Syllabus for 18.06 Linear Algebra, Spring 2012

MWF 11-12 Room 54-100

Text: *Introduction to Linear Algebra, 4th Edition, Gilbert Strang*

The three midterm exams will be held in Walker during lecture hours: Closed-Book. All grading is by your recitation instructor!

W	2/8	The Geometry of Linear Equations	1.1–2.1
F	2/10	Elimination with Matrices	2.2–2.3
M	2/13	Matrix Operations and Inverses	2.4–2.5
W	2/15	LU and LDU Factorization	2.6
F	2/17	Transposes and Permutations	2.7
T	2/21	Vector Spaces and Subspaces	3.1
W	2/22	The Nullspace: Solving $Ax = 0$	3.2
F	2/24	Rectangular $PA = LU$ and $Ax = b$	3.3–3.4
M	2/27	Row Reduced Echelon Form	3.3–3.4
W	2/29	Basis and Dimension	3.5
F	3/2	The Four Fundamental Subspaces	3.6
M	3/5	Graphs and Networks	8.2
W	3/7	Orthogonality	4.1 (and Review)
F	**3/9**	**Exam 1: Chapters 1 to 3.5**	
M	3/12	Projections and Subspaces	4.2
W	3/14	Least Squares Approximations	4.3
F	3/16	Gram-Schmidt and $A = QR$	4.4
M	3/19	Properties of Determinants	5.1
W	3/21	Formulas for Determinants	5.2
F	3/23	Applications of Determinants	5.3
M-F	3/26-30	HOLIDAY	
M	4/2	Eigenvalues and Eigenvectors	6.1
W	4/4	Diagonalization	6.2
F	4/6	Markov Matrices	8.3
M	4/9	*Review for Exam 2*	
W	**4/11**	**Exam 2: Chapters 1–5, 6.1, 8.2**	
F	4/13	Differential Equations	6.3
M	4/16	HOLIDAY	
W	4/18	Symmetric Matrices	6.4
F	4/20	Positive Definite Matrices	6.5
M	4/23	Matrices in Engineering	8.1
W	4/25	Similar Matrices	6.6
F	4/27	Singular Value Decomposition	6.7
M	4/30	Fourier Series, FFT, Complex Matrices	8.5, 10.2–10.3
W	5/2	Linear Transformations	7.1–7.2
F	5/4	*Course Review*	
M	**5/7**	**Exam 3: Chapters 1–6, 8.1, 2, 3, 5**	
W	5/9	Choice of Basis	7.3
F	5/11	Linear Programming	8.4
M	5/14	Numerical Linear Algebra	9.1–9.3
W	5/16	Computational Science	18.085
		Final Exam	

18.06 Linear Algebra, Spring 2012

Current (Spring 2012) class schedule and syllabus (PDF file)

Home | PSets and Exams | Matlab | Videos | Extras | Past Courses | Stellar

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Professor:	Gilbert Strang. Office 2-240, email: `gs[hieroglyph]math.mit.edu`
Lectures:	MWF 11 am, in **54-100**.
Course Administrator:	Niels Martin Moller. Office 2-588, email: `moller[hieroglyph]math.mit.edu`
Textbook:	Gilbert Strang's, *Introduction to Linear Algebra*, 4th edition.

OBS: 18.06 Spring 2012 starts with the first lecture on Wednesday February 8, 2012 (no recitations on Feb 7th 2012).

Lectures/Recitations

	Time	Room	Instructor	Office	OH	Phone	Email @math.mit.edu
Lec.	MWF 11	54-100	Gilbert Strang	2-240	TBA	3-4383	gs
Rec. #							
1	T 11	TBA	Geoffroy Horel	2-091	TBA	2-1196	ghorel
2	T 11	TBA	Niels Martin Moller	2-588	W 11	3-4110	moller
3	T 11	TBA	Geoffroy Horel	2-091	TBA	2-1196	ghorel
4	T 12	TBA	Jennifer Park	2-491	TBA	3-4091	jmypark
5	T 1	TBA	Dimiter Ostrev	2-229	TBA	3-1589	ostrev
6	T 1	TBA	Uhi Rinn Suh	2-229	TBA	3-1589	ursuh
7	T 1	TBA	Rune Haugseng	2-588	TBA	3-4110	haugseng
8	T 2	TBA	Rune Haugseng	2-588	TBA	3-4110	haugseng
9	T 2	TBA	Dimiter Ostrev	2-229	TBA	3-1589	ostrev

Checking grade records, changing recitations: online on Stellar Course Management System.

Grades: Problem sets 15%, three one-hour exams 45%, final exam 40%. The lowest problem set score will be dropped at the end of semester.

Exams: There will be three one-hour closed book written exams at class times on:

- Friday March 9th: Walker Memorial 11am-12.
- Wednesday April 11th: Walker Memorial 11am-12.
- Monday May 7th: Walker Memorial 11am-12.

Final Exam:

- TBA, 3 hours written exam

Concerns about homework, grading, exams: see your recitation instructor.

Tutoring, extra help: see the Learning Center

You are visitor number 1483856 since October 1, 1996. Welcome!

18.06 Lecture 1

2/8

Gilbert Strang

web.mit.edu / 18.06

ocw.mit.edu and YouTube - 10 years ago

- More modern textbook now

Syllabus online

Recitations Tue

HW on their

HW due Thur

Please ask qus

This is not calculus

Everything is straight - no curves + surfaces

Any dimensions

L like 5 dimensions $\mathbb{R}^5$ $\begin{bmatrix} 1 \\ 3 \\ 2 \\ \pi \\ e \end{bmatrix}$

What are the main ops on vectors

$$6 \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 18 \\ 12 \end{bmatrix}$$

②

$$\begin{bmatrix}1\\3\\2\end{bmatrix} + \begin{bmatrix}4\\1\\0\end{bmatrix} = \begin{bmatrix}5\\4\\2\end{bmatrix}$$

Also linear combination

$$6\begin{bmatrix}1\\3\\2\end{bmatrix} + 2\begin{bmatrix}4\\1\\0\end{bmatrix} = \begin{bmatrix}14\\20\\12\end{bmatrix}$$

Will go beyond $\mathbb{R}$ space

Anything we can add or multiply - is a matrix

Matricies

Matrix ∘ Vector

more insighted way → linear combo

The vectors become the column of the matrix

$$\begin{bmatrix}1 & 4\\3 & 1\\2 & 0\end{bmatrix}\begin{bmatrix}6\\2\end{bmatrix}$$

A_{3x2} ↑ matrix X ↑ vecto

③ Same linear combo!

$= 6 \cdot \text{col } 1 + 2 \cdot \text{col } 2$

But you do it compont by component

$$\begin{bmatrix} 1 \cdot 6 + 4 \cdot 2 \\ 3 \cdot 6 + 1 \cdot 2 \\ 2 \cdot 6 + 0 \cdot 2 \end{bmatrix} = \begin{bmatrix} 14 \\ 20 \\ 12 \end{bmatrix}$$

Its a combo. of columns

In matlab $A * x =$

We use Matlab in P-Sets
Use print out of result

Central problem of class - 1st half

$Ax = b$

nicest square matrix
invertable → $x = A^{-1} b$

So exactly 1 sol

④

Use from $\overset{\text{textbook section}}{1.3}$ example

$$\underbrace{\begin{bmatrix}1&0&0\\-1&1&0\\0&-1&1\end{bmatrix}}_{A}$$

- square 3x3
- triangular (lower)
- invertable ¡¡

$$\begin{bmatrix}1&0&0\\-1&+1&0\\0&-1&1\end{bmatrix}\underbrace{\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}}_{x}=\begin{bmatrix}x_1-0\\x_2-x_1\\x_3-x_2\end{bmatrix}$$

- so <u>difference</u> matrix
  ↳ the linear algebra equiv of a derivitive

Suppose given #s

$$=\underbrace{\begin{bmatrix}1\\3\\5\end{bmatrix}}_{b}$$

⑤

Now go backwards - find x

Have 3 eq w/ 3 unknowns

└ 3x3 matrix

Start w/ $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

Find the Xs from the bs

$X_1 = ?$

$X_2 = ?$

$X_3 = ?$

$$X = A^{-1} b$$

$X_1 = b_1$ easy since matrix is triangular

$X_2 = b_1 + b_2$ Can do w/ some substitution

$X_3 = b_1 + b_2 + b_3$ └ if not triangular, make it triangular

1st 4 lecture -> systematic process called elimination

We can see a lot about the matrix

Don't be decieved that its mechanical

⑥

18.06 is not just mechanical solving
But here it's the idea + way of thinking about it

Now we can recognize is a matrix on RHS

$$\begin{aligned} x_1 &= b_1 \\ x_2 &= b_1 + b_2 \\ x_3 &= b_1 + b_2 + b_3 \end{aligned} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$x = A^{-1} \cdot b$$

↑ this is A^{-1}!
- square
- lower triangular
  - inverse of lower triang. = lower triang
- Sum matrix

↗
Linear algebra analog of fund. theorm of calculus

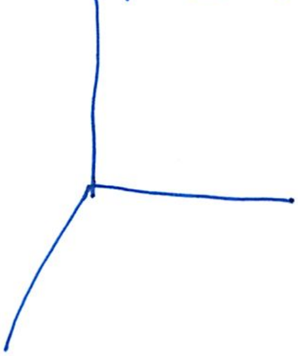
⑦

Pictures

Row picture

-familiar

$ax+by+cz=d$



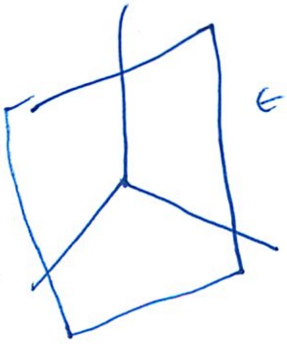
$1x+0y+0z=b_1$

$-1x+1y+0z=b_2$

$0x-1y+1z=b_3$

Graps are planes

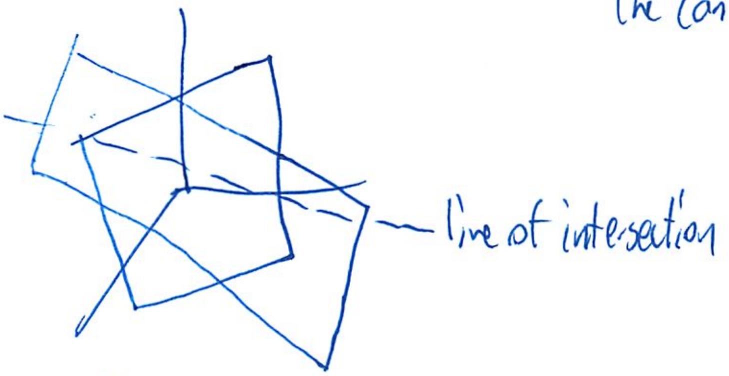
One from every row



Column Picture

-one he likes

If 2 planes – hopefully intersect in a line
(he can't draw it)

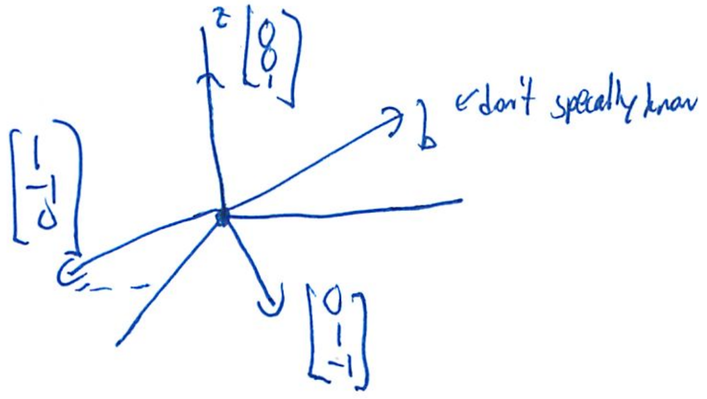


w/ 3 planes – get a single pt → the solution
x_1, x_2, x_3

Column picture

We are looking for combo of column that gives us b

The whole vector at once



Combine the columns to get b

⑨

When could column pic fail?
 └ when you <u>can't</u> combine them to get b

If all in the same plane

For example

$$\begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

not triangular anymore
 — not necess. bad
 but are in a plane - ie <u>dependent</u>

Elimination will show what is wrong

All cols are permutations of each other

3rd column is a combo of the 1st two

$$\text{Col } 3 = \underline{(-1)}\,\text{Col } 1 \cdot \underline{(-1)}\;\text{Col } 2$$

Or if just add all 3 - get 0 column

If b is in plane → too many sol
 b not in plane → no sols

⑩

Dep or ind? cols

$$\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

→ they lie in a plane = dep

- One of them is a combo of the other

- no 0s so harder to answer

How will we decide?

Clear when lots 0

like $\begin{bmatrix} 4 & 0 & 0 \\ -1 & 7 & 0 \\ 0 & -1 & 9 \end{bmatrix}$

Can always solve system top to bottom

$$4x_1 - 0 = b_1$$
$$7x_2 - x_1 = b_2$$
$$9x_3 - x_2 = b_3$$

$\boxed{\text{Is invertable}}$

As long as no 0
just solve going down

So ind - can always solve

⑪

−Col 1 + 2 col 2 = Col 3

So are in a plane

(¿¿ does that mean you can't always solve ¿)

Could find w/ determinant

- if b is not in plane
- then we would be in trouble
- b might still exist
  (need to check)

From: Niels Martin Møller <moller@MIT.EDU>
Sent: Thursday, February 09, 2012 7:52 PM
To: Niels Martin Moeller
Subject: [18.06] Welcome! [+ Problem Set #1]
Attachments: Ideas-To-You-From-Lectures.jpg

Follow Up Flag: Follow up
Flag Status: Flagged

Hello 18.06!

Greetings from your course admin. on 18.06 Linear Algebra! Welcome aboard, everybody!

18.06 QUICK FACTS:

- EVERYTHING you will need is on (or found via) the 18.06 WEB: http://web.mit.edu/18.06/

- You had the first LECTURE with Professor Strang yesterday (Wednesday). They are MWF11. Several of your TAs went to Gil's first lecture too - and we all found it to be very, very good! ff

- Your first RECITATION (where you will get to dig into problems yourself, and get to ask questions) will take place Tuesday Feb 14th. See 18.06 WEB for specifics.

- VIDEOS of the lectures can be found on the 18.06 WEB.
- OFFICE HOURS of the individual TAs will be decided with each class (from next week posted on the 18.06 WEB).
- Your first EXAM is on Friday March 9th (see 18.06 WEB)
- Weekly HOMEWORK (except exam weeks).

- PROBLEM SET #1 has been posted on the 18.06 WEB!

QUESTIONS? (On course material, or practical):

- Generally, ask your TAs (or me) at recitations, in office hours or by email. They are all very nice and friendly people - we are only here to help you best learn Linear Algebra.

Best - and again: Welcome!
Niels
P.S. 18.06 WEB: http://web.mit.edu/18.06/

18.06 Lecture 2

2/6

HW 1 posted, Due Thur

Recs meet Tue

How do we systematically find facts about matrices?

$$\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 10 \end{bmatrix}$$

method: Elimination

We can get good idea about matrix

But hard when have 6 dimensions

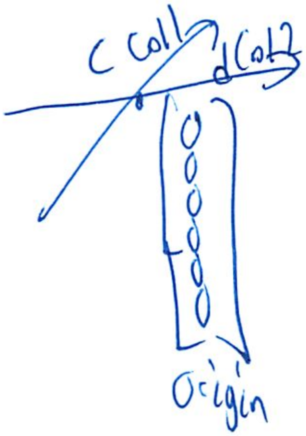
What are all the combinations?

If take $\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \\ 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$ What are all the combos?

$\underline{4}$ col 1 + $\underline{9}$ col 2 = ?

②

What do we get?



When do all the combos?

Don't have word for it?

All the multiples of col 1 and col 2.

Hyperplane?
└ in 6 dimension

2 Dimensional plane in $\mathbb{R}^6$

If take a random line in R6 - does it go through 2D plane? Very high odds it doesnt

2 dimensional subspace of whole space $\mathbb{R}^6$

Because origin is in it

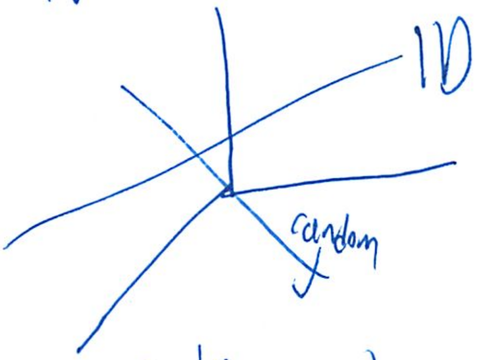
Random line - does it hit ~~origin~~ 2D subspace
- will prob not

We need to turn it into algebra

③

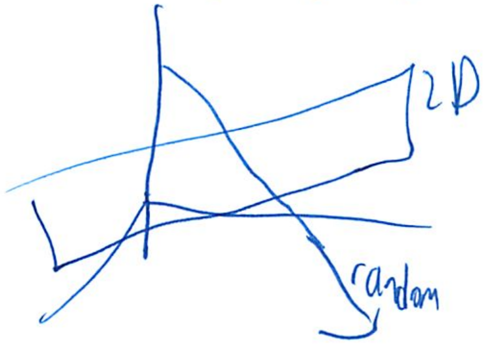
It will prob hit a 5D subspace

$\mathbb{R}^3$



Will prob not hit the line
└ very small chance

But if 2D Plane



– is likely to hit plane

We can do algebra
But we should also think about geometry

④

Mechanics of elimination

$$\text{pivot} \rightarrow \begin{bmatrix} \enclose{circle}{1} & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

Subtract multiples of 1
So get 0s in col 1
Row 1 wont change
Row 2 will start w/ 0

What is the multiple here to get 0 ? 2

$$\begin{bmatrix} 1 & 4 & 7 \\ 0 & -3 & -6 \\ 0 & \underset{\substack{\uparrow \\ ?}}{\boxed{-6}} & -12 \end{bmatrix} \begin{matrix} \leftarrow \text{multiplier } 2 \\ \leftarrow \text{mutiple } 3 \end{matrix}$$

3 4's
away from
6

Now next pivot: -3

$$U = \begin{bmatrix} 1 & 4 & 7 \\ 0 & \enclose{circle}{-3} & -6 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} \text{) keep} \\ \leftarrow \text{subtract multiple to get 0 here} \end{matrix}$$

subtract 2 away of previous row

⑤

Tells us all the cols lie in a plane

Want $Ax = b$

Like
$$\begin{cases} 1x_1 + 4x_2 + 7x_3 = b_1 \\ 2x_1 + 5x_2 + 8x_3 = b_2 \\ 3x_1 + 6x_2 + 9x_3 = b_3 \end{cases}$$

$$1x_1 + 4x_2 + 7x_3 = b_1$$
$$0x_1 - 3x_2 - 6x_3 = b_2 - 2b_1$$
$$0x_1 - 6x_2 - 12x_3 = b_3 - 3b_1$$

Same as matrix thing

$$1x_1 + 4x_2 + 7x_3 = b_1$$
$$0x_1 - 3x_2 - 6x_3 = b_2 - 2b_1$$
$$0x_1 + 0x_2 + 0x_3 = b_3 - 3b_1 - 2(b_2 - 2b_1)$$
$$= b_3 - 2b_2 + b_1$$

(Forward) Elimination Complete

⑥

Purpose of elimination: get to simple matrix

Is there always a solution?

No A is not always invertable

Have 0s so can't always be done

What is needed?

$b_3 - 2b_2 + b_1 = 0$

$\hookrightarrow$ must $= 0$

That is a whole plane of possible vectors

- 3 components
- as long as they add to 0

No sol unless b lies on plane

is exactly the Bs that <u>are</u> combos of the column

The bs must be in the plane

---

⑦ Since a pivot is 0 - not invatable

⑦

Messages

Elimination makes it clear what is up

2nd matrix out of trouble

Eliminate it

$$\begin{bmatrix}1 & 4 & 7\\ 0 & -3 & -6\\ 0 & -6 & -11\end{bmatrix}$$

↑3 7s away from 10

↓

$$\begin{bmatrix}1 & 4 & 7\\ 0 & -3 & -6\\ 0 & 0 & 1\end{bmatrix} = U$$

↑2 -6s from -11

3 pivots 1
-3
1

Pivots are good so can solve eqn w/ back substitution

$0x_1 + 0x_2 + 1x_3 = b_3 - 2b_2 + b_1$

No boring pratices on p-set, but try some

Another example

(very confused)

$$\begin{bmatrix} 1 & 4 & 7 \\ 2 & 8 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

Want example where after pivot 1, 2nd pivot = 0
Then need to do something different

$$\begin{bmatrix} 1 & 4 & 7 \\ 0 & 0 & -6 \\ 0 & -6 & -12 \end{bmatrix}$$

Remember this is 3 eqns

What now?

Can swap
└ another op of elimination

$$u = \begin{bmatrix} 1 & 4 & 7 \\ 0 & -6 & -12 \\ 0 & 0 & -6 \end{bmatrix}$$

Swap RHS as well

What can we say?

Solve like before now that 3 pivots

1

-6

-6

Test for invertable matrix

If do row ops + exchanges when needed

We get pivots (that ≠ 0)

This is what elimination does

How do we express those steps in a nice way

↳ Elimination by matrices

Can multiply $E_{21} \cdot A$

Elim matrix

← matrix that produces $\begin{bmatrix} - & - & - \\ 0 & - & - \\ - & - & - \end{bmatrix}$

what matrix produces that 0 when multiplied

Matrix that leaves stuff alone

Identity matrix $= I =$ eye(3)
in Matlab

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$IA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A \end{bmatrix} = \begin{bmatrix} A \end{bmatrix}$$

Several ways to think about matrix multiplication
See by rows

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \text{row } 1 \\ \text{row } 2 \\ \text{row } 3 \end{bmatrix} = \begin{bmatrix} 1 \cdot \text{row } 1 & + \, 0 \, \text{row } 2 & + \, 0 \, \text{row } 3 \\ 0 \quad '' & 1 \quad '' & 0 \quad '' \\ 0 \quad '' & 0 \quad '' & 1 \quad '' \end{bmatrix}$$

$$= \begin{bmatrix} \text{row } 1 \\ \text{row } 2 \\ \text{row } 3 \end{bmatrix}$$

Now fix matrix - so subtract a multiple of row 1 from row 2

$$\begin{bmatrix} \text{row } 1 & \\ \text{row } 2 & - \underset{\uparrow}{l} \text{ row } 1 \\ \text{row } 3 & \end{bmatrix}$$

$l = 2$ in our example

So $\begin{bmatrix} - & - & - \\ 0 & - & - \\ - & - & - \end{bmatrix}$

So $\begin{bmatrix} 1 & 0 & 0 \\ -l & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ = elementary matrix that does 1st step of elimination

Now E_{31}

- suppose E_{21} is done

$$\begin{bmatrix} 1 & 0 & 0 \\ -l & 1 & 0 \\ -m & 0 & 1 \end{bmatrix}$$

↑ Here $m = 3$ since $\begin{bmatrix} 6 & - & - \\ - & - & - \\ 3 & - & - \end{bmatrix}$

So

$$E_{31} E_{21} A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A$$

But what about swapping 2 rows?

Need a permutation matrix

$$\begin{bmatrix} \text{row } 1 \\ \text{row } 2 \\ \text{row } 3 \end{bmatrix} \rightarrow \begin{bmatrix} \text{row } 1 \\ \text{row } 3 \\ \text{row } 2 \end{bmatrix}$$

so $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

Steps for elimination

$$P\,(E_{31}\,(E_{21}\,A)) = U$$

↑

since

$\begin{bmatrix} - & - & - \\ - & 0 & - \\ - & - & - \end{bmatrix}$

we needed a P

So have a nice way to describe elimination

Most important fact about matrix multiplication

~~$P(E_{31}(E_{21}A)) = U$~~

As long as keep those in order

Can change order we do it

$$(PE_{31}E_{21})A = U$$

Associative $\Rightarrow A(BC) = (AB)C$

"moving the parenthesis"

But

$$AB \neq BA$$

for matrices

but can

Elimination

- row ops and row exchange
- matrices that does that
- have a matrix that does all at once

18.06 Inverse Day

2/13

Sq matrix: is it invertable? Singular?

Know rules for taking inverse?
when matrix has an inverse?

$$A^{-1}A = AA^{-1} = I$$

$$E = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ -\ell & 0 & 0 & 1 \end{bmatrix} = E_{4,1}$$ ← sometimes named after gauss

subtract ℓ x row 1 from row 4

E^{-1} = add back ℓ, row 1 to row 4

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \ell & 0 & 0 & 1 \end{bmatrix}$$

Inverse of a product $(E_1 E_2)$

$(E_1 E_2)^{-1}$ = inves product in reverse order

$= E_2^{-1} E_1^{-1}$

↑ so like Coat on top of sweater
when leave put sweater on 1st

~~A~~ Eliminiation

Remember $E_{32}E_{31}E_{21}A = U$ that was upper triangular

But cold put parentheses in

$$(E_{32}E_{31}E_{21})A = U$$

But what is inverse step? How do I get back to A?

$$A = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}U$$

↑ remember in opposet order

$$A = (E_{21}^{-1}E_{31}^{-1}E_{32}^{-1})U = LU$$

↑ lower triangular

↑ Great statement of elimination in linear algebra

Since

$$L = \begin{bmatrix} 1 & & \\ l_{21} & 1 & \\ l_{31} & l_{32} & 1 \end{bmatrix}$$

In this order the ls don't get jumbled up!
They just fall in place

③

When does a $\overset{A}{\text{matrix}}$ have an inverse? ~~[illegible]~~ Invertable?

L "invertable = good"

Book gives 10+ ways to answer

Determinant - "I hate the determinant"
"For a 10×10 matrix - nightmare"

$\det(A) \neq 0$

gets exponentially worse

A has n pivots
- product of the pivots are det
- find using elimination

Are the columns ind?

But what does this mean?

$Ax = 0$ has only solution $x = 0$

Why/How?

Multiply by A^{-1}

$AxA^{-1} = x$

^its like diving but not $\frac{1}{A}$ - instead A^{-1}

$$A^{-1}(Ax) = (A^{-1}A)x = Ix = x$$

But what do we get ?

$$X = \underbrace{A^{-1} 0}_{\substack{??\\ \text{Unsure}}} = 0$$

Is There is a ~~vector~~ another sol to $Ax = 0$ then that kills invetublity

<u>Only</u> 1 sol allowed

<u>example</u>

$$\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

Show matrix is <u>singular</u>

- Cols ~~and~~ are dep
- det = 0

Want

$$\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \underset{\uparrow}{\begin{bmatrix} \\ \\ \\ \end{bmatrix}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

So one possibility $\begin{bmatrix}1\\1\\1\end{bmatrix}$

Example 2

$$\begin{bmatrix}2 & -1 & -1\\-1 & 2 & -1\\-4 & -1 & 5\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Matrix is singular/not invertable

Could we prove that it does have an inverse? No

Since $x=\begin{bmatrix}1\\1\\1\end{bmatrix}$

Suppose wanted to compute A^{-1}

$$\begin{bmatrix}2 & -1 & -1\\-1 & 2 & -1\\-4 & -1 & 6\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}\neq\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Why would you want to find?

If wanted to find

$Ax=b$ then $x=A^{-1}b$

⑥

Inverse gives us the sol

Many times we don't actually compute A^{-1}

└ like on 100 × 100 matrix

~~I~~ Prof would do elim instead

How to compute A^{-1}?

"Gauss - Jordan"

$$AA^{-1} = I$$

$$A\left[\text{unknown}\right] = I$$

$$A\begin{bmatrix} x_1 & x_2 & \cdots & x_n \\ & \text{cols} & & \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & & 1 \end{bmatrix}$$

want to know the cols

So what is x_1? - a column vector

What eqn tells us x_1?

((can do matrix multiplication by column)

$$Ax_1 = \begin{bmatrix}1\\0\\0\\0\\\vdots\end{bmatrix} = b_1$$

$$\vdots$$

$$Ax_n = \begin{bmatrix}0\\0\\\vdots\\1\end{bmatrix}$$

So solve n equations

L each one for a col of A^{-1}

A lot of work!

L Not as much since same A in each eqn

Suppose want to solve 1st eqn:

$$\left[\begin{array}{c|c} A & \begin{matrix}1\\0\\0\\0\end{matrix}\end{array}\right]$$

L augment A with a vector of 0s (the RHS)

Now do elimination - so doing same to RHS as LHS at the same time

$$\left[\begin{array}{cccc|c} d_1 & & U & & \\ & d_2 & & & c \\ 0 & & d_3 & & \\ & & & d_n & \end{array}\right]$$

← pivots

U = upper triangular
C = RHS gets carried along) upper triangular System

Now $\overset{d_n = \text{pivots}}{\text{back}}$ substitution
- go back up to final sol

$$A^{-1}\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \text{1st column of } A^{-1}$$

<u>Example</u> Guass Jordan Col 1

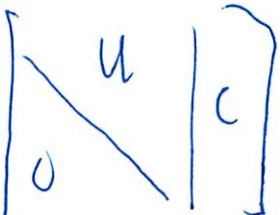
$$\begin{bmatrix} 1 & -1 & 0 & 1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & 0 \end{bmatrix}$$

↓ elimination

$$\begin{array}{r} \text{leave alone} \rightarrow \\ \text{add 1st 2 rows} \rightarrow \\ \\ \end{array} \begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & \boxed{-1} & 2 & 0 \end{bmatrix}$$

↓ next step elimination, (3,2) trying to kill -1

$$\begin{array}{r} \\ \\ \text{add 2nd two rows} \rightarrow \end{array} \begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} =$$



Now back substitution

└ Answer will be 1st col of A^{-1}

That is

$$\begin{aligned} x_1 - x_2 \quad &= 1 \\ x_2 - x_3 &= 1 \\ x_3 &= 1 \end{aligned}$$

So $x_3 = 1$

$$x_2 - 1 = 1$$
$$x_2 = 2$$

$$x_1 - 2 = 1$$
$$x_1 = 3$$

So answer $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

This is col 1

Jodan's idea

do it all at once
↑all the equations

put in not just 1st col, but all of the cols

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right] \leftarrow \text{"block matrices" short hand}$$

$$\begin{array}{cc} A & I \end{array}$$

$$\downarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & +1 & -1 & 1 & 1 & 0 \\ 0 & \boxed{-1} & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right]$$

Now back substitution

answer

$$A^{-1} = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

↑ 1st col c (as before)

↑ look at 2nd col of c

3rd col c

$$\begin{array}{lll} x_1 - x_2 & = 0 & = 0 \\ x_2 - x_3 & = 1 & = 0 \\ x_3 & = 1 & = 1 \end{array}$$

↑ changed

18.06 Reading

2/13

(Are all my books this semester written by MIT profs)

6.033 ✓ 18.06 ✓

6.006 ✓

Chap 1 Vectors

Linear combo (addition)

$$cv + dw = c\begin{bmatrix}1\\1\end{bmatrix} + d\begin{bmatrix}2\\3\end{bmatrix} = \begin{bmatrix}c & +2d\\c & +3d\end{bmatrix}$$

~~Fills 2D plane~~

cv = one line

When w not on that line

↳ Fills a 2D plane

Column Vector

$$v = \begin{bmatrix}v_1\\v_2\end{bmatrix}$$

- addition (as above)
- scalar multiplication

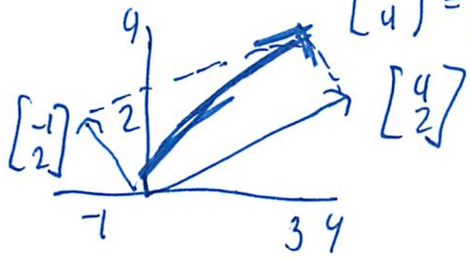
② (This book's org is weird...) and a fair bit of redundancy

$0v + 0w$ = zero vector

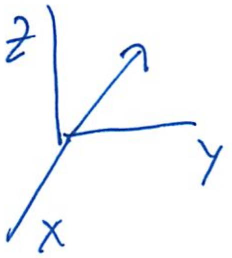
$cv + 0w$ = vector cv in direction of v

Vecotor addition is head to tail

$\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} -1+4 \\ 2+2 \end{bmatrix}$



3D vectors



$\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ same as $(0, 1, 1)$

(x, y, z)

cv = line

$cv + dv$ = plane

$cv + dv + ew$ = 3d space

Unless a special case

(ah so any c ~~fill~~ makes it a line)
not just refering to a certain c

(skipping looking at worked examples for now)

③

1.2 Lenght + Dot Products

$$V \circ W = V_1 W_1 + V_2 W_2$$

When dot product of 2 vectors $= 0$ → perpendicular

$$V \circ W = W \circ V$$

Like (p_1, p_2, p_3) (q_1, q_2, q_3) is $p_1 q_1 + p_2 q_2 + p_3 q_3$

price quantity

lenght $= ||v|| = \sqrt{v \circ v}$

often $\sqrt{V_1^2 + V_2^2}$

Unit vector → lenght $= 1$

$$i = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad j = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad u = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

$u = \dfrac{V}{||V||}$ = unit vector in same dir as v

Perpendicular

$$||v||^2 + ||w||^2 = ||v - w||^2$$

right angle → dot product $= 0$

$v \circ w > 0$ means $\theta < 90°$

$v \cdot w < 0$ means $\theta > 90°$

To find angle

$$u \cdot U = \cos\theta$$

↑ ↑
unit vectors

What is large U?

Can divide to get unit vectors

$$u = \frac{v}{\|v\|} \qquad U = \frac{w}{\|w\|}$$

So if v, w non zero $\dfrac{v \circ w}{\|v\| \cdot \|w\|} = \cos\theta$

Schwarz inequality

$$|v \circ w| \leq \|v\| \|w\|$$

Triangle inequality

$$\|v + w\| \leq \|v\| + \|w\|$$

(we didn't cover any of this in class!)

1.3 Matices

So we can do linear combo

$$c\begin{bmatrix}1\\-1\\0\end{bmatrix}+d\begin{bmatrix}0\\1\\-1\end{bmatrix}+e\begin{bmatrix}0\\0\\1\end{bmatrix}=\begin{bmatrix}c\\d-c\\e-d\end{bmatrix}$$

But can also rewrite as a matrix

$$\begin{bmatrix}1&0&0\\-1&1&0\\0&-1&1\end{bmatrix}\begin{bmatrix}c\\d\\e\end{bmatrix}=\begin{bmatrix}c\\-c+d\\-d+e\end{bmatrix}$$

$$Ax=\begin{bmatrix}u&v&w\end{bmatrix}\begin{bmatrix}c\\d\\e\end{bmatrix}=cu+dv+ew$$

(I'm not writing the bold for matricies)

* Matrix A acts on the vector x *

$$Ax=\begin{bmatrix}1&0&0\\-1&1&0\\0&-1&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}x_1\\-x_1+x_2\\-x_2+x_3\end{bmatrix}=\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}=b$$

Found book mistake!

x = input
b = output

We ~~[illegible] basically [illegible]~~ could also mutiply one column at a time — linear combo of
But most humans do rows

⑥

Linear Eqn's

If b is known and looking for x

$$\begin{aligned} x_1 &= b_1 \\ -x_1 + x_2 &= b_2 \\ -x_2 + x_3 &= b_3 \end{aligned} \rightarrow \begin{aligned} x_1 &= b_1 \\ x_2 &= b_1 + b_2 \\ x_3 &= b_1 + b_2 + B_3 \end{aligned}$$

(we saw this in class)

This was easy since A was lower triangular

invertable: from b we can recover x

Inverse Matrix

$$A = \text{diff matrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$S = \text{sum matrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

1. For every b, there is one sol to $Ax = b$
2. S produces $x = Sb$

$Ax = b$ is solved by $x = A^{-1}b = Sb$

Calculus $\quad Ax$ roughly derivative

$\qquad\quad Sx$ " integral

└ but instead of + C, start from $x(0) = 0$

[I don't get this centered diff stuff...)

Cyclic Differences

$$Cx = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 - x_3 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix} = b$$

Matrix is not triangular

Not so simple to solve for x when given b

└ Actually impossible

Either ∞ly many sols or no sols

$$\begin{bmatrix} x_1 - x_3 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

is solved by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} C \\ C \\ C \end{bmatrix}$$

Any C works

$$\begin{bmatrix} x_1 - x_3 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$$

no combo of x_1, x_2, x_3 works

⑧

For no sol can say that combo does not fill entire 3D space
(I don't fully get)

Independence w is not in plane of u and v
Depedence w is " " "

Is w a linear combo of u and v?
└ if it is - get no new vectors by adding w

On our Lower triangular - it was Ind
- So got 3D space

Ind No combo except $0u + 0v + 0w = 0$ gives $b=0$
Dep Other combos (ie $u+v+w$) give $b=0$

Ind Lower triangler example
Dep Cyclic example w/ $b = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$
Ind $Ax=0$ has 1 sol
Dep $Ax=0$ has man sols
Ind A is invertable
Dep A is singular

9

WP: Invertable matrix

- non singular
- if there exists B such that

$$AB = BA = I_n$$

Singular matrix

- not invertable
- det = 0 (if sq matrix)
- hard to pick a singular matrix

Lots of ways to find inverse

- Guass Jordan (we saw today in class)
- Analytic/Cramer's rule
- $\frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ for 2x2
  - $\det(A) = ad - bc$
- much more complicated for 3x3
- Neumann series

(what was that elimination thing he talked often about -oh is in 2.2)

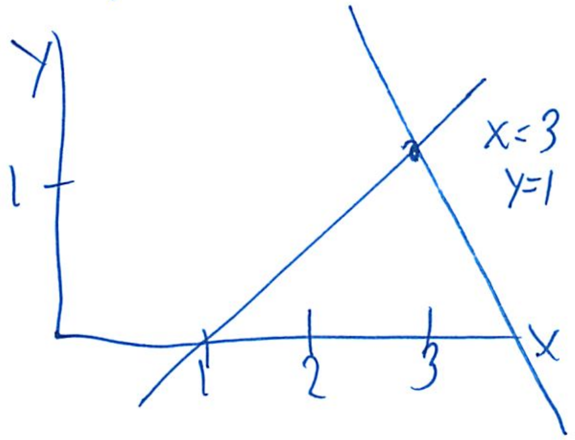
Chap 2 Solving Linear Eqns

2.1 Vector + Linear Eqns

Goal is to solve a system of eqns

$$\left.\begin{aligned} x - 2y &= 1 \\ 3x + 2y &= 11 \end{aligned}\right\} \text{2 eqns 2 unknowns}$$

Could graph



(This did not come to me naturally -but is so true -I should think like this)...

Want the point that [illegible] lives on both lines

In vectors

$$x\begin{bmatrix}1\\3\end{bmatrix} + y\begin{bmatrix}-2\\2\end{bmatrix} = \begin{bmatrix}1\\11\end{bmatrix} = b$$

[illegible]

$3(\text{col } 1) + 1(\text{col } 2) = b$

$3\begin{bmatrix}1\\3\end{bmatrix} + 1\begin{bmatrix}-2\\2\end{bmatrix} = \begin{bmatrix}1\\11\end{bmatrix}$ Ohhh! Duh!

Coefficent matrix

$$A = \begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}$$

(Last chap makes more sense now!)

$$\begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}$$

---

4 Steps of Elimination

1. Goes from A to upper triangular U by a seq of steps E_{ij}
2. The inverse matrix E_{ij}^{-1} in reverse order brings U back to the original A.
3. In matrix lang $A = LU$ (that reverse order is)
4. Elimination suceeds if A is invertable.
   (might need row exchanges)

---

3 eq/3 unknowns

- 3 3D planes ← one per eqn meeting at a single pt
  - two planes meet at a line
  - three planes meet at a point

⑫

The 3 pts meet at the solution $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

(I don't exactly when this breaks down....)

Also linear combo

$$x\begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix} + y\begin{bmatrix} 2 \\ 5 \\ -3 \end{bmatrix} + z\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}$$

Can see

$x=0$
$y=0$
$z=2$

Or matrix form

$$Ax=b$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 6 & -3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}$$

* So

Mutiplication by rows

$$Ax = \begin{bmatrix} \text{row 1} \cdot x \\ \text{row 2} \cdot x \\ \text{row 3} \cdot x \end{bmatrix}$$

Multiplication by columns

$$Ax = x(\text{col 1}) + y(\text{col 2}) + z(\text{col 3})$$

Identity matrix $Ix = x$

Notation

$$a_{\underset{\uparrow \text{row}}{i}\,\underset{\uparrow \text{col}}{j}} \quad \text{or} \quad A(i,j)$$

2.2 Elimination

(finally)

Systematic way to solve linear eqn's

$$\begin{aligned} x - 2y &= 1 \\ 3x + 2y &= 11 \end{aligned} \quad \rightarrow \quad \begin{aligned} x - 2y &= 1 \\ \underset{\uparrow \text{no } x}{} 8y &= 8 \end{aligned}$$

Ohhh - its similar to what I would have done before - right ??

try soling "normally"/old fasioned way

$$x = 1 + 2y$$

$$3(1+2y) + 2y = 11$$

$$3 + 6y + 2y = 11$$

$$8y = 8$$

so yeah how I solved before

⑭

The goal is the upper triangular system U.

$$\begin{bmatrix} 1 & -2 \\ 0 & 8 \end{bmatrix}$$

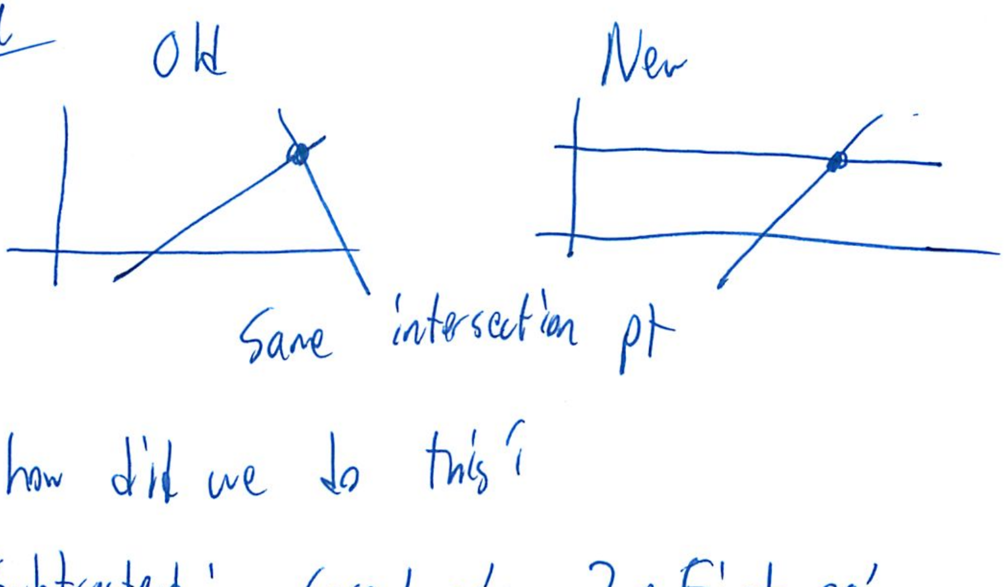
Then solve w/ back subsitution

$$8y = 8$$
$$y = 1$$
$$x - 2(1) = 1$$
$$x = 3$$

easy.

~~But how did we get from~~

Visually



Same intersection pt

But how did we do this?

- Subtracted: Second eq'n − 3 • First eq'n

And wanted 3 so that 3x disappeared

So multiple was $\ell = 3$

This all makes so much sense now!

First pivot was 1 (the coefficent of x)

Second eqn had $3x$ so multiple is 3

$\uparrow 3x - 3x = 0$

If first eqn $= 4x - 8y = 4$

Then $\ell = \frac{3}{4}$ ← divide coeff 3 by pivot 4

So $\begin{bmatrix} \textcircled{4} & \cdots \\ 3 & \cdots \end{bmatrix}$

↳ becomes $\ell = \frac{3}{4}$

before $\begin{bmatrix} 1 & \cdots \\ 3 & \cdots \end{bmatrix}$

$\ell = 3$

---

* <u>Pivot</u> = first non zero in the row that does the elimination

* <u>Mutiplier</u> = (entry to eliminate) divided by (pivot) $= \frac{3}{4}$

* Pivots end up on the diagonal of the triangle after elimination *

18.06 Recitation 1

2/14

R03

Jenn Park jmpark@math.mit.edu
- 3rd year
- # theory

OH
2-491
4th floor lange of building 2
the tower
Wed 7-8PM
L p-set due Thur

She grades the P-sets + Exams

No scratch work - final draft

Will attend Mon, Wed lectures

(seems like a very good TA)

From Canada

All things straight

- easy to visualize
- even if you suck at Calculus - could could do good here

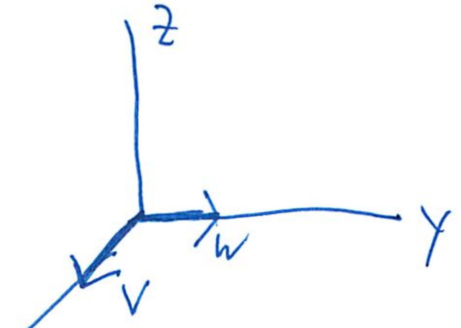
Linear Combinations

- One of the key concepts
- $\mathbb{R}^3$
  (3D space
- Let v, w be 2 vectors
- A linear combo of v, w is any vector in $\mathbb{R}^3$ that can be written as

$$c\bar{v} + d\bar{w}$$

Example

$v = (1,0,0)$
$w = (0,1,0)$



③

$(3,4,0)$ <u>is</u> a linear combo

$3v + 4w$

$(1,0,1)$ is **not** a linear combo

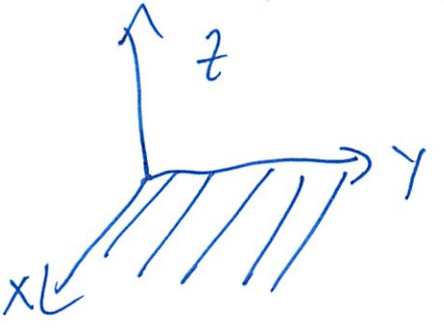
since can't get z

$(\pi, e, 0)$

$\pi v + e w$

$(c, d, 0)$

Can refer to any thing on the x, y plane



Called "v and w span the x, y plane"

if 3 vectors

then 3D space

Could span entire space

really like

④

$v = (1,0,0)$

$w = (2,0,0)$

would span a line

since w depends on v

↳ v and w are dependent

3 vectors to a point → is related to solving eqs - put that thought away for now (go back in book and clarify

We really want vectors that are ind

- like the unit vectors

Can look at matrices

- can look at each col as a vector
- if all cols ind → then matrix invertable
- then we can do elimination
- so we can solve a system of eqns

<u>example</u>

Solve

$$2x + y + z = 2$$
$$y + z = 0$$
$$-4x + 3y + 4z = -3$$

each of these eqs is a plane
They intersect in 1 point

Could write as a matrix

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 1 \\ -4 & 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underset{\uparrow u}{\begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix}}$$

Now can use elimination

* Make sure also doing operations to b vector

To not forget - you can use <u>augmented matrix</u>

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ -4 & 3 & 4 & -3 \end{array}\right]$$

⑥

Goal : Make matrix upper triangular - so make a pivot

1st pivot

$$\overset{\downarrow \text{pivot}}{\left[\begin{array}{ccc|c} \boxed{2} & 1 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ -4 & 3 & 4 & -3 \end{array}\right]} \xrightarrow{\text{row } 3 \to \text{row } 3 + 2(\text{row } 1)} \left[\begin{array}{ccc|c} 2 & 1 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 5 & 6 & 1 \end{array}\right]$$

2nd pivot

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 2 \\ 0 & \boxed{1} & 1 & 0 \\ 0 & 5 & 6 & 1 \end{array}\right] \xrightarrow{\text{row } 3 \to \text{row } 3 - 5(\text{row } 2)} \left[\begin{array}{ccc|c} 2 & 1 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

3rd pivot

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & \boxed{1} & 1 \end{array}\right]$$ don't do anything

3 pivots = invertable

* Pivots are always non 0

Now back substitute

$z = 1$

$y + (1) = 0$

$y = -1$

$2x + (-1) + (1) = 2$

$x = 1$

If the last pivot is 0?

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 1 \end{array}\right]$$

↑ but this is $0=1$ ⊗ No! "Inconsistent"

System does not have a solution

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \end{array}\right]$$

↑ $0=0$ ✓ Works "Consistent"

System does not have unique sol

3eq w/ 2 unknowns

∞ly many sols since can put anything on the previous choice

E_{21} ← elementary elimination matrices

↑ trying to get $(\underset{\uparrow\text{row}}{2}, \underset{\uparrow\text{col}}{1})$ entry to be 0

In our case, we didn't do anything

So just identity matrix $E_{21} = I = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}$

⑧

$E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$ → makes (3,1) entry 0

* always lower triangular
with 1s on the diagonal
the non 0 entry is in (3,1)

$E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{bmatrix}$ → makes (3,2) entry 0

So $\underbrace{E_{32} \cdot E_{31} \cdot E_{21}}_{\text{lower triangular matrix}} \cdot A = \underbrace{U}_{\text{get upper triangular matrix}}$

invertable can say $(E_{32} \cdot E_{31} \cdot E_{21})x = 0$ has a unique sol
& so can write

$$A = (E_{32} \cdot E_{31} \cdot E_{21})^{-1} U$$

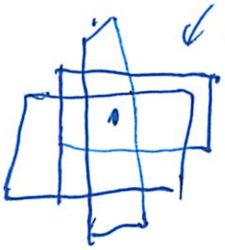
$(AB)^{-1} = B^{-1} A^{-1}$

⑨

Back to my q.

$$\left.\begin{aligned} 2x + y + z &= 2 \\ y + z &= 0 \\ -4x + 3y + 4z &= -3 \end{aligned}\right\} \text{ are } 3 - 3D \text{ planes}$$

row picture →



↙ 3 3D planes that meet in a single pt (the unique sol)

a particular line

? Not if dep - then ^ only span 2D plane

So then does not have a unique sol (a point)

but can have 0 sol or ∞ly many sols

(I'm still confused - isn't a 3D plane covering entire 3D space - look in book)

? TA: it does - I'm thinking of col picture

Above is the row picture

Exercise

$$A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 3 \\ 1 & 0 & -1 \end{bmatrix}$$

Solve a) $A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

b) $Ay = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

c) $Az = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

⑩

Can solve all at once
↓

$$\left[\begin{array}{ccc|ccc} \textcircled{1} & 2 & 1 & 1 & 0 & 0 \\ -1 & 0 & 3 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & \textcircled{2} & 4 & 1 & 1 & 0 \\ 0 & -2 & -2 & -1 & 0 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 1 & 1 & 0 \\ 0 & 0 & \textcircled{2} & 0 & 1 & 1 \end{array}\right]$$

Now have 3 systems → One for each row

Backwards sub. <u>individually</u> for each

$$x = \begin{pmatrix} 0 \\ 1/2 \\ 0 \end{pmatrix} \qquad y = \begin{pmatrix} 1/2 \\ -1/2 \\ 1/2 \end{pmatrix} \qquad z = \begin{pmatrix} -3/2 \\ -1 \\ 1/2 \end{pmatrix}$$

Why did we do this?

So its easy to find the inverse → using Gauss-Jordan method

d) What is A^{-1}?

$$A^{-1} = \begin{bmatrix} | & | & | \\ c_1 & c_2 & c_3 \\ | & | & | \end{bmatrix}$$ it satisfies $A A^{-1} = I$

$$A c_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad A c_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad A c_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

this is the system we solved!

⑪

$$A^{-1} = \begin{bmatrix} 0 & 1/2 & -1/2 \\ 1/2 & -1/2 & -1 \\ 0 & 1/2 & 1/2 \end{bmatrix}$$

More Reading

2/14

Breakdown of Elimination

What if we are asked to ÷ by 0?

$0y = 8$

No solution

System has no 2nd pivot

Like parallel lines that never meet

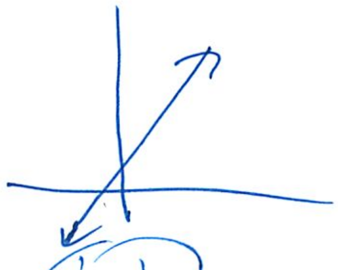
Singular

$0y = 0$

∞ solutions

y is "free"
L can be anything

Same line for both eq's



Singular

Or we can fix it w/ a row exchange

- When the 1st eq has no pt involving x

example

invertable

$$\begin{aligned} 0x + 2y &= 4 \\ 3x - 2y &= 5 \end{aligned} \updownarrow \text{flip} \quad \begin{aligned} 3x - 2y &= 5 \\ 0x + 2y &= 4 \end{aligned}$$

now we are already triangular
just back substitute!

②

Can do it w/ <u>3 eqn 3 unknowns</u>

$$2x + 4y - 2z = 2$$
$$4x + 9y - 3z = 8$$
$$-2x - 3y + 7z = 10$$

① First pivot = 2
Want to eliminate y) so $\frac{4}{2} = 2 = l_{21}$
Multiple line 1 by l_{21} and subtract this from line 2

(1b) Also for line 3
$$l_{31} = \frac{-2}{2} = -1$$
Subtract $(-1 \cdot \text{line } 1)$ from line 3

② Now have $1y + 1z = 4$ ← line 2
$1y + 5z = 12$ ← line 3

$$l = \frac{1}{1} = 1$$

Subtract $(1 \cdot \text{line } 2)$ from line 3

③ Done
$$2x + 4y - 2z = 2$$
$$1y + 1z = 4$$
$$4z = 8$$

③

④ Back substitute like normal

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$$

4x4 Problem

Col. 1. Use 1st eqn to create zeros below 1st pivot

Col 2. Use the new eqn 2 to create zeros below 2nd pivot

Col 3-n. keep going to find all n pivots and the triangular U

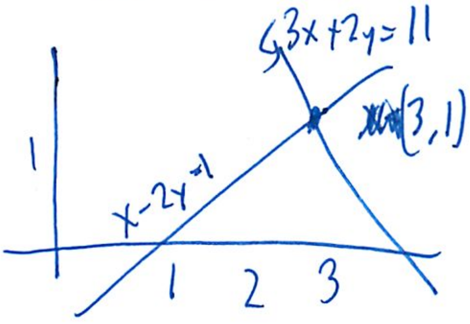
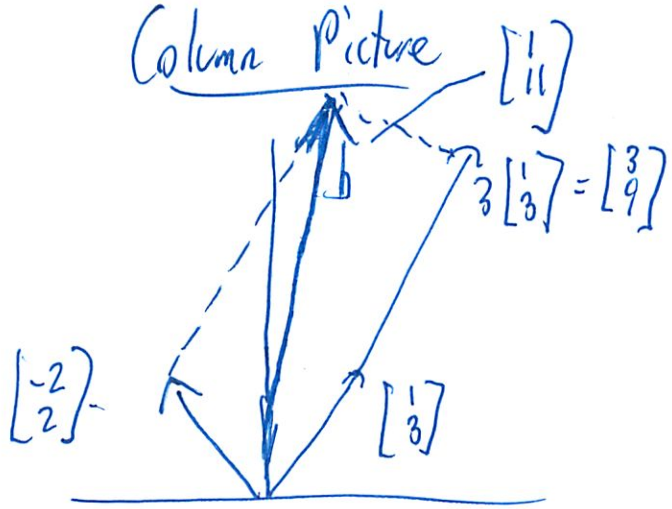
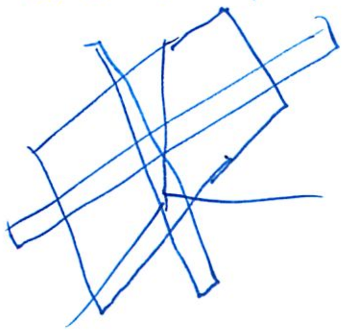
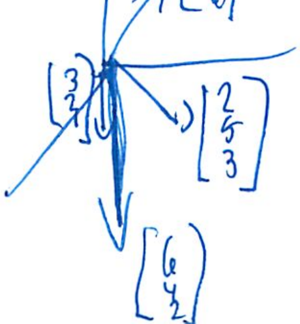
After Col 2 $$\begin{bmatrix} \textcircled{x} & x & x & x \\ 0 & \textcircled{x} & x & x \\ 0 & 0 & x & x \\ 0 & 0 & x & x \end{bmatrix}$$

We want $$\begin{bmatrix} \textcircled{x} & x & x & x \\ & \textcircled{x} & x & x \\ & & \textcircled{x} & x \\ & & & \textcircled{x} \end{bmatrix}$$

If non singular (ie invertable) this works!

Row Pictures vs Column Pictures

Was confused about in recitation

	Row Picture	Column Picture
	 	
2	Shows 2 lines meeting at a single pt (the sol)	Combines col vectors on the LHS to produce b on the RHS
	$x-2y=1$ $3x+2y=11$	$x\begin{bmatrix}1\\3\end{bmatrix}+y\begin{bmatrix}-2\\2\end{bmatrix}=\begin{bmatrix}1\\11\end{bmatrix}=b$ $\begin{bmatrix}1&-2\\3&2\end{bmatrix}\begin{bmatrix}3\\1\end{bmatrix}=\begin{bmatrix}1\\11\end{bmatrix}$
	"normal line picture"	"using vectors"
3	3 planes meeting at a single pt 	Combine 3 columns 

⑤

each row produces a "plane" in 3 D space

ie $x+2y+3z=6$

crosses x at $(6,0,0)$
crosses y at $(0,3,0)$
crosses z at $(0,0,2)$

those pts all solve eqn and determine the plane

Note does not have to include $(0,0,0)$

~~Other~~ 2nd planes intersect in a line w/ this

So a 3 variable eqn makes a plane?

I guess 2 variable eqn makes a line.

$2x+5y=7$

Yes that is a line!!

(I was thing plane for some reason... silly)

Yes!

Makes more sense now...

So solution makes a point here $\begin{bmatrix}0\\0\\2\end{bmatrix}$

This is $\begin{bmatrix}x\\y\\z\end{bmatrix}$ =

So here we see

$$0\begin{bmatrix}1\\2\\6\end{bmatrix}+0\begin{bmatrix}2\\5\\-3\end{bmatrix}+2\begin{bmatrix}3\\2\\1\end{bmatrix}=\begin{bmatrix}6\\4\\2\end{bmatrix}$$

So $\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\2\end{bmatrix}$

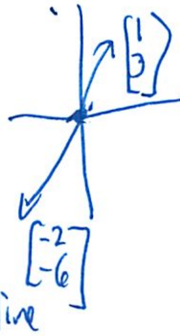
T: So this seems much more interesting....

$$Ax = \begin{bmatrix}\text{row}1 \cdot x\\ \text{row}2 \cdot x\\ \text{row}3 \cdot x\end{bmatrix}$$

$$Ax = x(\text{col }1) + y(\text{col }2) + z(\text{col }3)$$

No sol case

parallel

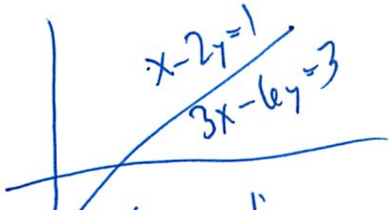


same line

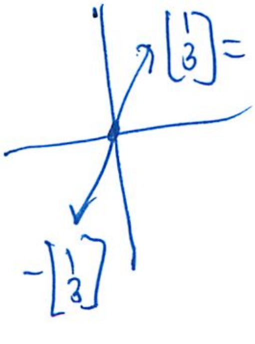
$b=\begin{bmatrix}1\\11\end{bmatrix}$

can't do

∞ly many sol case



same line



$\begin{bmatrix}1\\3\end{bmatrix}$ = b as well

2.3 Elimination using Matrixes

So can represent as a matrix
(we saw this in class, right?)

$$\begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}$$

$\uparrow$ unknown

So can represent elimination step w/ a matrix

$$l_{21} = -2$$

$$\downarrow$$

$$E = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

~~So first~~ ls

* Make sure to do it to b as well:

$$\begin{bmatrix} b_1 \\ b_2 - 2b_1 \\ b_3 \end{bmatrix}$$

$$E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -l & 0 & 1 \end{bmatrix}$$

⑧

Matrix multiplication

Associative	Yes	$A(BC) = (AB)C$
Commutative	No	$AB \neq BA$

~~Multiply by~~

(skipping since I know...)

Row Exchange

$P_{\underset{\substack{\uparrow\\ \text{row }1}}{i}\underset{\substack{\uparrow\\ \text{row }2}}{j}}$ = permutation matrix

ie. $P_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ ⟲ swap

Augmented Matrix

have b inside

$[A \;\; b]$

ie $\left[\begin{array}{ccc|c} 2 & 4 & -2 & 2 \\ 4 & 9 & -3 & 8 \\ -2 & -3 & 7 & 10 \end{array}\right]$

^ line is not actually there
its like 1 big matrix
nothing special

18.06 OH

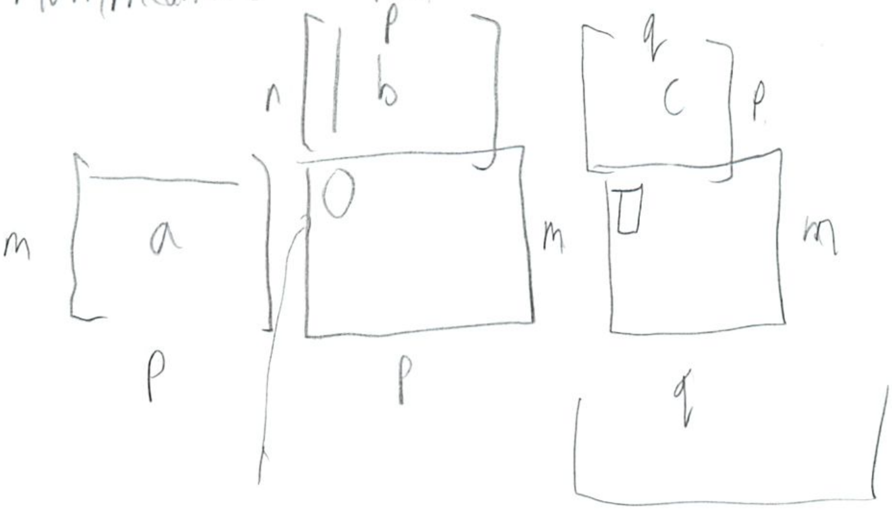
2/15

2.4 #36

(5 other people here)

Said she was worried at my P-set - not many augment matrices
- but other student pointed out not that many problem

Multiplication trick



n multiplications = n computations

if multipling the same thing

Trying to see how multiplication is faster

So n computations · m rows · p columns

$(mp)n$

For here □ is p multiplications

$(mq)p$

This is like a - but order switched

②

N component vector

$u, v, w \in \mathbb{R}^N$

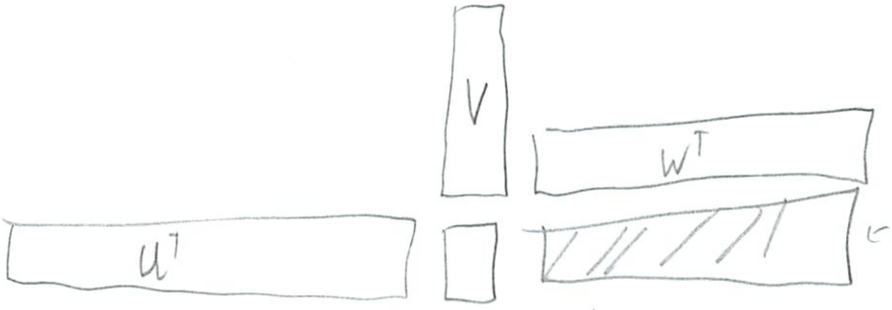
We have 3 of these u, v, w

Want $u^T v w^T$

Which order is eaiser

N ← all the same size N

So



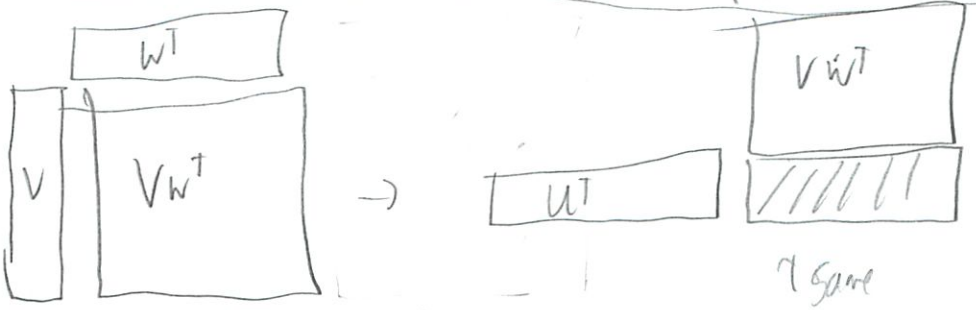
← n multiplications

= n + n multiplications

↑ n multiplications

↑ much eaiser

or



↑ do the same thing for this

↑ bit harder

③

c) Goes back to the general question

$(AB)C \rightarrow nmp + mpq$ computations

$A(BC) \rightarrow mnq + npq$ 〃

When is the 1st run faster than the 2nd?

$$mnp + mpq < mnq + npq$$

(asking when is (AB)C faster

Divide out $mnpq$

$$\frac{1}{q} + \frac{1}{n} < \frac{1}{p} + \frac{1}{m}$$

So when is that?

---

6. When is a matrix invertable?

When a matrix has n non-zero matrix

Permutation matrix

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

④

Show its invertable

So do elimination on it

We can swap the rows back

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow$$ Now simple back sub works ✓

Sol #2 Can also do $Ax = 0$ has a unique sol

$$P \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0 \xrightarrow{\text{get}} \begin{array}{l} x_4 = 0 \\ x_3 = 0 \\ x_1 = 0 \\ x_2 = 0 \end{array}$$

Sol #3 For both: prove w/ more generality

— no proof by example

$$P = \underbrace{\begin{bmatrix} - & r_1 & - \\ - & r_2 & - \\ & \vdots & \\ - & r_n & - \end{bmatrix}}_{n} \Big\} n$$

Then make an argument using this.

⑤

b) Prove $PP^T = I$

$$P = \begin{bmatrix}0&0&0&1\\0&0&1&0\\1&0&0&0\\0&1&0&0\end{bmatrix} \quad \text{So} \quad P^T = \begin{bmatrix}0&0&1&0\\0&0&0&1\\0&1&0&0\\1&0&0&0\end{bmatrix}$$

Multiply

$$\begin{bmatrix}0&0&1&0\\0&0&0&1\\0&1&0&0\\1&0&0&0\end{bmatrix}$$

$$\begin{bmatrix}0&0&0&1\\0&0&1&0\\1&0&0&0\\0&1&0&0\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

row 1 · col 1 - are the same thing!

det of transpose matrix

1 will have 1×1s

Will have $(n-1) \times 0$s

If dot product w/ itself get 1

If dot product w/ other rows gets 0

(That level of detail they want)

10. (Every one is confused on this. I think I got it)

$$W_1 = 100W + 0S + 0T$$
$$W_2 = 50W + 0S + 50T$$
$$W_3 = 30W + 50S + 20T$$

Oh I didn't notice $W + S + T = 100$

Thats what thed add to 100 is

But my answer had that

$$W = (45, 50, 5)$$

So Bardx eqn is

$$W = 45W + 50S + 5T$$

So we want

$$W = aW_1 + bW_2 + cW_3$$

$$\begin{pmatrix} 45 \\ 50 \\ 5 \end{pmatrix} = a\begin{pmatrix} 100 \\ 0 \\ 0 \end{pmatrix} + b\begin{pmatrix} 50 \\ 0 \\ 50 \end{pmatrix} + c\begin{pmatrix} 30 \\ 50 \\ 20 \end{pmatrix}$$

Thats what I had I think

Try and find a, b, c

⑦

$$\begin{bmatrix} 100 & 50 & 30 \\ 0 & 0 & 50 \\ 0 & 50 & 20 \end{bmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 45 \\ 50 \\ 5 \end{pmatrix}$$

(12+ people at OH now)

b) Then add a d

$$+ d \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

"must" add to 100

We can guess what to put in there

$$\begin{bmatrix} 100 & 50 & 30 & x \\ 0 & 0 & 50 & y \\ 0 & 50 & 20 & z \end{bmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 45 \\ 50 \\ 5 \end{pmatrix}$$

A lot of things work

c) 4 unknows w/ 3 eqns

Can still do elimination

⑧

(pushy people)

5. $q = \pm 8$

has to be $0=0$

$qq = 64q$

Solve that

(I didn't really follow)

but 2 eq $\overset{4}{8}$ unknowns

But both same eq ...

Lots of possibilites

if $q=0$ then $0=0$ → then only 1 eq'n

But want all qs

2/16

18.06 Spring 2012 – Problem Set 1

This problem set is due Thursday, February 16th, 2012 at 4pm (hand in to Room 2-106). The textbook problems are out of the 4th edition. For computational problems, please include a printout of the code with the problem set (for MATLAB in particular, `diary(''filename'')` will start a transcript session, `diary off` will end one.)

Every problem is worth 10 points.

1. Do Problem 8 from Section 1.3.
2. Do Problem 8 & Problem 32 from Section 2.2. two problems....
3. Do Problem 22 from Section 2.3.
4. Do Problem 19 & Problem 36 from Section 2.4.
5. For which values of q (if any) is the following system consistent (= solvable)?

$$\begin{aligned} x + 4y + 3z &= 1, \\ q^3x + 4q^3y + 3q^3z &= 64q. \end{aligned}$$

6. A permutation matrix P comes from permuting the rows of the identity matrix I_n. If the entries of P are labelled p_{ij}, the matrix A having entries $a_{ij} = p_{ji}$ is the transpose, $A = P^T$.
   (a) Is P invertible, and if yes *why*? How would we proceed in Gaussian elimination on P?
   (b) Explain why the product $C = PP^T$ is the identity matrix. Think about where the 1's and 0's are.
   (c) Since the answer to (a) was "yes", what is the inverse to P?
7. (a) Give examples of non-zero (meaning: not all entries zero) 2×2 and 4×4 matrices A, one of each, such that $A^2 = O$ (recall O means the zero matrix). Hint: You only need to use one 1, and the rest of the entries can be 0's!
   (b) Are there any invertible $n \times n$ matrices A such that $A^2 = O$?
8. Given the three vectors $\mathbf{a}_1 = (1, 2, 3)$, $\mathbf{a}_2 = (1, 0, -1)$ and $\mathbf{a}_3 = (0, 0, 1)$, find (if possible) numbers x_1, x_2 and x_3 such that:

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Your solution should involve Gaussian elimination on $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix}$ (the matrix with $\mathbf{a}_i$'s as columns).

9. (a) Using MATLAB, perform the matrix products A^2, A^3 and A^6 of the following lower-triangular matrix:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 7 & 2 & 0 & 0 \\ 5 & 1 & 3 & 0 \\ 3 & 2 & -1 & 4 \end{bmatrix}$$

   (b) Explain the rule for *diagonal* entries of A^k, for a lower-triangular matrix A.

   (c) Guess a rule for the $(2, 1)$ entry of A^k, for a lower-triangular matrix A.

10. A chemistry professor claimed on live TV that he could, by mixing, obtain *any* wine with given contents of water (W), sugar (S) and tannic acid (T), labelled by vectors $w = (W, S, T)$ such that $W + S + T = 100\%$. Due to a lack of research funding, his stock was quite limited:

    - Laboratory water supply: $w_1 = (100, 0, 0)$.
    - Budget wine: $w_2 = (50, 0, 50)$.
    - Plum tea concentrate: $w_3 = (30, 50, 20)$.

    (a) If a Chateaux Bordeaux 1915 has $(W, S, T) = (45, 50, 5)$, why was the professor *not* able to obtain this wine by mixing w_1, w_2, w_3? Explain by computing the mixing ratios needed (by MATLAB or by hand).

    (b) Help the professor restore honor, by adding any new wine w_4 that will enable him to make the Chateaux Bordeaux 1915 (a Chateaux Bordeaux 1915 not allowed!).

    (c) Are the mixing ratios unique after addition of the fourth wine?

Michael Plasmeier ~~R03~~ 18.06 P-set 1 83 2/16

you are in R04 in R03 as of 2.24.12

1.3 #8. Find the 4 components x_1, x_2, x_3, x_4 in $Ax = b$
Then write as $S = A^{-1}$

$$Ax = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

$$x_1 = b_1$$
$$-x_1 + x_2 = b_2$$
$$-x_2 + x_3 = b_3$$
$$-x_3 + x_4 = b_4$$

Now write as $x_1 = $:

$$x_1 = b_1$$
$$x_2 = b_2 + x_1$$

want just bs

$$= b_2 + b_1$$
$$x_3 = b_3 + x_2$$
$$= b_3 + b_2 + b_1$$
$$x_4 = b_4 + x_3$$
$$= b_4 + b_3 + b_2 + b_1$$

So can write

$$S = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

✓

Note $Ax = b$ is solved by $x = A^{-1} b = Sb$

Thus $A^{-1} = S$

2. 2.2 #8 For which 3 #s k does elimination break down? Can it be fixed, or is it 0 or ∞ sols?

$$kx + 3y = 6$$
$$3x + ky = -6$$

So k is our pivot

Want $\ell = \frac{3}{k}$

Want $\ell \neq 0$

$k = 0$ is bad - divide by 0 error

But can we fix?

$$3x + ky = -6$$
$$kx + 3y = 6$$

does not matter ~~Now we can do $\ell = \frac{k}{3}$~~

~~But ℓ is 0~~ which can be solved w/ only back substitution

③

What else?

No sol if $0y$ after division

Which is $k-\left(\frac{3}{k} \cdot 3\right)$

$$0 = k - \frac{9}{k}$$

$$k = \pm 3$$

both have no soln

Test out

$$3x + 3y = 6$$
$$3x + 3y = -6$$

$l = \frac{3}{3} = 1$

x) $3 - (1 \cdot 3) = 0$

y) $3 - (1 \cdot 3) = 0$

b) $-6 - (1 \cdot 6) = -12$

$0y = -12$ ⊗ No sol

But -3?

Then $l = -1$ so same

⑨

3.2.2 #32 Start w/ 100 eq $Ax=0$ for 100 unknowns

Suppose it reduces to $0=0$, so "singular"

a) Elimination takes linear combo. Singular property of <u>rows</u>

?? not sure what they are looking for ✓

-2

dependent

b) ∞ly many sols → <u>columns</u>

? ⟶ the solutions is one of the inputs

-2

(from figure 2.7)

c) Invent a 100 × 100 singular matrix

One in every entry

$$\begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & 1 & \cdots & 1 \\ \vdots & & & \ddots & & \vdots \\ 1 & 1 & 1 & 1 & \cdots & 1 \end{bmatrix}$$

d) For your matrix describe row, col pic of $Ax=0$

- infitly many sols

<u>row pic</u> - fills the entire space

<u>col pic</u> - sol lies on one of the column matrices

(5)

2.322) The entries of A and x are a_{ij} and x_j

So the 1st component of Ax is

$$\sum a_{1j} x_j = a_{11} x_1 + \dots + a_{1n} x_n$$

If E_{21} subtracts row 1 from row 2 write a formula for

a) the third component of Ax

ith component is $a_{i1} x_1 + \dots + a_{in} x_n$ ✓

3rd " $a_{31} x_1 + \dots + a_{3n} x_n$

b) The (2,1) entry of $E_{21} A$

So (2,1) entry of E_{21} is $-\ell$ $\boxed{a_{21} - a_{11}}$

Multiplied by A

(-1) $E_{21} A_{11}$ for any column

'' how write this - think interpreting wrong

c) (2,1) entry of $E_{21}(E_{21} A)$

∴ just do it again

$-l^2 A_{11}$ $a_{21} - 2a_{11}$

(-1)

∴?

d) The 1st component of EAx

$-\cancel{l}\left(\underline{a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n}\right) = \sum_j a_{1j}x_j$

✓

∴ I don't get the point of this qu...

2.4 # 19. The entries of A are a_{ij}. Assuming zeros don't appear

a) First pivot ?

a_{11} ✓

b) The multiplier l_{31}

$$\frac{a_{31}}{a_{11}}$$ ✓

c) The entry that replaces a_{32} after that subtraction

$$a_{32} - \left(\frac{a_{31}}{a_{11}}\right) a_{31}$$ ✓

d) Second pivot

a_{22}

? which is fom original matrix

$$a_{22} - \left(\frac{a_{21}}{a_{11}}\right) a_{22}$$ ✓

2.4 # 36 Pratical Challenge Qu Suppose A is m by n

B is $n \times p$ C is p by q

Then the multiplication count for (AB)C is

$mnp + mpq$

⑧

The same ans comes from $A(BC)$ which is $mnq + npq$

a) If $A = 2 \times 4$ which is better?

$B = 4 \times 7$

$C = 7 \times 10$

$A = m \times n$

$B = n \times p$

$C = p \times q$

So $m = 2$

$n = 4$

$p = 7$

$q = 10$

$\underline{(AB)C}$	$\underline{A(BC)}$
$mnp + mpq$	$mnq + npq$
$2(4)7 + 2(7)10$	$2(4)(10) + 4(7)10$
196	360
↑ less # of multiplications	

b) With N component vectors would you choose

$(u^T v) w^T$ or $u^T (v w^T)$?

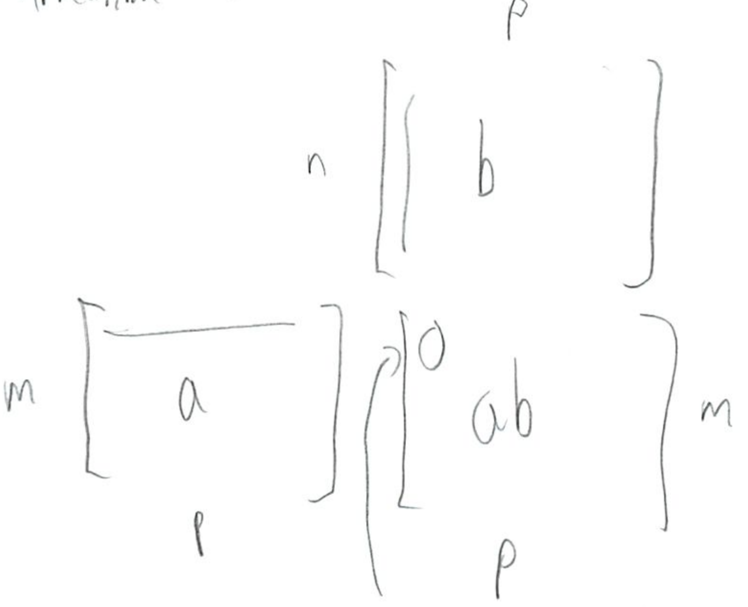
? diff letters....

Would want smallest multiplication early on

Whichever that is...

8b

Multiplication trick



need n multiplications

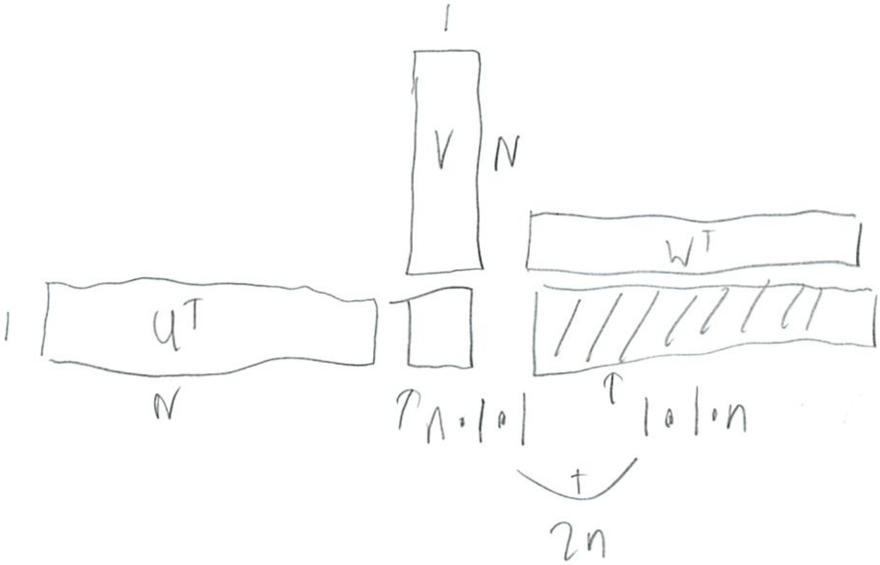
So total = (mp)n

So here

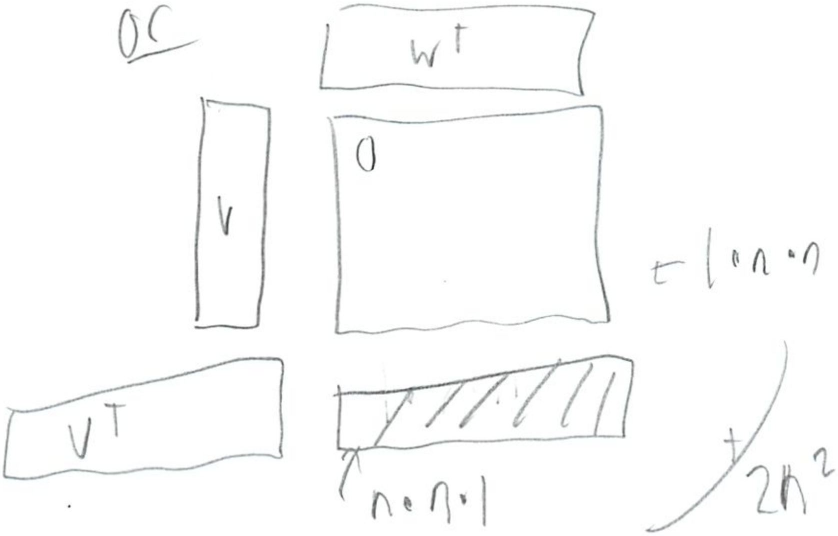
$u = v = w =$

$u^T = v^T = w^T =$

8c



OC



$2n < 2n^2$

⑨

c) Divide by mnpq to show that (AB)C is faster when

$$n^{-1} + q^{-1} < m^{-1} + p^{-1}$$

So (AB)C is $nmp + mpq$

A(BC) is $mnq + npq$

$$mnp + mpq \underset{\substack{\uparrow \\ \text{what question is asking}}}{<} mnq + npq$$

divide out $mnpq$

$$\frac{1}{q} + \frac{1}{n} < \frac{1}{p} + \frac{1}{m}$$

□ QED

⑩

5. For which values of q (if any) is the system solvable?

$$x + 4y + 3z = 1$$

$$q^3x + 4q^3y + 3q^3z = 64q$$

~~? So when second pivot not 0?~~
~~but 2 eq 3 unknowns~~
~~↳ So never? trick question?~~

One possibility

$q = 0$ ✓

Then $0 = 0$

Then 1 eq, 3 unknowns

Basically 2 eq 4 unknowns
So lots of possibilites

Also $q = \pm 8$ ✓

$8^3 = 512$

$64 \cdot 8 = 512$

Can divide by 512
Now eqns are the same → many possible sols.

106

Never proved that there are not any others.
How would you do that?

⑪

6. A permutation matrix P comes from permutating the Identity

$A = P^T$

└ where $a_{ij} = p_{ji}$

a) Is P invertable?

So if $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$ flipped is $\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$ ✓

Then transposed $\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$ — same

That is invertable ✓

I can't think of any counter examples

b) Explain why $C = PP^T$ is the identity

Since $P = P^T = P^{-1}$

and $A A^{-1} = I$

Try a 3, 3 example ✓

$C = \begin{bmatrix}1 & 0 & 0\\0 & 0 & 1\\0 & 1 & 0\end{bmatrix}$

MATLAB $C = C' = inv(C)$

c) $P^{-1} = P$

(11b)

? Should probably show that it is invertable

Method #1 Can you do elimination on it?

You can always reswap the rows back to the identity

Then just simple back subs

Method #2 Can you also $Ax=0$ has a unique sol

$$P\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = 0 \xrightarrow{\text{get}} \begin{matrix}x_4=0\\x_3=0\\x_2=0\\x_1=0\end{matrix}$$

Method 1b

Could argue like this

$$P = \underbrace{\begin{bmatrix} — & r_1 & — \\ — & r_2 & — \\ & \vdots & \\ — & r_n & — \end{bmatrix}}_{n} n$$

Just reorder back to identity

How more explicit about it?

(11c)

b Take 2) You always are multipling the same thing

since $\underset{P}{\text{row } 1} = \underset{P^T}{\text{col } 1}$

So $(\text{row } 1\ P)^2 = \text{row } 1\ P$

$$\begin{array}{ccccc} & & & & 0\\ & & & & 0\\ & & & & 0\\ & & & & 0\\ & & & & 1\\ 0&0&0&1&\boxed{1}\end{array}$$

the 1 ? will be in the same position always

So a larger example

$$\begin{array}{cc} & \begin{bmatrix} 0&0&1&0\\ 0&0&0&1\\ 0&1&0&0\\ 1&0&0&0 \end{bmatrix}\\ \begin{bmatrix} 0&0&0&1\\ 0&0&1&0\\ 1&0&0&0\\ 0&1&0&0 \end{bmatrix} & \begin{bmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1 \end{bmatrix} \end{array}$$

For n - always have 1 of 1s

(n-1) of 0s

$$1\ \underset{n}{\begin{bmatrix} \qquad\qquad \end{bmatrix}}$$

Dot product w/ any other row is 0

7. Give examples of non zero 2x2 and 4x4 matrices Such that $A^2 = \underset{\uparrow \text{zero}}{0}$

So I want 1s off set

$$\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}1&0\\0&1\end{bmatrix} = \begin{bmatrix}1&0\\0&1\end{bmatrix}$$

No

$$\begin{bmatrix}1&0\\0&0\end{bmatrix}$$

No

Only 4 choices!

$$\begin{bmatrix}0&0\\1&0\end{bmatrix}\begin{bmatrix}0&0\\1&0\end{bmatrix} = \begin{bmatrix}0\cdot0+0\cdot1 & 0\cdot0+0\cdot0\\ 1\cdot0+0\cdot1 & 0\cdot0+0\cdot0\end{bmatrix} \quad ✓$$

✓

Multiply correctly!
need to be good at doing it visually...

Now 4x4

$$\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\1&0&0&0\end{bmatrix} \quad ✓$$

b) Are there any invertable n×n matrices A such that $A^2 = 0$

I have a feeling, no - but what proof?

(-2)

Need n non-zero pivots (allong diagonal)
 └ such as I_3

$I_3^2 = I_3$

$A = A^{-1}A^2 = A^{-1}0 = 0$ so **No**

⑬

8. Given 3 vectors $a_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ $a_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$a_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

Find x_1, x_2, x_3 such that

$$x_1 a_1 + x_2 a_2 + x_3 a_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

This sounds easy....

$$\begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 0 \\ 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 2 & 0 & 0 & 1 \\ 3 & -1 & 1 & 1 \end{array}\right]$$

$l_{21} = \frac{2}{1}$ $l_{31} = \frac{3}{1}$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & -2 & 0 & -1 \\ 0 & -4 & 1 & -2 \end{array}\right] \begin{array}{l} \leftarrow \text{row 1} \\ \leftarrow \text{row 2} - (2 \cdot \text{row 1}) \\ \leftarrow \text{row 3} - (3 \cdot \text{row 1}) \end{array}$$

$l_{32} = \frac{-4}{-2} = 2$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & -2 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right] \begin{array}{l} \leftarrow \text{new row 1} \\ \leftarrow \text{new row 2} \\ \leftarrow \text{new row 3} - (2 \cdot \text{row 2}) \end{array}$$

⑭

$z = 0$

? This is a plausable answer - right? Im pretty sure

$-2y = -1$

$y = \frac{-1}{-2} = \frac{1}{2}$

$x + \left(\frac{1}{2}\right) = 1$

$x = \frac{1}{2}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ 0 \end{bmatrix}$$ ✓

No tricks?

Oh just saw line about Gaussian elimination on $A = [a_1 \; a_2 \; a_3]$

I basically did that???

I'm pretty sure...

9. Matlab question

a) See pg 15b

b) Rule for lower triangular matrix - diagonal entries

So A^2 $1 \to 4 \to 9 \to 16$

$1^2 \to 2^2 \to 3^2 \to 4^2$

-2

$A3$ $1 \to 8 \to 27 \to 64$

$1^3 \to 2^3 \to 3^3 \to 4^3$

$A6$ $1^6 \to 2^6 \to 3^6 \to 4^6$

c) Guess a rule for (2,1) entry of A^k

-2

$A \to 7$ $7 \cdot 1$ $7 \cdot 1$

$A^2 \to 21$ $7 \cdot 3$ what's this pattern? $7 \cdot 3$

$A3 \to 49$ $7 \cdot 7$ $7 \cdot 7$

$A^6 \to 441$ $7 \cdot 63$ $7 \cdot 7 \cdot 9$

15b

```
A = [1 0 0 0; 7 2 0 0 ; 5 1 3 0; 3 2 -1 4]

A =

     1     0     0     0
     7     2     0     0
     5     1     3     0
     3     2    -1     4

A^2

ans =

     1     0     0     0
    21     4     0     0
    27     5     9     0
    24    11    -7    16

A^3

ans =

     1     0     0     0
    49     8     0     0
   107    19    27     0
   114    47   -37    64

A^6

ans =

           1           0           0           0
         441          64           0           0
        3927         665         729           0
        5754        2681       -3367        4096

diary off
```

✓ ✓ ✓

(16)

10. Chem prof

$w = (W, S, T)$

$W + S + T = 100\%$

? 100% of what

Constraints

$w_1 = (100, 0, 0)$

$w_2 = (50, 0, 50)$

$w_3 = (30, 50, 20)$

a) Why could the prof not get $(45, 50, 5)$

So word problem - challenge is to set it up

Column view

$$b_1 \begin{bmatrix} 100 \\ 0 \\ 0 \end{bmatrix} + b_2 \begin{bmatrix} 50 \\ 0 \\ 50 \end{bmatrix} + b_3 \begin{bmatrix} 30 \\ 50 \\ 20 \end{bmatrix} = \begin{bmatrix} 45 \\ 50 \\ 5 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 100 & 50 & 30 & 45 \\ 0 & 0 & 50 & 50 \\ 0 & 50 & 20 & 5 \end{array}\right]$$

elimination....

already done (w/ row swap)

①

$50 b_3 = 50$

$b_3 = 1$

$50 b_2 + 20(1) = 5$

$50 b_2 = -15$

4

-3

⊗ Can't have negitive quantites

b) Add a w_4 that fits

↑ any way to do mathmatically
lots of possibilities....

-2

So just need 5 independent $\begin{bmatrix}0\\100\\0\end{bmatrix}$

1st get T locked in w/ b_3

$b_3 = \frac{1}{4}$ $\quad \frac{1}{4}\begin{bmatrix}30\\50\\20\end{bmatrix} = \begin{bmatrix}7.5\\12.5\\5\end{bmatrix}$

2nd add b_1 to get W

$45 - 7.5 = 37.5$

$\frac{37.5}{100} = .375 = b_1$

$.375\begin{bmatrix}100\\0\\0\end{bmatrix} = \begin{bmatrix}37.5\\0\\0\end{bmatrix}$

⑱

3rd Now add our new component

$$50 - 12.5 = 37.5 =$$

$$b_4 = .375$$

$$.375 \begin{bmatrix} 0 \\ 100 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 37.5 \\ 0 \end{bmatrix}$$

4th add it up

$$\begin{bmatrix} 7.5 \\ 12.5 \\ 5 \end{bmatrix} + \begin{bmatrix} 37.5 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 37.5 \\ 0 \end{bmatrix} = \begin{bmatrix} 45 \\ 50 \\ 5 \end{bmatrix}$$

¿how to do this mathmatically?

c) Is ratio now unique?

Well we've fixed $\begin{bmatrix} 0 \\ 100 \\ 0 \end{bmatrix}$

But within that?

Will answer if matrix invertable

See if any cols multiples of each other (dependent)

↳ No, doesn't look like it

↳ Unique.

But if we've not fixed it → plenty of solutions

⑲

b) Could we do this more rigerously?

$$\begin{bmatrix} 100 & 50 & 30 & X \\ 0 & 0 & 50 & Y \\ 0 & 50 & 20 & Z \end{bmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 45 \\ 50 \\ 5 \end{pmatrix}$$

Can pick X, Y, Z kinda randomly,

Lets keep (0, 100, 0)

$$\begin{bmatrix} 100 & 50 & 30 & 0 \\ 0 & 0 & 50 & 100 \\ 0 & 50 & 20 & 0 \end{bmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 45 \\ 50 \\ 5 \end{pmatrix}$$

Pivots

$$\left[\begin{array}{cccc|c} 100 & 50 & 30 & 0 & 45 \\ 0 & 0 & 50 & 100 & 50 \\ 0 & 50 & 20 & 0 & 5 \end{array}\right]$$

<u>1st</u> one already good

<u>2nd</u> row swap

$$\left[\begin{array}{cccc|c} 100 & 50 & 30 & 0 & 45 \\ 0 & 50 & 20 & 0 & 5 \\ 0 & 0 & 50 & 100 & 50 \end{array}\right]$$

So we are good for back sub

We have more rows than we can do (problem c)

⑳

$50c + 100d = 50$

We can assume $d = 1$ No! Then it wald be ⊖

$d = 0$ No That world be a

$d = .25$

$50c = 25$

$c = .5$

$50b + 20(.5) = 5$

$50b + 10 = 5$

× No, Go back. Is there a more rigeras way to do.

Did they say to guess at OH

$d = .1$

$50c + 100(.1) = 50$

$50c = 40$

$c = .8$

$50b + 20(.8) = 5$

× Wrong way

$d = .5$

$50c + 100(.5) = 50$

$c = 0$

(21)

$50b + 20(0) = 5$

$b = .1$

$100(a) + 50(.1) + 30(0) = 45$

$100a = 40$

$a = .4$

So $\begin{pmatrix} a \\ b \\ c \\ \hline d \end{pmatrix} = \begin{pmatrix} .4 \\ .1 \\ 0 \\ .5 \end{pmatrix}$

Another way, another answer?

Test

$$.4\begin{bmatrix} 100 \\ 0 \\ 0 \end{bmatrix} + .1\begin{bmatrix} 50 \\ 50 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 30 \\ 20 \\ 50 \end{bmatrix} + .5\begin{bmatrix} 0 \\ 0 \\ 100 \end{bmatrix} = \begin{bmatrix} 40+5 \\ 5 \\ 50 \end{bmatrix}$$

c) So there are other answers even within $\begin{bmatrix} 0 \\ 100 \\ 0 \end{bmatrix}$ as I just showed 2 possibilites

18.06 Spring 2012 – Problem Set 1 - Solutions

This problem set is due Thursday, February 16th, 2012 at 4pm (hand in to Room 2-106). The textbook problems are out of the 4th edition. For computational problems, please include a printout of the code with the problem set (for MATLAB in particular, `diary('filename')` will start a transcript session, `diary off` will end one.)

Every problem is worth 10 points.

1. Do Problem 8 from Section 1.3.

   Solution to 1.3.8:

$$\begin{aligned} x_1 - 0 &= b_1 & x_1 &= b_1 \\ x_2 - x_1 &= b_2 & x_2 &= b_1 + b_2 \\ x_3 - x_2 &= b_3 & x_3 &= b_1 + b_2 + b_3 \\ x_4 - x_3 &= b_4 & x_4 &= b_1 + b_2 + b_3 + b_4 \end{aligned}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = A^{-1}\mathbf{b}$$

2. Do Problem 8 & Problem 32 from Section 2.2.

   Solution to 2.2.8:

   If $k = 3$, then elimination must fail: No solution. If $k = -3$, elimination gives $0 = 0$ in equation 2: Infinitely many solutions. If $k = 0$ a row exchange is needed: Exactly one solution.

   Solution to 2.2.32:

   The question deals with 100 equations $Ax = 0$ when A is singular.

   (a) Some linear combination of the 100 rows is the row of 100 zeros.

   (b) Some linear combination of the 100 columns is the column of zeros.

   (c) A very singular matrix has all ones: $A = \mathbf{eye}(100)$. A better example has 99 random rows (or the numbers $1^i, \ldots, 100^i$ in those rows). The 100th row could be the sum of the first 99 rows (or any other combination of those rows with no zeros).

   (d) The row picture has 100 planes meeting along a common line through 0. The column picture has 100 vectors all in the same 99-dimensional hyperplane.

3. Do Problem 22 from Section 2.3.

   Solution to 2.3.22:

   (a) $\sum a_{3j}x_j$.

   (b) $a_{21} - a_{11}$.

(c) $a_{21} - 2a_{11}$.

(d) $(EAx)_1 = (Ax)_1 = \sum_j a_{1j}x_j$.

4. Do Problem 19 & Problem 36 from Section 2.4.

Solution to 2.4.19:

(a) a_{11}.

(b) $l_{31} = a_{31}/a_{11}$.

(c) $a_{32} - \left(\frac{a_{31}}{a_{11}}\right) a_{12}$.

(d) $a_{22} - \left(\frac{a_{21}}{a_{11}}\right) a_{12}$.

Solution to 2.4.36:

Multiplying $AB = (m \text{ by } n)(n \text{ by } p)$ needs mnp multiplications. Then $(AB)C$ needs mpq more. Multiply $BC = (n \text{ by } p)(p \text{ by } q)$ needs npq and then A(BC) needs mnq.

(a) If m, n, p, q are $2, 4, 7, 10$ we compare $(2)(4)(7)+(2)(7)(10) = 196$ with the larger number $(2)(4)(10) + (4)(7)(10) = 360$. So AB first is better, so that we multiply that 7 by 10 matrix by as few rows as possible.

(b) If u, v, w are N by 1, then $(u^T v)w^T$ needs $2N$ multiplications but $u^T(vw^T)$ needs N^2 to find vw^T and N^2 more to multiply by the row vector u^T. Apologies to use the transpose symbol so early.

(c) We are comparing $mnp + mpq$ with $mnq + npq$. Divide all terms by $mnpq$: Now we are comparing $q^{-1}+n^{-1}$ with $p^{-1}+m^{-1}$. This yields a simple important rule. If matrices A and B are multiplying v for ABv, don't multiply the matrices first.

5. For which values of q (if any) is the following system consistent (= solvable)?

$$x + 4y + 3z = 1,$$
$$q^3x + 4q^3y + 3q^3z = 64q.$$

Solution: We write the system as a matrix equation

$$\begin{bmatrix} 1 & 4 & 3 \\ q^3 & 4q^3 & 3q^3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 64q \end{bmatrix}.$$

In a one-step elimination, we get for the augmented matrix $[A \mid b]$:

$$\left[\begin{array}{ccc|c} 1 & 4 & 3 & 1 \\ 0 & 0 & 0 & 64q - q^3 \end{array}\right]$$

The equation $0 = 64q - q^3 = q(64 - q^2)$ holds if either $q = 0$ or $64 - q^2 = 0$, so:

$$\boxed{\text{Only consistent when either } q = 0,\ q = -8 \text{ or } q = 8.}$$

6. A permutation matrix P comes from permuting the rows of the identity matrix I_n. If the entries of P are labelled p_{ij}, the matrix A having entries $a_{ij} = p_{ji}$ is the transpose, $A = P^T$.

   (a) Is P invertible, and if yes *why*? How would we proceed in Gaussian elimination on P?

   (b) Explain why the product $C = PP^T$ is the identity matrix. Think about where the 1's and 0's are.

   (c) Since the answer to (a) was "yes", what is the inverse to P?

   Solution:

   (a) Yes. To proceed we would swap all rows back in their correct place and obtain the identity. Hence P is invertible.

   (b) Look at the entry c_{ij} in C, which is the dot product of the i'th row in P and the j'th column of P^T, the latter of which is simply the j'th row of P.

   For the identity matrix, each row dotted with itself gives 1, while no two (different) rows have a non-zero dot product - these properties are not changed when we swap the rows, so c_{ij} is 1 when $i = j$, and zero whenever $i \neq j$. So, we see $C = I$.

   (c) Using (b), we see $P^{-1} = P^T$.

   Note: This exercise says a permutation matrix is *orthogonal*: $PP^T = P^T P = I$.

7. (a) Give examples of non-zero (meaning: not all entries zero) 2×2 and 4×4 matrices A, one of each, such that $A^2 = O$ (recall O means the zero matrix). Hint: You only need to use one 1, and the rest of the entries can be 0's!

   (b) Are there any invertible $n \times n$ matrices A such that $A^2 = O$?

   Solution:

   (a) In both cases, putting a 1 in the top right corner and the rest of the entries to 0 works.

   (b) No. Since then $A = A^{-1}A^2 = A^{-1}O = O$.

8. Given the three vectors $\mathbf{a}_1 = (1, 2, 3)$, $\mathbf{a}_2 = (1, 0, -1)$ and $\mathbf{a}_3 = (0, 0, 1)$, find (if possible) numbers x_1, x_2 and x_3 such that:

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

   Your solution should involve Gaussian elimination on $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix}$ (the matrix with $\mathbf{a}_i$'s as columns).

   Solution:

$$\boxed{\text{The answer is: } x_1 = 1/2,\ x_2 = 1/2 \text{ and } x_3 = 0.}$$

9. (a) Using MATLAB, perform the matrix products A^2, A^3 and A^6 of the following lower-triangular matrix:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 7 & 2 & 0 & 0 \\ 5 & 1 & 3 & 0 \\ 3 & 2 & -1 & 4 \end{bmatrix}$$

(b) Explain the rule for *diagonal* entries of A^k, for a lower-triangular matrix A.

(c) Guess a rule for the $(2, 1)$ entry of A^k, for a lower-triangular matrix A.

Solution:

(a) The MATLAB output looks like this:

```
>> A^2

ans =

     1     0     0     0
    21     4     0     0
    27     5     9     0
    24    11    -7    16

>> A^3

ans =

     1     0     0     0
    49     8     0     0
   107    19    27     0
   114    47   -37    64

>> A^6

ans =

           1           0           0           0
         441          64           0           0
        3927         665         729           0
        5754        2681       -3367        4096
```

(b) For a lower-triangular (or upper-) matrix A, the rule

$$(A^k)_{ii} = (a_{ii})^k$$

holds.

(c) Deriving is maybe better than guessing? Let us for brevity write $b_k = (A^k)_{21}$. Hence $b_1 = a_{21}$. Since $A^k = AA^{k-1}$ we compute that:

$$b_k = (A^k)_{21} = a_{21}a_{11}^{k-1} + a_{22}(A^{k-1})_{21} = b_1 a_{11}^{k-1} + a_{22}b_{k-1}.$$

Baby case. If we had $a_{22} = 1$, we could more easily see what would happen:

$$b_k = b_{k-1} + b_1 a_{11}^{k-1}.$$

Thus we have $b_3 = b_2 + b_1 a_{11}^2 = b_1 + b_1 a_{11} + b_1 a_{11}^2$ and so on, leading to:

$$(A^k)_{21} = b_k = b_1 \sum_{s=0}^{k-1} a_{11}^s = b_1 \frac{1 - a_{11}^k}{1 - a_{11}} = a_{21}\frac{1 - a_{11}^k}{1 - a_{11}}.$$

In the second-to-last equality we used the sum formula for a finite geometric series, valid when $a_{11} \neq 1$ (we leave the case $a_{11} = 1$ to the reader!).

General case. Note that we can reduce to the special case by scaling: We let $C = \frac{1}{a_{22}}A$ (and leave the special case $a_{22} = 0$ to the reader!). Then, using our formula above (that works since $c_{21} = 1$) we see:

$$(A^k)_{21} = a_{22}^k (C^k)_{21} = a_{22}^k c_{21} \frac{1 - c_{11}^k}{1 - c_{11}} = a_{22}^{k-1} a_{21} \frac{1 - (\frac{a_{11}}{a_{22}})^k}{1 - \frac{a_{11}}{a_{22}}}.$$

Thus, we finally see:

$$\boxed{(A^k)_{21} = a_{21}\frac{a_{22}^k - a_{11}^k}{a_{22} - a_{11}} \quad (\text{when} \quad a_{11} \neq a_{22})}$$

CHECK: For example, in the above MATLAB output,

$$(A^6)_{21} = 7\frac{2^6 - 1^6}{2 - 1} = 441. \quad \checkmark$$

10. A chemistry professor claimed on live TV that he could, by mixing, obtain *any* wine with given contents of water (W), sugar (S) and tannic acid (T), labelled by vectors $w = (W, S, T)$ such that $W + S + T = 100\%$. Due to a lack of research funding, his stock was quite limited:

    - Laboratory water supply: $w_1 = (100, 0, 0)$.
    - Budget wine: $w_2 = (50, 0, 50)$.
    - Plum tea concentrate: $w_3 = (30, 50, 20)$.

    (a) If a Chateaux Bordeaux 1915 has $(W, S, T) = (45, 50, 5)$, why was the professor *not* able to obtain this wine by mixing w_1, w_2, w_3? Explain by computing the mixing ratios needed (by MATLAB or by hand).

    (b) Help the professor restore honor, by adding any new wine w_4 that will enable him to make the Chateaux Bordeaux 1915 (a Chateaux Bordeaux 1915 not allowed!).

(c) Are the mixing ratios unique after addition of the fourth wine?

Solution:

(a) The result is $(W, S, T) = (3/10, -3/10, 1)$. Since you would need to be able to *subtract* an amount of *Plum tea concentrate*, which is physically intractable, there is no mixing that will work.

(b) We can for example pick $w_4 = (40, 60, 0)$ (note that it sums to 100%, hence is an admissible wine).

The wine matrix $A = [w_1 \quad w_2 \quad w_3 \quad w_4]$ then reads:

$$A = \begin{bmatrix} 100 & 50 & 30 & 40 \\ 0 & 0 & 50 & 60 \\ 0 & 50 & 20 & 0 \end{bmatrix}.$$

But we can now also forget entirely about, say, the second wine w_2 (see the Figure 1 on the last page of these solutions), and consider instead the square matrix $A_2 = [w_1 \quad w_3 \quad w_4]$ which is:

$$A_2 = \begin{bmatrix} 100 & 30 & 40 \\ 0 & 50 & 60 \\ 0 & 20 & 0 \end{bmatrix}.$$

Using Gauss elimination on $[A_2 \mid \mathbf{b}]$ to solve $A_2\mathbf{x}_2 = \mathbf{b}$, where $\mathbf{b} = (45, 50, 5)$, we find:

$$\mathbf{x}_2 = \begin{bmatrix} 1/8 \\ 1/4 \\ 5/8 \end{bmatrix}.$$

Note that all the solution's entries automatically sum to 1.

(c) No - in this situation, $A\mathbf{x} = \mathbf{b}$ will have infinitely many solutions, and also infinitely many solutions that are admissible (i.e. have positive entries).

Later, after a few more weeks of 18.06, you will know how to obtain the complete solution to $A\mathbf{x} = \mathbf{b}$. We record it here for insight, and later reference:

$$\mathbf{x} = \begin{bmatrix} 1/8 \\ 0 \\ 1/4 \\ 5/8 \end{bmatrix} + s \begin{bmatrix} -7/12 \\ 1 \\ -5/2 \\ 25/12 \end{bmatrix}, \qquad s \in \mathbb{R}.$$

Note that all these sum to 100%. Here we can in fact pick any s in the interval $s \in [0, 3/14]$ and still have non-negative entries in $\mathbf{x}$.

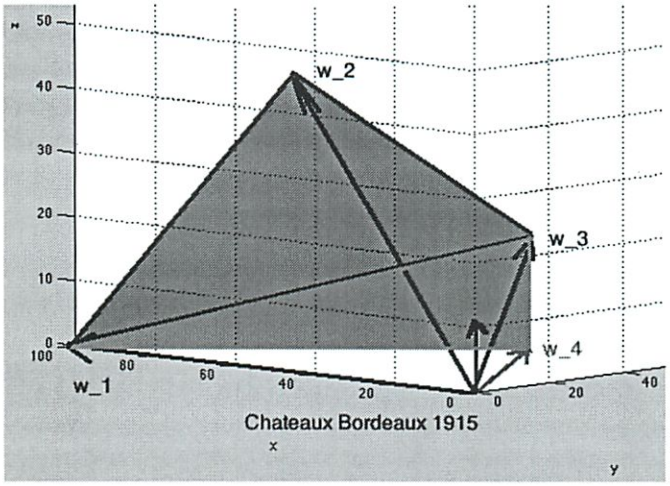


Figure 1: Problem 10. The grey and salmon-colored triangles are subsets of the plane $x + y + z = 100$ (i.e. admissible wines) with only positive mixing amounts of the w_i's.

<u>18.06 L4</u> OCW

2/15

(watching on OCW since missed lecture)
└ 2005

How to get inverse $A \cdot B$

Product will be elimination matrices

Gausian elimination

$$A = L U$$

↑ elimination matrix

Suppose A, B invertable → what is $(AB)^{-1}$
└ big question if invertable

Multiply in reverse order

$$AB(B^{-1}A^{-1}) = I$$

Can do multiplications any way we want

$$B^{-1}A^{-1}AB = I$$

If transpose

~~A~~ ↓

②

Well $AA^{-1} = I$

$I^T = I$

$(AA^{-1})^T = I = (A^{-1})^T A^T$

Need to transpose in reverse order

If you transpose a matrix → what is the inverse of the result?

$(A^{-1})^T = (A^T)^{-1}$

$A = LU$

L most basic factorization of a matrix

think about the underlying algebra - not just row operations

How is A related to U

There is a matrix L that converts them

$$A = \begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix}$$

↑ not 4 - so 2nd matrix is not singular - since wouldn't have 2nd Pivo

③

So operate on it w/ our 2nd matrix

$$E_{21} = \begin{bmatrix} \quad \end{bmatrix}$$

So then

$$E_{21} \quad A \quad = \quad U$$

$$\begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$$

produces a 0 here

$$A \quad = \quad L \quad U$$

$$\begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$$

want E_{21}^{-1}
easy to do
just flip the sign

Why L?
U = upper triangular - pivots on diagonal
L = lower triangular
- 1s on the diagonal

Can also write

$$= \overset{L}{\begin{bmatrix}1 & 0\\ 4 & 1\end{bmatrix}} \overset{p \leftarrow \text{pivot}}{\begin{bmatrix}2 & 0\\ 0 & 3\end{bmatrix}} \overset{U}{\begin{bmatrix}1 & 1/2\\ 0 & 1\end{bmatrix}}$$

now more balanced ↑ 1s also on the diagonal here

Larger Matrices

With 3x3 there is a more significant difference

~~$E_{21}A$~~

$E_{32}E_{31}E_{21}A = U$ (w/ typical case: no row exchanges)

↓

$$A = \underbrace{E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}}U$$

$$= \underset{\uparrow \text{nicer}}{L}U$$

If $E_{21} = \begin{bmatrix}1 & 0 & 0\\ -2 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$

$E_{31} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & -5 & 1\end{bmatrix}$

↑ done need

Then $E_{31}E_{21} = \begin{bmatrix}1&0&0\\-2&1&0\\10&-5&1\end{bmatrix}$

↙ nothing moves upward like in Gauss Jordan

? Why is the 10 there?

its right

but row 1 effected row 3

Now inverses

↳ remember in reverse order

$$E_{21}^{-1} \quad E_{32}^{-1} = L$$

$$\begin{bmatrix}1&0&0\\2&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&5&1\end{bmatrix} = \begin{bmatrix}1&0&0\\2&1&0\\0&5&1\end{bmatrix}\begin{matrix}\\ \leftarrow \text{from } E_{21}\\ \leftarrow \text{from } E_{32}\end{matrix}$$

just flip sign

? the rows just fint into L multiplier

$EA = U$

$A = LU$

<u>A=LU</u> If no row exchanges, multipliers go directly into L

? how to look at elimination

⑥

If doing it right you can throw away A as you create LU

When you've created a new row of U and multipliers

you can forget A – its all in LU

* new insight in elimination

How many operations on a $n \times n$ matrix A?

1000×1000

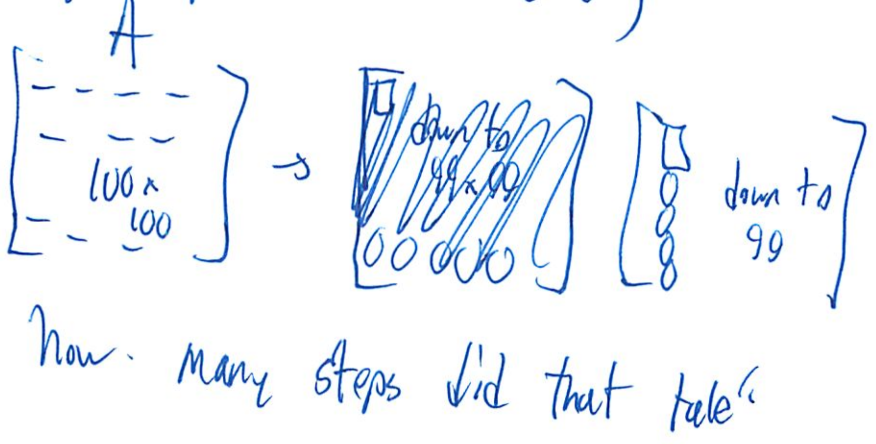
– minute?

– hour?

How much more expensive as $n \uparrow$

Say $n = 100$.

How many operations are we doing



how many steps did that take?

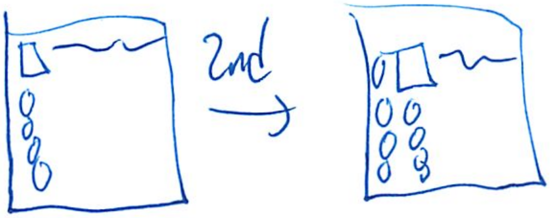
⑦

1 operation = + − × ÷

count × then − as <u>one</u> op

So this cost $\sim 100^2$

- Thats how many #s are there



This costs ~99

So this pattern repeats

$$\sum_{m=1}^{100} (n-m)^2$$

$$100^2 + 99^2 + \dots + 1^2$$

$= \text{Order of } N^3$

$\approx \frac{1}{3} N^3$ on A

↑ we can tell since $\int x^2 \approx \frac{1}{3} n^3$

⑧

How about extra column vector B?

- just 1 column
- elimination
- back sub

= n^2

Most fund. algorithm for elim

- what if there were row exchanges
  - when 0s in row position
- transposes
- permutations

<u>Permutations</u>

The matrices we need to do row exchanges

What are all the 3x3 permutation matrix?

$$\underset{\text{identity - no rows}}{\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}} \quad \underset{P_{12}}{\begin{bmatrix} & 1 & \\ 1 & & \\ & & 1 \end{bmatrix}} \quad \underset{P_{13}}{\begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix}} \quad \underset{P_{23}}{\begin{bmatrix} 1 & & \\ & & 1 \\ & 1 & \end{bmatrix}} \quad \underset{\text{cycle}}{\begin{bmatrix} & 1 & \\ & & 1 \\ 1 & & \end{bmatrix}} \quad \begin{bmatrix} & & 1 \\ 1 & & \\ & 1 & \end{bmatrix}$$

⑨

~~Each thing is its own inverse~~

$$P^{-1} = P^{T}$$

Some (not all) are their own inverse

4x4 → 24 Permutation matrices

Will use in next (OCW) lecture

18.06 Transpose

2/17

(Subsitute Prof: Niels)

└ The TA + Course Admin

Today Transpose

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}^T = \begin{bmatrix} 3 & 2 & 1 \end{bmatrix}$$

$$\begin{array}{cc} & n \\ m & \begin{bmatrix} 1 & -1 \\ 2 & 4 \\ 3 & 0 \end{bmatrix}^T \end{array} = \begin{array}{cc} & m \\ n & \begin{bmatrix} 1 & 2 & 3 \\ -1 & 4 & 0 \end{bmatrix} \end{array}$$

$(A^T)_{ij} = A_{ji}$ for each entry (def)

Rule: $(AB)^T = B^T A^T$

(saw on Wed's OCW lecture)
(coverage must have shifted...)

Why:

$$((AB)^T)_{ij} = (AB)_{ji} = \sum_{k=1}^{n} \underset{\uparrow}{A_{jk}} B_{kj}$$

^ where you finger runs as you do the motion

← list of #s getting summed up

$$= \sum B_{kj} A_{jk}$$

$$= \sum (B^T)_{ik} (A^T)_{kj}$$

Def Symmetric $A^T = A$

└ A must be square

example $A = \begin{bmatrix} 1 & -1 & 3 \\ & 4 & 2 \\ & & 0 \end{bmatrix}$

↑ can only fill in from here one way

must be symetric over the diagonal

$= \begin{bmatrix} 1 & -1 & 3 \\ -1 & 4 & 2 \\ 3 & 2 & 0 \end{bmatrix}$

[illegible]

$= A^T$

For any A, then $A^T A$ is symetical

$(A^T A)^T = A^T A^{TT}$

↑ mechanically apply rule

↓ note $A^{TT} = A$

$= A^T A$

So symmetrical

③

(Normal prof is back)

We're cleaning up chap 2

Remember $(AB)^T = B^T A^T$

We'll see $A^T A$ a lot

└ and don't forget its symmetric

If 2 symmetric matrices A, B - both symmetric

$A + B \rightarrow$ Symmetric

$A \circ B \rightarrow$ not usually symmetric → note $BA \neq AB$ most of the time

Could be if $A = B^T$

test by taking transpose $(AB)^T = B^T A^T$

$= B A$

Would be symmetric when $AB = BA$

A^{-1} ← if invertable

→ Yes Symmetric

We know $AA^{-1} = I$

Take transpose both sides

$(AA^{-1})^T = I^T = I$

$(A^{-1})^T A^T = I$

④

$(A^{-1})^T$ is the inverse of $A^T = A$

$(A^{-1})^T = A^{-1}$

key point about transposes that connects to inner products

inner product $x^T y = y^T x$

defn of dot product

$= x_1 y_1 + x_2 y_2 + \dots + x_n y_n$

~~Example~~

$$\begin{bmatrix}1 & 2 & 4\end{bmatrix}\begin{bmatrix}2\\1\\3\end{bmatrix} = 16 = \begin{bmatrix}2 & 1 & 3\end{bmatrix}\begin{bmatrix}1\\2\\4\end{bmatrix}$$

$x^T x = x_1^2 + x_2^2 + \dots + x_n^2$

$= 21$

Its the size of the vector

$= \|x\|^2$

↑ the <u>norm</u> of x

↑ in Matlab

⑤

So

$$\begin{bmatrix}1 & 2 & 4\end{bmatrix} \xrightarrow{\text{length}} \sqrt{21}$$

$$\begin{bmatrix}2\\1\\3\end{bmatrix} \rightarrow \sqrt{14}$$

So

$$\|x\|\,\|y\|\cos\theta = X^T y$$

↑ dot product tells us the angle b/w x and y $= \theta$

~~So here $\cos\theta = 1$~~

~~↑ it~~

If $x = y$

└ the same

$\cos\theta = 1$

$\theta = 0$

If $x \cdot y = 0$

Then perpendicular

$\cos\theta = 0$

$\theta = 90°$

ⓔ

So $\|x\| \|y\| \geq |x^T y|$

$\sqrt{21}\ \sqrt{14} \geq 16$

Since $\cos\theta \leq 1$

$21 \cdot 14 > 16 \cdot 16$

$294 > 256$

Been discovered 3 times

Schwarz Inequality

Coschi - French (spelling way wrong)

Bunoschari - Russian

What abat dot product of

Ax w/ y

$(Ax)^T y$

examples Work in Physics

Groceries ↓

⑦

$$\begin{array}{ccc} \lbrack\ 1 & 3 & 4\ \rbrack \\ \uparrow & \uparrow & \uparrow \\ \text{cereal} & \text{bread} & \text{milk} \end{array}$$

is $[\$4 \quad \$2 \quad \$70] = \$290$

quantity · price = total price

Now bring Matrix A into it

$$(Ax)^T y = (x^T A^T) y$$

Now dot product of x with $A^T y$

$$= x^T (A^T y)$$

↑ true reason to transpose a matrix

moving A off left and putting on right we get a transpose matrix

Suppose $\frac{d}{dt} x(t) \; y(t)$ ← moved up to functions

Inner product 2 functions

$$x(t)^T y(t)$$

⑧

$x(t)^T y(t)$

$0 \leq t \leq 1$

Paralles b/w discrete + contineous are nice to see

~~$x(t)^T y(t)$~~

If $\cos t = x(t)$
$\sin t = y(t)$

$x(t)^T y(t) = \cos t \sin t$

integrate

- started w/ summation series

$\sim \int \cos t \sin t$
$0 \leq t \leq 1$

So same - we moved w/ functions

(I didn't really get this last thing...)

18.06 Vector Space

2/21

Vector space - must be able to find all linear combos

$$c\vec{u} + d\vec{v}$$

- can be matrices, functions, etc
- default in class → vector space R^n
  - ↑ column vectors of n components

- other possibilities -
  - Complex C^n
    - Complex # instead
  - Rational
    - fractions
  - Matrices $M^{m \times n}$
    - can modify w/ c, d
    - does all the things matrix space requires
  - Functions All $f(x)$
    - $0 \leq x \leq 1$
    - sin x
    - cos x
    - etc

- Will talk about dimensions latter

Is - All functions in certain range
- All matrices of a possible size

Subspace

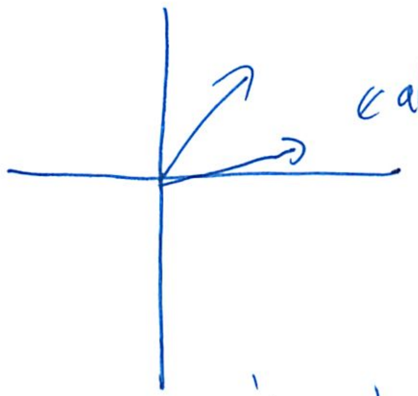
Some vector space inside a bigger space

$S \subseteq R^n$

S is a part of R^n
Could be whole R^n space

By itself its a vector space

Say R^2 space



← all vectors w/ constant components

- is sub ✓
- is it a vector space X
  ↳ are all combos $cu+dv$ ok
  ↳ stay in the space

③

No since if multiply by -1, you're out of the spae

How about?

Need some vectors whose sum is not in the space

$$\begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} -2 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

No - not a vector space

Can't construct w/ $\overset{\text{sum}}{\text{inequalites}}$

What vector belongs to every vector space?

$\begin{bmatrix} 0 \end{bmatrix}$ and every vector subspace

Biggest subspace of $R^3 = R^3$

Next Plane through $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Next line " "

Finally Zero vector alone → subspace Z
(not empty, has 0 vector

④

Subspaces of 3×3 Matrices $M^{3\times3}$

Some examples

Symetric matrices

- Check: add some → still symetric?
  multiply by scalar → " " ?
- how many dimensions? 6

Diagional matrices

- how many dimensions? 3

How do you get Subspaces

Column Space of $A = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 5 & 7 \\ 3 & 4 & 7 \end{bmatrix}$

inside $\mathbb{R}^3$

↑ columns are in $\mathbb{R}^3$

↑ a 3×3 matrix

The smallest subspace containing all the columns of A

Must pass $cv+dw$ test

Need a bunch of vectors where possible to take a subspace?

Note This A is dep

- So don't need 3rd column

What is the subspace?

Take their linear combo

Smallest subspace w/ all cols of A → all linear combo of the columns

So for

Can fix by including all vectors we need

So this is $\mathbb{R}^2$

Is a vector space

So here start w/ cols. Take all combos

Get subspace. ~~D~~

It can't be smaller - or won't be a subspace

⑥

It would be a plane R^2

If $A_{33} = 17$, not 7

Now its R^3

Since every vector is a combo of those 3 combos

Means matrix is invertable

$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ in R3 ← any one is a combo of the above

Aka

$$\begin{bmatrix} 1,4,5 \\ 2,5,7 \\ 3,4,17 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \text{ in } R3$$

$Ax = b$

Always has a solution for every b

$\boxed{\text{When } C(A) = \text{Whole space in } R}$ Every column is a combo of cols in A

It didn't work w/ $A_{33} = 7$ - was only R^2

Add. Subspaces of $M^{3\times 3}$

Upper triangular matrices dimension = 6

$$\begin{bmatrix} - & - & - \\ 0 & - & - \\ 0 & 0 & - \end{bmatrix}$$

Suppose you put all these spaces together

Diagional matrices are a subspace of both symmetric and upper triangular matrices

~~Every matr~~

$$\underset{\substack{\uparrow \\ \text{every} \\ \text{matrix}}}{M} = \underset{\substack{\uparrow \\ \text{sym.}}}{S} + \underset{\substack{\uparrow \\ \text{upper triangular}}}{U}$$

(he said figure out a math q/ w/ a specific example)

So a specific matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} & 4 & 7 \\ 4 & & 8 \\ 7 & 8 & \end{bmatrix} + \begin{bmatrix} & -2 & -4 \\ & & -2 \\ 8 & 0 & \end{bmatrix}$$

⑧ Many possible sols.

Could do

$$\begin{bmatrix} 1 & 4 & 7 \\ 4 & 5 & 8 \\ 7 & 8 & 9 \end{bmatrix} + \begin{bmatrix} 0 & -2 & -4 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

All upper triangular $\cap$ all symmetric

$\uparrow$ intersect

($\longrightarrow$

both are subspaces

What are intersection (are in both?)

= diagonal

$\uparrow$ is a subspace

Dimensions $6 + 6 = 12$

too many - in 9 space

So 3 left over = diagonal

(very confused!)

C(A) = Col space of a matrix

⑨

$S + \underline{\underline{\quad}} = R^3$

↑ strictly upper triangular (0s on the diagonal)

$\begin{bmatrix} 0 & - & - \\ 0 & 0 & - \\ 0 & 0 & 0 \end{bmatrix}$ now dimension = 3

Now every $M = S + U$

↑ strictly upper triangular

$A = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 3 & 2 & 3 & 7 \\ 4 & 0 & 4 & 11 \end{bmatrix}$

Is $C(A) = R^3$?

First is $\begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$ - is in col space - keep (pivot col)

Second as well

Third - we already have - don't keep it

$\begin{bmatrix} 1 \\ 13 \\ 4 \end{bmatrix}$ is also a combo $1\begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} + 5\begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$ - don't keep

Fourth - keep

⑩

How can we change a col so smaller than R^3

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 3 & 2 & 3 & 7 \\ 4 & 0 & 4 & 8 \end{bmatrix}$$

✓ ✓ x X

↑ is a combo of what we've got

So $C(A) \neq R^3$ ↑

$= $ Plane in R^3

↑ Stay in R3 if in R^3

HW has some permutation matrices w/ combos

18.06 Lecture

2/22

$$\left.\begin{array}{l} b \text{ in the col space } C(A) \\ \quad\Updownarrow \\ b \text{ is a combo of the columns} \\ \quad\Updownarrow \\ Ax = b \text{ has (1 or more) solutions} \end{array}\right\} \text{Core idea}$$

Today: Nullspace $N(A)$ = all sols to $Ax = 0$ subspace of R^n

---

$C(A) \subseteq R^n$

$$A = m\begin{array}{c}\begin{bmatrix} | & | & | \end{bmatrix} \\ 1 \quad n \quad n\end{array} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} n = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

but null space → look at x's

want

$$Ax = 0$$

So why a subspace of R^n?

$Ax = 0$

$Ay = 0$; true that $A(x+y) = 0$

↳yes since $Ax + Ay = 0$

$0 + 0 = 0$ ☺

~~check? A(cx)~~

$A(cx) = c(Ax) = c0$

$Ax = 0$

Today: Figure out what's in a subspace

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 2 & 6 & 5 \end{bmatrix}$$

Always contains 0 vector

is anyone else in that null space

Do elimination

$$\begin{bmatrix} 1 & 3 & 3 \\ 0 & 0 & -1 \end{bmatrix}$$

↳ 2 3s from 5

? 2 3s away from 6

But the 0 in 2,3

Call col 1 → a pivot col

∟ since has a pivot in it

Col 2 → non pivot col aka free col

∟ since no pivot

Col 3 → pivot col

We can still call this U

- on + above diagonal

$$\begin{bmatrix} 1 & 3 & 3 \\ 0 & 0 & -1 \end{bmatrix}$$

Can go further w/ elimination

Can make pivot 2,3 +1

Why switch?

Since $Ax = 0$

$Ax = 0$ $\quad x_1 + 3x_2 + 3x_3 = 0$

$+ x_3 = 0$

④

Can change all the pivots to 1
  └ dividing by the pivot

Or clean out col they are in

$$\hookrightarrow \begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} = R \leftarrow$$

Can't reduce any further

reduced row echelon form
$R = rref(A)$
↑
matlab

~~R = rref(B)~~

Have the identity in the pivot col

$$\begin{bmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \begin{matrix} 3 \\ 0 \end{matrix} & \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{bmatrix}$$

Ready to figure null space

$$x_1 + 3x_2 = 0$$
$$x_3 = 0$$

Looking for sols now
- ∞ ly many sols
- x_2 is free - can give <u>any</u> value

⑤

Steps

1. Give free variables any values
2. This decides the pivot variable

X in the null space

$$X = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} \leftarrow \text{free variable - assign 1 to get "special sol"s}$$

One of many vectors in the null space

$$x = c\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$$

— Can call it either

$$x = x_2\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$$

This subspace is a line

 ∟ all multiples give line

* How many special sols?

 ∟ # of free cols

⑥

↓ pivots

$$P = \begin{bmatrix} \textcircled{1} & 3 & 0 & 0 & ? \\ 0 & 0 & \textcircled{1} & 0 & ? \\ 0 & 0 & 0 & \textcircled{1} & ? \end{bmatrix}$$

we don't know these → assume $\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$

know this is 0
since echelon

cols 1 3 4 = pivot cols

2 5 = free cols

R has 3 pivot cols → the rank of the matrix

└ imortant property of matrix

there
3x5 matrix
rank = 3
└ "full row rank"

~~A~~ rank ≤ # of rows

Now special sols to
$Rx = 0$
$Ux = 0$
$Ax = 0$
nullspace
} same

$$x = \begin{bmatrix} 1 \\ \\ 0 \end{bmatrix}$$

fill in free variables
now solve for pivot cols
└ are determined

⑦

$$x = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{array}{l} \\ \leftarrow \text{free } x_2 \\ \\ \\ \leftarrow \text{free } x_5 \end{array}$$

Now a diff sol

$$x = \begin{bmatrix} -4 \\ 0 \\ -5 \\ -6 \\ 1 \end{bmatrix} \begin{array}{l} \\ \leftarrow \text{free} \\ \\ \\ \leftarrow \text{free} \end{array}$$

← note the sign switch b/w matrix + special sol

What is in the null space here?

linear combonation ← the magic words

$$c_1 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -4 \\ 0 \\ -5 \\ -6 \\ 1 \end{bmatrix}$$

$$= x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -4 \\ 0 \\ -5 \\ -6 \\ 1 \end{bmatrix}$$

2 Dimensional subspace

⑧

Pratice

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \rightarrow R = ? \rightarrow \text{Special sol}$$

So elim

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ \multicolumn{4}{c}{\cancel{\text{2nd pivot}}} \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad ✓$$

? no pivots ← confused

So r is

$$\begin{array}{cccc} \downarrow \text{pivot} & \swarrow f & f & \text{free } ✓ \\ \hline 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}$$

how special sol I don't know how to get

$$\text{free} \begin{matrix} \\ \rightarrow \\ \rightarrow \\ \rightarrow \end{matrix} \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix} \leftarrow \text{but how do you get this I didn't see}$$

Are there more?

Special sols ← solve to find pivot variables

3 special sols since 3 free cols

$$\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

← I was kinda thinking this close for never seeing this

Now all linear combos of this $c_1 \begin{bmatrix}-1\\1\\0\\0\end{bmatrix} + c_2 \begin{bmatrix}-1\\0\\1\\0\end{bmatrix} + c_3 \begin{bmatrix}-1\\0\\0\\1\end{bmatrix}$

rank = 1

special sols = 3 = dim. of null space

$$= \underset{\#\text{cols}}{\overset{\uparrow}{n}} - \underset{\#\text{pivots}}{\overset{\uparrow}{r}}$$ ✱ key fact

So

$$3 = 4 - 1$$

Practice 2

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \end{bmatrix}$$

elim

$$\begin{array}{c} \\ 21 \\ 31 \end{array} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & -2 & -4 & -6 \end{bmatrix}$$ ✓

$$23 \quad \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ he got $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ I think he was able to switch signs here

⑩

$$A = \begin{array}{c} \begin{array}{cccc} \text{pivot} & & \text{free} & \\ \downarrow & \downarrow & \downarrow & f \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

(See what I got wrong)

$r = 2$

$n - r = 2$

So 2 special sols

$$x = c_1 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} \leftarrow \text{free cols}$$

null space is all linear combos of this

18.06 Reading

2/21

Chap 3 Vector Spaces + Subspaces

Matrix calculations involve vectors

Ax, AB } linear combos of n vectors → The Columns of A

Vector spaces

$\mathbb{R}^1, \mathbb{R}^2, \mathbb{R}^3, \mathbb{R}^4, \ldots$

each $\mathbb{R}^n$ contains a whole collection of vectors

Consists of all column vectors v w/ n components

(How do you visualize ?)

Components of v are real #s → $\mathbb{R}$

$\mathbb{C}$ = Complex space

$\mathbb{R}^2$ = x, y plane

- each vector v in $\mathbb{R}^2$ has two components
- "Space" thinks about all vectors - whole plane
- each vector gives the x, y coord ~~ax~~ of a pt in the plane $\vec{v} = (x, y)$

$\mathbb{R}^3$ → in 3D space

$\mathbb{R}^1$ → line (like the x axis)

$\begin{bmatrix}4\\ \pi\end{bmatrix} \to \mathbb{R}^2$

$(1, 1, 0, 1, 1) \to \mathbb{R}^5$

Oh so $\mathbb{R}^2$ is any point in the plane

↑ and thus ~~any~~ the vector from origin to that pt

So any vector

So basically $\begin{bmatrix}x\\y\end{bmatrix}$ w/ any x, y

Not just $c\begin{bmatrix}1\\1\end{bmatrix}$ ← this is a line

↑ any c ↑ fixed x, y

We can add any vectors in $\mathbb{R}^n$ and multiply vector v by a scalar

↳ result stays in the space

$w + v = v + w$

$c(v + w) = cv + cw$

$0 + v = v$

③

A real vector space is set of vectors together w/ rules for vector add. and multiplication by real #s

8 conditions
(not really listed)

Other vector spaces
(covered in class)

Z = zero dimensional

Subspaces

The vectors we need most are the ordinary col vectors
These have n components
are the impotant vectors inside $\mathbb{R}^n$

Start w/ $\mathbb{R}^3$
Choose a plane through (0,0,0)
└ That plane is a vector space
if add 2 vectors - their sum is a plane

* A plane in 3D space is not $\mathbb{R}^2$

└ but it looks like it!

If vector have 3 components → $\mathbb{R}^3$

plane is a vector space <u>inside</u> $\mathbb{R}^3$

(just think about things this way)

* A subspace of a vector space is a set of vectors (including $\vec{0}$) that satisfies 2 requirements:

If v and w are vectors in the subspace and c is any scalar then

i) $v + w$ is in the subspace

ii) cv is in the subspace

Basically all linear combos stay in the subspace

Facts

1. Every subspace contains the zero vector
2. Lines through the origin are also subspaces
3. All of $\mathbb{R}^3$ = a subspace

↓

⑤

Possible subspaces of $\mathbb{R}^3$

$\vec{L}$ Any line through (0,0,0)

$\vec{P}$ Any plane through (0,0,0)

$\mathbb{R}^3$ The entire space

$\vec{Z}$ The zero vector

Not subspaces

Quarter plane

two quarter planes

(this is straight out of lecture!)

Two subspaces of M 2x2

$\vec{U}$ All upper triangular matrices $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$

[illegible]

$\vec{D}$ All diagonal matrices $\begin{bmatrix} d & 0 \\ 0 & d \end{bmatrix}$

Since add any two matrices in $\vec{U} = \vec{U}$

(I don't fully get all the rules
P-Set will prob explain
Too bad no recitation this week...)

Column Space of A

$Ax = b$

If A not invertable, System only solvable for certain bs)
The "Good" bs form the column space of A.

(What is the point of all this?
Maybe that is why I do poorly in Math...)

Ax = combo of cols of A

to get every possible b, we use every possible x

So take all ~~combos~~ linear combos of A's cols
L Produces col space of A
L A's vector space made up of col vectors

* C(A) contains not just the n cols of A,
but their linear combos Ax *

aka

The col space consists of all linear combos of the cols.
The combos are all possible vectors Ax
They fill col space $C(A)$

⑦

To solve $Ax = b$ is to express b as a combo of the columns

The system $Ax=b$ is solvable iff b is in the col space of A

$A = m \times n$ matrix

↳ Cols have m components so R^{m}

$S =$ set of vectors in V

$SS =$ all combos of vectors in S

$= \text{all } c_1 \vec{V}_1 + \ldots C_N \vec{V}_N$

$=$ The subspace of V spanned by S

~~When~~ Always SS is smallest subspace containing S

The subspace SS is the "span" of S containing all combos of vectors in S

Review

2/21

Symmetric $A = A^T$

Transpose Rotate cols

Inverse The long process

-1

$A A^{-1} = I$

18.06 OH

2/22

2.6 #13

Is do it the long way

2.7 # 38

Not always identity

Permutation rearrange rows

So rearrange rows

How many way can you permute n!

So if had $(n+1)!$ then 2 of them will be =

$3! = 6$

So if had 7

P^7 - 2 of them had to be =

Can be

$P^3 = P^7$ w/ some #s in 3x3 case

$(P^{-1})^3 P^3 = (P^{-1})^3 P^7$

$I = P^4$

$P^k = n+1$

That does not guarentee smallest matrix must prove $P^5 \neq I$ etc

Think of exchanging rows

②

$P \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ rearranges rows

PP " rearranges again

PPP " changes again

$P3 = I$ here

Show the smaller thing is not the identity

P^{-1} is still a permutation matrix

$(P^{-1})^3 P^3 \neq (P^{-1})^3 P^{5}_{2}$ we don't know this
which on this is
must be a n+1

$P \circ P =$ [illegible] a diff permutation

P^3

P^4

$\vdots$

P^{n+1}

) etc) One must $= P$

③

$p^{\square} = I$

$p^a = p^b \quad a < b$

$p^{b-a} = I$

9. ~~Not R^3~~ More abstract than this

~~$M \times N \to R^N$~~

$$\left[M = m \left[\quad {}_{n} \quad \right] \right]$$

dim = m·n

each element is its own space

$\sim R^{mn}$ = (long row of #s)

So if 1x3 matrix $[-, -, -] \to R^3$

Vector space → a space you can do linear combos in and its the same

Since in LA we add + scale space

So we care about the space ans is in

Not in vector space

in vector space

M is just a very high dimensional space

$$7\left[\qquad\right]_{11} \rightarrow R^{77} = (-,-,\ldots,-)$$

Want to be able to add vectors

elements in there are called vectors
- each one is a vector
- Or each matrix could be called a vector

A 3x3 matrix could be called a vector
L has 9 components
- but only in a 3x3 space

Must be able to add + multiply - still in set

Only multiply entire matrix - by scalar

⑤

Is the set of <u>all</u> 7×11 matices

Here is ~~my~~ the vecto space ↳ it satisfies

Here is my space

b) Want to find 77 matrices $M_1, \ldots, M_{77}$

So can write any matrix of linear combo of this

just 1 in each position

$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 00 & \cdots & & \end{bmatrix}$ move 1 to each pos $\begin{bmatrix} 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & \\ & & \vdots & \end{bmatrix}$

etc 77 times total

again a) If add 2 $_7\begin{bmatrix} & \end{bmatrix}_{11}$ matrix → still $_7\begin{bmatrix} & \end{bmatrix}_{11}$

Subspace - a smaller vector space

Col space - one example of a vector space

⑥

10. .5 = One half chance 1, 0

Do elim - which of 0, 1, 2 pivots

Have already made it to upper triangular

5. Permutation switches the rows

2 $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ only 2 options

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

3x3 $\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$ $\begin{bmatrix} & & 1 \\ 1 & & \\ & 1 & \end{bmatrix}$ $\begin{bmatrix} & 1 & \\ & & 1 \\ 1 & & \end{bmatrix}$ $\begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix}$

$\begin{bmatrix} 1 & & \\ & & 1 \\ & 1 & \end{bmatrix}$ $\begin{bmatrix} & 1 & \\ 1 & & \\ & & 1 \end{bmatrix}$ thats it

will have n! - diff ways to arrange 3 #s exclusivly

Can't be diff > n! From stats

Can do because matrix is invertable
- otherwise not

9b) (TA does not like q^)

∞ly many of them!

any lines that go through matrix

Symetric - more restrictions

Vector space

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

R^3 can be vector space - since 3 components

can - add
- multiply by scalar

So R^3 is legit vector space

Col space is a subspace of R^3

look at each col as a vector

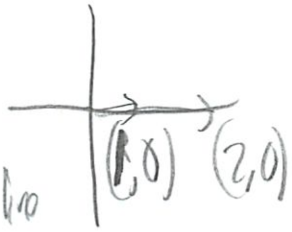
← col space = smallest vector space that contains [illegible]

⑧

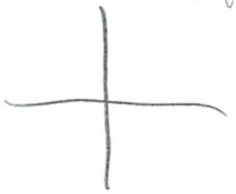
$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$ vector space

↑ just R^2 – 2 components

Col space → is just the line



If $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ then vector space } both R^2
col space



Null space

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$x_1 = 0$
$x_2 = 0$ just the origin is null space

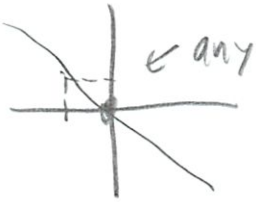
⑨

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$x_1 + 2x_2 = 0$$

$$0 = 0$$

any x_1, x_2 that satisfys is in null space



← any vector that falls on line = null space

r = rank = # col of pivots

6 more focus) $M_1, \ldots\ldots, M_4$

R^6

- What are all subspaces of R^6

dim 6 → R^6

dim 5 → hyperplane - plane 1 dimension less then the Space its in

dim 4 → the smallest 4D subspace

~~th~~ 4 ind vector that <u>span</u> the vectors

even though looks like 4 components - it actually has 6

⑩

So 3x3 matrix

How many free matices there are?

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$$

~~need to put 2 somewhere else - in 2 places b/c Symetric~~

Symetic

So howmany are free? 6

- since 3 are ~~a~~ fixed by requiring symetric

$V_1, V_2, V_3, \dots V_6 \in \mathbb{R}^6$

So any $V = a_1 V_1 + \dots + a_6 V_6$

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

So convient bases for earlier matices

⑪

So do for each pivot

$$M = a_1 M_1 + \dots + a_6 M_6$$

$$\begin{bmatrix} 1 & \\ & 0 \end{bmatrix} + etc$$

So describe vector space

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ ← since these two are fixed

For qv I use this one - Matrix qv

Writing as $a_1 V_1$ as more basic, alt form

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$S_{3 \times 3}$ ← space 3,3

$= \{(a_1, \dots, a_6)\}$

can write as parameters

this here is symetric

because of our function

Need to explain why

By showing M

2/22

Remember

$$L = \begin{bmatrix} 1 & & \\ l_{21} & 1 & \\ l_{31} & l_{32} & 1 \end{bmatrix}$$

U = ans from elim

$$A = L \cdot V$$

Reading

2/22

p112 Textbook

$A = LU$

$= (E_{21}^{-1} \dots E_{ij}^{-1} \dots) U$

↑ inverted elim step

brings U back to A

$L =$ has 1s on diagonal

This does not always work

need row ex sometimes needed to make pivot

Compress all into single P

So get factorization for every invertable A

Can do

① in advance

$PA = LU$ ← we focus on

② after of elim

$A = L_1 P_1 U_1$

the pivot rows are in a strange order

P_1 puts them in correct triangular order in U_1

we focus on method 1 - but method 2 more elegant
└ both should not take much time

Most important case $P = I$ → no row ex needed

<u>Sign</u> of P tells us if row ex is even ← $\oplus$
odd # of row ex is $\ominus$

At start P is I and sign is $\oplus$
Then w/ each row ex the sign is flipped
Final value of sign = <u>determinant</u>

LU(A) to make this

Michael Plasmeier   18.06 PS2   2/21

2.6 #7 T or F?

Wrong qu

a) Block matrix is auto symmetric?

Yes $\begin{bmatrix}0&A\\A&0\end{bmatrix}^T = \begin{bmatrix}0&A\\A&0\end{bmatrix}$ True

b) If A, B sym. then A·B sym?

? Didn't we show this last time?

$$\begin{bmatrix}0&A\\A&0\end{bmatrix}\begin{bmatrix}0&A\\A&0\end{bmatrix} = \begin{bmatrix}0+A^2&0+0\\0+0&A^2+0\end{bmatrix} = \begin{bmatrix}A^2&0\\0&A^2\end{bmatrix}$$

$$\begin{bmatrix}A^2&0\\0&A^2\end{bmatrix}^T = \begin{bmatrix}A^2&0\\0&A^2\end{bmatrix}$$ Still sym true

c) If A is not sym. then A^{-1} is not sym.

Can't we say something about the rules of singular matrices

True

②

d) When A, B, C sym, $(ABC)^T = CBA$

Well we generally multiply back in reverse

But that is $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

Relevant here?

I don't think so

if $A = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$

$A^3 = \begin{bmatrix} 0 & 8 \\ 8 & 0 \end{bmatrix}$

$(A^{-1})^3 = \begin{bmatrix} 0 & 1/8 \\ 1/8 & 0 \end{bmatrix}$

Wrong q ✓

False

③

Wrong qu

9. If P_1, P_2 permutation matrices, so is P_1P_2
This sill has rows of I in some order
Give examples w/ $P_1 P_2 \neq P_2 P_1$
$P_3 P_4 = P_4 P_3$

So if $P_1 = \begin{bmatrix} 1 & & \\ & & 1 \\ & 1 & \end{bmatrix}$ $P_2 = \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix}$

$P_1 \cdot P_2 = \begin{bmatrix} & & 1 \\ 1 & & \\ & 1 & \end{bmatrix}$ $\neq$

$P_2 \cdot P_1 = \begin{bmatrix} & 1 & \\ & & 1 \\ 1 & & \end{bmatrix}$

Now for the same:
If they are the same:

$P_1^2 = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} = P_3 = P_4$

You never said $P_3 \neq P_4$!

⑦

2.7 # 38 Why are some P^k eventually $= I$?

Because $P \cdot P = I$

(I think we showed last time)

So P^3 is back to P

P^4 is back to I

But not for all - because of rest of problem

Find 5×5 P where $P^6 = I$

?smallest

$$\begin{bmatrix} & & & 1 & 1 \\ & & & 1 & \\ & & 1 & & \\ & 1 & & & \\ 1 & & & & \end{bmatrix}$$

⊗ No

Oh Throw out/

Replace

18.06 Spring 2012 – Problem Set 2

This problem set is due Thursday, February 23rd, 2012 at 4pm (hand in to Room 2-106). The textbook problems are out of the 4th edition. For computational problems, please include a printout of the code with the problem set (for MATLAB in particular, `diary('filename')` will start a transcript session, `diary off` will end one.)

1. Do Problems 7 & 9 from Section 2.6.
2. Do Problem 13 & 23 from Section 2.6.
3. Do Problem 6 from Section 2.7.
4. Do Problem 22 from Section 2.7.
5. Do Problem 38 from Section 2.7.
6. Do Problems 17 from Section 3.1.
7. Do Problem 23 from Section 3.1.
8. Do Problems 30 & 31 from Section 3.1.
9. This problem is about the vector space of matrices for a fixed number of rows and columns.
    (a) Explain carefully why the set of all 7×11 matrices forms a vector space (What is $cA + dB$? Which matrix is the zero vector?). Describe the simplest list of matrices you can think of which, allowing arbitrary linear combinations, will yield *all* 7×11 matrices. There should be 77 different matrices in your answer.
    (b) How many real number-valued parameters would you use to (unambiguously) describe the vector space $S_{3\times3}$ of 3×3 symmetric matrices (e.g. the set of all 3×3 matrices A such that $A^T = A$)? Identify *all* vector subspaces of $S_{3\times3}$ (it may be convenient to refer to the parameters you've introduced).
    (c) The 2×2 matrices with equal row sums ($a + b$ and $c + d$ are the same number), and equal column sums ($a + c$ and $b + d$), is a vector space. Find two matrices so that all these matrices are linear combinations of those two.
10. The MATLAB command `A = double(rand(2,2) < 0.5)` gives a random 2×2 matrix where each entry is either 0 or 1 (with equal probabilities).
    (a) Make a plot of the distribution of the number of pivots of the row-reduced versions (in MATLAB, the command `rank(A)` gives this number) of these random matrices. Here's some sample code that you can copy-paste into MATLAB:

```
clear; N=1000; num_zeros=0; num_ones=0; num_twos=0;
for i = 1:N
 A = double(rand(2,2) < 0.5);
 if rank(A)==2
```

```
  num_twos = num_twos + 1; %Then add one to that counter!
 end
 if rank(A)==1
  num_ones = num_ones + 1;
 end
 if rank(A)==0
  num_zeros = num_zeros + 1;
 end
end
distrib = [num_zeros num_ones num_twos]/N
bar([0 1 2], distrib, 0.1)
```

(b) Compare this to the exact probabilities of each value for the pivot number. Compute these by writing down all 16 possibilities and counting pivots.

(c) Extend the code in (a) to work for 5 × 5 matrices, and again show a histogram plot.

(d) For the 2 × 2 examples, what do you think the probability of having 2 pivots would be, if we took each matrix entry distributed continuously (and uniformly) in the *interval* $[0, 1]$? (No need to compute - but explain why!)

Michael Plasmeier

18.06 PS 2

65/100

2/21

2.6 #7 What 3 elimination matrices E_{21}, E_{31}, E_{32} put A into its upper triangular form

8/10

$$E_{32} E_{31} E_{21} A = U?$$

Umm the 3 elim matrices gotten through elimination

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 2 \\ 3 & 4 & 5 \end{bmatrix}$$

$$l = \frac{2}{1}$$

$$E_{21} = \begin{bmatrix} 1 & & \\ -2 & 1 & \\ & & 1 \end{bmatrix}$$ ✓

$$E_{31} = \begin{bmatrix} 1 & & \\ & 1 & \\ -3 & & 1 \end{bmatrix}$$ ✓

$$E_{32} = \begin{bmatrix} 1 & & \\ & 1 & \\ & -2 & 1 \end{bmatrix}$$ ✓

②

$$U = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$ ✓

Then find L 2 matlab $(E_{21})^{-1}(E_{31})^{-1}(E_{32})^{-1}$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$$ ✓

$A = LU$

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 2 \\ 0 & 4 & 0 \end{bmatrix}$$

2.6 #9 When a zero appears in a pivot pos $A=LU$ is not possible

Show why

So you would have $0 = \frac{2}{0}$ which is a divide by 0 error and thus impossible ✓

? anything else?

for which matrix?

To show this directly, you just need to multiply matrices.

−2

③

2.6 #13. Compute L, U for Symetric matrix A

8/10

$$A = \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix}$$

So $E_{21} = \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \rightarrow \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ & \vdots & & \end{bmatrix}$

since $\frac{a}{a} \rightarrow$

$E_{31} = \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 1 & & 1 & \\ 0 & & & 1 \end{bmatrix} \rightarrow \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & b-a & c-a & c-a \\ 0 & b-a & c-a & d-a \end{bmatrix}$

etc

$E_{41} \longrightarrow$

TA: need to do it the long way

$E_{32} \rightarrow \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & - & - & - \end{bmatrix}$

$\frac{b-a}{b-a} = 1$

$b-a-(b-a)$

$c-a-(b-a)$

$c-b$

③b.

$E_{42} = \frac{b-a}{b-a} = 1$

$$\begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & \boxed{c-b} & c-b \\ 0 & 0 & c-b & d-c-c+b \end{bmatrix} \leftarrow d-a-(c-b)$$

$E_{43} = \frac{c-b}{c-b} = 1$

$$\begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & 0 & \uparrow \end{bmatrix}$$

$d-a-2c+2b$

$d-a-(c-b)-(c-b)$

$d-a-2c+2b$

? Seems wrong?

yes, it's wrong!

-1

Find the 4 conditions to get the 4 pivots

$d-a-2c+2b \neq 0$

$b-a \neq 0$

$c-b \neq 0$

$a \neq 0$

And L is $(E_{43})^{-1}(E_{42})^{-1}(E_{32})^{-1}$

$(E_{41})^{-1}(E_{31})^{-1}(E_{21})^{-1}$

which is? -1

④

2.6 #23 If A has pivots 5, 9, 3 w/ no row ex

What are the Pivots for upper left 2x2

Sub matrix A_2

$$\begin{bmatrix} \boxed{\begin{matrix}5 & \\ & 9\end{matrix}} & \\ & 3 \end{bmatrix}$$

Umm 5 and 9

Well l_{31} would affect but just 3,1

So could change ?

But not given enough info ✓

So I will say 5,9

⑤.

2.7 6. $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$

4/10

$M^T = \begin{bmatrix} A & C \\ B & D \end{bmatrix}$

Under what conditions is this symetric

$\llcorner M = M^T$

When $B = C$ X

2.7 22. About $PA = LU$

3/10

$A = L_1 P_1 U_1$

Find the $PA = LU$ factorization

$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 3 & 4 \end{bmatrix}$

So need to do row ex

$P = \begin{bmatrix} & & 1 \\ 1 & & \\ & 1 & \end{bmatrix}$

⑥.

$$AP = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 3 & 4 & 2 \end{bmatrix}$$ eek! you want to swap rows, not columns!

So now elimination

$l_{21} = \frac{0}{1}$ So same

$l_{31} = \frac{3}{1}$ $$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

$l_{32} = \frac{1}{1}$ $$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = U$$

So L = inverse of this

Or a shortcut?

$$\begin{bmatrix} 1 & & \\ 0 & 1 & \\ 3 & 1 & 1 \end{bmatrix} = L$$

$\uparrow l$ $\uparrow$ all l_s

(oh I never realized this)

Test

$$LU = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 3 & 4 & 2 \end{bmatrix}$$ ✓

How about $A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 1 \\ 1 & 1 & 1 \end{pmatrix}$?

⑦

2.7 #38 If you take powers of a permutation matrix, why is some P^k eventually = to I? Find a 5x5 permutation P so smallest power to = I is P^6

10/10

~~So $P \cdot P = I$ since $P = P^{-1}$~~ $P^{-1} = P^T$

You have n! possible permutations for $n \times n$ P

$n=2$ $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $2! = 2 \cdot 1 = 2$

$n=3$ $\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & & \\ & & 1 \\ & 1 & \end{bmatrix}$ $\begin{bmatrix} & 1 & \\ 1 & & \\ & & 1 \end{bmatrix}$ $\begin{bmatrix} & & 1 \\ 1 & & \\ & 1 & \end{bmatrix}$

$\begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix}$ $\begin{bmatrix} & 1 & \\ & & 1 \\ 1 & & \end{bmatrix}$ $3! = 3 \cdot 2 \cdot 1 = 6$

So if you had $(n!+1)$ permutations then it would eventually overflow and have the same P be = to each other again.

⑧ Now find a 5×5

So $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$ possible combos

¿Can you solve besides G+V?

So not $P = P^T$

Can make 2 rows alternate w/ each other and then make 3 rows in a certain pattern.

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Then $P^6 = \underline{I}$

Can use Matlab to check.

⑨.

3.1 #17 a) Show that the set of invertable matrices in M is not a subspace

4/10

So vector space means linear combo — addable — Scalable

Subspace can be any subset, including the entire vector space, so M must not be a vector space, so not addable, scalable to some space.

Well a lot of Matrices are inertable. The rules are in the back of the book

Well is it since no free rows - so no "special" sols, thus can't build w/ linear combo of rows

⑩.

b) Show that the set of singular matices M is not a subspace

Well this is the opposite of earlier.
But must be same case where it is not always true.

There is some rank - so some free - so some special solutions.

But not always? "The set" requires that all singular matrices be linearly combinable

Can combine two singular matrices into an invertable matrix so we've exited the subspace.

While I understand (at least vaguely) your argument, there are easier ways to show this more clearly. please see the posted sol'n.

⑪

3.1 # 23 If we add a extra col b to matrix A then the col space gets larger unless ____. Give an example where col space gets larger and an example where it doesnt. Why is $Ax = b$ solvable exactly when the col space doesn't get larger - it is the same for A, $[A\ b]$?

10/10

So gets larger unless col space is a multiple of another col or linear combo of multiple cols
- Can't reach anywhere new ✓

Ax is $\begin{bmatrix} 1 & 0 \\ 4 & 3 \\ 2 & 3 \end{bmatrix}$

larger $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ ✓

not larger $\begin{bmatrix} 2 \\ 8 \\ 4 \end{bmatrix}$ ✓

Solvable since cols are linear combos of each other
- multiple special sols ✓
- can build any space

↳ sols of $Ax = b$

⑫ .

3.1 #30 Suppose S, T are two subspaces of V

6/10

a) def: Sum $S+T$ contains all sums $s+t$ of a vector s in S and a vector t in T

Show $S+S$ satisfies all the req.

: Umm since S satisfies, T satisfies, a linear combo of the two also satisfies

Satisfies what? see sol'n. ⓘ $\boxed{-4}$

b) S, T lines in $\mathbb{R}^m$ What is diff $S+T$ and $S \cup T$

: So $S+T$ is both S and T!

Span = if linear combos fill the space

So Col space is spanned by the cols

If S and T are diff lines - then
$S \cup T$ are two lines (not a subspace) ✓
but $S+T$ is the whole plane they span ✓

(: still unclear on this!) - Union them just X (cross)
- the + will span the space

⑬.

3.1 #31. If S is col space of A and T is $C(B)$ then $S+T$ is the col space of what matrix M?

$S = C(A)$ $T = C(B)$

$S+T$ is what

A, B are all in $\mathbb{R}^m$

So the unique cols from A and B
^ not multiples or linear combos of other cols
(? But that is not a challenge problem)

Basically $M = [A \ B]$ ✓

? how does that tell us anything?

⑭.

9.a) Why the sets of all 7×11 matices form a vector space.

5/10

$M = 7\begin{bmatrix} & & \\ & & \end{bmatrix}_{11}$

Any linear combo (add/multiply) still produces ~~X~~ another 7×11 matrix.

So the simplest is 77 versions of 1 in each position ⊖1

$$\begin{bmatrix}1 & 0 & \cdots \\ 0 & 0 & \cdots \\ \vdots & & \end{bmatrix} + \begin{bmatrix}0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & & & \end{bmatrix} \cdots \text{etc}$$

77 total times ✓

b) How many real # valued params to use to describe S_{3x3} Of symetric matices $(A^T = A)$

So Symetric $\begin{bmatrix}1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6\end{bmatrix}$ ✓

Subspaces? ⊖2

So can combine 6 parameters

$$a\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix} + b\begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix} + c\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix} + d\begin{bmatrix}0&0&1\\0&0&0\\1&0&0\end{bmatrix} + e\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix} + f\begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix}$$

⑮ .

9c) 2×2 matrices w/ = row sums
= col sums is a vector space

(are you defining that as a vector space now?)

Find two matrices so all these matrices
are linear combos of the two

$$\underset{\underset{=}{\circlearrowright}}{\begin{bmatrix} a & b \\ c & d \end{bmatrix}} \curvearrowleft =$$

Find the 2 component matrices

$$\overbrace{\begin{bmatrix} a & \\ & d \end{bmatrix} + \begin{bmatrix} & b \\ c & \end{bmatrix}}$$

don't have equal
col / row sums.

−2

will always be =

⑯

10. Matlab

7/10 a) $\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$ plot? -2

b) $[\ .0640 \quad .5730 \quad .3630\ ]$

So all possibilites of 2×2
(in Matlab)

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ rank = 0

rank = how many ind cols there are

0 is special case
- doesn't count towards rank

$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} 1, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} 1, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} 1, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} 2$

$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} 1, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} 1, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} 2, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} 1$

$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} 1, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} 2, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} 2, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} 2, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} 2$

$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} 1, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} 1$

So

#	rank	%
0	1	.0625
1	9	.5625
2	6	.3750

✓

Umm close - since prob.

⑰ .

c) Extend for 5x5

$$\left[\underset{\overset{\uparrow}{\text{zeros}}}{0} \quad 0 \quad 6 \quad 90 \quad 552 \quad \underset{\overset{\uparrow}{\text{fives}}}{354}\right]$$

plot?, (-1)

d) What if continuous $[0,1]$

Much more $\sim\!>$ $\underset{\overset{\uparrow}{100\%}}{1}$ to the limit

Since very $\underset{\overset{\uparrow}{\text{to no}}}{\text{few}}$ values would be 0

So everything will be a pivot

✓

```
A = double(rand(2,2) < .5)

A =

     0     1
     0     0

clear;
clear; N=1000; num_zeros=0; num_ones=0; num_twos=0;

for i = 1:N
A=double(rand(2,2) < .5);
if rank(A)==2
num_twos = num_twos + 1;
end
if rank(A) == 1
num_ones = num_ones + 1;
end
if rank(A) == 0
num_zeros = num_zeros + 1;
end
end

distrib = [num_zeros, num_ones, num_twos]/N

distrib =

    0.0640    0.5730    0.3630

bar([0 1 2], distrib, .1)
rank( [0 0; 0 0])

ans =

     0

rank( [0 0; 0 1])

ans =

     1

rank( [0 0; 1 1])

ans =

     1

rank( [0 1; 0 1])

ans =
```

```
     1

rank( [1 0; 0 1])

ans =

     2

rank( [0 1; 0 0])

ans =

     1

rank( [1 1; 0 0])

ans =

     1

rank( [0 1; 1 0])

ans =

     2

rank( [1 0; 1 0])

ans =

     1

rank( [0 0; 1 0])

ans =

     1

rank( [0 0; 1 0])

ans =

     1

rank( [1 1; 0 1])

ans =

     2

rank( [1 1; 1 0])
```

```
ans =

     2

rank( [0 1; 1 1])

ans =

     2

rank( [1 0; 1 1])

ans =

     2

rank( [1 0; 0 0])

ans =

     1

rank( [1 1; 1 1])

ans =

     1

1/16

ans =

    0.0625

9/16

ans =

    0.5625

6/16

ans =

    0.3750


 for i = 1:N
A=double(rand(5,5) < .5);
if rank(A)==5
num_fives = num_fives+1;
end
```

```
if rank(A)==4
num_fours = num_fours+1;
end
if rank(A)==3
num_threes = num_threes+1;
end
if rank(A)==2
num_twos = num_twos + 1;
end
if rank(A) == 1
num_ones = num_ones + 1;
end
if rank(A) == 0
num_zeros = num_zeros + 1;
end
end


distrib = [num_zeros, num_ones, num_twos, num_threes, num_fours, num_fives]

distrib =

     0     0     6    90   552   354


bar([0 1 2 3 4 5] , distrib, .1)
diary off
```

Solutions

18.06 Spring 2012 – Problem Set 2

This problem set is due Thursday, February 23rd, 2012 at 4pm (hand in to Room 2-106). The textbook problems are out of the 4th edition. For computational problems, please include a printout of the code with the problem set (for MATLAB in particular, `diary('filename')` will start a transcript session, `diary off` will end one.)

1. Do Problems 7 & 9 from Section 2.6.

2.6.7. Given

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 2 & 2 \\ 3 & 4 & 5 \end{pmatrix} \text{ and } L = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1},$$

what three elimination matrices E_{21}, E_{31}, E_{32} put A into its upper triangular form $E_{32}E_{31}E_{21}A = U$? Multiply by E_{32}^{-1}, E_{31}^{-1}, and E_{21}^{-1} to factor A into L times U.

Solution.

$$E_{32}E_{31}E_{21}A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 2 & 2 & 2 \\ 3 & 4 & 5 \end{pmatrix}$$

and this gives

$$U = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Taking the inverses of the elimination matrices, and then putting them together gives:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix} U.$$

☐

2.6.9. Showing why $A = LU$ is not possible.

Solution. The 2×2 case: Multiplying the two matrices on the right shows that we must have $d = 0$, which is not allowed.

The 3×3 case: Again, multiply the two matrices on the right to get $d = 1, e = 1, g = 0, l = 1$. Then we need $f = 0$, which is not allowed. ☐

2. Do Problem 13 & 23 from Section 2.6.

2.6.13.

Solution.

$$\begin{pmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & 0 & d-c \end{pmatrix}$$

this works when $a \neq 0$, $a \neq b$, $b \neq c$, $c \neq d$ to get four pivots. $\square$

2.6.23

Solution. A_2 has the pivots 5 and 9, because elimination on A starts in the upper left corner, with elimination on A_2. $\square$

3. Do Problem 6 from Section 2.7.

   The transpose of a block matrix $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ is $M^T = \left(\begin{smallmatrix} A^T & C^T \\ B^T & D^T \end{smallmatrix}\right)$. Test an example. For this matrix to be symmetric, we need $A = A^T, D = D^T$, and $B = C^T$ (and hence $C = B^T$).

4. Do Problem 22 from Section 2.7.

   Find the $PA = LU$ factorizations (and check them) for

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 3 & 4 \end{pmatrix} \quad \textbf{and} \quad \begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

Solution.

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

$\square$

5. Do Problem 38 from Section 2.7.

   If you take powers of a permutation matrix, why is some P^k eventually equal to I? find a 5 by 5 permutation matrix P so that the smallest power to equal I is P^6.

Solution. Since there are only finitely many permutation matrices (in fact, exactly $n!$ of them), there must be two powers P^a and P^b that are the same, with $a > b$. Then since P is invertible by pset 1, $P^{a-b} = I$. $\square$

6. Do Problems 17 from Section 3.1.

   Solution to 3.1.17:

   (a) The zero matrix is not invertible. Therefore, the set of invertible matrices is not closed under multiplication by scalars, since multiplying anything by 0 gives the zero matrix. Therefore it is not a subspace of $\mathbf{M}$.

   (b) The matrices $\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right)$ are both clearly singular, but their sum is the identity matrix, which is obviously invertible. Thus the set of singular matrices is not closed under addition and so is not a subspace of $\mathbf{M}$.

7. Do Problem 23 from Section 3.1.

   Solution to 3.1.23:

   If we add an extra column $\mathbf{b}$ to a matrix A, then the column space gets larger unless $\mathbf{b}$ was already in the column space. If

$$A = \left(\begin{smallmatrix}1&0\\0&1\\0&0\end{smallmatrix}\right) \quad \text{and} \quad \mathbf{b} = \left(\begin{smallmatrix}0\\0\\1\end{smallmatrix}\right)$$

   then the column space gets larger. If

$$A = \left(\begin{smallmatrix}1&0\\0&1\\0&0\end{smallmatrix}\right) \quad \text{and} \quad \mathbf{b} = \left(\begin{smallmatrix}1\\1\\0\end{smallmatrix}\right)$$

   then it does not. The column space of A is the same as that of $[A\ \mathbf{b}]$ precisely when $\mathbf{b}$ can be written as a linear combination of the columns of A, i.e. when there exists a vector $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{b}$.

8. Do Problems 30 & 31 from Section 3.1.

   Solution to 3.1.30:

   (a) Suppose $\mathbf{a}$ is in $\mathbf{S}+\mathbf{T}$; then by definition there exist $\mathbf{s} \in \mathbf{S}$ and $\mathbf{t} \in \mathbf{T}$ such that $\mathbf{a} = \mathbf{s}+\mathbf{t}$. For λ a scalar we have $\lambda\mathbf{a} = \lambda(\mathbf{s}+\mathbf{t}) = \lambda\mathbf{s}+\lambda\mathbf{t}$. Since $\mathbf{S}$ and $\mathbf{T}$ are subspaces of $\mathbf{V}$, the vector $\lambda\mathbf{s}$ is in $\mathbf{S}$ and the vector $\lambda\mathbf{t}$ is in $\mathbf{T}$. Thus we have written $\lambda\mathbf{a}$ as a sum of a vector in $\mathbf{S}$ and a vector in $\mathbf{T}$, which by definition means that $\lambda\mathbf{a} \in \mathbf{S}+\mathbf{T}$. This proves that $\mathbf{S}+\mathbf{T}$ is closed under multiplication by scalars.

   Now suppose $\mathbf{a}$ and $\mathbf{a}'$ are in $\mathbf{S}+\mathbf{T}$; by definition there exist $\mathbf{s}, \mathbf{s}' \in \mathbf{S}$ and $\mathbf{t}, \mathbf{t}' \in \mathbf{T}$ such that $\mathbf{a} = \mathbf{s}+\mathbf{t}$ and $\mathbf{a}' = \mathbf{s}'+\mathbf{t}'$. Then since $\mathbf{V}$ is a vector space we have $\mathbf{a}+\mathbf{a}' = (\mathbf{s}+\mathbf{t})+(\mathbf{s}'+\mathbf{t}') = (\mathbf{s}+\mathbf{s}')+(\mathbf{t}+\mathbf{t}')$. As $\mathbf{S}$ and $\mathbf{T}$ are subspaces of $\mathbf{V}$, the vector $\mathbf{s}+\mathbf{s}'$ is in $\mathbf{S}$ and the vector $\mathbf{t}+\mathbf{t}'$ is in $\mathbf{T}$. Thus we have written $\mathbf{a}+\mathbf{a}'$ as a sum of a vector in $\mathbf{S}$ and a vector in $\mathbf{T}$, which by definition means that $\mathbf{a}+\mathbf{a}' \in \mathbf{S}+\mathbf{T}$. This proves that $\mathbf{S}+\mathbf{T}$ is closed under addition.

   (b) $\mathbf{S}\cup\mathbf{T}$ is the set of vectors that lie in either $\mathbf{S}$ or $\mathbf{T}$, whereas $\mathbf{S}+\mathbf{T}$ is the set of sums of vectors in $\mathbf{S}$ and $\mathbf{T}$. These are clearly not the same — for example, $\mathbf{S}\cup\mathbf{T}$ will not be a vector space unless $\mathbf{S}$ and $\mathbf{T}$ are the *same* line through the origin, while we proved above that $\mathbf{S}+\mathbf{T}$ *is* a vector space. The *span* of a subset of a vector space $\mathbf{V}$ is the set of vectors that can be written as linear combinations of elements of the set; since $\mathbf{S}$ and $\mathbf{T}$ are subspaces of $\mathbb{R}^m$, the span of $\mathbf{S}\cup\mathbf{T}$ is $\mathbf{S}+\mathbf{T}$.

Solution to 3.1.31:

The space $\mathbf{C}(A) + \mathbf{C}(B)$ consists of those vectors in $\mathbb{R}^m$ that can be written as a sum of a linear combination of the columns of A and a linear combination of the columns of B. This is the same thing as the vectors that are linear combinations of the columns of A together with the columns of B, so we can take $M = [A\ B]$.

9. This problem is about the vector space of matrices for a fixed number of rows and columns.

   (a) Explain carefully why the set of all 7×11 matrices forms a vector space (What is $c\,A + d\,B$? Which matrix is the zero vector?). Describe the simplest list of matrices you can think of which, allowing arbitrary linear combinations, will yield *all* 7×11 matrices. There should be 77 different matrices in your answer.

   (b) How many real number-valued parameters would you use to (unambiguously) describe the vector space $S_{3\times3}$ of 3×3 symmetric matrices (e.g. the set of all 3×3 matrices A such that $A^T = A$)? Identify *all* vector subspaces of $S_{3\times3}$ (it may be convenient to refer to the parameters you've introduced).

   (c) The 2×2 matrices with equal row sums ($a + b$ and $c + d$ are the same number), and equal column sums ($a + c$ and $b + d$), is a vector space. Find two matrices so that all these matrices are linear combinations of those two.

Solution:

(a) $cA+dB$ is the matrix whose (i, j)-component is $cA_{ij}+dB_{ij}$. Since scalar multiplication and addition are defined component-wise, the associativity and commutativity of addition of matrices, as well as the associativity of scalar multiplication and its distributivity over addition, all follow from the same properties of $\mathbb{R}$. The zero matrix is the zero vector in this vector space. Let E^{ij} be the matrix with components

$$E^{ij}_{kl} = \begin{cases} 1 & k = i, l = j, \\ 0 & \text{otherwise.} \end{cases}$$

Then any 7×11 matrix can be written as a linear combination of these 77 matrices:

$$A = \sum_{i=1}^{7} \sum_{j=1}^{11} A_{ij} E^{ij}.$$

Moreover, no subset of these matrices spans the set 7×11 matrices, since only those matrices A such that $A_{kl} = 0$ can be written as a linear combination of E^{ij}'s not including E^{kl}.

(b) A symmetric 3×3-matrix A is uniquely determined by its 6 upper-triangular components A_{ij} with $j \geq i$. A subspace of $S_{3\times3}$ is determined by some (independent) linear equations in these parameters.

(c) Consider $I = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $J = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$; these clearly have equal row sums and equal column sums. Suppose $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ where $a + b = c + d$ and $a + c = b + d$. Then

$a = -b + c + d$ and $a = b - c + d$; adding these we get $2a = 2d$, so $a = d$. Then $a + b = c + a$ so $b = c$ also. Thus

$$A = \left(\begin{smallmatrix} a & b \\ b & a \end{smallmatrix}\right) = aI + bJ.$$

10. The MATLAB command `A = double(rand(2,2) < 0.5)` gives a random 2×2 matrix where each entry is either 0 or 1 (with equal probabilities).

    (a) Make a plot of the distribution of the number of pivots of the row-reduced versions (in MATLAB, the command `rank(A)` gives this number) of these random matrices. Here's some sample code that you can copy-paste into MATLAB:

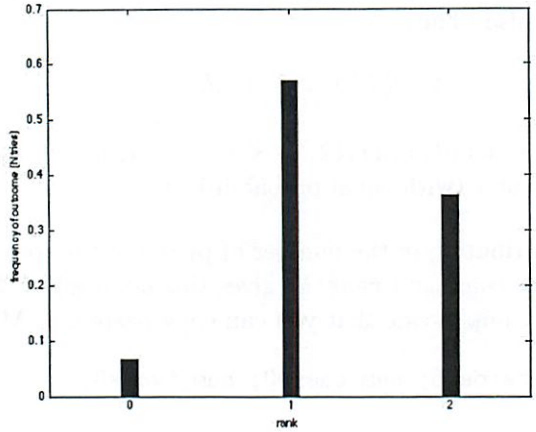
```
clear; N=1000; num_zeros=0; num_ones=0; num_twos=0;
for i = 1:N
 A = double(rand(2,2) < 0.5);
 if rank(A)==2
  num_twos = num_twos + 1; %Then add one to that counter!
 end
 if rank(A)==1
  num_ones = num_ones + 1;
 end
 if rank(A)==0
  num_zeros = num_zeros + 1;
 end
end
distrib = [num_zeros num_ones num_twos]/N
bar([0 1 2], distrib, 0.1)
```

    (b) Compare this to the exact probabilities of each value for the pivot number. Compute these by writing down all 16 possibilities and counting pivots.

    (c) Extend the code in (a) to work for 5×5 matrices, and again show a histogram plot.

    (d) For the 2×2 examples, what do you think the probability of having 2 pivots would be, if we took each matrix entry distributed continuously (and uniformly) in the *interval* $[0, 1]$? (No need to compute - but explain why!)

Solution:



(a)

(b) Only the zero matrix has no pivots, so the probability of 0 pivots is 1/16. The matrices

$$\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)\left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right)$$

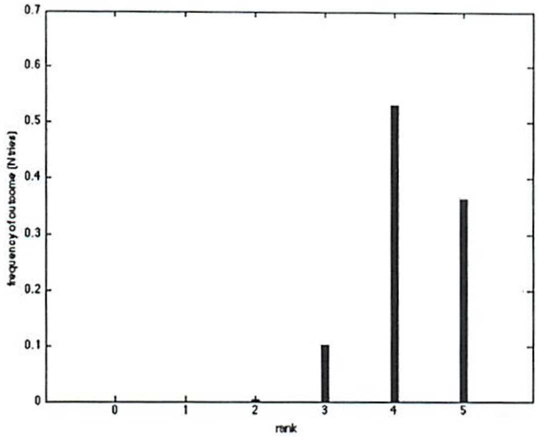
$$\left(\begin{smallmatrix}1&1\\0&0\end{smallmatrix}\right)\left(\begin{smallmatrix}0&1\\0&1\end{smallmatrix}\right)\left(\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\right)\left(\begin{smallmatrix}0&0\\1&1\end{smallmatrix}\right)\left(\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right)$$

have one pivot, so the probability of this is 9/16. The matrices

$$\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)\left(\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right)\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right)$$

have two pivots, so the probability of this is $6/16 = 3/8$.

(c)
```
clear; N=1000;
matsize=5;
nums=zeros(1, matsize+1);
for i = 1:N
 A=double(rand(matsize,matsize)<0.5);
 index = rank(A) + 1;
 nums(index) = nums(index)+1;
end
distrib = nums/N;
bar([0:1:matsize], distrib, 0.1)
xlabel('rank')
ylabel('frequency of outcome (N tries)')
```



(d) We can think of the entries of our matrices as points in the "hypercube" $[0,1] \times [0,1] \times [0,1] \times [0,1] \subseteq \mathbf{R}^4$. Since the entries are distributed uniformly, the probability that a matrix picked at random lies in some region in this subset of $\mathbf{R}^4$ equals the " 4-dimensional volume" of this region. A matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ has less than two pivots precisely when it is singular, i.e. when its entries satisfy the equation $ad - bc = 0$. But the space where this equation holds is 3-dimensional, so its 4-dimensional volume is 0 (just like a curve in the plane has no area, or a surface in space has no volume). Thus a random matrix has 2 pivots with probability 1.

18.06 Lecture

2/24

Last time $Ax=0$ vectors in the null space

Special sols $x = c_1 s_1 + \dots + c s$

$$A \rightarrow \text{rref}(A) = R = \left[\begin{array}{cc} I_r & F \\ 0 & 0 \end{array}\right]\begin{array}{l} r \\ m \text{ rows} \end{array}$$

n cols

pivot cols = r

free cols = n − r

F, I could be mixed in/reversed

rank = r = # pivot cols = # pivot rows

$= I_r \leftarrow$ identity

special sol = n − r

one for each free variable

Today ~~Today~~ Steps to get complete~~r~~ solutions $Ax=b$

$$x = x_{\text{particular}} + c_1 s_1 + \dots + c s$$

~~Call~~ get all the sols

On HW

If start w/ A

Could you arrive at diff rs.

- could row ex
- multiply by other #s
- So diff stages along the way

typical case $r=n$

no special sols, free cols

the zero vector is in null space

└ only thing

rank = n $N(A) = \{0\}$

"good"

Cols are ind

② $Ax = b$

~~All matrix eqs have the form~~

~~[illegible] particular [illegible]~~

?'if any sols at all for $Ax = b$ (call it $x = x_{particular}$)'

all soln's are $x = x_p + x_n$

(remember from 18.03 - diff eq)

particular sol

$$A x_p = b$$

$$A x_n = 0$$

Add

$$A(x_p + x_n) = b + 0$$

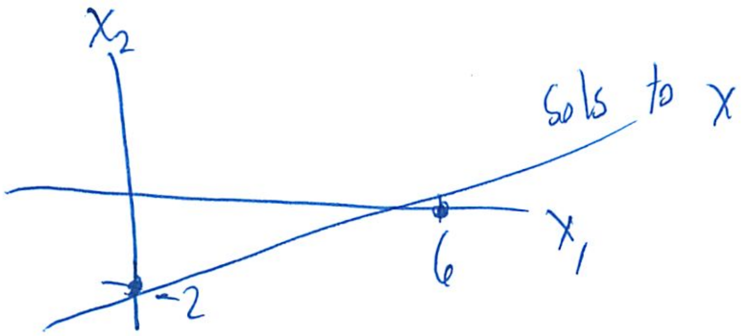
The front cover of the book

$x_n \rightarrow$ Could be many x_n

$x_p \rightarrow$ one particular soln

Example

$x_1 - 3x_2 = 6$



$$\begin{bmatrix}1 & -3\end{bmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix} = 6$$

Row Reduced Echelon Form = R

$$= \begin{bmatrix}\text{same matrix}\end{bmatrix}$$

Since 1 pivot
that is already 1

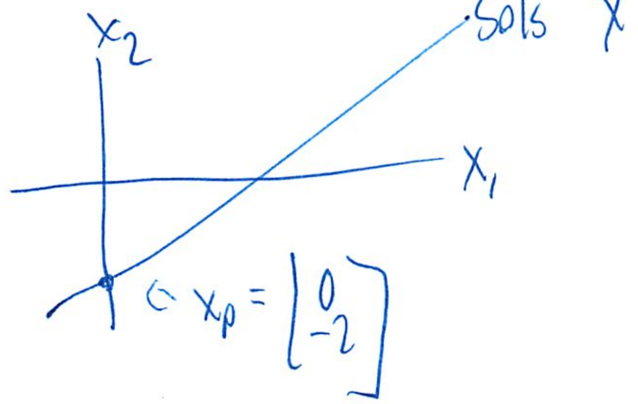
Rank = 1
n = 2 ← # cols

So special sol
- put a 0 in

$$s = \begin{bmatrix}3 \\ 1\end{bmatrix}$$

So $As = 0$

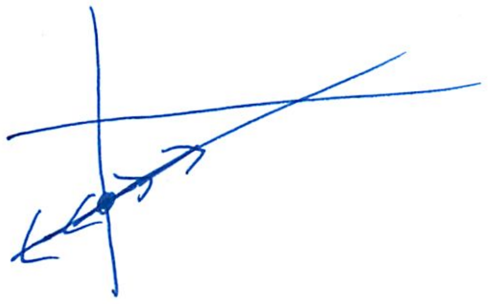
④



Go to x_p

then Go in the direction of s

→ all multiples of $\begin{bmatrix}3\\1\end{bmatrix}$



Complete / General sol

$$x = \begin{bmatrix}0\\-2\end{bmatrix} + c\begin{bmatrix}3\\1\end{bmatrix}$$

Would like to see this vector + anything in null space

(missed)

Tack on to matrix A, augment it w/ b

$[A \ b]$

Since what we do to A, we must also do to b

Example

$$\overset{A}{\begin{bmatrix} 1 & -3 \\ 3 & -9 \end{bmatrix}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ b_2 \end{bmatrix}$$

$x_1 - 3x_2 = 6$

$3x_1 - 9x_2 = b_2$

b_2 must be 18 or no sol

How get 18?

$$\begin{bmatrix} 1 & -3 & 6 \\ 3 & -9 & b_2 \end{bmatrix}$$

$$Rx = d \qquad \xrightarrow{\text{Do elim}} \begin{bmatrix} 1 & -3 & 6 \\ \underbrace{0 \quad 0}_{R} & & b_2 - 18 \end{bmatrix} \quad \text{rank} = 1$$

So went from $Ax = b$ to $Rx = d$

Hope to read off the answer

$x = x_p + x_n$

Case $b_2 = 18$
(If)

then problem is

$$\begin{array}{ccc} \text{pivot} & \text{free} & \\ \hline \end{array}$$

$$\left[\begin{array}{ccc} 1 & -3 & 6 \\ 0 & 0 & 0 \end{array}\right]$$

$x_1 - 3x_2 = 6$

$0 = 0$ ✓ No problem

How to write the answer

The complete solution

(a particular sol)

We will just pick one → Free variables = 0

Could pick another one
- can be anything

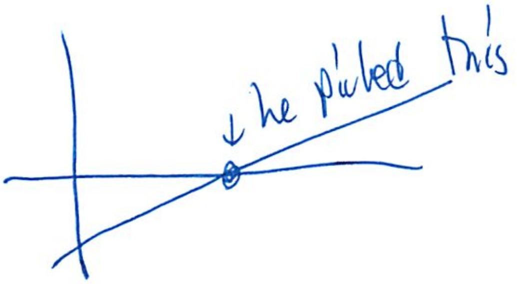
$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \end{bmatrix} + c \begin{bmatrix} 3 \\ 1 \end{bmatrix} \leftarrow \text{free}$$

↑ special sol

Now find x_1
the pivot col
$x_1 = 6$

⑦

So



If $b_2 = 100$

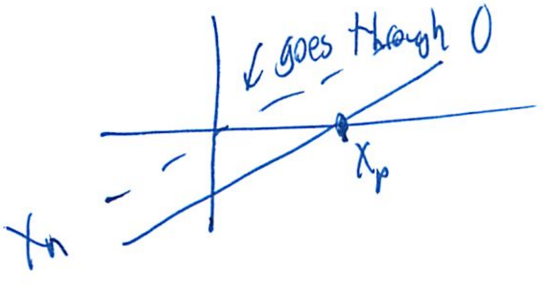
$$\begin{bmatrix} 1 & -3 & 6 \\ 0 & 0 & 82 \end{bmatrix}$$

No sol

In lang of subspaces

Vector $\begin{bmatrix} 6 \\ 100 \end{bmatrix}$ is not in col space

The ~~col~~ null space C(A)



then $X_{particular}$ moves it off 0

from having a b

X_p + anything x_n = line through x_p

Two stories

- null spaces
- Col spaces

- They lie in diff dimensions
- null space is in n dimensions
  n Components $N(A) \subseteq R^n$

- Col space is in m dimensions $C(A) \subseteq R^m$

HW expands on this

Now General case

$A \rightarrow$ m by n matrix

rank = r

special sols = n - r

[illegible]

If r = n

full column rank 0 or 1 sols

then $n \leq m$

⑨

rectangular → so can't say if invertable

$R = \begin{bmatrix} I \\ 0 \end{bmatrix} \begin{matrix} \leftarrow n \\ \leftarrow m-n \end{matrix}$ 0 rows

~~[illegible]~~

$N(A) = 0$ ← No free variables ← col are ind

~~Suppose r=n~~

If $Ax=b$ has a sol, then that solution is unique

↑ there is only 1

If $r=m$

full row rank

~~[illegible]~~ m ind. rows

$R = [I \ F]$ no zero rows

$Ax=b$ has a sol'n

↳ Many sols if $r<n$) 1 or ∞ sols

~~[illegible]~~

Next week: dimension, basis

18.06 Lecture

2/27

(full day ahead of schedule)

Today:
independence
span
basis
dimension
} 4 words

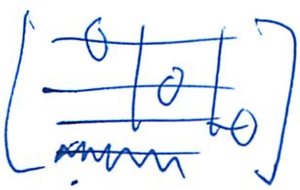
Last time: $Ax = 0$

Suppose $m < n$
more cols than rows

Example A is 3×5
What do we know about $Ax = 0$?

The cols of A are dependent
There is a non-0 solution to $Ax = 0$
↑ non-trivial
Since $x = 0$ always a sol

Since we reduce to R matrix
Only 3 rows
≤ 3 pivots
So ≥ 2 free cols


or


②

Linear independence

(for the cols of a matrix)

For the cols of A this means $Ax = 0$

has No non-zero solution

aka The Only solution is $x = 0$

(So this is that cols are not multiples of each other?)

For any set of vectors $V_1, \ldots, V_n$

the only combo of $V_1, \ldots, V_n$ $(x_1 V_1 + \ldots + x_n V_n = 0)$

is
$x_1 = 0$
$x_2 = 0$
$\vdots$
$x_n = 0$

Non col vector examples

- we can take combos w/ col vectors

Examples → dep or ind?

2×2 matrices, all entries 1, 0 → 16 possibilites

can have $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ then no way ~~have~~ be ind

③

If $\begin{bmatrix} - & 0 \\ - & 0 \end{bmatrix}$ then dep

So revise problem

2x2 matrices, all entries 1 or 0 except $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

<u>dep</u>

Now show it w/ an example

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

linear combo
w/ one of each

Want Vectors ($\overset{2\times 2}{\text{matrices}}$)

1. Ind
   - not going to keep 15
   - throw some away
   - what is right #?
   - have enough to produce <u>all</u> 2×2 matrices
   - just right $\downarrow$
2. Span all 2×2 matrices
   - every 2×2 matrix is a combo of my basis
   - should be ind
   - but keep enough to retain space

④

One possibility of a basis

$$\begin{bmatrix}1&0\\0&0\end{bmatrix}, \begin{bmatrix}0&1\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\1&0\end{bmatrix}, \begin{bmatrix}0&0\\0&1\end{bmatrix}$$ (the standard basis)

So to get $\begin{bmatrix}a&b\\c&d\end{bmatrix}$

$$a\begin{bmatrix}1&0\\0&0\end{bmatrix} + b\begin{bmatrix}0&1\\0&0\end{bmatrix} + c\begin{bmatrix}0&0\\1&0\end{bmatrix} + d\begin{bmatrix}0&0\\0&1\end{bmatrix} = \begin{bmatrix}a&b\\c&d\end{bmatrix}$$

Are there other possibilites of basises?

? sp

For $\mathbb{R}^2$

$$\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}$$

$$\begin{bmatrix}1\\5\end{bmatrix}, \begin{bmatrix}3\\7\end{bmatrix}$$

? not $\begin{bmatrix}1\\5\end{bmatrix}, \begin{bmatrix}8\\0\end{bmatrix}, \begin{bmatrix}2\\10\end{bmatrix}$

must go in diff dir

$\downarrow$

But does this span the space?

$$x_1\begin{bmatrix}1\\5\end{bmatrix} + x_2\begin{bmatrix}3\\7\end{bmatrix} = \begin{bmatrix}a\\b\end{bmatrix}$$ for any a, b?

There is a 2×2 matrix here!

⑤

$$\begin{bmatrix}1 & 3\\ 5 & 7\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}a\\ b\end{bmatrix}$$

T is invertable ← (key)

So yes is a basis!

Span: $V_1, \dots, V_n$ Span the space V when for every u in V, there are #s $a_1, \dots a_N$

Every u in space in V so that

$$a_1 V_1 + a_2 V_2 + \dots + a_n V_n = u$$

is a combo

Must be ind and span the space

The pivot cols are a basis for the col space $C(A)$
of A
not R

ex
$$A = \begin{bmatrix}1 & 1 & 1\\ 2 & 2 & 2\end{bmatrix}$$

One possible → basis (pivot col) $= \begin{bmatrix}1\\ 2\end{bmatrix}$ for $C(A)$

don't mix basis w/ space

also $= \begin{bmatrix}2\\ 4\end{bmatrix}, \begin{bmatrix}12\\ 24\end{bmatrix}$ ← any pt on line except $\begin{bmatrix}0\\ 0\end{bmatrix}$

⑥ $$R = \begin{array}{c} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \text{pivot} & \text{free} & \end{array} \end{array} \quad \text{count} = 1$$

Basis → bunch of vectors

1. Ind
2. Span, Space

very efficent way to say what is in vector space

Quiz Q

Basis for space of 3x3 symmetic matrices

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} & & \\ & 1 & \\ & & \end{bmatrix} \begin{bmatrix} & & \\ & & \\ & & 1 \end{bmatrix} \leftarrow \text{basis for digonal}$$

$$\begin{bmatrix} & 1 & \\ 1 & & \\ & & \end{bmatrix} \begin{bmatrix} & & 1 \\ & & \\ 1 & & \end{bmatrix} \begin{bmatrix} 4 & 7 & 2 \\ 7 & 9 & 1 \\ 2 & 1 & 6 \end{bmatrix} \leftarrow \text{other synetlic}$$

↑ be troublesome
need to subtract
using others to get
back to basis looking for

Dimension

Is a #, not a matix

Every basis has the same # of vectors in it for a vector space V

The # of vectos in a basis

For our 3 x 3 matrix

dim = 6

Since 6 vectors were in its basis

n-dimensional space has dim n

└ convince by providing a basis

usually the st. basis

ex dim of Col space C(A) is r

$\boxed{A \text{ is } m \times n \text{ rank } r}$

So r is crucial #. Tells you how many cols are actually ind

So w/ our $\begin{bmatrix}1&1&1\\2&2&2\end{bmatrix}$ matix rank = 1

$\downarrow$

dim = 1

Next time: Understand why this more

18.06 Recitation

2/28

Matrix Factorization

1. $A = LU$

row operations $\leftrightarrow$ $E_{ij} = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ -\ell & & 1 \end{bmatrix}$

2. $A = LDU$

ex $A = LU = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & 2 \\ 0 & 5 & 2 \\ 0 & 0 & 7 \end{bmatrix}$

L same → 1s on diagonal

↓ factor out

$$= \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} 3 & & \\ & 5 & \\ & & 7 \end{bmatrix} \begin{bmatrix} 1 & 1/3 & 2/3 \\ & 1 & 2/5 \\ & & 1 \end{bmatrix}}_{U}$$

← factoring out

↑D ↑U' ← upper triangular 1s on diagonal so more symmetric

3. If A is <u>symmetric</u>

then $A = LDU$ → ~~[illegible]~~

↪ $A = LDL^T$

↑since when flips becomes upper triangular

4. $PA = LU$

↳ was on P-set

for temp failure of elim (row swap)

↙ 1st pivot, but can't be 0

$$A = \begin{bmatrix} \textcircled{0} & 1 & 3 \\ 0 & 2 & 2 \\ 1 & 2 & 5 \end{bmatrix}$$

So swap row 1 w/ row 3

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Sometimes have matrix A w/ elim

$$EEPEEA$$

↳ can't swap matrices
but can still swap before hand

5. We will learn about <u>Reduced Row Echelon Form</u>
- is a big part of P-set
- refers to the U part

$$A = L\,\textcircled{U}$$

- Reduced - means simple
  - every pivot = 1
  - each pivot col contains exactly 1 of 1

③

Everything else $_\wedge$ is 0
in the pivot col

Examples: RREF or not

a) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 7 \end{bmatrix}$ ⊗ No 2nd col ~~should be~~ must be 0, not 2

Can convert by subtracting 2·row 1 from row 2

→ row 1 → row 1 − 2 row 2 →

$\begin{bmatrix} 1 & 0 & -11 \\ 0 & 1 & 7 \end{bmatrix}$ ✓ RREF

b) $\begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 6 \end{bmatrix}$ ✓ yes

c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ ⊗ No

Pivots must be 1

Can divide each row by 1, 2, 3

(4)

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{array}{l}\leftarrow \text{divided row by } 1\\ \leftarrow \qquad\qquad\quad 2\\ \leftarrow \qquad\qquad\quad 3\end{array}$$

(✓) Yes

d) $$\begin{array}{c}\begin{bmatrix}1&0&0\\0&0&1\end{bmatrix}\\ \begin{array}{ccc}\uparrow&\quad&\uparrow\end{array}\end{array}$$ (✓) Yes

pivot cols

So when non squ matrix

$$\begin{array}{l}\begin{bmatrix}0&&&\\ \vdots&&&\\ &&&\end{bmatrix}\\ \;\;\uparrow\end{array}$$

if all 0 → move to next row
if one is non 0 → row swap

Example Find RREF of

$$A=\begin{bmatrix}1&2&2&4&6\\1&2&3&6&9\\0&0&1&2&4\end{bmatrix}$$

⑤

If matrix invertable, only RREF of sq matrix is identity $\begin{bmatrix}1&&\\&1&\\&&1\end{bmatrix}$

If not invertable
Could also $\begin{bmatrix}1&0\\0&0\end{bmatrix}$ ✓RREF

Will always have same rank as original matrix

$\boxed{\text{invertable } \# \text{ cols} = \text{rank}}$

So RREF

Want upper triangular form
Unclear when non square

1,1 is pivot
~~subtract~~ row 2 → Row 2 − Row 1
nothing to 3rd row - already 0

$$\begin{bmatrix}1&2&2&4&6\\0&0&1&2&3\\0&0&1&2&4\end{bmatrix}$$

roughly upper triangular
? is 0, look below, so move on

⑥

Now only look b/w row 2 and 3

row 3 → row 3 − row 2

↑ since $\ell_{32} = 1$

$$\begin{bmatrix} \textcircled{1} & 2 & 2 & 4 & 6 \\ 0 & 0 & \textcircled{1} & 2 & 3 \\ 0 & 0 & 0 & 0 & \textcircled{1} \end{bmatrix}$$

✓ is roughly in upper triangular form

↑ is 0 and nothing to swap

↑ so choose this one

Went through all rows

Done

✓ upper triangular for non sq matrix

But is it RREF?

No. Pivot cols should have 0s elsewhere

Now need to eliminate upwards.

⑦

Suggestion: ~~Can subtract 2 row 2 from Row 1~~

People generally start from bottom

Row 2 → Row 2 − 3 Row 3

$$\begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

So want to get rid of 6

So take 6 of row 3, subtract from row 1

Row 1 → Row 1 − 6 Row 3

$$\begin{bmatrix} \textcircled{1} & 2 & 2 & 4 & 0 \\ 0 & 0 & \textcircled{1} & 2 & 0 \\ 0 & 0 & 0 & 0 & \textcircled{1} \end{bmatrix}$$

One more step

Row 1 → Row 1 − 2 Row 2

$$\begin{bmatrix} \textcircled{1} & 2 & 0 & 0 & 0 \\ 0 & 0 & \textcircled{1} & 2 & 0 \\ 0 & 0 & 0 & 0 & \textcircled{1} \end{bmatrix}$$

So this RREF is eaiser to solve

Don't need to back sub

Can just read off answer

Vector spaces, subspaces, col spaces, and null spaces

Problem a) Show that the space V of all 3x 3 matrices is a vector space

b) Show that " " W of all 3x3 symmetric matrices is a subspace of V

← won't be vector space

Usually linear

Since if add vectors or multiply by a scalar

So vectors will go out of circle

⑨

Sol'n (a) Need to show that addition and scalar multiplication makes sense

addition Let A, B be any 3×3 matrices

So $A+B$ = still in space

= still a 3×3 matrix

↑half of proof

In general, if other space → $A, B \in V$

Show $A+B \in V$

↑are in

Scalar multiplication

Let A be a 3×3 matrix

and a scalar c = a real #

So cA is still in space

is still 3×3 vector

Generally, take $A \in V$, $c \in R$

Show $cA \in V$

Subspace

to show is a subspace

① - must show $W \subseteq V$
  ↑ can =

② - W is a vector space in its own right
- Same way as before - add
  and show still - Scaler multiply
  Symmetric
  ↑ if trying to say Symmetric ones are subspace

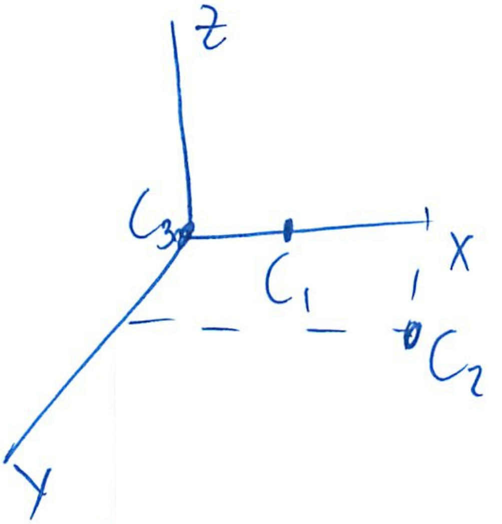
Column Spaces

Problem Find the col space of

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

in which space shall it belong to?
We are looking in 3D - since cols have 3 components

$$\begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



So smallest col space = x,y plane
Since all 3 lie on x,y plane
So that is the col space

Null space

Prob a) Find null space of

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

Sol'n All x's satisfying $Ax = 0$

(I get this stuff in concept - but can't do the qui!)

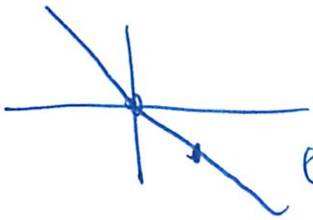
$$\begin{bmatrix}1&2\\0&0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

$x+2y=0$

$0=0$

Tells us what x, y satisfy

Pts on $x+2y=0$ lie on 0 line



anything on line = our null space

b) Find complete sol to

$$\begin{bmatrix}1&2\\0&0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}5\\0\end{bmatrix}$$

$x+2y=5$

Guess specific sols 1st

$$x=\begin{bmatrix}3\\1\end{bmatrix}$$

have $Ax=\begin{bmatrix}5\\0\end{bmatrix}$ $Ax_i=\begin{bmatrix}0\\0\end{bmatrix}$

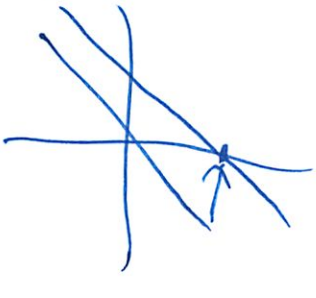
⑬

If guess spcial sol
add to it the x_1's
Such that $Ax_1 = 0$
We found x_s in null space

⋮

So $A(x + x_1) = Ax + Ax_1$
$= \begin{bmatrix} 5 \\ 0 \end{bmatrix}$

So all sols to this are sols to our problem $\begin{bmatrix} x \\ y \end{bmatrix}$
just shifting line up to 5



18.06 PS4
Reading

2/28

3.2 Nullspace $Ax = 0$

Subspace containing all sols to $Ax = 0$

Square or rectangular

One intermediate sol → $x = 0$

- invertable → only sol
- non invertable → are non zero sol to $Ax = 0$

These vectors x are in $\mathbb{R}^n$ - so subspace of $\mathbb{R}^n$

$N(A)$ ← null space

Check sol vectors form a subspace!

$$\begin{matrix} Ax = 0 \\ Ay = 0 \end{matrix} \rightarrow \begin{matrix} A(x+y) = 0 + 0 \\ A(cx) = c\,0 = 0 \end{matrix}$$

$Ax = b$, $b \neq 0$ then is <u>NOT</u> a subspace

is set of sols $\neq$ inc 0 then not a subspace

Ex $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ Apply to $Ax = 0$

$$\begin{matrix} x_1 + 2x_2 = 0 \\ 3x_1 + 6x_2 = 0 \end{matrix} \xrightarrow{\text{elim}} \begin{matrix} x_1 + 2x_2 = 0 \\ 0 = 0 \end{matrix}$$ ← anything on this line

②

One pt on the line is the <u>special sol</u>
All other pts on line is multiple of this

So here $x_2 = 1$ (a special choice)

then $x_1 + 2(1) = 0$
$x_1 = -2$

So $\underset{\substack{\uparrow \\ \text{special} \\ \text{sol}}}{s} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \leftarrow$ all multiples of this

nullspace is

When 3 ~~eqn~~ components

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$s_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \qquad s_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

↑ null space is all ~~a~~ combos on this plane

Note last 2 components "free"
So ~~choice~~ choose 1, 0 specially here

$$\begin{bmatrix} \underset{\uparrow}{1} & \underset{\uparrow}{2} & \underset{\uparrow}{3} \end{bmatrix}$$
pivot free ← for free values

③

Remember if A is invertable -> all cols of A have pivots

and only sol to $Ax=0$ is 0

$$\left\langle\begin{matrix}x_1\\x_2\end{matrix}\right\rangle \overset{\wedge}{=} \begin{bmatrix}0\\0\end{bmatrix}$$

Extra rows = more conditions on $\overset{\text{vectors}}{\wedge}$ x in nullspace

Extra cols = more free cols = more special sols

So what are the special sols again

Put a 1 in each free col 1 at a time
then 0s in the rest

ie $\begin{bmatrix}1\\0\\0\end{bmatrix}$ $\begin{bmatrix}0\\1\\0\end{bmatrix}$ $\begin{bmatrix}0\\0\\1\end{bmatrix}$ for 3 free cols?

At upper triangular U

Can still reduce more to <u>reduced row echlon form</u>

1. Produce zeros above the pivots by eliminating upward
2. Produce ones in the pivots by dividing the whole row by its pivots

(we talked about today in recitation)

So $U = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & 2 & 0 & 4 \end{bmatrix} \rightarrow$ becomes $\rightarrow R = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}$

$\uparrow \uparrow$

Pivot cols
Contain I

An . row 1 ← row1 − row 2
row 2 ← $\frac{1}{2}$row 2

Special sols are still same

eaiser to see w $Rx = 0$

invertable → on $x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ sol → independent
no cols free, all are pivots

Solving $Ax = 0$ by Elimination

A is rectangular and we still use elim

m eq in n unknowns when $b = 0$

Steps

1. Forward elim takes A to triangular U
   or RREF R
2. Back sub in $Ux = 0$ or $Rx = 0$ produces x
   - a difference when A, U have < n pivots

⑤

If a col has no pivot → go on to next col

So in recitation ; if 0 and all rows below it are 0

√ Same in textbook

$$A = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 2 & 2 & 8 & 10 \\ 3 & 3 & 10 & 13 \end{bmatrix}$$

$a_{11} = 1$

$$A = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 4 & 4 \end{bmatrix} \quad \begin{matrix} \text{row 2} \leftarrow \text{row 2} - 2 \cdot \text{row 1} \\ \text{row 3} \leftarrow \text{row 3} - 3 \cdot \text{row 1} \end{matrix}$$

↑ empty

↪ go on

$a = 4$

$$U = \begin{array}{cccc} \Big[\ \textcircled{1} & 1 & 2 & 3 \\ 0 & 0 & \textcircled{4} & 4 \\ 0 & 0 & 0 & 0\ \Big] \\ \uparrow & \uparrow & \uparrow & \uparrow \\ \text{pivot} & \text{free} & \text{pivot} & \text{free} \end{array} \quad \begin{matrix} \leftarrow \text{row 3} \leftarrow \text{row 3} - 4 \text{ row 2} \\ \nwarrow \text{last col} \Rightarrow 0 = 0 \end{matrix}$$

So $\begin{bmatrix} \_ \\ 1 \\ \_ \\ 0 \end{bmatrix}$ $\begin{bmatrix} \_ \\ 0 \\ \_ \\ 1 \end{bmatrix}$ are special sols :

⇒ solve + fill in

⑥

Yes so x_2, x_4 can be given any value

But special sols

$x_2 = 1$ $\quad$ $x_2 = 0$ $\quad$ ✓ As I said

$x_4 = 0$ $\quad$ $x_4 = 1$

then by back sub

$x_3 = 0$ $\quad$ $x_3 = -1$ $\quad$ ← should actually try

$x_1 = -1$ $\quad$ $x_1 = -1$

So $x_1 + x_2 + 2x_3 + 3x_4 = 0$

$4x_3 + 4x_4 = 0$

If $x_2 = 1$ $\quad$ If $x_2 = 0$

$x_4 = 0$ $\quad$ $x_4 = 1$

$x_1 + 1 + 2x_3 = 0$ $\quad$ $x_1 + 2x_3 + 3 = 0$

$4x_3 = 0$ $\quad$ $4x_3 + 4 = 0$

$x_3 = 0$ $\quad$ $x_3 = -1$

$x_1 = -1$ $\quad$ $x_1 = -1$

$x_3 = 0$

✓ $\quad$ ✓

Then put together for complete sol

$$x = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -x_2 - x_4 \\ x_2 \\ -x_4 \\ x_4 \end{bmatrix}$$

c All sols are linear combos of s_1, s_2

~~Special sol~~

Special sols are in the Nullspace $N(A)$ and their combos fill the whole null space

MatLab: nullbasis()

L makes nullspace matrix N

Complete sol is combo of these cols

Echelon Matrices

Forward Elim $A \to U$

Acts by row ~~exchanges~~ ops from col to col

The staircase U is an echelon matrix

Ex $U = \begin{bmatrix} p & x & x & x & x & x & x \\ 0 & p & x & x & x & x & x \\ 0 & 0 & 0 & 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

3 pivot variables x_1 x_2 x_6

4 free variables 3, 4, 5, 7

4 special sols in $N(U)$

⑧

What is col space? Nullspace?

4 components → R^4

$C(U)$ consists of all vectors of form $(b_1, b_2, b_3, 0)$

↑ since 4th component = 0

For those vectors we can solve $Ux = b$ by back sub

↑ all possible combo of 7 columns

Nullspace $N(U)$ is a subspace of R^7

Sols are combo of the 4 special sols - One for each free variable

1. Col 3, 4, 5, 7 no pivots - free
2. Set free to 1 and oter to 0
3. Solve $Ux = 0$ for pivot variables 1, 2, 6

Gives 1 of the 4 special sols

(this seems like reviews)

So like a staircase pattern

With $n > m$ will always be > 1 variable

$Ax = 0$ has ≥ 1 special sol

~~Dimension = # of free variables~~

$n > m$ always has non zero vectors in the null space

Must be at least $n - m$ free variables

row can have 0 or 1 pivots

So at most m pivots

dim = # of free variables

RREF R

Pivot rows contain I

0s above pivots from upward elimination

If A is invertible RREF $R = I$

└ since nullspace $= Z$

Easy to find sols w/ this

(10)

3.3 Rank + Row Reduced Form

m, n give size of matrix
- but not true size of a linear eq'n
  - 0 = 0 does not count
  - 2nd of identical rows disappears in elimination

true size of A = rank

rank of A = # of pivots = r

$$A = \begin{array}{c} \begin{bmatrix} 1 & 1 & 2 & 4 \\ 1 & 2 & 2 & 5 \\ 1 & 3 & 2 & 6 \end{bmatrix} \\ \begin{array}{cccc} \uparrow & \uparrow & \uparrow & \uparrow \\ \text{pivot} & \text{pivot} & & 3 \cdot \text{First} + \text{Second} \\ & & 2 \cdot \text{First} & \text{No} \\ & & \text{No} & \text{Free} \\ & & \text{Free} & \end{array} \end{array}$$

Free = combo of earlier pivot cols

Can find special sol's w/ combo of pivot cols
(This seems repeat of earlier info...)

$Ax = 0$ equivilant $Ux = 0$ and $Rx = 0$
(only when 0, right?)

⑪

$$\overset{A}{\begin{bmatrix} 1 & 1 & 2 & 4 \\ 1 & 2 & 2 & 5 \\ 1 & 3 & 2 & 6 \end{bmatrix}} \rightarrow \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & 0 & 2 \end{bmatrix} \rightarrow \overset{U}{\begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}}$$

↑ rank = 2 ↑

$$\rightarrow \overset{R}{\begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}} \leftarrow \text{row1} = \text{row1} - \text{row2}$$

↑ 3(col 1) + col 2

Move all cols to left side

$$-3(\text{col } 1) - (\text{col } 2) + (\text{col } 4) = 0$$

$$s = \begin{pmatrix} -3 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

(I don't get what we did here...)

; Trying to show every free col is combo of earlier pivot cols;

And another way to get special sol s

⑫

Rank One

Only 1 pivot

When elim has 0 in first col → zero in all cols

└ Since every row is a multiple of pivot row

(oh cool!)

$$A = \begin{bmatrix} 1 & 3 & 10 \\ 2 & 6 & 20 \\ 3 & 9 & 30 \end{bmatrix} \rightarrow R = \begin{bmatrix} 1 & 3 & 10 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So col space = 1D

$$A = u v^T$$

$$\begin{bmatrix} 1 & 3 & 10 \\ 2 & 6 & 20 \\ 3 & 9 & 30 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 3 & 10 \end{bmatrix}$$

So easy to see sols

$$Ax = 0$$

$$u(v^T x) = 0$$

$$v^T x = 0$$

All vectors x in null space must be orthogonal to v in the row space

row space = line, null space = perp plane

(I am confused about row space + col space)

Row space - $C(A^T)$ = all combo of rows in A

↑ (Col vectors by convention

(I realize this, it just does not feel natural))

Col space $C(A)$ = space of all combos of the cols of A

Example

Pivot row $[1\ 3\ 10]$

x_1 pivot

x_2, x_3 free

$$s_1 = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} \quad s_2 = \begin{bmatrix} -10 \\ 0 \\ 1 \end{bmatrix}$$

Null space = all combos s_1, s_2

Produces plane $x + 3y + 10z = 0$

└ perp to row $(1, 3, 10)$

* Null space (plane) $\perp$ to row space (line) *

14

rank r is the dimension of the col space

also the dimension of the row space

r also reveals dim of nullspace

Pivot Cols

have 1 in pivots and 0 elsewhere

r pivot cols together contain r x r identity I

it sits above m-r rows of zeros

pivot cols same in A, U, R

^harder to see in A

* pivot cols = not combos of earlier cols *

Special Sols

Each special sol to $Ax=0$ $Rx=0$ has one free

variable = 1

The other free variables in x are 0

(we saw this before)

⑮

Null space N contains 3 special sols so

$$AN = \text{zero}$$

Null space matrix $N = \left[\begin{array}{ccc|l} -3 & -2 & 1 & \text{not free} \\ 1 & 0 & 0 & \text{free} \\ 0 & -4 & 3 & \text{not free} \\ 0 & 1 & 0 & \text{free} \\ 0 & 0 & 1 & \text{free} \\ \uparrow & \uparrow & \uparrow & \end{array}\right]$

$n - r = 5 - 2 \Rightarrow$

↳ 3 special sols

each col a special sol

So special sol for every free variable

Note no cols dep on one another - all ind

$Ax = 0$ has r ind eq

$n - r$ ind sols

Easy to see special sols for $Rx = 0$

$$R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} \begin{array}{l} r \text{ pivot rows} \\ m-r \text{ zero rows} \end{array}$$

r pivot cols $n-r$ free cols

↳ Null space Matrix $N = \begin{bmatrix} -F \\ I \end{bmatrix} \begin{array}{l} r \text{ pivot variables} \\ n-r \text{ free variables} \end{array}$

pivot variables in $n-r$ special sols come from changing F to $-F$

Check $RN = 0$

$\llcorner$ first row $(I \cdot -F) + (F \cdot I) = 0$

$Rx = 0$

$$I \begin{bmatrix} \text{pivot variables} \end{bmatrix} = -F \begin{bmatrix} \text{free variables} \end{bmatrix}$$

What is F?

- doesn't seem to say ...

In each special sol, free variables are a col of I
Pivot variables are a col of $-F$
Those give nullspace matrix N

ie $N = \begin{bmatrix} -F \\ I \end{bmatrix} = \begin{bmatrix} -2 & -3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$

rank $r = 1$

2 special sols

Can reduce to R differently - but same result

More Reading

2/28

3.4 Complete Sol to $Ax = b$

Last section solved $Ax = 0$

Elim converted to $Rx = 0$

Free variables given special values [1, 0]

Pivot variables found w/ back sub

RHS = 0

Solved for x → null space

Now $b \neq 0$

Row ops LHS → must also be on RHS

$Ax = b$ reduced to $Rx = d$ — also do elim steps here

So can add b as col (augment)

$$\begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 1 & 3 & 1 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 7 \end{bmatrix}$$

$$\downarrow$$

$$\left[\begin{array}{cccc|c} 1 & 3 & 0 & 2 & 1 \\ 0 & 0 & 1 & 4 & 6 \\ 1 & 3 & 1 & 6 & 7 \end{array}\right] = \begin{bmatrix} A & b \end{bmatrix}$$

One Particular Sol'n

For an easy x sol, choose free variables to be $x_2 = x_4 = 0$

Then the 2 non zero eq give pivot variables

$x_1 = 1 \qquad x_3 = 6$

So $x_p = \begin{bmatrix} 1 \\ 0 \\ 6 \\ 0 \end{bmatrix}$ for $\begin{matrix} Ax = b \\ Rx = d \end{matrix}$

↑ * free variables = zero, pivot variables from d method always works

* For a sol to exist, zero rows in R must also be zero in d

Since I is in the pivot rows + cols of R the pivot variable in x_{part} comes from d

↑ the b in RREF

③

Combine the Sols

X particular $\quad Ax_p = b$ ← this chap

X null space $\quad Ax_n = 0$ ← last chap

Complete Sol

$$x = \underset{\text{one}}{\underset{\uparrow}{x_p}} + \underset{\text{many}}{\underset{\uparrow}{x_n}}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 6 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ -4 \\ 1 \end{bmatrix}$$

Square matrices

$m = n = r$

One and only sol $A^{-1}b$

$R = I$ has no zero rows

$x_n = 0$

So $\quad x = x_p + x_n = A^{-1}b + 0$

↑ this was chap 2

So only nullspace when not sq – Yeah we added non sq, here in this chap

④

Full Col Rank

- all cols have pivots
- no free variables $n = r$
- So no special sols
- So $x_n = 0$
- So only sol is $Ax = b$
  $Rx = d$

Matrix tall + thin $(m \geq n)$

Row reduction puts I at top
When A is reduced to R w/ rank n

Full col rank $R = \begin{bmatrix} I \\ 0 \end{bmatrix} = \begin{bmatrix} \text{n by m identity matrix} \\ \text{m - n row of zeros} \end{bmatrix}$

No free cols or free variables

Key facts $r = n$ full ~~Row~~ Col rank

1. All cols of A are pivot cols
2. There are no free variables or special sols
3. The null space $N(A)$ has only zero vector $x = 0$
4. If $Ax = b$ has a sol → then it has only 1 sol

essentially A has ind. cols

$Ax = b$ is unique (if exists)

Complete Sol

full row rank ← was that not the last section?

$Ax=b$ has one or ∞ly many sols

A <u>short and</u> wide $(m \leq n)$ ⟵ <u>new</u>

full row rank if $r = m$ "ind. rows"

Every row has a pivot and here is an example

$n = 3$ unknowns but only $m = 2$ eqs

Full row rank (yeah think of as eqn

$$x + y + z = 3$$
$$x + 2y - z = 4$$

rank $= r = m = 2$

two planes in xyz space

insersect in a line

- what elim will find

Particular sol will be 1 point on the line

Adding null space moves us along line

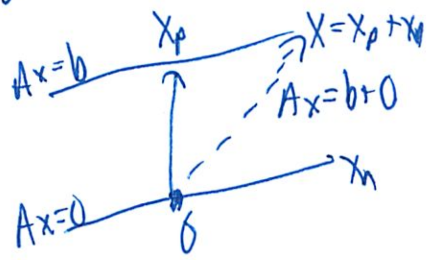
$x = x_p + x_n$ gives whole line of sols

$x_{particular}$ → from d on RHS

└ add dummy $x_3 = 0$?

$x_{special}$ → comes from 3rd col (the free col) of R

$$x = x_p + x_n = \begin{bmatrix}2\\1\\0\end{bmatrix} + x_3\begin{bmatrix}-3\\2\\1\end{bmatrix}$$



Note: Any pt on line could have been chosen

└ we chose $x_3 = 0$

The particular sol is not multiplied by a constant

So if $m < n$ then $Ax = b$ is underdetermined

(ie many sols)

So rows are linearly ind

So cols A^T ind

Key facts $r = m$ full row rank

1. All rows have pivots and R has no 0 pivots
2. $Ax = b$ has a sol for every RHS b
3. The col space is whole space R^m
4. There are $n - r = n - m$ special sols in null space of A

4 possibilities for linear eqn

$r = m$	$r = n$	Sq + invertable	$Ax = b \to 1$ sol
$r = m$	$r < n$	Short + wide	$Ax = b \to \infty$ sols
$r < m$	$r = n$	Tall + Thin	$Ax = b \to 0$ or 1 sol
$r < m$	$r < n$	No full rank	$Ax = b \to 0$ or ∞ sols

Reduced R in same cat as A

For $Rx = d$ (and $Ax = b$) to be solvable, d must end in $m - r$ sols

4 Types	$R = [I]$	$[I \; F]$	$\begin{bmatrix} I \\ 0 \end{bmatrix}$	$\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$
Their ranks	$r = m = n$	$r = m < n$	$r = n < m$	$r < m, r < n$

⑧

↙ what exactly does this mean?

Case 1+2 → full row rank $r=m$

1+3 → full col rank $r=n$

4 → most general in theory

↳ rare in pratice

18.06 Lecture

2/24

Independence + Span = **Basis** / Dimension

? Last time: Definitions

Ind → null space has only 0 vector

Span → every vector in the space (and all combos) spans the entire covered space

1 vector

↑ will be a basis

basis →

dimension → a #,

Smallest vector space → zero vector alone {0}
(has dim = 0 by convention

②

Today: 2 big facts

Key pt # 1: all bases for a vector space have

- are all ind by def
- all span the space by def
- the same # of vectors (the count) = dim
- takes 2 vectors to be basis for R^2
  - not 1 $\xrightarrow{\text{vectors}}$ just line
  - not 3 vectors → one of the vectors will be dep

Theorem

v_1, v_2, v_3 in R^2 are not ind

since $A = \underset{\text{put into cols}}{[v_1 \ v_2 \ v_3]}$ Reason

So 2 x 3 matrix

Can't have 3 pivots

at most 2 ind

Check their combos

- look at their null space

$$[v_1 \ v_2 \ v_3] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- is there a sol besides 0

So can do elim
See pivot variables
Free variables

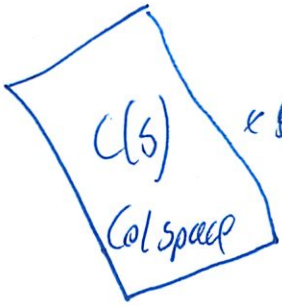
Can be set to 1
Get $x \neq 0$ in $N(A)$

So dim $R^n = n$
since $R^2 = 2$
$R^3 = 3$

if $n > R^n$
then do what we just did
would show vectors are dep

the 2 most imp facts in LA

2. About the col space and the row space
The big pic of Linear Algebra
$m \times n$ matrix / rank r / $r \leq m$, $r \leq n$

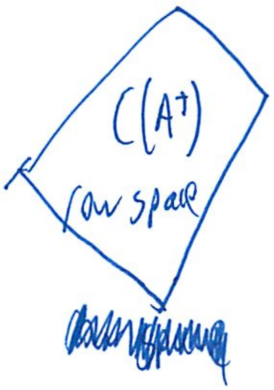


← extends to ∞

← sits in larger space R^m

④

row space = all combo of rows



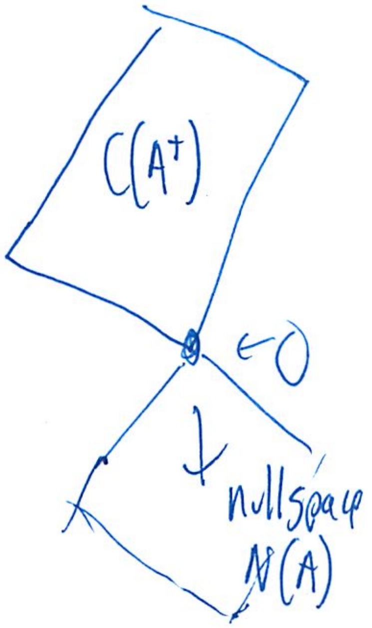
← don't want diff symbol
so $C(A^T)$

So now rows are col vectors

So keep vectors → col vectors convention

in $\mathbb{R}^n$ space

What other subspace is in $\mathbb{R}^n$

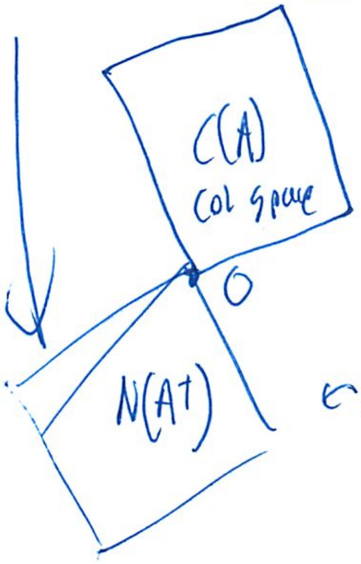


This pic on front of book

Need still 4th fund. space

(5)

So fourth fund. space



← he says math is beutiful

What do I ask about space

1. Basis
2. Dimension

dim of col space = r

- by identifying pivot cols - of A
  - maybe not after elim

basis of null space:
Special sols

dim of null space:
n - r

⑥

2. $C(A)$ and $C(A^T)$ have the <u>same</u> dimension r,

Col space   row space

These are the 2 key ideas linear algebra is built on

- Actual dim
- $C(A)$, $C(A^T)$ have same dim
    - diff spaces
    - diff basis
    - but same dim
- dim $N(A^T) = m - r$

  ↑ since transpose will be free

- Square $n = m$

  dim same

  Spaces <u>not</u> same

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$$

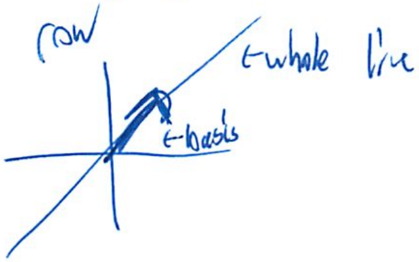
$m = n = 2$

$r = 1$

①

Discibe the row space

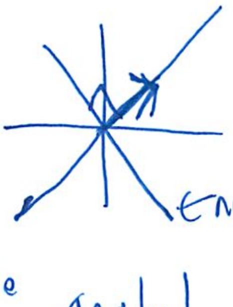
- line
- 1D inside R^2
- basis $\begin{bmatrix}1\\2\end{bmatrix}$



- null space

$$\begin{bmatrix}1&2\\3&6\end{bmatrix}\underset{\overset{\uparrow}{\begin{bmatrix}-2\\1\end{bmatrix}}}{\begin{bmatrix}\quad\\\quad\end{bmatrix}}=\begin{bmatrix}0\\0\end{bmatrix}$$

- Thats a special sol


←null space $N(A)$ $n-r=2-1=1$

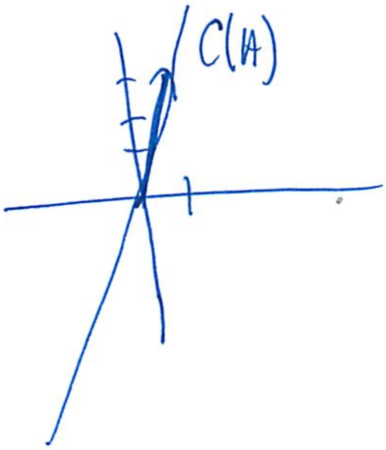
90° angle!

- even when >3 dim

can find angle w/ dot product

Other space ; Col Space

$A^T = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$



Now transpose of null space

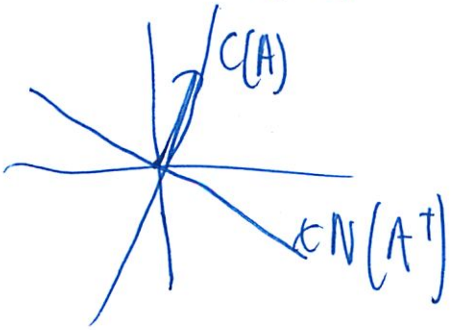
(Some authors don't even put in book

Next time ; Networks
Kirchoffs' Law
Most basic part of applied math)

$A^T = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} \quad \end{bmatrix} = [0\,0$

$\uparrow$

$\begin{bmatrix} -3 \\ 1 \end{bmatrix}$



⑨

Essay on 18.06 website: The 4 Lines
- explains this
- and expands to eigenvalues

2. Why? Proof:

If 100 × 1000 matrix, would it still work?

MATLAB ~~Let~~ A = rand(100, 1000)

rank(A)

↳ most likely ans: 100

because its random

but if only 0 or 1:
since rand gives [0, 1]

Proof:

$$A \xrightarrow{elim} R = \begin{bmatrix} \overset{\text{size } r\times r \text{ — have pivots}}{I} & \overset{\text{free cols – no pivots}}{F} \\ 0 & 0 \end{bmatrix}$$

$\dim C(R) = r$ ← col space
$\dim C(R^T) = r$ ← row space

0 rows not going into basis

Row space is <u>spanned</u> by all the rows
(just means have enough, even 0s

⑩

Could you say rows of r are a basis for (missed)

No. There the zero rows don't count

Rest of proof:

Show did not change dim $A \to R$

18.06 OH

2/20

Going to 3PM OH instead

@ 2e. If now on permutations

Def in this class don't fully determine

- like if go down till <u>first</u> non zero
- but not explicit like ~~it~~ that, in textbook

Would lead to diff RREF

Other TA: Unique

if have matrix A

null space A

Then free variables are same as A;

$$\begin{matrix}0&0&1\\1&2&0\\1&3&0\end{matrix}$$

But ans is always same

* Book says final R is always the same

②

3.2 #20 Special sol depends on 5th col

Other 4 are pivots

Col 1 + 3 + 5 = 0

$$\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

So trying to find <u>which</u> cols add up to 0

~~the~~ set free col to 1

read off which linear combo leads to 0

4 pivots → lets us know 1D

since dimension = n − r

3.2 #23

~~Size of matrix~~

null space → 3D

↓

says matrix has 3 cols

Col space 3 components

↓

matrix 3 rows

→ 3x 3 matrix

So $\begin{bmatrix}1&0&-\\1&3&-\\5&1&-\end{bmatrix}\begin{bmatrix}1\\1\\2\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

↑ solve for

Sheet

10 c) 2x2 example

$\underset{c}{\begin{bmatrix}1&0\\0&1\end{bmatrix}}$ ← Only null space is x=0

← the ⟷ invertable

$\overset{D}{\begin{bmatrix}1&1\\1&1\end{bmatrix}}\overset{c}{\begin{bmatrix}1&0\\0&1\end{bmatrix}}$

? So c does not change D ? No ...?

So $=\left[c\begin{bmatrix}-1\\1\end{bmatrix}\right]$

$\begin{bmatrix}1&1\\0&1\end{bmatrix}\begin{bmatrix}1&0\\0&1\end{bmatrix}$

? again does not change? No ...?

* remember order $CD \neq DC$

multiply identity by that

④ In some cases null space changes
In other cases it does not

<u>Random Qu</u> kernals

$$A\underbrace{B\ x}_{?} = \overline{0}$$

$$\underset{\uparrow}{A}\ \underset{\uparrow}{y} = 0$$

invertable
so no kernal

$$y = 0$$

So $Bx = 0$

says x in kernal of B
So x ~~A~~ is in kernarl of AB
if A invertable

$\begin{bmatrix}1&1\\1&1\end{bmatrix}$ not invertable - can A kernal

$\begin{bmatrix}1&1\\0&1\end{bmatrix}$ can be invertable - so matrix (invertable) = same null space

3.2 # 35 Is the null space the same

$A =$ 4 components

$B =$ 8 components

\# components = # cols

So null space not the same

rect matrices w/ rows < cols have null space

- So many possible sol to an eqn

Textbook gives algorithm for RREF

- Swap rows
- first 4 x 4
  - Squ inv → identity matrix
- then 2 halfs

<u>is</u> identity matrix twice

<u>Same</u> row ops

$R = \begin{bmatrix} I & I \end{bmatrix}$

⑥

So now find nullspace

$$\begin{bmatrix} I & I \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

but actually

$$4\,\overset{8}{\underset{\text{pivot} \qquad \text{free}}{\begin{bmatrix} \\ \\ \end{bmatrix}}}\;{}_{8}\overset{1}{\begin{bmatrix} \\ \\ \\ \\ \end{bmatrix}} = \begin{bmatrix} 0\\0\\0\\0\\0\\0\\0\\0 \end{bmatrix}$$

4 special sols

$$\begin{bmatrix} - \\ - \\ - \\ - \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} - \\ - \\ - \\ - \\ 0 \\ 1 \\ 0 \end{bmatrix} + \dots \text{ 2 more}$$

$$\begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \dots$$

Null space = all possible linear combos

②

So then add it up

$$X = \begin{bmatrix} y \\ -y \end{bmatrix}$$ if y is 4×1 matrix

- any value for y

$$a \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + d \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Get $$\begin{pmatrix} -a \\ -b \\ -c \\ -d \\ a \\ b \\ c \\ d \end{pmatrix}$$ So components can be any real #

⑧

3.3 #28

One way to look at row ops
Take row w/ pivot
Do stuff till get 0s
Brings to RREF

$$\begin{bmatrix} 1 & 2 & & & \\ & & 1 & 3 & \\ & & & & \dots \end{bmatrix}$$

So take pivot → till get 0s on the rows

So only pivots left
If allow col switches
So pivots in diagonal in beginning

$$\begin{matrix} 1 & & & & 0 \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ \cancel{0} & \cancel{0} & & & 0 \end{matrix}$$ elsewhere

⑨

RREF Unique

Unique by purpose

The RREF tracks how cols are linear combos of each oter

$$\begin{matrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{matrix}$$

span of 1st   span of 3rd

↑   ↑

Unique   unique

So row ops don't change properties

So unique in the begining

2/29

Terms

full row rank = if r=m ind rows ←rows are linearly ind

full col rank = if r=n ←all cols ind

Reading 3.5

2/28

3.5 Independence, Basis, Dimension

- about the true size of a subspace
- n cols in m x n matrix
- but true dime $\overset{\text{might}}{\neq}$ n
- dim = # ind cols

  ?didn't we define this already?

- true dime of col space is rank r
- idea of ind applies to any vectors $v_1, \ldots v_n$ in any vector space

Goal understand basis → ind vectors that span the space

* Every vector in the space is a unique combo of basis vectors *

4 key ideas

1. Ind vectors -> no extra vectors
2. Spanning a space -> enough vectors to produce the rest
3. Basis for a space -> not too many or too few
4. Dimension of a space -> the # of a vectors in a basis

Linearly ind

[unconventional def] ↓

The cols of A are lin ind when only sol to $Ax = 0$ is $x = 0$.
No other combo of Ax of the cols gives the zero vector.

when null space is only 0 vector
(ie when invertable)

③

So 3 vector example in R^3

1. If 3 vectors are not on same plane → ind

   No combo of V_1, V_2, V_3 give 0 except $0v_1 + 0v_2 + 0v_3$

2. If are in same plane → they are dep

So lin ind iff only zero vector is $0v_1 + \dots + 0v_n$

* Must say seq of vectors not matrix

Remember when $R^x \leftarrow$ is # of components of vector (which is # of rows [ ])

But what combo of vectors give 0?

$\begin{bmatrix}1\\0\end{bmatrix} \begin{bmatrix}0\\1\end{bmatrix}$ ind

$\begin{bmatrix}1\\0\end{bmatrix} \begin{bmatrix}1\\.0001\end{bmatrix}$ ind

$\begin{bmatrix}1\\1\end{bmatrix} \begin{bmatrix}-1\\-1\end{bmatrix}$ dep

$\begin{bmatrix}1\\1\end{bmatrix} \begin{bmatrix}0\\0\end{bmatrix}$ dep b/c of 0 vector

④

dep exactly when there is a nonzero vector in nullspace
↳ saw before

3 vectors in $\mathbb{R}^2$ can't be ind

- yeah not all in diff dir ??
- the matrix A w/ 3col must have a free variable and a special sol $Ax=0$
- Some combo will produce 3rd vector

But?

← $\mathbb{R}^2$
2 sep
?but I guess one can be linear combo of 3rd
(Think matrix addition)

3 vectors in $\mathbb{R}^3$

if one is a multiple of another one these vectors are dep

But complete test involves all 3 vectors at once

Try to solve $Ax=0$

⑤

$$\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 5 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$$ is

$$-3\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 3 \\ 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

ⓧ so another sol $\begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$

$r = 2$

ind means full col rank $r = n = 3$

└ that means rows are also dep

(did we talk about this before?)

For sq matrix dep cols $\overset{\text{imply}}{\Rightarrow}$ dep rows

Can use elim to find sol to $Ax = 0$

Full col rank Cols $\overset{A}{\text{are}}$ ind when rank $r = n$

There are n pivots and no free variables

Only $x = 0$ in the nullspace

Any set of n vectors in $\mathbb{R}^m$ must be lin dep if $n > m$

⑥

Vectors that Span a Subspace

1st subspace in book → col space

Starting w/ cols $v_1, \ldots, v_n$ space filled out w/

$x_1 v_1 + \ldots + x_n v_n$

Col space consists of all combos Ax of the cols

↑ **Spanned**

*A set of vectors span a space if their linear combo fills the space

- seems so obvious - but complex usage!

Cols of a matrix span its col space

· They might be dependent

Examples

$v_1 = \begin{bmatrix}1\\0\end{bmatrix}$ $v_2 = \begin{bmatrix}0\\1\end{bmatrix}$ span full 2D space $\mathbb{R}^2$

↳ since can make any matrix combo of this

$v_1 = \begin{bmatrix}1\\0\end{bmatrix}$ $v_2 = \begin{bmatrix}0\\1\end{bmatrix}$ $v_3 = \begin{pmatrix}4\\7\end{pmatrix}$ also spans

↳ but unneeded!

⑤

$w = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $w_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ only spans a line

↑ since that is duplicate

Think of 2 vectors coming from $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ in 3D space

generally they span a plane w/ linear combo

[So how is all this relevant again? to solving eqn]

Cols span the col space

New subspace: combo of rows produce the "row space"

Row space of a matrix is the subspace of R^n spanned by the rows.

[illegible] $= C(A^T)$

Rows of $m \times n$ matrix have n components

$$m\begin{bmatrix} \quad n \quad \\ \\ \end{bmatrix}$$

↑ are vectors in R^n – or would be if written as col vectors ↖ Before it was R^m

⑧

Basis for Vector Space

Two vectors can't span all of R^3 - even if ind
(makes sense)

We want enough ind vectors to span the space and not more

* A basis is just right *

A basis for a vector space is a seq of vectors w/ 2 properties:
- basis vectors are lin ind and they span the space

* Combo of properties fund. to linear algebra

Every vector v in the space is a combo of the basis vectors (just enough and not more)

The combo that produces v is unique b/c basis vectors are ind

⑨

* There is one and only way to write v as a combo of the basis vectors *

Standrd basis for $\mathbb{R}^n$ is $n \times n$ identity matrix

---

May find other bases

Basis is not unique

└ But vectors are an unique combo of the basis

---

* The cols of every invertable $n \times n$ matrix gives a basis for $\mathbb{R}^n$

Invertable

Singular

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \qquad B = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \end{pmatrix}$$

Only sol $Ax = 0$

is $x = A^{-1}0 = 0$

Cols are ind

Span whole space - every b is a combo of the cols

* The vectors $v_1 \dots v_n$ are a basis for R^n exactly when they are the cols of an $n \times n$ invertable matrix.

Thus R^n has ∞ many diff bases
!!!

When cols are dep - only keep pivot col
- these are ind + span col space
basis for col space
So is pivot rows of echelon form R

Don't really get point of next section
bases A ≠ bases R ?
row space was the same
∞ly many bases to choose from

$A = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix}$ $R = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$

Basis for col space $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$

Basis for row space $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

(more about in next chaps

How do you find a basis for the space they span?
Given 5 vectors in R^7

1st → Make them rows of A, elim, find non zero rows of R

At → Put vectors as col. Elim. use col basis

All bases for vectors contain the same # of vectors

Dimension of a Vector Space

Many choices for basis vectors, but # of basis vectors does not change

If $v_1, \dots, v_m$ and $w_1 \dots w_n$ are both bases for the same vector space → $m=n$

Dim of R^n space is n

* dimension = # of vectors in every basis

⑫

Note terms

Not
- → the rank of a space
- → dim of a basis
- → basis of a matrix

Instead
- → dim col space
- → rank of a matrix

18.06 Spring 2012 – Problem Set 3

This problem set is due Thursday, March 1st, 2012 at 4pm (hand in to Room 2-106). The textbook problems are out of the 4th edition. For computational problems, please include a printout of the code with the problem set (for MATLAB in particular, `diary('filename')` will start a transcript session, `diary off` will end one.)

Every problem is worth 10 points.

1. Without asking anyone for help, write down an accurate definition of what it means for a matrix to be in reduced row echelon form (RREF).
2. TRUE or FALSE? (No need for explanation):
   (a) Every upper-triangular matrix is in reduced row echelon form?
   (b) Every lower-triangular matrix is in reduced row echelon form?
   (c) Every permutation matrix is in reduced row echelon form?
   (d) The following matrix is in reduced row echelon form?
   $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$
   (e) The reduced row echelon form of A is unique?
   (f) The full solution set of $Ax = b$, where A is $m \times n$ and $b \in \mathbb{R}^m$, is always a vector subspace of $\mathbb{R}^n$?
   (g) The difference $\mathbf{a} = \mathbf{x}_1 - \mathbf{x}_2$, between any two solutions $\mathbf{x}_1$ and $\mathbf{x}_2$ to $A\mathbf{x} = \mathbf{b}$, is a vector that belongs to the null space $N(A)$? (Apply the rule $A(\mathbf{x} + \lambda\mathbf{y}) = A\mathbf{x} + \lambda A\mathbf{y}$ to $A(\mathbf{x}_1 - \mathbf{x}_2)$ to answer the question).
3. Do Problems 20 & 23 from Section 3.2.
4. Do Problem 35 from Section 3.2.
5. Do Problems 3 & 8 from Section 3.3.
6. Do Problems 17 & 28 from Section 3.3.
7. Do Problems 5 & 16 from Section 3.4.
8. Do Problems 24 & 33 from Section 3.4.
9. Do Problem 9 from Section 3.5.

   (See Problem 10 on next page!)

10. In this exercise, we try MATLAB's function `null(A)` for finding a basis (i.e. a minimal set of spanning vectors = a maximal set of independent vectors) for the null space of a matrix. We also try `rref(A)` for finding the reduced row echelon form.

```
B = [1  0  0  0;
     0  0  1  0;
     0  0  0  1;
     0  1  0  0];

C = [1  2  1 -2;
     0  0  1  5;
     0  0  0  0;
     0  0  0  0];

D = [1  2  0  1;
     0  2  2  1;
     0  0  3  3;
     1  0  0  4];
```

(a) Using `null()`, find a basis of each of $N(B), N(C)$ and $N(D)$ (the column vectors in the matrix MATLAB outputs are the basis vectors). Same for $N(BC)$ and $N(DC)$.

(b) Figure out whether $N(C)$ and $N(DC)$ are the same subspaces of $\mathbb{R}^4$, as follows: $\longrightarrow$ MATLAB can easily perform this, if we make use of the following two facts, for V and W subspaces of $\mathbb{R}^n$ with given collections of vectors used for spanning them, respectively $\mathbf{v}_1, \ldots, \mathbf{v}_k$ spanning V and $\mathbf{w}_1, \ldots, \mathbf{w}_l$ spanning W.

<u>**Fact 1**</u>: A vector $\mathbf{b} \in \mathbb{R}^4$ belongs to V *if and only if* the system $A\mathbf{x} = \mathbf{b}$ has at least one solution, where $A = [\mathbf{v}_1 \ \mathbf{v}_2 \ \ldots \ \mathbf{v}_k]$ is the matrix which as columns has a collection of vectors we use to span V.

Example (2×2): In MATLAB we create the augmented matrix $[A|\mathbf{b}]$ and use the command `rref`.

```
A = [1 2;
    -1 -2];
b = [1;
     1];

>> A_aug_b = [A b]

A_aug_b =
      1      2  |  1
     -1     -2  |  1
```

↓ Correction A_aug - b

```
>> rref(A_aug)

ans =
       1      2    | 0
       0      0    | 1
```

(Note: The augmentation bars in the output will not show in MATLAB).

Notice the zero row that has a non-zero entry to the right of the bar: This system $A\mathbf{x} = \mathbf{b}$ has no solution. Hence, $\mathbf{b} = [1, 1]^T$ is not in the subspace spanned by the columns of A.

<u>Fact 2</u>: Two subspaces are the same, $V = W$, *if and only if:*

- i. Vectors spanning V lie in W, that is $\mathbf{v}_1, \ldots, \mathbf{v}_k \in W$ (so $V \subseteq W$), and
- ii. Vectors spanning W lie in V, that is $\mathbf{w}_1, \ldots, \mathbf{w}_k \in V$ (so $W \subseteq V$).

Example: Referring to the previous example, the subspace V spanned by the vectors $\mathbf{b}$ and $[0, 1]^T$ cannot be the same as the subspace W spanned by the columns of A (since we saw $\mathbf{b} \notin W$).

Now, for using Fact 1 & Fact 2 in MATLAB to determine if $N(C)$ and $N(DC)$ are in fact the same, you will need the ":" option:

```
>> A(:,2)  %Example: Gives you the 2nd column from matrix A
```

Then proceed as in the examples, checking each basis vector from one space for membership of the other space.

(c) Which property of the square matrix D explains the result of your comparison of $N(C)$ and $N(DC)$? State this as a general rule, and put a box around it. Apply your rule to explain why $N(DC)$ and $N(BC)$ are the same subspace.

(d) Is $N(CB)$ the same as $N(C)$? Either use the method from (b) again (you can do it all at once using `rref([null(CB) null(C)])`, if you carefully read off the result!), or simply try applying CB to the basis vectors you found for $N(C)$, and vice versa.

Michael Plasmeier   18.06 P-Set 3   77/100   2/28

1. RREF

9/10 Upper triangular U

w/ 1. zeros above the pivots (eliminate upwards)

2. Ones in the pivots (divide whole row by pivot)

So zeros above pivot and below

+ in echelon form

If A = Invertable → R = Identity

Easy to read special sols off

2. T or F

8/10 a) No! No! No!

☺

b) "

c) ~~Yes~~ - is just 1s and 0s. Only 1 one per col

but must be upper triangular so actually No

d) No - Since the 2

e) ~~Hmm...?~~

↓

⑯.

So it says this in the book.

The RREF will be unique - always gets to same outcome True

This makes sense - the RREF has all the "entropy" removed and it can't be simplified further

② .

f) Full sol set $Ax = b$

$A = m \times x$

$$\begin{array}{cc} & n\text{ cols} \\ m\text{ rows} & \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \end{array}$$

Cols have m components

So cols belong to R^m ← col space

When b is in col space - is a combo of cols

So I would say False - R^m instead

? Or do we not use col space here?

g) $a = x_1 - x_2$ is a vector in the subspace $N(A)$

So null space only if $b = 0$ ←

So normally difference is a linear combo

But

So False

X

③ .

3. 3.2 #20 Suppose col 1 + 3 + 5 = 0 in a 4×5

9/10 Matix w/ 4 pivots.

Which col is sure to have no pivot?

Special sol

Null space

pivots = 4 = rank

So 1 free col

Nullspace = All combos of special sols

This is

special + special = complete sol

$Ax = 0$

So solve $Ux = 0$

¿ Or since we know 3, 5 togeter can off set 1

then things are lin dep?

④ .

Ans Col 5 has no pivot ✓ since combo of earlier cols

free variable?

↑ saw earlier kinda

~~ahh~~ but knew that only 1 ind, since 4 pivot cols

Then w/ pivots in 4 other cols

$$s = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$ ← How ✓ did they get this?

Null space contains all multiples of this vector ✓ s

↳ a line in R^5

↑ as expected

But how did they get s?

Ah so col 1 + 3 + 5 = 0

So s is take 1 of each col to = 0

Clever - need to think that way...

---

* Trying to find which cols add up to 0

See free cols to 1

read off which linear combo leads to

↑ which they directly told us!

⑤ .

3.2 #23 Construct a matrix

$$\begin{bmatrix} 1 & 0 \\ 1 & 3 \\ 5 & 1 \end{bmatrix} \qquad \text{null space} \quad \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

Col space

?so have special sols = this
Or include these

But null space

? But how to get there?

? add a row to A?

We know null space is 3D
↓
So matrix has 3 col

Col space has 3 components
└each col has 3 items
↓
matrix has 3 rows

3 x 3 matrix

5b.

So basically $$\begin{bmatrix} 1 & 0 & \_ \\ 1 & 3 & \_ \\ 5 & 1 & \_ \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

↑ solve for

$1 \cdot 1 + 0 \cdot 1 + \_ \cdot 2 = 0$

$+1 + 2x_1 = 0$

$2x_1 = -1$

$x_1 = -\frac{1}{2}$

$1 \cdot 1 + 3 \cdot 1 + \_ \cdot 2 = 0$

$2x_2 = -4$

$x_2 = -\frac{1}{2}$

$5 \cdot 1 + 1 \cdot 1 + 2x_3 = 0$

$2x_3 = -6$

$x_3 = -\frac{1}{3}$

$$\begin{bmatrix} -1/2 \\ -1/2 \\ -1/3 \end{bmatrix}$$ ✓

⑥ .

3.2 #35 If A is a 4 × 4 invertable, describe all vectors in the null space of the 4 × 8 matrix $B = [A\ A]$

~~Key here → invertable ?~~

~~But the cols would be similar to the cols already found - so the new ones would be linear multiples of , So null space does not change ? - ans seems too simple~~

~~Ans~~

~~Contains all vectors $x = \begin{bmatrix} y \\ -y \end{bmatrix}$ for y in R^4~~

~~? What does this tell us ?~~

A = 4 components
B = 8 components

\# components = # cols

So null space is *not* the same

⑥b

Rect matrices w/ rows < cols have null space

— so many possible sol to an eq'n

So do RREF

The first 4 x 4 will just be iddentity

- Since A invertable

$R=I$

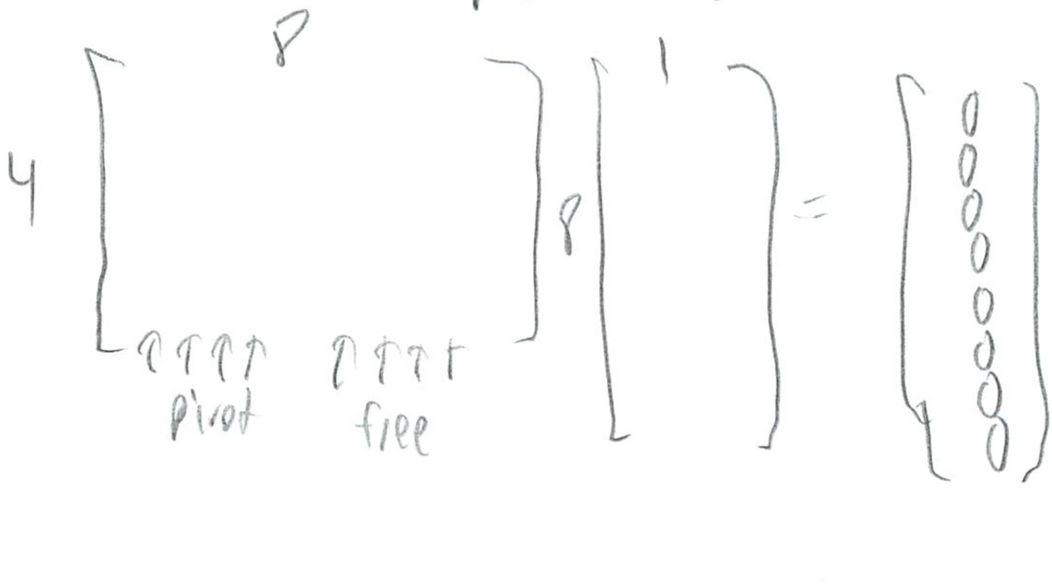
Then 2 horizontally since we do the same row ops

$R=[I\ I]$

Then basically do

$$[I\ I]\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

but w/ proper scale



4c.

So 4 special sol

10/10

$$\begin{bmatrix} - \\ - \\ - \\ - \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} - \\ - \\ - \\ - \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \cdots \text{2 more}$$

Then solve

$$\begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \cdots$$

Null space = <u>all</u> possible linear combos

$$a\begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c\begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -a \\ -b \\ -c \\ -d \\ a \\ b \\ c \\ d \end{bmatrix}$$

So components can be any real #s

$$\begin{bmatrix} -y \\ y \end{bmatrix}$$

⑦ .

3.3 #3 Find reduced R for each of the blocks

8/10

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 3 \\ 2 & 4 & 6 \end{bmatrix} \quad B = \begin{bmatrix} A & A \end{bmatrix} \quad C = \begin{bmatrix} A & A \\ A & 0 \end{bmatrix}$$

So finally a basic, pratice question

First normal elim

Row switch

L it does exchange rows. what was that eek thing? On last P-set

$$\begin{bmatrix} 2 & 4 & 6 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} \leftarrow \text{still not perfect}$$

~~$0 = \frac{0}{2}$~~ So this is U

Lets do 2 pivots = rank

$$U = \begin{array}{c} \begin{bmatrix} \boxed{2} & 4 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & \boxed{3} \end{bmatrix} \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \text{Pivot} & \text{free} & \text{pivot} \\ & 2 \cdot \text{col } 1 & \end{array} \end{array}$$

Now RREF

1. Zeros ~~wa~~ above the pivots by elim ?
2. Produce ones in the pivots by dividing the whole row by its pivots.

⑧.

So get rid of the 6

row 1 ← row 1 − 2 row 3

$$\begin{bmatrix} 2 & 4 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Make pivots 1 by mutiding rows

row 1 ← row 1 ∘ $\frac{1}{2}$

row 3 ← row 3 ∘ $\frac{1}{3}$

$$R_A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ ✓ Matches book I belive

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$ ✓

b) Now $B = [A \; A]$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 3 \\ 2 & 4 & 6 & \underbrace{2 \;\; 4 \;\; 6}_{\text{now all repeat}} \end{bmatrix}$$

all free

but how to represent ?

will steps ya do same

-same pivots

So ends up same ✓

$R_B = [R_A \; R_A]$ ⓪

⑨.

c) $C = \begin{bmatrix} A & A \\ A & 0 \end{bmatrix}$ ← so what does this do

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 3 \\ 2 & 4 & 6 & 2 & 4 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 2 & 4 & 6 & \underbrace{0 \quad 0 \quad 0} & & \end{bmatrix}$$

now this col is **not** a multiple of previous col - so different ;

So swap (permutate) busy rows uptop

$$\begin{bmatrix} \textcircled{2} & 4 & 6 & 2 & 4 & 6 \\ 2 & \textcircled{4} & 6 & 0 & 0 & 0 \\ 0 & 0 & \textcircled{3} & 0 & 0 & 3 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ & \uparrow & & \uparrow & \uparrow & \uparrow \end{bmatrix}$$

actually duplicate free

free

Still 2 pivot cols ⇒ rank = 2

⑩

$\ell_{21} = \frac{2}{2} = 1$

$$\begin{bmatrix} 2 & 4 & 6 & 2 & 4 & 6 \\ 0 & 0 & 0 & -2 & -4 & -6 \\ 0 & 0 & 3 & 0 & 0 & 3 \\ 0 & 0 & 3 & 0 & 0 & 0 \end{bmatrix}$$

↑ thats how row falls out

But how does RREF change?

$$\begin{bmatrix} 1 & 0 & & - - \\ & 0 & & \\ & 0 & 1 & \end{bmatrix}$$

...

? same

Book $R_C = \begin{bmatrix} R_A & 0 \\ 0 & R_A \end{bmatrix}$ ✓

?, how come aren't those rows 0

well like above w/ negitive rows

Ohh thats why on earlier ans was

$\begin{bmatrix} F \\ -F \end{bmatrix}$?

⑪ .

3.3 # 8 Fill out these matrices so the have rank = 1

$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & & \\ 4 & & \end{bmatrix}$ $B = \begin{bmatrix} & 9 & \\ 1 & & \\ 2 & 6 & -3 \end{bmatrix}$ $M = \begin{bmatrix} a & b \\ c & \end{bmatrix}$

rank 1 = only 1 pivot col
? each ind'i

$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \\ 4 & 8 & 16 \end{bmatrix}$ ✓

$B = \begin{bmatrix} 3 & 9 & -9/2 \\ 1 & 3 & -3/2 \\ 2 & 6 & -3 \end{bmatrix}$ ✓

$2 \cdot ? = -3$

$? = \frac{-3}{2}$

$2 \cdot \frac{-3}{2} = -3$

$3 \cdot -\frac{3}{2} = \frac{-9}{2}$

$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

↑ ? hmm can't be 0, or valid row swap

then say $a = e\, b$
$c = e\, d$ e = constant

a, c are already fixed, though.

$\boxed{-2}$

↑ cheap, but legal?

⑫

3.3 #17 Suppose col j of B is combo of previous combo

$\frac{9}{10}$

Show that col j of AB is same combo of AB

So $B = \begin{bmatrix} & \\ & \end{bmatrix}$

↑ col j
linear combo of

Then AB can not have new pivot cols so
rank (A B) ≤ rank (B)

I think I get it - but don't get why

So matrix multiplication

$$\underset{\text{↑ can be any A}}{\begin{bmatrix} a & b \end{bmatrix}} \underset{\text{↑ col j}}{\begin{bmatrix} c \\ d \end{bmatrix}} = \underset{\text{↑ linear combo of B col j}}{\begin{bmatrix} ac + bd \end{bmatrix}}$$

Should do this in full generality: nxn instead of 2x2

But more complex

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \underset{\text{↑ col j}}{\begin{bmatrix} c & e \\ d & f \end{bmatrix}} = \underset{\text{↑ still linear combo col j}}{\begin{bmatrix} ac + bd & ae + bf \\ cc + dd & ce + df \end{bmatrix}}$$

⑬.

I don't think this is sufficent to prove...

But now AB can not have new pivot cols

So $\text{rank}(AB) \leq \text{rank}(B)$

Since, as shown, each rank item is preserved

b) Find A_1 and A_2 so that $\text{rank}(A_1 B) = 1$

$\text{rank}(A_2 B) = 0$

for $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

So

$$\overset{A_1}{\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}} \overset{B}{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}} = \text{rank } 1 \; \begin{array}{cc} \downarrow \text{pivot} & \downarrow \text{free} \\ \multicolumn{2}{c}{\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}} \end{array}$$

✓

↑ RREF, I believe

$$\overset{A_2}{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}} \overset{B}{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}} = \text{rank } 0 \; \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

“ seems too simple

⑭.

3.3 #28 Suppose you allow elementary col operations on A as well as ele. row ops (which get to R). What is the "row + col reduced form" for an m x n matrix of rank r?

; What is the col reduced form of something ;

~~Ans Always~~ ~~$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$~~

~~I is r by r~~

So RREF shows row reduced form

- fill 0s on the rows
- Only pivots left on the rows

So can do same idea w/ col rows

But then will be left w/ only the pivots in diagonal form

$$\begin{array}{ccccc} 1 & & & & \\ & 1 & & & 0 \\ & & 1 & & \\ & & & 1 & 0 \\ 0 & 0 & 0 & 0 & \end{array}$$

0s at the edges

✓

⑮ .

3.4 #5 Under what condition is system solable

Find all sol where condition holds

3/10

$$\left[\begin{array}{ccc|c} 1 & 2 & -2 & b_1 \\ 2 & 5 & -4 & b_2 \\ 4 & 9 & -8 & b_3 \end{array}\right]$$

So what is the rank of this 2

$m = 3$

$n = 3$ Use table

So $r < m$ $r < n$ No full rank

$Ax = b \rightarrow 0$ or ∞ sols

So when is this solvable?

$$R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$$

When last $m-r$ eqns reduce to $0=0$ ✓

$3-1 = 1$ eqn

$4x + 9y - 8z = b_3$

So all sols where $b_3 = 0$

But the matrix is not in RREF

⑯

I guess set free col to 1

$$4x + 9y - 8 = 0$$

$$4x + 9y = 8$$

? need to solve matrix more yes...

Well prob just asks which cols

Previous ans sufficent ?

Or actually solve ?

- can't w/ other bs

(−5)

I think you should look at the model sol'n and see how they write it up... While I see exactly what you're thinking, some of what you write isn't really relevant to the pset!

About full rank matrices $r = m$ or $r = n$

3.4 # 16 The largest possible rank of a 3x5 matrix is $\underline{\ ?\ }$

So largest is full row ✓ rank

$r = m = 3$

↑ since most constrained

Then there pivot in every row ✓ of U and R

The sol to

$r = 3$
$n = 5$
$m = 3$

$m = r \quad r < n$ short + wide

$Ax = b \rightarrow \infty$ sols

$$\begin{bmatrix} I & F \end{bmatrix}$$

So sol is always exists

not is unique

Col space of A is whole space $R^{\textcircled{m}}$?

↑ all of the combos of the col ✓

An example is $A = \begin{bmatrix} 1 & & & 2 & 3 \\ & 1 & & & \\ & & 1 & & \end{bmatrix}$

↑ does that have a col-by story

⑱ .

3.4 # 24 Give an example of matrices A for which the # of sols to $Ax = b$ is

9/10

a) 0 or 1 depending

read off chart!

$r < m$ $r = n$ tall + thin

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$ ✓

b) ∞ regardless of b

$r = m$ $r < n$ short + wide

$$\begin{bmatrix} 1 & 0 & 0 & 2 & 3 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$ ✓

c) 0 or ∞

$r < m$ $r < n$ No full rank

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ ✓

d) 1 regardless of b

$r = m$ $r = n$ square + inertable

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$ ✓

⑲

3.4 #33 The complete sol to $Ax = \begin{bmatrix}1\\3\end{bmatrix}$

is $x = \begin{bmatrix}1\\0\\0\end{bmatrix} + c\begin{bmatrix}0\\0\\1\end{bmatrix}$ Find A

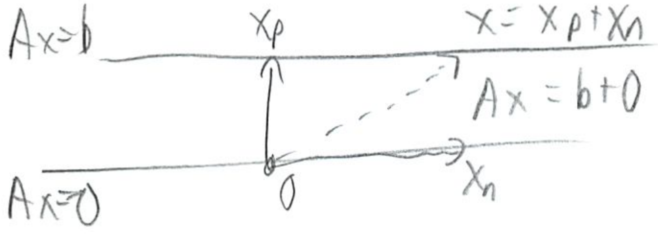
Grrr, I don't like work backwards qu ☹

So one x_p and many x_n

↳ any multiple of s

So this is full row rank

$m < n \rightarrow Ax = b$ underdetermined



Short + wide

more unknowns than eq'ns

So $d = \begin{matrix}1\\0\end{matrix}$

So 0,1 is 3rd col of A $\begin{matrix}0\\1\\0\end{matrix}$

$$\left[\begin{array}{ccc|c} - & - & 0 & 1 \\ - & - & 1 & 0 \end{array}\right]$$

↳R ↳d

Then the other cols are pivots
One will be a repeat

Oh told us b

So ??? can we say

$$\left[\begin{array}{ccc|c}1&0&0&1\\0&1&1&0\end{array}\right]$$

R d

↙ reverse elim $E_{32}^{-1} E_{21}^{-1}$

$$\left[\begin{array}{ccc|c}&&&1\\&&&3\end{array}\right]$$

? can we do this w/ just this info
We don't know pivots

So what's the matrix?

−1

㉑

3.5 #9 Suppose V_1, V_2, V_3, V_4 are vectors in R3

10/10 a) 4 vectors are dep since:

Only R^3 space. One must repeat

pg 170

$n=4$ vectors

$m=3$ ✓

$4 > 3 \to$ lin dep

b) Two vectors V_1 V_2 will be dep if:

: One was a multiple of the other

$$\begin{bmatrix} a \\ b \end{bmatrix} = c \begin{bmatrix} a \\ b \end{bmatrix}$$ ✓

: c is any constant

: too easy ☺

c) Vectors V_1, $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ ✓ dep because:

You can always multipy V_1 times 0

$$0V_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$ ✓

㉒

10. Matlab

2/10

Null = finding basis

N(B) = empty matrix 4 by 0

$$N(C) = \begin{bmatrix} 0 & .92 \\ .5659 & -.31 \\ -.8085 & -.21 \\ .1617 & .04 \end{bmatrix}$$

N(D) = empty matrix 4 by 0

$$N(BC) = \begin{bmatrix} 0 & -.82 \\ .5659 & .31 \\ -.8085 & .21 \\ .1617 & -.04 \end{bmatrix}$$

? same   ? -1·before

$$N(CD) = \begin{bmatrix} 0 & -.97 \\ .76 & -.05 \\ -.63 & -.03 \\ .08 & .21 \end{bmatrix}$$

$$N(DC) = \begin{bmatrix} -.03 & .92 \\ -.55 & -.33 \\ .81 & -.18 \\ -.16 & .03 \end{bmatrix}$$

㉓ ·

b) Figure out whether N(C) and N(DC) are same subspace

Fact 1 from 3.5 book

Ans $= \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 1 \end{array}\right)$

what is your conclusion?

c) What property of sq matrix D explains this

Remembe order $CD \neq DC$

what?

Sometimes it does not change

X

㉔. rref([I null(C·B) null(C)])

d) $\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} = I_4$

Yeah - identity vector does not change it...

What?

Be more precise when you're writing up solutions!

```
>> B=[1 0 0 0; 0 0 1 0; 0 0 0 1; 0 1 0 0]

B =

     1     0     0     0
     0     0     1     0
     0     0     0     1
     0     1     0     0

>> C=[1 2 1 -2; 0 0 1 5; 0 0 0 0; 0 0 0 0]

C =

     1     2     1    -2
     0     0     1     5
     0     0     0     0
     0     0     0     0

>> D = [1 2 0 1; 0 2 2 1; 0 0 3 3; 1 0 0 4]

D =

     1     2     0     1
     0     2     2     1
     0     0     3     3
     1     0     0     4


>> null(B)

ans =
                                        ✓
   Empty matrix: 4-by-0

>> null(C)

ans =

         0    0.9245                  ✓
    0.5659   -0.3142
   -0.8085   -0.2115
    0.1617    0.0423

>> null(D)

ans =
                                        ✓
   Empty matrix: 4-by-0

>> null(B*C)

ans =
```

```
        0   -0.9245
   0.5659    0.3142      ✓
  -0.8085    0.2115
   0.1617   -0.0423

>> null(C*D)        ?

ans =

        0   -0.9744
   0.7682   -0.0511
  -0.6348   -0.0336
   0.0828    0.2162

>> null(D*C)

ans =

  -0.0331    0.9239
  -0.5543   -0.3343    ✓
   0.8155   -0.1824
  -0.1631    0.0365

>> A = [1 2; -1 -2]

A =

     1     2
    -1    -2

>> b = [1; 1]

b =

     1
     1

>> A_aug_b = [A b]

A_aug_b =

     1     2     1
    -1    -2     1

>> rref(A_aug_b)

ans =

     1     2     0
     0     0     1
```

```
>> rref([null(C*B) null(C)])

ans =

     1     0     0     0
     0     1     0     0
     0     0     1     0
     0     0     0     1
```

Solutions

18.06 Spring 2012 – Problem Set 3

This problem set is due Thursday, March 1, 2012 at 4pm (hand in to Room 2-106). The textbook problems are out of the 4th edition. For computational problems, please include a printout of the code with the problem set (for MATLAB in particular, `diary('filename')` will start a transcript session, `diary off` will end one.)

1. Without asking anyone for help, write down an accurate definition of what it means for a matrix to be in reduced row echelon form (RREF).

   Solution. $m \times n$ matrix R is in RREF means

   (a) R is in echelon form.
   (b) Every pivot is 1.
   (c) Columns with a pivot have no other nonzero entry.

   ☐

2. TRUE or FALSE? (No need for explanation):

   (a) Every upper-triangular matrix is in reduced row echelon form?
   (b) Every lower-triangular matrix is in reduced row echelon form?
   (c) Every permutation matrix is in reduced row echelon form?
   (d) The following matrix is in reduced row echelon form?

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

   (e) The reduced row echelon form of A is unique?
   (f) The full solution set of $Ax = b$, where A is $m \times n$ and $b \in \mathbb{R}^m$, is always a vector subspace of $\mathbb{R}^n$?
   (g) The difference $\mathbf{a} = \mathbf{x}_1 - \mathbf{x}_2$, between any two solutions $\mathbf{x}_1$ and $\mathbf{x}_2$ to $A\mathbf{x} = \mathbf{b}$, is a vector that belongs to the null space $N(A)$? (Apply the rule $A(\mathbf{x} + \lambda\mathbf{y}) = A\mathbf{x} + \lambda A\mathbf{y}$ to $A(\mathbf{x}_1 - \mathbf{x}_2)$ to answer the question).

   Solution. (a) No. The rows of all zeros must be below all the other rows. This is not true, for instance, of

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

   (b) No. For instance,

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

   is not.

(c) No. For instance, for the matrix

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

the pivot of the second row is to the left of the pivot of the first row.

(d) No. The leading coefficient of the second row is not a one.

(e) Yes. This will be explained in class, though you do not need to know a proof. (The proof-oriented reader should read e.g. http://web.gccaz.edu/ wkehowsk/225-Linear-10-11-Sp/yuster-rref-unique.pdf .)

(f) No. For instance, the solution set of

$$\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

contains a unique vector, $[0, 1]^T$. This is not a vector subspace of $\mathbb{R}^2$.

(g) Yes. $A(\mathbf{x}_1 - \mathbf{x}_2) = A(\mathbf{x}_1) - A(\mathbf{x}_2) = \mathbf{b} - \mathbf{b} = \mathbf{0}$.

$\square$

3. Do Problems 20 & 23 from Section 3.2.

Solution to 3.2.20:

Let A be the matrix in the problem.
The column 5 does not have pivot. If not, since $(A)_{45} = c \neq 0$ is a pivot and $(A)_{4i} = 0$ for any $i \neq 5$, column 1 + column 3 + column 5 = $(*, *, *, c)^T \neq \mathbf{0}$. In other words, the fifth variable x_5 is the only free variable. We have

$$A \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \text{column } 1 \cdot 1 + \text{column } 3 \cdot 1 + \text{column } 5 \cdot 1 = \mathbf{0}.$$

Hence the special solution is $(1, 0, 1, 0, 1)^T$ and the null space is $\{(x_5, 0, x_5, 0, x_5)^T : x_5 \in \mathbf{R}\}$.

Solution to 3.2.23:

$$(a, b, c) = \left(-\frac{1}{2}, -2, -3\right)$$

satisfies the equation

$$\begin{bmatrix} 1 & 0 & a \\ 1 & 3 & b \\ 5 & 1 & c \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = 0.$$

Hence

$$\begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 1 & 3 & -2 \\ 5 & 1 & -3 \end{bmatrix}$$

is a matrix we wanted.

4. Do Problem 35 from Section 3.2.

   Solution. The nullspace of $B = [A\,A]$ contains all vectors $\mathbf{x} = \begin{bmatrix} \mathbf{y} \\ -\mathbf{y} \end{bmatrix}$ for all $\mathbf{y}$ in $\mathbb{R}^4$.

   ☐

5. Do Problems 3 & 8 from Section 3.3.

   Solution to 3.3.3:

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 3 \\ 2 & 4 & 6 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 4 & 6 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = RREF(A).$$

Using the same elimination or permutation operators as in the case A, we get

$$RREF(B) = [RREF(A)RREF(A)].$$

$$C = \begin{bmatrix} A & A \\ A & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} A & A \\ 0 & -A \end{bmatrix} \longrightarrow \begin{bmatrix} A & 0 \\ 0 & -A \end{bmatrix} \longrightarrow \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} RREF(A) & 0 \\ 0 & RREF(A) \end{bmatrix} = RREF(C).$$

Solution to 3.3.8:

If the matrix has rank 1, every column is constant multiple of any other nonzero columns. So

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \\ 4 & 8 & 16 \end{bmatrix}, B = \begin{bmatrix} 3 & 9 & -\frac{9}{2} \\ 1 & 3 & -\frac{3}{2} \\ 2 & 6 & -3 \end{bmatrix}.$$

For M, if $a \neq 0$,

$$M = \begin{bmatrix} a & b \\ c & \frac{bc}{a} \end{bmatrix}$$

and if $a = 0$,

$$M = \begin{bmatrix} 0 & b \\ 0 & d \end{bmatrix} \text{ for any } (b, d) \neq (0, 0), \text{ or } M = \begin{bmatrix} 0 & 0 \\ c & d \end{bmatrix} \text{ for any } (c, d) \neq (0, 0).$$

6. Do Problems 17 & 28 from Section 3.3.

   Solution to 3.3.17:

   (a) By matrix multiplication, each column of AB is A times the corresponding column of B. So if column j of B is a combination of earlier columns, then column j of AB is the same combination of earlier columns of AB. Thus rank $(AB) \leq$ rank (B). There are no new pivot columns!

   (b) The rank of B is $r = 1$. Multiplying by A cannot increase this rank. The rank of AB stays the same for $A_1 = I$ and $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. It drops to zero for $A_2 = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$.

   Solution to 3.3.28:

   The row-column echelon form is always $\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$; I is the $r \times r$ identity matrix.

7. Do Problems 5 & 16 from Section 3.4.

   Solution to 3.4.5: Consider the augmented matrix

$$\begin{bmatrix} 1 & 2 & -2 & b_1 \\ 2 & 5 & -4 & b_2 \\ 4 & 9 & -8 & b_3 \end{bmatrix}.$$

   The operations those make the first 3×3 matrix to RREF change our augmented matrix to

$$\begin{bmatrix} 1 & 0 & -2 & 5b_1 - 2b_2 \\ 0 & 1 & 0 & -2b_1 + b_2 \\ 0 & 0 & 0 & -2b_1 - b_2 + b_3 \end{bmatrix}.$$

   Hence this equation is solvable when $-2b_1 - b_2 + b_3 = 0$ and the set of solutions is $\{(5b_1 - 2b_2 + 2z, -2b_1 + b_2, z) : z \in \mathbf{R}\}$.

   Solution to 3.4.16:

   The largest possible rank of a 3 by 5 matrix is 3. Then there is a pivot in every row of U and R. The solution of $Ax = b$ always exists. The column space of A is $\mathbf{R}^3$. An example of A is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

8. Do Problems 24 & 33 from Section 3.4.

   Solution to 3.4.24:

   (a) $\begin{bmatrix} 1 \\ 1 \end{bmatrix} [x] = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ has 0 or 1 solutions, depending on $\mathbf{b}$.

(b) $\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [b]$ has infinitely many solutions for every b.

(c) There are 0 or ∞ solutions when A has rank $r < m$ and $r < n$: the simplest examples is a zero matrix.

(d) *One* solution for all $\mathbf{b}$ when A is square and invertible (like $A = I$).

Solution to 3.4.33:

If the complete solution to $A\mathbf{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ c \end{bmatrix}$ then $A = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}$.

9. Do Problems 9 from Section 3.5.

Solution. (a) the dimension of $\mathbf{R}^3$ is 3 and 3 is the biggest possible number of independent vectors in $\mathbf{R}^3$.

(b) there exists $(c_1, c_2) \neq (0, 0)$ such that $c_1 \cdot v_1 + c_2 \cdot v_2 = 0$.

(c) $0 \cdot v_1 + 1 \cdot (0, 0, 0) = (0, 0, 0)$.

$\square$

(See Problem 10 on next page!)

10. In this exercise, we try MATLAB's function `null(A)` for finding a basis (i.e. a minimal set of spanning vectors = a maximal set of independent vectors) for the null space of a matrix. We also try `rref(A)` for finding the reduced row echelon form.

```
B = [1  0  0  0;
     0  0  1  0;
     0  0  0  1;
     0  1  0  0];

C = [1  2  1 -2;
     0  0  1  5;
     0  0  0  0;
     0  0  0  0];

D = [1  2  0  1;
     0  2  2  1;
     0  0  3  3;
     1  0  0  4];
```

(a) Using `null()`, find a basis of each of $N(B), N(C)$ and $N(D)$ (the column vectors in the matrix MATLAB outputs are the basis vectors). Same for $N(BC)$ and $N(DC)$.

(b) Figure out whether $N(C)$ and $N(DC)$ are the same subspaces of $\mathbb{R}^4$, as follows:
$\longrightarrow$ MATLAB can easily perform this, if we make use of the following two facts, for V and W subspaces of $\mathbb{R}^n$ with given collections of vectors used for spanning them, respectively $\mathbf{v}_1, \ldots, \mathbf{v}_k$ spanning V and $\mathbf{w}_1, \ldots, \mathbf{w}_l$ spanning W.
Fact 1: A vector $\mathbf{b} \in \mathbb{R}^4$ belongs to V *if and only if* the system $A\mathbf{x} = \mathbf{b}$ has at least one solution, where $A = [\mathbf{v}_1\ \mathbf{v}_2\ \ldots\ \mathbf{v}_k]$ is the matrix which as columns has a collection of vectors we use to span V.
Example (2 × 2): In MATLAB we create the augmented matrix $[A|\mathbf{b}]$ and use the command `rref`.

```
A = [1 2;
    -1 -2];
b = [1;
     1];

>> A_aug_b = [A b]

A_aug_b =
     1     2  |   1
    -1    -2  |   1
```

```
>> rref(A_aug_b)

ans =
        1     2   | 0
        0     0   | 1
```

(Note: `A_aug_b` is only a variable name. The augmentation bars in the output will not show in MATLAB).

Notice the zero row that has a non-zero entry to the right of the bar: This system $A\mathbf{x} = \mathbf{b}$ has no solution. Hence, $\mathbf{b} = [1, 1]^T$ is not in the subspace spanned by the columns of A.

Fact 2: Two subspaces are the same, $V = W$, *if and only if:*

i. Vectors spanning V lie in W, that is $\mathbf{v}_1, \dots, \mathbf{v}_k \in W$ (so $V \subseteq W$), and
ii. Vectors spanning W lie in V, that is $\mathbf{w}_1, \dots, \mathbf{w}_k \in V$ (so $W \subseteq V$).

Example: Referring to the previous example, the subspace V spanned by the vectors $\mathbf{b}$ and $[0, 1]^T$ cannot be the same as the subspace W spanned by the columns of A (since we saw $\mathbf{b} \notin W$).

Now, for using Fact 1 & Fact 2 in MATLAB to determine if $N(C)$ and $N(DC)$ are in fact the same, you will need the ":" option:

```
>> A(:,2)  %Example: Gives you the 2nd column from matrix A
```

Then proceed as in the examples, checking each basis vector from one space for membership of the other space.

(c) Which property of the square matrix D explains the result of your comparison of $N(C)$ and $N(DC)$? State this as a general rule, and put a box around it. Apply your rule to explain why $N(DC)$ and $N(BC)$ are the same subspace.

(d) Is $N(CB)$ the same as $N(C)$? Either use the method from (b) again (you can do it all at once using `rref([null(CB) null(C)])`, if you carefully read off the result!), or simply try applying CB to the basis vectors you found for $N(C)$, and vice versa.

Solution. (a) Bases for the null spaces are as follows:

```
>> null(B)

ans =

   Empty matrix: 4-by-0

>> null(C)

ans =

         0    0.9245
    0.5659   -0.3142
```

```
   -0.8085   -0.2115
    0.1617    0.0423

>> null(D)

ans =

   Empty matrix: 4-by-0

>> null(B*C)

ans =

         0   -0.9245
    0.5659    0.3142
   -0.8085    0.2115
    0.1617   -0.0423

>> null(D*C)

ans =

   -0.0331    0.9239
   -0.5543   -0.3343
    0.8155   -0.1824
   -0.1631    0.0365
```

(b) Here we get, for example:

```
nC=null(C);
nC_1 = nC(:,1);
nC_2 = nC(:,2);

nDC = null(D*C);

redux1 = rref([nDC nC_1])
redux2 = rref([nDC nC_2])

>> redux1 =

    1.0000         0   | -0.9994
         0    1.0000   | -0.0358
         0         0   |       0
         0         0   |       0
```

```
redux2 =

    1.0000         0    | -0.0358
         0    1.0000    |  0.9994
         0         0    |    0
         0         0    |    0
```

Here we looked at the system $A\mathbf{x} = \mathbf{b}_i$, with A's columns being the basis of $N(DC)$ and for $i = 1, 2$ let $\mathbf{b}_1 =$ `nC_1`, $\mathbf{b}_2 =$ `nC_1` be the two basis vectors we got for $N(C)$ in (a). Since in both cases the system is consistent (the zero rows in the left compartment of the above RREF of the augmented matrix has a corresponding zero in the right compartment).

Thus, since $\mathbf{b_1}, \mathbf{b_2} \in N(DC)$ and since $N(C)$ is spanned by its two basis vectors $\mathbf{b_1}, \mathbf{b_2}$, we conclude that: $N(C) \subseteq N(DC)$.

Note: $N(C) \subseteq N(DC)$ is true for *any* matrices D and C! Why? Because if $\mathbf{x} \in N(C)$, meaning $C\mathbf{x} = 0$, then also $DC\mathbf{x} = D\mathbf{0} = \mathbf{0}$ meaning $\mathbf{x} \in N(DC)$.

Similar code, reversing the roles of C and DC checks for us that also (which is not always true - see below): $N(DC) \subseteq N(C)$.

Thus, we have checked that: $N(DC) = N(C)$.

(c) The property the square matrix D has is: Invertible. Here's the rule:

$$\boxed{\text{If } D, C \text{ are any } n \times n \text{ matrices, and } D \text{ invertible, then } N(DC) = N(C).}$$

We saw the invertibility of D above in (a): The basis for the null space was $\emptyset$ (the empty set), so $N(D) = \{0\}$ (the subspace only consisting of the zero vector). Thus, if we reduced D to its RREF matrix R we would obtain the 4×4 identity I (since D is square!). But this means that D is invertible.

This also explains why B is invertible, using (a). Now, we may use our new rule: $N(DC) = N(D(B^{-1}B)C) = N((DB^{-1})BC) = N(BC)$.

(d) No, $N(CB)$ and $N(C)$ are not the same (in this example).

```
nC=null(C);
nCB=null(C*B);
BigMat = rref([nCB nC]);

BigMat =

     1     0  |   0     0
     0     1  |   0     0
     0     0  |   1     0
     0     0  |   0     1
```

Note that we have solved all the four systems at once by using the augmentation. Reading left-to-right, you can see that none of the two basis vectors MATLAB chose for us for $N(C)$ belong to $N(CB)$. Reading right-to-left, we see that reversely the $N(CB)$ basis we chose is not in $N(C)$. So these subspaces are not identical.

Alternatively, we can try:

```
C*B*nC_1

ans =

   -2.5870
    2.9913
         0
         0
```

Since that's not the zero vector, we have that the vector `nC_1` from $N(C)$ is not in $N(CB)$. So, these two subspaces are not the same.

□

18.06 Lecture

3/2

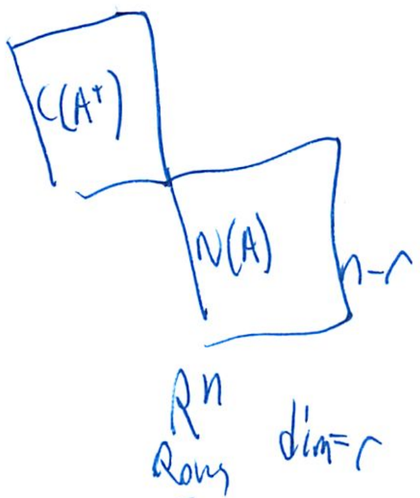
Today: Matrices he sees all the time

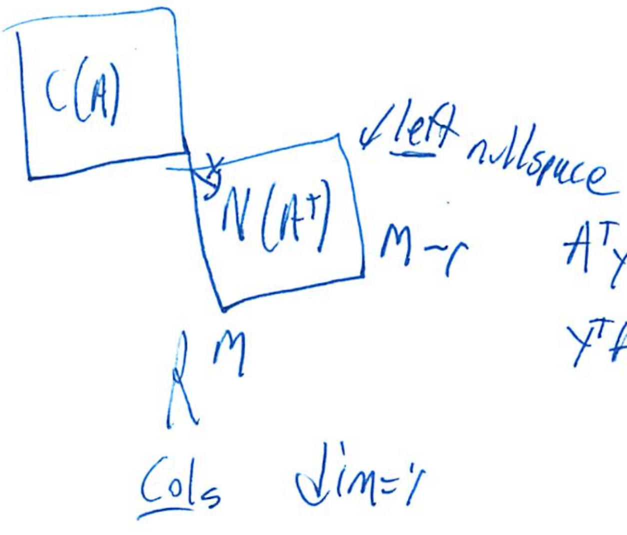
Large

Connect to applications

Beutiful examples

Row space = $C(A^T)$





$A^T y = 0$ cols

$y^T A = 0^T$ rows

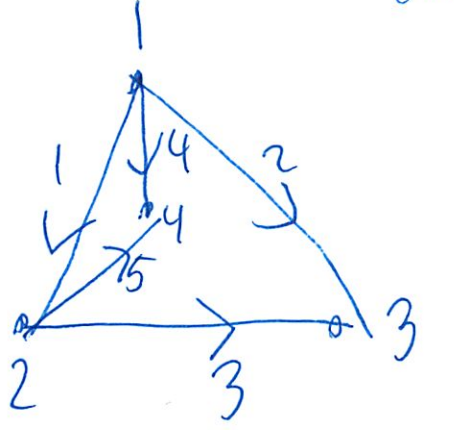
Last time: dimension for both spaces

~~Important~~

②

Matrix starts w/ a <u>graph</u>

- Some nodes connected
- Some are not (not complete)



#s from physical assignment

Adjecency matix n x ~~m~~n m = 5 edges
n = 4 nodes

Incidunce " ~~nxm~~ m x n

Laplicican " n x ~~m~~ n

For example: the internet

~~the~~ nodes = websites

lines = links

Google's Page Rank

Or Phone company studying their network

③

He is taking it as undirected

Allow to say what is ±

Could also be network of friends
- 6 Degrees of Connection

Not incidence is m x n
- [This is part of the case
  - not just a sidebar]

Adjacency

$$\begin{array}{cc} & 4 \\ 4 & \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Would allow us to reconstruct [illegible] graph

- but not the #s
- or the arrow direction

④

Incidence

5x4

$$\begin{array}{ccccl} \text{node } 1 & 2 & 3 & 4 & \\ \downarrow & \downarrow & \downarrow & \downarrow & \\ -1 & 1 & 0 & 0 & \leftarrow \text{edge } 1 \\ -1 & 0 & 1 & 0 & \leftarrow 2 \\ 0 & -1 & 1 & 0 & \leftarrow 3 \\ -1 & 0 & 0 & 1 & \leftarrow 4 \\ 0 & -1 & 0 & 1 & \leftarrow 5 \end{array}$$

-1 since edge leaving node

Elim

→ combo of rows

- if some rows dep, there will be som 0 rows in R
- is exactly $A^T y = 0$

  $y^T A = 0$ from earlier

- tells what combo of the rows to give 0 row

  cols

- $Ax=0$ ← combo of cols to get 0 col
- like reversing row pic and col pic

So back to our incidence poster

rank =

nullspace : col1 + col 2 + col 3 + col4 = zero col

$$x = c\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$$

↑ nullspace is a line

Only combo that gives 0

$$\text{So rank} = \underset{\substack{\uparrow\\ n\\ \text{cols}}}{4} - \underset{\substack{\uparrow\\ \text{null space dim}\\ n-r}}{1} = 3$$

basis for col space

First 3 pivot cols?

Can do elim to try

Or think about it

Col1, col 2, col 3

Can we see 1st 3 cols are ind

(6)

How to tell 1st 3 cols are indi

- forget the 4th for now

$$\begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

How do you convince?

Look at last row

$$0x_1 + -1x_2 + x_3 = 0$$

Then know x_2 must be 0

Look at row 4

x_1 must be 0

Like it had $\begin{bmatrix} 2 & 5 & 6 \\ 0 & 3 & 3 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

3rd row x_3 must be 0
2nd row x_2 must be 0
1st row x_1 must be 0
so invertable, full rank

⑦

Triangular matrix not invertible

$$\begin{bmatrix} 5 & ? & ? \\ 0 & 0 & ? \\ 0 & 0 & 9 \end{bmatrix}$$

? have 0 on the pivots - at least 1 of them will be dep

- any of them

$$\begin{bmatrix} 5 & ? & ? \\ 0 & 7 & ? \\ 0 & 0 & 0 \end{bmatrix}$$ are dep

+ 3rd col will be combo of 1st two

So can we find a non tivial

$$\begin{bmatrix} 5 & ? & ? \\ 0 & 7 & ? \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Col 1, 2, 3 are not the only basis *

Could have been diff cols

Or vectors that aren't even cols

(8)

Remember

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

Are we going to find vectors in the null space of A^T?

$A^T y = 0$

↑ A was 5 x 4

A^T is 4 x 5

$$A^T = \begin{bmatrix} -1 & -1 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 & -1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \overset{y =}{\begin{bmatrix} y_1 \\ x_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

y is in dim 2

= m - r

5 - 3 = 2

(8)

Remember

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

Are we going to find vectors in the null space of A^T?

$A^T y = 0$

A was 5 x 4

A^T is 4 x 5

$$A^T = \begin{bmatrix} -1 & -1 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 & -1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \overset{y=}{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

y is in dim 2

= m-r

5-3=2

⑨

So looking for 2 sol

And the meaning for this eq

5 componets = 5 edges

Tells us the flows on the edges

Like $x \rightarrow$ voltages at nodes

$y \Rightarrow$ currents on edges

If had row 2, node 2

$$y_1 - y_3 - y_5 = 0$$

+ flow in = flow out at each node

Kirchhoff's ... ← (Prof: don't spell this wrong!)

Basics of applied math

Law of conservation

How to send flow so that in = out at every edge

If $y_1 = 1$
$y_2 = 0$
$y_3 = 0$
$y_4 = 1$
$y_5 = 1$

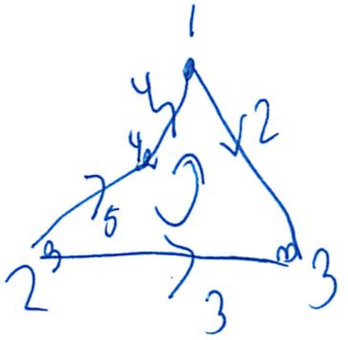
to send current to a certain node from a certain node

⑩

See if the cols of A^T add up to this

2nd special sol

Send current around a loop



$$\begin{bmatrix} 0 \\ -1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

3rd special sol

Add both loops

$$\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Euler

\# nodes − # edges + # loops

4 − 5 + 2 = 1

"only for 2D graphs"

18.06 Lecture

3/5

Today: Start of Orthagonality

- of spaces we know - so not useless

Fri: Quiz

- on ideas
- not just row reduction
- saw ideas over + over
- will use the words we introduced

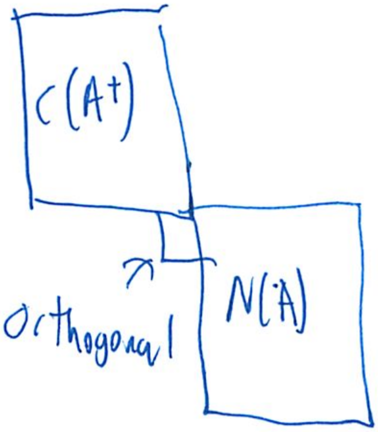
→ Orthogonal vectors

$$x^T y = 0 \quad (\text{when } \theta = 90°)$$

Orthogonial subspaces

ex Rowspace $C(A^T)$

Null space $N(A)$

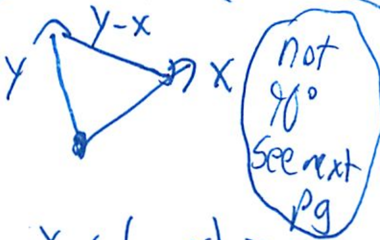


②

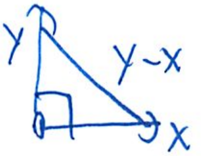
Orthogonal complements

Pythagoras

Looking at triangle



$x + (y-x) = y$



2D question

$$\|x\|^2 + \|y\|^2 = \|y-x\|^2$$

lenght of vector

$$\cancel{x^Tx} + \cancel{y^Ty} \overset{?}{=} (y-x)^T(y-x)$$

$$= \cancel{y^Ty} - y^Tx$$

$$+ \cancel{x^Tx} - x^Ty$$

$$0 = -y^Tx - x^Ty$$

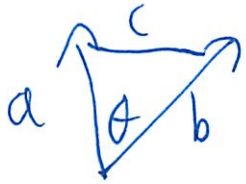
are really the same

$$0 = y^Tx$$

③

If not right angle - get law of Cosines

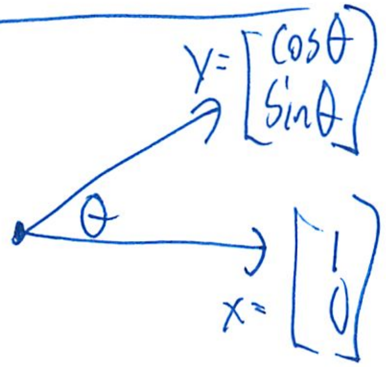
$$c^2 = a^2 + b^2 - 2ab\cos\theta$$



Can apply similar rules to other

$$\cos\theta = \frac{x^T y}{\|x\| \|y\|}$$

Where cosθ comes from



$$y^T x = \cos\theta$$

④

Orthogonial Subspaces

Subspaces V, W are orthogonal when
every v in V is orth. to every w in W

Like floor, wall are both subspaces
Are they orthogonial?

No!

Line that runs along bottom are in both
So it can't be orthogonal

V dim = 2
W dim = 2 ⓧ can't in R^3

What would make this correct?

Define W = A vertical line
└ the z-axis
└ taking the whole line - not just a vector

V dim = 2
W dim = 1 ✓ can in R^3 2+1

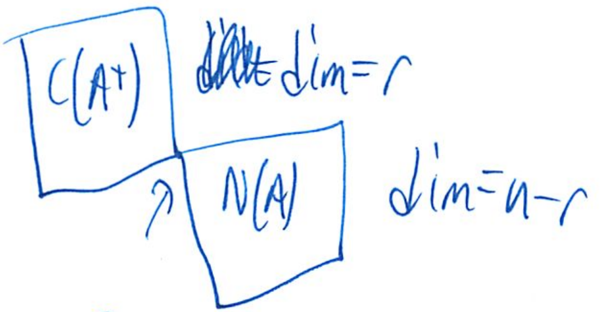
0 subspace is orthogonal to every other subspace

Orthogonal complements

dimensions of V, W add to full dimensions

$2+1=3$

So



need to know stuff for quiz

are orthogonal complements

$r+n-r=n$

---

null space contains every vector orthogonal to the row space

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

~~span~~

rank = 2

span row space but not basis for row space

$n-r=3-2=1$

So null space

$$\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{bmatrix}\begin{bmatrix}1\\ -2\\ 1\end{bmatrix}$$

$$\left.\begin{array}{l}2(\text{row } 2) = \text{row } 1 + \text{row } 3\\ 2(\text{col } 2) = \text{col } 1 + \text{col } 3\end{array}\right\}$$ not symmetric but $C(A) = C(A^T)$ for some reason

Why is Null space orthogonal to the row space?

$$N(A) \perp C(A^T)$$

Reason Look at $Ax = 0$

$$\begin{bmatrix}\text{row } 1\\ \text{row } 2\\ \vdots\\ \text{row } m\end{bmatrix}\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix} = \begin{bmatrix}0\\ \vdots\\ 0\end{bmatrix}$$

Col space = plane

See $x_1 \perp$ row 1 in the null space

$$\boxed{\text{perpendicular} = \text{orthogonal} = \perp}$$

⑦

Since $\text{row}_1 \; x_1 = 0$

$\text{row}_2 \; x_2 = 0$

$\vdots$

$\text{row}_m \; x_n = 0$

So we checked is perpendicular to row
but everyone in the row space?
Lots of other stuff in row space
↳All of their combos!

So any combo gives 0.

$c_1(\text{row}1) + \dots + c_m(\text{row}m) \circ x = 0$

Why?

$c_1(\text{row}1) \circ x + c_2(\text{row}2) \circ x + \dots$

$c_1 \cdot 0 + c_2 \cdot 0 + \dots = 0$

So $N(x)$ has EVERY x w/ $Ax=0$

↳full dim $n-r$

$r+(n-r)=n$ ✓Complement

* know the words *

⑧

Orthogonal basis

Basis $v_1, \dots, v_k$ for V is orthogonal basis

if they span the space V

↑ every vector in the space is a combo of these

but is more to basis!

Here, Each pair is orthogonal $(v_i)^T(v_j) = 0 \quad i \neq j$

Now: one more thing

Form matrix Q

$$Q = \begin{bmatrix} v_1 & \dots & v_k \end{bmatrix}$$

↑ use Q when cols are w/ orthogonal

One improvement on an orthogonal basis:

Normalize to unit length 1

$$V_i^T V_j = 1$$

A orthonormal matrix

↑Q only means unit vectors

---

normalize Vs to be 1

$$Q^T Q = \begin{bmatrix} V_1^T \\ \vdots \\ V_k^T \end{bmatrix} \begin{bmatrix} V_1 & \cdots & V_k \end{bmatrix} \approx \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I_{k \times k}$$

$$Q^T Q = I$$

So ready to go in this chap

Wed: Review

- up to chap 3.5
- ~~except~~ not new material in 3.6 (ie row space
- hw uses that stuff, but comes from 3.5

EXAM 1 REVIEW

JENNIFER PARK

This is an unofficial summary of topics we have covered so far in class, and what I think is important for the upcoming exam. Please do NOT take this as a complete list of things to study, but rather a rough guideline on the things that you absolutely must know for the exam.

1. DEFINITIONS

Make sure you can define these things correctly for the exam:

- The **transpose** of a matrix is...
- A matrix is in **echelon form** if...
- A matrix is in **reduced row echelon form** if... (see the model solution for pset 3 - many people missed some parts of the definition!)
- The **rank** of a matrix is...
- An **elimination matrix is**... A **permutation matrix** is...
- A **vector space** is...
- A **subspace** of a vector space is...
- A **linear combination** of vectors $v_1, \cdots v_n$ is...
- $v_1, \cdots, v_n$ are **linearly dependent** if... They are **linearly independent** if...
- The **column space** of a matrix A is...
- The **null space** of a matrix A...
- Vectors $v_1, \cdots, v_n$ **span** the vector space V if...
- $v_1, \cdots, v_n$ are a **basis** for the vector space V if...
- The **dimension** of a vector space V is...

2. SOLVING MATRIX EQUATIONS

We have used several techniques in solving matrix equations of the form $Ax = b$. Please look back to your notes, and make sure you understand the following: Again, this is not meant to be a complete list!

- Do you understand the row and the column pictures of a matrix equation?
- Solving by elimination and back substitution
- Using elimination and permutation matrices. Make sure you can do the following factorizations: $A = LU, A = LDU, PA = LU, A = LDL^T$ (the last one only applies when A is symmetric)
- Reduced Row Echelon Form
- when does a matrix equation have 0, 1, or ∞-many solutions?
- Can you describe the complete set of solutions in a matrix equation?
- When is a square matrix invertible? How can you find the inverse matrix?

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18.06 Recitation

3/6

Exam 1 Fri

Review Session from other TA

4:30 - 5:30

Thur 2 - 190

Lecture tomorrow: Review Session

OH 7-9 Wed instead of 7-8 - bonus hour

Or via email jmypark@math.mit.edu

Row + Col Pictures

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

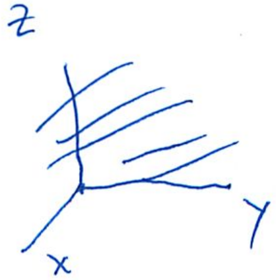
Row pic → look at row eqn

$x = 0$ → yz plane

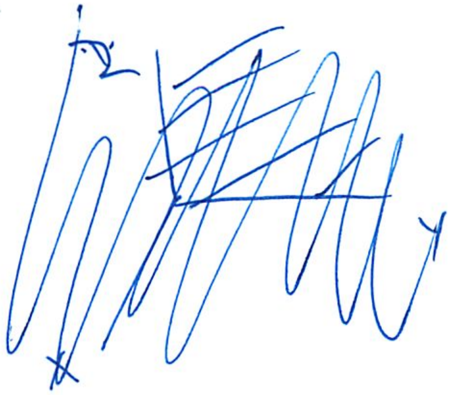
$y = 0$

$z = 0$

since x is 0
y, z can be anything



②



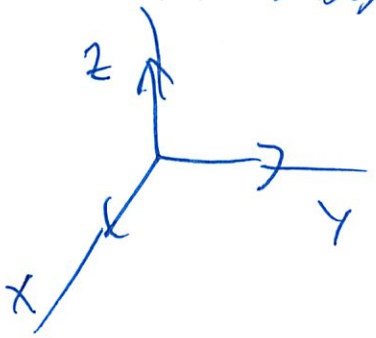
Y=0 is xz plane
since y is 0

Col Picture

look at cols as a vector

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

just our 3 basis vectors



Anything straight + through origin = subspace

③

Finding Solution to Ax =b

from 18.03

If had diff eq

$$y'' + x\,y' + z\,y = f(x)$$

Guess 1 sol

Find homogeneous sol

Use pincipal of superposition

Also have particular sol

$$\underset{\substack{\uparrow\\ \text{superpos}}}{y} = \underset{\substack{\uparrow\\ \text{homo}}}{y_h} + \underset{\substack{\uparrow\\ \text{particular}}}{y_p}$$

Special sols

$$A\,\underset{\text{nullspace}}{x_s} = b$$

Null space $\quad A\,x_{null} = b$

So complete sol $X = X_s + X_{null}$

Example

Solve $\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ -2 & -3 \end{bmatrix} x = \begin{bmatrix} 2 \\ 2 \\ -4 \end{bmatrix}$

For particular sol
- want 1 sol
- like normal

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 1 & 2 & 2 \\ -2 & -3 & -4 \end{array}\right]$$

$\downarrow$ row 2 = row 2 -2 row 1
row 3 = row 3 +2 row 1

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{array}\right]$$

$\downarrow$ row 3 = row 3 + row 2

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

So $x_s = \begin{bmatrix} 2 \\ 10 \end{bmatrix}$

Now find nullspace $N(A)$

Instead of $\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$ have $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

or $\begin{bmatrix} 2 \\ 2 \\ -4 \end{bmatrix}$ originally

$N(A) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

(If free cols, put in 1s -
- can do for both
- technique of solving)

Complete solution

$= \begin{pmatrix} 2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$= \begin{pmatrix} 2 \\ 0 \end{pmatrix}$

⑥

If $N(A) = \begin{pmatrix} z \\ 0 \end{pmatrix}$

Then complete sol $= \begin{pmatrix} 2+z \\ 0 \end{pmatrix}$

Matix not invertable → not sq

Full col rank → so no nullspace oter than $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Spans

The span of vectors $v_1, \dots, v_n$ is the set of vectors

such that

$\{ v \mid x_1 v_1 + \dots + x_n v_n = v_1$ for $x_1, \dots, x_n \in \mathbb{R} \}$

A set of vectors spans a space if their span is a space

Example

(a) is $\begin{pmatrix}2\\3\end{pmatrix}$ in the span of $\begin{pmatrix}1\\1\end{pmatrix}$ and $\begin{pmatrix}3\\3\end{pmatrix}$

⊗ So can't since can't linear combine $\begin{pmatrix}1\\1\end{pmatrix}$ and $\begin{pmatrix}3\\3\end{pmatrix}$ to be $\begin{bmatrix}2\\3\end{bmatrix}$

Proof: Suppose by contridiction, we could put in a span. Then by def

$$\begin{pmatrix}2\\3\end{pmatrix} = \text{a linear combo } \begin{pmatrix}1\\1\end{pmatrix} \text{ and } \begin{pmatrix}3\\3\end{pmatrix}$$

$$\begin{pmatrix}2\\3\end{pmatrix} = x_1\begin{pmatrix}1\\1\end{pmatrix} + x_2\begin{pmatrix}3\\3\end{pmatrix}$$

Could write as matrix eqn

$$\begin{bmatrix}1 & 3\\1 & 3\end{bmatrix} x = \begin{bmatrix}2\\3\end{bmatrix}$$

$$\left[\begin{array}{cc|c}1 & 3 & 2\\1 & 3 & 3\end{array}\right] \text{ elimination}$$

$$\downarrow$$

$$\left[\begin{array}{cc|c}1 & 3 & 2\\0 & 0 & 1\end{array}\right] \otimes \text{ contridiction } 0 \neq 1$$

b) Is $\begin{pmatrix}6\\11\\11\end{pmatrix}$ in col space of $\begin{bmatrix}1 & 4\\2 & 7\\3 & 5\end{bmatrix}$

Same problem. Col space is <u>span</u> of the cols

<u>Prove</u>

$$\begin{pmatrix}6\\11\\11\end{pmatrix} = x_1\begin{bmatrix}1\\2\\3\end{bmatrix} + x_2\begin{pmatrix}4\\7\\5\end{pmatrix}$$

$$\begin{bmatrix}1 & 4\\2 & 7\\3 & 5\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}6\\11\\11\end{bmatrix}$$

Here x_1, x_2 <u>does</u> exist

So <u>is</u> in the col space

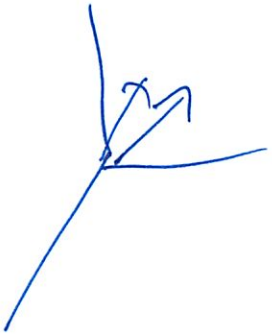
<u>Linear Independence</u>

The second big concept in linear algebra

Vectors $v_1, \ldots, v_n$ are <u>lin. indp.</u> if whenever we have $x_1 v_1 + \cdots + x_n v_n = 0$

Then $x_1 = \cdots = x_n = 0$

⑨



When add - <u>should not</u> lie on same line

So for vector in R3

if 2 form a plane

if 3rd outside → ind

if 3rd on that plane → dep

- we didn't need that 3rd vector
- Could have got other 2 from plane

Examples

a) Are $\begin{pmatrix}1\\1\end{pmatrix}$ and $\begin{pmatrix}3\\3\end{pmatrix}$ linear ind?

No! linearly dep

So we could have

$$x_1\begin{pmatrix}1\\1\end{pmatrix} + x_2\begin{pmatrix}3\\3\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

⑩

$$\begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & 3 & 0 \\ 1 & 3 & 0 \end{array}\right]$$

$$\Big\downarrow \text{ elimination / solve}$$

$$\left[\begin{array}{cc|c} 1 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

So <u>can</u> find non 0 x_1, x_2

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -3 \\ -1 \end{pmatrix}$$

So <u>Not</u> lin. ind.

Fat matrix

$$\left[\qquad\qquad\qquad\qquad\right]$$

<u>Can</u> be lin ind

But not cols

b.c)

Can have it transposed

Can have at most 1 unique sol

b) Are $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$ lin ind?

Yes!

$$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

Only sol $\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$

Basis + Dimension

Vectors $v_1, \ldots, v_n$ are a basis for a space if

$\left.\begin{array}{l}\text{1) must be lin. ind}\\ \text{2) must span the space}\end{array}\right\}$ w/ no extra

<u>Ex</u> Find the basis for the row space

$$A = \begin{bmatrix} ① & 2 & 0 & 3 \\ 0 & 0 & ② & 4 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$\downarrow$ Do elim

$$\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \left.\begin{matrix} \\ \\ \end{matrix}\right\} \text{ind} \\ \} \text{ lin dep} \end{matrix}$$

So the first 2 span the row space
are lin ind

So $\underbrace{\text{basis}}_{\text{for the row space}}$ = 1st 2 rows

Dim row space = # of vectors in the basis
= 2

So this is a real $\underset{\substack{\uparrow \\ \text{2 vectors} \\ \text{in basis} \\ = \text{dim}}}{\text{2D}}$ plane $\underset{\uparrow \text{4 components}}{\mathbb{R}^4}$

⑬

Basis for col space

She likes row ops
So does transpose
then finds basis for that row space

Or if identify pivot col

$$\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$
$$\begin{array}{cccc} \uparrow & & \uparrow & \end{array}$$

these are pivot cols

these are the basis

dim col space = 2

18.06 Lecture

Exam Review

3/7

Exam: Fri, 11AM, Walker

ideas: basis, rank, span, ... col space, null space

Questions use the hw

review: could look at other problems in the book

Computation takes a back seat to ideas

Prove the rank $(A+B) \leq$ rank (A) + rank (B)

(need to fully remember rank term)

example of equality

$$A = \begin{bmatrix} 1 & 1 \\ 3 & 3 \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$$

must be same shape to add them

Could do w/ $B = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

both rank 1 - then together rank 2

②

How to approach

<u>rank</u> = # of pivot cols (or # pivots)

= dim of the col space

↑ another def = # of vectors in a basis for the col space

$C(A)$ has basis $v_1, \dots, v_r$

$C(B)$ has basis $w_1, \dots w_R$ $\quad r \neq R$

So $\quad \underbrace{\text{rank}(A)}_{r} + \underbrace{\text{rank}(B)}_{R}$

So A+B So $r+R$

may or may not be a basis

So its **not** a basis

Do these $r+R$ vectors $v_1, \dots v_r, w_1, \dots, w_R$ **span** col space of $C(A+B)$

↳ is every col of $A+B$ is a combo of those

③

Every col of A + B is a combo of v's w's

$A \cdot B = C$

rank(C) $\leq$ rank(A)

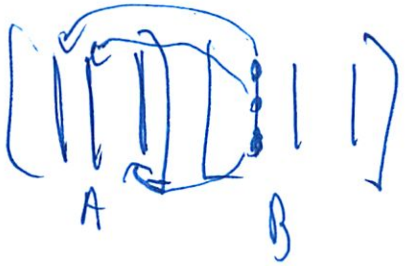
Since $C(C) = C(AB)$ is contained in the col space A

$$C(AB) \leq C(A)$$

every col of AB is in the col space of A

Why is the first col of AB in col space A

Because its a linear combo



Since row * col multiplication

~~AB in row space of B~~

The row space of AB is in row space of B

④ (The answers make sence when I look at it for a while - but need to actuly come up w/ it)

rank = minimum # of vectors that span col space

Suppose

$$R = \begin{bmatrix} 1 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

What can you say about the cols of A?

- 3 pivots
- 3 ind cols 1st, 4th, 5th
- 4 col 1 = col 2
- col 3 of A was all 0s

T or F?

Every lower triangular matrix (sq) w/ 1s on diagonal invertable

true

⑤

Usually "triangular" is square → but would specify

If $PA = LU$

is there only one possible P, L, U?

any investable A has this

if L has 1s on diagonal False

L, U set for a P

but could have multiple P

Matlab tries for largest P

-other programs might do something else

If a vector space dim n, than no spanning set could have $> n$ False

Set of investable 4x4 forms a subspace of all 4x4 False

↳since 0-vector is not in the space

Adding 2 invertables could be neg of each other - cancel out

⑥

A System of n eq unknowns is solvable for every RHS if the cols are ind.

True

- if the matrix is invertable

$\downarrow$

Which is if cols are ind given ✓ true

$r = n$

In the vector space of all 3 x 3 matrices

$\dim = 3^2 = 9$

What subspaces are spanned by all possible R

?RREF

- upper triangular + are all

If A is a 3 x 5 matrix, what info do you have on its null space

└ Contains at least 2 vectors ind

has at least 2 special sols

⑦

Suppose have m × n matrix

① $Ax = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ has no sol

② $Ax = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ has exactly 1 sol

What info about rank r, m, n do we have

① know m = 3
 ↳ cols have 3 components

① know $r \leq 2$

② Nullspace is zero vector $N(A) = \{0\}$
 ↳ since exactly 1 sol

also cols of A are ind

So since cols are ind
 ↳ $n \leq 2$

Could not pick $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 4 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

Complete sol $X = X_p + X_n$

Only the zero vector is in the null space

⑧ Left nullspace will not be on there

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

U

rank of A = 2
└ 2 ind cols
  1 and 3rd cols

Basis for col space
  Not col1, col3 from A
  Need to multiply

$$\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

What is the null space of A ? N(A)

3×5
$m \times n$

Looking for 5−2 = 3 items in null space

④

Nullspace (A) = Nullspace (U)

Since that is how elimination works

$$\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

OH 1

3/7

3.5 #7

$$c_1 W_1 + c_2 W_2 + c_3 W_3 = [0]$$

If ind $\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

~~$(W_2 - W_3)\ W_1 + 2W$~~

$$c_1 V_1 + c_2 V_2 + c_3 V_3 = [0]$$

$$c_1(W_2 - W_3) + c_2(W_1 - W_3) + c_3(w_1 - W_2) = [0)$$

is there anything that could make that

(if #s - elim)

but here group terms

$$W_1(c_2 + c_3) + W_2(c_1 - c_3) + W_3(-c_1 - c_2) = 0$$

lin ind - so constraints

$c_2 + c_3 = 0$

$c_1 - c_3 = 0$

$-c_1 - c_2 = 0$

②

$c_3 = -c_2$

$c_1 - (-c_2) = 0$

$c_1 + c_2 = 0$

$c_1 = -c_2$

$-c_2 - c_3 = 0$

$-c_2 = c_3$

Family of sols

$$\begin{bmatrix} c_2 + c_3 \\ c_1 - c_3 \\ -c_1 - c_2 \end{bmatrix}$$

have a non [illegible] 0 sol

So if $c_1 = 1$ then $c_2 = -1$, $c_3 = 1$

which is non 0 sol → <u>dep</u>

③

Finding matrix A in

$$[v_1\ v_2\ v_3] = [w_1\ w_2\ w_3]\, A$$

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} A$$

$$A = \cancel{\begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}}$$

read off as <u>cols</u> not rows

$$= \begin{array}{ccc} \left[\begin{matrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{matrix}\right] \\ \begin{matrix} \uparrow & \uparrow & \uparrow \\ v_1 & v_2 & v_3 \end{matrix} \end{array}$$

Reading

3/7

(3.5 in progress)

Basis for Matrix Spaces + Function Spaces

Can ask if matrices are ind

└ does some linear combo of them $=0$?

$\dim = n^2 = m \cdot n$

So for Vector space $M =$ 2 x 2 matrices $\boxed{n^2 = mn}$

$\dim = 4$

$$A_1, A_2, A_3, A_4 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Lin ind - looking at whole matrix

$$C_1 A_1 + C_2 A_2 + C_3 A_3 + C_4 A_4 = \begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix} = A$$

The 3 matrices A_1 A_2 A_4 are the basis for $\boxed{\frac{1}{2}n^2 - \frac{1}{2}n}$

a subspace - the upper triangular matrix

Or diagonal A_1, A_4 $\boxed{n}$

②

Functional Spaces

$$\frac{d^2y}{dx^2} = 0 \qquad \frac{d^2y}{dx^2} = -y \qquad \frac{d^2y}{dx^2} = y$$

$y'' = 0$ solved by linear $y = cx + d$

$y'' = -y$ 〃 $y = c\sin x + d\cos x$

$y'' = y$ 〃 $y = ce^{x} + de^{-x}$

2 basis functions $\sin x$, $\cos x$

dim $= 2$

Empty set is basis for Z

zero vector never in basis

③

3.6 Dimensions of Four Subspaces

rank = # of pivots ← so actual?

dim = # of vectors in the basis
↑ so min #?

---

rank of A reveals the dim of ~~the~~ all 4 fundenental subspaces

4 Fund Subspaces

1. Row space $C(A^T)$ is subspace of R^n
2. Col space $C(A)$ R^m
3. Null space $N(A)$ R^n
4. Left null space $N(A^T)$ R^m

Left nullspace

$A^T y = 0$ ← n by m
↑ Nullspace (A^T)

④

The vectors y go on the left side of A when the eqn is written as $y^T A = 0$

✱ The row and col space have same dim r ✱

$N(A)$ is $n-r$ | $N(A^T)$ is $m-r$

The 4 subspaces fit together to understand $Ax=b$

---

4 Subspaces for R

Suppose A is reduced to row echelon form R

└ 4 subspaces easy to identify
pick basis

2 subspaces change when look back at A

✱ the 4 dims are the same for A and R ✱

⑤

example

$$\begin{array}{c} \begin{bmatrix} 1 & 3 & 5 & 0 & 7 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{ccccc} \uparrow & & & \uparrow & \end{array} \end{array} \begin{array}{l} \leftarrow \\ \leftarrow \\ \end{array}$$

pivot rows

pivot cols

$r = 2$

pivot rows = # pivot cols

1. Row space has dim 2

Since 1st 2 rows are basis

2. Col space has dim $r=2$

2 ~~rows~~ cols unique

the other 3 cols are combo of other cols

3. Null space dim $n - r = 5 - 2 = 3$

= 3 free vaiables

= 3 special sols $Rx = 0$

(I think I am kinda getting this stuff)

So to find special sols

free x_2 x_3 x_5

So

$x_2 = 1$ $\quad \bar{1}$

0 ← solve for

$\bar{0}$

$$\begin{bmatrix} 1 & 3 & 5 & 0 & 7 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ 1 \\ 0 \\ x_4 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

⑥

$X_1 + 3 + 0 + 0 + 0 = 0$

$X_1 = -3$

$X_4 - 0 = 0$

$X_4 = 0$

So one special sol $\begin{bmatrix}-3\\1\\0\\0\\0\end{bmatrix}$ ✓

$X_3 = 1$ $\begin{array}{c}\hline 0\\1\\ \hline 0\end{array}$

$X_1 + 5 = 0$

$X_1 = -5$

$X_4 = 0$

$\begin{bmatrix}-5\\0\\1\\0\\0\end{bmatrix}$ ✓

$X_5 = 1$ $\begin{array}{c}\hline 0\\0\\ \hline 1\end{array}$

$X_1 + 7 = 0$

$X_1 = -7$

$X_4 + 2 = 0$

$X_4 = -2$

$\begin{bmatrix}-7\\0\\0\\-2\\1\end{bmatrix}$ ✓

Special sols form a basis

4. Left null space (nullspace A^T) dim $m - r = 3 - 2$

Fund Theorm Linear Algebra Part 1

Col + row space both have dim r

nullspaces dim $n - r$ $\quad \underset{\text{left}}{m} - r$

The ys in the left null space combine the rows to give the zero row

---

Matricies of Rank One

Last example rank = 1
(I didn't write)

This is special

Since dim(row space) = dim (col space)

When $r = 1$ every row is a multiple of the same row

$A = uv^T$ the

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ -3 & -6 & -9 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -3 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = v^T$$

Col · row
$4 \cdot 1$ $\quad$ $1 \cdot 3 = 4 \cdot 3$

Row space line in $\mathbb{R}^n$
Col " line $\mathbb{R}^m$

⑧

Every rank one matrix has $A = u v^T$

Col ∘ row

$u \quad v^T$

Nullspace = plane perp to v

$$Ax = 0$$

$$u(v^T x) = 0$$

$$v^T x = 0$$

Office Hrs 2

3/7

Looks like R^4

Want polynomials

$$1$$

$$x$$

$$1 + x + x^2 + x^3$$

$$(1-x), (x-x^2)(x^2-x^3)(x^3-1)$$

No significance hor or vertical

#4 e Identity matrix ∈ both row + col ~~span~~ space are the same

$$\begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$$

row space spaned $[1\ 3]$ $[0\ 0]$

col space $\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}3\\0\end{bmatrix}$) So spaces are not =

②
If =, ~~the~~ it must be sq for dim match
possible say about RREF

↑ care about rows

messes up col space

Row space

Space spaned by rows

↑ all linear combos of

$$\begin{bmatrix}1 & 3\\ 0 & 0\end{bmatrix}$$

row sp = $\{x_1\begin{pmatrix}1\\3\end{pmatrix} + x_2\begin{pmatrix}0\\0\end{pmatrix}\}$

$x_1, x_2 \in R$

Col sp $\{x_1\begin{pmatrix}1\\0\end{pmatrix} + x_2\begin{pmatrix}3\\0\end{pmatrix}\}$

↑ multiples of each other

$= \begin{pmatrix}x_1 + 3x_2\\ 0\end{pmatrix}$

$= \begin{pmatrix}y\\0\end{pmatrix}$ $y \in R$

③

So it like x dim in $\mathbb{R}^{y}$

? how many dims

? what space is it in

Null space + col space

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 2 & 0 & 0 & 5 \\ 0 & 1 & 3 & 4 \end{bmatrix}$$

Row space is contained in $\mathbb{R}^4$ ← 4 components in

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 2 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 0 \\ 5 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 3 \\ 4 \end{pmatrix}$$

Col space is contained in $\mathbb{R}^3$

(c)

Null space spans such that $Ax=0$

x should have 4 components

Vector x spans null spaces

So R^4

Left null space

Set of $xA=0$

$= (xA)^T$

$= A^T x^T$

So in R^3

Row space dim 3

- each row unique

Finding So doing row ops does not change row space

Can do elim to RREF

Can easily see dim

$$\text{elim} \to \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 3 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{So ind}$$

⑤ And since ind → are basis for row space

So dim = 3 $\begin{bmatrix}1\\0\\0\\2\end{bmatrix} \begin{bmatrix}0\\1\\3\\4\end{bmatrix} \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$

Alt method: Transpose $\begin{bmatrix}1 & 2 & 0\\0 & 0 & 1\\0 & 0 & 3\\2 & 5 & 4\end{bmatrix}$

- done in class

Then elim on this

$$\begin{bmatrix}1 & 2 & 0\\0 & 0 & 1\\0 & 0 & 3\\0 & 1 & 4\end{bmatrix} \rightarrow \begin{bmatrix}\textcircled{1} & 2 & 0\\0 & \textcircled{1} & 4\\0 & 0 & \textcircled{1}\\0 & 0 & 0\end{bmatrix} \begin{matrix}\leftarrow\\\leftarrow\\\leftarrow\\ \end{matrix}$$

Find pivots → 3 rank = 3

So can find these back in the original matrix

take rows 1, 2, 3

and know in original ~~cols~~ rows 1, 2, 3 are basis

Next chap $\boxed{\text{row}^{\text{dim}}\ \text{space} = \text{col}^{\text{dim}}\ \text{space}}$ ← by this rule

col space dim = 3

---

dim null space $= n - r$

$4 - 3 = 1 =$ dim nullspace

So $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ One degree of freedom

And left nullspace dim

$m - r$ ~~[illegible]~~

$3 - 3 = 0$

$n > r$ $m = r$

short + wide

has ∞ sols

~~[illegible]~~ ↓ more specifically

sols form a 1 D space

3.6 #32 How to show it

Don't do proof by example

(She ~~[illegible]~~'s still trying to figure it true)

Row spaces in $\mathbb{R}^n$

~So should be identical

Example $\begin{bmatrix} 1 & 0 & x & x & \cdots \\ 0 & 1 & x & x & \cdots \\ 0 & 0 & 0 & 0 & \end{bmatrix}$

⑦ Could say row, col, null, leftnull space $\overset{?}{=}$ r

in <u>RREF</u> this is true

Since in last P-set showed RREF is unique

3.5 #45 (for got this problem)

In Linear Algebra, you always pick a basis for your subspace

V W in $\mathbb{R}^n$

dim $= n_1$ $\to$ basis $v_1, \ldots v_{n_1}$

dim $= n_2$ $\to$ basis $w_1 \ldots, w_{n_2}$

know $n_1 + n_2 > n$

If there is a vector v in both V and W

v should be a linear combo of <u>both</u> the sets of bases

⑧

Means $v = x_1\vec{v}_1 + \ldots + x_{n_1}\vec{v}_{n_1}$

$= y\vec{w}_1 + \cdots + y_{n_2} \cdot \vec{w}_{n_2}$

So can rearrange

$$x_1\vec{v}_1 + \ldots + x_n\vec{v}_{n_1} - (y\vec{w}_1 + \ldots + y_{n_2}\vec{w}_{n_2}) = 0$$

So there is a relationship b/w our 2 sets of bases

Since basis is a spanning set

We still don't know our vector v

Since it same, subtracting them $= 0$

So they are dep

So if could find some linear dep b/w them

$$C_1 V_1 + \cdots + C_{n_1} V_{n_1} + d_1 W_1 + \cdots + d_{n_2} W_{n_2} = 0$$

So vector v should be in our space $V_1 \ldots V_n$

But this should kinda be $v - v = 0$

So can say

$$v = C_1 V_1 + \cdots + C_{n_1} V_{n_1}$$

So $v - v = 0$

And v is in $V_1, \dots V_n$

So the 2 spaces are the same

Use condition $n_1 + n_2 > n$

If v's and ws togeter - we know is a dependence relation

So they must be dep

So can get $c_1 v_1 \dots c_{n_1} v_{n_1} + d_1 w_1 + \dots d_{n_2} v_{n_2} = 0$

$v - v = 0$

(Don't get)

- v is in w

Can scalar multiple

bases are lin ind

but more than enough so lin dep

⑩

Review x_p , x_n

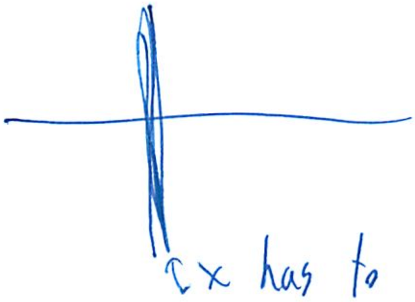
So in 18.03

↑ multiple sol

but w/ initial conditions → particular sol

on - only 1 line

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x = 0 \rightarrow x = 0$$

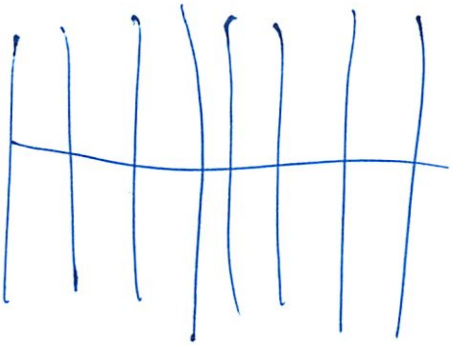


↑ x has to be 0

y can be anything

⑪

Solve $Ax = b = \begin{pmatrix}3\\ \partial\end{pmatrix}$ solving ~~special sol~~ → give 1 lie
↑ particar sol → to



pins down family

~~Does not~~

Add to null space → get complete set of sol

But then p 159 table

Skinny 0 or 1 sols) but still same process to find
fat 0 or ∞ sols

REF vs RREF
↑ kinda upper triangular
↑ only 1 or 0 on pivots

⑱ Finding RREF

$$\begin{bmatrix} 1 & 0 & 7 & 3 \\ 2 & 0 & 4 & 0 \\ 2 & 1 & 0 & 0 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 0 & 7 & 3 \\ 0 & 0 & 4 & 6 \\ 0 & 1 & 0 & -6 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 0 & 7 & 3 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 4 & 6 \end{bmatrix} \quad \text{REF}$$

↑ pivots can be anything

$$\downarrow$$

$$\begin{bmatrix} 1 & 0 & 7 & 3 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & - \end{bmatrix} \quad \text{RREF}$$

↑ 0s below

↑ 1s for pivots

⑦

Wrong problem

#3.5 # 35 Find a basis for the space of polynomials $p(x)$ degree ≤ 3, Find a basis for the subspace w/ $p(1) = 0$

So $a_0 + a_1 x + a_2 x^2 + a_3 x^3$

~~$(a_0, a_1, a_2, a_3) = (1000), (0100), (0010), (0001)$~~

$1, x, x^2, x^3$

b) $p(1) = 0$

$a_0 + a_1 (1) + a_2 (1)^2 + a_3 (1^3) = 0$

$a_0 + a_1 + a_2 + a_3 = 0$

Now everything must add to 0

? $(1-x), (x-x^2), (x^2-x^3) (x^3-x)$

$(a_0, a_1 a_2 a_3) =$ ~~$(1,-1,0,0), (0 1 -1 0), (0 0 1 -1)$~~ $(-1,0,0,1)$

? is that the min required?

but certainly all will add to 0

i can use elim to see

$$\begin{pmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} \ldots$$

⑰

Wrong problem

~~3.6 #22 Construct $A = UV^T + WZ^T$, whose col space has bais $\begin{bmatrix}1\\2\\4\end{bmatrix}$ $\begin{bmatrix}2\\2\\1\end{bmatrix}$~~

~~and whose row space has bais $\begin{pmatrix}1\\0\end{pmatrix}$ $\begin{pmatrix}1\\1\end{pmatrix}$~~

~~Write A as $(3 \times 2) \times (2 \times 2)$~~

~~$$A = {}_{3}\overset{2}{\underset{UV^T}{\begin{bmatrix} & \\ & \end{bmatrix}}} + \times 2 \overset{2}{\underset{WZ^T}{\begin{bmatrix} & \\ & \end{bmatrix}}}$$~~

18.06 Spring 2012 – Problem Set 4

This problem set is due Thursday, March 8th, 2012 at 4pm (hand in to Room 2-106). The textbook problems are out of the 4th edition. For computational problems, please include a printout of the code with the problem set (for MATLAB in particular, `diary('filename')` will start a transcript session, `diary off` will end one.)

Every problem is worth 10 points.

p6

1. Do Problem 5 from Section 3.5.
2. Do Problem 7 from Section 3.5.
3. Do Problems 15 & 18 from Section 3.5.
4. Do Problem 28 from Section 3.5.
5. Do Problem 45 from Section 3.5.
6. Do Problem 4 from Section 3.6.
7. Do Problem 9 from Section 3.6.
8. Do Problem 16 from Section 3.6.
9. Do Problem 18 from Section 3.6.
10. Do Problem 32 from Section 3.6.

18.06 Wisdom. The first exam is coming up (next Friday, March 9th, Walker Memorial, 11am-12 noon). It will include concepts - not only numbers. Your most valuable study resources will be the book, the problems you've solved so far and the old exams (with solutions!) on the course web: `http://web.mit.edu/18.06/www/old.shtml`. If we were you, we would spend this weekend brushing up on concepts and trying to solve as many old exams as possible.

Michael Plasmeier PS 4 77/100 3/8

3.5 #5 Dep or ind

10/10

$$\begin{pmatrix}1\\3\\2\end{pmatrix}\begin{pmatrix}2\\1\\3\end{pmatrix}\begin{pmatrix}3\\2\\1\end{pmatrix}$$

Can try elimination

$$\begin{bmatrix}1 & 2 & 3\\3 & 1 & 2\\2 & 3 & 1\end{bmatrix}$$

(can't check ans <u>before</u> exam -> bad pcatice)

$l=3$

$$\begin{bmatrix}1 & 2 & 3\\0 & -5 & -7\\0 & -4 & -6\end{bmatrix}$$

$l = \frac{-4}{-5} = \frac{4}{5}$

$$\begin{bmatrix}1 & 2 & 3\\0 & -5 & -7\\0 & 0 & -\frac{58}{5}\end{bmatrix}$$

$-4 + -5 \cdot \frac{4}{5} = -4+4=0$

$-6 + -7 \cdot \frac{4}{5} = -6+\frac{28}{5}$

③ Pivot cols -> is ind ✓

②

b) $\begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}$

$$\begin{bmatrix} 1 & 2 & -3 \\ -3 & 1 & 2 \\ 2 & -3 & 1 \end{bmatrix}$$

$\ell = -3 \quad \ell = 2$

$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & 7 & -7 \\ 0 & -7 & 7 \end{bmatrix}$$

$\ell = -1$

$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & 7 & -7 \\ 0 & 0 & 0 \end{bmatrix} \qquad 7 - -7, -7$$

✓

ⓧ Not independent → dependent

(Seem eaiser)

3.5 #7 If w_1, w_2, w_3 are ind. show the difference

5/10

$V_1 = W_2 - W_3$

$V_2 = W_1 - W_3$ are dependent

$V3 = W_1 - W_2$

Find a combo of the V_{is} that gives 0

Which matrix A in $\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} W_1 \\ W_2 \\ W_3 \end{bmatrix}$ A is singular

③

(not so easy!)

So hmm,

8 is related

Are these dep just within themselves or w/ the ws

Independence is

$$c_1 W_1 + c_2 W_2 + c_3 W_3 = [0]$$

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

So we're asking about

$$c_1 V_1 + c_2 V_2 + c_3 V_3 = [0]$$

$$c_1 (W_2 - W_3) + c_2 (W_1 - W_3) + c_3 (W_1 - W_2) = [0]$$

We know it is dependent.

Is there anything here that can make that?

(If it were #s - we could do elimination)

But here we can group terms

$$W_1 (c_2 + c_3) + W_2 (c_1 - c_3) + W_3 (-c_1 - c_2) = 0$$

(3b).

if it were linear ind, the constraints would be

$$C_2 + C_3 = 0$$
$$C_1 - C_3 = 0$$
$$-C_1 - C_2 = 0$$

$$C_3 = -C_2$$
$$C_1 = -C_2$$

So if $C_1 = 1$ then $C_2 = -1$, $C_3 = 1$

which is a non-0 sol → <u>dependent</u>

great writing!
But you missed the
second half of the
problem: which
matrices are invertible?
(-5)

(4.)

3.5 #15 If $V_1, \ldots V_n$ are lin ind, the space they span has dimension $\underline{n}$ ✓

7.5/10

These vectors are a <u>basis</u> ✓ for that space.

If vectors are the col of a $m \times n$ matrix than

m is $\underline{\geq}$ n -0.5

↑ So this means full col space $(r=n)$

So tall and thin

$m > n$

If $m=n$ the matrix is <u>square + invertable</u> ✓

3.5 #18 Suppose $V_1, V_2, \ldots V_6$ are six vectors in R^4

↑ 6 vectors w/ 4 components

a) These vectors <u>might</u> ✓ span R^4

↑ Could if the lin dep matrices span R^4?

So if ≥ 4 lin dep matrices

b) Those vectors <u>might</u> ✗ -2 be linear ind

We don't know that the matrices currently <u>span</u> the space

When you have n vectors in $\mathbb{R}^m$, with $n > m$, they are <u>always</u> dependent. This is because the basis of the space only needs m elements.

⑤

c) Any 4 of these vectors might be a basis for R^4 ✓

3.5 # 28 Find a basis for the space of all 2 × 3 matrices whose column add to 0.

10/10

Ahh

$$\begin{bmatrix}1&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&1&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&1\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\1&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&1&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&1\end{bmatrix},$$

↑ No ← want where Cols add to 0

So if each one has col add to 0

$$\begin{bmatrix}1&0&0\\-1&0&0\end{bmatrix}, \begin{bmatrix}0&1&0\\0&-1&0\end{bmatrix}, \begin{bmatrix}0&0&1\\0&0&-1\end{bmatrix}$$ ✓ ← So just a basis whose Cols happen to add to 0

Now whos rows add to 0

$$\cancel{\begin{bmatrix}1&-1&0\\0&0&0\end{bmatrix}}, [$$

$$\begin{bmatrix}1&-1&0\\-1&1&0\end{bmatrix}\begin{matrix}=0\\=0\end{matrix} \quad \begin{bmatrix}1&0&-1\\-1&0&1\end{bmatrix}$$ ✓

↑ ↑ ↑
ind

So we can build anything in 2×3 space w/ these 2 matrices?

⑥

~~Lets try it $\begin{bmatrix} 5 & 4 & 3 \\ 2 & 3 & 4 \end{bmatrix}$~~

~~$-4 \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -4 & 4 & 0 \\ 4 & -4 & 0 \end{bmatrix}$~~

~~$+9 \begin{bmatrix} 1 & 0 & -1 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 0 & -9 \\ -9 & 0 & 9 \end{bmatrix}$~~

~~$\begin{bmatrix} 5 & 4 & -9 \\ -5 & -4 & 9 \end{bmatrix}$~~

~~∴ So we were not able to create <u>any</u> 3×2 matrix~~

Or the question whose subspace <u>whose rows add to 0</u>

So the matrix to try must add to 0

~~$\begin{bmatrix} 5 & -2 & -3 \\ 3 & 4 & -7 \end{bmatrix}$~~

~~$+2 \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -2 & 0 \\ -2 & 2 & 0 \end{bmatrix}$~~

~~$3 \begin{bmatrix} 1 & 0 & -1 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & -3 \\ -3 & 0 & 3 \end{bmatrix}$~~

$\begin{bmatrix} 5 & -2 & -3 \\ -5 & 2 & 3 \end{bmatrix}$ ← still adds to 0

∴ but not all matrices can be made of it!

(6b)

Key: **Both** rows + cols add to 0! Read carefully!

$$\begin{bmatrix} 5 & 6 & -11 \\ -5 & -6 & 11 \end{bmatrix}$$

$$-6\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -6 & 6 & 0 \\ 6 & -6 & 0 \end{bmatrix}$$

$$11\begin{bmatrix} 1 & 0 & -1 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 11 & 0 & -11 \\ -11 & 0 & 11 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 6 & -11 \\ -5 & -6 & 11 \end{bmatrix}$$ ✓ worked

①

3.5 #45 Inside $\mathbb{R}^n$ suppose dim (V) + dim (W) $> n$

5/10

Show that some non zero vector is in both V and W.

If there is a vector v in both V and W, v should be a linear combo of both sets of bases ✓

$$V = x_1 \vec{v}_1 + \dots + x_n \vec{V}_n$$
$$= y_1 \vec{w}_1 + \dots + y_n \vec{w}_n$$

What are the v's and w's ?

If we take Vs and ws together - we know it is a dependence relationship

So they must be dep since dim(V) + dim(W) $> n$ ✓

So the vector is in both V and W

⑧

3.6 #4 Construct the matrix w/ the required property or explain why impossible.

10/10

a) Col space contains $\begin{bmatrix}1\\1\\0\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix}$ ✓

row space $\begin{bmatrix}1\\2\end{bmatrix}\begin{bmatrix}2\\5\end{bmatrix}$

So This means that these rows or cols could be linear comboed from our matrix

$$\begin{bmatrix}1&0\\1&0\\0&1\end{bmatrix} \leftarrow \text{just that!}$$ ✓ ✓

↑↑
The 2 cols

Row space is that transposed

$$\begin{bmatrix}1&1&0\\0&0&1\end{bmatrix}$$

↑So $\begin{bmatrix}1\\2\end{bmatrix}$ is 1·col 1 + 2·col 2

$\begin{bmatrix}2\\5\end{bmatrix}$ 2 " 5 "

T?: ~~What format is this written in?~~

its fine either way

⑨

b) Col space has basis $\begin{bmatrix}1\\1\\3\end{bmatrix}$

null " " " $\begin{bmatrix}3\\1\\1\end{bmatrix}$

$\begin{bmatrix}1\\1\\3\end{bmatrix}$

↑only one basis vector.

nullspace = all combos of special sols

$$\begin{bmatrix}1\\1\\3\end{bmatrix}\overset{\swarrow\text{can be neg}}{\begin{bmatrix}-3\\-1\\-1\end{bmatrix}} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

↑need more cols

$$\begin{bmatrix}1 & -3 & 0\\1 & -1 & 0\\3 & -9 & 0\end{bmatrix}\begin{bmatrix}3\\1\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

↑but now diff col space basis

Impossible ✓ ✓

Also we know $\underset{\overset{\uparrow}{1}}{r} + \underset{\overset{\uparrow}{1}}{(n-r)}$ must be 3

So here $\mp 1 \neq 3$

(I didn't explicitly see this before)

⑩.

c) Dim of nullspace = (1 + dim of the left nullspace)

? thought no problems about left nullspace

lame

But left nullspace is just $N(A^T)$

So $\dim \text{col} + \dim \text{null} = n$

$\dim \text{row} + \dim \text{l. null} = m$

So $r + n - r = n$

$m + m - r = m$

$n - r = (1 + m - r)$ ← our constraint

So if $r = 2$

$m = 3$

$n - 2 = 1 + 3 - 2$

$n - 2 = 2$

$n = 4$

$$\begin{array}{c c} & n=4 \\ m=3 & \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \quad r = 2$$

✓

⑪

d) Left nullspace contains $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$, row space contains $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

all $A^T y$ row space

So $A^T y = 0$ left nullspace

$$\begin{bmatrix} 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x = 0$

$y = 0$

$$\begin{bmatrix} 3 & 1 \\ -6 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$3x + y = 0$

$-6x - 2y = 0$

old fashioned way

$y = -3x$

~~$-6x - 2(-3x) = 0$~~ useless

~~$-6x + 6x = 0$~~

Could do elim or just pick one

$x = 1$

$y = -3$

↑ but negitive

So No ?

(11b).

Wo I think I did something wrong

book $\begin{bmatrix}-9 & -3\\ 3 & 1\end{bmatrix}$

how is that diff from what I had?

$-9x + -3y = 0$

$-9x = 3y$

$-3x = y$

So if $x = 1$

then $y = -3$ as before

$\begin{bmatrix}1\\ -3\end{bmatrix}$ is not in same space as $\begin{bmatrix}1\\ 3\end{bmatrix}$

Or did I do that wrong

$-9x - 3y = 0$ 2eq = 2 unknowns

$3x + y = 0$

$y = -3x$

$-9x - 3(-3x)$

$0 = 0$ ✓

But one repeats

So ∞ sols

put 1 for free -> I did

Hmm...

⑫ .

e) Row space = col space, null space ≠ left null space
? so that is exact same space, not just same dim

That means must be square $m = n$

Then $m - r = n - r$ so the null spaces have the same dimension

?, but that just says dim =, the spaces could still be diff ... yes!

ah but not if row space = col space

(why? you should explain this point more fully).

⑬

3.6 #9 Which subspaces are the same for matrices of diff sizes?

5/10

a) $[A]$ and $\begin{bmatrix} A \\ A \end{bmatrix}$

If the first matrix has its col. space in $\mathbb{R}^n$, the second matrix has its col. space in $\mathbb{R}^{2n}$. So they're definitely different!

So col spaces same (well same # of them) (and space is now more dimensional

Row spaces its like $\boxed{[A\ A]}$ so now more free cols

did you mean $\begin{bmatrix} A \\ A \end{bmatrix}$?

but actual space does not change

b) $\begin{bmatrix} A \\ A \end{bmatrix}$ and $\begin{bmatrix} A & A \\ A & A \end{bmatrix}$

here same # of cols in col space and actual space the same ✓

(just more free cols

Row spaces does not change

$\begin{bmatrix} A \\ A \end{bmatrix}$ $\boxed{[A\ A]}$ $\to \begin{bmatrix} A & A \\ A & A \end{bmatrix}$

more dimensions

⑭

c) Prove that all 3 of those matrices have same rank r.

Rank is # of pivots

I think I mentioned this above

$[A] \to \begin{bmatrix} A \\ A \end{bmatrix}$ ← just extra free rows/cols in A^T

$\begin{bmatrix} A \\ A \end{bmatrix} \to \begin{bmatrix} A & A \\ A & A \end{bmatrix}$ ✓

More duplicate/free cols

⑮ .

3.6 #16 Explain why $v = (1, 0, -1)$ can not be a row of A and also be in the null space

10/10

$$A = \begin{bmatrix} 1 & 0 & -1 \\ & \cdots & \end{bmatrix}$$

Nullspace

$$\begin{bmatrix} 1 & 0 & -1 \\ & \cdots & \end{bmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$1 \cdot 1 + 0 \cdot 0 + -1 \cdot -1 = 2 \neq 0$$

(16)

3.6 #18 (Left nullspace) Add the extra col b and reduce A to echalon form

6/10

$$[A\ b] = \begin{bmatrix} 1 & 2 & 3 & b_1 \\ 4 & 5 & 6 & b_2 \\ 7 & 8 & 9 & b_3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 & b_1 \\ 0 & -3 & -6 & b_2 - 4b_1 \\ 0 & 0 & 0 & b_3 - 2b_2 + b_1 \end{bmatrix}$$

A combo of rows in A produced zero row

(row 3) − 2 (row 2) + (row 1) ✓

What are the vectors of the null space

$A b = 0$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 - 4b_1 \\ b_3 - 2b_2 + b_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$b_1 + 2(b_2 - 4b_1) + 3(b_3 - 2b_2 + b_1) = 0$

$-3(b_2 - 4b_1) - 6(b_3 - 2b_2 + b_1) = 0$

$b_1 + 2b_2 - 8b_1 + 3b_3 - 6b_2 + 3b_1 = 0$

$-4b_1 - 4b_2 + 3b_3 = 0$

$-3b_2 + 12b_1 - 6b_3 + 12b_2 - 6b_1 = 0$

$6b_1 + 9b_2 - 6b_3 = 0$

⑰ .

$2b_1 + 3b_2 - 2b_3 = 0$

2 eq 3 unknowns

↓ ~~full row rank 2 ind rows~~

~~actually~~ ?

So can set anything to b_3

b_3 = anything (so ∞ sols)

say $b_3 = 1$

Then $-4b_1 - 4b_2 = -3$

$6b_1 + 9b_2 = 6$

2 eq 2 unknowns - better

$9b_2 = 6 - 6b_1$

$b_2 = \frac{2}{3} - \frac{2}{3}b_1$

$-4b_1 - 4\left(\frac{2}{3} - \frac{2}{3}b_1\right) = -3$

$-4b_1 - \frac{8}{3} + \frac{8}{3}b_1 = -3$

$-\frac{4}{3}b_1 = -\frac{1}{3}$

$b_1 = \frac{-1/3}{-4/3} = \frac{1}{4}$

$b_2 = \frac{2}{3} - \frac{2}{3}\left(\frac{1}{4}\right) = \frac{1}{2}$

$\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ ← pivot, ← pivot, ← free ∞ sols

↑ book says $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ sign error somewhere

(1.8)

Now for left nullspace.

Basically we transpose and do something?

$$\begin{bmatrix} 1 & 4 & 7 & b_1 \\ 2 & 5 & 8 & b_2 \\ 3 & 6 & 9 & b_3 \end{bmatrix}$$ elim

$$\downarrow$$

$$\begin{bmatrix} 1 & 4 & 7 & b_1 \\ 0 & -3 & -6 & b_2-2b_1 \\ 0 & -6 & -12 & b_3-3b_1 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 4 & 7 & b_1 \\ 0 & -3 & -6 & b_2-2b_1 \\ 0 & 0 & 0 & b_3-2b_2-3b_1 \end{bmatrix} \qquad \frac{-6}{-3} = 2$$

Then the same reduction

$$b_1 + 4(b_2-2b_1) + 7(b_3-2b_2-3b_1) = 0$$

$$-3(b_2-2b_1) + -6(b_3-2b_2-3b_1) = 0$$

$$b_1 + 4b_2 - 8b_1 + 7b_3 - 14b_2 - 21b_1 = 0$$

$$-3b_2 + 6b_1 - 6b_3 + 12b_2 + 18b_1 = 0$$

$$-28b_1 - 10b_2 + 7b_3 = 0$$

$$24b_1 + 9b_2 - 6b_3 = 0$$

⑲.

$b_3 = 1$

$-28b_1 - 10b_2 = -7$

$24b_1 + 9b_2 = 6$

$b_2 = \frac{6}{9} - \frac{24}{9}b_1$

$= 2 - \frac{8}{3}b_1$

$-28b_1 - 10(2 - \frac{8}{3}b_1) = -7$

$-28b_1 - 20 + \frac{80}{3}b_1 = -7$

$-\frac{4}{3}b_1 = 13$

$b_1 = -9.75$

$b_2 = 2 - \frac{8}{3}(9.75) = 28$

$$\begin{bmatrix} -39 \\ 112 \\ 4 \end{bmatrix}$$

Way off - should be same?? yes!!

⑳

3.6 #32 Suppose have $m \times n$ matrix A, B have the same 4 subspaces

8/10

If They are both in RREF prove F = G

$$A = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} I & G \\ 0 & 0 \end{bmatrix}$$

Well we proved last time RREF was unique for a matrix,

Something about a similar claim if all 4 subspaces the same (- would just be a different permutation of what we have)

So they are identical

See sol'n.
In fact, you only need equal row spaces in RREF!

Solutions

18.06 Spring 2012 – Problem Set 4

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Every problem is worth 10 points.

1. Do Problem 5 from Section 3.5.

   Solution. (a) The vectors are independent. One way to see this is to arrange the vectors as columns of a matrix, do elimination, and observe that there are 3 pivots.

   (b) The vectors are dependent. One way to see this is to observe that they sum to zero.

   ☐

2. Do Problem 7 from Section 3.5.

   Solution. To show that the differences are dependent, observe that $\mathbf{v}_1 + \mathbf{v}_3 - \mathbf{v}_2 = \mathbf{0}$. In the equation

$$\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3\end{bmatrix} = \begin{bmatrix}\mathbf{w}_1 & \mathbf{w}_2 & \mathbf{w}_3\end{bmatrix}\begin{bmatrix}0 & 1 & 1\\ 1 & 0 & -1\\ -1 & -1 & 0\end{bmatrix}$$

   the matrix $\begin{bmatrix}\mathbf{w}_1 & \mathbf{w}_2 & \mathbf{w}_3\end{bmatrix}$ is invertible and the other two matrices are singular. ☐

3. Do Problems 15 & 18 from Section 3.5.

   Solution. Solution to 3.5.15 If $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent, the space they span has dimension n. These vectors are a basis for that space. If the vectors are columns of an m by n matrix, then m is greater than or equal to n. If $m = n$, that matrix is invertible *Solution to 3.5.18*

   (a) might not

   (b) are not

   (c) might be

   ☐

4. Do Problem 28 from Section 3.5.

Solution. The following is a basis for the space of 2 by 3 matrices where the entries of each column add to zero:

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

The following is a basis for the subspace of the above where the entries of each row also add to zero:

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

☐

5. Do Problem 45 from Section 3.5.

Solution. Take a basis $\mathbf{v_1}, \ldots, \mathbf{v_k}$ for $\mathbf{V}$ and a basis $\mathbf{w_1}, \ldots, \mathbf{w_l}$ for $\mathbf{W}$. We know that

$$k + l = dim(\mathbf{V}) + dim(\mathbf{W}) > n$$

Therefore, there exists a nontrivial linear combination of $\mathbf{v_1}, \ldots, \mathbf{v_k}, \mathbf{w_1}, \ldots, \mathbf{w_l}$ that gives zero; let this combination be

$$\sum a_i \mathbf{v_i} + \sum b_j \mathbf{w_j}$$

Then the vector

$$\sum a_i \mathbf{v_i} = -\sum b_j \mathbf{w_j}$$

is a non-zero vector that is in both $\mathbf{V}$ and $\mathbf{W}$. ☐

6. Do Problem 4 from Section 3.6.

Solution.

(a) We can just take the matrix

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Its column space clearly contains $(1, 1, 0)$ and $(0, 0, 1)$ and its row space contains the standard basis vectors $(1, 0)$ and $(0, 1)$ for $\mathbb{R}^2$, so the row space contains *all* vectors in $\mathbb{R}^2$.

(b) For the column space and the null space both to be in $\mathbb{R}^3$ we need a 3×3 matrix. But then the dimensions of the column space and the null space must add to 3, so they can't both be 1-dimensional.

(c) Take the matrix

$$\begin{pmatrix} 0 & 0 \end{pmatrix}.$$

Its null space is all of $\mathbb{R}^2$ and its left null space is all of $\mathbb{R}$.

(d) Consider the matrix

$$\begin{pmatrix} 9 & 3 \\ -3 & -1 \end{pmatrix}.$$

Its row space clearly contains $(3, 1)$, and

$$\begin{pmatrix} 1 & 3 \end{pmatrix} \begin{pmatrix} 9 & 3 \\ -3 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \end{pmatrix}.$$

(e) This is impossible — we know that the left null space is the space of vectors orthogonal to the column space and the null space is the space of vectors orthogonal to the row space. Therefore if the row space and the column space are the same, the null space and the left null space must also be the same. ☐

7. Do Problem 9 from Section 3.6.

Solution. (a) The row spaces of $[A]$ and $\left[\begin{smallmatrix} A \\ A \end{smallmatrix}\right]$ are clearly the same, since the rows of the second matrix are the same as those of the first, just repeated. Since the null space consists of the vectors orthogonal to the row space, they also have the same null space. Since the rank equals the dimension of the row space, they have the same rank.

(b) The column spaces of $\left[\begin{smallmatrix} A \\ A \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} A & A \\ A & A \end{smallmatrix}\right]$ are clearly the same, since the columns of the second matrix are the same as those of the first, just repeated. Since the left null space consists of the vectors orthogonal to the column space, they also have the same left null space. Since the rank equals the dimension of the column space, they have the same rank, which by (a) is also the rank of $[A]$. ☐

8. Do Problem 16 from Section 3.6.

Solution. If $\mathbf{v}$ is a row of A then it is orthogonal to every vector in the null space. But the only vector that is orthogonal to itself is the zero vector; since $\mathbf{v} \neq \mathbf{0}$ it therefore can't be in the null space if it is a row. ☐

9. Do Problem 18 from Section 3.6.

Solution. The combination (row 3) - 2(row 2) + (row 1) is zero. Therefore $(1, -2, 1)$ is in the left null space. Since we see from the echelon form that A has rank 2 (there are two pivot columns), the null space has dimension 1 and so consists of vectors of the form $c \cdots (1, -2, 1)$ for all scalars c. From the echelon form we can also see that (column 3) - 2(column 2) + (column 1) is zero, so $(1, -2, 1)$ is also in the null space. Since the rank of A is 2 its null space has dimension 1 too, so for this matrix the null space and the left null space happen to be the same. ☐

10. Do Problem 32 from Section 3.6.

Solution. Since A and B have the same column space, the two identity matrices must have the same size, say $k \times k$. Therefore the matrices F and G have the same size too. Because A and B have the same row spaces, the ith row of A must be a linear combination of the rows of B. We can clearly ignore the rows that are all zeros, so we have a linear combination $a_1(\text{row } 1) + a_2(\text{row } 2) + \cdots + a_k(\text{row } k)$. But the first k components of this row vector are $(a_1, a_2, \ldots, a_k)$, so to get the ith row of A we must have $a_i = 1$ and $a_j = 0$ for $j \neq i$. In other words, the ith row of B equals the ith row of A. ☐

18.06 Wisdom. The first exam is coming up (next Friday, March 9th, Walker Memorial, 11am-12 noon). It will include concepts - not only numbers. Your most valuable study resources will be the book, the problems you've solved so far and the old exams (with solutions!) on the course web: `http://web.mit.edu/18.06/www/old.shtml`. If we were you, we would spend this weekend brushing up on concepts and trying to solve as many old exams as possible.

18.06 Review Session

3/8

Vector Spaces

- addition
- scalar multiplication

eg $\mathbb{R}^3$ $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ 3 components

2 x 2 matrices $\begin{bmatrix} a & c \\ b & d \end{bmatrix}$

Subspaces

Closed under
- addition
- scalar multiplication

$$\begin{bmatrix} 2a \\ a \\ 3a \end{bmatrix} \text{ or } \begin{bmatrix} 2a+b \\ b \\ 3a \end{bmatrix} \text{ or } \cancel{\begin{bmatrix} 1+a \\ a \\ 2a \end{bmatrix}}$$

does not contain 0
so no!

- Upper triangular
- Symmetric
- $\begin{bmatrix} 2a+b & a \\ b & a+3b \end{bmatrix}$ ← can be any scalar
- ~~invertable~~ ← add 2 inv matrices → not invertable

②

Vector space V

Set of vectors $V_1, \ldots, V_n$ in V

Can be called

lin ind - no

→ Linear Combo $V_1, \ldots, V_n$

is $a_1 V_1 + \cdots + a_n V_n$

Lin Ind

No non trival linear combo is 0

$$a_1 V_1 + \cdots + a_n V_n = 0$$

$$a_1 = a_2 = \cdots = a_n = 0$$

Otherwise lin dep

Could

Span V - every vector in V is a linear combo of $V_1, \ldots, V_n$

ie $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix}, \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$

- do span the space
- lin dep

③

Basis → $v_1, \ldots, v_n$ span and are lin ind

dim V = # of basis elements

So if look at first 3

Can see lin ind

but does not span

If you take any 4 of them

are lin ind and span

Dim = 4 = # basis elements

---

m×m matrices

- dim = m · n ← ii

Basis for set $\begin{bmatrix} 2a+b & a \\ b & a+3b \end{bmatrix}$

Set 1 parameter for 1 and other to 0

$\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix}$ ← is

④

These bases are <u>not</u> unique

Actually bases are <u>never</u> unique

This is a basis as long as $\begin{bmatrix}a\\b\end{bmatrix} + \begin{bmatrix}c\\d\end{bmatrix}$ are lin ind.

<u>Matrix factorization</u> (I forgot this stuff)

$A = LU$

$PA = LDU$ ← (sq matrices only)

$$A = \begin{bmatrix} \textcircled{1} & 2 & 1 \\ \boxed{-1} & -1 & 3 \\ \boxed{1} & 0 & 1 \end{bmatrix}$$

↓ factorize to $A = LU$ $\quad \underset{E_{31}}{\begin{bmatrix}1&0&0\\0&1&0\\-1&0&1\end{bmatrix}} \underset{E_{21}}{\begin{bmatrix}1&0&0\\1&1&0\\0&0&1\end{bmatrix}}$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & \textcircled{-1} & 4 \\ 0 & \boxed{-2} & -2 \end{bmatrix}$$

$\underset{E_{32}}{\begin{bmatrix}1&0&0\\0&1&0\\0&+2&1\end{bmatrix}}$ ← since its -1

↓

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 4 \\ 0 & 0 & 6 \end{bmatrix} = U$$

↑ $E_{32} E_{31} E_{21} A = U$

Are same E_{ij} entries but −

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 4 \\ 0 & 0 & 6 \end{bmatrix} \quad LU$$

⑤

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

LDU

just a factorization to know

So then when add $Ax=b$

augment w/ b

$$[A|b] = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ -1 & -1 & 3 & 1 \\ 1 & 0 & -1 & 3 \end{array}\right]$$

do same elim steps

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & 4 & 2 \\ 0 & 0 & 6 & 6 \end{array}\right]$$

$$\underbrace{}_{U}$$

2 possibilites to solve for x

① Gausian elimination

$$x_1 + 2x_2 + x_3 = 1$$

$$x_2 + 4x_3 = 2$$

$$6x_3 = 6$$

↑ solve for x

⑥

② RREF

$$\left[\begin{array}{ccc|c} 1 & 2 & \boxed{1} & 1 \\ 0 & 1 & \boxed{4} & 2 \\ 0 & 0 & \textcircled{1} & 1 \end{array}\right]$$

$\downarrow$ Get rid of pivots above diagonal

$$\left[\begin{array}{ccc|c} 1 & \boxed{2} & 0 & 0 \\ 0 & \textcircled{1} & 0 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

[I removed this row

$\downarrow$ remove element again

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

? Can read off $\begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}$

Not always Identity

We will do some special cases

⑦

If A is square and invertable

$$[A \mid I] \rightarrow [I \mid A^{-1}] \checkmark$$

(didn't follow)

how computers find A^{-1}

Suppose

$$\left[\underbrace{\begin{matrix}1&0&3\\0&1&2\\0&0&0\end{matrix}}_{R}\;\middle|\;\underbrace{\begin{matrix}4\\-2\\1\end{matrix}}_{b'}\right] \leftarrow \text{no solution}$$

<u>Problem</u> b' is <u>not</u> in the col space of R

$$\begin{array}{ccc}\begin{bmatrix}\textcircled{1}&0&3\\0&\textcircled{1}&2\\0&0&0\end{bmatrix}\\ \uparrow\;\uparrow\;\;\uparrow \\ \text{pivot}\;\;\text{free}\end{array}$$

Basis for $C(R)$

$$\begin{pmatrix}1\\0\\0\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix}$$

$C(A)$ is a subspace of R^m

= Space spanned by col vectors

<u>basis</u> = pivot cols

When pivoting → relationship b/w cols don't change
'' '' rows does ''

So in general $C(A) \neq C(R)$

~~$\begin{bmatrix}1&0&5\\0&1&2\\0&0&0\end{bmatrix}$ Basis~~

$$\begin{bmatrix}1&1&5\\0&1&2\\1&0&3\end{bmatrix}$$

Basis $C(B)$

$$\begin{pmatrix}1\\0\\1\end{pmatrix}\begin{pmatrix}1\\1\\0\end{pmatrix}$$

note - diff than before!

$N(A) \subseteq \mathbb{R}^n$

$N(A) = N(R)$

So if have $\left[\begin{array}{ccc|c}1&0&3&4\\0&1&2&-2\\0&0&0&0\end{array}\right]$

∞ sols

since $N(A) = (N(R))$

⑨

basis given by special sols → free/variable cols

To find *special sol*

$$\begin{bmatrix}1&0&3\\0&1&2\\0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$ ← set free variables to 1

$x_1=-3$

$x_2=-2$

Also need *particular* sol

$$\begin{bmatrix}1&0&3\\0&1&2\\0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\0\end{bmatrix}=\begin{bmatrix}4\\-2\\0\end{bmatrix}$$

← set free variable to 0 solve for other 2

← b that we were given

$x_1=4$

$x_2=-2$

General sol

$$\begin{bmatrix}4\\-2\\0\end{bmatrix}+\underbrace{c\begin{bmatrix}-3\\-2\\1\end{bmatrix}}_{\text{null space = vector space}}$$

← *note* These vectors do NOT for a vector space

↑ in that form when ∞ # of sols

⑩

What if we had

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}}_{R}$$

In general, can have 0, 1, or ∞ sols

To always have at least 1 sol

① $\hookrightarrow C(A) = \mathbb{R}^m$

To a have at most 1 sol

② $\llcorner N(A) = \{0\}$

review w/ my intuition

① $\text{rank}(A) = m$

or

not "Full row rank"

~~com~~

② $N(A) = \{0\}$

- "Full column rank"
- Cols of A are all lin. ind.
- $\text{rank}(A) = n$

$\text{rank}(A) = \text{rank}(R)$

$= \#$ pivots

$= \dim C(A)$

Free col = linear combo of the pivot cols that are to its left

⑪

In general rank $(A) \leq \min(m, n)$

So

① Need
$m \leq n$
Short + wide

② $m \geq n$
tall and wide

* Need to make sure you understand this
$\dim' N(A) = n - r$

$$\begin{bmatrix} 1 & 2 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

rank$(A) = 3$
$\dim N(A) = 2$
full col rank ⊗ No
$C(A) = \mathbb{R}^m$

So ∞ many sol

⑫

If $\begin{bmatrix} 1 & 2 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ how many sols?

$N(A) \neq (000)$ so can have 0 or ∞ man sols

$C(A) \neq R^m$ since can't span $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \text{has no sol}$$

So can have 0 or ∞ sols

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ rank = 2 → full col rank $\Big)$ 0 or 1 sol

$C(A) \neq R3$

Advice Conceptual qu

- Pick small $\left[\begin{matrix}1 & 0\\ 0 & 1\end{matrix}\right)$ 2x2 0,1 matrices
- actually try it
- See what is going on

- write down everything you know
  - see what you can do w/ ~~this~~ that

- do conceptual qu from the hw

Post I think I got it. I just need to review that dim stuff and factorization stuff more.

18.06 Review

3/8

Exam 1

On my own

Review of textbook

Linear combo

(only write stuff I don't understand - save time)

line → plane → 3D space

dot product
- not matrix multiplication ;
- order does not matter

$0 =$ perpendicular

lenght $= ||v|| = \sqrt{v \cdot u}$

unit vector $u = \frac{v}{||v||}$

Matrix multiplication.

$$Ax = \begin{bmatrix} u & v & w \end{bmatrix} \begin{bmatrix} c \\ d \\ e \end{bmatrix} = uc + vd + we$$

0

Inverse

$Ax = b$

$x = A^{-1} b$

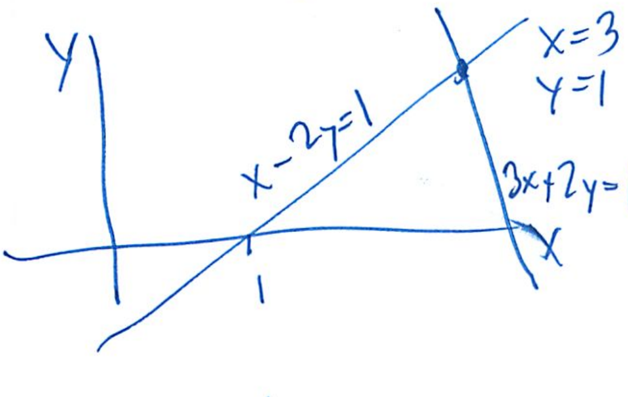
$= S b$

↑sum

$$\begin{aligned} x_1 &= b_1 \\ x_2 &= b_1 + b_2 \\ x_3 &= b_1 + b_2 + b_3 \end{aligned} = \underset{\underset{S}{A^{-1}}}{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

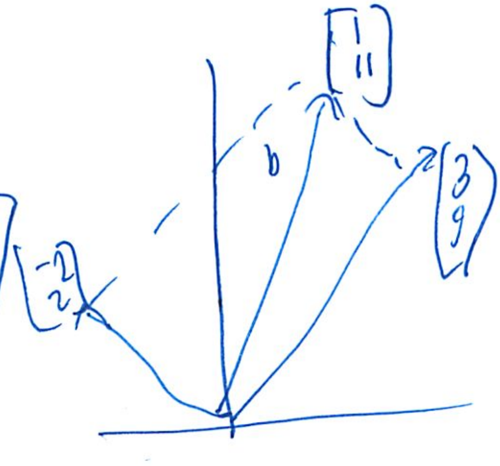
Chap 2 Solving Linear Eqn

$$\begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}$$



⇐ solution

row pic



Col pick

③

(I still don't fully get that pic...)

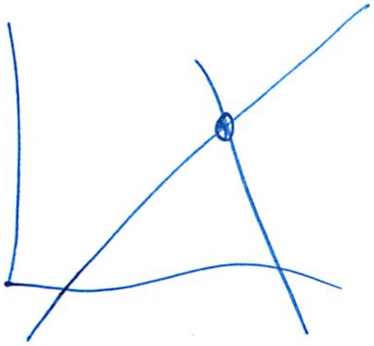
In 3D the 2 planes meet at a pint

Combine vectors in 3D

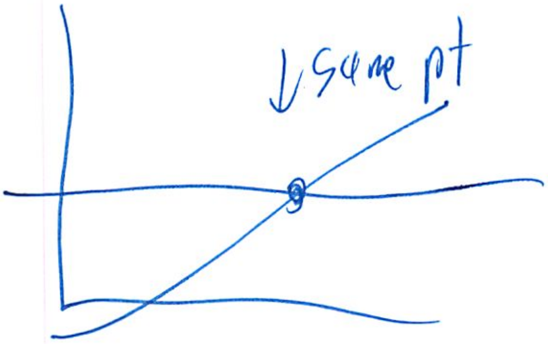
$$Ax = \begin{bmatrix} (row1) \cdot x_1 \\ (row2) \cdot x_2 \\ (row3) \cdot x_3 \end{bmatrix}$$

(This stuff seems very familar)

Elimination

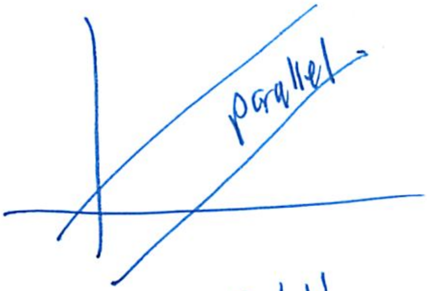


Before Row Pic



After Row pic

But



no solution

$x - 2y = 1$

$0y = 8$ ← false

∞ly many sols

$x - 2y = 1$

$0y = 0$

← anything works for y

Elim matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ -l_{21} & 1 & 0 \\ -l_{31} & -l_{23} & 1 \end{bmatrix}$$

Permutation

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Augmented $\begin{bmatrix} A & b \end{bmatrix}$

⑤

$AB \neq BA$

$C(A+B) = CA + CB$

$(A+B)C = AC + BC$

$A(BC) = (AB)C$ parenthesis don't matter

Inverse

$AA^{-1} = A^{-1}A = I$

$x = A^{-1}b$

If A is invertiable $Ax = 0$ can only have zero sol $x = A^{-1}0 = 0$

* only if square *

$(AB)^{-1} = B^{-1}A^{-1}$

For elim → change l $\overset{\text{sign}}{}$

And from review session

$L =$ Elim matrices w/ sign change on l $= E^{-1}$

Correct?

⑥

Gauss Jordan elimination

to find A^{-1}

elim ~~goes~~ to x

~~elim~~ we use it here to find $AA^{-1} = I$

$$A\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = I$$

So basically elim w/ $b = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} -2 & 1 & 0 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{bmatrix}$$

↓ elim

$$\begin{bmatrix} -2 & 1 & 0 & 1 & 0 & 0 \\ 0 & \frac{2}{3} & -1 & \frac{1}{2} & 1 & 0 \\ 0 & 0 & \frac{4}{3} & \frac{1}{3} & \frac{2}{3} & 0 \end{bmatrix}$$

↓ Go to REF

↓ why was I thinking below?

- produce zeros **above** the pivots

$$\begin{bmatrix} I & \begin{matrix} \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{matrix} \end{bmatrix}$$

I $\qquad$ K^{-1}

¿ so how is k^{-1} related to A^{-1} ?

It seems like k is a specific instance of A

NOT a form (like L, U)

Invertable → Square

full set of pivots

(so many different def ?!

row and cols ind

full rank $r = n = \underset{\uparrow \text{since square}}{m}$

Elim = Factorization $A = LU$

Saw before $L = E^{-1}$

basically ℓ not $-\ell$

$A = LU$ ie $\begin{bmatrix}1&0\\3&1\end{bmatrix}\begin{bmatrix}2&8\\0&5\end{bmatrix}$

or $A = L\underset{\uparrow \text{diagonal}}{D}U$ $\begin{bmatrix}1&0\\3&1\end{bmatrix}\begin{bmatrix}2&\\&5\end{bmatrix}\underset{\uparrow \text{1s on diagonal}}{\begin{bmatrix}1&4\\0&1\end{bmatrix}}$

⑧

Transpose

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix} \to A^T = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 3 & 4 \end{bmatrix}$$

$(A+B)^T \to A^T + B^T$

$(AB)^T \to B^T A^T$

$(A^{-1})^T = (A^T)^{-1}$

~~$\ast\ {}^{T}A^{T} = A$~~

Symmetric $A^T = A$

Permutations

$$\underset{\text{ahh}}{P}A = LU$$

just adding this

(There was less in this chap than I remember

Chap 3 Vector Spaces + Subspaces

$\mathbb{R}^n$ = all col vectors v w/ n components

$$\begin{array}{cc} & n \\ m & \begin{bmatrix} \quad & \quad \end{bmatrix} \end{array}$$

↳ m components

Yeah seems to be same as other part of book

M = matrices

F = functions

Z = zero vector

Subspaces

$\left.\begin{array}{l} V+W \\ cV \end{array}\right\}$ contains all linear combos

Col Space

Vector space made of Col vectors

* Ax = b only solvable if b is in col space

18:06

(so many diff ways to write same thing)

⑩

Nullspace Solving $Ax = 0$
↖all sols to this

ex $N\left(\begin{bmatrix}1 & 2\\ 3 & 6\end{bmatrix}\right)$ is all multiples of $\begin{bmatrix}-2\\ 1\end{bmatrix}$

Since $x_1 + 2x_2 = 0$
$3x_1 + 6x_2 = 0$
↓ really only 1 eq
$0 = 0$

Special sol Pick $x_2 = 1$ ← free col
Then $x_1 = -2$ hence

So # free cols = # special sols

Can solve w/ elim
Go to U
or furter to R ← RREF

Or back sub (original method)

Echelon Matrices

$$\begin{bmatrix} p & x & x & x & x & x & x \\ 0 & p & x & x & x & x & x \\ 0 & 0 & 0 & 0 & 0 & p & x \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Stair case = U

Col space $R^4 \Leftarrow 4$ components

Null space is subspace of R^7

Sol $Ux = 0$

↑ 4 special sols

So sol has non zero sols

(will do more here later)

RREF

← so is both

Zeros above and below pivots

pivot rows 1 so I

⑫

Rank # of pivots

free cols are combo of previous cols

can elim to find

or even to R

Rank 1

$$A = \text{col} \cdot \text{row} = U V^T$$

$$\begin{bmatrix} 1 & 3 & 10 \\ 2 & 6 & 20 \\ 3 & 9 & 30 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 3 & 10 \end{bmatrix}$$

pivot cols are not combos of previous col

null space matrix

$n - r = 5 - 2$

$= 3$ special sols

$$N = \begin{array}{l} \begin{bmatrix} 3 & -2 & 1 \\ 1 & 0 & 0 \\ 0 & -4 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array} \begin{array}{l} \text{pivot} \\ \text{free} \leftarrow \\ \text{pivot} \\ \text{free} \leftarrow \\ \text{free} \leftarrow \end{array} \text{ notice } I$$

So $AN = 0$

⑬

Complete Sol $Ax = b$

here pay attention to b

from OH its like 18.03

where

multiple curves

The "b" picks a certain curve

Aubment $[A\ b]$

$\downarrow$ elim

$[R\ d]$

(just a naming thing I think

One particular sol

Choose free variables $x = 0$ w/ our b

to get x_p

$$\boxed{A x_p = b}$$

⑭

Then combine w/ $A x_n = 0$

So $x = x_p + x_n$

$$x = \underbrace{\begin{bmatrix} 1 \\ 0 \\ 6 \\ 0 \end{bmatrix}}_{x_p} + \underbrace{x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ -4 \\ 1 \end{bmatrix}}_{x_n}$$

Remember

x_p must be in col space of A

---

full col rank $r = n$

all cols are pivots

$N(A)$ is only 0

no free variables → One sol or no sols

just extra 0 $R = \begin{bmatrix} I \\ 0 \end{bmatrix}$

2 unknows

3 eqs

tall + thin

$m \geq n$

Why is this not explicit?

---

full row rank $r = m$

all rows have pivots

$C(A)$ is whole space R^m → ∞ sols

$A x = b$ sol for every b

$x = x_p + x_n$

$R = [I \; F]$

3 unknows

2 eq

Short + wide

$m \leq n$

	$r = m$	$r = n$	Sq + invertable	$R = [I]$	1 sol
Full row rank	$r = m$	$r < n$	Short + wide	$R = [I\ F]$	∞ sol
full col rank	$r < m$	$r = n$	tall + thin	$R = \begin{bmatrix} I \\ 0 \end{bmatrix}$	0 or 1
	$r < m$	$r < n$	Not full rank	$R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$	0 or ∞

What is F?

looks like the free cols

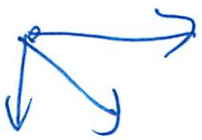
$$R = \begin{bmatrix} \underset{\substack{\uparrow \\ \text{pivot}}}{1} & \underset{\text{free}}{\underset{\uparrow\ \ \uparrow}{2\ \ 3}} \end{bmatrix}$$

$$I = \begin{bmatrix} \underset{\substack{\uparrow \\ \text{pivot}}}{1} \end{bmatrix} \qquad F = \begin{bmatrix} 2 & 3 \end{bmatrix}$$

Lin ind

When only sol to $Ax = 0$ is 0

$$x_1 v_1 + \dots + x_n v_n = 0$$



⑯

Any set of vectors in R^m must be lin dep
if $n \geq m$

<u>Span</u> if linear combos fills the space
↳ Cols of matrix <u>span</u> its col space

<u>Row space</u>
$C(A^T)$
↳ vectors in R^n.

<u>Basis</u> enough ind vectors to span the space
an no more (lin ind)
↳ pivot cols of A are a <u>basis</u> for col space

↳ all bases contain same # of vectors
↳ but are never unique ?
So something like

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

⑰

dim # vectors in every basis

---

row space ↙for example

$r = 2 = \dim = 2$

=

col space

$r = 2 = \dim = 2$

nullspace

$n - r = 5 - 2 = 3$ free variables

(find special sols)

↳ forms a basis of the null space

left nullspace

$m - r$

(not on exam)

key terms

3/8

rank = r = # of pivots

$$\boxed{\text{dim Nullspace} = n - r}$$

independence = no extra vectors

Lin ind → only sol $Ax = 0$ is 0

---

full col rank

$r = n$

all cols are pivots

$N(A)$ is only 0

no free variables

0 or 1 sols

$R = \begin{bmatrix} I \\ 0 \end{bmatrix}$

[tall + thin]

---

full row rank

$r = m$

all rows have pivots

$C(A)$ is whole space $\mathbb{R}^m$

∞ sols

$AX = b$ sol for every b

$x = x_p + x_n$

[Short + wide]

$R = [\,I \; F\,]$

②

<u>Spans</u> if linear combos fills the space

<u>Basis</u> just enough vectors to span the space and no more (ie lin ind)

<u>dim</u> # vectors in every basis

18.06 Quiz 1 Professor Strang February 28, 2011

Your PRINTED name is ______________________ **1.**

Your Recitation Instructor (and time) is ______________________ **2.**

Instructors: (Pires)(Hezari)(Sheridan)(Yoo) **3.**

Pratice 3/8

1. (a) By elimination find the **rank** of A and the pivot columns of A (in its column space):

R i same

$$A = \begin{bmatrix} 1 & 2 & 1 & 4 \\ 3 & 6 & 3 & 9 \\ 2 & 4 & 2 & 9 \end{bmatrix}.$$

p f f p

2

(b) Find the special solutions to $Ax = 0$ and then find **all solutions** to $Ax = 0$.

(c) For which number b_3 does $Ax = \begin{bmatrix} 3 \\ 9 \\ b_3 \end{bmatrix}$ have a solution?

↑the bs

Write the **complete solution** x (the general solution) with that value of b_3.

Augment

$$\left[\begin{matrix} A & \begin{matrix} 3 \\ 9 \\ b_3 \end{matrix} \end{matrix}\right] \to \text{reduce}$$

find b_3

$$X = X_p + X_n$$

[ ]

m n full row rank

2. Suppose A is a 3 by 5 matrix and the equation $Ax = b$ has a solution for every b. What are $(a)(b)(c)(d)$? (If you don't have enough information to answer, tell as much about the answer as you can.)

(a) Column space of A

3 pivots in $\mathbb{R}^3$

(b) Nullspace of A

$n - r$ $5-3=2$ ✓

(c) Rank of A

$= m = 3$ ✓

(d) Rank of the 6 by 5 matrix $B = \begin{bmatrix} A \\ A \end{bmatrix}$.

? same

elim $\begin{bmatrix} A \\ 0 \end{bmatrix}$

so same $= 3$

3. (a) When an odd permutation matrix P_1 multiplies an even permutation matrix P_2, the product $P_1 P_2$ is ~~I~~ odd (EXPLAIN WHY).

? always

P_1 odd # row ex to I

P_2 even # so odd + even = odd

(b) If the columns of B are vectors in the nullspace of A, then AB is ________ (EXPLAIN WHY).

$B = [B_1 | \cdots | B_n]$

$A \cdot B_i = 0$ then $AB = [AB_1 | \cdots | AB_n] = [0 | \cdots | 0] = 0$

(c) If $c = 0$, factor this matrix into $A = LU$ (lower triangular times upper triangular):

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 9 \\ 1 & 8 & c \end{bmatrix}.$$

elim

(d) That matrix A is invertible unless $c =$ ________ .

pivot can't be 0

$c = 21$

Wow quiz over [illegible] already!

18.06 Quiz 1 Professor Strang February 28, 2011

Your PRINTED name is ____________________ 1.

Your Recitation Instructor (and time) is ____________________ 2.

Instructors: (Pires)(Hezari)(Sheridan)(Yoo) 3.

1. (a) By elimination find the **rank** of A and the pivot columns of A (in its column space):

$$A = \begin{bmatrix} 1 & 2 & 1 & 4 \\ 3 & 6 & 3 & 9 \\ 2 & 4 & 2 & 9 \end{bmatrix}.$$

(b) Find the special solutions to $Ax = 0$ and then find **all solutions** to $Ax = 0$.

(c) For which number b_3 does $Ax = \begin{bmatrix} 3 \\ 9 \\ b_3 \end{bmatrix}$ have a solution?

Write the **complete solution** x (the general solution) with that value of b_3.

(a) $$\begin{bmatrix} 1 & 2 & 1 & 4 \\ 3 & 6 & 3 & 9 \\ 2 & 4 & 2 & 9 \end{bmatrix} \xrightarrow[-2R_1+R_3]{-3R_1+R_2} \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\frac{1}{3}R_2+R_3} \begin{bmatrix} \textcircled{1} & 2 & 1 & 4 \\ 0 & 0 & 0 & \textcircled{-3} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$r = \text{rank}(A) = 2$, pivot columns are $\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ 9 \\ 9 \end{bmatrix}$

(b) Special solutions : $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$. $N(A) = \left\{ c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

all solutions $c_1, c_2 \in \mathbb{R}$

(c) $$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 4 & 3 \\ 3 & 6 & 3 & 9 & 9 \\ 2 & 4 & 2 & 9 & b_3 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 2 & 1 & 4 & 3 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 1 & b_3-6 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 2 & 1 & 4 & 3 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & b_3-6 \end{array}\right]$$

Hence to have a Solution we need $b_3 - 6 = 0 \Rightarrow \boxed{b_3 = 6}$

For this value of b_3, a particular solution is given by $x_p = \begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

complete solution : $x_c = x_p + x_n = \begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

nullsolution

2. Suppose A is a 3 by 5 matrix and the equation $Ax = b$ has a solution for every b. What are $(a)(b)(c)(d)$? (If you don't have enough information to answer, tell as much about the answer as you can.)

(a) Column space of A : Since $Ax = b \Rightarrow b = x_1 A_1 + \cdots + x_n A_n$ where $A_1 \cdots A_n$ are the columns of A, every b in $\mathbb{R}^3$ is in $C(A)$. So $C(A) = \mathbb{R}^3$.

(b) Nullspace of A Since $C(A) = \mathbb{R}^3$, we have $r = 3$ and therefore # free variables = # special solutions $= n - r = 5 - 3 = 2$. Hence $N(A)$ is a plane in $\mathbb{R}^5$.

(c) Rank of A

By (a) $C(A) = \mathbb{R}^3$ therefore $r = 3$. We can also argue by saying that $r = m$ or we have a constraint on b.

(d) Rank of the 6 by 5 matrix $B = \begin{bmatrix} A \\ A \end{bmatrix}$.

We can use elimination to obtain

$$\begin{bmatrix} A \\ A \end{bmatrix} \longrightarrow \begin{bmatrix} A \\ 0 \end{bmatrix}.$$

But $\begin{bmatrix} A \\ 0 \end{bmatrix}$ has rank 3. Therefore $\text{rank}(B) = 3$.

3. (a) When an odd permutation matrix P_1 multiplies an even permutation matrix P_2, the product P_1P_2 is odd (EXPLAIN WHY).
P_1 applies an odd number of row exchanges to I and P_2 applies an even number. Hence P_1P_2 applies $\underline{\text{odd}} + \underline{\text{even}} = \underline{\text{odd}}$ number of row exchanges.

(b) If the columns of B are vectors in the nullspace of A, then AB is 0 matrix (EXPLAIN WHY). Let $B = [B_1|\cdots|B_K]$, where $B_1,\dots,B_K$ are the columns of B. Since each B_i is in $N(A)$ we have $AB_i = 0$.

Then $AB = [AB_1|\cdots\cdots|AB_K] = [0|\cdots|0] = 0$

(c) If $c = 0$, factor this matrix into $A = LU$ (lower triangular times upper triangular):

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 9 \\ 1 & 8 & c \end{bmatrix}.$$

(d) That matrix A is invertible unless $c =$ 21.

(b) $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 9 \\ 1 & 8 & c \end{bmatrix} \xrightarrow[-R_1+R_3]{-R_1+R_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 6 \\ 0 & 6 & c-3 \end{bmatrix} \xrightarrow{-3R_2+R_3} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 6 \\ 0 & 0 & c-21 \end{bmatrix}$

Hence A is invertible unless $c-21=0 \Rightarrow \boxed{c=21}$

(a) when $c=0$ we get $A = LU$

where $L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 3 & 1 \end{bmatrix}$ and $U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 6 \\ 0 & 0 & 0 \end{bmatrix}$

18.06 Quiz 1 March 1, 2010 Professor Strang

Your PRINTED name is: ______________________ **1.**

Your recitation number or instructor is ______________________ **2.**

3.

4.

Weird backwards work

1. Forward elimination changes $A\mathbf{x} = \mathbf{b}$ to a row reduced $R\mathbf{x} = \mathbf{d}$: the complete solution is

$$\mathbf{x} = \begin{bmatrix}4\\0\\0\end{bmatrix} + \mathbf{c}_1 \begin{bmatrix}2\\1\\0\end{bmatrix} + \mathbf{c}_2 \begin{bmatrix}5\\0\\1\end{bmatrix}$$

(a) **(14 points)** What is the 3 by 3 reduced row echelon matrix R and what is $\mathbf{d}$?

d

$$\begin{bmatrix}1 & -2 & 5\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$$

(b) **(10 points)** If the process of elimination subtracted 3 times row 1 from row 2 and then 5 times row 1 from row 3, what matrix connects R and $\mathbf{d}$ to the original A and $\mathbf{b}$? Use this matrix to find A and $\mathbf{b}$.

2. Suppose A is the matrix

$$A = \begin{bmatrix} 0 & 1 & 2 & 2 \\ 0 & 3 & 8 & 7 \\ 0 & 0 & 4 & 2 \end{bmatrix}.$$

(a) **(16 points)** Find all special solutions to $Ax = 0$ and describe in words the whole nullspace of A.

(b) **(10 points)** Describe the column space of this particular matrix A. "All combinations of the four columns" is not a sufficient answer.

(c) **(10 points)** What is the reduced row echelon form $R^* = \text{rref}(B)$ when B is the 6 by 8 block matrix

$$B = \begin{bmatrix} A & A \\ A & A \end{bmatrix} \text{ using the same } A?$$

3. **(16 points) Circle the words** that correctly complete the following sentence:

(a) Suppose a 3 by 5 matrix A has rank $r = 3$. Then the equation $Ax = b$

(always / sometimes but not always)

has (a unique solution / many solutions / no solution).

(b) What is the column space of A? Describe the nullspace of A.

4. Suppose that A is the matrix

$$A = \begin{bmatrix} 2 & 1 \\ 6 & 5 \\ 2 & 4 \end{bmatrix}.$$

(a) **(10 points)** Explain in words how knowing all solutions to $A\mathbf{x} = \mathbf{b}$ decides if a given vector $\mathbf{b}$ is in the column space of A.

(b) **(14 points)** Is the vector $\mathbf{b} = \begin{bmatrix} 8 \\ 28 \\ 14 \end{bmatrix}$ in the column space of A?

18.06 Professor Johnson Quiz 1 March 2, 2009

Your PRINTED name is: ____________________

Please circle your recitation:

(R01)	M2	2-314	Qian Lin
(R02)	M3	2-314	Qian Lin
(R03)	T11	2-251	Martina Balagovic
(R04)	T11	2-229	Inna Zakharevich
(R05)	T12	2-251	Martina Balagovic
(R06)	T12	2-090	Ben Harris
(R07)	T1	2-284	Roman Bezrukavnikov
(R08)	T1	2-310	Nick Rozenblyum
(R09)	T2	2-284	Roman Bezrukavnikov

Grading

1
2
3
Total:

1 (20 pts.) Your classmate, Nyarlathotep, performed the usual elimination steps to convert A to echelon form U, obtaining:

$$U = \begin{pmatrix} 1 & 4 & -1 & 3 \\ 0 & 2 & 2 & -6 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

(a) Find a set of vectors spanning the nullspace $N(A)$.

(b) If $U\mathbf{y} = \begin{pmatrix} 9 \\ -12 \\ 0 \end{pmatrix}$, find the complete solution $\mathbf{y}$ (i.e. describe all possible solutions $\mathbf{y}$).

(c) Nyarla gave you a matrix

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 3 & 1 \end{pmatrix}$$

and told you that $A = LU$. Describe the complete sequence of elimination steps that Nyarla performed, assuming that she did elimination in the usual way starting with the first column and eliminating downwards. That is, Nyarla first subtracted ______ times the first row from the second row, then subtracted ______ times the first row from the third row, then subtracted ____________________________________.
(Be careful about signs: *adding* a multiple of a row is the same as subtracting a *negative* multiple of that row.)

(d) If $A\mathbf{x} = \begin{pmatrix} 0 \\ 2 \\ 6 \end{pmatrix}$, then $U\mathbf{x} =$ ______.

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2 (20 pts.) Which of the following (if any) are subspaces? For any that are *not* a subspace, give an example of how they violate a property of subspaces.

(I) Given some 3×5 matrix A with full row rank, the set of all solutions to $A\mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

(II) All vectors $\mathbf{x}$ with $\mathbf{x}^T\mathbf{y} = 0$ *and* $\mathbf{x}^T\mathbf{z} = 0$ for some given vectors $\mathbf{y}$ and $\mathbf{z}$.

(III) All 3×5 matrices with $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ in their column space.

(IV) All 5×3 matrices with $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ in their nullspace.

(V) All vectors $\mathbf{x}$ with $\|\mathbf{x} - \mathbf{y}\| = \|\mathbf{y}\|$ for some given fixed vector $\mathbf{y} \neq 0$.

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3 (20 pts.) A is a matrix with a nullspace $N(A)$ spanned by the following three vectors:

$$\begin{pmatrix} 1 \\ 2 \\ -1 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 4 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 3 \\ 1 \end{pmatrix}.$$

(α) Give a matrix B such that its column space $C(B)$ is the same as $N(A)$. (There is more than one correct answer.) [Thus, any vector $\mathbf{y}$ in the nullspace of A satisfies $B\mathbf{u} = \mathbf{y}$ for some $\mathbf{u}$.]

(β) Give a different possible answer to (α): another B with $C(B) = N(A)$.

(γ) For some vector $\mathbf{b}$, you are told that a particular solution to $A\mathbf{x} = \mathbf{b}$ is

$$\mathbf{x}_p = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}.$$

Now, your classmate Zarkon tells you that a second solution is:

$$\mathbf{x}_Z = \begin{pmatrix} 1 \\ 1 \\ 3 \\ 0 \end{pmatrix},$$

while your other classmate Hastur tells you "No, Zarkon's solution can't be right, but here's a second solution that is correct:"

$$\mathbf{x}_H = \begin{pmatrix} 1 \\ 1 \\ 3 \\ 1 \end{pmatrix}.$$

Is Zarkon's solution correct, or Hastur's solution, or are both correct? (Hint: what should be true of $\mathbf{x} - \mathbf{x}_p$ if $\mathbf{x}$ is a valid solution?)

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18.06 Quiz 1 March 1, 2010 Professor Strang

Your PRINTED name is: ____________________________ **1.**

Your recitation number or instructor is ____________________ **2.**

3.

4.

1. Forward elimination changes $A\mathbf{x} = \mathbf{b}$ to a row reduced $R\mathbf{x} = \mathbf{d}$: the complete solution is

$$\mathbf{x} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + \mathbf{c}_1 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + \mathbf{c}_2 \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}$$

(a) **(14 points)** What is the 3 by 3 reduced row echelon matrix R and what is $\mathbf{d}$?

Solution: First, since R is in reduced row echelon form, we must have

$$\boxed{\mathbf{d} = \begin{bmatrix} 4 & 0 & 0 \end{bmatrix}^T}$$

The other two vectors provide special solutions for R, showing that R has rank 1: again, since it is in reduced row echelon form, the bottom two rows must be all 0, and the top row is $\begin{bmatrix} 1 & -2 & -5 \end{bmatrix}^T$, i.e. $\boxed{R = \begin{bmatrix} 1 & -2 & -5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}$.

(b) **(10 points)** If the process of elimination subtracted 3 times row 1 from row 2 and then 5 times row 1 from row 3, what matrix connects R and $\mathbf{d}$ to the original A and $\mathbf{b}$? Use this matrix to find A and $\mathbf{b}$.

Solution: The matrix connecting R and $\mathbf{d}$ to the original A and $\mathbf{b}$ is

$$E = E_{31}E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -5 & 0 & 1 \end{bmatrix}}$$

That is, $R = EA$ and $E\mathbf{b} = \mathbf{d}$. Thus, $A = E^{-1}R$ and $\mathbf{b} = E^{-1}\mathbf{d}$, giving

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 5 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -2 & -5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \boxed{\begin{bmatrix} 1 & -2 & -5 \\ 3 & -6 & -15 \\ 5 & -10 & -25 \end{bmatrix}}$$

$$\mathbf{b} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 5 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} = \boxed{\begin{bmatrix} 4 \\ 12 \\ 20 \end{bmatrix}}$$

2. Suppose A is the matrix

$$A = \begin{bmatrix} 0 & 1 & 2 & 2 \\ 0 & 3 & 8 & 7 \\ 0 & 0 & 4 & 2 \end{bmatrix}.$$

(a) **(16 points)** Find all special solutions to $Ax = 0$ and describe in words the whole nullspace of A.

Solution: First, by row reduction

$$\begin{bmatrix} 0 & 1 & 2 & 2 \\ 0 & 3 & 8 & 7 \\ 0 & 0 & 4 & 2 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 2 & 2 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 4 & 2 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

so the special solutions are

$$\boxed{s_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, s_2 = \begin{bmatrix} 0 \\ -1 \\ -\frac{1}{2} \\ 1 \end{bmatrix}}$$

Thus, $N(A)$ is a plane in $\mathbb{R}^4$ given by all linear combinations of the special solutions.

(b) **(10 points)** Describe the column space of this particular matrix A. "All combinations of the four columns" is not a sufficient answer.

Solution: $C(A)$ is a plane in $\mathbb{R}^3$ given by all combinations of the pivot columns, namely

$$\boxed{c_1 \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 8 \\ 4 \end{bmatrix}}$$

(c) **(10 points)** What is the reduced row echelon form $R^* = \text{rref}(B)$ when B is the 6 by 8 block matrix

$$B = \begin{bmatrix} A & A \\ A & A \end{bmatrix} \text{ using the same } A?$$

Solution: Note that B immediately reduces to

$$B = \begin{bmatrix} A & A \\ 0 & 0 \end{bmatrix}$$

We reduced A above: the row reduced echelon form of of B is thus

$$\boxed{B = \begin{bmatrix} rref(A) & rref(A) \\ 0 & 0 \end{bmatrix}, rref(A) = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}}$$

3. **(16 points) Circle the words** that correctly complete the following sentence:

(a) Suppose a 3 by 5 matrix A has rank $r = 3$. Then the equation $Ax = b$

(always / sometimes but not always)

has (a unique solution / many solutions / no solution).

Solution: the equation $Ax = b$ $\boxed{\text{always}}$ has $\boxed{\text{many solutions}}$.

(b) What is the column space of A? Describe the nullspace of A.

Solution: $\boxed{\text{The column space is a 3-dimensional space inside a 3-dimensional space}}$, i.e. it contains all the vectors, and $\boxed{\text{the nullspace has dimension } 5 - 3 = 2 > 0 \text{ inside } \mathbb{R}^5}$.

4. Suppose that A is the matrix

$$A = \begin{bmatrix} 2 & 1 \\ 6 & 5 \\ 2 & 4 \end{bmatrix}.$$

(a) **(10 points)** Explain in words how knowing all solutions to $A\mathbf{x} = \mathbf{b}$ decides if a given vector $\mathbf{b}$ is in the column space of A.

Solution: The column space of A contains all linear combinations of the columns of A, which are precisely vectors of the form $A\mathbf{x}$ for an arbitrary vector $\mathbf{x}$. Thus, $\boxed{A\mathbf{x} = \mathbf{b} \text{ has a solution if and only if } \mathbf{b} \text{ is in the column space of } A}$.

(b) **(14 points)** Is the vector $\mathbf{b} = \begin{bmatrix} 8 \\ 28 \\ 14 \end{bmatrix}$ in the column space of A?

Solution: $\boxed{\text{Yes}}$. Reducing the matrix combining A and $\mathbf{b}$ gives

$$\left[\begin{array}{cc|c} 2 & 1 & 8 \\ 6 & 5 & 28 \\ 2 & 4 & 14 \end{array}\right] \to \left[\begin{array}{cc|c} 2 & 1 & 8 \\ 0 & 2 & 4 \\ 0 & 3 & 6 \end{array}\right] \to \left[\begin{array}{cc|c} 2 & 1 & 8 \\ 0 & 2 & 4 \\ 0 & 0 & 0 \end{array}\right]$$

Thus, $\mathbf{x} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ is a solution to $A\mathbf{x} = \mathbf{b}$, and $\mathbf{b}$ is in the column space of A.

18.06 Quiz 1 Solution

Hold on Monday, 2 March 2009 at 11am in Walker Gym.
Total: 60 points.

Problem 1: Your classmate, Nyarlathotep, performed the usual elimination steps to convert A to echelon form U, obtaining:

$$U = \begin{pmatrix} 1 & 4 & -1 & 3 \\ 0 & 2 & 2 & -6 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

(a) Find a set of vectors spanning the nullspace $N(A)$.

(b) If $U\vec{y} = \begin{pmatrix} 9 \\ -12 \\ 0 \end{pmatrix}$, find the complete solution $\vec{y}$ (i.e. describe all possible solutions $\vec{y}$).

(c) Nyarla gave you a matrix

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 3 & 1 \end{pmatrix}$$

and told you that $A = LU$. Describe the complete sequence of elimination steps that Nyarla performed, assuming that she did elimination in the usual way starting with the first column and eliminating downwards. That is, Nyarla first subtracted ______ times the first row from the second row, then subtracted ______ times the first row from the third row, then subtracted ______________________________________. (Be careful about signs: *adding* a multiple of a row is the same as subtracting a *negative* multiple of that row.)

(d) If $A\vec{x} = \begin{pmatrix} 0 \\ 2 \\ 6 \end{pmatrix}$, then $U\vec{x} =$ ______.

$\boxed{\text{Solution}}$ (20 points = 5+5+5+5)

(a) The pivots are in the first two columns of U, so x_3 and x_4 are the free variables. Setting $x_3 = 1, x_4 = 0$, we get (from the second row of $U\vec{x} = 0$) $x_2 = -1$

and (from the first row) $x_1 = 1 - 4x_2 = 5$; setting $x_3 = 0, x_4 = 1$, we get (from the second row) $x_2 = 3$ and (from the first row)$x_1 = -3 - 4x_2 = -15$. Hence, $N(A)$ is spanned by two special solutions as follows.

$$N(A) = x_3 \begin{pmatrix} 5 \\ -1 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -15 \\ 3 \\ 0 \\ 1 \end{pmatrix} \qquad \text{for all } x_3, x_4 \in \mathbb{R}.$$

(b) First, we need to find a particular solution. For this, we may set the free variables to $y_3 = y_4 = 0$. Thus, (from the second row of $U\vec{y} = b$) $y_2 = -6$ and (from the first row) $y_1 = 9 - 4y_2 = 33$. Hence, all the solution to the equations are given by the sum of the particular solution and any vector in the nullspace (all linear combinations of the special solutions):

$$\vec{y} = y_3 \begin{pmatrix} 5 \\ -1 \\ 1 \\ 0 \end{pmatrix} + y_4 \begin{pmatrix} -15 \\ 3 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 33 \\ -6 \\ 0 \\ 0 \end{pmatrix} \qquad \text{for all } y_3, y_4 \in \mathbb{R}$$

(c) Nyarla first subtracted **2** times the first row from the second row, then subtracted **−1** time s the first row from the third row, then subtracted **3 times the second row from the third row.**

There are a couple of ways to solve this problem. The easiest is to remember that the L matrix, the product of the inverses of the elimination matrices, is simply composed of the multipliers for each of the elimination steps below each column. Under the first column of L we have 2 and -1, and these are thus the multiples of the first row that get subtracted from rows 2 and 3. Under the second column of L we have a 3, and this is the multiple of the second row that gets subtracted from the third row.

The other way to solve it is to just multiply L by U to get $A = LU$, and re-do the elimination process. Obviously, this is a bit more work, but is not too bad.

(d) Applying the same elimination operations in (c) to $A\vec{x}$ should give $U\vec{x}$. So, we have

$$\begin{pmatrix} 0 \\ 2 \\ 6 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 0 \\ 2 \\ 6 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$$

Alternatively, we can just solve $U\vec{x}$ from $A\vec{x}$ as follows. Let $\vec{v} = U\vec{x}$. Then $L\vec{v} = UL\vec{x} = A\vec{x} = \begin{pmatrix} 0 \\ 2 \\ 6 \end{pmatrix}$. Thus, we can solve from the top as follows. $v_1 = 0$, $v_2 = 2 - 2v_1 = 2$, and $v_3 = 6 - 3v_2 + v_1 = 0$. Hence, $U\vec{x} = \vec{v} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$.

REMARK: Some students realized that $U\vec{x} = L^{-1}(A\vec{x})$. But several of these students did not get $L^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -3 & 1 \end{pmatrix}$ correctly. Be careful that the inverse of $\begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix}$ is *not* $\begin{pmatrix} 1 & 0 & 0 \\ -l_{21} & 1 & 0 \\ -l_{31} & -l_{32} & 1 \end{pmatrix}$; the lower left entry should be $l_{21}l_{32} - l_{31}$. (Only for elimination matrices, which have nonzero entries below only a single diagonal, can you always invert just by flipping signs.) More generally, if you find yourself inverting a matrix, you should realize that there is probably an easier way to do it: to multiply $\vec{v} = L^{-1}(A\vec{x})$, it is easier to solve $L\vec{v} = A\vec{x}$ for $\vec{v}$ by elimination (especially since L is triangular, so you can just do forward substitution as above).

Problem 2: Which of the following (if any) are subspaces? For any that are *not* a subspace, give an example of how they violate a property of subspaces.

(I) Given some 3×5 matrix A with full row rank, the set of all solutions to $A\vec{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

(II) All vectors $\vec{x}$ with $\vec{x}^T\vec{y} = 0$ *and* $\vec{x}^T\vec{z} = 0$ for some given vectors $\vec{y}$ and $\vec{z}$.

(III) All 3×5 matrices with $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ in their column space.

(IV) All 5×3 matrices with $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ in their nullspace.

(V) All vectors $\vec{x}$ with $\|\vec{x} - \vec{y}\| = \|\vec{y}\|$ for some given fixed vector $\vec{y} \neq 0$.

$\boxed{\text{Solution}}$ (20 points = 4+4+4+4+4)

(I) No. This is not a vector space because $\vec{x} = 0$ is not in this subspace.

(II) Yes. (This is actually just the left nullspace of the matrix whose columns are $\vec{y}$ and $\vec{z}$.)

(III) No. For example, the zero matrix is not in this subset.

(IV) Yes. If the nullspaces of A_1 and A_2 contain $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, then any linear combination of these matrices does too:

$$(\alpha_1 A_1 + \alpha_2 A_2)\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \alpha_1 A_1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \alpha_2 A_2 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 0; \text{ for all } \alpha_1; \alpha_2.$$

(V) No. For example, $2\vec{y}$ satisfies the condition (because $\|2\vec{y} - \vec{y}\| = \|\vec{y}\|$) but $\vec{y}$ does not satisfies the condition (because $\|\vec{y} - \vec{y}\| = 0 \neq \|\vec{y}\|$). This violates the fact that a subspace is preserved under multiplication by scalars.

REMARK: A common problem we saw in the grading is that some students do not know how to express a counterexample. A counterexample is simply a single specific element of the set that violates a specific property of subspaces, or a specific element that should be in the set but isn't (as in the case of the sets missing $\vec{0}$ above). One such example is all that is needed to disqualify a set as a subspace; no further abstract argument is necessary. If you were asked to find an "example" and you find yourself writing a long, abstract essay, you are probably making a mistake!

Problem 3: A is a matrix with a nullspace $N(A)$ spanned by the following three vectors:

$$\begin{pmatrix}1\\2\\-1\\3\end{pmatrix}, \quad \begin{pmatrix}0\\1\\1\\4\end{pmatrix}, \quad \begin{pmatrix}-1\\-1\\3\\1\end{pmatrix}.$$

(α) Give a matrix B such that its column space $C(B)$ is the same as $N(A)$. (There is more than one correct answer.) [Thus, any vector $\vec{y}$ in the nullspace of A satisfies $B\vec{u} = \vec{y}$ for some $\vec{u}$.]

(β) Give a different possible answer to (α): another B with $C(B) = N(A)$.

(γ) For some vector $\vec{b}$, you are told that a particular solution to $A\vec{x} = \vec{b}$ is

$$\vec{x}_p = \begin{pmatrix}1\\2\\3\\4\end{pmatrix}.$$

Now, your classmate Zarkon tells you that a second solution is:

$$\vec{x}_Z = \begin{pmatrix}1\\1\\3\\0\end{pmatrix},$$

while your other classmate Hastur tells you "No, Zarkon's solution can't be right, but here's a second solution that is correct:"

$$\vec{x}_H = \begin{pmatrix}1\\1\\3\\1\end{pmatrix}.$$

Is Zarkon's solution correct, or Hastur's solution, or are both correct? (Hint: what should be true of $\vec{x} - \vec{x}_p$ if $\vec{x}$ is a valid solution?)

$\boxed{\text{Solution}}$ (20 points = 5+5+10) (α) Since the nullspace is spanned by the given three vectors, we may simply take B to consist of the three vectors as columns, i.e.,

$$B = \begin{pmatrix}1 & 0 & -1\\2 & 1 & -1\\-1 & 1 & 3\\3 & 4 & 1\end{pmatrix}.$$

B need not be square (many students insisted on square solutions).

(β) For example, we may simply add a zero column to B:

$$B = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 2 & 1 & -1 & 0 \\ -1 & 1 & 3 & 0 \\ 3 & 4 & 1 & 0 \end{pmatrix}.$$

Or, we could interchange two columns. Or we could multiply one of the columns by -1. For example:

$$B = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ -1 & 1 & -3 \\ 3 & 4 & -1 \end{pmatrix}.$$

Or we could replace one of the columns by a linear combination of that column with the other two columns (any invertible column operation). Or we could replace B by $-B$ or $2B$. There are many possible solutions. In any case, the solution shouldn't require any significant calculation!

(γ) Since any solution $\vec{x}$ to the equation $A\vec{x} = \vec{b}$ is of the form $\vec{x}_p + \vec{n}$ for some vector $\vec{n}$ in the nullspace, the vector $\vec{x} - \vec{x}_p$ must lie in the nullspace $N(A)$. Thus, we want to look at:

$$\vec{x}_Z - \vec{x}_p = \begin{pmatrix} 0 \\ -1 \\ 0 \\ -4 \end{pmatrix}, \quad \vec{x}_H - \vec{x}_p = \begin{pmatrix} 0 \\ -1 \\ 0 \\ -3 \end{pmatrix}.$$

To determine whether a vector $\vec{y}$ lies in the nullspace $N(A)$, we can just check whether it is in the column space of B, i.e. check whether $B\vec{z} = \vec{y}$ has a solution. As we learned in class, we can check this just by doing elimination: if elimination produces a zero row in B, it should produce a zero row in the right-hand side. In terms of B from part (α) augmented by the right-hand side, this gives:

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 2 & 1 & -1 & -1 \\ -1 & 1 & 3 & 0 \\ 3 & 4 & 1 & a \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 1 & 2 & 0 \\ 0 & 4 & 4 & a \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & a+4 \end{pmatrix}$$

We can get a solution if and only if $a = -4$. So Zarkon is correct.

REMARK: Several students apparently just stared at the nullspace vectors and found a linear combination that gave $\vec{x}_Z - \vec{x}_p$:

$$\vec{x}_Z - \vec{x}_p = \begin{pmatrix} 0 \\ -1 \\ 0 \\ -4 \end{pmatrix} = 1\begin{pmatrix} 1 \\ 2 \\ -1 \\ 3 \end{pmatrix} - 2\begin{pmatrix} 0 \\ 1 \\ 1 \\ 4 \end{pmatrix} + 1\begin{pmatrix} -1 \\ -1 \\ 3 \\ 1 \end{pmatrix}.$$

Then they stared at Hastur's solution, couldn't find such a combination, and concluded that it was not a solution. This conclusion is correct *in this case*, and was awarded full marks because you were not asked to justify your solution. However, doing elimination is much more systematic and reliable, and ensures that there isn't a linear combination that you simply missed. Use elimination next time!

REMARK: Some students saw the zero components of $\vec{x}_Z - \vec{x}_p$, didn't see any corresponding zero components in the given nullspace vectors, and concluded that $\vec{x}_Z - \vec{x}_p$ was not in the nullspace. This is wrong: the key point is that $\vec{x}_Z - \vec{x}_p$ can be any vector in the nullspace, which means any linear combination of the given nullspace vectors. There are plenty of ways to combine nonzero vectors to get vectors with zero components!

18.06 Quiz 1 Professor Strang March 9, 2012

Please PRINT your name Michael Plasmeier

1. 32
2. 9
3. 18
4. 10

69

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r1	T	11	4-159	Ailsa Keating
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[✓ r3]	T	12	4-159	Jennifer Park
r4	T	12	36-153	Rune Haugseng
r5	T	1	4-153	Dimiter Ostrev
r6	T	1	4-159	Uhi Rinn Suh
r7	T	1	66-144	Ailsa Keating
r8	T	2	66-144	Niels Martin Moller
r9	T	2	4-153	Dimiter Ostrev
r10	ESG			Gabrielle Stoy

1. **(36 pts.)** Suppose the 4 by 4 matrix A (with 2 by 2 blocks) is already reduced to its rref form

$$A = \begin{bmatrix} I & 3I \\ 0 & 0 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Find a basis for the column space $C(A)$.

9

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

(b) Describe all possible bases for $C(A)$.

5

all linear combos of $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ and, $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

(c) Find a basis (special solutions are good) for the nullspace $N(A)$.

9

$$\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 + 3x_3 = 0$
$x_2 + 3x_4 = 0$

free cols x_3, x_4

$\underline{x_3 = 1}$
$x_1 = -3$
$x_2 = 0$

$\underline{x_4 = 1}$
$x_1 = 0$
$x_2 = -3$

$$\rightarrow \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

9

(d) Find the complete solution x to the 4 by 4 system

$b =$

$$Ax = \begin{bmatrix} 5 \\ 4 \\ 0 \\ 0 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 + 3x_3 = 5$
$x_2 + 3x_4 = 4$
$x_3 = 0$
$x_4 = 0$

I can you do!

$x_1 = 5$
$x_2 = 4$

$$x_p = \begin{bmatrix} 5 \\ 4 \\ 0 \\ 0 \end{bmatrix}$$

$$x = \overset{x_p}{\begin{bmatrix} 5 \\ 4 \\ 0 \\ 0 \end{bmatrix}} \overset{+}{+} x_3 \overset{x_n}{\begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}} + x_4 \begin{bmatrix} 0 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

Simple qu! did the practice!

Otherwise

$x_1 = 5 - 3x_3$

$x_2 = 4 - 3x_4$

I guess need to pick x_3, x_4 since ∞ # sols ... ?

Say $x_3 = 1$

$x_4 = 1$

$x_1 = 2$

$x_2 = 1$

$$\begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{array}{cc} & n \\ m & \begin{bmatrix} & A & \end{bmatrix} \end{array} \text{rank}=r \qquad \begin{array}{cc} & N \\ M & \begin{bmatrix} & B & \end{bmatrix} \end{array} \text{rank}=r$$

2. **(16 pts.)** Suppose the matrix A is m by n of rank r, and the matrix B is M by N of rank R. Suppose the column space $C(A)$ is contained in (possibly equal to) the column space $C(B)$. (This means that every vector in $C(A)$ is also in $C(B)$.) What relations must hold between m and M, n and N, and r and R? $C(A) \subseteq C(B)$

It might be good to write down an example of A and B where all the columns are different.

One example $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$m < M$
$n < N$
$r < R$

$m \leq M$ ← could be = or larger
? ? you can just add extra 0s?
?depends on tech, def of contained in... otherwise $m = M$

$n \leq N$ ← could be = or larger

? if col space =

$r \leq R$ ← could be = or larger

9

Also $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$m = M$
$n = N$
$r \leq R$

$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$m = M$
$n < N$
$r < R$

$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

$m = M$
$n = N$
$r = R$

$$[A]\,\overbrace{[B]}^{\text{dep}} = [C]$$

3. (a) **(16 pts.)** Suppose three matrices satisfy $AB = C$. If the columns of B are dependent, show that the columns of C are dependent.

way too little space!

(b) **(12 pts.)** If A is 5 by 3 and B is 3 by 5, show using part (a) or otherwise that $AB = I$ is impossible. Next Pg!

Matrix multiplication is

$$\begin{bmatrix} a & b & c \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} ax + by + cz \end{bmatrix}$$

6/16

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}\underbrace{\begin{bmatrix} x & i \\ y & j \\ z & k \end{bmatrix}} = \begin{bmatrix} ax + by + cz & ai + bj + ck \\ \underbrace{dx + ey + fz} & \underbrace{di + ej + fk} \end{bmatrix}$$

Not general!!

if $\begin{bmatrix} x \\ y \\ z \end{bmatrix}\begin{bmatrix} i \\ j \\ k \end{bmatrix}$. are lin dep

Then these columns are still lin dep since they are multiples of the same values a,b,c, d,e,f

b) $${}_{5}\begin{bmatrix} \overset{3}{A} \end{bmatrix} {}_{3}\begin{bmatrix} \overset{5}{B} \end{bmatrix} \neq I$$

The answer will be $5\begin{bmatrix} \overset{5}{\ } \end{bmatrix}$ - So not b/c of that

Somehow related to part a ??

dim col space = dim of row space

dim ≈ rank

size of basis # pivot cols

Answer: So max r in A is 3 - (full col rank)

max r B is 3 - (full row rank) ✓

That means 2 of the cols in B will be dep.

As we saw in a), if some cols of B are dep, then some cols will be dep in AB, so it can't be I ■ ✓

12/12

nice!

4. **(20 pts.)** Apply row elimination to reduce this invertible matrix from A to I. Then write A^{-1} as a product of three (or more) simple matrices coming from that elimination. Multiply these matrices to find A^{-1}. $A=LDU$

normally $Ax=B$

$A^{-1}=xB$;

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 4 & 0 & 1 \end{bmatrix}.$$

elim

Permute $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 4 & 0 & 1 \end{bmatrix} \xrightarrow[\text{row 3 = row 3} - 4\,\text{row 1}]{\ell_{31}=4} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = U$

$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

RREF remove 1s above + below the 1

row 2 = row 2 − row 3

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I = R$$

did not study RREF

10

$A = LU$

$$A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}}_{U}$$

$L = (E_{31}^{-1})$

$AA^{-1} = I$

$A = A^{-1} I$

A^{-1}

$$\overset{A}{\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 4 & 0 & 1 \end{bmatrix}} = \overset{A^{-1}}{\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}} \overset{I}{\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}}$$

$A^{-1} = AI$

$$\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 4 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$$

$A^{-1} = LUI$

(next pg)

1st: 15 to go

$$A^{-1} = \overset{L}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix}} \overset{U}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}} \overset{I}{\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} \Big[$$

I should be A

$$A^{-1} = \overset{A}{\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 4 & 0 & 1 \end{bmatrix}} \overset{I}{\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}}$$

$$A^{-1} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 4 & 0 & 1 \end{bmatrix}$$

that does not make any sense

$$A^{-1} = U^{-1} L^{-1}$$

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$$

or for 3 terms

$$A^{-1} = U^{-1} D^{-1} L^{-1}$$

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & -1 \\ -4 & 0 & 1 \end{bmatrix}$$

plausable

Solution

18.06 Quiz 1 Professor Strang March 9, 2012

Please PRINT your name ______________________________

Please Circle your Recitation:

r1	T	11	4-159	Ailsa Keating
r2	T	11	36-153	Rune Haugseng
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r9	T	2	4-153	Dimiter Ostrev
r10	ESG			Gabrielle Stoy

1.
2.
3.
4.
______

1. **(36 pts.)** Suppose the 4 by 4 matrix A (with 2 by 2 blocks) is already reduced to its rref form

$$A = \begin{bmatrix} I & 3I \\ 0 & 0 \end{bmatrix}.$$

(a) Find a basis for the column space $C(A)$.

(b) Describe all possible bases for $C(A)$.

(c) Find a basis (special solutions are good) for the nullspace $N(A)$.

(d) Find the complete solution x to the 4 by 4 system

$$Ax = \begin{bmatrix} 5 \\ 4 \\ 0 \\ 0 \end{bmatrix}.$$

Solution.

(a) The column space is spanned by the vectors $(1,0,0,0), (0,1,0,0), (3,0,0,0), (0,3,0,0)$. We then put them in a matrix and do a Gaussian elimination to find independent vectors. This tells us that the basis for the column space is $\{(1,0,0,0),(0,1,0,0)\}$

(b) The column space can be described by

$$C(A) = \{(x,y,0,0) \mid x,y \in \mathbb{R}\},$$

so the basis of $C(A)$ is the set of any two independent vectors $(x_1,x_2,0,0)$ and $(x_3,x_4,0,0)$. This means that the matrix

$$A = \begin{pmatrix} x_1 & x_3 \\ x_2 & x_4 \end{pmatrix}$$

has full rank (in other words $x_1x_4 - x_2x_3 \neq 0$ must hold).

(c) We observe that $(3,0,-1,0)$ and $(0,3,0,-1)$ are two independent vectors belonging to the null space. Since the column space has dimension 2, the null space has dimension $4-2=2$, so any basis of $N(A)$ has two elements. Hence, $\{(3,0,-1,0),(0,3,0,-1)\}$ is a basis for $N(A)$.

(d) We start by looking for $x_{\text{particular}}$ via elimination. Note that the matrix is already in a reduced row echelon form:

$$\left(\begin{array}{cccc|c} 1 & 0 & 3 & 0 & 5 \\ 0 & 1 & 0 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

So $x_{\text{particular}} = (5,4,0,0)$. Then the complete solution is given by

$$\begin{aligned} x &= x_{\text{particular}} + x_{\text{nullspace}} \\ &= (5,4,0,0) + (3a,3b,-a,-b) \\ &= (5+3a, 4+3b, -a, -b) \end{aligned}$$

for any $a,b \in \mathbb{R}$.

□

2. **(16 pts.)** Suppose the matrix A is m by n of rank r, and the matrix B is M by N of rank R. Suppose the column space $C(A)$ is contained in (possibly equal to) the column space $C(B)$. (This means that every vector in $C(A)$ is also in $C(B)$.) What relations must hold between m and M, n and N, and r and R?

   It might be good to write down an example of A and B where all the columns are different.

   Solution. The column space of A is contained in $\mathbb{R}^m$, and the column space of B is contained in $\mathbb{R}^M$. If $C(A) \subseteq C(B)$, this means they are contained in the same Euclidean space, so $M = m$. The dimension of the column space is the rank of the matrix, so if $C(A) \subseteq C(B)$, this means $\dim C(A) \leq \dim C(B)$, hence $r \leq R$. There are no relations between N and n; $n = N$ if $A = B$, $n \leq N$ if $B = [A\ A]$, and $n \geq N$ if $A = [B\ B]$.

   ☐

3. (a) **(16 pts.)** Suppose three matrices satisfy $AB = C$. If the columns of B are dependent, show that the columns of C are dependent.

(b) **(12 pts.)** If A is 5 by 3 and B is 3 by 5, show using part (a) or otherwise that $AB = I$ is impossible.

Solution. (a) The columns of B being dependent means by definition that there is a vector $\mathbf{x} \neq 0$ such that $B\mathbf{x} = 0$. But then we also have

$$C\mathbf{x} = (AB)\mathbf{x} = A(B\mathbf{x}) = A(\mathbf{0}) = \mathbf{0},$$

which means that the same $\mathbf{x} \neq 0$ works to show that the columns of C are dependent.

(b) The columns of B are dependent, since these are five vectors in $\mathbb{R}^3$, and $5 > 3$. Thus, by part (a), the columns of AB must be dependent. However, columns of I are independent, so AB can never equal I. [Note: Switching the order matters here. One can indeed find a 3×5 matrix A, and a 5×3 matrix B such that $AB = I$ is the 3×3 identity - hence any "proof" that is insensitive to the order of A and B must be flawed].

☐

4. **(20 pts.)** Apply row elimination to reduce this invertible matrix from A to I. Then write A^{-1} as a product of three (or more) simple matrices coming from that elimination. Multiply these matrices to find A^{-1}.

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 4 & 0 & 1 \end{bmatrix}.$$

Solution. Swapping rows 1 and 2 corresponds to

$$P := \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Subtracting 4 times row 1 from row 3 corresponds to

$$E_{31} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{pmatrix}.$$

Subtracting row 3 from row 2 corresponds to

$$E_{23} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Putting them together, we get

$$E_{23}E_{31}PA = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} A = I.$$

Hence, $A^{-1} = E_{23}E_{31}P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 4 & -1 \\ 0 & -4 & 1 \end{pmatrix}.$

☐

From: Niels Martin Møller <moller@MIT.EDU>
Sent: Tuesday, March 13, 2012 7:40 PM
To: Niels Martin Moeller
Subject: [18.06] Exam #1 solutions posted

Follow Up Flag: FollowUp
Flag Status: Flagged

Dear 18.06,

- All grades should be approved and visibly in Stellar shortly (if not, email you TA about it).

- The solutions to Exam #1 have been posted on the 18.06 website:

http://web.mit.edu/18.06/www/psets.shtml

- Here are some statistics for your first test:
Average: 78.15
Standard deviation: 14.70

We do know many of you will wonder what this means - and some of you have expressed worries about your test scores. We won't get into thinking about your letter grades until much later in the course.

Now is the time to leave the first exam behind you (after you checked the solutions, and pondered what you could have done differently, of course), and keep working more on grasping the concepts and tools of everything linear.

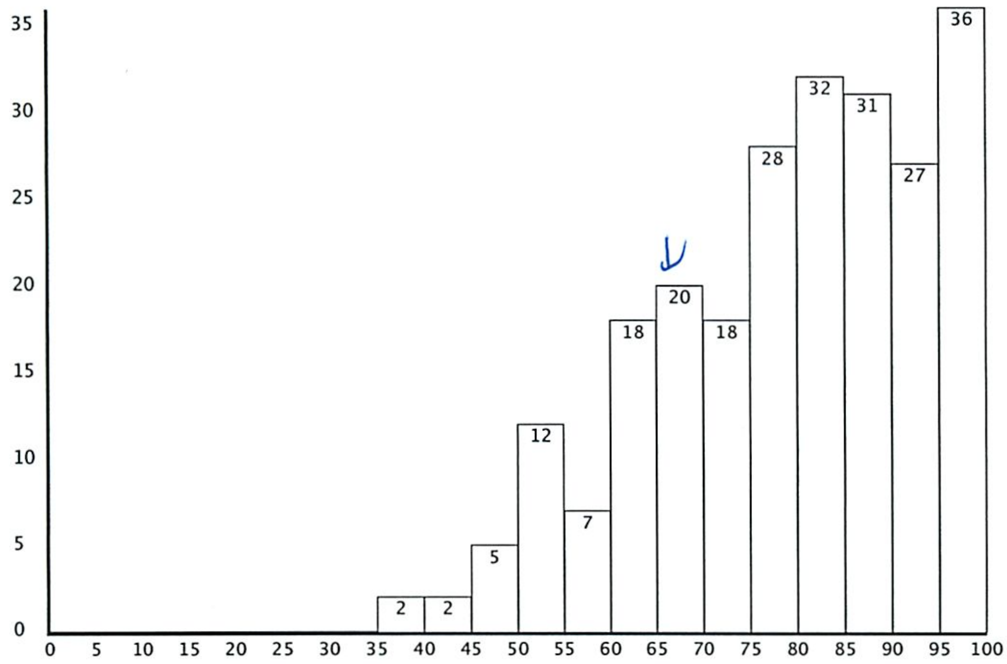
Enjoy learning and understanding more of this exciting subject!

Best,
Niels

18.06 Linear Algebra

Dashboard | Students | Assignments

Grading Summary for Exam 1



Number of Scores:	238
Average:	78.47
Standard Deviation:	14.68