 4\,\delta_2(t)\,dt = 4$

Convolution

ex $\quad \dfrac{2s}{(s^2+4)^2} = \dfrac{2}{s^2+4} \cdot \dfrac{s}{s^2+4}$

Looks like multiplication of 2 LaPlace transforms we know

$= \mathcal{L}(\sin 2t)\,\mathcal{L}(\cos 2t)$

$\neq \cancel{\mathcal{L}(\sin 2t \cos 2t)}$ unfortunately not that easy

But what fn combines these to?

$= \mathcal{L}(\text{some combo } \sin 2t, \cos 2t)$

(9)

Its called convoltion ($*$)

$\mathcal{L}(\sin 2t * \cos 2t)$

But what in all wold does this mean?

Define $f*g$ as

$$\mathcal{L}(f*g) = \mathcal{L}(f)\cdot\mathcal{L}(g)$$

normal multiplication

"f convolved with g" $(f*g)(t)$

functions at t

Need examplees

$$\mathcal{L}(f*0) = 0$$

$$\mathcal{L}(f*\delta) = \mathcal{L}(f)\cdot\mathcal{L}(\delta) = \mathcal{L}(f)\cdot 1$$

What we're saying by uniqueness

$$f*\delta = f$$

Remember def δ $\int_a^b f(t)\,\delta(t)\,dt = f(0)$ if $0\in(a,b)$

$$(f*\delta)(t) = \int_0^t f(\tau)\cdot\delta(t-\tau)\,d\tau$$

kinda like negative of delta fn picks off value at t

$$= f(t)$$

10

In general

$$f * g(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau$$

Lecture 25 11/14

Plan: Solve ODEs using Laplacian.

E.g. spring model with impulsive force

(w/o impulsive force): $x'' + 9x = 0$

$$\mathcal{L}(x'') = s^2 \mathcal{L}(x) - s\,x(0) - x'(0)$$

so $$\mathcal{L}(x'' + 9x) = (s^2+9)\mathcal{L}(x) - s\,x(0) - x'(0)$$

Since $\mathcal{L}(0) = 0$ then RHS $= 0$. Solve for $\mathcal{L}(x)$:

$$\mathcal{L}(x) = \frac{s\,x(0) + x'(0)}{s^2+9}$$

$$x(t) = \mathcal{L}^{-1}\left(\frac{s\,x(0)}{s^2+9} + \frac{x'(0)}{s^2+9} \right)$$

$$= x(0)\cos 3t + \frac{x'(0)}{3}\sin 3t$$

(w/ impulsive force) $$x'' + 9x = 4\underbrace{\delta_2(t)}_{\delta(t-2)}$$

$\delta(t-2)$ impulse @ $t=2$.

$$(s^2+9)\mathcal{L}(x) - s\,x(0) - x'(0) = 4 \cdot \underbrace{\mathcal{L}(\delta_2(t))}$$

remember exponential shift rule:

$\mathcal{L}(\delta(t)) = 1$

so $\mathcal{L}(\delta(t-2)) = e^{-2s}$

$$\mathcal{L}(x) = \frac{s\,x(0) + x'(0)}{s^2+9} + \frac{4e^{-2s}}{s^2+9}$$

$$x(t) = \mathcal{L}^{-1}\left(\frac{s\,x(0) + x'(0)}{s^2+9} \right) + \mathcal{L}^{-1}\left(\frac{4e^{-2s}}{s^2+9} \right)$$

What is $\mathcal{L}^{-1}\left(e^{-2s} / s^2+9 \right)$

Rough idea : e^{-2s} appears as result of shift by 2 in t var.

something like $\frac{1}{3} \sin(3(t-2))$

In general, given $e^{-as} F(s)$ where $F(s) = \mathcal{L}(f)$:

Guess
$\mathcal{L}^{-1}(e^{-as} F(s))$
$\approx f(t-a)$

Then to check our guess : $e^{-as} F(s) = e^{-as} \int_0^\infty e^{-s\tau} f(\tau)\, d\tau$

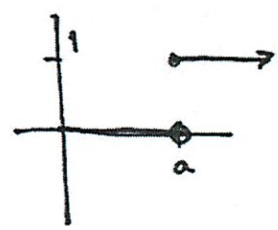
$$= \int_0^\infty e^{-s(\tau+a)} f(\tau)\, d\tau \qquad t = \tau + a$$

$$= \int_a^\infty e^{-st} f(t-a)\, dt$$

not exactly Laplace transform. Wrong lower bound.

STUPID = FIX $\int_0^\infty e^{-st} f(t-a)\, u(t-a)\, dt$

where $u(t-a)$ is the unit step function



so $\mathcal{L}^{-1}(e^{-as} F(s)) = f(t-a)\, u(t-a)$

Conclusion : $\mathcal{L}^{-1}\left(e^{-2s} / s^2+9 \right) = \frac{1}{3} \mathcal{L}^{-1}\left(e^{-2s} \cdot \frac{3}{s^2+9} \right)$

$$= \frac{1}{3} \sin(3(t-2))\, u(t-2)$$

book uses notation: $u_2(t)$ as well.

(see section 4.5 of book for more examples)

– Mention no need to worry about duplication of roots in Laplace methods

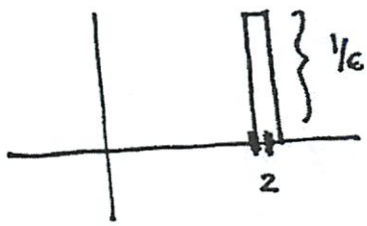
What did $4\,\delta_2(t)$ mean in ODE? How can we have a sol'n $x(t)$ to an ODE w/o generalized functions? $\delta_2(t)$ not a function!

$\delta_2(t)$: idealized model for impulsive force. at $t=2$ (hammer strike)

acts over short time interval compared to time interval for which we care about solution.

Only using well-defined properties of δ-function when we solve by Laplace transform.

If we work with :



$\delta_{2,\varepsilon}(t) = \frac{1}{\varepsilon}\left[u_2(t) - u_{2+\varepsilon}(t)\right]$

unit step funct. @ 2 and $2+\varepsilon$, resp.

Since $u_2(t) = u_2(t)\cdot 1$ then $\mathcal{L}(u_2(t)) = e^{-2s}\cdot\frac{1}{s}$

so $$\mathcal{L}(\delta_{2,\varepsilon}(t)) = e^{-2s}\cdot\underbrace{\frac{1}{s\cdot\varepsilon}(1-e^{-s\varepsilon})}_{\approx 1 \text{ if } \varepsilon \text{ small.}}$$

$(\longrightarrow 1$ as $\varepsilon \longrightarrow 0)$

Where does the "4" in $4\delta_2(t)$ come from?

Calculate the impulse of our force : $\int_2^{2+\varepsilon} f(t)\,dt = 4$

hammer strike

then we have $$\int_{2^-}^{2+\varepsilon} 4\,\delta_2(t)\,dt = 4.$$

Convolution : Often encounter products of functions we want to take inverse transform of .

Example :
$$\frac{2s}{(s^2+4)^2} = \frac{2}{s^2+4} \cdot \frac{s}{s^2+4} = \mathcal{L}(\sin 2t)\,\mathcal{L}(\cos 2t)$$

But
$$\mathcal{L}^{-1}\left(\frac{2s}{(s^2+4)^2}\right) \neq \sin 2t \cos 2t = \frac{1}{2}\sin 4t$$
$$\downarrow \mathcal{L}$$
$$\frac{2}{s^2+16} \quad \text{(WRONG!)}$$

Given $f(t) = \sin 2t$, $g(t) = \cos(2t)$

is there some operation "$*$" to produce :

$$\mathcal{L}(f * g) = \mathcal{L}(f)\cdot\mathcal{L}(g).$$

To guess how to define "$*$" notice WANT
$$\mathcal{L}(f * \delta) = \mathcal{L}(f)\underbrace{\mathcal{L}(\delta)}_{1} = \mathcal{L}(f).$$

By uniqueness of Laplace transform,
$$f = f * \delta .$$

Recall : δ defined by $\int_a^b f(t)\,\delta(t)\,dt = f(0)$ if $0 \in (a,b)$

So define
$$f * \delta = \int_0^t f(\tau)\,\delta(t-\tau)\,d\tau$$

In general ,
$$f * g = \int_0^t f(\tau)\,g(t-\tau)\,d\tau$$

check : $\sin 2t * \cos 2t = \frac{1}{2}\sin 4t$ as desired in ex. above.

Heaviside's Practice

11/14

$$\frac{5x+6}{(x^2+4)(x-2)} = \frac{Ax+B}{x^2+4} + \frac{C}{x-2}$$

↑ since can't decompose

So get C first

$x=2$

$$\frac{10+6}{8\cdot 0} = \frac{A(2)+B}{2^2+4} + \frac{C}{0}$$

↑ I think we treat this as 1 ?

$$1 = \frac{2A+B}{8} + C$$

No

Try again from basics

Basically multiply other side I think

← to get C

$$\frac{5x+6\,\cancel{(x-2)}}{(x^2+4)\cancel{(x-2)}} = \frac{(Ax+B)(x-2)}{x^2+4} + \frac{C\,\cancel{(x-2)}}{\cancel{(x-2)}}$$

$$\frac{5x+6}{x^2+4} = \frac{(Ax+B)(x-2)}{x^2+4} + C$$

(2)

Now sub in x so that $(x-2) \to 0$

$x = 2$

Now sub in

$$\frac{5(2)+6}{(2)^2+4} = 0 + C$$

$$\frac{16}{8} = 2 = C$$

Ohhh that makes more sense than "cancel"

Just do it this way

18.03 Recitation

11/15

Convolution

Convolution

↓ book uses Tau for z

$$(f * g)(t) = \int_0^t f(z)\, g(t-z)\, dz$$

f, g fns of t

z dummy variable

$$0 * g = \int_0^t 0 \cdot g(t-z)\, dz = 0$$

$$1 * g = \int_0^t 1 \cdot g(t-z)\, dz =$$

change of variable

$$= \int_0^t g(z)\, dz$$

$$1 * t^n = \int_0^t 1 \cdot t^n (t-z)$$

$$= \int_0^t z^n\, dz$$

$$= \frac{t^{n+1}}{n+1}$$

(2)

$$f * 1 = \int_0^t f(z)\, dz$$

$$= 1 * f$$

$$\neq f$$

Find $t^n * t^m$

Find $t * t$ ← easier

$$\int_0^t t \cdot t(t-z)\, dz$$

$$\int_0^t t^3 - t^2 z\, dz$$

$$t^3 z \quad \frac{t^2 z^2}{2} \Big|_0^t$$

$$\cancel{t^3 z \quad \frac{t^2 z^2}{2}} \qquad t^3 t \quad \frac{t^2 t^2}{2}$$

$$\cancel{\frac{t^5 z^3}{2}} \qquad \frac{t^8}{2}$$

wrong - see next pg

(3)

$$\int_0^t z\,(t-z)\,dz$$

$f(t)=t$

$f(z)=z$

$f(t-z)=t-z$

? when function is just t our function is t confused?

$$=\int_0^t zt - z^2\,dz$$

$$=\frac{1}{2}z^2 t-\frac{z^3}{3}\Big|_0^t$$

$$=\frac{t^3}{6}$$

Helps w/ inverse Laplace transform

$$\mathcal{L}(f*g)=\mathcal{L}(f)\,\mathcal{L}(g)$$

Proof write what both sides say

next pg

$$\int_0^\infty e^{-st}\int_0^t f(t)\,g(z-t)\,dz\,dt = \cancel{\int_0^\infty e^{-st_1} f(t_1)\,dt_1}$$

(4)

$$= \int_0^\infty e^{-st_1} f(t_1)\, dt_1 \int_0^\infty e^{-st_2} g(t_2)\, dt_2$$

Convert

$$= \int_0^\infty \int_0^\infty e^{-s(t_1+t_2)} f(t_1) g(t_1)\, dt_1\, dt_2$$

$t = t_1 + t_2$

$z = t_1$

So $dt_1\, dt_2 = dt\, dz$

$$= \int_0^\infty dt\, e^{-st} \int_0^t \underbrace{f(z) g(t-z)}\, dz$$

need to put condition on how fast functing grows to be 100% correct

Why $f * g = g * f$;

$y = t - z$

$dy = -dz$

$$\int_0^t f(z) g(t-z)\, dz$$

$$= \int_0^t f(y-t)\, g(y)\, dy$$

once make change of variables can see

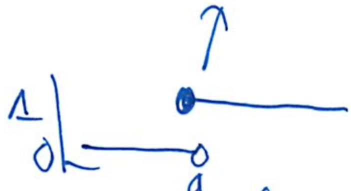
5

Convolution essentially designated to make $\mathcal{L}(f*g)=\mathcal{L}(f)\mathcal{L}(g)$

So when product of two things we can take inverse LaPlace transform of each thing individually

---

ex $\mathcal{L}(u(t-a))$



Step fn

$$\mathcal{L}\big(u(t-a)\underbrace{f(t-a)}_{f(t-a)}\big)=e^{-as}\cdot\mathcal{L}\big(\underbrace{f(t)}_{f(t)}\big)$$

---

example

+ five

$$x''+5x'+4x=\sin t+\delta(t-7)$$

Current approach: LaPlace Transforms

$$\cancel{(s^2-5s+4)\mathcal{L}(x)=\mathcal{L}(\sin t)+\mathcal{L}(\delta(t-7))+(s-1)x(0)+x'(0)}$$

See later

First RHS

moved already

6

$$\mathcal{L}(\sin t)$$

$$\stackrel{!!}{=} \int_0^\infty \sin t \, e^{-st} \, dt$$

can compute if you forget

$$= \int_0^\infty \frac{e^{it} - e^{-it}}{2i} e^{-st} \, dt$$

$$= \frac{1}{2i}\left(\frac{1}{s-i} - \frac{1}{s+i}\right)$$

$$= \frac{1}{s^2+1}$$

$$\mathcal{L}(\delta(t-7))$$

See next pg

Remember $\mathcal{L}(x') = s\mathcal{L}(x) - x(0)$

$$\mathcal{L}(x'') = s\mathcal{L}(x') - x'(0)$$

$$= s^2 \mathcal{L}(x) - s(x(0)) - x'(0)$$

7

$$\mathcal{L}(x'') = s^2\mathcal{L}(x) - sx(0) - x'(0)$$

$$5\mathcal{L}(x') = 5s\mathcal{L}(x) - 5x(0)$$

$$4\mathcal{L}(x) = 4\mathcal{L}(x)$$

$$(s^2+5s+4)\mathcal{L}(x) - (s+5)x(0) - x'(0)$$

So I was doing it wrong
but did not end up mattering
since $x'(0) = 0$

$$(s^2+5s+4)\mathcal{L}(x) = \frac{1}{s^2+1} + \mathcal{L}(\delta(t-7)) + (s+5)x(0) + x'(0)$$

$$\boxed{\mathcal{L}(\delta(t-7)) = \int_0^\infty e^{-st}\delta(t-7)\,dt = e^{-7s}}$$

$$\mathcal{L}^{-1}\left(\frac{\frac{1}{s^2+1} + e^{-7s} + (s+5)x(0) + x'(0)}{(s+4)(s+1)}\right)$$

factor

(8)

He uses partial fractions - not Heavysides

$$\mathcal{L}^{-1}\left(\frac{1}{(s^2+1)(s+4)(s+1)}\right) + \mathcal{L}^{-1}\left(\frac{e^{-2s}}{(s+4)(s+1)}\right)$$

$$+ \mathcal{L}^{-1}\left(\frac{(s+5)\,x(0) + x'(0)}{(s+4)(s+1)}\right)$$

Remark: each thing is $\mathcal{L}^{-1}$(product of things whose each of whose LaPlace Transforms we know)

So can find each's LaPlace Transform w/ Convolution

$$\frac{1}{(s+4)(s+1)} = \frac{A}{s+4} + \frac{B}{s+1}$$

partial fractions (basically same thing)

$A + B = 0$

$A + 4B = 1$

$B = \frac{1}{3}$

$A = -1/3$

9

$$-\frac{1}{3}\mathcal{L}^{-1}\left(\frac{e^{-7s}}{s+4}\right)+\frac{1}{3}\mathcal{L}^{-1}\left(\frac{e^{-7s}}{s+1}\right)$$

now how to do this
can do w/ convolution
not always a good idea

$$\mathcal{L}^{-1}\left(\frac{e^{-7s}}{s+1}\right)=\mathcal{L}^{-1}\left(e^{-7s}\right)*\mathcal{L}^{-1}\left(\frac{1}{s+1}\right)$$

def of convolution

$$=\delta(t-7)*e^{-t}$$

$$=\int_0^t \delta(z-7)\,e^{-(t-z)}\,dz$$

have to actually $\int$

$$=\begin{cases}0 & \text{if } t<7\\ e^{-(t-7)} & \text{if } t\geq 7\end{cases}$$

$e^{-(t-z)}$
at $z=7$

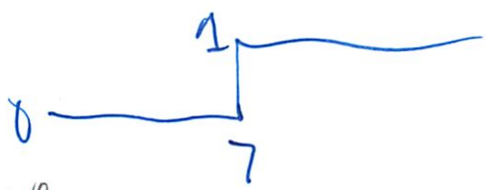
write as

$$=u(t-7)\,e^{-(t-7)}$$

Heavyside's step fn

(10)

$u(t-7)$ =



So in general have $\mathcal{L}^{-1}(e^{-cs} F)$

$$\delta(t-c) * \mathcal{L}^{-1}(F)$$

$$= \int_0^t \delta(z-c)\, \mathcal{L}^{-1}(F)(t-z)\, dz$$

$$= \begin{cases} 0 & t < c \\ e^{-(t-c)} & t \geq c \end{cases}$$

Convolution Practice

1/15

Remember can split +

But <u>not</u> multiplication

Need to do convolution

$$\mathcal{L}(\cos t \sin t) \neq \mathcal{L}(\cos t)\mathcal{L}(\sin t)$$

$$(f * g)(t) = \int_0^t f(t)\, g(t-\tau)\, d\tau$$

has property that

$$\mathcal{L}(h(t)) = H(s) = F(s)G(s)$$

$$\mathcal{L}(f * g) = \mathcal{L}(f) \cdot \mathcal{L}(g)$$

defn

Sub in $u = t - \tau$

$$f(t) * g(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau$$

$$= \int_t^0 f(t-u)\, g(u)\, (-du)$$

$$= \int_0^t g(u)\, f(t-u)\, du$$

$$= g(t) * f(t)$$

proof

(2) book
example

$(\cos t) * (\sin t)$

$= \int_0^t \cos \tau \sin(t-\tau)\, d\tau$

$\boxed{\cos A \sin B = \frac{1}{2}\left[\sin(A+B) - \sin(A-B)\right]}$

$= \int_0^t \frac{1}{2}\left[\sin t - \sin(2\tau - t)\right] d\tau$

$= \frac{1}{2}\left[\tau \sin t + \frac{1}{2}\cos(2\tau - t)\right]_{\tau=0}^{t}$

$= \frac{1}{2} t \sin t$

So must actually do integration

$\mathcal{L}\left(\frac{1}{2} t \sin t\right) = \frac{s}{(s^2+1)^2}$ ← like we had written

but when doing reverse we start w/ this??

When do we use it?
When taking the inverse, right?

(3)

$$\mathcal{L}^{-1}\left(F(s)\cdot G(s)\right) = \int_0^t f(\tau)\, g(t-\tau)\cdot d\tau$$

Like if have

$$\mathcal{L}^{-1}\left(\frac{2}{(s-1)(s^2+4)}\right) = \sin 2t \quad e^t$$

of this — or this

¿ So what do we start w/ ?

intermediate step

$$= \int_0^t e^{t-\tau} \sin 2\tau \, d\tau$$

$$= e^t \int_0^t e^{-\tau} \sin 2\tau \, d\tau$$

$$= e^t \left[\frac{e^{-\tau}}{5} \left(-\sin 2t - 2\cos 2\tau\right) \right]_0^t$$

¿ wait to go over in class or ask tomorrow

Recitation example

So can see in recitation notes

Will be asked to take $\mathcal{L}^{-1}\left(\frac{e^{-7s}}{s+1}\right)$

Can easily see $= \mathcal{L}^{-1}(e^{-7s}) * \mathcal{L}^{-1}\left(\frac{1}{s+1}\right)$

Convolution of

(4)

So can't just write $\delta(t-7) \cdot e^{-t}$

product

But must convolve

$$\delta(t-7) * e^{-t}$$

but do write like this

Then convolve

$$\int_0^t \delta(z-7)\, e^{-(t-z)}\, dz$$

have to actually integrate

(did we skip step here)

$$= \begin{cases} 0 & t < 7 \\ e^{-(t-7)} & t \geq 7 \end{cases}$$ or see pattern

Write as

$$u(t-7)\, e^{-(t-7)}$$

Heavyside's step Fn

~~So in general have~~

5 So in general

We have: $\mathcal{L}^{-1}(e^{-cs} f)$

$$\delta(t-c) * \mathcal{L}^{-1}(f)$$

$$= \int_0^t \delta(z-c)\, \mathcal{L}^{-1}(f)(t-z)\, dz$$

$$= \begin{cases} 0 & t < c \\ e^{-(t-c)} & t \geq c \end{cases}$$

:: so I guess can memorize that rule ::

Office Hrs

11/16

↓ 11/9 lecture

So for

$$\mathcal{L}(y''-4y'+3y)$$

$$=\mathcal{L}(y'')-4\mathcal{L}(y')+3\mathcal{L}(y)$$

↑

$$\mathcal{L}(y'')=s^2\mathcal{L}(y)-s(y(0))-y'(0)$$

$$\mathcal{L}(y')=s\mathcal{L}(y)-y(0)$$

So try plugging in myself

$$s^2\mathcal{L}(y)-s(y(0))-y'(0)-4\left[s\mathcal{L}(y)-y(0)\right]+3\mathcal{L}(y)$$

now collect terms

$$\mathcal{L}(y)(s^2-4s+3)-s\,y(0)-y'(0)+4\,y(0)$$

↑ chara eqn

$$\mathcal{L}(y)(s^2-4s+3)+(4-s)\,y(0)-y'(0)$$

Given $y(0)=1$ $y'(0)=2$

(2)

$\mathcal{L}(y)(s^2-4s+3)+(4-s)(1)-2$

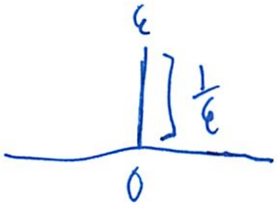
Same as in class

So you do have to apply it twice

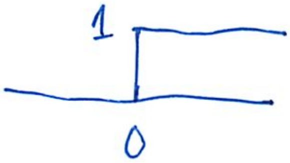
need to re-do P-set

Review convolve more

δ



$\int \delta$



$\delta \cdot f$ does not mean anything

$\int \delta \cdot f$ depends on limits of integrating

- if integrating over region containing 0

  L then answer is always $f(0)$ no matter the input

  "like strobe light at $t = x = 0$" $x = t =$ anything

  ind. variable

B

$$\int_a^b \delta(x) f(x)\,dx = \begin{cases} 0 & \text{if } 0 \notin (a,b) \\ f(0) & \text{if } 0 \in (a,b). \end{cases}$$

change variable

$$\int_a^b \delta(x-c) f(x)\,dx = \begin{cases} 0 & \text{if } c \notin (a,b) \\ f(c) & \text{if } c \in (a,b). \end{cases}$$

delta fn is easiest thing in world to integrate

output 0 unless 7 is b/w 0 and t

the c value

So t must be $\geq$ 7 in order to fit within the limits of integration

Whether its $\geq$ or $>$ depends on exact def'n ($\geq$ or $\leq$)

Just ignore the $=$ to possibility for now

Will not always be δ for convolution

└ then we have to actually integrate

18.03 Lecture 26 11/16

Weight Function

P-Set due Fri
Test Tue night
Today: last day of content
Fri: Connections Unit 2 + Unit 3
Mon: Review
next P-Set out week from Mon (post thanksgiving)
due week " Fri

Practice exams + solutions posted tonight

Lots of ways to think about delta function
Blackbird ~~pattern~~ poem
Good to think about diff ways to think about δ
5 items which will be on slide

δ - means 0 - det (including impulsive force)
#4 + #5 just starting to learn about
δ is like the identity

②

#5 - shouldn't be too unfamiliar

1. Physical
2. Basic def
3. tricky way w/ Laplace
4. + 5. More specific ways

$*$ = convolution

Want $\mathcal{L}(f * g) = \mathcal{L}(f) \cdot \mathcal{L}(g)$

define $*$ such that

Based on some investigation w/ delta fn that

$$f * g = \int_0^t f(x)\, g(t-x)\, dx$$

f, g are fns of t

x is internal integration variable

So lets check if guess is true

<u>Ex 0</u> $f(t) = 1 = g(t)$

(3)

$$f * g = \int_0^t 1 \cdot 1 \, dx = t \Big|_0^t = t$$

<u>Check</u> So

$$\mathcal{L}(f * g) = \mathcal{L}(t) = \frac{1}{s^2}$$

Compare

$$\mathcal{L}(f) = \mathcal{L}(1) = \frac{1}{s}$$

$$\mathcal{L}(g) = \mathcal{L}(1) = \frac{1}{s}$$

So $\mathcal{L}(f) \cdot \mathcal{L}(g) = \frac{1}{s} \cdot \frac{1}{s} = \frac{1}{s^2}$ ✓

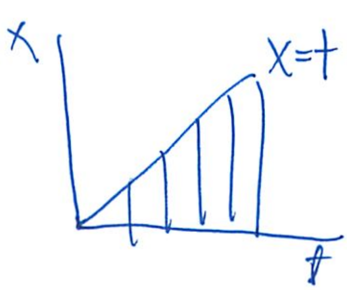
<u>Proof</u>

$$\mathcal{L}(f * g) = \int_{0^-}^{\infty} (f * g)(t) \, e^{-st} \, dt$$

Becomes $\int$ within integral

$$= \int_0^{\infty} \int_0^t f(x) \, g(t - x) \, dx \, e^{-st} \, dt$$

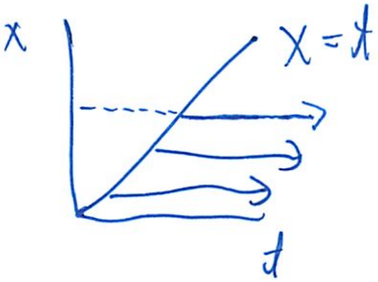
So domain of integration



pick a t
integrate $0 \to x$

(4)

Could also



inner now in terms of t

$$= \int_0^\infty \int_x^\infty \left(g(t-x) e^{-st}\, dt \right) f(x)\, dx$$

Change of variables $t' = t - x$

new integral in t'

$$= \int_0^\infty \int_0^\infty g(t') \underbrace{e^{-s(t'+x)}}_{e^{-st}\, e^{-sx}} dt'\, f(x)\, dx$$

double interal → do 1 at a time

inner + outter

$$= \int_0^\infty \underbrace{\left(\int_0^\infty g(t') \cdot e^{-st'}\, dt' \right)}_{\text{Can pull outside of integral}} \underbrace{f(x)\, e^{-sx} dx}$$

$G(s) = \mathcal{L}(g)$ $\qquad$ $F(s) = \mathcal{L}(f)$

$$= \mathcal{L}(g) \cdot \mathcal{L}(f)$$

⑤

As soon as we know - easy to check convolution satisfies all the nice properties of multiplication

1. Commutative $f * g = g * f$ ← since multiplication $\mathcal{L}(g) \cdot \mathcal{L}(f)$ works, then multiplication works
2. Associative $f * g * h = (f * g) * h$
3. Distributive $f * (g+h) = f * g + h$
4. Identive $f * \delta = \delta * f = f$

$$\mathcal{L}(f \cdot \delta) = \mathcal{L}(f) \cdot \mathcal{L}(\delta)$$

by above

$$= \mathcal{L}(f) \cdot 1$$

Since LaPlace transforms are unique

$$f * \delta = f$$

(only works on fns defined on ⊕ real line)

How does this help us to solve ODEs?

↓

6

ODEs w/ convolution

$$\underbrace{p(D)}_{\substack{\text{operator}\\ \text{(polynomial)}\\ \text{in } D=\frac{d}{dt}}} x = f(t)$$ ← basic/std diff eq

limits from the left (0^-)

Suppose $x(0^-) = x'(0^-) = \dots\ x^{(n)}(0^-)$

↑ call this <u>rest condition</u>

or say $x(t) = 0$ for $t < 0$

if fn

then rest condition

all derivs will be 0

↳ computed up to 0, on left

How do we use to solve ODEs?

↓

(2)

1. Take Laplace transform of everything in site

$$\mathcal{L}(p(D)\,x) = \mathcal{L}(f)$$

since rest conditions

$$p(s)\,\mathcal{L}(x) = \mathcal{L}(f)$$

characteristic eqn

So rewrite

$$\mathcal{L}(x) = \frac{1}{p(s)}\,\mathcal{L}(f)$$

if not rest, more terms are here

Define/Let $w(t) = \mathcal{L}^{-1}\left(\frac{1}{p(s)}\right)$ weight function

Then $\mathcal{L}(x) = \underbrace{\mathcal{L}(w)\,\mathcal{L}(f)}_{\mathcal{L}(w*f)}$

call

By uniqueness of Laplace transforms

$$x = w*f$$

(8)

ie

$$x(t) = \int_0^t f(u)\, w(t-u)\, du$$ ← dummy variable

Input Response Models

Input $f(t)$ → $\boxed{\int_0^t \text{Convolves w/ } w(t-u)\, du}$ → $x(t)$ Output

$\boxed{\int_0^t (\cdots)\, w(t-u)\, du}$ ← Greens' function

So don't need to solve diff eq again + again for every f

Can code up integration into box

Record response

If initial conditions → 2nd convolutions

- remains constant inside (like Homogeneous)

More about why useful on Friday

$\mathcal{L}^{-1}\left(\frac{1}{p(s)}\right) \overset{\text{def}}{=} w(t)$ ← positive real axis only

1.

2. Could also think about as sol to ODE

$p(D)\, w = \delta$

(9)

Why Same?

Take LaPlace of both sides

$\boxed{w: \text{rest conditions}}$ $\quad p(s)\,\mathcal{L}(w) = \mathcal{L}(\delta) = 1$

So $\quad \mathcal{L}(w) = \frac{1}{p(s)}$

$$w = \mathcal{L}^{-1}\left(\frac{1}{p(s)}\right)$$

Example

$p(D) = D^2 + 25$ $\quad$ can do automatically

then $\quad w(t) = \mathcal{L}^{-1}\left(\frac{1}{s^2+25}\right)$

$$= \frac{1}{5}\sin(5t)$$

(or better $\frac{u(t)}{5}\sin(5t)$)

where $u(t)$ = unit step fn

ie $\quad w(t) = \frac{1}{5}\sin 5t$

is sol'n to $\quad (D^2+25)w = \delta$

10

Before we could plug in fn to check if soln

But w/ generalized functions?

Yes we can using generalized derivatives

Example $u_a(t)$ unit step fn at $t=a$

$= U(t-a)$

What is $u'_a(t)$?

Normal deriv - does not exist

Generalized deriv - defined @ u_a integration by parts

$$\int_{-\infty}^{\infty} u_a'(t)\,\phi(t)\,dt$$

ϕ: has good decay @ $\pm\infty$

$$= -\int_{-\infty}^{\infty} u_a(t)\,\phi'(t)\,dt + 0$$

extra term

but how do we do that at $\pm\infty$ - said good decay

(11)

$$\mathcal{L}(u_a(t)) = \frac{e^{-as}}{s}$$ calculated last time

- from def
- do change of variables

+ $\mathcal{L}_-(f') = sF(s) - \underbrace{f(0^-)}$

=0 for fns only supported on real line

=0 if $f(t)=0$ for $t<0$,

$$\mathcal{L}(u_a'(t)) = s\mathcal{L}(u_a(t)$$

$$= s \cdot \frac{e^{-as}}{s}$$

$$= e^{-as}$$

also Laplace transform of delta fn at a

$$= \mathcal{L}(\delta_a(t))$$

So conclude deriv is the delta fn

(12)

L unique: $u'_a(t) = \delta_a(t)$

? same proof works for generalizated fns

integration by parts

got deriv $= \delta$

back to example

$$w(t) = \frac{u(t)}{5} \sin 5t$$

$$w'(t) = u(t) \cos 5t + \xrightarrow{\to 0}{\cancel{\frac{u'(t)}{5}}} \sin 5t$$

Do again

$$w(t) = -5 \sin 5t \; u(t) + \xrightarrow{\to \delta(t)}{\cancel{u'(t)}} \cos 5t$$

take deriv again?

Add it together

Stuff cancles

$\delta(t)$ left

Lecture 26

Last time: Wanted to define operation "$*$" (convolution)

such that $\mathcal{L}(f * g) = \mathcal{L}(f) \cdot \mathcal{L}(g)$

Using δ-function, made guess using integral involving $f \cdot g$

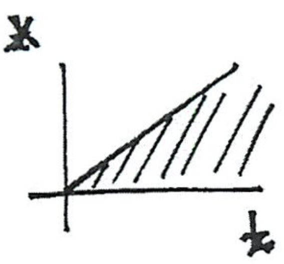
$$f * g = \int_0^t f(x)\, g(t-x)\, dx$$

Example: $f = g = 1$ then $1 * 1 = \int_0^t 1\, dx = t$

$\mathcal{L}(1) = \frac{1}{s}$. $\mathcal{L}(t) = \frac{1}{s^2}$ so indeed $\mathcal{L}(1)\mathcal{L}(1) = \mathcal{L}(1*1)$

(or, if we're being very careful, replace 1 by $u(t)$: unit step function which is 0 for all $t<0$)

Let's prove it (in general): $\mathcal{L}(f*g) = \int_0^\infty \int_0^t f(x)\, g(t-x)\, dx\, e^{-st}\, dt$



Domain of integration

$$= \int_0^\infty \int_x^\infty g(t-x)\, e^{-st}\, dt\, f(x)\, dx$$

$$t' = t - x$$

$$= \int_0^\infty \int_0^\infty g(t')\, e^{-s(t'+x)}\, dt' f(x)\, dx$$

$$= \int_0^\infty g(t')\, e^{-st'}\, dt' \int_0^\infty f(x)\, e^{-sx}\, dx$$

$$= G(s)\, F(s) \quad ✓$$

Since $*$ behaves like multiplication via Laplace transform, (1A)

it satisfies all same properties of mult:

(1) commutative: $f * g = g * f$

(2) associative: $(f * g) * h = f * (g * h)$

(3) distributive: $f * (g + h) = f * g + f * h$

(4) identity: $f * \delta = \delta * f = f$.

prove directly from definition.

or note that

$$\mathcal{L}(f * g) = \mathcal{L}(f) \cdot \mathcal{L}(g)$$
$$= \mathcal{L}(g) \cdot \mathcal{L}(f)$$
$$= \mathcal{L}(g * f)$$

(then $f * g = g * f$ as functions on $(0, \infty)$)

How does this improve our understanding of ODEs?

Suppose we have ODE: $P(D)\,x = f(t)$ with $P(D)$ polynomial operator.

Suppose further that $x(0) = x'(0) = \cdots = x^{(n)}(0) = 0$

(i.e. "rest conditions" — true if $x(t) = 0$ for $t < 0$.)

Then $\mathcal{L}(P(D)x) = P(s)\,\mathcal{L}(x)$ ← usually we get add't'l terms from initial conditions. Here we've assumed all 0.

so $\mathcal{L}(x) = \dfrac{\mathcal{L}(f)}{P(s)}$

Find function $w(t)$: weight function $\mathcal{L}^{-1}\left(\frac{1}{P(s)}\right) = w(t)$

then $\mathcal{L}(x) = \mathcal{L}(w)\,\mathcal{L}(f)$ so $x = w * f$

i.e. $x(t) = \displaystyle\int_0^t w(z)\,f(t-z)\,dz = \int_0^t f(z)\,w(t-z)\,dz$

For solving input-response models, think of convolution as black box for calculating response:

Green's function

INPUT: $f(t)$ → $\boxed{\int_0^t (\cdots)\, w(t-z)\,dz}$ → RESPONSE: $x(t) = f * w$

Two ways to think about weight function $w(t)$:

$\mathcal{L}^{-1}\left(\frac{1}{P(s)}\right) = w(t)$ or as solution to ODE. $P(D)\,w = \delta$ w/ rest conditions: $w(t) = 0$ for $t < 0$

(solve: $\mathcal{L}(P(D)w) = \mathcal{L}(\delta) = 1 \ldots$)

<u>Example</u> : $P(D) = D^2 + 25$. then $w(t) = \mathcal{L}^{-1}\left(\frac{1}{s^2+25}\right)$

Since $\mathcal{L}(\sin bt) = \frac{b}{s^2+b^2}$ then $\mathcal{L}^{-1}\left(\frac{1}{s^2+25}\right) = \frac{1}{5}\sin(5t)$

(or more properly, $\frac{u(t)}{5}\sin 5t$, with $u(t)$: unit step function)

Using <u>second formulation of</u> $w(t)$: $w(t) = \frac{1}{5}\sin 5t$ is solution to

$P(D)\cdot w = \delta$ with $P(D) = D^2+25$

$w(0) = w'(0) = 0$.

Can we check our solution? In ODEs with smooth functions, we just computed derivatives. Regard this as identity of generalized functions. Derivatives defined via integration by parts.

<u>Example</u> : $u_a(t)$: unit step function at $t = a$

$$= \begin{cases} 1 & \text{if } t \geq a \\ 0 & \text{else} \end{cases}$$

$\mathcal{L}(u_a(t)) = e^{-as}/s$ (simple change of vars. from $\mathcal{L}(u_0(t)) = \frac{1}{s}$)

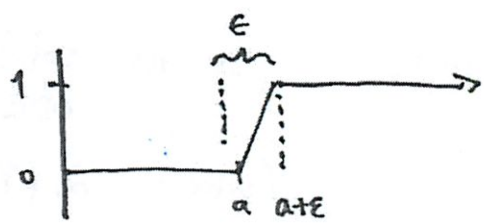
What is $u_a'(t)$? Recall that $\mathcal{L}_-(f') = s\,F(s) - \underbrace{f(0^-)}_{=0}$

so expect $\mathcal{L}(u_a'(t)) = s\cdot\mathcal{L}(u_a(t)) = e^{-as}$

$= \mathcal{L}(\delta_a(t))$

so then $u_a'(t) = \delta_a(t)$.

<u>visually</u> : $u_{a,\epsilon}(t)$:



$$= \begin{cases} 1 & \text{if } t \geq a+\epsilon \\ \frac{t-a}{\epsilon} & \text{if } t \in (a, a+\epsilon) \\ 0 & \text{else} \end{cases}$$

So $u_{a,\varepsilon}(t) \approx u_a(t)$ and has derivative $= \frac{1}{\varepsilon}$ on $(a, a+\varepsilon)$

as $\varepsilon \to 0$, this is precisely $\delta_a(t)$.

Back to checking our example: $w(t) = \frac{u(t)}{5} \sin 5t$

solves $w'' + 25w = \delta$.

$w'(t) = u(t) \cos 5t$. since no jump at $t=0$ in $w(t)$

$w''(t) = -5 \sin 5t \cdot u(t) + \delta(t)$ since, there is a jump of ht. 1 in $w'(t)$ at $t=0$.

so indeed $25\,w(t) + w''(t) = \delta(t)$ ✓.

Being careful with $u_a(t)$: $y'' + y = \delta + \delta_\pi$ with rest conditions $y(0) = y'(0) = 0$.

$$\Rightarrow \quad (s^2+1)\,Y(s) = 1 + e^{-\pi s}$$

$$\mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) = \sin t \cdot u(t)$$

$$\mathcal{L}^{-1}\left(\frac{e^{-\pi s}}{s^2+1}\right) = \sin(t-\pi)\, u(t-\pi)$$

Address physical motivation here, for why solution is one hump of sine function.

Why no more complicated ODEs? Bessel's equation of order 0:

$$t x'' + x' + t x = 0.$$

Solution is not easily described function.

$$\mathcal{L}^{-1}(F(s)) = \frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} F(s)\, e^{st}\, ds \qquad F(s) = \frac{1}{\sqrt{s^2+1}}$$

Math 18.03 : Differential Equations

Lecture 26 Supplemental Notes

Wednesday, November 16, 2011

One Object, Many Views

By now, you've probably realized that having several ways to understand the same (mathematical) object can be incredibly powerful.

Let's compare Wallace Stevens' "Thirteen ways of looking at a blackbird" to the many faces of the delta function $\delta(t)$.

Blackbirds

I.
Among twenty snowy mountains,
The only moving thing
Was the eye of the blackbird.

V.
I do not know which to prefer,
The beauty of inflections
Or the beauty of innuendoes,
The blackbird whistling
Or just after.

IX.
When the blackbird flew out of sight,
It marked the edge
Of one of many circles.

Delta Functions

I. As a "skyscraper function" approximating an impulsive force.

II. As a generalized function – defined according to integration:

$$\int_a^b \delta(t)f(t)\,dt = f(0) \quad \text{if } 0 \in (a,b)$$

III. As the function whose Laplace transform $\mathcal{L}_-$ is equal to 1:

$$\mathcal{L}_-(\delta) = \int_{0^-}^{\infty} \delta(t)e^{-st}\,dt = 1$$

IV. As the identity element in the convolution product $*$:

$$f * \delta = \delta * f = \int_0^t f(\tau)\delta(t-\tau)\,d\tau = f$$

V. The generalized derivative of the unit step function.

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V. The generalized derivative of the unit step function.

18.03 Recitation

11/17

Warmup: Solve $x'' + x = \delta(t-3)$

1. Laplace Transform everything

$$\mathcal{L}(x''+x) = \mathcal{L}(\delta(t-3))$$

$$(s^2+1)\mathcal{L}(x) - x(0)s - x'(0)$$

$$\mathcal{L}(x) = \frac{\mathcal{L}(\delta(t-3)) + x(0)s + (x'(0))}{s^2+1}$$

could find - but could also leave

$$\mathcal{L}^{-1}\left(\frac{\mathcal{L}(\delta(t-3))}{s^2+1}\right) + \mathcal{L}^{-1}\left(\frac{x(0)\,s + x'(0)}{s^2+1}\right)$$

$\mathcal{L}(\delta(t-3)) = e^{-3s}$

$$= \mathcal{L}^{-1}(\mathcal{L}(\delta - 3), \mathcal{L}(\mathcal{L}^{-1}(\tfrac{1}{s^2+1})))$$

Because of convolution

$$\boxed{\begin{aligned}&\mathcal{L}(f*g) = \mathcal{L}(f)\cdot\mathcal{L}(g)\\ &\text{Use by } f*g = \mathcal{L}^{-1}(\mathcal{L}(f)\mathcal{L}(g))\end{aligned}}$$

$$= \delta(t-3) * \sin t$$

do convolution/integration

$$= x(0)\cos t + x'(0)\sin t$$

Partial fractions (if huge polynomial on bottom [aka it does not meet pattern)

②

$$= \int_0^t \delta(z-3)\sin(t-2)\,dz$$

$$= \begin{cases} 0 & t<3 \\ \sin(t-3) & t>3 \end{cases}$$

↑ since was $0 \to t$ integration bands

$$= U(t-3)\sin(t-3)$$

Another one $\quad x''+2x'+x=\delta(t-7)$

$$x'(0)=x(0)=0$$

$$\mathcal{L}(x) = \mathcal{L}\left(\frac{\delta(t-7)}{s^2+2s+1}\right)$$

$$x = \delta(t-7) * \mathcal{L}^{-1}\left(\frac{1}{(s+1)^2}\right)$$

$$* \; -te^{-t}$$

$$\boxed{\begin{cases} P(D)\,x = \delta(t) \\ x(0)=x'(0)=x''(0)=\cdots=0 \end{cases}}$$

$\mathcal{L}$ of

$$P(s)\cdot\mathcal{L}(x) = \mathcal{L}(\delta(t)) = \underline{1}$$

③ Sidebar

$x(t) = \mathcal{L}^{-1}\left(\frac{1}{p(s)}\right)$

$P(D)x = f(t)$ ← solving this has 2 steps

~~cara~~

$P(s) \cdot \mathcal{L}(x) = \mathcal{L}(f(t))$

① Solve for f as δ

② Convolve

$$\mathcal{L}(x) = \frac{\mathcal{L}(f(t))}{p(s)}$$

$$x = \mathcal{L}^{-1}\left(\frac{1}{p(s)}\right) * f(t)$$

$p(s)\,\mathcal{L}(x) - q(s) = \mathcal{L}(f(t))$

↑ involves $x(0)$, $x'(0)$

$$x = \left(f(t) + \mathcal{L}^{-1}(q(s))\right) * \mathcal{L}\left(\frac{1}{p(s)}\right)$$

$\mathcal{L}^{-1}(1) = \delta$

$\mathcal{L}^{-1}(s) = \pm\delta' + \dots$

$\mathcal{L}^{-1}(s^2) = \delta'' + \dots$

(4) Exercise
Compute using table $\mathcal{L}^{-1}$ of

$$1, s, s^2, s^3, \ldots$$

(did not see because worked on paper for other class)

18.03 FALL 2011 – Problem Set 8

Due Friday 11/18/11, high noon in 2-106

To encourage you to keep up with homework as it appears in lecture, both Part I and Part II problems are listed with the accompanying lecture in which the material will be covered.

Part I (24 points)

Lecture 24. Wed. Nov. 9: Solving ODEs with Laplace, delta function
READ: EP 4.2, 4.3 HW: Notes: 3A-9, 10; 3B-1ace, 6.

Friday, Nov. 11 – No Class (Veteran's Day)

Lecture 25. Mon. Nov. 14: Convolution
READ: EP 4.4, Notes CG HW: Notes 3D-1, 4, 6.

Lecture 26. Wed. Nov. 16: Step functions, transfer function
READ: EP 4.5, 4.6 HW: Notes 3C-1b, 2a, 5.

Lecture 27. Fri. Nov. 18: Pole diagrams
READ: Mathlet on Pole Diagrams HW: To be assigned with next pset.

Part II (34 points)

0. (3 points) Write the names of all the people you consulted or with whom you collaborated and the resources you used, or say "none" or "no consultation". This includes visits outside recitation to your recitation instructor. If you don't know a name, you must nevertheless identify the person, as in, "tutor in Room 2-102," or "the student next to me in recitation." Optional: note which of these people or resources, if any, were particularly helpful to you.

1. (Last week, 7 pts) E-P Section 8.5: 4, 7, 11

2. (Wed., Nov. 9, 6 pts) Solve the initial value problem

$$y'' - 3y' + 2y = f(t) \quad y(0) = 1, y'(0) = 0$$

where

a) $f(t) = e^{5t}$

b) $f(t)$ is an impulsive force acting on an extremely short time interval ϵ around $t = 2$ with

$$\int_2^{2+\epsilon} f(t)\,dt = -1.$$

c) $f(t) = 0$ for $0 \le t \le 2$. Use this to compute the constants $z_0 = y(2)$ and $z_0' = y'(2)$. Then solve the ODE above with $f(t) = 0$ but now using $y(2) = z_0, y'(2) = z_0' - 1$ for $2 \le t \le \infty$. Compare with your answer from (b).

3. (Mon., 6 pts)

a) Calculate the convolutions $t * t^2$ and $e^t * e^t$ directly from the definition of convolution.

b) Calculate $e^{at} * e^{bt}$ for $a \neq b$ using the Laplace transform.

c) Take the limit of your answer in (b) as $b \to a$ to find $e^{at} * e^{at}$. Compare with your answer to (a), which is the special case $a = 1$.

d) Find $\cos t * \cos t$ using complex exponentials and your formulas in (a) and (b) for complex numbers.

3. (Wed., Nov. 16, 12 pts) Let $u(t)$ denote the unit step function:

$$u(t) = \begin{cases} 1 & t \ge 0 \\ 0 & t < 0 \end{cases}$$

a) Show that if we assume $\mathcal{L}(f') = sF(s) - f(0^+)$, then using $u' = \delta$ and $u(0^+) = 1$ gives $\mathcal{L}(\delta) = 0$. (Here f can be a generalized function, and so f' is a generalized derivative.)

b) Show that if we define

$$\mathcal{L}_+(f) = \int_{0^+}^{\infty} f(t)e^{-st}\,dt, \quad \mathcal{L}_-(f) = \int_{0^-}^{\infty} f(t)e^{-st}\,dt$$

then we get different formulas for $\mathcal{L}_\pm(f')$ (write them down) and thus different values for $\mathcal{L}_\pm(\delta)$ (again, write them down).

c) Let

$$f(t) = \begin{cases} \sin(2t) & t \ge 0 \\ 0 & t < 0. \end{cases}$$

and let $F(s) = \mathcal{L}(f)$ as usual. Show that

$$\mathcal{L}_-(f'') = s^2 F(s)$$

by calculating each side separately. (Again, f'' is the generalized derivative here.)

d) Use the Laplace transform to show that for f and g generalized functions satisfying $f(t) = g(t) = 0$ for all $t < 0$, then

$$P(D)(f * g) = (P(D)f) * g = f * (P(D)g)$$

for any polynomial operator in D.

e) Let w be the weight function for $P(D)$. Find the differential equation (for generalized functions) satisfied by w. Use your answer from part (d) to show that

$$P(D)(w * f) = f.$$

Michael Plasmeier 18.03 P-Set 8 −20 38/58 11/14

Part 1

Lecture 24 Solving ODEs w/ LaPlace, Delta function

3A-1 By using table of formulas find

a) $\mathcal{L}(e^{-t}\sin 3t)$

; Warmup

(A) $\mathcal{L}(e^{at}\cdot f(t)) = F(s-a)$

So $a=-1$

$f(t) = \sin 3t$

$\mathcal{L}(f(t)) = \mathcal{L}(\sin 3t) = \dfrac{3}{s^2+9}$

but $s-1$

$$= \frac{3}{(s+1)^2+9}$$ ✓

(2)

b) $\mathcal{L}\left(e^{2t}\left(t^2-3t+2\right)\right)$

$f(t) = t^2 - 3t + 2$

Split it up

$t^n \to \frac{n!}{s^{n+1}}$

$\frac{2!}{s^{2+1}} - 3\frac{1!}{s^{1+1}} + 2 \cdot \frac{1}{s}$

Then $s \to (s-2)$

$\frac{2}{(s-2)^3} - \frac{3}{(s-2)^2} + \frac{2}{(s-2)}$ ✓

10) Find $\mathcal{L}^{-1}(F(s))$

a) $\mathcal{L}^{-1}\left(\frac{3}{(s-2)^4}\right)$

? must split? w/ Heavysides

L according to solutions, no

– but how to know?

(3)

So $e^{at} \to \frac{1}{s-a}$

So 4 of those?? No

$e^{2t} \mathcal{L}^{-1}\left(\frac{3}{s^4}\right)$

So do inner → rule (A)

works in reverse too!

$\mathcal{L}(e^{at} \cdot f(t)) = F(s-a)$

So $f(t)$ is $\frac{3}{s^4}$

× (-.5)

$= e^{2t} \frac{3}{s^4}$ (?) Done

---

b) $\frac{1}{s(s-2)}$

Split this w/ Heavysides / Partial Fractions

Solutions say yes

~~$\frac{1}{s(s-2)} = \frac{A}{s} + \frac{B}{s-2}$~~

~~now set $s=0$~~

hmm - seems like bad plan

~~$\frac{1}{2(2-0)} = \frac{A}{2} + \frac{B}{2-2}$~~

~~set $s=2$~~

(4)

$\frac{A}{2} = 1$

$A = 2$

Now try other one

$$\frac{1}{s(s-2)} = \frac{2}{s} + \frac{B}{s-2}$$

multiply so denom disappears

$$\frac{1\,\cancel{s}(\cancel{s-2})}{\cancel{s}(\cancel{s-2})} = \frac{2\,(\cancel{s})(s-2)}{\cancel{s}} + \frac{B\,(s)(\cancel{s-2})}{(\cancel{s-2})}$$

$1 = 2(s-2) + Bs$

Coefficent of LHS constant $= 1$

$s = 0$

$1 = 2s - 4 + Bs$

$1 = -4$??

$0 = 2 + B$

$B = 0$

(x) No - should be $\frac{1}{2}$

That last part not right!

(5)

Try Heavysides over (studied more)

$$\frac{1}{s(s-2)} = \frac{A}{s} + \frac{B}{s-2}$$

? try to get rid of

$$\frac{1}{s\cancel{(s-2)}}\cancel{(s-2)} = \frac{A(s-2)}{s} + \frac{B\cancel{(s-2)}}{\cancel{(s-2)}}$$

$$\frac{1}{s} = \frac{A(s-2)}{s} + B$$

Pick $s=2$ so A term disappers

$$\frac{1}{2} = 0 + B$$

$$B = \frac{1}{2}$$

$$\frac{1}{\cancel{s}(s-2)}\cancel{s} = \frac{A\cancel{(s)}}{\cancel{s}} + \frac{B(s)}{(s-2)}$$

$$\frac{1}{s-2} = A + \frac{B(s)}{(s-2)}$$

set $s=0$

$$-\frac{1}{2} = A + 0$$

So $$\frac{-\frac{1}{2}}{s} + \frac{\frac{1}{2}}{s-2}$$ ✓ On Progress

6

Now the Laplace stuff

$$\mathcal{L}^{-1}\left(\frac{-\frac{1}{2}}{s}\right) = -\frac{1}{2}$$

$$\mathcal{L}^{-1}\left(\frac{\frac{1}{2}}{s-2}\right) = e^{2t}\,\frac{1}{2}$$

$$\frac{1}{2}e^{2t} - \frac{1}{2}$$ ✓ Bingo

✓

c) $\dfrac{s+1}{s^2-4s+5}$

? factor $\dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

$$= \frac{4 \pm \sqrt{16-4(1)(5)}}{2(1)}$$

$$= \frac{4 \pm \sqrt{-4}}{2}$$

$$= 2 \pm i$$

hmm - I don't think we deal w/ those here

Solutions: Complete the square

$$\frac{s+1}{(s-2)^2+1}$$ ← I guess ya could do that

7

Now want top to match

$$\frac{s-2}{(s-2)^2+1} + \frac{3}{(s-2)^2+1}$$

So can say

$$e^{2t} \cos 1t + 3 e^{2t} \sin 1t$$

rule A ; b=1 ; take out in front ; rule A ; b=1

✓

3B-1a) Solve IVPs w/ LaPlace

a) $y' - y = e^{3t}$ $\quad y(0) = 1$

I will do it long way for 1st time

LHS

$\mathcal{L}(y' - y) = \mathcal{L}(y') - \mathcal{L}(y)$

$\mathcal{L}(y') = s F(s) - f(0)$

$= s \mathcal{L}(y) - f(0)$

given as $y(0)=1$

$= s\mathcal{L}(y) - f(0) - \mathcal{L}(y)$

(8)

$$= s\mathcal{L}(y) - \mathcal{L}(y) - 1$$

$$= (s-1)\mathcal{L}(y) - 1$$

? charastic polynomial ✓ Matches

? was this what was called "junk"?? ? can't find reference

RHS,

$$\mathcal{L}(e^{3t}) = \frac{1}{s-3}$$

So now write it

$$(s-1)\mathcal{L}(y) - 1 = \frac{1}{s-3}$$

Write in terms of $\mathcal{L}(y)$

$$\mathcal{L}(y) = \frac{\frac{1}{s-3} + 1}{s-1}$$

Make proper function

? How again - dividing everything $\frac{1}{(s-3)(s-1)} + \frac{1}{(s-1)}$

(9)

Common factors

$$= \frac{1}{(s-3)(s-1)} + \frac{(s-3)}{(s-1)(s-3)}$$

$$= \frac{1+s-3}{s^2-4s+3}$$

Now take inverse

$$y = \mathcal{L}^{-1}\left(\frac{1+s-3}{s^2-4s+3}\right)$$

But need to heavysides it

$$\frac{1+s-3}{s^2-4s+3} = \frac{A}{s-1} + \frac{B}{s-3}$$

Multiply by $s-1$

$$\frac{(1+s-3)\cancel{(s-1)}}{(s-3)\cancel{(s-1)}} = \frac{A\,(s-1)}{\cancel{(s-1)}} + \frac{B\,(s-1)}{s-3}$$

$s=1$

$$\frac{1+1-3}{1-3} = A + 0B$$

(10)

$$A = \frac{-1}{-2} = \frac{1}{2}$$

Now B

$$\frac{(1+s-3)\cancel{(s-3)}}{\cancel{(s-3)}(s-1)} = \frac{A(s-3)}{(s-1)} + \frac{B\cancel{(s-3)}}{\cancel{s-3}}$$

$$s=3$$

$$\frac{1+3-3}{3-1} = B$$

$$\frac{1}{2} = B$$

$$= \frac{\frac{1}{2}}{s-1} + \frac{\frac{1}{2}}{s-3}$$

? what was point of having it in this form? numerator separated?

$$y = \mathcal{L}^{-1}\left(\frac{\frac{1}{2}}{s-1} + \frac{\frac{1}{2}}{s-3}\right)$$

$$y = \frac{1}{2}e^{t} + \frac{1}{2}e^{3t}$$ ✓ Nice

✓

---

c) $y'' + 4y = \sin t$ $\quad y(0)=1$, $y'(0)=0$

Now try shortcut

$$\underbrace{(s^2+4)}_{\text{characteristic}}\mathcal{L}(y) - s\,y(0) - y'(0)$$

(11)

$$\mathcal{L}(\sin t) = \frac{1}{s^2+1^2}$$

$$(s^2+4)\,\mathcal{L}(y) - (s \cdot 1) - 0 = \frac{1}{s^2+1}$$

$$\mathcal{L}(y) = \frac{\frac{1}{s^2+1} + s}{s^2+4}$$

Proper fraction

$$\frac{1}{(s^2+1)(s^2+4)} + \frac{s}{(s^2+4)}$$ ✓ check point

$$\frac{(s)(s^2+1)}{(s^2+1)(s^2+4)}$$

Heavysides

$$\frac{(s)(s^2+1)}{(s^2+1)(s^2+4)} = \frac{Ax+B}{s^2+1} + \frac{Cx+D}{s^2+4}$$

? can you break up furter

(X) Sols do something different

12

Treat s^2 as just v

? How do we know this is possible
How could we think of doing
I guess Heavysides would fail

Lets do Heavysides for fun

$$\frac{(s)(s^2+1)\cancel{(s^2+1)}}{\cancel{(s^2+1)}(s^2+4)} = \frac{(Ax+B)\cancel{(s^2+1)}}{\cancel{s^2+1}} + \frac{(Cx+D)(s^2+1)}{s^2+4}$$

~~s can be ± 1~~
~~say +1~~ (: do both later ??)

$$\cancel{\frac{(1)(1+1)}{(1+4)}} = Ax+B$$

No can't set s so
Heavysides would not work

$$\underbrace{\frac{1}{(v+1)(v+4)}}_{\text{for just 1st part}} = \frac{A}{(v+1)} + \frac{B}{(v+4)}$$

(13)

A

$$\frac{1}{u+4} = A + 0$$

$u = -4$

$$\frac{1}{3} = A$$

B

$$\frac{1}{u+1} = B$$

$u = -4$

$$B = \frac{-1}{3}$$

I can't do in head
- should write at

$$\mathcal{L}^{-1}\left(\frac{\frac{1}{3}}{u+1} - \frac{\frac{1}{3}}{u+4} + \frac{s}{s^2+4}\right)$$

↑ So basically want bite sized pieces that match our pattern

Must restore s^2

$$= \mathcal{L}^{-1}\left(\frac{\frac{1}{3}}{s^2+1} - \frac{\frac{1}{3}}{s^2+4} + \frac{s}{s^2+4}\right)$$

$$= \frac{1}{3}\sin 1t - \frac{1}{6}\sin 2t + \cos 2t$$ ✓

↑ since numerator should be 2

(14)

e) $y'' - 2y' + y = e^t$ $\quad y(0) = 1$, $y'(0) = 0$

$$(s^2 - 2s + 1)\mathcal{L}(y) - (s-2)y(0) - y'(0)$$

forgot originally $\quad$? don't need to add extra - already taken care of

$$(s^2 - 2s + 1)\mathcal{L}(y) - (s-2) = \frac{1}{s}$$

$$\mathcal{L}(y) = \frac{\frac{1}{s} + s - 2}{s^2 - 2s + 1}$$

$$= \frac{\frac{1}{s} + s - 2}{(s-1)(s-1)}$$

$$= \frac{1}{(s)(s-1)(s-1)} + \frac{s-2}{(s-1)(s-1)}$$

$$= \frac{s^2 - 2s}{(s)(s-1)(s-1)}$$ ← didn't propagate fix further

(15)

Hevinsides!

$$\frac{s^2}{(s)(s-1)(s-1)} = \frac{A}{s} + \frac{B}{(s-1)^2} + \frac{C}{s-1}$$

need to keep squared

$$\frac{s^2(s)}{(s-1)^2} = A + \frac{B(s)}{(s-1)^2} + \frac{C(s)}{s-1}$$

$s=0$

$$\frac{0}{1} = A$$

$$\frac{s^2\cancel{(s-1)^2}}{(s)(s-1)^2} = \frac{A(s-1)^2}{s} + \frac{B\cancel{(s-1)^2}}{\cancel{(s-1)^2}} + \frac{C\cancel{(s-1)}(s-1)}{\cancel{(s-1)}}$$

$s=1$

$$\frac{1^2}{1} = 0 + B + 0$$

$1 = B$

$$\frac{(s^2)\cancel{(s-1)}}{s(s-1)\cancel{(s-1)}} = \frac{A(s-1)}{s} + \frac{B\cancel{(s-1)}}{(s-1)^2} + \frac{C\cancel{(s-1)}}{\cancel{(s-1)}}$$

$\frac{1^2}{0}$ ← invalid

$s=1$

(16)

B can't be found w/ coverup method

Try subbing in any # like $x = 2$

$$\frac{2^2}{2(2-1)^2} = 0 + \frac{1}{(s-1)^2} + \frac{C}{(s-1)}$$

$$2 = 1 + \frac{C}{1}$$

$$C = 1$$

So

$$\mathcal{L}^{-1}\left(\frac{1}{(s-1)^2} + \frac{1}{(s-1)}\right)$$

$$= te^t + e^t$$

(X) I think we lost the $\frac{1}{(s-1)^3}$ term somewhere

↳ goes to $\frac{t^2}{2}e^t$

Ohh so 1st conversion

$e^t \to \frac{1}{s-1}$ Not $\frac{1}{s} \xrightarrow{\text{which is inverse}} 1$

(17)

$$\mathcal{L}(y) = \frac{\frac{1}{s-1} + s}{(s-1)(s-)}$$

$$\frac{1}{(s-1)^3} + \frac{s}{(s-1)^2}$$

?could expand? no

and does not match

? Now missing somemore terms??

I had forgotton $(s-2)$ term → just had s

-2 what did you just do?

(18)

6) If $y(t)$ is a solution to IVP

$$y'' + ty = 0 \qquad y(0) = 1, \quad y'(0) = 0$$

What ODE is satisfied by fn

$$Y(s) = \mathcal{L}(y(t))?$$

The solution $y(t)$ is called an Airy Function

The ODE it solves is an Airy equation

(this sounds like a part B question)

LaPlace it

$$(s^2 + t)\mathcal{L}(y) - s\,y(0) - y'(0) = \mathcal{L}(0)$$

$$(s^2 + t)\mathcal{L}(y) - s = 0$$

$$\mathcal{L}(y) = \frac{s}{s^2 + t}$$

+ (-2)

? can we decompose?

No - just look at it!

$$y = \cos\sqrt{t}\,dt \;?$$

(X) Book gets $-s$

? I don't know how

(19) 11/17

Lecture 25 Convolution

3D-1) Solve the IVP $y'' + 2y' + y = \delta(t) + u(t-1)$

$y(0) = 0$

Write ans in "cases" format $y'(0^-) = 1$

$$y(t) = \begin{cases} \cdots & 0 \le t \le 1 \\ \cdots & t > 1 \end{cases}$$

1st step: Take LaPlace of

screwed up previously

$$(s^2 + 2s + 1)\mathcal{L}(x) - (s+2)(0) - 1 = \mathcal{L}(\delta(t-1))$$

$$\mathcal{L}(x) = \frac{\mathcal{L}(\delta(t-1)) + (s+2)(0) - 1}{s^2 + 2s + 1}$$

$$x = \mathcal{L}^{-1}\left(\frac{\mathcal{L}(\delta(t-1))}{s^2+2s+1}\right) - \mathcal{L}^{-1}\left(\frac{1}{s^2+2s+1}\right)$$

$$= \mathcal{L}^{-1}(\mathcal{L}(\delta(t-1))) * \mathcal{L}^{-1}\left(\frac{1}{(s+1)^2}\right)$$

(20)

$\downarrow$

Now this is convolution

$$\mathcal{L}(f * g) = \mathcal{L}(f) \cdot \mathcal{L}(g)$$

use by $f * g = \mathcal{L}^{-1}\left(\mathcal{L}(f)\,\mathcal{L}(g)\right)$

$$= \delta(t-1) * t e^{-t}$$

$$= \int_0^t \delta(t-1)\,(t-z)e^{-(t-z)}\,dz$$

$$= \begin{cases} 0 & t < 1 \\ (t-1)\,e^{-(t-1)} & t > 1 \end{cases}$$

$\downarrow$

$= t e^{-t}$

Combined

$$= \begin{cases} 0 & t < 1 \\ (t-1)\,e^{-(t-1)} + t e^{-t} & t > 1 \end{cases}$$

(X) Oh in first step illegally combined $\delta(t) + u(t-1)$

x -2

(21)

4. Find $\mathcal{L}^{-1}$ using Convolution

a) $\mathcal{L}^{-1}\left(\frac{s}{(s+1)(s^2+4)}\right)$

Need to split w/ partial fractions

$$\frac{s}{(s+1)(s^2+4)} = \frac{A}{s+1} + \frac{B}{(s^2+4)}$$

$$\frac{s\cancel{(s+1)}}{\cancel{(s+1)}(s^2+4)} = \frac{A\cancel{(s+1)}}{\cancel{(s+1)}} + \frac{B(s+1)}{(s^2+4)}$$

$s=-1$

$$\frac{-1}{1+4} = A + 0$$

$$A = -\frac{1}{5}$$

$$\frac{s\cancel{(s^2+4)}}{(s+1)\cancel{(s^2+4)}} = \frac{A(s^2+4)}{(s+1)} + \frac{B\cancel{(s^2+4)}}{\cancel{(s^2+4)}}$$

s = can't do it

(22)

Must do other way

$$\frac{s}{(s+1)(s^2+4)} = \frac{-\frac{1}{5}}{(s+1)} + \frac{B}{s^2+4}$$

So set s to anything $\neq -\frac{1}{5}$

$s=1$

$$\frac{1}{(2)(5)} = \frac{-\frac{1}{5}}{2} + \frac{B}{5}$$

$$\frac{1}{10} = -\frac{1}{10} + \frac{B}{5}$$

$$\frac{2}{10} = \frac{B}{5}$$

$B=1$

$$\mathcal{L}^{-1}\left(\frac{-1/5}{(s+1)} + \frac{1}{s^2+4}\right)$$

$$\mathcal{L}^{-1}\left(\frac{-\frac{1}{5}}{(s+1)}\right) + \mathcal{L}\left(\frac{1}{s^2+4}\right)$$

$$-\frac{1}{5}e^{-t} + \frac{1}{2}\sin 2t$$

but I don't think we were supposed to do it that way (X) and its wrong

I did partial fractions wrong

(23) Retry

$$= \mathcal{L}^{-1}\left(\frac{1}{s+1} \cdot \frac{s}{s^2+4}\right)$$

$$= \mathcal{L}^{-1}\left(\frac{1}{s+1}\right) * \mathcal{L}^{-1}\left(\frac{s}{s^2+4}\right)$$

$$= e^{-t} * \cos 2t$$

$$= \int_0^t -e^{-(t-z)} \cos 2z \, dz$$

$$= \underbrace{e^{-t}}_{\text{move out}} \int_0^t e^{z} \cos 2z \, dz$$

trig identity

$$= e^{-t}\left[\frac{e^t}{5}(\cos 2t + 2\sin 2t) - \frac{1}{5}\right]$$

$$= \frac{1}{5}\cos 2t + \frac{2}{5}\sin 2t - \frac{1}{5}e^{-t}$$

(✓) Better.

b) $\mathcal{L}^{-1}\left(\frac{1}{(s^2+1)^2}\right)$

$$= \mathcal{L}^{-1}\left(\frac{1}{(s^2+1)} \cdot \frac{1}{(s^2+1)}\right)$$

$$= \mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) * \mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right)$$

24

$= \sin t * \sin t$

$= \int_0^t \sin(t-z)\sin z \, dz$

~~sin t constant~~ no does not work

trig identity

$= \int_0^t \frac{1}{2}\left[\cos(t-2z) \cdot \cos z\right] dz$

✓ repeated root handled

$= \frac{\sin t}{2} - \frac{t\cos t}{2}$ ✓

---

6. Prove that $f(t) * g(t) = g(t) * f(t)$

Is in ~~textbook~~ recitation

$y = t - z$

$dy = -dz$

can see when make change in variable

$\int_0^t f(z)\, g(t-z)\, dz = -\int_t^0 g(y-t)\, f(y)\, dy$

Book ↑ limits

$z=0 \to y=t$

$z=t \to y=0$

So negitive

↑ ↑ and these switch

did wrong in recitation ?

change of var. (-2)

Lecture 26 Step Function, Transfer Function

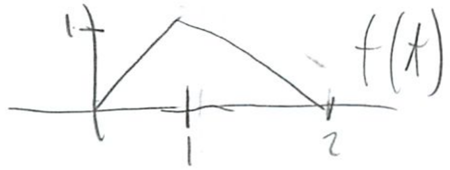
3C-1b

Find the LaPlace Transform of each of the functions in terms of table

b) $f(t) = \begin{cases} t & 0 \le t \le 1 \\ 2-t & 1 \le t \le 2 \\ 0 & \text{otherwise} \end{cases}$

So $u(t-a) = \begin{cases} 0 & t < a \\ 1 & t \geq a \end{cases}$

$u(t-a)\, f(t-a) = \begin{cases} 0 & t < a \\ f(t-a) & t \geq a \end{cases}$



So this is additive like before

$t\, u(t) - u(t-1) \cdot 2(t-1) + u(t-2)\,(t-2)$

apply at 0 — remove — remove all

(26)

Then find $\mathcal{L}(\text{of})$

Rules

$$\mathcal{L}(u(t-a)f(t-a)) = e^{-as}F(s)$$

$$\mathcal{L}^{-1}(e^{-as}F(s) = u(t-a)f(t-a)$$

$$\mathcal{L}(u(t-a)) = \frac{e^{-as}}{s}$$

So

$$\frac{1}{s^2}\cdot\frac{e^{-0s}}{s} - \frac{e^{-1s}}{s}2\left(\frac{1}{s^2}-\frac{1}{s}\right) + \frac{e^{-2s}}{s}\left(\frac{1}{s^2}-\frac{2}{s}\right)$$

$$\frac{1}{s^3} - 2\frac{e^{-s}}{s^3} - 2\frac{e^{-s}}{s^2} + \frac{e^{-2s}}{s^3} - 2\frac{e^{-2s}}{s^2}$$

$$\frac{1}{s^2}\left(\frac{1}{s} - \frac{2e^{-s}}{s} - 2e^{-s} + \frac{e^{-2s}}{s} - 2e^{-2s}\right)$$

(X) Book

-2

$$\frac{1}{s^2}\left(1-2e^{-s}+e^{-2s}\right)$$

must have had too many terms in there somewhere...

27

2a] Find $\mathcal{L}^{-1}\left(\frac{e^{-s}}{s^2+3s+2}\right)$

¿ Convolution ?

$$\mathcal{L}^{-1}\left(\frac{e^{-s}}{x+1}\right) * \mathcal{L}^{-1}\left(\frac{1}{x+2}\right)$$

Book uses partial fractions

$$\frac{e^{-s}\cancel{(x+1)}}{\cancel{(x+1)}(x+2)} = A + \frac{B(x+1)}{(x+2)}$$

$x=-1$

$$\frac{e^{-s}}{1} = A$$

$$\frac{e^{-s}}{(x+1)} = \frac{A(x+2)}{(x+1)} + B$$

$x=-2$

$$-e^{-s} = B$$

$$\mathcal{L}^{-1}\left(\frac{e^{-s}}{x+1}\right) + \mathcal{L}^{-1}\left(\frac{e^{-s}}{x+2}\right)$$

(28)

So e^{-s} is $\delta(t)$

$$\mathcal{L}^{-1}\left(\frac{1}{x+1}\right) = e^{-t}$$

we did convolution inside there

So
$$\begin{cases} 0 & t<0 \\ e^{-t} & t>0 \end{cases}$$

$$+ \begin{cases} 0 & t<0 \\ e^{-2t} & t>0 \end{cases}$$

So
$$\begin{cases} 0 & t<0 \\ e^{-t}+e^{-2t} & t>0 \end{cases}$$

~~Ohhh only the denominator gets partial fractionidized - duh in retrospect - what I screwed up earlier~~

Sols
$$\mathcal{L}^{-1}\left(\frac{e^{-s}}{s^2+3s+2}\right) = u(t-1)\, f(t-1)$$

?calc above

-1

$$= u(t-1)\left(e^{(1-t)} - e^{2(1-t)}\right)$$

So I got it wrong - oh during "convolution" did not make it $(1-t)$ - so I did above right (inc partial fractions)

but convolution wrong

28

5. Solve the IVP $y'' - 3y' + 2y = r(t)$

$y(0) = 1 \qquad y'(0) = 0 \qquad r(t) = u(t)\,t$

So $y'' - 3y' + 2y = u(t)\,t$

$$(s^2 - 3s + 2)\,\mathcal{L}(x) - (s-3)\,y(0) - y'(0) = \mathcal{L}(\sigma(t)\,t)$$

$\frac{1}{s^2}$

$$\mathcal{L}(x) = \frac{\frac{1}{s^2} + (s-3)}{s^2 - 3s + 2}$$

split

$$= \frac{(s-3)}{(s-2)(s-1)} + \frac{1}{s^2(s-2)(s-1)}$$

get common denom

$$= \frac{s^2(s-3) + 1}{s^2(s-2)(s-1)}$$

$$= \frac{s^3 - 3s^2 + 1}{s^2(s-2)(s-1)}$$

Partial fraction it at

30

$$\frac{s^3-3s^2+1}{s^2(s-2)(s-1)} = \frac{A}{s^2} + \frac{B}{s-2} + \frac{C}{s-1}$$

$$\frac{(s^3-3s^2+1)}{(s-2)(s-1)} = A + \frac{Bs^2}{s-2} + \frac{Cs^2}{s-1}$$

$s=0$

$$\frac{1}{2} = A$$

$$\frac{(s^3-3s^2+1)}{s^2(s-1)} = \frac{A(s-2)}{s^2} + B + \frac{C(s-2)}{s-1}$$

$s=2$

$$\frac{(8-12+1)}{(4)(1)} = B$$

$$-\frac{3}{4} = B$$

$$\frac{(s^3-3s^2+1)}{s^2(s-2)} = \frac{A(s-1)}{s^2} + \frac{B(s-1)}{s-2} + C$$

$s=1$

$$\frac{(1-3+1)}{(1)(-1)} = C$$

$C=1$

(31)

Oh did it wrong

$$\frac{A}{s^2} + \frac{D}{s} + \cdots$$ ← call it D

$$\frac{(s^3-3s^2+1)\cancel{s}}{s^{\cancel{2}}(s-1)(s-2)} = D + \frac{A\cancel{s}}{s^{\cancel{2}}} + \frac{Bs}{s-2} + \frac{Cs}{s-1}$$

$s=0$

$$\frac{1}{2} = D + \frac{A}{0}$$ ← error

Other way

$$\frac{s^3-3s^2+1}{s^2(s-1)(s-2)} = \frac{\frac{1}{2}}{s^2} + \frac{-\frac{3}{4}}{s-2} + \frac{1}{s-1} + \frac{D}{s}$$

$s=3$ ← anything

$$\frac{27-27+1}{(9)(2)(1)} = \frac{\frac{1}{2}}{9} + \frac{-\frac{3}{4}}{1} + \frac{1}{2} + \frac{D}{s}$$

$$\frac{1}{18} = \frac{1}{18} - \frac{3}{4} + \frac{1}{2} + \frac{D}{3}$$

$$\frac{1}{4} = \frac{D}{3}$$

$$4D = 3$$

$$D = \frac{3}{4}$$

(32)

$$= \frac{\frac{1}{2}}{s^2} + \frac{\frac{3}{4}}{s} + \frac{-\frac{3}{4}}{s-2} + \frac{1}{s-1}$$

(Inverse) Laplace transform it

$$y = \frac{1}{2}t + \frac{3}{4} - \frac{3}{4}e^{2t} + e^{t}$$

✓ Woot figured it out on my own ✓

Part 2

0. ~~No one yet~~
Kimberly Aziz

1. 8.5 #4 Boundry Value problem

$$U_t = U_{xx} \quad 0 < x < \pi \quad t > 0$$
$$U(0,t) = U(\pi,t) = 0$$
$$U(x,0) = 4\sin 4x \cos 2x$$

(asking problems from last week - no fail!)

(33)

$$u(t,x) = \sum_{n=1}^{\infty} b_n e^{\frac{-n^2\pi^2 t}{L^2}} \sin \frac{n\pi x}{L}$$

going to skip ahead - this was last weeks work!

$$f(x) = \frac{4(4 \sin 4x \cos 2x)}{\pi} \sum_{n \text{ odd}} \frac{1}{n} \sin \frac{n\pi x}{L}$$

$$4 \sin 4x \cos 2x = 2 \sin 2x + 2 \sin 6x$$

Does it match for any n?

yes at $n = 2, 6$ b_2 and $b_6 = 2$

$$u(x,t) = 2e^{-4t} \sin 2x - 2e^{-36t} \sin 6t$$

#7. $3u_t = u_{xx}$ $0 < x < 2$ $t > 0$ $u_x(0,t) = u_x(2,t) = 0$ $u(x,0) = \cos^2 2\pi x$

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-\frac{n^2\pi^2 k t}{L^2}} \cos \frac{n\pi x}{L}$$

$a_8 = 1$

$\frac{a_0}{2} = \frac{1}{2}$

(34)

$$u(x,t) = \frac{1}{2} + \sum_{n=1}^{\infty} a_n \, e^{-\frac{n^2\pi^2 kt}{3\cdot 4}} \cos\frac{n\pi x}{2}$$

$$\cos^2 2\pi x = \frac{1 + \cos 4\pi x}{2}$$

$$a_0 = 1 \qquad a_5 = \frac{1}{2} \qquad a_n = 0$$

$$U(x,t) = \frac{1}{2} + \frac{1}{8} e^{-\frac{64\pi^2 t}{12}} \cos\frac{8\pi x}{2}$$

$$= \frac{1}{2} + \frac{1}{8} e^{-\frac{16}{3}\pi^2 t} \cos 4\pi x$$

-.5

---

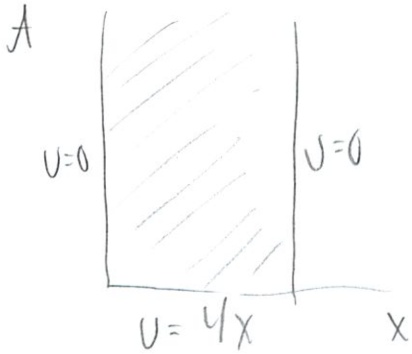
11. $5u_t = u_{xx}$

$0 < x < 10$

$t > 0$

$u_x(0,t) = u_x(10,t) = 0$

$u_x(x,0) = 4x$



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$k = \frac{1}{5}$

$L = 10$

$U(x,t) =$

Super position of solutions

$$T = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} e^{\lambda_n t} + b_n \sin \frac{n\pi x}{L} e^{\lambda_n t}$$

$$= \frac{1}{2} a_0 + \sum \left(a_n \cos \frac{n\pi x}{t} + b_n \sin \frac{n\pi x}{L} \right)$$

? did I make mistake in copying

∴ using example 2

$$f(t) = \frac{4}{\pi} \sum_{n \text{ odd}} \frac{1}{n} \sin \frac{n\pi t}{L} = \begin{cases} +1 & 0 < t < L \\ -1 & -L < t < 0 \end{cases}$$

So Fourier Sine series $f(x) = 4x$

$$f(x) = \frac{4 \cdot (4x)}{\pi} \sum_{n \text{ odd}} \frac{1}{n} \sin \frac{n\pi x}{L}$$

For $0 < x < L$. Hence

$$b_n \begin{cases} \frac{4(4x)}{n\pi} & \text{for } n \text{ odd} \\ 0 & \text{" } n \text{ even} \end{cases}$$

(36)

So rod's temp

$$u(t, x) = \frac{4 u_0}{\pi} \sum_{n \text{ odd}} \frac{1}{n} e^{\frac{-n^2 \pi^2 k t}{L^2}} \sin \frac{n\pi x}{L}$$

Fill in values

$U_0 = 4x$

etc

-1.

37

2. Solve the IVP

$$y'' - 3y' + 2y = f(t) \qquad y(0)=1$$
$$y'(0)=0$$

a) $f(t) = e^{5t}$

(I've done this enough times ...)

$$(s^2 - 3s + 2)\mathcal{L}(y) - (s-3)y(0) - y'(0) = \frac{1}{s-5}$$

$$\mathcal{L}(y) = \frac{\frac{1}{s-5} + (s-3)}{s^2-3s+2}$$

$$\mathcal{L}^{-1}\left(\frac{\frac{1}{s-5}}{s^2-3s+2}\right) + \mathcal{L}^{-1}\left(\frac{s-3}{s^2-3s+2}\right)$$

Convolution

$$\mathcal{L}^{-1}\left(\frac{1}{s-5}\right) * \mathcal{L}^{-1}\left(\frac{1}{s-2}\right) * \mathcal{L}^{-1}\left(\frac{1}{s-1}\right) + \mathcal{L}^{-1}(s-3) * \mathcal{L}^{-1}\left(\frac{1}{s-2}\right) * \mathcal{L}^{-1}\left(\frac{1}{s-1}\right)$$

Or partial fractions

$$\frac{1}{(s-5)(s-2)(s-1)} + \frac{s-3}{(s-2)(s-1)}$$

← can split here, trick

(38)

$$\frac{1}{(s-5)(s-2)(s-1)} + \underbrace{\frac{\cancel{s-2}}{(\cancel{s-2})(s-1)}}_{e^t} + \frac{1}{(s-2)(s-1)}$$

So partial fractions on each one

Using WA since I did enough practice

$$-\frac{1/3}{(s-2)} + \frac{1/4}{(s-1)} + \frac{1/12}{(s-5)} + e^t + \frac{1}{(s-2)} - \frac{1}{(s-1)}$$

$$-\frac{1}{3}e^{2t} + \frac{1}{4}e^t + \frac{1}{12}e^{5t} \cancel{+e^t} + e^{2t} \cancel{-e^t}$$

$$y = \frac{1}{12}e^{5t} + \frac{2}{3}e^{2t} + \frac{1}{4}e^t$$

-1

---

b) $f(t)$ = impulse around $t=2$

$$\int_2^{2+\epsilon} f(t)\,dt = -1$$

So basically $-1\,\delta(t-2)$

$$\mathcal{L}(\;''\;) = -1e^{-2s}$$

(39)

Similar

$$\mathcal{L}(y) = \frac{-e^{-2s} + (s-3)}{s^2-3s+2}$$

$$= \mathcal{L}^{-1}\left(\frac{-e^{-2s}}{s^2-3s+2}\right) + \mathcal{L}^{-1}\left(\frac{s-3}{s^2-3s+2}\right)$$

convolution

did last time

$$\mathcal{L}^{-1}(\mathcal{L}(\delta(t-2))) * \mathcal{L}^{-1}\left(\frac{1}{s^2-3s+2}\right)$$

did last time?

-use WA

$$\delta(t-2) * \left[e^t(e^t-1)\right]$$

$\int$ etc

but pattern same

$$\begin{cases} 0 & t<2 \\ e^t(e^t-1) & t>2 \end{cases} + e^{2t} - e^t$$

$$u(t-2)\, e^t(e^t-1) + e^{2t} - e^t$$

(40)

c) $f(t) = 0$ for $0 \le t \le 2$

Use this to compute constants

$z_0 = y(2) \qquad z_0' = y'(2)$

Then solve ODE above w/ $f(t) = 0$

But now using $y(2) = z_0$

$y'(2) = z_0' - 1$ for $2 \le t \le \infty$

? did we do this in class?

-2

(41)

3. Calculate the convolutions $t * t^2$ directly from def'n

$$\underset{f}{t} \quad \underset{g}{t^2}$$

$$f * g = \int_0^t f(x)\, g(t-x)\, dx$$

$$\int_0^t x\,(t-x)^2\, dx$$

$$u = t-x$$
$$du = -dx$$

$$\int_0^t u^2(u-t)\, du$$

$$\int_0^t (u^3 - u^2 t)\, du$$

$$\int u^3\, du - t\int u^2\, du$$

$$\frac{u^4}{4} - t\,\frac{u^3}{3}$$

Sub back

$$\frac{(t-x)^4}{4} - t\,\frac{(t-x)^3}{3}\Bigg|_0^t$$

$$\frac{1^4}{4} - t\,\frac{(1)^3}{3}$$

$$= \frac{t^4}{12}$$ ✓

(12)

$e^t * e^t$

$= \int_0^t e^x e^{(t-x)} \, dx$

$= \int_0^t e^x e^t e^{-x} \, dx$

$= \int_0^t e^t \, dx$

$e^t x |_0^t$

$= t e^t$

b) Calc $e^{at} * e^{bt}$ for $a \neq b$

$\int_0^t e^{ax} e^{b(t-x)} \, dx$

$= \int_0^t e^{ax} e^{bt} e^{-bx}$

?, how to split?

Or do w/ LaPlace transform

$e^{at} * e^{bt}$

$\mathcal{L}(e^{at}) \cdot \mathcal{L}(e^{bt})$

$\frac{1}{s-a} \cdot \frac{1}{s-b}$

(43)

Common denom

$$\frac{1}{(a-b)}\left(\frac{1}{s-a}-\frac{1}{s-b}\right)$$

Laplace inverse

$$\frac{1}{a-b}\left(e^{at}-e^{bt}\right)$$

c) Take lim of ans as $b \to a$ to find $e^{at} * e^{at}$

So

$$\frac{1}{a-a}$$

? problem!

but lim mens $\frac{1}{\text{very small}}$ So basically go to 0 ?

X (-2)

X

d) Find $\cos t * \cos t$ using complex exponentials

$$\cos t = \frac{e^{it}+e^{-it}}{2}$$

$\cos t * \cos t$

$$\frac{e^{it}+e^{-it}}{2} * \frac{e^{it}+e^{-it}}{2}$$

(4)

By properties similar to multiplication - proved in class

$$= \frac{e^{it} * e^{it} + 2e^{it} * e^{-it} + e^{-it} * e^{-it}}{4}$$

do convolution

$$= \frac{1}{4}\left(t e^{it} + t e^{it}\right) + \frac{1}{2}\,\frac{e^{it} - e^{-it}}{2i}$$

$$= \frac{1}{2} t \cos t + \frac{1}{2} \sin t$$

---

3. Let $u(t)$ denote step fn as usual

a) Show that if we assume $\mathcal{L}(f') = s(F(s)) - F(0^+)$ then using $u' = \delta$ and $u(0^+) = 1$ gives $\mathcal{L}(\delta) = 0$

f = generalized function

$$\mathcal{L}(\delta) \overset{\text{def}}{=} \mathcal{L}(u') \overset{\text{as saw in class}}{=} s\,\mathcal{L}(u) - u(0^+)$$

$$= s\,\mathcal{L}(u) - 1 \quad \text{given}$$

Now what?

-2

(45)

b) Show that if we define

$$\mathcal{L}_+(f) = \int_{0^+}^{\infty} f(t)\, e^{-st}\, dt$$

$$\mathcal{L}_-(f) = \int_{0^-}^{\infty} f(t)\, e^{-st}\, dt$$

Then we can get diff formulas for $\mathcal{L}_\pm(f')$

So this is for delta function - right?
It depends if we include it?

X

-2

(4c)

These are clearly the Challenge q's

c) Let $f(t) = \begin{cases} \sin(2t) & t \geq 0 \\ 0 & t < 0 \end{cases}$

and let $F(s) = \mathcal{L}(f)$ as usual,

Show that $\mathcal{L}_-(f'') = s^2 F(s)$

by calc each side sepraty

$$f(t) = u(t) \sin 2t$$

$$f'(t) = 2u(t) \cos 2t$$

$$f''(t) = -4u(t) \sin 2t + 2\delta(t)$$

$$\mathcal{L}_-(f'') = \mathcal{L}_-(-4u(t) \sin 2t + 2\delta(t))$$

$= -4$? need values for - did we find earlier?

not done! (-2)

(97)

d) Use the LaPlace Transform to show that for f and g (generalized functions) satisfying $f(x) = g(x) = 0$ for all $x < 0$, then

$$P(D)(f * g) = (P(D)f) * g = g * (P(D)g)$$

for any polynomial operator in D

; didn't we do this before

$$\mathcal{L}(P(D)\, f) = P(s)\, \mathcal{L}(f)$$

$$\mathcal{L}(f * g) = \mathcal{L}(f)\, \mathcal{L}(g)$$

$$\mathcal{L}(P(D)\, f * g) = \mathcal{L}(P(D)\, f * P(D)\, g)$$

multiplication rules

= ; now it can be applied to either ;

$$= \mathcal{L}(P(D)\, f * g)$$

$$= \mathcal{L}(f * P(D)\, g)$$

(48)

c) Let w be the weight function in $P(D)$

Find the diff eq (for generalized fns)

Satisfied by w.

$$P(D)(w * f) = f$$

?: by multiplication rules this is not true

?: but it drops out when integrating?

But what is a weight function for $P(D)$?

$$P(D)\, w * P(D) f$$

$$\mathcal{L}(P(D)\, w * P(D f)$$

$$P(s)\, \mathcal{L}(w) \cdot \mathcal{L}(f)$$

? what is ??

So if say $\mathcal{L}(w) = \frac{1}{P(s)}$

then get $\mathcal{L}(f)$

then take $\mathcal{L}^{-1}$

$= f$ ✓

✓

FALL 2011 18.03 HW 8 SOLUTIONS

2. Consider the equation $y'' - 3y' + 2y = f(t)$. Taking Laplace transforms we have

$$(s^2 - 3s + 2)\mathcal{L}(y) = \mathcal{L}(f(t)) + y'(0) + (s-3)y(0)$$

or rearranging

$$y = \mathcal{L}^{-1}\left(\frac{\mathcal{L}(f(t)) + y'(0) + (s-3)y(0)}{(s-1)(s-2)}\right)$$

We would like to expand

$$\frac{y'(0) + (s-3)y(0)}{(s-1)(s-2)} = \frac{A}{s-1} + \frac{B}{s-2}$$

which leads to the equation $As - 2A + Bs - B = y'(0) + (s-3)y(0)$. Solving this we find $A = 2y(0) - y'(0)$ and $B = y'(0) - y(0)$. That is,

$$\frac{y'(0) + (s-3)y(0)}{(s-1)(s-2)} = \frac{2y(0) - y'(0)}{s-1} + \frac{y'(0) - y(0)}{s-2}$$

Taking inverse Laplace transforms and plugging in to the formula for y gives

$$y = \mathcal{L}^{-1}\left(\frac{\mathcal{L}(f(t))}{(s-1)(s-2)}\right) + (2y(0) - y'(0))e^t + (y'(0) - y(0))e^{2t}$$

Similarly we expand

$$\frac{1}{(s-1)(s-2)} = \frac{-1}{s-1} + \frac{1}{s-2}$$

and trade multiplication for convolution to obtain

$$y = f(t) \star (e^{2t} - e^t) + (2y(0) - y'(0))e^t + (y'(0) - y(0))e^{2t}$$

We are given $y(0) = 1$ and $y'(0) = 0$, plugging these in we find

$$y = f(t) \star (e^{2t} - e^t) + 2e^t - e^{2t}$$

We turn to the question at hand.

(a) Take $f(t) = e^{5t}$. Having computed below the general formula for convolving exponentials, we find

$$y = \frac{e^{5t} - e^{2t}}{3} - \frac{e^{5t} - e^t}{4} + 2e^t - e^{2t}$$

(b) Take $f(t) = -\delta(t-2)$. Computing the convolution,

$$y = -u(t-2)(e^{2(t-2)} - e^{t-2}) + 2e^t - e^{2t} = (2 + u(t-2)e^{-2})e^t - (1 + u(t-2)e^{-4})e^{2t}$$

(c) Again take $f(t) = -\delta(t-2)$. Between 0 and 2 the ODE behaves as the homogenous system $y'' - 3y' + 2y = 0$. Its general solution is $Ae^t + Be^{2t}$, and in order that $y(0) = 1$ and $y'(0) = 0$ we must have $A + B = 1$ and $A + 2B = 0$, i.e. $A = 2$ and $B = -1$ and

$$y(t)_{(0,2)} = 2e^t - e^{2t}$$

We find $y(2) = 2e^2 - e^4$ and $y'(2) = 2e^2 - 2e^4$. Then an impulse $-\delta(t)$ is applied, so at $t = 2_+ = 2 + \epsilon$ we have $y(2_+) = 2e^2 - e^4$ and $y'(2_+) = 2e^2 - 2e^4 - 1$. For

$t > 2_+$ the ODE again behaves as the homogenous system, but now we must have $Ae^2+Be^4 = 2e^2-e^4$ and $Ae^2+2Be^4 = 2e^2-2e^4-1$. Solving gives $B = (-e^4-1)/e^4$ and $A = (2e^2+1)/e^2$ and hence

$$y(t)_{(2,\infty)} = (2-e^{-2})e^t - (1+e^{-4})e^{2t}$$

Gluing the solutions with the step function $u(t-2)$ gives the same result as in (b).

3. (Wherein we calculate some Laplace transforms.)

a.

$$t \star t^2 = \int_0^t z^2(t-z)dz = [tz^3/3 - z^4/4|_0^t = t^4/12$$

$$e^t \star e^t = \int_0^t e^z e^{t-z} dz = \int_0^t e^t dz = te^t$$

b.

$$e^{at} \star e^{bt} = \mathcal{L}^{-1}(\mathcal{L}(e^{at})\mathcal{L}(e^{bt})) = \mathcal{L}^{-1}(\frac{1}{(s-a)(s-b)}) = \mathcal{L}^{-1}(\frac{1}{a-b}(\frac{1}{s-a} - \frac{1}{s-b})) = \frac{e^{at}-e^{bt}}{a-b}$$

c.

$$\lim_{a\to b} \frac{e^{at}-e^{bt}}{a-b} = \lim_{h\to 0} \frac{e^{(b+h)t}-e^{bt}}{h} = \frac{d}{db}e^{bt} = te^{bt}$$

d.

$$\cos t \star \cos t = \frac{e^{it}+e^{-it}}{2} \star \frac{e^{it}+e^{-it}}{2} = \frac{1}{4}(te^{it} + 2\frac{e^{it}-e^{-it}}{2i} + te^{-it}) = \frac{1}{2}(t\cos t + \sin t)$$

3 (sic). Let $u(t)$ be 0 for $t < 0$ and 1 for $t \geq 0$. Consider the Laplace transforms $\mathcal{L}_+$ and $\mathcal{L}_-$ in which the lower limit of the defining integral is 0_+ and 0_- respectively.

(a) We have $\mathcal{L}(\delta) = \mathcal{L}(u') = s\mathcal{L}(u) - u(0^+) = s(1/s) - 1 = 0$.

(b)

$$\mathcal{L}_\pm f' = \int_{0_\pm}^\infty f'(t)e^{-st}dt = [f(t)e^{-st}|_{0_\pm}^\infty + \int_{0_\pm}^\infty f(t)se^{-st}dt = s\mathcal{L}_\pm(f) - f(0_\pm)$$

Plugging in $u = f$ gives $\mathcal{L}_\pm(\delta) = 1 - u(0_\pm)$, i.e., $\mathcal{L}_+(\delta) = 0$ and $\mathcal{L}_-(\delta) = 1$.

(c) I cannot imagine any way in which this is not answered by (b).

Henceforth take $\mathcal{L} = \mathcal{L}_-$.

(d) From (b) note $\mathcal{L}(P(D)h) = P(s)\mathcal{L}(h)$. Thus the three terms are just different ways of writing $\mathcal{L}^{-1}(P(s)\mathcal{L}(f)\mathcal{L}(g))$.

(e) From (d) observe $\mathcal{L}(P(D)\mathcal{L}^{-1}(1/P(s))) = P(s)\mathcal{L}L^{-1}(1/P(s)) = 1 = \mathcal{L}(\delta)$. Inverting the Laplace transform, $P(D)\mathcal{L}^{-1}(1/P(s)) = \delta$. Convolving with f gives $P(D)\mathcal{L}^{-1}(1/P(s)) \star f = f$, and applying (c) gives $P(D)(\mathcal{L}^{-1}(1/P(s)) \star f) = f$.

18.03 Lecture 27

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Pole Diagrams

Formula page on exam posted on website - under handouts

Warmup example qu

could be qu 1 or 2

$$y'' + y = \delta + \delta_{\pi} \quad \text{w/ rest conditions}$$

[can't just guess

$y(0^-) = y'(0^-) = 0$

LHS

$$\mathcal{L}(y'' + y) = (s^2+1)\mathcal{L}(y) - \text{junk}$$

but they're 0

(polynomial in s)

RHS

$$\mathcal{L}(\delta) = 1$$

$$\mathcal{L}(\delta_{\pi}) = \mathcal{L}(\delta(t-\pi)) = e^{-\pi s}$$

$$\mathcal{L}(y) = \frac{1 + e^{-\pi s}}{s^2+1}$$

$$y = \mathcal{L}^{-1}\left(\frac{1+e^{-\pi s}}{s^2+1}\right)$$

can split - can use table

$$= \mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) + \mathcal{L}\left(\frac{e^{-\pi s}}{s^2+1}\right)$$

(2)

Use table

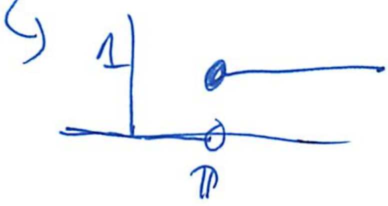
↓ see exponential piece: see shift

$$= (\sin t)\, u(t) \quad + \quad \sin(t-\pi)\; u(t-\pi)$$

since only know $t \geq 0$

shift here too (I don't do this usually!)

$u(t-\pi)$



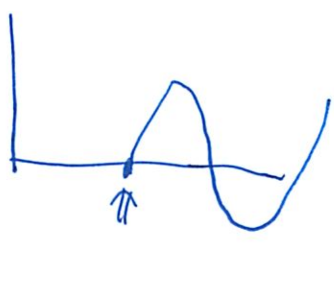
$\sin(t-\pi)$

but it also goes back

so, just start normal sin here

$u(t-\pi)\sin(t-\pi)$

what we want! →

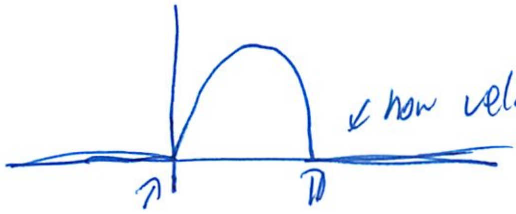


Hanh

(3)

What does function look like?

$$\sin(t)\,u(t) + \sin(t-\pi)\,u(t-\pi)$$



← now velocity in other direction so it stops! dead

<u>Weight functions</u> $\quad w(t) = \mathcal{L}^{-1}\left(\dfrac{1}{p(s)}\right)$ (oh forgot on hw)

in ODE $\quad p(D)y = f(t)$ w/ rest conditions

Useful b/c solution

$$x(t) = w * f(t)$$ ← magic ans w/ input response formula

$$= \int_0^t w(t-u)\,f(u)\,du$$

<u>In our example</u> $\quad x'' + x = \delta + \delta_\pi$

$$w(t) = \mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) = \sin t\,(u(t))$$

No matter fn on right

$$x = \sin t \cdot u(t) * f$$ ✓ no matter the f

here $f = \delta + \delta_\pi$

(4)

$\sin t\, u(t) * (\delta + \delta_\pi)$

distributive law

$= \sin t\; u(t) * \delta + \sin t\, u(t) * \delta_\pi$

δ acts like identity in convolution

since $\delta(t-z-\pi)$

$\delta((t-z)-\pi)$

$= \sin t\; u(t) + \int_0^t \sin z\; u(z)\, \delta_\pi (t-z)\, dz$

$=$ " $+ \sin(t-\pi)\, u(t-\pi)$

Circuit example

↳ can treat as black box

if know transfer functions

then plug in input response model

So don't need to work w/ individual pieces

instructions to visualize convolution

[We also did something like this in 6.02

basically it smoothes stuff out

square waves

(5)

If have peaks all too close together → would mess stuff up

Example w/ Damping

$$y'' + 4y' + 5y = U_5(t) \quad + \text{rest conditions}$$

damping

$$\mathcal{L}\left((s^2 + 4s + 5)\,\mathcal{L}(y) \right) = \mathcal{L}\left(u_5(t)\right) = \frac{e^{-5s}}{s} \leftarrow \text{fhe } s$$

$$\mathcal{L}(y) = \frac{e^{-5s}}{(s)\,(s^2+4s+5)}$$

How to take inverse Laplace transform?

Don't bother w/ exponential stuff now (e^{-5s})

Do partial fractions on $\dfrac{1}{s(s^2+4s+5)}$

~~Usually partial~~

$$\frac{1}{(s)(s^2+4s)+5} = \frac{A}{s} + \frac{Bs+C}{s^2+4s+5}$$

ok but will make your life harder later

6

Find roots of quadratic factor first

$\hookrightarrow -2 \pm i \quad 1i$

Can rewrite (s^2+4s+5) as completing the square

$(s+2)^2+1^2$

So do partial fractions for

before! think ahead to this

$$= \frac{A}{s} + \frac{B(s+2)+C}{(s+2)^2+1}$$

Will end up having nice Laplace Transform

$$\mathcal{L}^{-1}\left(\frac{s}{s^2+1}\right) = \cos t \cdot u(t)$$

only care about values $t \geq 0$

shift by a

don't really need this unless shift matters

$$\mathcal{L}^{-1}\left(\frac{s+a}{(s+a)^2+1}\right) = e^{-at}\cos t\, u(t)$$

So for our example

$$\mathcal{L}^{-1}\left(\frac{B\cdot(s+2)}{(s+2)^2+1}\right) = Be^{-2t}\cos t\, u(t)$$

⑦

$$\mathcal{L}^{-1}\left(\frac{C}{(s+2)^2+1^2}\right) = Ce^{-2t}\sin t \cdot u(t)$$

To find A, B, C use heavy sides

$$\frac{A}{s} + \frac{B(s+2)+C}{(s+2)^2+1}$$

So get $A = 1/5$

~~BW~~ B+C identity of complex ~~#~~

can plug in complex root 2+i

$$\frac{1}{-2+i} = B(-2+i+2)+C$$

After clearing quadratic denom

$$\frac{1}{s} = \cancel{\frac{A(s^2+4s+5)}{s}} + B(s+2)+C$$

at $s = -2+i$

Real parts cancle

get pure imag for B

get pure real for C

(8)

Multiply by conjugate to get nice form

(ii)

So $B = -\frac{1}{5}$ $\quad C = -\frac{2}{5}$

Now time to worry about e^{-5s}

↳ to get translation in t

$$\frac{1}{5} u_5(t) - \frac{1}{5} e^{-2(t-5)} \cos(t-5) u_5(t)$$

sin part ↓ $- \frac{2}{5} e^{-2(t-5)} \sin(t-5) u_5(t)$

This was the very verbose way of doing

Poles

Solve ODEs w/ LaPlace transform

Need $\mathcal{L}^{-1}$ (rational function)

eg) $p(D) y = \delta$ ~~xxxxt~~

↓

$\mathcal{L}^{-1}\left(\frac{1}{p(s)}\right)$ w/ rest conditions

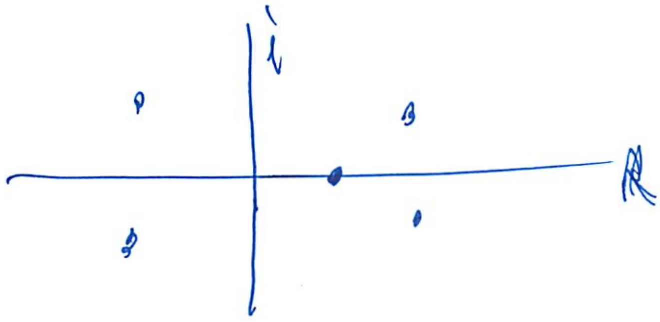
9

So what maters are 0_s of denom $\hookrightarrow \frac{1}{P(s)}$

Called "poles"

L where rational fn is undefined

eg roots of denom $1, 2 \pm i, -1 \pm i$



So denom $= s(s^2+4s+5) \cdot (s^2-2s+2)$

So inverse Laplace $= 1,\ e^{2t}\sin t,\ e^{-t}\sin t$
$e^{2t}\cos t \quad e^{-t}\cos t$

L So these fns appear in sol

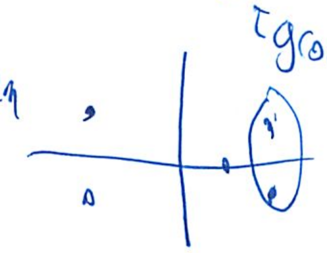
Which dominates in long term

$e^{2t}\sin t$

$e^{2t}\cos t$

L grow huge

Far right on diagram



Warm-up example :

Lecture 27

11/18

$y'' + y = \delta + \delta_\pi$ with rest conditions $y(0) = y'(0) = 0$.

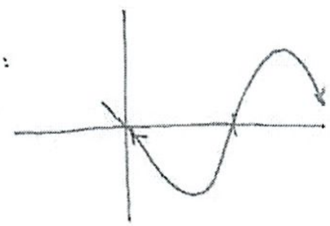
Take Laplace transform: $\mathcal{L}(y'' + y) = (s^2+1)\mathcal{L}(y)$.

$$\mathcal{L}(\delta + \delta_\pi) = \mathcal{L}(\delta) + \mathcal{L}(\delta_\pi) = 1 + e^{-\pi s}$$

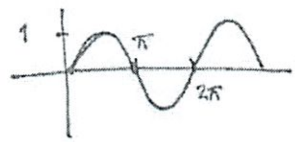
so $$y = \mathcal{L}^{-1}\left(\frac{1+e^{-\pi s}}{s^2+1}\right) = \underbrace{\mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right)}_{\sin t \cdot u(t)} + \underbrace{\mathcal{L}^{-1}\left(\frac{e^{-\pi s}}{s^2+1}\right)}_{\sin(t-\pi)\, u(t-\pi)}$$

very important to include this unit step function at π.

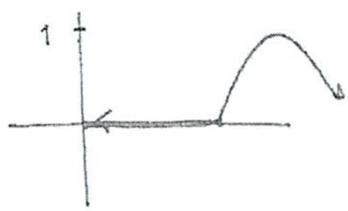
$\sin(t-\pi)$:



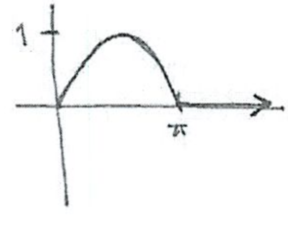
since $\sin(t)$:



$\sin(t-\pi)u(t-\pi)$:



so $y(t) = \sin t \cdot u(t) + \sin(t-\pi)u(t-\pi) =$



(discuss physical significance)

With weight function : remember that $w(t) = \mathcal{L}^{-1}\left(\frac{1}{p(s)}\right)$ ← only depends on LHS of ODE. not on input function.

in ODE $p(D) \cdot x = f(t)$ with rest conditions.

In our problem: $w(t) = \mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) = \sin(t)\, u(t)$.

Then $x(t) = w(t) * f(t) = \int_0^t f(u)\, w(t-u)\, du$

In our example: $x(t) = (\sin t\; u(t)) * (\delta + \delta_\pi)$

Example <u>with damping</u> $y'' + 4y' + 5y = u_5(t)$ w/ rest cond.

unit step function at $t = 5$

$\mathcal{L}$ of both sides : $(s^2 + 4s + 5)\,\mathcal{L}(y) = \mathcal{L}(u_5)$

$= e^{-5s}/s$

so $\mathcal{L}(y) = 1 \dfrac{e^{-5s}}{s\cdot(s^2+4s+5)}$. ← Know how to use shifts to deal with e^{-5s} later, so we do partial fractions for : $\dfrac{1}{s(s^2+4s+5)}$

Could do : $\dfrac{A}{s} + \dfrac{Bs+C}{s^2+4s+5}$ but then inverse Laplace transform is messy. Better to do more clever partial fraction

roots of $s^2+4s+5 = -2 \pm i$

$\qquad\qquad\qquad \| \qquad$ note relationship.

$(s+2)^2 + 1$

so in partial fractions expansion

<u>So write</u> : $\dfrac{A}{s} + \dfrac{B(s+2)+C}{\underbrace{s^2+4s+5}_{(s+2)^2+1}}$

Good because $\mathcal{L}^{-1}\left(\dfrac{s}{s^2+1}\right) = \cos t \cdot u(t)$

So $\mathcal{L}^{-1}\left(\dfrac{s+a}{(s+a)^2+1}\right) = e^{-at}\cos t\; u(t)$

in <u>our</u> case : $\mathcal{L}^{-1}\left(\dfrac{s+2}{(s+2)^2+1}\right) = e^{-2t}\cos t\; u(t)$

<u>similarly</u> : $\mathcal{L}^{-1}\left(\dfrac{1}{(s+2)^2+1}\right) = e^{-2t}\sin t\; u(t)$.

(Knew this should appear since roots were $-2 \pm i$)

So just need to solve for A, B, C then done.

Remember, we find these by cover up. So $A = \left.\dfrac{1}{s^2+4s+5}\right|_{s=0}$

$= \boxed{\dfrac{1}{5}}$

B, C are found by complex cover up.

$$\frac{1}{s(s^2+4s+5)} = \frac{A}{s} + \frac{B(s+2)+C}{(s+2)^2+1}$$

plug in $-2+i$ after multiplying both sides by s^2+4s+5.

$$\frac{1}{-2+i} = Bi + C$$

(if you do it correctly, B multiplied by i will always be pure imaginary. Of course, C real.)

$$\frac{1}{-2+i} = \frac{-2-i}{5} \quad \text{so} \quad B = -\frac{1}{5}$$

$$C = -\frac{2}{5}$$

Solution:

$$y(t) = \mathcal{L}^{-1}\left(\frac{5e^{-5s}}{s} - \frac{1}{5}e^{-5s}\left(\frac{(s+2)+2}{(s+2)^2+1}\right)\right)$$

$$= \underbrace{\frac{1}{5}}\,u_5(t) - \frac{1}{5}\underbrace{e^{-2(t-5)}\cos(t-5)\,u_5(t)}$$

$$- \frac{2}{5}\underbrace{e^{-2(t-5)}\sin}(t-5)\,u_5(t).$$

this piece of solution dominates.

these exhibit damping

When we solve ODEs, comes down to computing

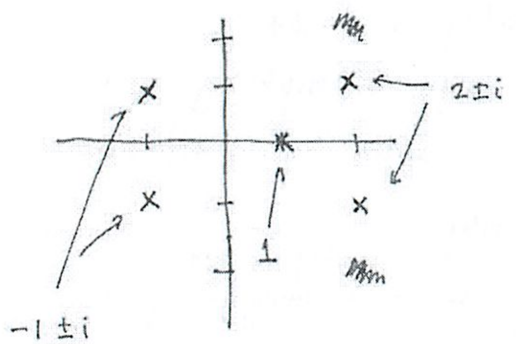
$\mathcal{L}^{-1}$ (rational function).

e.g. $p(D)x = \delta$ then compute $\mathcal{L}^{-1}\left(\frac{1}{p(s)}\right)$

Zeros of denominator : "poles" (isolated points where cx. function is undefined.)

Plot these poles on complex plane. Complex roots in denom. come in pairs $a \pm bi$.

If we drew poles of some rational function and they looked like:



roots of denom. : $1, \; 2 \pm i, \; -1 \pm i$

then denom : $(s-1)(s^2 - 4s + 5)(s^2 + 2s + 2)$

and inverse Laplace transform : e^t, $e^{2t} \sin t$, $e^{-t} \sin t$
$e^{2t} \cos t$, $e^{-t} \cos t$

rt. 1 — rts $2 \pm i$ — rts $-1 \pm i$

Show mathlet on pole diagrams.

Play around with possible roots.

which roots dominate in long term behavior of solution?

Answer: One with largest real part (rightmost pole)

Sheet of formulas + Sample exams posted on website.

18.03 Lecture 28 11/21

Exam 3 Review

Exam in 26-100 tomorrow @ 7:30

1 addition to formula sheet $\mathcal{L}(u_a(t)) = \frac{e^{-as}}{s}$

Pole Diagram

L "poles" = zeros of denoms of rational functions

eg: $p(D)x = 0$

guess $x = e^{rt}$ → sols have r as root of $p(r) = 0$

To solve IV problem → solve system of eq'ns

Laplace transform: 1. Take La P. T. of each side

$$\mathcal{L}(p(D)x) = \mathcal{L}(0) = 0$$

$$p(s)\mathcal{L}(x) = 0$$

$$\mathcal{L}(x) = \frac{0}{p(s)}$$

if rest conditions

2. But if told inital conditions $x(0) = A, x'(0) = B, \ldots$

$$p(s)\mathcal{L}(x) - As^{d-1} - \ldots = 0$$

$$x = \mathcal{L}^{-1}\left(\frac{As^{d-1} + \ldots}{p(s)}\right)$$

②

Each method provides same sol

Solve w/ Partial fractions

└ pieces correspond to the poles

(poles still sols to characteristic eqns)

So poles should be what would you guess

So guess method is a good way to check

poles $\Longleftrightarrow$ inv. Laplace transform of rational fractions

$$1 \longleftrightarrow \frac{\cancel{A}}{s-1} \Leftrightarrow e^{t}$$

can graph

see which roots furthest to right

that is what contributes the most

If RHS $\neq 0$, similar - you are still dividing by $P(s)$

like homogeneous part, inhomogeneous pf

$$x = \mathcal{L}^{-1}\left(\frac{As^{d-1}+\cdots}{P(s)}\right) + \mathcal{L}^{-1}\left(\frac{\mathcal{L}(f(t))}{P(s)}\right)$$

(3)

We'll only see constant coefficient linear ODEs

Not doing $t x'' + x' + t x = 0$

where fns of ind variables

Could solve w/ Laplace transform

But end w/ $x = \mathcal{L}^{-1}\left(\frac{1}{\sqrt{s^2+1}}\right)$

Could also do power series sol

But we would ~~have to mem~~ not be able to recognize

Fourier Series

1. Make sure can compute Fourier series of basic fns
   - square wave
   - ~~Saw~~ triangle wave
   - finite sum of trig functions

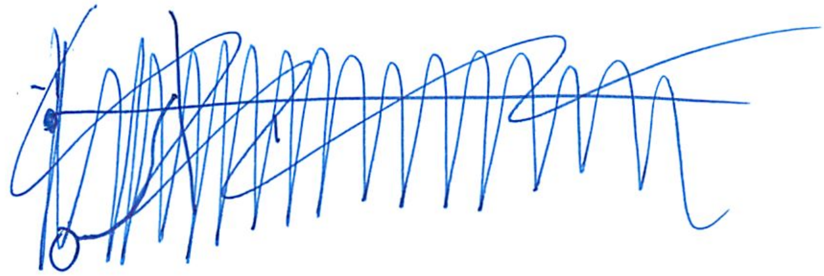
Ex $f(t) = t^3$

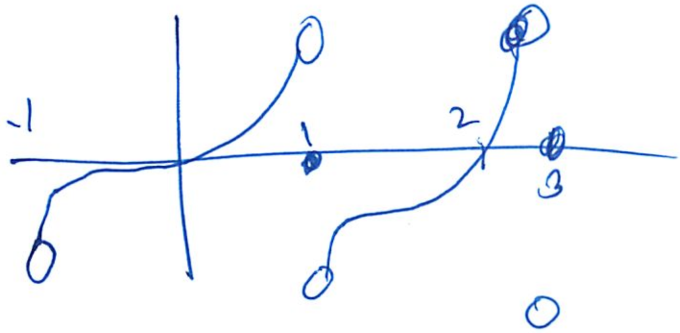
No real Fourier expansion - since not periodic

(4)

So make it periodic

ex 2 $f(t) = \begin{cases} t^3 & \text{on } (-1,1) \\ 0 & t = 1 \end{cases}$ repeat





Fourier expansion symmetry (f odd) cosine turns 0

Now sine

$$b_n = \frac{1}{2}\int_{-1}^{1} \overbrace{f(t)}^{t^3} \sin\left(\frac{n\pi}{1} t\right) dt$$

do definite integral

└ hardest part

integration by parts 3 times!

└ would prob only see int. by parts 1x

- know square wave - be able to do

5

Fourier Series in an ODE

$$y'' + 7y = \sum_{n=1}^{\infty} \frac{\cos nt}{n^2}$$

RHS

only even derivs

already packaged in fourier series

So can conclude that y is expressible as a cosine series

$$y = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt$$

since nt above
if was $7t$
would be 7π

period = 2π here implicitly

LHS

Substitute in guess and solve

$$y'' = \sum_{n=1}^{\infty} -n^2 a_n \cos nt$$ ← take 2 derivs each term

$$\frac{7a_0}{2} + \sum_{n=1}^{\infty} (7 - n^2)\, a_n \cos nt$$

Combine y'', $7y$

(6) $= RHS$

$$= \sum_{n=1}^{\infty} \frac{\cos nt}{n^2}$$

$\therefore$ Every coefficient of $\cos nt$ must match

math coeffs !

$a_0 = 0$

$$a_n = \frac{1}{n^2} \cdot \frac{1}{(7-n^2)}$$

So final sol is

$$y(t) = \sum_{n=1}^{\infty} \left(\frac{1}{7n^2 - n^4} \right) \cos nt$$

This problem was easier due to symmetry

<u>Worse</u>

$$y'' + 2y' + 3y = \sum_{n=1}^{\infty} \frac{\cos nt}{n^2}$$

1st deriv term

x's even fn

but can't guess RHS is made of cosines - must guess full boat - cos + sin terms

$$y(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + \sum_{n=1}^{\infty} b_n \sin nt$$

(7)

What happens w/ resonance?

- As on pratice exam

If have $y'' + ky = \ldots$

For what values of k does resonance occur?

k has purly imag roots i

Produces sin, cos

So if terms $\boxed{n = \sqrt{k}}$

whatever is on RHS

Resonance = fail to be periodic fn

L sol involves $t \sin t$

Endpoint problems

Have some diff eq $y'' + 7y = \overset{f(t)}{t}$ w/ $y(0) = y(1) = 0$

Solve for $y(t) \in [0,1]$

Can artifically introduce F.S. to solve

L since want periodic fn and satisfy endpoints

(8)

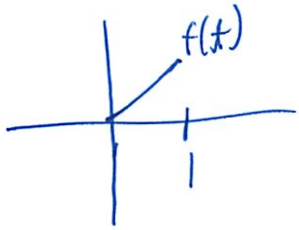
So extend def $f(t)$ to $[-1, 1]$ w/ symmetry

Then make periodic

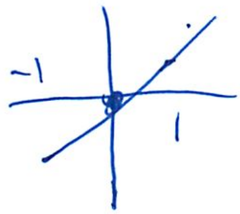
So resulting F.S. involves only fn that obey endpoint conditions

Make periodic $\rightsquigarrow$ Fourier Series involves only $\sin n\pi t$

Have



Extend so still odd fn



Make periodic fn

We can do this because we <u>have not</u> changed $[0,1]$

(9)

Fourier expansion of t for $t \in (-1,1)$

$\llcorner$ repeated periodically

$$= \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\sin n\pi t}{n}$$

? would have actually taken for triangle fn

am shortcutting that step here

? cleverly rigged so get right ans

Now can solve like earlier problem from today

Questions long - will only ask 1

4 qu on exam - not 5

If damping factor - harder to guarentee solved endpoint problems

(10)

Heat Eqn

- will be a problem on exam

Want

$U(x,t)$ s.t. $\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$

← temp fn

extend f to odd $(-L,L)$

$\boxed{\text{frozen}}$ $\boxed{\text{Sine}}$

$u(0,t) = u(L,t) = 0$ ← endpoint conditions

$U(x,0) = f(x)$

Could also see $\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t)$

$\boxed{\text{Cosine}}$ $\boxed{\text{insulated ends}}$

← extend to even $(-L,L)$

Solve endpoint accordrding to which end point condition you have

Take Fourier Series

...

What do you do with the result?

It solves heat eqn because generic sol is = to

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-k\left(\frac{n^2\pi^2}{L^2}\right)t} \sin\left(\frac{n\pi x}{L}\right)$$

← will be given / don't need to derive / ← (for Sine one) for cos: have cos

(11)

To solve inital conditions, say what b_n are

$$\cancel{\ldots}\ u(x,0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} = f(x)$$

So b_n: Fourier coefficents of extented function

Plug back into original function

That is your solution

---

key Just need to recognize sin or cos

extend odd or even

find

plug in to big mess

Tricky Qu which can be asked

Harder problem $f(x) = \sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$

Find $\int_0^{2\pi} f(x) \cos^2(3x)\,dx$

(12)

What if just

$$\int_0^{2\pi} f(x)\cos(3x)\,dx$$

reminds us of integral to initially deriv

So ans

$$= \pi \cdot \frac{1}{9}$$

3rd Fourier coefficient

Could also write

$$\int_{-\pi}^{\pi} f(x)\cos 3x\,dx$$

Lecture 28

11/21

Last word about pole diagrams : "poles" $\to$ places where denom. of rational function is 0.

e.g. $p(D)x = 0$ (homogeneous equation)

Guess method : $x = e^{rt}$ $\Rightarrow$ $p(r) = 0$ for a solution $x = e^{rt}$

i.e. roots of char. poly.

To solve I.V.P., then solve system of equations.

Laplace trans. method : $p(D)x = 0$ $\quad x(0) = A,\ x'(0) = B, \ldots$

then $p(s)\mathcal{L}(x) - As^{d-1} - \cdots = 0$

$$x = \mathcal{L}^{-1}\left(\frac{As^{d-1} + \cdots}{p(s)}\right)$$

Do partial fractions according to roots of char. poly. $p(s)$

pole diagram : picture of poles.

"poles" of rational function.

(lots of things read off pole diagram, e.g. practical resonance, but we'll be content with knowing that right-most pole(s) dominate long-term behavior)

Aside : Why never did more complicated linear ODEs? (non-const. coeffs.)

e.g. Bessel equation : $tx'' + x' + tx = 0$ $\leftarrow$ use either Laplace transform or power series

(don't expect periodic solus here)

get $X(s) = \dfrac{1}{\sqrt{s^2+1}}$ $\leftarrow$ not a function whose $\mathcal{L}$ we recognize...

i.e. $x(t) = \mathcal{L}^{-1}\left(\dfrac{1}{\sqrt{s^2+1}}\right)$.

There is a closed form integral for $\mathcal{L}^{-1}(F(s))$:

$$\mathcal{L}^{-1}(F(s)) = \frac{1}{2\pi i}\int_{6-i\infty}^{6+i\infty} F(s)\, e^{st}\, ds \qquad 6 \gg 0 \text{ so that } \mathcal{L}(f) \text{ converges.}$$

so answer: $$\frac{1}{2\pi i}\int_{6-i\infty}^{6+i\infty} \frac{1}{\sqrt{s^2+1}}\, e^{st}\, ds.$$

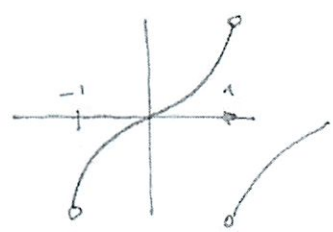
---

fourier series: ① Can you compute Fourier series of basic functions? square wave, triangle wave, finite sum of trig functions with common period

Ex: $f(t) = t^3$ on $[-1,1)$

0 at 1

extend to periodic function of period 2



By symmetry ($f(t)$ is odd) the cosine part of Fourier series $= 0$.

Other coeffs: $$f(t) = \sum_{n=1}^{\infty} b_n \sin \pi n t$$ in general: $\sin: \frac{\pi n}{L} t$, L: ½-period

with $$b_n = \frac{1}{L}\cdot\int_{-L}^{L} f(t)\sin\frac{\pi n t}{L}\, dt$$

so $$b_n = \frac{1}{1}\cdot\int_{-1}^{1} t^3 \sin \pi n t\, dt$$

$u = \pi n t$

$dt = \pi n\, du$

$$= \frac{1}{\pi^2 n^2}\int_{-\pi n}^{\pi n} u^3 \sin u\, du$$

Remember:

$$\int u^k \sin u\, du = -u^k\cos u + k\int u^{k-1}\cos u\, du$$

$$= \frac{1}{\pi^2 n^2}\Big[-u^3\cos u + 3u^2\sin u + 6u\cos u - 6\sin u\Big]_{-\pi n}^{\pi n} \dots$$

(2) Fourier series in ODEs. (discuss when solution is periodic / resonant)

$$y'' + 7y = \sum_{n=1}^{\infty} \frac{\cos nt}{n^2} \quad \text{or} \quad y'' + 2y' + 3 = \sum_{n=1}^{\infty} \frac{\cos nt}{n^2}$$

---

(3) Heat equation : By guessing we know general sol'n to heat equation : with 0-endpt. temp.

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-k(n^2\pi^2/L^2)t} \sin\left(\frac{n\pi}{L}\right)x$$

Given $u(x,0) = f(x)$, then solve for $b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi}{L}\right)x\,dx$

---

(4) Endpoint problems. $y'' + 7y = t$ with $y(0) = y(1) = 0$.

complete t to an odd periodic function of period 2.

Find Fourier sine series of t : $\frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\sin n\pi t}{n}$

then solve ODE as before.

E.g. $u(x,0) = \sin \pi x$ in rod of length 1. for (3)

Harder problems : Given $f(x) = \sum_{n=1}^{\infty} \frac{\cos nt}{n^2}$, find

$$\int_0^{4\pi} f(x)\sin x\,dx, \quad \int_0^{2\pi} f(x)\cos 3x\,dx, \quad \int_0^{2\pi} f(x)\cos^2 3x\,dx$$

11/21

Integration by Parts

Integration by parts is a technique for performing indefinite integration $\int u\,d\,v$ or definite integration $\int_a^b u\,d\,v$ by expanding the differential of a product of functions $d\,(u\,v)$ and expressing the original integral in terms of a known integral $\int v\,d\,u$. A single integration by parts starts with

$$d\,(u\,v) = u\,d\,v + v\,d\,u, \tag{1}$$

and integrates both sides,

$$\int d\,(u\,v) = u\,v = \int u\,d\,v + \int v\,d\,u. \tag{2}$$

Rearranging gives

$$\int u\,d\,v = u\,v - \int v\,d\,u. \tag{3}$$

For example, consider the integral $\int x\cos x\,d\,x$ and let

$$u = x \quad d\,v = \cos x\,d\,x \tag{4}$$
$$d\,u = d\,x \quad v = \sin x, \tag{5}$$

so integration by parts gives

$$\int x\cos x\,d\,x = x\sin x - \int \sin x\,d\,x \tag{6}$$
$$= x\sin x + \cos x + C, \tag{7}$$

where C is a constant of integration.

The procedure does not always succeed, since some choices of u may lead to more complicated integrals than the original. For example, consider again the integral $\int x\cos x\,d\,x$ and let

$$u = \cos x \quad d\,v = x\,d\,x$$
$$d\,u = -\sin x\,d\,x \quad v = \tfrac{1}{2}x^2, \tag{8}$$

giving

$$\int x\cos x\,d\,x = \tfrac{1}{2}x^2\cos x - \tfrac{1}{2}\int x^2(-\sin x)\,d\,x \tag{9}$$

$$= \quad \tfrac{1}{2}x^2 \cos x + \tfrac{1}{2}\int x^2 \sin x \, dx, \tag{10}$$

which is more difficult than the original (Apostol 1967, pp. 218-219).

Integration by parts may also fail because it leads back to the original integral. For example, consider $\int x^{-1}\, dx$ and let

$$\begin{aligned} u &= x & dv &= x^{-2}\, dx \\ du &= dx & v &= -x^{-1}, \end{aligned} \tag{11}$$

then

$$\int x^{-1}\, dx = -1 - \int (-x^{-1})\, dx + C = \int x^{-1}\, dx + C - 1, \tag{12}$$

which is same integral as the original (Apostol 1967, p. 219).

The analogous procedure works for definite integration by parts, so

$$\int_a^b u\, dv = [u\, v]_a^b - \int_a^b v\, du, \tag{13}$$

where $[f]_a^b = f(b) - f(a)$.

Integration by parts can also be applied n times to $\int f^{(n)}(x)\, g(x)\, dx$:

$$\begin{aligned} u &= g(x) & dv &= f^{(n)}(x)\, dx \\ du &= g'(x)\, dx & v &= f^{(n-1)}(x). \end{aligned} \tag{14}$$

Therefore,

$$\int f^{(n)}(x)\, g(x)\, dx = g(x)\, f^{(n-1)}(x) - \int f^{(n-1)}(x)\, g'(x)\, dx. \tag{15}$$

But

$$\int f^{(n-1)}(x)\, g'(x)\, dx = g'(x)\, f^{(n-2)}(x) - \int f^{(n-2)}(x)\, g''(x)\, dx \tag{16}$$

$$\int f^{(n-2)}(x)\, g''(x)\, dx = g''(x)\, f^{(n-3)}(x) - \int f^{(n-3)}(x)\, g^{(3)}(x)\, dx, \tag{17}$$

so

$$\int f^{(n)}(x)\,g(x)\,dx = g(x)\,f^{(n-1)}(x) - g'(x)\,f^{(n-2)}(x) + g''(x)\,f^{(n-3)}(x) - \ldots + (-1)^n \int f(x)\,g^{(n)}(x)\,dx. \quad (18)$$

Now consider this in the slightly different form $\int f(x)\,g(x)\,dx$. Integrate by parts a first time

$$\begin{aligned} u &= f(x) & dv &= g(x)\,dx \\ du &= f'(x)\,dx & v &= \int g(x)\,dx. \end{aligned} \quad (19)$$

so

$$\int f(x)\,g(x)\,dx = f(x)\int g(x)\,dx - \int \left[\int g(x)\,dx\right] f'(x)\,dx. \quad (20)$$

Now integrate by parts a second time,

$$\begin{aligned} u &= f'(x) & dv &= \int g(x)\,dx \\ du &= f''(x)\,dx & v &= \iint g(x)\,(dx)^2, \end{aligned} \quad (21)$$

so

$$\int f(x)\,g(x)\,dx = f(x)\int g(x)\,dx - f'(x)\iint g(x)\,(dx)^2 + \int \left[\iint g(x)\,(dx)^2\right] f''(x)\,dx. \quad (22)$$

Repeating a third time,

Cut off

$$\int f(x)\,g(x)\,dx = f(x)\int g(x)\,dx - f'(x)\iint g(x)\,(dx)^2 + f''(x)\iiint g(x)\,(dx)^3 - \int \left[\iiint g \right. \quad (23)$$

Therefore, after n applications,

$$\int f(x)\,g(x)\,dx =$$

$$f(x)\int g(x)\,dx - f'(x)\iint g(x)\,(dx)^2 + f''(x)\iiint g(x)\,(dx)^3 - \ldots + (-1)^{n+1} f^{(n)}(x) \underbrace{\int \cdots \int}_{n+1} g \quad (24)$$

If $f^{(n+1)}(x) = 0$ (e.g., for an nth degree polynomial), the last term is 0, so the sum terminates after n terms and

$$\int f(x)\,g(x)\,dx = f(x)\int g(x)\,dx - f'(x)\iint g(x)\,(dx)^2 + f''(x)\iiint g(x)\,(dx)^3 - \ldots + (-1)^{n+1} \tag{25}$$

CG. Convolution and Green's Formula

1. Convolution. A peculiar-looking integral involving two functions $f(t)$ and $g(t)$ occurs widely in applications; it has a special name and a special symbol is used for it.

Definition. The **convolution** of $f(t)$ and $g(t)$ is the function $f * g$ of t defined by

$$(1) \qquad [f * g](t) = \int_0^t f(u)g(t-u)\,du.$$

Example 1 below calculates two useful convolutions from the definition (1). As you can see, the form of $f * g$ is not very predictable from the form of f and g.

Example 1. Show that

$$(2) \qquad e^{at} * e^{bt} = \frac{e^{at} - e^{bt}}{a-b}, \quad a \neq b; \qquad\qquad e^{at} * e^{at} = t\,e^{at}$$

Solution. We do the first; the second is similar. If $a \neq b$,

$$e^{at} * e^{bt} = \int_0^t e^{au}e^{b(t-u)}du = e^{bt}\int_0^t e^{(a-b)u}du = e^{bt}\frac{e^{(a-b)u}}{a-b}\Big]_0^t = e^{bt}\frac{e^{(a-b)t}-1}{a-b} = \frac{e^{at}-e^{bt}}{a-b}.$$

The convolution gives us an expressive formula for a particular solution y_p to an inhomogeneous linear ODE. The next example illustrates this for the first-order equation.

Example 2. Express as a convolution the solution to the first-order constant-coefficient linear IVP (cf. Notes IR (3))

$$(3) \qquad y' + ky = q(t); \quad y(0) = 0.$$

Solution. The integrating factor is e^{kt}; multiplying both sides by it gives

$$(y\,e^{kt})' = q(t)e^{kt}.$$

Integrate both sides from 0 to t, and apply the Fundamental Theorem of Calculus to the left side; since the particular solution y_p we want satisfies $y(0) = 0$, we get

$$y_p\,e^{kt} = \int_0^t q(u)e^{ku}\,du\;; \qquad\qquad (u \text{ is a dummy variable.})$$

Moving the e^{kt} to the right side and placing it under the integral sign gives

$$y_p = \int_0^t q(u)e^{-k(t-u)}\,du$$
$$y_p = q(t) * e^{-kt}.$$

?why # from 0 ?

We see that the solution is the convolution of the input $q(t)$ with the solution to the IVP (3) where $q = 0$ and the initial value is $y(0) = 1$. This is the simplest case of **Green's formula**, which does the same thing for higher-order linear ODE's. We will describe it in section 3.

2. Physical applications of the convolution.

The convolution comes up naturally in a variety of physical sitations. Here are two typical examples.

Example 3. Radioactive dumping. A radioactive substance decays exponentially:

$$R = R_0 e^{-kt}, \tag{4}$$

where R_0 is the initial amount, $R(t)$ the amount at time t, and k the decay constant.

A factory produces this substance as a waste by-product, and it is dumped daily on a waste site. Let $f(t)$ be the rate of dumping; this means that in a relatively small time period $[t_0, t_0 + \Delta t]$, approximately $f(t_0)\Delta t$ grams of the substance is dumped.

Find a formula for the amount of radioactive waste in the dump site at time t, and express it as a convolution. Assume the dumping starts at time $t = 0$.

Solution. Divide up the time interval $[0, t]$ into n equal intervals of length Δu, using the times

$$u_0 = 0,\ u_1,\ u_2,\ \ldots,\ u_n = t.$$

$$\text{amount dumped in the interval } [u_i, u_{i+1}] \ \approx\ f(u_i)\Delta u\ ;$$

by time t, this amount will have decayed for approximately the length of time $t - u_i$; therefore, according to (4), at time t the amount of waste coming from what was dumped in the time interval $[u_i, u_{i+1}]$ is approximately

$$f(u_i)\Delta u \cdot e^{-k(t-u_i)}.$$

Adding up the radioactive material in the pile coming from the dumping over each time interval, we get

$$\text{total amount at time } t \ \approx\ \sum_0^{n-1} f(u_i)e^{-k(t-u_i)}\Delta u\ .$$

As $n \to \infty$ and $\Delta u \to 0$, the sum approaches the corresponding definite integral and the approximation becomes an equality. So we conclude that

$$\text{total amount at time } t \ =\ \int_0^t f(u)e^{-k(t-u)}\,du \ =\ f(t) * e^{-kt};$$

i.e., the amount of waste still radioactive at time t is the convolution of the dumping rate and the decay function.

Example 4. Bank interest. On a savings account, a bank pays the continuous interest rate r, meaning that a sum A_0 deposited at time $u = 0$ will by time $u = t$ grow to the amount $A_0 e^{rt}$.

Suppose that starting at day $t = 0$ a Harvard square juggler deposits every day his take, with deposit rate $d(t)$ — i.e., over a relatively small time interval $[u_0, u_0 + \Delta u]$, he deposits approximately $d(u_0)\Delta u$ dollars in his account. Assuming that he makes no withdrawals and the interest rate doesn't change, give with reasoning an approximate expression (involving a convolution) for the amount of money in his account at time $u = t$.

Solution. Similar to Example 3, and left as an exercise.

3. Weight and transfer functions and Green's formula.

In Example 2 we expressed the solution to the IVP $y' + ky = q(t)$, $y(0) = 0$ as a convolution. We can do the same with higher order ODE's which are linear with constant coefficients. We will illustrate using the second order equation,

$$y'' + ay' + by = f(t), \qquad y(0) = 0,\ y'(0) = 0. \tag{5}$$

The Laplace transform of this IVP, with the usual notation, is

$$s^2Y + asY + bY = F(s) \ ;$$

solving as usual for $Y = \mathcal{L}(y)$, we get

$$Y = F(s)\,\frac{1}{s^2 + as + b} \ ;$$

using the convolution operator to take the inverse transform, we get the solution in the form called **Green's formula** (the function $w(t)$ is defined below):

$$y = f(t) * w(t) = \int_0^t f(u)w(t-u)\,du \ . \tag{6}$$

In connection with this form of the solution, the following terminology is often used. Let $p(D) = D^2 + aD + b$ be the differential operator; then we write

$$\begin{aligned} W(s) &= \frac{1}{s^2 + as + b} && \text{the } \textbf{transfer function} \text{ for } p(D), \\ w(t) &= \mathcal{L}^{-1}(W(s)) && \text{the } \textbf{weight function} \text{ for } p(D), \\ G(t,u) &= w(t-u) && \text{the } \textbf{Green's function} \text{ for } p(D). \end{aligned}$$

The important thing to note is that each of these functions depends only on the operator, not on the input $f(t)$; once they are calculated, the solution (6) to the IVP can be written down immediately by Green's formula, and used for a variety of different inputs $f(t)$.

The weight $w(t)$ is the unique solution to either of the IVP's

$$y'' + ay' + by = 0; \qquad y(0) = 0,\ \ y'(0) = 1; \tag{7}$$

$$y'' + ay' + by = \delta(t); \qquad y(0) = 0,\ \ y'(0^-) = 0; \tag{8}$$

in (8), the $\delta(t)$ is the Dirac delta function. It is an easy Laplace transform exercise to show that $w(t)$ is the solution to (7) and to (8). In the next section, we will give a physical interpretation for the weight function and Green's formula.

Let us check out Green's formula (6) in some simple cases where we can find the particular solution y_p also by another method.

Example 5. Find the particular solution given by (6) to $y'' + y = A$, $y(0) = 0$.

Solution. From (7), we see that $w(t) = \sin t$. Therefore for $t \geq 0$, we have

$$y_p(t) = \int_0^t A\sin(t-u)\,du = A\cos(t-u)\Big]_0^t = A(1-\cos t).$$

Here the exponential response formula or the method of undetermined coefficients would produce the particular solution $y_p = A$; however, $A - A\cos t$ is also a particular solution, since $-A\cos t$ is in the complementary function y_c; the extra cosine term is required to satisfy $y(0) = 0$.

Example 6. Find the particular solution for $t \geq 0$ given by (6) to $y'' + y = f(t)$, where $f(t) = 1$ if $0 \leq t \leq \pi$, and $f(t) = 0$ elsewhere.

Solution. Here the method of Example 5 leads to two cases: $0 \leq t \leq \pi$ and $t \geq \pi$:

$$y_p = \int_0^t f(u)\sin(t-u)\,du = \begin{cases} \displaystyle\int_0^t \sin(t-u)\,du = \cos(t-u)\Big]_0^t = 1 - \cos t, & 0 \leq t \leq \pi; \\ \displaystyle\int_0^\pi \sin(t-u)\,du = \cos(t-u)\Big]_0^\pi = -2\cos t, & t \geq \pi\,. \end{cases}$$

Terminology and results analogous to (6) hold for the higher-order linear IVP's with constant coefficients (here $p(D)$ is any polynomial in D)

$$p(D)y = f(t), \quad y(0) = y'(0) = \ldots = y^{(n-1)}(0) = 0;$$

Green's formula for the solution is once again (6), where the weight function $w(t)$ is defined to be the unique solution to the IVP

$$(9) \qquad p(D)y = 0, \qquad w(0) = w'(0) = \ldots w^{(n-2)}(0) = 0, \quad w^{(n-1)}(0) = 1.$$

Equivalently, it can be defined as the unique solution to the analogue of (8).

4. Impulse-response; interpretation of Green's formula.

We obtained Green's formula (6) by using the Laplace transform; our aim now is to interpret it physically, to see the "why" of it. This will give us further insight into the weight function and the convolution operation.

We know the weight function $w(t)$ is the solution to the IVP

$$(7) \qquad y'' + ay' + by = 0, \qquad y(0) = 0, \quad y'(0) = 1.$$

We think of this as modeling the motion of the mass in a spring-mass-dashpot system (we will take the mass $m = 1$, for simplicity). The system is initially at rest, but at time $t = 0$ the mass is given a kick in the positive direction, which imparts to it unit velocity:

$$y'(0^+) = 1.$$

According to physical mechanics, to impart this unit velocity to a unit mass, the kick must have unit impulse, which is defined for a constant force F to be

$$(10) \qquad \text{Impulse} = (\text{force } F)(\text{length of time } F \text{ is applied})\,.$$

A kick is modeled as a constant force F applied over a very short time interval $0 \leq t \leq \Delta u$; according to (10), for the force to impart a unit impulse over this time interval, it must have magnitude $1/\Delta u$:

$$1 = (1/\Delta u)(\Delta u)\,.$$

Input: force $(1/\Delta u)$ applied over time interval $[0, \Delta u]$ Response: $w(t)$

If the kick is applied instead over the time interval $u \le t \le u + \Delta u$, the response is the same as before, except that it starts at time u:

Input: force $(1/\Delta u)$ applied over time interval $[u, u + \Delta u]$ Response: $w(t - u)$

Finally, if the force is not a constant $(1/\Delta u)$, but varies with time: $F = f(u)$, by (10) the impulse it imparts over the time interval $[u, u + \Delta u]$ is approximately $f(u)\Delta u$, instead of 1, so the response must be multiplied by this factor; our final result therefore is

(11) Input: force $f(u)$ applied over $[u, u + \Delta u]$ Response: $f(u)\Delta u \cdot w(t - u)$.

From this last it is but a couple of steps to the physical interpretation of Green's formula. We use the

Superposition principle for the IVP (5): *if $f(t) = f_1(t) + \ldots f_n(t)$, and $y_i(t)$ is the solution corresponding to $f_i(t)$, then $y_1 + \ldots y_n$ is the solution corresponding to $f(t)$.*

In other words, the response to a sum of inputs is the sum of the corresponding responses to each separate input.

Of course, the input force $f(t)$ to our spring-mass-dashpot system is not the sum of simpler functions, but it can be approximated by such a sum. To do this, divide the time interval from $u = 0$ to $u = t$ into n equal subintervals, of length Δu:

$$0 = u_0, u_1, \ldots, u_n = t, \qquad u_{i+1} - u_i = \Delta u\ .$$

Assuming $f(t)$ is continuous,

$$f(t) \approx f(u_i) \quad \text{over the time interval } [u_i, u_{i+1}]$$

Therefore if we set

$$(12) \qquad f_i(t) = \begin{cases} f(u_i), & u_i \le t < u_{i+1}; \\ 0, & \text{elsewhere,} \end{cases} \qquad i = 0, 1, \ldots, n-1\ ,$$

we will have approximately

$$(13) \qquad f(t) \approx f_0(t) + \ldots + f_{n-1}(t), \qquad 0 < u < t\ .$$

We now apply our superposition principle. According to (10), the response of the system to the input $f_i(t)$ described in (12) will be approximately

$$(14) \qquad f(u_i)w(t - u_i)\Delta u;$$

Applying the superposition principle to (13), we find the response $y_p(t)$ of the system to the input $f(t) = \sum f_i(t)$ is given approximately by

$$(15) \qquad y_p(t) \approx \sum_0^{n-1} f(u_i)w(t - u_i)\Delta u\ .$$

We recognize the sum in (15) as the sum which approximates a definite integral; if we pass to the limit as $n \to \infty$, i.e., as $\Delta u \to 0$, in the limit the sum becomes the definite integral and the approximation becomes an equality and we get Green's formula:

$$(16) \qquad y_p(t) = \int_0^t f(u)\, w(t-u)\, du \qquad \textit{system response to } f(t)$$

> In effect, we are imagining the driving force $f(t)$ to be made up of an infinite succession of infinitely close kicks $f_i(t)$; by the superposition principle, the response of the system can then be obtained by adding up (via integration) the responses of the system to each of these kicks.

Green's formula is a very remarkable one. It expresses a particular solution to a second-order differential equation directly as a definite integral, whose integrand consists of two parts: a factor $w(t-u)$ depending only on the left-hand-side of (5) — that is, only on the spring-mass-dashpot system itself, not on how it is being driven — and a factor $f(t)$ depending only on the external driving force. For example, this means that once the unit impulse response $w(t)$ is calculated for the system, one only has to put in the different driving forces to determine the responses of the system to each.

Green's formula makes the superposition principle clear: to the sum of input forces corresponds the sum of the corresponding particular solutions.

Still another advantage of Green's formula is that it allows the input force to have discontinuities, as long as they are isolated, since such functions can always be integrated. For instance, $f(t)$ could be a step function or the square wave function.

Exercises: Section 2H

Official Exam 3 Formula Sheet

USEFUL FORMULAS:

If the Fourier series of a function $f(x)$ with period $2L$ is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right],$$

then

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx \text{ and } \quad b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

The convolution of functions $f(t)$ and $g(t)$ is defined to be

$$[f * g](t) = \int_0^t f(u)g(t-u)du.$$

We have $f * g = g * f$. The Laplace transform of the function $f(t)$ is defined to be

$$F(s) = \mathcal{L}[f(t)] = \int_0^{\infty} f(t)e^{-st}dt \quad \text{(defined for } s \gg 0\text{)}.$$

We have

Linearity:	$\mathcal{L}[af(t) + bg(t)] = aF(s) + bG(s).$
t-Shift:	$\mathcal{L}[u(t-a)f(t-a)] = e^{-as}F(s).$
s-Shift:	$\mathcal{L}[e^{at}f(t)] = F(s-a).$
t-Derivatives:	$\mathcal{L}[f'(t)] = sF(s) - f(0).$
	$\mathcal{L}[f''(t)] = s^2F(s) - sf(0) - f'(0).$
s-Derivative:	$\mathcal{L}[tf(t)] = -F'(s).$
Convolution:	$\mathcal{L}[f(t) * g(t)] = F(s)G(s).$

Also

$$\mathcal{L}[1] = \frac{1}{s} \qquad \mathcal{L}[e^{at}] = \frac{1}{s-a} \qquad \mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}[\cos(at)] = \frac{s}{s^2+a^2} \qquad \mathcal{L}[\sin(at)] = \frac{a}{s^2+a^2}$$

$$\mathcal{L}[\delta(t-a)] = e^{-as}.$$

18.03 Quiz 3 Review

11/21

Integration by parts

$\int u \, dv$

L expand difference and express original integral in terms of $\int v \, du$

(was catch phrase in OH?)

$\cancel{u \, dv = u \, dv + v \, du}$

$\int u \, dv \cancel{=}$

$d(uv) = u \, dv + v \, du$

Integrate

$\int d(uv) = uv = \int u \, dv + \int v \, du$

$\int u \, dv = uv - \int v \, du$

②

So review

$$\frac{d}{dx}(uv) = u\,dv + v\,du$$

$$uv = \int u\,dv + \int v\,du$$

$$\int u\,dv = uv - \int v\,du$$

So try

$$\int x \cos x \, dx$$

Recognize ~~with~~ $u = x$

$dv = \cos x$

Calculate $du = 1$

$v = \sin x$

Plug in

$$x \sin x - \int \sin x \cdot 1$$

$$x \sin x - -\cos x$$

$$x \sin x + \cos x + C$$ ✓ See can do it

(3)

So 1 more time

$\int u\,dv = uv - \int v\,du$

Does not always work

- can be more complex
- can be back to same integral

Fourier Series For Periodic Functions

periodic $f(x) = f(x + P)$

$\uparrow$ period

want smallest period

(since $2P, 3P, -P, -2P, \ldots$ also work

$\sin(x), \cos(x) \rightarrow$ period $= 2\pi$

$\sin(mx), \cos(mx) \rightarrow \dfrac{2\pi}{m}$

Constant $\rightarrow$ periodic

Can add or multiply

(4)

$$f(x) = \underbrace{\frac{a_0}{2}}_{\text{constant}} + a_1 \cos(x) + a_2 \cos(2x) + \dots + a_n \cos(nx)$$
$$+ b_1 \sin(x) + b_2 \sin(2x) + \dots + b_n \sin(nx)$$

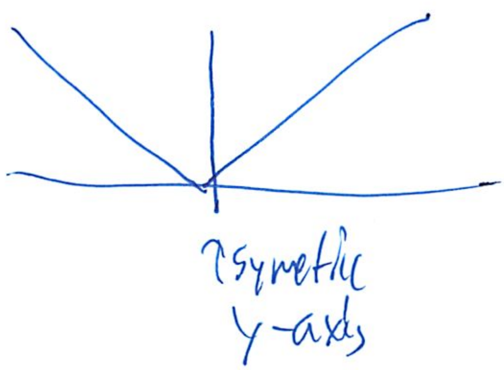
Describes values for 1 period – window $[-\pi, \pi)$

a_i are infront of even fns

b_i " odd "

<u>Even</u>

$f(x) = f(-x)$

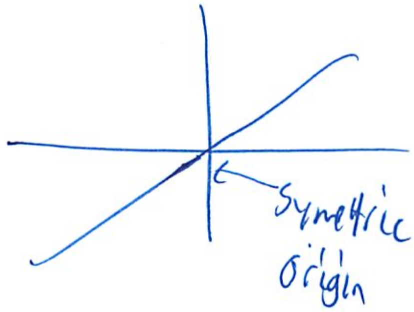


$\cos(x)$

$\cos(2x)$

<u>Odd</u>

$f(x) = -f(x)$



$\sin(x)$

$\sin(2x)$

Remember $\cos(2x)$ means period $\frac{1}{2}$ of before
– otherwise no change

(5)

even + even = even

odd + odd = odd

even · even = even

even · odd = odd

odd · odd = even

Must memoize!

- flash cards!

Calculation is main thing

Lits like Power Series how it converges

(this was on last exam - will not study)

Figure out a_i, b_i s w/ integration

<u>Prelims</u>

1. $\int_{-\pi}^{\pi} \sin(mx)\,dx = -\frac{\cos(mx)}{m}\Big|_{-\pi}^{\pi} = 0$

since this is odd

since cos

Cos is even

$\int_{-c}^{c} \text{odd} = 0$

6

2. $\int_{-\pi}^{\pi} \cos(mx)\, dx = \frac{\sin(mx)}{m}\Big|_{-\pi}^{\pi} = 0$

0 since $\sin(m\pi)$ is 0 for all integers m

? confused how are these different?

- well both are 0
- So I guess are same?

3. $\int_{-\pi}^{\pi} \cos(mx) \cos(nx)\, dx = \begin{cases} 2\pi & \text{if } m=n \\ 0 & \text{if } m \neq n \end{cases}$

4. $\int_{-\pi}^{\pi} \sin(mx)\, \sin(nx)\, dx = \begin{cases} \pi & \text{if } m=n \\ 0 & \text{if } m \neq n \end{cases}$

5. $\int_{-\pi}^{\pi} \sin(mx) \cos(nx)\, dx = 0$

Sin = odd

Cos = even

$e \cdot e = e$

$o \cdot o = e$

$e \cdot o = o$

7

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(mx)\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(mx)\,dx$$

And the general form - I did not realize was the same earlier

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$L = \frac{\text{Period}}{2}$

So when pd $= 2\pi$

$L = \frac{2\pi}{2} = \pi$

$\cos\left(\frac{n\cancel{\pi}x}{\cancel{\pi}}\right) = \cos(nx)$

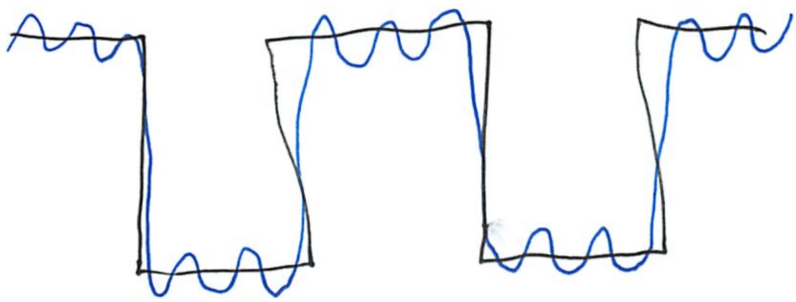
$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

So key fact can write

$$f(x) = \frac{a_0}{2} + a_1\cos x + a_2\cos(2x) + \cdots$$
$$+ b_1 \sin x + b_2\sin(2x) + \cdots$$

(8)

Essentially trying to approx square wave



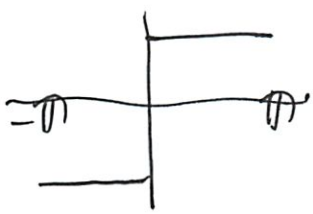
L20]

Think of sq wave as superposition (sum) of sinusoidal wave

Straight line is a wave ossilating ∞ often

Square wave

$$Sq(x) = \begin{cases} 1 & \text{if } x \in (0, \pi) \\ -1 & x \in (-\pi, 0) \end{cases}$$



I will try

period $= 2\pi$

$L = \pi$

$$a_0 = \frac{1}{L} \int_{-\pi}^{\pi} \underbrace{f(x)}_{\text{odd}} \, dx = 0$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \text{odd} = 0$$ ← I never usually take this shortcut!

(9)

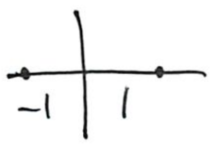
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\sin\left(\frac{n\pi x}{L}\right)$$

Think need to split If I remember

$$= \frac{1}{\pi}\int_{-\pi}^{0} -1\,\sin(n\,x)\,dx + \frac{1}{\pi}\int_{0}^{\pi} 1\,\sin(n\,x)\,dx$$

$$= \frac{1}{\pi}\;-\cos(n\,x)\Big|_{-\pi}^{0} + \frac{1}{\pi}\cos nx\Big|_{0}^{\pi}$$

$$= \frac{1}{\pi}\left(-\cos(0) + \cos(-\pi x)\right) + \frac{1}{\pi}\left(\cos(\pi n) - \cos(0)\right)$$



left out the n

$$= \frac{1}{\pi}(-1 + -1) + \frac{1}{\pi}(-1 - 1)$$

So it depends if n is odd or even

$$= \frac{-2}{\pi} + \frac{-2}{\pi}$$

$$= \frac{-4}{\pi}$$

$\frac{2}{n}$ if n odd

0 if n even

So $\cancel{f(x) =}$ $b_n = \begin{cases} \frac{4}{n\pi} & \text{if } n \text{ odd} \\ 0 & n \text{ even} \end{cases}$

(10)

So write that like

$$= \frac{4}{\pi} \sin x + \frac{4}{3\pi} \sin 3x + \dots$$

$$= \sum_{n=1}^{\infty} \frac{4}{\pi} \frac{\sin (2n-1) x}{2n-1}$$

↑ Remember $2n-1$ is code for odd

I believe can also do

$$= \sum_{n \text{ odd}} \frac{4}{\pi n} \sin (nx)$$

<u>Piecewise smooth</u>

↳ means <u>piecewise continuous</u>

↑ finitly many jump discontinuites on a bounded finite integral

So for squarewave

$Sq'(x) = 0$ unless x is a multiple of π

↳ not defined there

(11)

In general <u>jump discontinuity</u>

Can't be ∞ly many discontinuities

$$g(x) = \begin{cases} 1 & \text{if } x = \text{rational} \\ 0 & x = \text{irrational} \end{cases}$$

If piecewise smooth - than fourier series of f
Converges to f at every pt that f is contineous

<u>L23</u>

$$b_1 = \frac{1}{1} \int_{-1}^{1} \underbrace{\underbrace{f(t)}_{\text{even}} \underbrace{\sin(n\pi t)}_{\text{odd}}}_{\text{odd}}$$

$$\int \text{odd} = 0$$

<u>Applications</u> Spring w/ square way input

$$mx'' + kx = S_q(t)$$

(12)

Use what we just did $\left(x_p(t) = \sum_{n \text{ odd}} b_n (\sin n\pi t) \right)$

to plug guess in

find 2nd deriv

plug that in too & wish cald sep

Here it is

get 2nd deriv $\left[X_p''(t) = \sum_{n \text{ odd}} -n^2 \pi 2 b_n \sin(n\pi t) \right.$

Now find coeff ~~coeff~~ $\sin(n\pi t)$

write LHS $\left[\underbrace{-mn^2 \pi b_n}_{\text{2nd deriv}} + \underbrace{k b_n}_{\text{regular}} \right.$

Solve FS from before $\left[= \text{Coeffs RHS} \right.$

$\frac{4}{\pi} \cdot \frac{1}{n}$ ← which is what we found last time

Now match coefficents

$$-m n^2 \pi^2 b_n + k \cdot b_n = \frac{4}{\pi} \cdot \frac{1}{n}$$

Equating coeffs of $\sin 3^{n}\pi t$

(13)

So solve for b_n

$$b_n = \frac{4}{\pi} \cdot \frac{1}{n(k - mn^2\pi^2)}$$

Have found particular sol'n

$$X_p(t) = \sum_{n \text{ odd}}^{\infty} \frac{4}{\pi} \frac{1}{n(k - mn^2\pi^2)} \sin n\pi t$$

Oh thats the ans?

Remember: trying to model spring

We don't need to visualize

Must be continuous or piecewise

└ No teleportation!

Can find Amplitude Gain

- graph 1st 10 terms
- take max
- Can find critical pt in original

└ where cos = 0

Will this be one of our qus?

(14)

So this is somehow connected w/ resonance

Fails if $n\pi = \sqrt{\frac{k}{m}}$ for any n

; So basically find $\sqrt{\frac{k}{m}}$

If any of the a_n, b_n terms $= \sqrt{\frac{k}{m}}$

└ then resonance!

So in class today they said could ask

Where is resonance

Don't fully get - look at later

2nd Application: Boundry Value Problems

eg $P(D)x = f(t)$ on finite interval $t \in (0, L)$

but $x(0) = x(L) = 0$

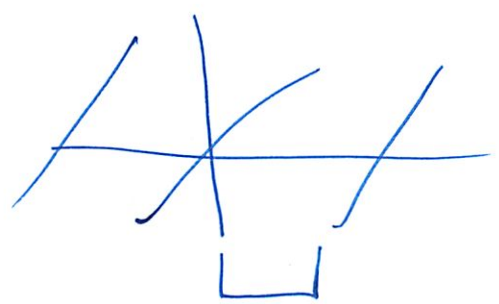
example $x'' + 9x = t$

on $t \in (0, 1)$ w/ $x(0) = x(1) = 0$

(15)

(Oh reviewed today in lecture)

Sneaky idea extend $f(x)$ to periodic fn of period $2L$ w/ boundry conditions $f(0)=f(1)=0$



as long as interval up want is unchanged

Solve FS as normal

But then must still solve like earlier today

└ Don't get

Is this the coefficent matching step

Revien in book or something

Used $\underbrace{X'' + \lambda X}_{f(x)} = \cancel{A}$ for $[0,1]$

So make ↑ periodic

where did we get from

write as $f(x) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin n\pi x$

found fourier series of triangle

16

Then make guess of undet. coeff

$$X_p(t) = \sum_{n=1}^{\infty} b_n \sin n\pi t$$ what undt coefficent on $x = ?$

Plug into/ ~~on~~ set = to

$$X_p(t) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sin n\pi t}{n(9-\pi^2 n^2)}$$

i don't get this

Notes are very different

So we know if $F(t) = F_0 \sin \omega t$ answer will be

$$X_p(t) = \frac{F_0}{\sqrt{(k-m\omega^2)^2 + (c\omega)^2}} \sin(\omega t - \alpha)$$

I remember this from last time

$$\alpha = \tan^{-1}\left(\frac{c\omega}{k-m\omega^2}\right)$$

So by superposition if

$$F(t) = \sum b_n \sin \frac{n\pi t}{L}$$

(17)

Then $F_{0,n} = b_n$

$\omega_n = \frac{n\pi}{L}$

$\alpha_n = \ldots$ for each n gives soln

So $$x_p(t) = \sum_{n=1}^{\infty} \frac{b_n}{\sqrt{(k-m\omega_n^2)^2 + (c\omega_n)^2}} \sin(\omega_n t - \alpha_n)$$

~~Whas~~ was that the right section?

Yeah - thats the matching

but it seems like it was desribed differently last time

Oh we calculated b_n via

$$\frac{1}{L} \int_{-L}^{L} f(t) \sin \frac{n\pi t}{L} \, dt = b_n$$

But to use - just memorize

$$x_p(t) = \sum_{n=1}^{\infty} \frac{b_n \sin(\omega_n t - \alpha_n)}{\sqrt{(k-m\omega_n^2)^2 + (c\omega_n)^2}}$$

That does not ~~seem~~ seem right

Mark to review in OH

(18)

Oh did in recitation

$p(D)\,x = q(t)$

just generic notation, right?

polynomial in D

Only care $0 \le t \le 1$

So make periodic

Looking for $x^{\text{periodic}}(t) = \frac{a_0}{2} + \sum a_n \cos \ldots + b_n \ldots$ etc

know $\sin = 0$ at boundries
$\cos = 1$

So sum a_n, b_n

Ahg TA confused too!

Was marked in book as confused as well

the $x(t) = \frac{F_0}{\text{etc}}$ was for damped

Gr forget it

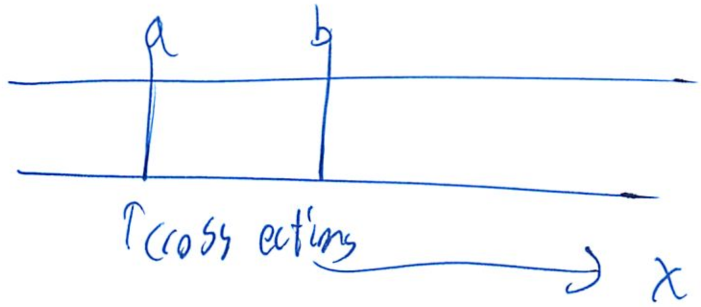
prob set me up badly for recitation ea

L23 | # Heat Equ

L 1st apps of Forier series

$T = T(x, t)$ time

temp position



C = heat capacity

F = heat flux

Heat change

$$\frac{d}{dt}\int_a^b cT\,dx\,A = F_A A - F_B B$$

$$= -A\int_a^b F_x\,dx$$

$$\int_a^b \left(c\frac{\partial T}{\partial t} + \frac{\partial F}{\partial x}\right)dx = 0$$ for any a, b

I don't get this stuff

20

Ficks' Law

$F = -\mu T_x$

$c T_t - \mu T_{xx} = 0$

$T_t = \mu T_{xx}$

heat diff

Need to solve for T periodic

Want ODE

$T = \varphi(x) e^{rt}$

$r \varphi = \varphi''$

find periodic

$r = -\lambda^2 \qquad \varphi = b_n \sin \lambda x + a_n \cos \lambda x$

$\lambda_n = \frac{n\pi}{L}$

$r_n = -\lambda_n^2 = -\frac{n^2 \pi^2}{L^2}$

21

Also need $\varphi = \frac{1}{2} a_0$ $c_0 = 0$

So now have tons of sols to heat eqn

$$T = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{L} x \, e^{\lambda_n t} + \left(b_n \sin \frac{n\pi}{L} x\right) e^{\lambda_n x}$$

When set $T = 0$

so same as we saw before

$$T(x,0) = T_0(x) = \frac{1}{2} a_0 + \sum \left(a_n \cos \frac{n\pi}{L} x \; b_n \sin \frac{n\pi}{L} x \right)$$

; So just need to pratice converting into this form;

WTF is T_{xx};

Read the textbook;

Heat eqn is partial differential eqn

dep variable that is fn of ≥ 2 ind variables

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$$

So x, t

(22)

Heated Rod

- uniform cross section
- area A

$\phi(x,t)$ = heat flux = rate of flow for heat

$$\phi = -k \frac{\partial u}{\partial x}$$

k: thermal conductivity

If $u_x > 0$, then $\phi < 0$ so heat flows $\Leftarrow$

$u_x < 0$ $\quad \phi > 0$

So rate of flow proportional to $\text{abs}(u_x)$

$Q(t)$ = heat content $= \int_x^{x+\Delta x} c \delta A u(x,t)\, dx$

Wire insulated, so

$$Q'(t) = kA[u_x(x+\Delta x, t) - u_x(x,t)] = A$$

(23)

So differentiate $Q(t)$, apply MVT for integrals

$$Q'(t) = \int_x^{x+\Delta x} c\,\sigma A\, u_t(x,t)\,dx$$

$$= c\sigma A\, u_t(\bar{x},t)\,\Delta x$$

for some $\bar{x}$ in $(x, x+\Delta x)$

Ohhh $\bar{x}$ is average

I remember that

but for some reason thought vector

$$c\sigma A\, u_t(\bar{x},t)\,\Delta x = kA\left[u_x(x+\Delta x,t) - u_x(x,t)\right]$$

$$u_t(\bar{x},t) = k\,\frac{u_x(x+\Delta x,t) - u_x(x,t)}{\Delta x}$$

where $k = \frac{k}{c\sigma}$ = thermal diffusivity

Take limit as $\Delta x \to 0$ so $\bar{x} \to x$

$$\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2}$$

(29)

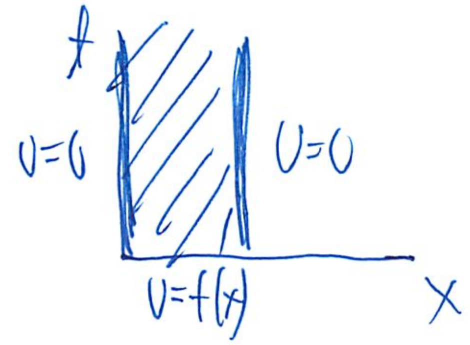
Boundry Conditions

So here for partial ~~dd~~ we have functions not constants

Can say points against ice

$$u(0,t) = u(L,t) = 0$$

Can make geometric interpertation



Linear - so super position

---

examples

The book does not say what u_{xx} is!

Took me several min of searching but

$$u_{xy} = \frac{\partial^2 u}{\partial x \partial y} \text{ so } u_{xx} \text{ prob} = \frac{\partial^2 u}{\partial x^2}$$

(25)

But what does 2nd deriv ~~att~~ ~~lll~~ mean?

Like I see $U_t = U_{xx}$

What does that mean?

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}$$

Remember heat eqn

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2}$$

Ok move on.... will prob get this wrong

Pre Reduced Values

~~Old~~ ~~Ht~~ Heated Rod w/ Zero Endpt Tem

$$U(x,t) = \sum_{n=1}^{\infty} b_n \cancel{} e^{\frac{-n^2\pi^2 k t}{L^2}} \sin\frac{n\pi x}{L}$$

Examle 2 - has some #s

100°C then ends in ice

What is tem at midpoint in half an hr

26e

So know BVs

$u_t = k u_{xx}$ ← heat eqn

$u(0,t) = u(L,t) = 0$ ← ice

$u(x,0) = u_0$ ← initall starting tem

Recall square wave seq

$$f(t) = \frac{4}{\pi} \sum_{n\,\text{odd}} \frac{1}{n} \sin \frac{n\pi t}{L} = \begin{cases} +1 & 0<t<L \\ -1 & -L<t<0 \end{cases}$$

; So only care $t > 0$

and need ~~the~~ inital condition

$$f(x) = \frac{4u_0}{\pi} \sum_{n\,\text{odd}} \frac{1}{n} \sin \frac{n\pi x}{L}$$

Change or different variable

Hence F. coeffients

$$b_n = \begin{cases} \frac{4u_0}{n\pi} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

Thats the jump I was wondering about before

27

So rod's temp is given by

$$u(x,t) = \frac{4u_0}{\pi} \sum_{n \text{ odd}} \frac{1}{n} e^{-\frac{n^2 \pi k t}{L^2}} \sin \frac{n\pi x}{L}$$

So this is following some formula I saw before, but I don't 100% get

Then to answer the question

$$u(25, 1800) = \frac{400}{\pi} \sum_{n \text{ odd}} \frac{(-1)^{n+1}}{n} e^{-\frac{18 n^2 \pi^2 k}{25}}$$

25: half of rod

1800: 30 min in seconds (must know units)

Now can find

$= 43.85\,°C$

So what did we talk about today?

So basically 2 types of problems

ends frozen $u(0,t) = u(L,t) = 0$ →

ends insulated $\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t)$ →

(28)

ends frozen → sine sol →

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{\frac{-n^2\pi^2 kt}{L}} \sin\frac{n\pi x}{L}$$

Fourier sine coeffs of rod's initial temp fn $f(x) = u(x,0)$

ends insulated → $$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{\frac{-n^2\pi^2 kt}{L}} \cos\frac{n\pi x}{L}$$

Fourier cosine coeffs of rod's initial temp fn $f(x) = u(x,0)$

Ok think →
I am starting to see
possible problems
The other stuff was filler

$$\lim_{t\to\infty} u(x,t) = \frac{a_0}{2} = \frac{1}{L}\int_0^L f(x)\,dx$$

Ok move on finally

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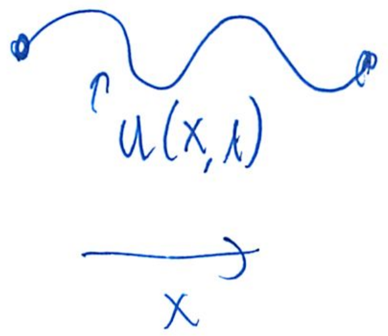
Another eq'n : Wave Eq'n

Sting under tension, Vibrating



$U = 0$ at $x = 0, L$

Look at conservation of angular motion

Don't think have to study

L23 LaPlace Transforms

Some what reminicent of operator method

$D = \frac{d}{dx}$

But now use integration as operator

$\mathcal{L}(f) = \int_0^\infty f(t)\, e^{-st}\, dt$

s is real variable
$s >> 0$ so that
integral converges

30

$\mathcal{L}(f) = F(s)$ ↓ the laplace transform

↑ $F(s)$ f is function of s

$\underline{\mathcal{L}(1)}$

$$\mathcal{L}(1) = \int_0^\infty 1\, e^{-st}\, dt$$

$$= \lim_{A \to \infty} \int_0^A e^{-st}\, dt$$

↑ fund theorm of calculus

$$= \lim_{A \to \infty} \left(-\frac{1}{s} e^{-st}\right) \Big|_{0=t}^{A=t}$$

$$= \frac{1}{s}$$

↑ don't need to know this stuff
- on formula sheet

Solving ODEs using LaPlace Transforms

1. Start w/ ODEs
2. LaPlace Transform both sides
   - this converts the derivitives to algebra
3. Do algebra to simplify problem
4. Take inverse Laplace Transform

Library of fns - not copying

Properties $\mathcal{L}(f) = F(s)$

A $\mathcal{L}[e^{at} \cdot f(t)] = F(s-a)$

B $\mathcal{L}(t \cdot f(t)) = -F'(s)$

C $\mathcal{L}[f'(t)] = sF(s) - f(0)$

D $\mathcal{L}(f''(t)) = s^2 F(s) - s f(0) - f'(0)$

? good on the formula sheet

32

Inverse LaPlace Transforms

$$\mathcal{L}^{-1}\left(\frac{1}{s^2+5s+4}\right)$$

→ when $\frac{P(s)}{Q(s)}$ ← polynomials can simplify w/ partial fractions

Try

Step 1. Factor $(s+4)(s+1)$

$$\frac{1}{s^2+5s+4} = \frac{A}{s+4} + \frac{B}{s+1}$$

$$\frac{1\,\cancel{(s+4)}}{\cancel{(s+4)}(s+1)} = A + \frac{B(s+4)}{s+1}$$

$s=-4$

$$\frac{1}{-3} = A$$

$$\frac{1}{s+4} = B + \frac{(s+1)A}{s+4} \quad s=-1$$

$B=\frac{1}{3}$

$$= \frac{-\frac{1}{3}}{s+4} + \frac{\frac{1}{3}}{s+1}$$

Now check patterns

$$-\frac{1}{3}e^{-4t} + \frac{1}{3}e^{-t}$$

Can same thing in spring model

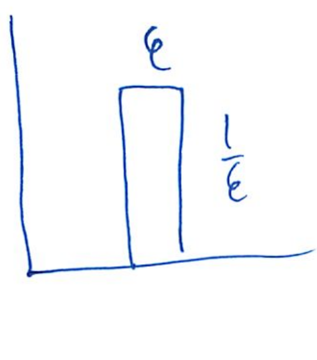
L24 Solving ODEs + Delta Fns

$$\mathcal{L}(f'') = s^2F(s) - sf(0) - f'(0)$$

* must have both terms
where did I get silly idea that
might only need one?

I've done these practice problems a lot recently

δ is delta function



ε = very small

(34)

$$\mathcal{L}(\delta) = \int_{0^-}^{\infty} \delta(t)\, e^{-st}\, dt$$

$$= e^{-st} \Big|_{t=0}$$

$$= \underline{1}$$

~~So just leave~~

? So just leave it like this?

$$\mathcal{L}(\delta') = \int_{0^-}^{\infty} \delta'(t)\, e^{-st}\, dt$$

$$\overset{\text{define}}{=} \int_{0^-}^{\infty} \delta(t)\, \frac{d}{dt}\left(e^{-st}\right) dt$$

$$= s \cdot \int_{0^-}^{\infty} \delta(t)\, e^{-st}$$

$$= s \cdot \underline{1}$$

$$= s$$

i just leave it as $\mathcal{L}(\delta)$ saw in recitation

Le wiser to deal w/

35

L35 | Convolution

So if have $x'' + 9x = 0$

$$\mathcal{L}(x'' + 9x) = \mathcal{L}(0) = 0$$

$$(s^2+9)\mathcal{L}(x) - s(x) - x'(0))$$

$$\mathcal{L}(x) = \frac{s\{x(0)\} + x'(0)}{s^2+9}$$

$$x = \mathcal{L}^{-1}\left(\frac{s x(0)}{s^2+9} + \frac{x'(0)}{s^2+9}\right)$$

Can't reduce any furter

$$x(t) = x_0 \cos 3t + \tfrac{1}{3} x'(0) \sin 3t$$

Can add impulsive force

$4\delta_2 t$

$\delta(t-2)$

So only starts 2 sec in

They have 2 pages of complex stuff

(36)

Convolution

$$\frac{2s}{(s^2+4)^2} = \frac{2}{s^2+4} \cdot \frac{s}{s^2+4}$$

$$= \mathcal{L}(\sin 2t * \cos 2t)$$

$$= \mathcal{L}(\sin 2t) * \mathcal{L}(\cos 2t)$$

I think have to write this

$= \sin 2t * \cos 2t$ but technically must actually $\int$ ← need this

$$\int_0^t f(z)\, g(t-z)\, dz$$

Is that right?

So our cos, sin example

$$= \int_0^t \cos 2z \; \sin(2(t-z))\, dz$$

Trig identity $\cos A \sin B = \frac{1}{2}\left[\sin(A+B) - \sin(A-B)\right]$

$$= \int_0^t \frac{1}{2}\left[\sin 2t - \sin 2(2z-t)\right] dz$$

$$= \frac{1}{2} z \sin 2t + \frac{1}{2} \cos 2(2z-t) \Big|_{z=0}^{t}$$

(37)

$$= \frac{1}{2} t \sin 2t + \frac{1}{2} \cos 2(2t - t)$$

$$- \frac{1}{2} \cdot 0 + \frac{1}{2} \cos 2(-t)$$

$$= \frac{1}{2} t \sin 2t + \frac{1}{2} \cos 2t + \frac{1}{2} \cos(-2t)$$

can we do this on circle
don't think so - leave for now

$$\mathcal{L}\left(\frac{1}{2} t \sin 2t\right) + \mathcal{L}\left(\frac{1}{2} \cos 2t\right) + \mathcal{L}\left(\frac{1}{2} \cos(-2t)\right)$$

$$\frac{1}{2} \frac{1}{s^2} \frac{2}{s^2+a^2} + \frac{1}{2} \frac{s}{s^2+2^2} + \frac{1}{2} \frac{s}{s^2+(-2)^2}$$

combine

$$+ \frac{s}{s^2+2^2}$$

Should be what had earlier

L26 Weight function

did we actually cover this?
it seems like we did other stuff in this lecture

Input Response Models

Input
$f(t) \rightarrow$ $\boxed{\int_0^t \text{convolves w/ } w(t-u)du}$ $\rightarrow$ Output $x(t)$

$\boxed{\int_0^t (\dots)\, w(t-u)\, du}$

↑ Greens' Function

↑ what in all world is this?

Green's Function - a fn to solve inhomogeneous diff eq that have a specific inital/boundry condition

WP:

$$LG(x,s) = \delta(x-s)$$

to solve diff eq of form

$$Lu(x) = f(x)$$

I don't think this will be focus

$\mathcal{L}^{-1}\left(\frac{1}{p(s)}\right) \underset{\sim}{\text{defined}} w(t)$

↑ positive real axis only

Can also think about as a sol to ODE

$p(D)\, w = \delta$

Why the same? – take Laplace of both sides

$p(s)\, \mathcal{L}(w) = \mathcal{L}(\delta) = 1$

So $\mathcal{L}(w) = \frac{1}{p(s)}$

$w = \mathcal{L}^{-1}\left(\frac{1}{p(s)}\right)$

<u>Example</u> $p(D) = D^2 + 25$

Then $w(t) = \mathcal{L}^{-1}\left(\frac{1}{s^2+25}\right)$

$= \frac{1}{5}\sin(5t)$

Or better $\frac{u(t)}{5}\sin(5t)$

ie $w(t) = \frac{1}{5}\sin 5t$ is sol to $(D^2+25)w = \delta$

40

Notes

$$f(t) \to \boxed{\int_0^t (\ldots)\, w(t-\tau)\, d\tau} \to x(t) = f * w$$

Weight function $w(t)$

$\mathcal{L}^{-1}\left(\frac{1}{p(s)}\right) = w(t)$ or $\mathcal{L}(p(D)\,w) = \mathcal{L}(\delta) = 1$

w/ rest conditions

Still don't understand motivation behind

L27 | Pole Diagrams

Solve ODEs w/ LaPlace transform

Need $\mathcal{L}^{-1}$(rational fn)

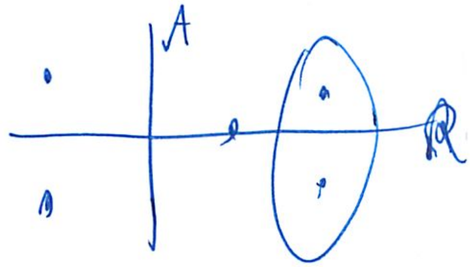
eg/ $p(D)\,y = \delta$

$\downarrow$

$\mathcal{L}^{-1}\left(\frac{1}{p(s)}\right)$ w/ rest conditions

(41)

eg roots of denom are $1,\ 2\pm i,\ -1\pm i$



↑ dominate in long term

"poles" = zero of denoms of rational functions

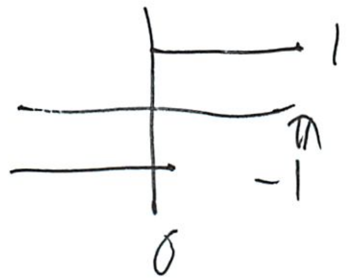
That's it - next lecture is today's review

18.03 Practice

1/22

Do some Fourier

for step fn



odd

So $\int_{-L}^{L} \text{odd} = 0$

↑ can't say now

period $= 2\pi$

$L = \pi$

$$a_n = \frac{1}{\pi} \underbrace{\int_0^{\pi} \underbrace{1}_{\text{even}} \underbrace{\cos(nx)}_{\text{even}}}_{\text{even}} dx + \underbrace{\int_{-\pi}^{0} \underbrace{-1}_{\text{even}} \underbrace{\cos(nx)\,dx}_{\text{even}}}_{\text{even}}$$

$$\frac{1}{\pi} - \sin(nx) + \frac{1}{\pi} \sin(nx)$$

0

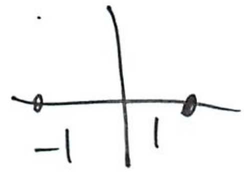
$$b_n = \frac{1}{\pi} \int_0^{\pi} \underbrace{\underbrace{1}_{\text{even}} \underbrace{\sin(nx)}_{\text{odd}}}_{\text{odd}} dx + \int_{-\pi}^{0} \underbrace{\underbrace{-1}_{e} \underbrace{\sin(nx)}_{o}}_{o} dx$$

↑ but can't use trick here

②

$$b_n = \frac{1}{\pi}\left(\cos(nx) \Big|_0^{\pi} + -\cos(nx)\Big|_{-\pi}^{0} \right)$$

$$= \frac{1}{\pi}\left((\cos(\pi x) - \cos(0) - \cos(0) + \cos(-\pi) \right)$$



$$\frac{1}{\pi}(-1 - 1 - 1 - 1)$$

$$= \frac{-4}{\pi}$$

Think forgot n

Same problem as before

Oh $\int \sin(nx) = -\underbrace{\frac{1}{n}}_{\text{forgot this}}\cos(nx)$

I did $\sin(x)$

Problem is $\sin(nx)$!

need to integrate inside

③

$$= \frac{1}{\pi}\left[\frac{1}{n}\cos nx\right]_{-\pi}^{0} + \frac{1}{\pi}\left[-\frac{1}{n}\cos nx\right]_{0}^{\pi}$$

Try from there

$$= \frac{1}{\pi}\cdot\left[\frac{1}{0}\cos(0x) - \frac{1}{-\pi}\cos(-\pi x)\right] + \frac{1}{\pi}\left[-\frac{1}{\pi}\cos\pi x - -\frac{1}{0}\cos 0x\right]$$

$= \frac{1}{\pi}\left[\right.$?! what to do $\frac{1}{0}$?

Ohh we are subbing for x!

Uuhhhh

$$= \frac{1}{\pi}\left[\frac{1}{n}\cos(n0) - \frac{1}{n}\cos(-\pi n)\right] + \frac{1}{\pi}\left[-\frac{1}{n}\cos(\pi n) - -\frac{1}{n}\cos(0n)\right]$$

$$= \frac{1}{\pi}\left[\frac{1}{n}\cdot 1 - \frac{1}{n}\cdot -1\right] + \left[-\frac{1}{n}\cdot -1 + \frac{1}{n}\cdot 1\right]$$

$$= \frac{1}{\pi}\left[\frac{2}{n}\right] + \left[\frac{2}{n}\right]$$

$-1 \quad 1$

$= \frac{4}{n\pi}$ ✓ but only n odd

(4)

Since

$$= \frac{2}{n\pi}\left(1 - \cos n\pi\right)$$

$$= \frac{2}{n\pi}\left(1 - (-1)^n\right)$$

So $\cos(n\pi) = (-1)^n$

I just put (-1)

Why?

Thats ~~the~~ just the value it

you put in WA

? So just know that value

$\sin(n\pi) = 0$

$\sin(0) = 0$

$\sin\left(\frac{n\pi}{2}\right) =$

n	out
1	1
2	0
3	-1
4	0
5	1

What do ya call this in closed form?

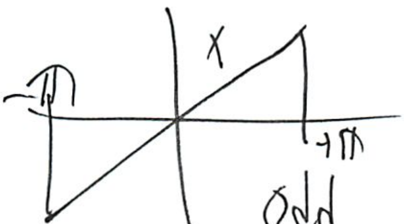
(5)

$$\begin{array}{cccccc} & 1 & 2 & 3 & 4 & 5 \\ (2n-1) = & 1 & 3 & 5 & 7 & 9 \end{array}$$

Ok use the integral formulas to make it easy

└ On the hw did it the hard way
Could have memorized these new things

Now triangle



odd

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{\underbrace{x}_{\text{odd}} \; \underbrace{\cos(nx)}_{\text{even}}}_{\text{odd}} \, dx$$

$$= 0$$

← much faster than I did on p-set!

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{\underbrace{x}_{\text{odd}} \; \underbrace{\sin(nx)}_{\text{odd}}}_{\text{even}} \, dx$$

(e)

$$= 2\int_0^{\pi} x \sin(nx)\, dx$$

int by parts

$$\int u\, dv = uv - \int v\, du$$

$u = x \rightarrow du = 1$

$dv = \sin nx \qquad v = \frac{1}{n}\cos nx$

$$= 2\left(x \cdot \frac{1}{n}\cos nx - \int_0^{\pi} \frac{1}{n}\cos nx \cdot 1 \right)$$

$$= 2\left(x \cdot \frac{1}{n}\cos nx - \frac{1}{n^2}\sin nx \Big|_0^{\pi} \right)$$

$\frac{d}{dx}\sin = +\cos$

$\frac{d}{dx}\cos = -\sin$

$$= \frac{2x\cos nx}{n} - \frac{\sin n\pi}{n^2} - - \frac{\sin 0}{n^2}$$

π; include this as well;

$$= \frac{2\pi\cos n\pi}{\pi} - \frac{2\,0\cos 0x}{0}$$

$$= 2(-1)^n$$; where can verify?

(7)

Write in form

$$= \sum_{n=1}^{\infty} 2(-1)^n \sin(n\pi)$$

$$= -2\sin(\pi) + 2\sin(2\pi) - 2\sin(3\pi) + 2\sin(4\pi) + \dots$$

8.3 Example 1

$$b_n = \frac{2L}{n\pi}(-1)^{n+1}$$

I had: $\frac{2\cancel{\pi}}{n\cancel{\pi}}(-1)^{n+1}$

divide n is ? +1 to exponent
or can't make that

Also a_n should not have been 0?

These are Calculus problems - not diff eq

(8)

Ok do a ~~[illegible]~~ LPT w/ Convolution

$$x'' + 2x' + x = \delta(t-7)$$

~~w/ rest cond~~
$x(0) = 1$
$x'(0) = 1$

$$(s^2 + 2s + 1)\,\mathcal{L}(x) - s - 1 - 1 = \mathcal{L}(\delta(t-7))$$

$$\mathcal{L}(x) = \frac{\mathcal{L}(\delta(t-7)) + s + 2}{s^2 + 2s + 1}$$

~~[illegible]~~ now ~~that~~ try to factor $(s+1)(s+1)$

But can't split - need partial fracs

No - can split

↓ I added them

$$x = \mathcal{L}^{-1}(\mathcal{L}(\delta(t-7))) + \mathcal{L}^{-1}\left(\frac{s+2}{(s+1)^2}\right)$$

No

so might not

$$+ \mathcal{L}^{-1}\left(\frac{s}{(s+1)^2}\right) + \mathcal{L}^{-1}(2)$$

(9)

I think I illegally split
i it split numerator like that

$\delta(t-7) * \mathcal{L}^{-1}\left(\frac{1}{(s+1)^2}\right)$ ← w/ rest conditions

$\delta(t-7) * \underbrace{-te^{-t}}_{\text{need to be able see}}$

↑ in this easy convo case

just $\begin{cases} -te^{-t} & t>0 \\ 0 & t<0 \end{cases}$ ← not in notes to verify

Michael E Plasmeier

From: Michael E Plasmeier
Sent: Tuesday, November 22, 2011 12:46 AM
To: Vivek Shende (vivek@math.mit.edu)
Subject: Tomorrow's Recitation

Could you please cover:

- Initial Value Problems (like Lecture 21 handout, last page) Could you show this completely from start to finish; the handout jumps steps
- Heat Equation problems – how to set up from word problem to something we know how to solve?
- Green's function – what is the motivation behind?; what types of questions using it would be on the exam?
- Pole diagrams – in what situations do we want to use them?

Thanks -Michael

P-Sets back

1803 Recitation

11/22

$y'' + 7y = t$ $\qquad y(0) = 0 \qquad y(1) = 0$

Look for periodic fn that meets this

period = ?

$\underbrace{y''}_{\text{odd}} + \underbrace{7y}_{\text{odd}} = \underbrace{t}_{\text{odd}}$

So y being ~~even~~ odd is good guess

Can assume $y(-1) = 0$

Any periodic odd fn of period 1

w/ $y(1) = 0$

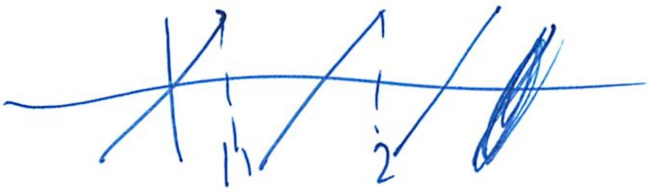
Satisfied these

②

So to solve w/ Fourier Series

replace t w/ periodic Function

└ only care $[-1, 1]$



Find fourier expansion for our periodic fn

$$f = \sum b_n \sin\left(\frac{\pi n t}{1}\right)$$

↑ find

Then set = to

$$y'' + 7y = \sum b_n \sin(n\pi t)$$

See can't be repeated roots

$$\sqrt{\frac{k}{m}} = \sqrt{\frac{7}{1}} = \sqrt{7} \neq \text{something w/ } \pi$$

(3) Solve Sepeètly

$$y_n'' + 7y_n = b_n(\pi n t)$$

@ previous exam

Add up solutions

Will be

$$A\sin(n\pi t) + B\cos(\pi m t)$$

$$y = \sum y_n$$

(still confused)

Heat eq is just boundry value problem

Lput in

plug + chug

$$u_t = 3u_{xx}$$

$$0 < x < \pi$$

$$u(x,0) = 4\sin 2x$$

$$t > 0$$

$$u(0,t) = u(\pi,t) = 0$$

4

$$u(t,x) = \sum f(t)\; g(x)$$

zero cond $0, \pi$

$$\sum \lambda_n(t)\sin(nx)$$

can build any eqn w/ this form

Need to find coeffs

$$\sum \lambda_n'(t)\sin(nx) = \cancel{\sum A B n^2}$$

$$= \sum -3n^2 \sin(nx)$$

$\lambda_n(t) = \cancel{}\; te^{-3n^2 t}$

solve for constant

$$\lambda_n'(t) = -3n^2 \lambda_n(t)$$

↳ solve for $\lambda_n(t)$

Can also say

$$y' = -3n^2 y$$

Only 1 sol to this since

1st order diff eq

5

↙ constant t is

$$x_n(0) = \begin{cases} 4 & \text{if } n = 2 \\ 0 & \text{otherwise} \end{cases}$$

$$u(x,t) = 4e^{-3n^2 t} \sin 2x$$

Here solved it from scratch

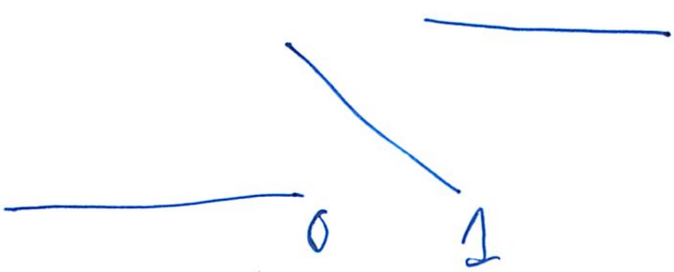
— presumably recognize formula

Practice Test 2c

— doesn't know if we need to know this

$$f(t) = \begin{cases} 0 & t < 0 \\ 1-t & 0 < t < 1 \\ 1 & t > 1 \end{cases}$$

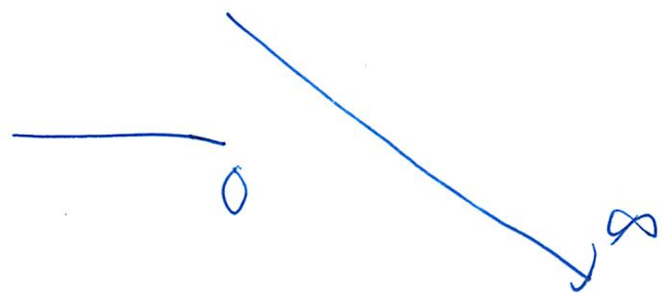
Take the derivitive



(6)

$u(t)(1-t)$

Would be



$u(t)(1-t) - u(t-1)(1-t) \quad + u(t-1)$

$\delta(0) + \delta(t-1) + (-1)(u(t-1) - u(t))$

$u(t)(1-t) + t\,u(t-1)$

deriv. $u'(t)(1-t) - u(t) + u(t-1) + t\,u(t-1)$

Replace $u' \to \delta$

$\delta(t)(1-t) - u(t) + u(t-1) + t\,\delta(t-1)$

(Hes not sure which is right...)

(Confused)

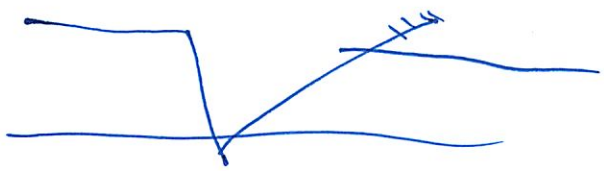
(7)

Anoter Qu $\mathcal{L}[f(t)] = \frac{1}{s-2} + \frac{1}{s-(-3+4i)} + \frac{1}{s-(-3-4i)}$

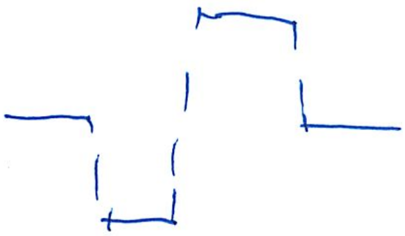
$$f(t) = e^{2t} + e^{(-3+4i)t} - e^{(-3-4i)t}$$

$$e^{a+bi} = \cos a + i \sin b$$

$f =$



$f' =$



$$f'' = -\delta(t+1) + 2\delta(t) - \delta(t-1)$$

$$f'' = \sum a_n \cos\left(\frac{\pi}{2} n t\right)$$

$$a_n = \frac{1}{2}\int_{-2}^{2} (-\delta(t+1) + 2\delta(t) - \delta(t-1)\cos\left(\frac{\pi}{2} n t\right) dt$$

(8)

δ fn easy to integrate

└ look if within limits of integration

(missed rest)

$$\frac{1}{2}\left(-\cos\left(-\frac{\pi}{2}n\right)+2-\cos\left(\frac{\pi}{2}n\right)\right)$$

$$= 1 - \cos\left(\frac{n\pi}{2}\right)$$

~~? always 0 if n is integer~~ Only 0 when n odd

when n is even its ± 1

$\cancel{= 1}$

~~(not the best nods~~

$$\cos\left(\frac{n\pi}{2}\right) = \begin{cases} 0 & \text{if } n \text{ odd} \\ (-1)^{n/2} & \text{if } n \text{ even} \end{cases}$$

an is something odd depending on what n is

$$(t-k)\,\delta(t-k) = 0$$

may or may not be true

he thinks its true

18.03

More Practice

11/22

Heat Eqn

Have our BVs

And our square wave seq

$$f(t) = \frac{4}{\pi} \sum_{n \text{ odd}} \frac{1}{n} \sin \frac{n\pi t}{L} = \begin{cases} +1 \\ -1 \end{cases}$$

Only care $t > 0$

$$f(x) = \frac{4u_0}{\pi} \sum_{n \text{ odd}} \frac{1}{n} \sin \frac{n\pi x}{L}$$

f coeffs $b_n \begin{cases} \frac{4u_0}{n\pi} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$

Rod's temp

$$u(x,t) = \frac{4u_0}{\pi} \sum_{n \text{ odd}} \frac{1}{n} e^{-\frac{n^2 \pi^2 k t}{L^2}} \sin \frac{n\pi x}{L}$$

L what is generic formula?

Oh diff formulas

frozen ends $\sum b_n e^{\frac{-n^2\pi^2 k t}{L}} \sin \frac{n\pi x}{L}$

ends insulate $\frac{a_0}{2} + \sum a_n e^{\frac{-n^2\pi^2 k t}{L}} \cos \frac{n\pi x}{L}$

2

∴ Match coeffs like before
└ practice next

Now mem this - not on formula sheet

frozen $\sum_{n=1}^{\infty} b_n \, e^{-\frac{n^2 \pi^2 k t}{L^2}} \sin \frac{n \pi x}{L}$

insulated $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \, e^{-\frac{n^2 \pi^2 k t}{L^2}} \cos \frac{n \pi x}{L}$

Fourier Series in an ODE

8.4 ex 1

mass 2 kg $k = 32$

$F = \begin{cases} 10 & 0 < t < 1 \\ -10 & 1 < t < 2 \end{cases}$

find $x_{sp}(t)$
"periodic"

$F(t) = \frac{40}{\pi} \sum_{n \text{ odd}} \frac{\sin \pi n t}{n}$ ← fourier series of sine wave found 8.1 sect 1

(3)

Substitute in

$$x = \sum_{n \text{ odd}} b_n \sin n\pi t$$

↑ Fourier sine series

Sub in trial solution w/ only odd terms

↑?how get??

$$\sum_{n \text{ odd}} b_n \left(-2n^2\pi^2 + 32\right) \sin n\pi t = \frac{40}{\pi} \sum_{n \text{ odd}} \frac{\sin n\pi x}{n}$$

Equate coeffs

$$b_n = \frac{20}{n\pi(16 - n^2\pi^2)} \quad \text{for } n \text{ odd}$$

Let me try this

What do we do to Σ? ignore it seems

$$b_n\left(-2n^2\pi^2 + 32\right) = \frac{40}{\pi n}$$

$$b_n = \frac{40}{\pi n\left(-2n^2\pi^2 + 32\right)}$$

$$= \frac{20}{n\pi\left(-n^2\pi^2 + 16\right)}$$

✓ for n odd

(4)

Then write as

$$X_{sp} = \frac{20}{\pi} \sum_{n\,odd} \frac{\sin n\pi t}{n(16 - n^2\pi^2)}$$

Try

$$X_{sp} = \sum_{n\,odd} b_n \sin n\pi t$$

$$= \sum \frac{20}{n\pi(n^2\pi^2 + 16)} \sin n\pi t \checkmark$$

then just simplify a bit more

2nd qu

$$x'' + 10x = F(t) \leftarrow 5t \qquad -2 < t < 2$$

F.S. for (found before)

$$= \frac{20}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi t}{2}$$

$$X_{sp} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi t}{2} \leftarrow L$$

Now what? set = to?

$$\frac{20}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi t}{2} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi t}{2}$$

5

Solve for b_n

$$\frac{20}{\pi} \frac{(-1)^{n+1}}{n} \sin \cancel{\frac{n\pi t}{2}} = b_n \sin \cancel{\frac{n\pi t}{2}}$$

? can I do ?

$$\cancel{b_n = \frac{20}{\pi} \frac{(-1)^{n+1}}{n}}$$

$$\cancel{X_{sp} = \sum_{n=1}^{\infty} \frac{20}{\pi} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi t}{2}}$$

? we had that before ??

Got b_n wrong

Oh forgot to plug in terms

$$\sum b_n \left(\underbrace{\frac{-n^2 \pi^2}{4} + 10}_{\text{Where did we get that ?}} \right) \sin \frac{n\pi t}{2} = \frac{20}{\pi} \sum \frac{(-1)^{n+1}}{n} \sin \frac{n\pi t}{2}$$

Ohhh plug in for x !!!

Not set = to

$x' =$ (? not in book anywhere) $\frac{1}{2} n \pi \sin^2 \left(\frac{n\pi t}{4} \right)$

$$x'' = \frac{-n^2 \pi^2}{4}$$

(5c)

need to fill in it cos

In lecture review $y = \frac{a_0}{2} + \sum a_n \cos nt$

$y'' = \sum -n^2 a_n \underbrace{\cos nt}$

in our case has

$\cos \frac{n\pi t}{2}$

This is P.O ing me that they don't have intermediate

; was it something we saw earlier ?

So

$\cancel{\sum} b_n \left(\frac{-n^2\pi^2}{4} + 10\right) \cancel{\sin \frac{n\pi t}{2}} = \frac{20}{\pi} \cancel{\sum} \frac{(-1)^{n+1}}{n} \cancel{\sin \frac{n\pi t}{2}}$

$$b_n = \frac{\frac{20}{\pi}\frac{(-1)^{n+1}}{n}}{\frac{-n^2\pi^2}{4} + 10}$$

Now need to make proper Fraction

$\frac{20}{\pi} \cdot \frac{(-1)^{n+1}}{n} \cdot \frac{4}{-n^2\pi^2} + \cancel{\frac{1}{10}}$

$b_n = \frac{8(-1)^{n+1}}{-n^3\pi^3}$? this prob illegal

(7)

forgot i

$$X_{sp} = \sum \frac{8(-1)^{n+1}}{-n^3\pi^3} \sin \frac{n\pi \cancel{x}}{2}$$

$$X_{sp} = \frac{80}{\pi} \frac{(-1)^{n+1}}{n(40-n^2\pi^2)} \frac{n\pi x}{2}$$

I did Something Wrong on multiplication prob

Can plug in actual value now

Heat Eq'n

ice $\sum b_n \, e^{\frac{-n^2\pi^2 kt}{L^2}} \sin \frac{n\pi x}{L}$

insulate $\frac{a_0}{2} + \sum a_n \, e^{\frac{-n^2\pi^2 kt}{L^2}} \cos \frac{n\pi x}{L}$

Plug in values $u_t = k\, u_{xx}$ other stuff BV

$u_x(0,t) = u_x(50,t) = 0$ ice

$u(x,0) = f(x)$ ← get the b_n from

8

Ok I think I got that for 1st time :)

Should have practiced more LaPlace

- w/ partial Facts

~~Conv~~ delta fn

~~dl~~ Convolutio

And Greens/weight/poles = total mystery

Didn't do any "real" practice problems

18.03 Study Guide and Practice Hour Exam III, April, 2010

Study Guide on Step, Delta, Convolution, Laplace

You can think of the step function $u(t)$ as any nice smooth function which is 0 for $t < -a$ and 1 for $t > a$, where a is a positive number which is much smaller than any time scale we care about in the context we are studying at the moment. Similarly, the best way for you to understand the "delta function" is to think of it as any smooth function which is zero except in the immediate neighborhood of $t = 0$ and which has integral 1.

So $\dot{u}(t-b) = \delta(t-b)$ and if $a < b$ then $\int_a^t \delta(\tau - a)\,d\tau = u(t-a)$.

A function $f(t)$ is "regular" or "piecewise smooth" if it can be broken into pieces each having all higher derivatives and such that at each breakpoint $f^{(n)}(a-)$ and $f^{(n)}(a+)$ exist. A "singularity function" is a linear combination of shifted delta functions. A "generalized function" $f(t)$ is a sum $f(t) = f_r(t) + f_s(t)$ of a regular function and a singularity function. Any regular function $f(t)$ has a "generalized derivative" $f'(t)$, with regular part $f_r'(t)$ the regular derivative of $f(t)$ whereever it exists, and singular part $f_s'(t)$ given by a sum of terms $(f(a+) - f(a-))\delta(t-a)$, one for each break in the graph of $f(t)$.

Then $\int_a^c f'(t)\,dt = f(c) - f(a)$. (To be more precise, $\int_{a-}^{c+} f'(t)\,dt = f(c+) - f(a-)$.)

For the rest of this unit, all "signals" (functions of t) are supposed to be zero for $t < 0$. We look for solutions to the differential equation $p(D)x = q(t)$, especially with "rest initial conditions," so that $x(t) = 0$ for $t < 0$ and x has as many derivatives as possible.

The *unit impulse response* or *weight function* of the operator $p(D)$ is the solution $w(t)$ to the equation $p(D)w = \delta(t)$, with rest initial conditions. If $p(s) = a_n s^n + \cdots + a_0$ with $a_n \neq 0$, and x is such that $p(D)x = 0$ and $x(0) = \cdots = x^{(n-2)}(0) = 0$ and $x^{(n-1)}(0) = \frac{1}{a_n}$, then $w(t) = u(t)x(t)$. The *unit step response* is the solution to $p(D)v = u(t)$ with rest initial conditions; for $t > 0$ this coincides with the solution to $p(D)x = 1$ such that $x(0) = \cdots = x^{(n-1)}(0) = 0$. Since $\dot{u}(t) = \delta(t)$, $\dot{v}(t) = w(t)$.

The response to a unit impulse determines the system response to any other input signal (with rest initial conditions): the solution to $p(D)x = q(t)$ is given by $x(t) = w(t) * q(t)$, where the asterisk indicates the *convolution product* $f(t) * g(t) = \int_0^t f(t-\tau)g(\tau)\,d\tau$.

The convolution product has properties analogous to the ordinary product: $f * (g * h) = (f * g) * h$, $f * (ag + bh) = a(f * g) + b(f * h)$, $f * g = g * f$. Also $f(t) * \delta(t) = f(t)$. If you feed the output of a system (with unit impulse response $g(t)$) into another system (with unit impulse response $f(t)$), you get a "composite system" with unit impulse response $f(t) * g(t)$.

The Laplace transform carries a generalized function (with $f(t) = 0$ for $t < 0$) to a function $F(s)$ of a complex variable s. It obeys a bunch of rules, the most important of which are *linearity* and the *t-derivative* rule $\mathcal{L}[f'(t)] = sF(s)$, where here $f'(t)$ denotes the generalized derivative. There are standard computations, too, including $\mathcal{L}[\delta(t)] = 1$ (the constant function of s). These already imply that the unit impulse response $w(t)$ of $p(D)$ satisfies $\mathcal{L}[w(t)] = \frac{1}{p(s)}$. $W(s) = \mathcal{L}[w(t)]$ is the *transfer function* of the operator $p(D)$. Also the solution to $p(D)x = f(t)$ with rest initial conditions has Laplace transform $X(s) = W(s)F(s)$. This relates to the formula $\mathcal{L}[f(t) * g(t)] = F(s)G(s)$.

If $f(t)$ and $g(t)$ have the same Laplace transform, then $f(a-) = g(a-)$ and $f(a+) =$

$g(a+)$ for every $a \geq 0$.

If $F(s)$ is a rational function (quotient of a polynomial by a polynomial), "continued fractions" provides standard way to find $f(t)$. "Coverup" is an efficient way to compute some of the coefficietns. This couples with completing the square and the s-shift rule.

The *pole diagram* of $F(s)$ is the set of complex numbers s at which $|F(s)|$ becomes infinite. This is the set of uncancelled zeros in the its denominator. The pole diagram of $F(s) = \mathcal{L}[f(t)]$ determines the region of convergence of the integral: it is the region to the right of the vertical line through the rightmost pole. The pole diagram also controls much of the behavior of $f(t)$ for large t (while saying nothing about behavior for small t). The rightmost poles dominate. A pole at $a + i\omega$ leads to exponential growth/decay like e^{at} (or some polynomial times e^{at}) and oscillation of circular frequency ω.

The transfer function occured earlier in the course: the exponential solution to $p(D)z = e^{rt}$ is $z_p = W(r)e^{rt}$. So the sinusoidal solution to $p(D)x = \cos(\omega t)$ is $x_p = |W(i\omega)|\cos(\omega t - \phi)$, where $\phi = -\mathrm{Arg}\,(W(i\omega))$: $W(i\omega)$ is the complex gain if e^{rt} itself is regarded as the input signal), and $|W(i\omega)|$ is the gain. The amplitude response curve is obtained by intersecting the graph of $|W(s)|$ with the plane above the imaginary axis.

Practice Hour Exam

1. Let ω be a positive constant. We drive a harmonic oscillator with a square wave of circular frequency ω: $\ddot{x} + 4x = \mathrm{sq}(\omega t)$.

(a) Write down a periodic solution to the equation, if ω is such that there is one.

(b) For what values of ω does there fail to be a periodic solution?

2. Let $f(t) = \begin{cases} 0 & \text{for } \; t < 0 \\ 1 - t & \text{for } \; 0 < t < 1 \\ 1 & \text{for } \; t > 1 \end{cases}$.

(a) Sketch the graph of $f(t)$.

(b) Sketch the graph of the generalized derivative $f'(t)$.

(c) Write down a formula for $f'(t)$ in terms of step and delta functions.

3. (a) Compute the convolution product $t * t^6$.

(b) A certain operator $p(D)$ has unit impulse response $w(t) = 2u(t)te^{-t}$. What is the solution to $p(D)x = e^{-t}$ with rest initial conditions?

4. (a) What is the Laplace transform of the solution to the equation $\ddot{x} + 2\dot{x} + 2x = 1$ having rest initial conditions?

(b) What function $f(t)$ has Laplace transform $F(s) = \dfrac{2s}{(s+1)(s^2+2s+5)}$?

5. (a) Sketch the pole diagram for the function $F(s) = \dfrac{2}{(s+1)(s^2+2s+5)}$.

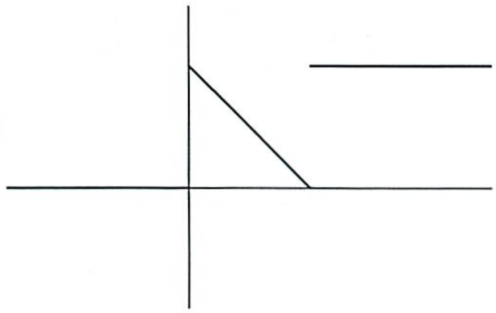
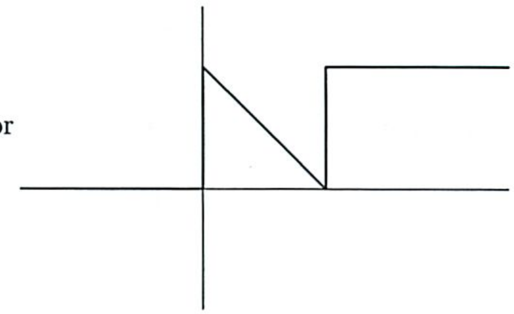
(b) Give an example of a function $f(t)$ whose Laplace transform has poles at $s = 2$ and $s = -3 \pm 4i$ and nowhere else.

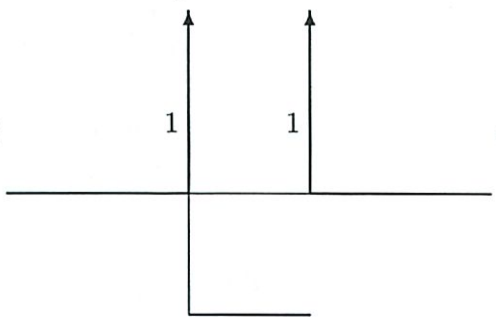
Solutions

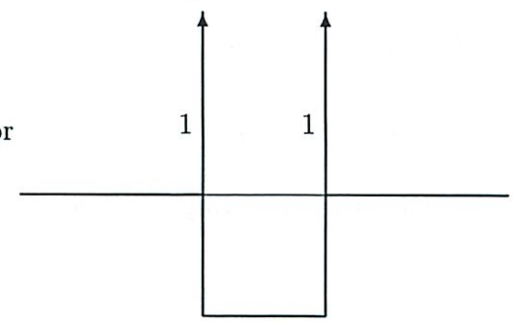
1. **(a)** $\operatorname{sq}(\omega t) = \dfrac{4}{\pi}\left(\sin(\omega t) + \dfrac{\sin(3\omega t)}{3} + \dfrac{\sin(5\omega t)}{5} + \cdots\right)$. Using superposition and the fact that $\ddot{x} + 4x = A\sin(\omega t)$ has solution $x_p = A\dfrac{\sin(\omega t)}{4-\omega^2}$,

$$x_p = \frac{4}{\pi}\left(\frac{\sin(\omega t)}{4-\omega^2} + \frac{\sin(3\omega t)}{3(4-9\omega^2)} + \frac{\sin(5\omega t)}{5(4-25\omega^2)} + \cdots\right)$$

(b) This solution does not exist if $\omega = 2/(\text{odd integer})$.

2. **(a)**  or 

2. **(b)** 

or 

(c) $f'(t) = -(u(t) - u(t-1)) + \delta(t) + \delta(t-1)$.

3. **(a)** $t * t^6 = \displaystyle\int_0^t (t-\tau)\tau^6\, d\tau = \int_0^t (t\tau^6 - \tau^7)\, d\tau = \left[t\frac{\tau^7}{7} - \frac{\tau^8}{8}\right]_0^t = t^8\left(\frac{1}{7} - \frac{1}{8}\right) = \frac{t^8}{56}$.

(b) $x(t) = w(t) * e^{-t} = \displaystyle\int_0^t 2(t-\tau)e^{-(t-\tau)}e^{-\tau}\, d\tau = e^{-t}\int_0^t 2(t-\tau)\, d\tau = e^{-t}\left[-(t-\tau)^2\right]_0^t = t^2 e^{-t}$

4. **(a)** $s^2X + 2sX + 2X = \dfrac{1}{s}$, so $X = \dfrac{1}{s(s^2+2s+2)}$.

(b) $\dfrac{2s}{(s+1)(s^2+2s+5)} = \dfrac{a}{s+1} + \dfrac{b(s+1)+c}{(s+1)^2+4}$.

By coverup, $a = \dfrac{2(-1)}{(-1)^2 + 2(-1) + 5} = -\dfrac{1}{2}$, $b(2i) + c = \dfrac{2(-1+2i)}{2i} = 2 + i$ so $b = \frac{1}{2}$ and $c = 2$. Thus $f(t) = -\frac{1}{2}e^{-t} + e^{-t}\left(\frac{1}{2}\cos(2t) + \sin(2t)\right)$.

5. **(a)** Poles at $s = -1$ and at $s = -1 \pm 2i$.

(b) $e^{2t} + e^{-3t}\sin(4t)$, or many others.

Properties of the Laplace transform

0. Definition: $\mathcal{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st}\,dt \qquad$ for $\operatorname{Re} s \gg 0$.

1. Linearity: $\mathcal{L}[af(t) + bg(t)] = aF(s) + bG(s)$.

2. Inverse transform: $F(s)$ essentially determines $f(t)$.

3. s-shift rule: $\mathcal{L}[e^{at}f(t)] = F(s-a)$.

4. t-shift rule: $\mathcal{L}[f_a(t)] = e^{-as}F(s), \qquad f_a(t) = \begin{cases} f(t-a) & \text{if } t > a \\ 0 & \text{if } t < a \end{cases}.$

5. s-derivative rule: $\mathcal{L}[tf(t)] = -F'(s)$.

6. t-derivative rule: $\mathcal{L}[f'(t)] = sF(s)$, where $f'(t)$ denotes the generalized derivative.

$\mathcal{L}[f_r'(t)] = sF(s) - f(0+)$ if $f(t)$ is continuous for $t > 0$.

7. Convolution rule: $\mathcal{L}[f(t) * g(t)] = F(s)G(s)$, $f(t) * g(t) = \int_0^t f(t-\tau)g(\tau)\,d\tau$.

8. Weight function: $\mathcal{L}[w(t)] = W(s) = 1/p(s)$, $w(t)$ the unit impulse response.

Formulas for the Laplace transform

$$\mathcal{L}[1] = \frac{1}{s} \qquad \mathcal{L}[e^{at}] = \frac{1}{s-a} \qquad \mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}[\cos(\omega t)] = \frac{s}{s^2+\omega^2} \qquad \mathcal{L}[\sin(\omega t)] = \frac{\omega}{s^2+\omega^2}$$

$$\mathcal{L}[t\sin(\omega t)] = \frac{2\omega s}{(s^2+\omega^2)^2} \qquad \mathcal{L}[t\cos(\omega t)] = \frac{s^2-\omega^2}{(s^2+\omega^2)^2}$$

Fourier coefficients for periodic functions of period 2π:

$$f(t) = \frac{a_0}{2} + a_1\cos(t) + a_2\cos(2t) + \cdots + b_1\sin(t) + b_2\sin(2t) + \cdots$$

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(mt)\,dt, \qquad b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(mt)\,dt$$

If $\operatorname{sq}(t)$ is the odd function of period 2π which has value 1 between 0 and π, then

$$\operatorname{sq}(t) = \frac{4}{\pi}\left(\sin(t) + \frac{\sin(3t)}{3} + \frac{\sin(5t)}{5} + \cdots\right)$$

18.03 Practice Hour Exam 3.3, Spring, 2011

1. Consider the ODE $x'' + kx = f(t)$ where

$$f(t) = \sin t + \frac{\sin 3t}{3} + \frac{\sin 5t}{5} + \ldots = \sum_{\substack{n>0 \\ n \text{ odd}}} \frac{\sin nt}{n}$$

[10] (a) For which values of k does the ODE have a periodic solution? For those values of k, write down a periodic solution in the form of a Fourier series.

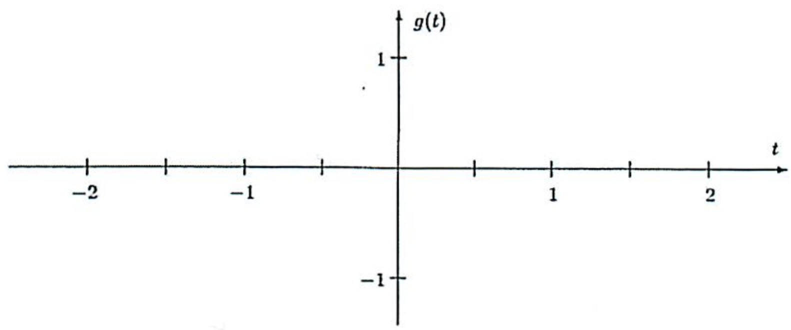
[5] (b) In the case $k = 10$, find the term in the periodic solution obtained in part (a) with the largest amplitude.

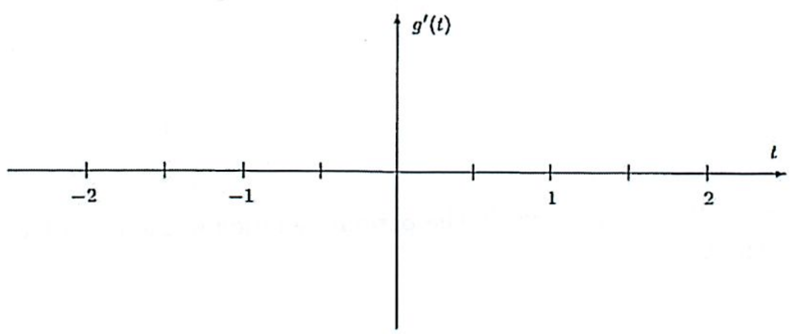
2. Consider a Fourier series of the form

$$g(t) = c_0 + c_1 \left(\cos \pi t - \frac{\cos 3\pi t}{3} + \frac{\cos 5\pi t}{5} - \dots \right)$$

[5] (a) Write down the (divergent) Fourier series of the generalized derivative $g'(t)$. Use the letters c_0 and c_1 if necessary.

[10] (b) Suppose further that $g(t) = \begin{cases} 1, & 0 < t < 1/2 \\ 0, & 1/2 < t < 1 \end{cases}$. Sketch the graphs of $g(t)$ and $g'(t)$ in the range $-2 < t < 2$.





[5] (c) Write a formula for $g'(t)$ in the range $-2 < t < 2$ in terms of delta functions.

[10] **2.** (d) Recall from the previous page that

$$g(t) = c_0 + c_1 \left(\cos \pi t - \frac{\cos 3\pi t}{3} + \frac{\cos 5\pi t}{5} - \ldots \right)$$

and that $g(t) = \begin{cases} 1, & 0 < t < 1/2 \\ 0, & 1/2 < t < 1 \end{cases}$. Evaluate c_0 and c_1.

<u>EXTRA CREDIT PROBLEM</u> (Warning: This is a harder problem and will be graded more stringently. Unless you know exactly what to do, go on to the rest of the exam before attempting it.)

[10*] (e) Compute the Fourier coefficients of $g'(t)$ directly from its expression in terms of delta functions obtained in part (c).

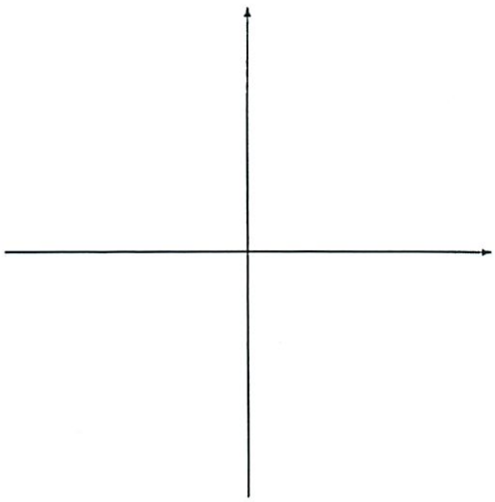
[15] 3. Find the Laplace transform of the solution to the initial value problem

$$x'' + 5x' + 10x = \cos 3t + \delta(t-2); \qquad x(0) = 1,\ x'(0) = 0$$

4. This problem concerns the function

$$F(s) = \frac{1}{(s+1)(2s^2+10s+25)}$$

[8] (a) Draw the pole diagram of $F(s)$. Label the poles.



[5] (b) Write $f(t) = \mathcal{L}^{-1}(F(s))$ as a linear combination of elementary functions using undetermined coefficients A, B, C, etc. **Do not evaluate these coefficients.**

[7] (c) Which term in the linear combination in part (b) becomes dominant as $t \to +\infty$? Evaluate its coefficient.

[5] 5. (a) Find $\mathcal{L}(\sin t * \cos t)$ without computing $\sin t * \cos t$.

[8] (b) Find $\mathcal{L}^{-1}\left(\frac{d}{ds}\frac{1}{s^2+1}\right)$. Use the result to compute $\sin t * \cos t$.

[7] (c) Find a function $f(t)$ and numbers a, b, c and d so that the initial value problem

$$x'' + ax' + bx = f(t); \qquad x(0) = c,\ x'(0) = d$$

has $\sin t * \cos t$ as its solution.

PRACTICE EXAM 3.3 SOLUTIONS, 18.03 SPRING '11

Problem 1. a) The equation has a periodic solution for all k except $k = n^2$, for n odd. To see this, we use the ERF for the complexified equation $z'' + kz = \sum_{n>0, n \text{ odd}} \frac{e^{int}}{n}$ to find

$$z_p(t) = \sum_{n>0, n \text{ odd}} \frac{e^{int}}{nP(in)} = \sum_{n>0, n \text{ odd}} \frac{e^{int}}{n(k - n^2)}$$

Taking the imaginary part, we find the periodic solution

$$\boxed{\sum_{n>0, n \text{ odd}} \frac{\sin(nt)}{n(k - n^2)}}$$

b) For $k = 10$, our solution is $\sum_{n>0, n \text{ odd}} \frac{\sin(nt)}{n(10 - n^2)}$. The term with the largest amplitude is therefore $\boxed{\frac{\sin(3t)}{3}}$.

Problem 2. We have $g(t) = c_0 + c_1(\cos(\pi t) - \frac{\cos(3\pi t)}{3} + \frac{\cos(5\pi t)}{5} - \ldots)$

a) Differentiating the series gives

$$\boxed{g'(t) = c_1\pi(-\sin(\pi t) + \sin(3\pi t) - \sin(5\pi t) + \ldots)}$$

b) see graph below.

c) Since $g(t) = \begin{cases} 1 & 0 < t < 1/2 \\ 0 & 1/2 < t < 1 \end{cases}$, and has period 2, we have

$$\boxed{g'(t) = -\delta_{-3/2} + \delta_{-1/2} - \delta_{1/2} + \delta_{3/2}}$$

d) $\boxed{c_0 = \frac{a_0}{2} = \frac{1}{2}\int_{-1}^{1} g(t)dt = \frac{1}{2}\int_{-1/2}^{1/2} dt = \frac{1}{2}}$. Further,

$$\boxed{c_1 = a_1 = \int_{-1}^{1} g(t)\cos(\pi t)dt = \int_{-1/2}^{1/2} \cos(\pi t)dt = \frac{2}{\pi}}.$$

e) We compute

$$b_n = \int_{-1}^{1} g'(t)\sin(\pi t)dt = \int_{-1}^{1} (\delta_{-1/2} - \delta_{1/2})\sin(\pi t)dt = \sin(\frac{-n\pi}{2}) - \sin(\frac{n\pi}{2}) = -2\sin(\frac{n\pi}{2})$$

Therefore we get that

$$\boxed{g'(t) \sim 2(-\sin(\pi t) + \sin(3\pi t) - \sin(5\pi t) + \ldots)}$$

in agreement with part a).

Problem 3. We have that

$$\mathcal{L}_+(x'' + 5x' + 10x) = (s^2 + 5s + 10)X - s - 5$$

while

$$\mathcal{L}_+(\cos(3t) + \delta(t-2)) = \frac{s}{s^2+3^2} + e^{-2s}$$

and so

$$\boxed{X = \frac{s+5+e^{-2s}}{s^2+5s+10} + \frac{s}{(s^2+3^2)(s^2+5s+10)}}$$

Problem 4. a) Poles occur at $\boxed{s=-1}$ and $\boxed{s = -\frac{5}{2} \pm \frac{5}{2}i}$.

b) We have $2s^2 + 10s + 25 = 2(s^2 + 5s + \frac{25}{4} - \frac{25}{4} + \frac{25}{2}) = 2((s+\frac{5}{2})^2 + \frac{25}{4})$. Therefore

$$F(s) = \frac{1}{2(s+1)((s+\frac{5}{2})^2 + (\frac{5}{2})^2)} = \frac{1}{2}\frac{A}{s+1} + \frac{1}{2}\frac{B(s+\frac{5}{2})+C}{((s+\frac{5}{2})^2+(\frac{5}{2})^2)}$$

and so

$$\boxed{\mathcal{L}^{-1}(F) = \frac{A}{2}e^{-t} + \frac{B}{2}e^{-5t/2}\cos(\frac{5t}{2}) + \frac{C}{5}e^{-5t/2}\sin(\frac{5t}{2})}$$

c) The term $\boxed{\frac{A}{2}e^{-t}}$ is dominant as $t \to \infty$. The heaviside cover-up with $s=-1$ yields $\boxed{\frac{A}{2} = \frac{1}{17}}$.

Problem 5. a) $\boxed{\mathcal{L}(\sin(t) * \cos(t)) = \mathcal{L}(\cos(t))\mathcal{L}(\sin(t)) = \frac{s}{(s^2+1)^2}}$.

b) Since $\mathcal{L}^{-1}(F'(s)) = -tf(t)$ we have $\mathcal{L}^{-1}(\frac{d}{ds}\frac{1}{s^2+1}) = -t\sin(t)$. On the other hand

$$\frac{d}{ds}\frac{1}{s^2+1} = \frac{-1}{(s^2+1)^2}2s = -2\mathcal{L}(\sin(t)*\cos(t))$$

and so

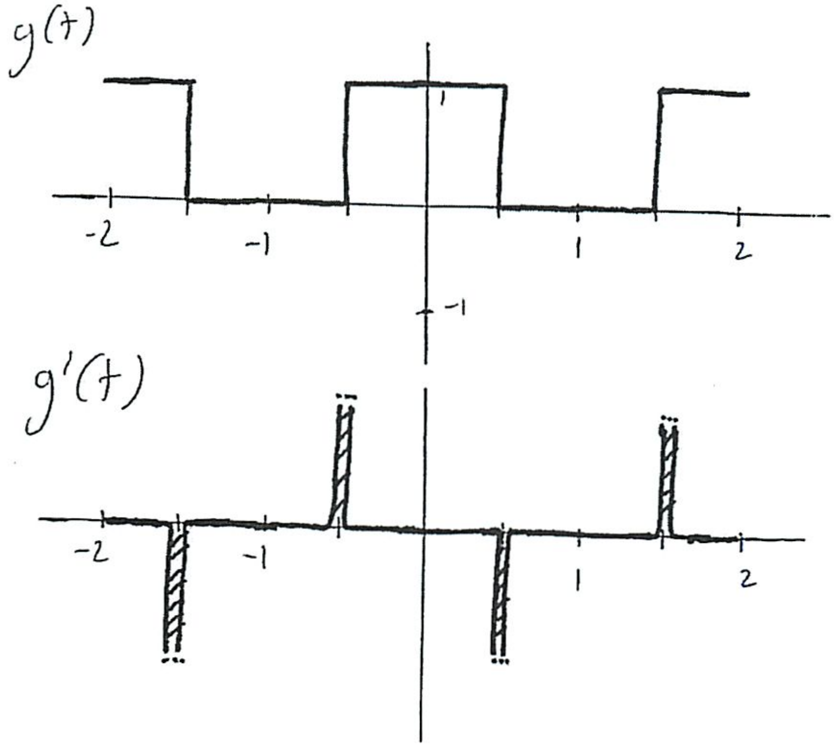
$$\boxed{\sin(t)*\cos(t) = -\frac{1}{2}\mathcal{L}^{-1}(\frac{d}{ds}\frac{1}{s^2+1}) = \frac{t}{2}\sin(t)}$$

c) We know that $\frac{te^{it}}{P'(i)}$ occurs as a particular solution when we apply ERF' to the complexified equation $z'' + z = e^{it}$. Since $P'(i) = 2i$ we find $z_p = \frac{te^{it}}{2i} = \frac{-t}{2}(-\sin(t) + i\cos(t))$, whose real part is $\frac{t}{2}\sin(t) = \sin(t)*\cos(t)$. So we should consider the equation

$$x'' + x = \cos(t)$$

such that $x(0) = x'(0) = 0$.

2 b)



Quiz Formula Sheet

 Name: Plasmeier

USEFUL FORMULAS:

If the Fourier series of a function $f(x)$ with period $2L$ is

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{L}\right)+b_n\sin\left(\frac{n\pi x}{L}\right)\right],$$

then

$$a_n=\frac{1}{L}\int_{-L}^{L}f(x)\cos\left(\frac{n\pi x}{L}\right)\quad\text{and}\quad b_n=\frac{1}{L}\int_{-L}^{L}f(x)\sin\left(\frac{n\pi x}{L}\right).$$

The convolution of functions $f(t)$ and $g(t)$ is defined to be

$$[f*g](t)=\int_0^t f(u)g(t-u)du.$$

Convolution is commutative: $f*g=g*f$. The Laplace transform of a function $f(t)$ is defined to be

$$F(s)=\mathcal{L}[f(t)]=\int_0^\infty f(t)e^{-st}dt\quad\text{(defined for } s\gg 0\text{)}.$$

It satisfies the following properties:

Linearity:	$\mathcal{L}[af(t)+bg(t)]=aF(s)+bG(s)$.
t-Shift:	$\mathcal{L}[u(t-a)f(t-a)]=e^{-as}F(s)$.
s-Shift:	$\mathcal{L}[e^{at}f(t)]=F(s-a)$.
t-Derivatives:	$\mathcal{L}[f'(t)]=sF(s)-f(0)$.
	$\mathcal{L}[f''(t)]=s^2F(s)-sf(0)-f'(0)$.
s-Derivative:	$\mathcal{L}[tf(t)]=-F'(s)$.
Convolution:	$\mathcal{L}[f(t)*g(t)]=F(s)G(s)$.

Laplace transforms of common functions:

$$\mathcal{L}[1]=\frac{1}{s}\qquad \mathcal{L}[e^{at}]=\frac{1}{s-a}\qquad \mathcal{L}[t^n]=\frac{n!}{s^{n+1}}$$

$$\mathcal{L}[\cos(at)]=\frac{s}{s^2+a^2}\qquad \mathcal{L}[\sin(at)]=\frac{a}{s^2+a^2}$$

$$\mathcal{L}[\delta(t-a)]=e^{-as}\qquad \mathcal{L}[u(t-a)]=\frac{e^{-as}}{s}.$$

18.03 Exam 3 Debrief

11/22

Screwed up #4 - did integration wrong
 L should have matched coefs
#2/3 did not know weights at all
Think got #1
Forgot the other one
 L think ok
But overall - not very good :(