18.06 Lecture

3/12

Quiz 1 being graded now

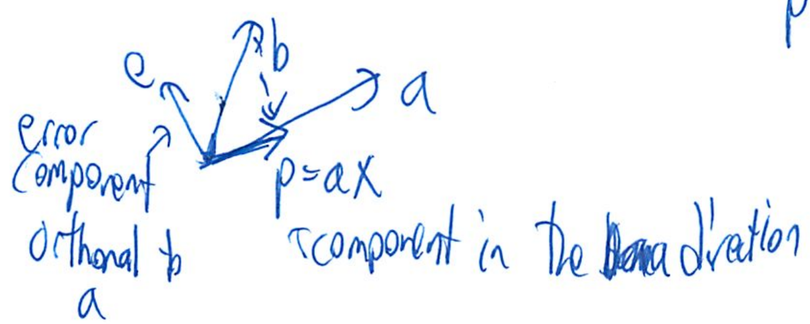
- problem 1 went well
- all possible basies - need word "independent"
- will get more tomorrow

Chap 4 Perpendicular/Orthoganity

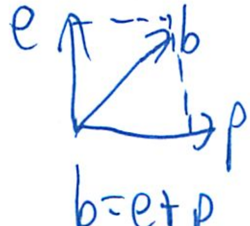
key

- projections p
- Projection matrix P

"projection onto a line Through a



b seperated into p and e



$b = e + p$

②

Don't use θ – mess up nice formulas w/ angles!

Key Fact $\perp$ $e = b - p$

$= b - ax$

(a number, not a vector)

a is perpendicular to e

$a^T(b - ax) = 0$

$a^Tb - a^Tax = 0$

$\boxed{x = \frac{a^Tb}{a^Ta}}$ ← There is a cos embedded in here

$p = ax = \frac{a\,a^Tb}{a^Ta}$

Projection of b on line through $2a$ = same since same line

$\frac{2a\,2a^Tb}{2a^T2a}$ cancles

Projection Matrix

For projection onto line through a

Take line · Projection Matrix = projection

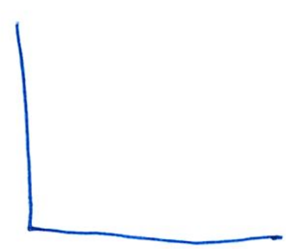
$$P = \frac{a a^T}{a^T a}$$

$$\frac{\begin{bmatrix} \\ \\ \end{bmatrix}\begin{bmatrix} & & \end{bmatrix}}{\begin{bmatrix} & & \end{bmatrix}\begin{bmatrix} \\ \\ \end{bmatrix}}$$

$$p = P b$$

Example

$\mathbb{R}^3$



$a = (2,0,0)$
$b = (b_1, b_2, b_3)$ $\Big\rangle$ $p = (b_1, 0, 0)$

$p = a x$

$p = (2,0,0)\frac{b_1}{2}$

(4)

Checking on formula for x

Dot product $2b_1$

$a^Ta = 4$

$$\frac{2b_1}{4} = \frac{b_1}{2}$$

What is the Projection Matrix?

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

"So simple!"

$p = Pb$

What about P^2

$P^2 b = p$ ← same

Since after 1st time → on line

So 2nd time → no change

$Pp = p$

5

key point: $P^2 = P$

$$P^2 = \frac{aa^T}{a^Ta}\frac{aa^T}{a^Ta}$$ = Put parenthesis in right place

$$= \frac{a\cancel{(a^Ta)}a^T}{\cancel{(a^Ta)}(a^Ta)}$$

$$= \frac{aa^T}{a^Ta} = P$$

Rank of $P = 1$

Col space of P = all ~~possible~~ outputs from P. any matrix

= all outputs from Px

= line through a

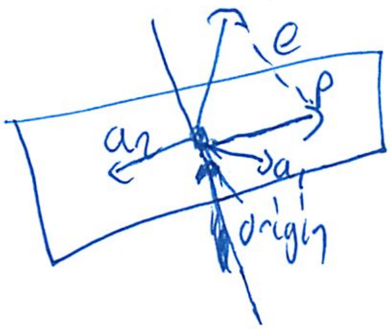
$P^T = P$ Symmetric

These are the crucial facts for projection on a line

(6)

But dim does not have to = 1

What if 3 Dimension



$b = p + e$ ← splitting vectors into 2 parts

plane: need a basis of 2 vectors

a_1, a_2

Subspace = all combinations

2D plane

We now have 2 a_is instead of 1

$p = a_1 x_1 + a_2 x_2$

e is perpendicular to the plane

$e = b - p$

$e^T \underline{a_1} = 0$
$e^T \underline{a_2} = 0$ } 2 key facts

7

aka

$$\boxed{\begin{aligned} a_1^T e &= 0 \\ a_2^T e &= 0 \end{aligned}}$$

can do in either order

2 eq, 2 unknowns

$\cancel{\begin{bmatrix} a_1^T e \\ a_2^T e \end{bmatrix}}$

$$\begin{bmatrix} a_1^T \\ a_2^T \end{bmatrix} \overset{e=}{\begin{bmatrix} b-p \end{bmatrix}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$A = \begin{bmatrix} a_1 & a_2 \end{bmatrix}$$

has the 2 basis vectors in it (tells us subspace)

Now can write the projection

$$p = a_1 x + a_2 x_2$$
$$= Ax$$

8

$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$p = Ax$

So $[b - p]$ is $[b - Ax]$

$\begin{bmatrix} \text{———} \cancel{a_1^T} \\ a_2^T \end{bmatrix}$ is A^T

So $A^T [b - Ax] = 0$

$\boxed{\boxed{A^T A x = A^T b}}$ ← biggest eq in stats
The "normal" eq

Revise our previous fact w/ capital A

$A^T(b - Ax) = 0$

Can then mutate that back to above formula

9

One more thing to note

plane = col space of A

e = left null space ~~of~~ of A

~~the~~ perpendicular to plane

So plane $A^T e = 0$

Key equations

$e \perp$ space

$A^T A x = A^T b$ ← key eq

$x = (A^T A)^{-1} A^T b$ ← its sol

$p = Ax = A(A^T A)^{-1} A^T b$ ← multiplied by A

↑ proj

L remember for 1 D

$p = a x = \dfrac{a a^T}{a^T a} b$

↑ division instead of inverse

10

Why not

$$A A^{-1}(A^{T})^{-1} A^{T} b = b$$

Since A is not square - its rectangular

Can't get an inverse in wrong direction

When is it correct?

projection $= b$

When you are projecting onto the entire space!

Can't break $(A^T A)^{-1}$ into $A^{-1}(A^T)^{-1}$!!!

Cardinal sin in Chap 4

$$\boxed{P = A(A^T A)^{-1} A^T}$$

What would P^2 be?

$$P^2 = P$$

$$P^2 = A(A^TA)^{-1}A^TA(A^TA)^{-1}A^T$$

$$A\cancel{(A^TA)^{-1}}\cancel{(A^TA)}(A^TA)^{-1}A^T$$

$$= A(A^TA)^{-1}A^T$$

$$= P$$

18.06 Recitations

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Exam 1 back

Solutions posted soon

(substitute TA - not as good)

~~Mean~~ Mean ~ 77
St dev ~ 15
) Prelim

B is usually 1 st dev

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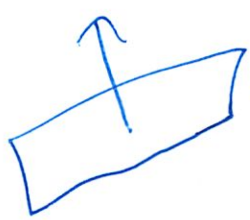
4 Fundamental Subspaces

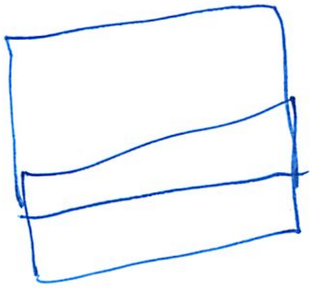
Chap 3.6

Graphs + Networks

Projections + Orthogonality



orthogonal



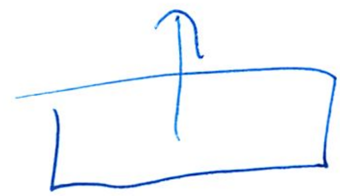
Not orthogonal
since where wall meets ceiling
same vector

2

In $\mathbb{R}^3$ zero vector orthagonal to all vectors

$V \subseteq \mathbb{R}^n = V + V^{\perp}$

$\uparrow$ k-dim plane line

you can write anything as 

$\mathbb{R}^n = \mathbb{R}^n + \{0\}$

Graphs

p ~~[illegible]~~ 426 summary

Randomly put orentations on
Put lables on nodes and lines

$\uparrow$ w/ circle

Write incidence matrix

$$A = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & +1 & -1 & 0 \\ 2 & +1 & 0 & -1 \\ 3 & 0 & -1 & +1 \end{array}$$

n nodes

m edges

-1 = Comes out of

1 = Goes into

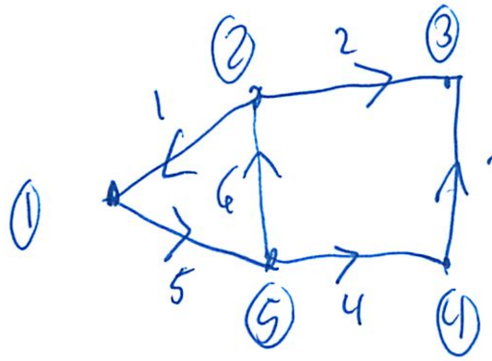
7

Look at what you have vs your convention of ⊕

Then can look for lots of ~~loops~~ loops in the graph

Where notion of ind loops comes at

Can use these formulas for the Null space



a) $N(A)$

b) $N(A^T)$

c) $C(A)$

d) $C(A^T)$

Handy to know dim

Find enough ind vectors to $=$ dim

So then spaned by the space

(6)

No connection b/w the 2 - so argument does not really work

$$N(A^T) = \{ \underbrace{A^T y}_{\mathbb{R}^n} = \underline{0} \mid \underline{y} \in \mathbb{R}^m \}$$

edges

takes current that runs around eges

spits out something in node world

L current in + out of node

$A^T y$ has n entries

each $= 0$

total current in + out $= 0$

in = out

Find y_0 such that $A^T y_0 = 0$

Imagine current flows

(3)

So null space of A

$$N(A) = \{ x \in \mathbb{R}^{n \text{ nodes}} \mid A\underline{x} = \underline{0} \}$$

$\mathbb{R}^m$

eating stuff on nodes

0 difference along each leg

- makes sense for voltages

(Confused)

$$= \{ (c, c, \ldots, c) \mid c \in \mathbb{R} \}$$

$$= \text{Span}\{ (1, 1, \ldots, 1) \}$$

$$\dim(N(A)) = 1$$

$$\text{rank}(A) = \dim C(A) = \dim C(A^T)$$

row space

$n - 1$

If graph is not connected

Argument still kinda works

(8) ~~Find~~ Find a basis for

$V^\perp$ orthg. complements

a) ~~xxx~~ $V = \text{span}\{(1,2,3),(3,2,1)\} \subseteq \mathbb{R}^3$

Find $V^\perp$ by think in linear algebra terms which eqns?

b) Find the projection matrix P on V

Check $P^2 = P$

(BR break) $P^T = P$

~~a~~) $N(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\} \subseteq \mathbb{R}^5$

$\dim C(A) = ?$

~~xx~~ $\dim C(A^T) = ?$

$\dim N(A^T) = ?$

⑨

~~dim~~ $\dim C(A) = 5 - 1 = 4$ always same

$\dim C(A^T) = 4$

$\dim N(A^T) = 6 - 4 = 2$

row space

Since its # cols (A) = # pivot cols + # free cols

What is difference b/w pair of nodes

Null space i b/w any pair of nodes - no voltage diff

Only 1 piece of circuit so dim = 1

2 unconnected pieces → dim = 2

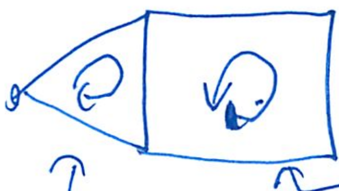
So

$N(A^T) = \text{span}\{[$

look for loops that give you 2 lin ind vectors

(11)

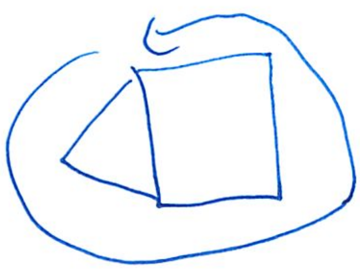
What if we go around



will be ind

$$N(A^T) = \text{span}\left\{ \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

So big loop is a linear combo of



the vectors we wrote above

The middle edge will disappear

$$\begin{bmatrix} | & | \\ b_1 & b_2 \\ | & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \text{blue} \\ \text{loop} \end{bmatrix}$$

2 bases as cols

(12)

So col space A

↳ 4 dimensional

Can use orthogonality ideas

$C(A) \perp N(A^T)$

$C(A) = (N(A^T))^\perp$

$Ax \in C(A)$

$A^T y = 0 \quad (y \in N(A^T))$

$b \cdot y = b^T y = (Ax)^T y = x^T A^T y = x^T 0 = 0$

Def

$x^T A y = (A^T x)^T y$

$x \cdot (Ay) = (A^T x) \cdot y$

P3

$v \in V^\perp \iff v \perp (1,2,3)$

and $v \perp (3,2,1)$

Since space is all linear combos of

will be $\perp$ to all things in their span

As a vector in $\mathbb{R}^3$

$$\text{rows} \rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Precicly orthogonal complement to the

Span of those rows

18.06 Lecture

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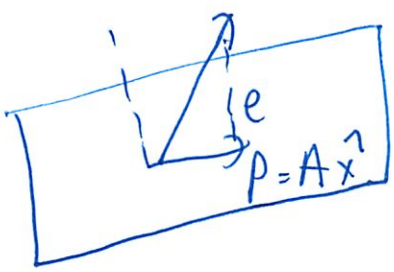
$A^T A \hat{x} = A^T b \quad e = b - A\hat{x}$

$p = A\hat{x}$

$= A(A^T A)^{-1} A^T b$

$= Pb$



$\leftarrow$ all Ax

(col space of A

Use what we learned from last time

$Ax = b$ - not a sol since b is not in the col space

m eqs - n unknowns

but $m >> n$

lots more measurements than variables

$A = \left[\quad \right]_{1000 \times 2}$ $\leftarrow$ chance it satisfies pretty low

but what is best sol?

②

Best "solution" $= Ax$

↑ least squares

$\min \|Ax-b\|^2 = \|e\|^2$

$Ax = b$

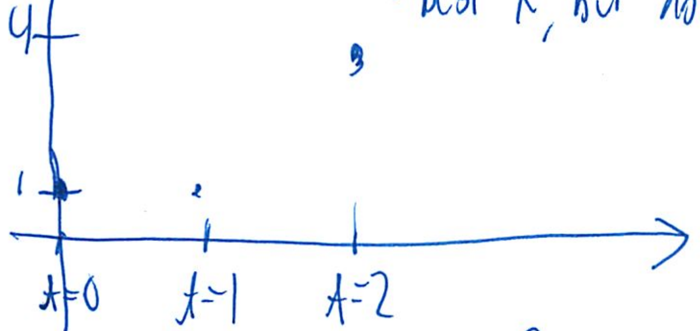
$A^TA\hat{x} = A^Tb \quad e = b - A\hat{x}$

The best x is $\hat{x}$

$\hat{x} = (A^TA)^{-1}A^Tb$

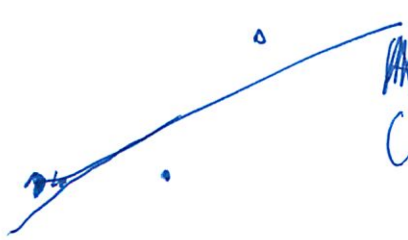
↑ is square / invertable

↑ best x, but not exact x



$m=3$

If fitting w/ straight line



$C+Dt$

(3)

This picture is in 2D

The original picture is in 3 space

$A = \begin{bmatrix} \\ \\ \\ \end{bmatrix} \quad B = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$

using items like

$C + D(2) = 4$

$C + D(1) = 1$

$C + D(0) = 1$

$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \quad b = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$

~~He is renaming A → C B → D Done~~

$\underline{\min}\ \|Ax - b\|^2 = \|e\|^2$ min at $\hat{x}$

Best x is $\hat{x} = (A^TA)^{-1}A^Tb$

(9)

Research : Don't square

Look for sparcity (most 0s)

Prof: Look for important stuff

So to find the best line

$$A^T A \hat{x} = A^T b$$

$$\overset{A^T}{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \hat{C} \\ \hat{D} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$$

Now actually do the its

$$\begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix}$$ invertable
Symetric

$$\begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} \hat{C} \\ \hat{D} \end{bmatrix} = \begin{bmatrix} 6 \\ 9 \end{bmatrix}$$

2 by 2

(5)

Could use elim

$$\begin{bmatrix} 3 & 3 \\ 0 & 2 \end{bmatrix}$$

$2\hat{D} = 3$

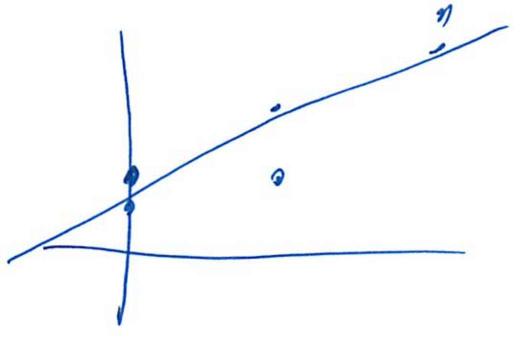
$\hat{D} = 1.5$

$3\hat{C} + 4.5 = 6$

$3\hat{C} = 1.5$

$\hat{C} = .5$

Now can go back to pic and ask if reasonable



$.5 + 1.5t$

Where is e (the error)

$e = b - A\hat{x}$

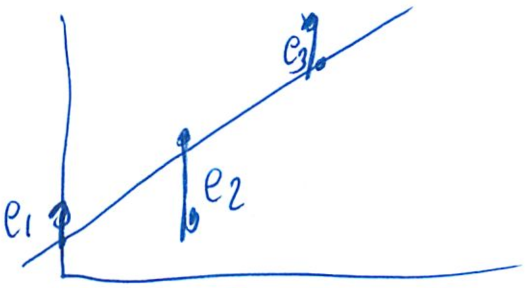
(6)

So what is this

$$e = b - A\hat{x}$$

$$= \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} - \begin{bmatrix} .5 \\ 2.0 \\ 3.5 \end{bmatrix}$$

$$= \begin{bmatrix} .5 \\ -1 \\ .5 \end{bmatrix} = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}$$



since

$$p = Ax$$

$$= \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} .5 \\ 1.5 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 2.0 \\ 3.5 \end{bmatrix}$$

True on a line

the line is "best"

the sum of squares of the e_is is the smallest

Could you best fit by a quadratic?

Yeah $C + Dt + Et^2$

$n = 3 \leftarrow 3$ parameters

7

Here 3 parameters 3 unknowns

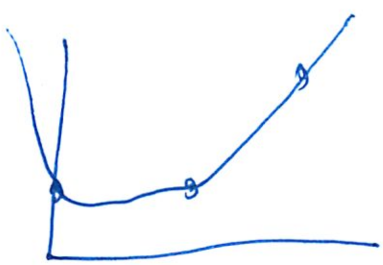
So would fit exactly

$$A = \begin{bmatrix} 1 & 0 & 0^2 \\ 1 & 1 & 1^2 \\ 1 & 2 & 2^2 \end{bmatrix} \begin{bmatrix} \hat{C} \\ \hat{D} \\ \hat{E} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$$

Now E is a square matrix

$t + t^{1} + t^{2}$

So parabola right through the points



But it will screw up a lot more if you add extra points

Fourier fits by sin, cos (Fourier Series)

⑧

2D Find the best plane

$C + Dx + Ey$

zero level slopes

To fit 4 data points $b_1 = 1$ at $x=2$ $y=3$

$= 2$ $x=3$ $y=0$

$= 1$ $x=0$ $y=3$

$= 1$ $x=1$ $y=50$

made up a problem

Can we find best C,D,E to "best" ~~all~~ fit the flat surface

$m = 4$ eqns

Is the matrix $A^T A$ invertable?

When is $A^T A$ invertable?

- Square
- Symmetric

always (continued)

(4)

It's invertable when A has ind cols

↳ rank of A is n

? The quiz qu

If A has dep cols, $A^T A$ is not any better

If A has dep cols → expect to have trouble

↳ The Nullspace has some solutions

$Ax=0$

$A^T A x = 0$ has <u>same</u> solutions

<u>Recap</u>: Formula for projection matrix

$$P = A(A^T A)^{-1} A^T$$

proj onto $C(A)$

m by m

$P^2 = P$

$P^T = P$

18.06 Lecture

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Gram-Schmidt Day

Start w/ ind. vectors $a_1, a_2, \dots a_n$

Produce orthonormal vectors $q_1, \dots, q_n$

$$q_i^T q_j = 0 \qquad q_i^T q_i = 1$$

Today: Finish Chap 4 (need to read)

Orthonormal - orthogonal

divide by lenght → unit vectors

Like elimination - but orthogonization

qr(A) = [Q, R]

↑ output

orthonormal

↓ not pivots

diagonal

$$A = \underbrace{\begin{bmatrix} a_1 & \cdots & a_n \\ | & & | \\ | & & | \end{bmatrix}}_{A_{m\times n}} = \underbrace{\begin{bmatrix} q_1 & \cdots & q_n \\ | & & | \\ | & & | \end{bmatrix}}_{Q_{m\times n}} \underbrace{\begin{bmatrix} \ddots & \ddots & \ddots \\ & \ddots & \ddots \\ 0 & & \ddots \end{bmatrix}}_{R_{n\times n}}$$

Q

If LU most used command

Then this is #2 used

Want to work in plane of these 2

$q_1 = \frac{a_1}{\|a_1\|}$

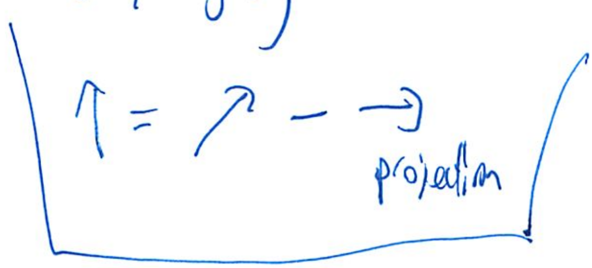
q_2 must be orthogonal

What using today i.e. e - the vector in the $\perp$ direction

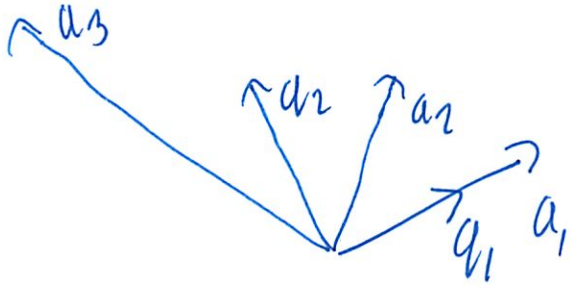
$$q_2 = \left(a_2 - a_1 \frac{a_1^T a_2}{a_1^T a_1}\right) / \| \qquad \|$$

the projection

Since its in the same dir we are already going



(3)



q_3 = want the part that sticks out of plane

= a_3 − proj of a_3 onto a_1, a_2
q_1, q_2

$\|\qquad\qquad\|$

Take each vector as it comes
Subtract out the part you have
We know how to create projection

If in matlab - get $\sqrt{\ }$
but if renormalize get nice #s

$q_3 =$ Could use projection matrix onto a_1, a_2 plane
but $P_{a_1,a_2} = A(A^TA)^{-1}A^T$
Lots of work!

(4)

Same plane

But $P_{q_1,q_2} = P_{r_1} = Q(Q^TQ)^{-1}Q^T$

$\uparrow I$ so do the work ahead of time

Suppose q_1, q_2 are orthonormal

Project b onto that q_1 q_2 plane

$$\begin{bmatrix} q_1^T \\ q_2^T \end{bmatrix} \begin{bmatrix} q_1 & q_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\underset{2\times5}{Q^T} \; \underset{5\times2}{Q} = \underset{2\times2}{I}$$

$$P = Q \; Q^T$$

$$\begin{bmatrix} \\ \\ \end{bmatrix} \begin{bmatrix} & & \end{bmatrix}$$

$$\begin{aligned} P^2 &= Q(Q^TQ)Q^T \\ &= QQ^T \\ &= P \end{aligned}$$

5

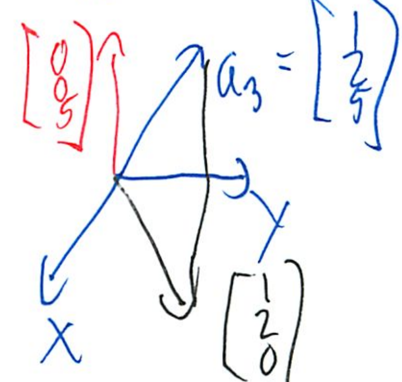
$$q_3 = a_3 - P_{a_3}$$

$$= a_3 - QQ^T a_3$$

$$= a_3 - \underbrace{\frac{a_3^T q_1}{q_1^T q_1} q_1}_{\text{projection } a_3 \text{ on } q_1 \text{ dir}} - \underbrace{\frac{a_3^T q_2}{q_2^T q_2} q_2}_{\text{proj } a_3 \text{ on } q_2 \text{ dir}}$$

$$= a_3 - a_3^T q_1 q_1 - a_3^T q_2 q_2$$

by using q's already orthogonal



Gram-Schmidt to 3 parts

- as far as we have to know

(6)

Gram-Schmidt q_4

a_1, a_2, a_3, a_4

$q_1, q_2, q_3, \hat{q}_4$

$$q_4 = a_4 - (a_4^T q_1) q_1 - (a_4^T q_2) q_2 - (a_4^T q_3) q_3$$

Expect q_4 to be orthogonal to q_1

$a_4 q_1 = 0$ True?

Inner product

$(a_4^T q_1) - (q_4^T q_1) - (\quad) 0 - 0$

Yes ✓

You can orthogonize - and Gram Schmidt found a way
Some one else found a better way, - used in Matlab
Householder

7

But really complex to explain, not straightforward

---

al example

- See QR

$$q_1 = \frac{a_1}{\|a_1\|}$$

$$a_1 = q_1 \|a_1\|$$

$$\underset{A}{\begin{bmatrix} \; & \; \end{bmatrix}} = \underset{Q}{\begin{bmatrix} \; & \; \end{bmatrix}} \underset{R}{\begin{bmatrix} \|a_1\| & \; \\ & \; \end{bmatrix}}$$

Triangular since we are marching forward

1st ones are directly connected
Only need that 1 number

8

$$A = \begin{bmatrix} 1 & 3 \\ 1 & 4 \\ 1 & 5 \\ 1 & 12 \end{bmatrix}$$

$$\begin{array}{cc} a_1 & a_2 \end{array}$$

$$q_1 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$$ normalize a_1

$\|a_1\| = 2$

$$q_2 = a_2 - \left(\frac{a_2^T q_1}{\| \ \|} \right) q_1$$

$$= \begin{bmatrix} 3 \\ 4 \\ 5 \\ 12 \end{bmatrix} - 12 \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} -3 \\ -2 \\ -1 \\ 6 \end{bmatrix}$$

should be orthogonal
need to normalize

$\frac{\quad}{\| \ \|}$

$= \sqrt{50}$

⑨

Tells us how to get back from Q to A

$A = LU$

$A = QR$

What would happen if original ~~other~~ a_is were dep

Would get 0

Would fail

Vectors must be ind to get orthonormal

Next week: determinants

18.06 Lecture

3/30

L18 on OCW

Originally 3/19 but missed

2nd half of course

Square matricies

determinents → to get → eigenvalues

assocated w/ every sq. matrix

det

packs in a lot of info

det ≠ 0 → matrix invertable

det = 0 → singular

| | vertical bars = det

3 properties

1. Det of I = 1

2. row ex → reverse sign of det

~~(wow cool!)~~ (oh no that is when adding)

3. ↳ so know all the det (permutation) = ± 1

even # row ex → (+)

odd # row ex → (−)

②

So ① $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \leftarrow I$

② $\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1 \leftarrow$ since 1 row ex from I

General case

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

(ohhh!)

③ key property

3a) If multiply one row by t
then t time det

$$\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

3b) keep 2nd row the same ← always
↑ the n-1 last rows

If $a+a'$ and $b+b'$ → lin combos
of the 1st row only
Then get sum of dets

$$\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$$

(3)

$\det(A+B) \neq \det(A) + \det(B)$

↑ well does not have to be

(3) Only need linear in each row

∟ could be in 2nd row

∟ but not both 2nd and 1st row!

So from these 3 properties - learn more about det

(4) if 2 rows are =, the det = 0

we can see easily 2 × 2 case

but more general ^larger case ?

↳ we can use (2) / row ex- ~~change~~

∟ we will get same matrix

det didn't change, but sign did change

So the only possible thing this could be = 0

also we know 2 = rows means duplicate, non ind

So rank < n

(4)

(5) <u>key</u> Elim step we are always doing

└ Subtract $l \times$ row i from ~~another row~~ row k

└ Can clean out matrix like this

└ get 0s below diagonal

Det does <u>not</u> change!

$$\begin{vmatrix} a & b \\ c-la & d-lb \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a & b \\ -la & -lb \end{vmatrix}$$

using (3b)

Now use (3a) - can factor out $-l$

$$= \begin{vmatrix} a & b \\ c & d \end{vmatrix} - l \begin{vmatrix} a & b \\ a & b \end{vmatrix}$$

Now by (4) we see is 0

$$= \begin{vmatrix} a & b \\ c & d \end{vmatrix} + 0$$

(nice proof!)

(5)

(6) Row of zeros → det A = 0

a complete

└ So can know its singular

$\begin{vmatrix} \cancel{a} & \cancel{b} \\ \cancel{0} & \cancel{0} \end{vmatrix}$

$\begin{vmatrix} 0 & 0 \\ c & d \end{vmatrix}$

↓ multiply by 5

$5 \begin{vmatrix} 0 & 0 \\ c & d \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ 5c & 5d \end{vmatrix}$ (3a)

det has to be 0

(7) So we can do elim (5) to upper triangular

So at $U = \begin{bmatrix} d_1 & * & * & * \\ 0 & d_2 & * & * \\ 0 & 0 & \ddots & d_n \\ 0 & 0 & 0 & \end{bmatrix}$ ← diagonal

~~[illegible]~~

det U = just the product of the dis (the pivots)

$= d_1 \cdot d_2 \cdot \dots \cdot d_n$

(6) (so much ~~eas~~ eaiser than I saw before!)

So that is how matlab computes the det

So must remember the $\oplus, \ominus$ signs

└ did row ex to get to U probaby

if no row ex $\oplus$

odd row ex $\ominus$

even row ex $\oplus$

So why does the * not make a difference?

└ by elimination

- Since we can kill those if we multiply by right value

$$\begin{bmatrix} d_1 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 \\ 0 & 0 & d_3 & 0 \\ 0 & 0 & 0 & d_4 \end{bmatrix}$$

But why is det that for U?

(3a) factor the d_1 out

$$= d_1 \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 \\ 0 & 0 & d_3 & 0 \\ 0 & 0 & 0 & d_4 \end{vmatrix}$$

7

Continue

$$= d_4 d_3 d_2 d_1 \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

? I

So 1 says this is 1

$$= d_4 d_3 d_2 d_1 \cdot 1$$

But what if a d is 0?

└ so w/ elim 5 we know can get row of 0s

and w/ rule 6 we know its 0

---

8 det (A) = 0 when A is singular

So surruget for invertability

Poof by elim 5 go from A to U

Then row of 0s so det is 0 6

8b if ~~matrix~~ det A ≠ 0 when A is invertable

└ from chap 2 → full set of pivots

8

$U \to D \to d_1 d_2 d_3 \ldots$

So we have a formula now!

What are pivots of 2×2

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$\downarrow$ elim $l = \frac{c}{a}$

$$\begin{vmatrix} a & b \\ 0 & d - \frac{c}{a} b \end{vmatrix}$$

So det $= a \cdot \left[d - \frac{c}{a} b\right]$

$= ad - cb$ (!) Nice!

As long as $a \neq 0$

└ then need to do the row ex

9

$\downarrow$ product

9 $\det(AB) = \det(A) \cdot \det(B)$

- very valuable property
- not linear $\det(A+B) \neq \det(A) + \det(B)$

  $\uparrow$ not ness.

So $\det(A^{-1}) = ?$ using 9

know $A^{-1}A = I$

so det both sides

$$\det(A^{-1}A) = 1$$

$$\det(A^{-1})\det(A) = 1$$

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

Check

$A = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$ det $= 6$ $\quad A^{-1} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix}$ det $= \frac{1}{6}$

✓

(10)

$\det(A^2) = (\det(A))^2$ (9) If det $\neq 0$, det $A^{-1} \neq 0$
L makes sense since invertible

$\det(2A) = \det(A+A) \neq 2\det(A)$

(3a) $= 2^n \det(A)$

since multiplying each row by 2
factoring out 2 from <u>each</u> row

its like volume!
2^n

(this is a good lecture!)

(10) $\text{Det}(A^T) = \text{Det}(A)$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & c \\ b & d \end{vmatrix}$$

$ad - bc = ad - cb$

We've been working w/ rows
But this adds cols
L If cols all 0 → det $= 0$
Exchanging cols is sign ex

(11)

Since we can transpose our rows to cols

(10) Proof (using (1-9))

$$|A^T| = |A|$$

$\downarrow$ $\downarrow$ almost every case; factor

$$|U^T L^T| = |L U|$$

using (9)

$$|U^T| |L^T| = |L| |U|$$

$\uparrow$ since triangular

In (2) we said row-ex causes sign swap

but permutation ~~...~~ either even or odd

So det is well defined by 1, 2, 3

What is the best way to describe a basis?

- a basis (I answered! :))

  ind + spanning set

- Smallest generating set for the subspace
  - aka spans space

    can't remove any vector + get same result

Q: Can 1 vector span a 2D space?

No! Need at least 2 vectors to span the space $(1 < 2)$

- biggest ind set for the space

  Anything more would be dep
  - Can five vectors in $\mathbb{R}^3$ be ind?
  - basis has 3 els
  - anything more than 3 is dep No
  - $(3 < 5)$

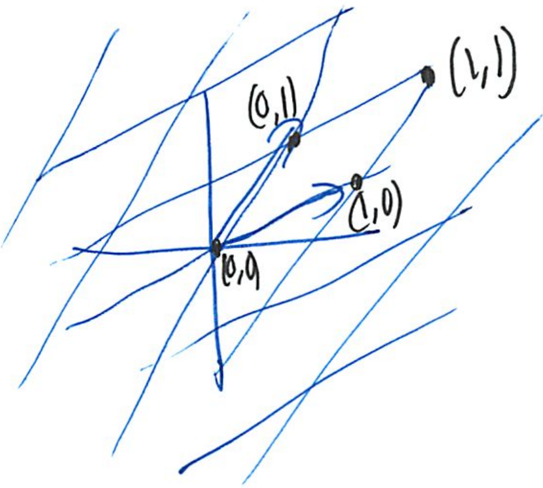
2

How to visualize a basis?

Say have 2 bases vectors

Expand grid

Start translating



Kinda hard to draw

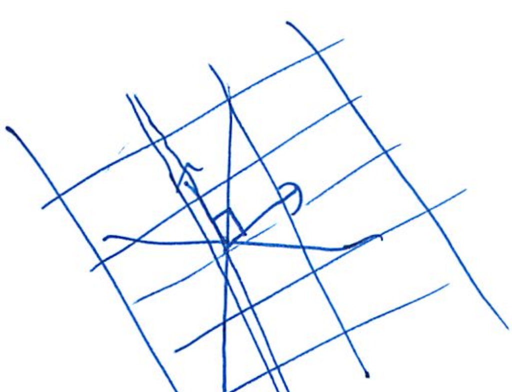
Nice set of bases?

↳ Yes ⇒ the orthonormal set of bases

Orthonormal does not imply

But will be lenghth $=1$

~~Bu~~ And $\perp$



This is what trying to get

Gram-Schmidt Process gives us this

↳ for ind basis vectors

(3)

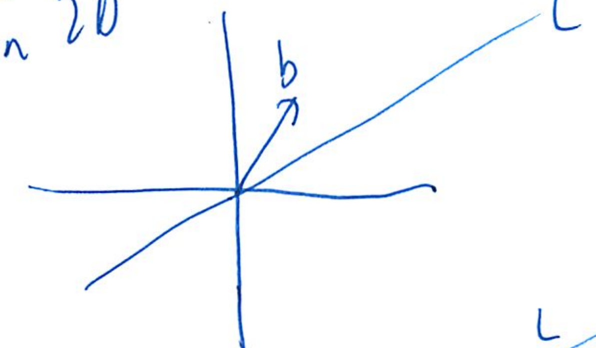
It will help us in computation

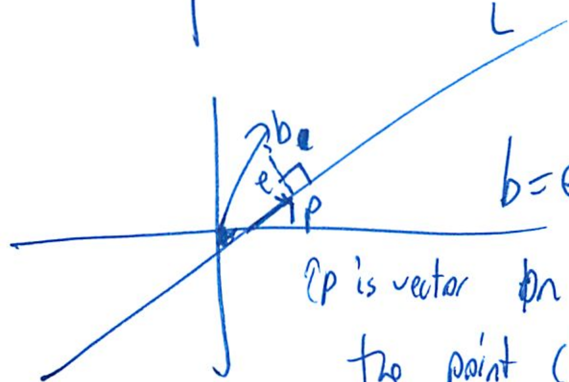
Since $Q^{-1} = Q^T$ if cols orthonormal

So don't need to find Q^{-1}

Projection

In 2D

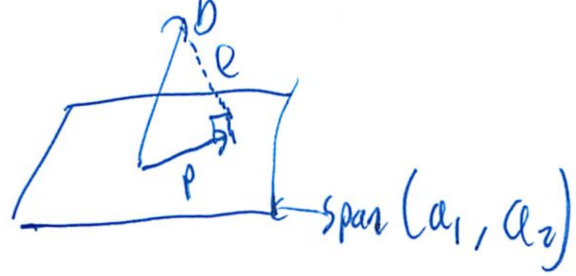




$b = e + p$

p is vector on L that is the point closest to b

Rember b ends at a specific point



Only person comfortable w/ projection formula

(4)

Say $L = \text{Span}(\vec{a})$

If forget formula on exam - write everything you know

$$\left[\begin{array}{l} p = xa \quad \uparrow \text{some multiple} \\ p \perp e \end{array}\right\} \text{can recover projection formula}$$

2D

$e = b - p$

$0 = p\,e$

$0 = p(b-p)$

$= xa(b - xa)$

$= x \cdot a \cdot b - x^2\, a \cdot a$

divide by x

$= a \cdot b - x \cdot a \cdot a$

We know a, b

So can solve for x

$x = \dfrac{a \cdot b}{a \cdot a} = \dfrac{a^T b}{a^T a}$ ← a is a col vector

⑤

$$p = xa$$
$$= \frac{a^T b}{a^T a} \, a$$

For 3D

$$A = \begin{bmatrix} | & | & \\ a_1 & a_2 & \cdots \\ | & | & \end{bmatrix}$$

$$p = A \underbrace{(A^T A)}_{\text{esentially } \frac{1}{a^T a} \text{ from 2D}}{}^{-1} A^T b$$

Examples

Project $(1,2,3)$ to the subspace of $\mathbb{R}^3$ spanned by $(1,0,0)$ and $(1,1,1)$

- this is a 3D case
  - not projecting onto a line

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$

subspace is spanned by these 2

⑥

$$p = A(A^TA)^{-1}A^Tb$$

$$b = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$A^T = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$

2 × 2 matrix

$$A^TA = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}$$

Invert matrix

- do have to invert since not orthonormal
- lot of work to orthonormalize, so don't really do

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{array}\right]$$

$\downarrow$ elim

7

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 2 & -1 & 1 \end{array}\right]$$

$\downarrow$? want identity

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & -\frac{1}{2} & \frac{1}{2} \end{array}\right]$$

$\downarrow$

$$\left[\begin{array}{cc|cc} 1 & 0 & 3/2 & -1/2 \\ 0 & 1 & -1/2 & \frac{1}{2} \end{array}\right]$$ row1 = row1 − row2

$$\underbrace{\begin{array}{cc} 3/2 & -1/2 \\ -1/2 & \frac{1}{2} \end{array}}_{(A^TA)^{-1}}$$

(skipping computation)

$$p = \begin{bmatrix} 1 \\ 5/2 \\ 5/2 \end{bmatrix}$$

(8)

Gram Schmidt Process to Orthonormalize

Input An abitrary basis of some subspace

- An abitrary grid 

Output Get an orthonormal basis for the same subspace

L the  grid

Example Find an orthonormal basis for the subspace of $\mathbb{R}^3$ spanned by $a_1 = \begin{pmatrix}2\\1\\3\end{pmatrix}$ $a_3 = \begin{pmatrix}0\\1\\1\end{pmatrix}$

$a_2 = \begin{pmatrix}2\\1\\-4\end{pmatrix}$

Standard basis already orthonormal

- ~~Stand~~ since all lenght 1
  - and $\perp$ to each other

9

Steps

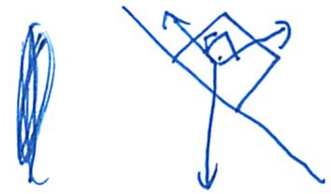
1. Take 1st vector
   - just accepted

$$A = a_1 = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$$

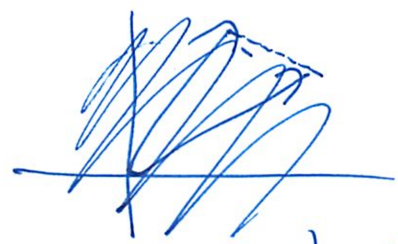
← orthogonalize 1st → (hard)
then normalize last (easy)

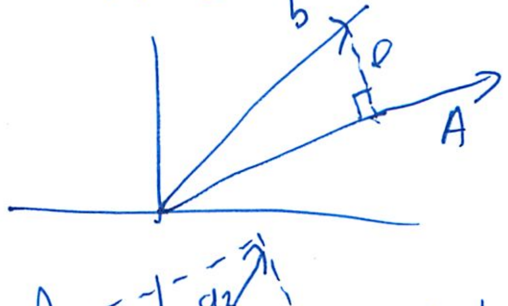
2. Isolate the component of the 2nd vector that is orthogonal to A.

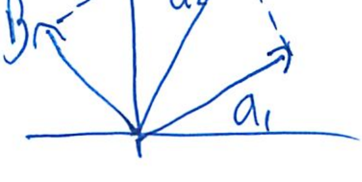
This is like decomposing vectors in physics



So







Proj of a_2 onto a_1

(10)

Want to take away projection part

Proj. formula

$$B = a_2 - \frac{A^T a_2}{A^T A} A$$

Now all computation

$$B = \begin{pmatrix} 2 \\ 1 \\ -4 \end{pmatrix} - \frac{(2\ 1\ 3)\begin{pmatrix} 2 \\ 1 \\ -4 \end{pmatrix}}{(2\ 1\ 3)\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}} \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$$

= (skip messy steps)

$$= \begin{pmatrix} 3 \\ 3/2 \\ -5/2 \end{pmatrix}$$

To check if correct

$A \cdot B$ should $= 0$

example

$$\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 3/2 \\ -5/2 \end{pmatrix} = 6 + \frac{3}{2} - 15/2$$

$$= 0$$ ✓

(11)

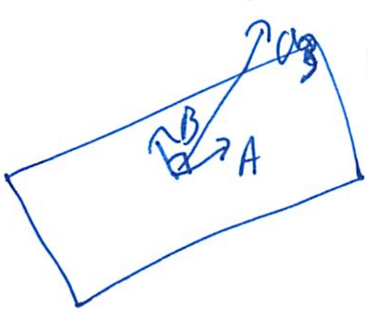
(3) Now want 3rd vector to be $\perp$ to first 2

Isolate the component of the 3rd vector that is $\perp$ to A, B

$$C = a_3 - \frac{A^T a_3}{A^T A} A - \frac{B^T a_3}{B^T B} B$$

(make sure to actually try it)

$$= \begin{pmatrix} -2/7 \\ 4/7 \\ 0 \end{pmatrix}$$



← only isolate the part that is $\perp$ to a_3

These vectors are orthogonal

Still need to make them normal

↓

(12)

(4) Normalize (each individually)

$$q_1 = \frac{A}{\|A\|} = \frac{\begin{pmatrix}2\\1\\3\end{pmatrix}}{\sqrt{2^2+1^2+3^2}} = \frac{\begin{pmatrix}2\\1\\3\end{pmatrix}}{\sqrt{14}}$$

$\|A\|$ ← length of A

lots of sq rts & ugly #s!
- don't get insecure!

$$q_2 = \frac{1}{\sqrt{20}}\begin{pmatrix}6\\3\\-10\end{pmatrix}$$

$$q_3 = \frac{1}{\sqrt{5}}\begin{pmatrix}-1\\2\\0\end{pmatrix}$$

QR factorization

$$Q = \begin{bmatrix} | & | & | \\ q_1 & q_2 & q_3 \\ | & | & | \end{bmatrix}$$

Fun fact

$$Q^T Q = I$$

since $Q^T = Q^{-1}$
since orthonormal

$$\begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

(13)

but $QQ^T \neq I$

Since rows are not orthonormal
Only the cols are

And only when Q is not a square matrix

$Q^{-1} = Q^T$

↑ no need to do augmented elim to find ~~the~~ inverse

$A = QR$

original, just found, ?

R) If square, invert Q and multiply on the other side

do → $Q^{-1}A = R$

$Q^{-1} = Q^T$, so $Q^TA = R$

R ← upper triangular

18.06 Chap 4
Reading

3/18

4.1 Orthogality the 4 Subspaces

Orthogonial = dot product = 0

$v \cdot w = 0$

$v^T w = 0$

So subspaces
basis vectors
Column vectors
} orthogonial

like $a^2 + b^2 = c^2$

$||v||^2 + ||w||^2 = ||v+w||^2$

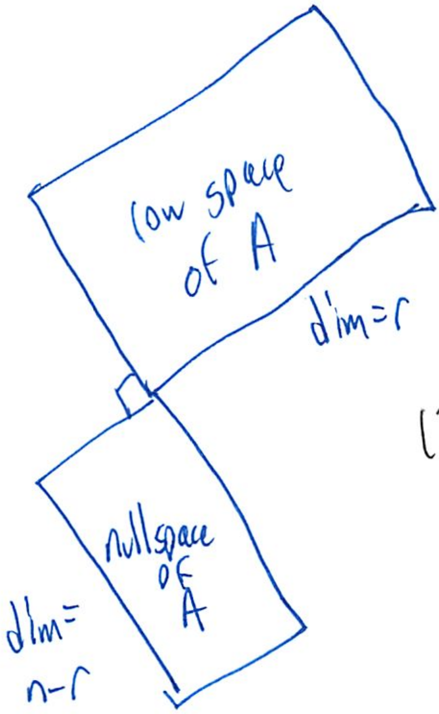
(confused by next line in book)

~~Axx~~ A times x

↳ The row space is ⊥ in the null space

Every row of A is ⊥ to every sol
of $Ax = 0$
(Fund Theorm Part 2)

2



(I don't get this picture)

(It is a big deal - is on the cover of the book)

Col space is $\perp$ to nullspace of A^T

When b is outside col space, (can't solve $Ax=b$)

Then null space A^T important

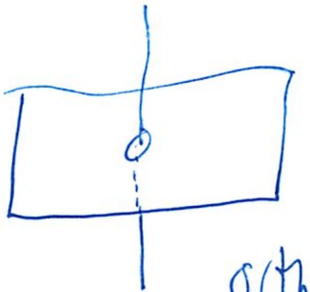
- Containing the error $e = b - Ax$ in the "least squares" sol

Two ~~vecto~~ subspaces are orthogonal if every vector v in V is $\perp$ to every w in W

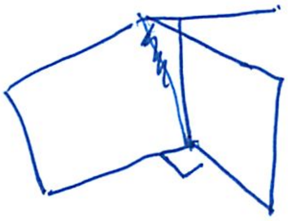
perpendicular

$v^T w =$ for all v in V or w in W

(3)



orthogonal line + plane



perpendicular, but
not orthogonial $v^T w \neq 0$

Since the line where they meet are in both subspaces

When a vector is in 2 orth. subspcaes
it must be 0
(instead: only 0 is in 2 orth. subspaces?)

Orthgonality is impossible when dim V + dim W > dim whole space

Zero is the only place where the null space meets the row space
They meet at $90°$
Since are orthogonial → $Ax = 0$

$$Ax = \begin{bmatrix} \text{row } 1 \\ \vdots \\ \text{row } m \end{bmatrix} \begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow \begin{matrix} (\text{row } 1)\, x = 0 \\ \vdots \\ (\text{row } m)\, x = 0 \end{matrix} \leftarrow \text{row 1 perp to } x$$

$C(A^T) \perp N(A)$

(4)

(finally a good explination...)

$$x^T(A^T y) = (Ax)^T y = 0^T y = 0$$

Every vector y in the nullspace of A^T is $\perp$ to every column of A.

The left nullspace $N(A^T)$ and $C(A)$ are orthogonal in $\mathbb{R}^m$

$$C(A) \perp N(A^T)$$

$$A^T y = \begin{bmatrix} (\text{col } 1)^T \\ \vdots \\ (\text{col } n)^T \end{bmatrix} \begin{bmatrix} y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Dot product of every col w/ $y = 0$

$$A^T y = 0$$

So	
$Ax = 0$	$A^T y = 0$
$N(A) \perp C(A^T)$	$C(A) \perp N(A^T)$

(The book is not symmetrical - hard to read)

(5)

Orthogonal Complements

The fundamental subspaces have dims or $\begin{cases} 2 \text{ and } 1 \\ 3 \text{ and } 0 \end{cases}$

Lamari null space
Col space
row space
left nullspace

Orthogonal complement of a subspace V

Contains every vector that is $\perp$ to V

This subspace is denoted by $V^{\perp}$ (V perp)

So null space is orth. comp. of the row space.

$$N(A)^{\perp} = C(A^{T})$$

$$r + (m - r) = m$$

(6)

Fund Theorm Lin Algebra Part 2

$N(A)$ is orth. comp. $C(A^T)$ in $\mathbb{R}^n$

$N(A^T)$ is orth. comp. $C(A)$ in $\mathbb{R}^m$

Part 1 = dims subspace

Part 2 = the 90° angle b/w them

So every x can be split into row space component x_r
and null space " x_n

$x = x_r + x_n$

$A x_n = 0$ $A x_r = A x$

(Some picture of this I don't get...)

Every b in col space comes from one and only vector in the row space

$A x_r = b$

Some mention of Single Value decomposition

(7)

Combining Bases from Subspaces

Any n ind. vectors in R^n must span R^n; → so basis

Any n vectors that span R^n must be ind → so basis

So if $A =$ $n \times n$ sq matrix

If the n cols of A are ind → they span R^n

So $Ax = b$ is solvable

If the n cols span R^n → are ind

So $Ax = b$ has only 1 sol.

Basically uniqueness implies existence and

existance → uniquness

Thus A is invertable

↓

If No free values → X unique

↓

n pivots

↓

Solves $Ax = b$

(8)

Pen

If $Ax=b$ can be solved for every b (existance of sols). → then elim produces no zero rows, there are n pivots and no free variables. ~~The~~

↓

The nullspace contains only $x=0$ (uniqueness)

$r + (n-r) = n$ vectors

↓

Right #

↓

So n vectors are ind

↓

Therefore they span $\mathbb{R}^n$.

$$\boxed{\text{Each } x \text{ is the sum } \underset{\text{rowspace}}{X_r} + \underset{\text{nullspace}}{X_n}}$$

(9)

4.2 Projections

When b is projected onto a line, its projection p is the part of b along that line

When b is projected onto a plane, p is the part of b in that plane.
L The projection p is Pb.
(they never described what a projection is ...)

So if $b = (2,3,4)$
One projection gives $p_1 = (0,0,4) \leftarrow z$ axis
$p_2 = (2,3,0) \leftarrow xy$ plane

Projection matrices

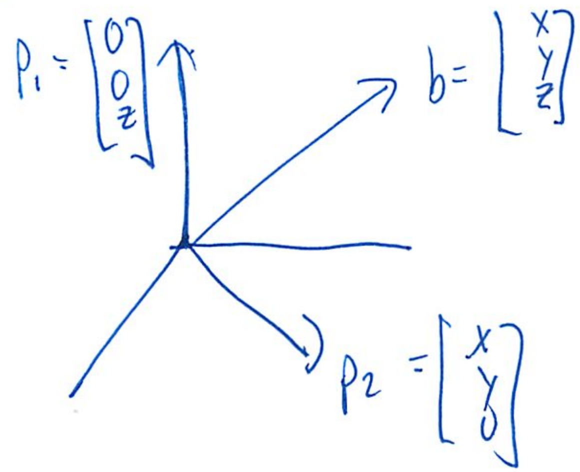
$$P_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad P_2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$p_1 = P_1 b \qquad p_2 = P_2 b$$

(10)

So p_1, p_2 are $\perp$

xy plane, z axis are orthogonal subspaces



And orthogonal complements

dim $1 + 2 = 3$

vectors $p_1 + p_2 = b$

matrices $P_1 + P_2 = I$

So want to find $p = Pb$

Every subspace of $\mathbb{R}^m$ has its own m by m projection matrix.

11

Best description of a subspace is a basis

└ a subspace may have many base

Want to project any b onto the col space of any m by n matrix.

<u>Project on a Line</u>

dim = 1 = n

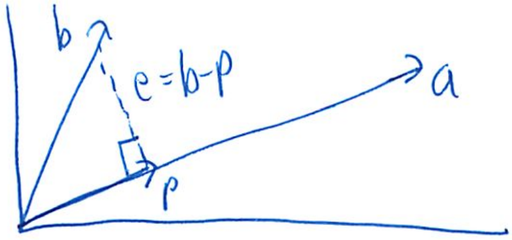
So A only has 1 col a

A line goes through the origin in the direction of $a = (a_1, \dots, a_m)$

Along this line we want the point p closest to $b = (b_1, \dots, b_m)$

* The line from b to p is $\perp$ to the vector a. *

This needs a picture



How is this the point p Closest to b?

⑫

The projection p is some multiple of a.

$$p = \hat{x}\, a$$

"x hat"

Computing $\hat{x}$ gives you p

vector

Then from the formula for p we read off P.

1. Find $\hat{x}$ (#)
2. Find p (vector)
3. Find P (matrix)

dotted line $p - b = e = b - \hat{x}a$

$\perp$ to a

$$\boxed{\hat{x} = \frac{a \cdot b}{a \cdot a} = \frac{a^T b}{a^T a}}$$

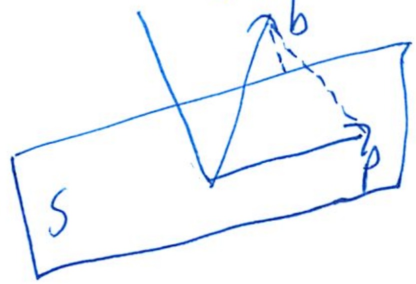
$a \cdot b - \hat{x} a = 0$

$a \cdot b - \hat{x} a \cdot a = 0$

$\boxed{\text{Note } a^T b = a \cdot b}$

13

So $\hat{x} = \dfrac{a^T b}{a^T a}$ gives $p = \hat{x} b$



$$\begin{aligned} p &= A\hat{x} \\ &= A(A^TA)^{-1}A^T b \\ &= Pb \end{aligned}$$

$$\boxed{p = \hat{x} a = \frac{a^T b}{a^T a} a}$$

Special Case 1 If $b = a$ then $\hat{x} = 1$

This is projection of a onto itself

$Pa = a$

Special case 2 If b is $\perp$ to a, then $a^T b = 0$

The projection is $p = 0$

(Good example in the book) p 288 - 209

When vector a is doubled, P stays the same

- still projects on same line

$P^2 = P$

Projecting a second time → no change

(14)

So diagonal entries add up

$$\frac{1}{9}(1+4+4) = \underline{1}$$

Matrix $I-P$ should be a projection as well

It produces the other side e of the triangle - the $\perp$ part of b.

Note that $(I-P)b = b-p$ which is e in the left nullspace

* So when P projects into one subspace
  $I-P$ " " the perpendicular " *

<u>Projection onto a subspace</u>

Start w/ n vectors $a_1, \ldots a_n$ in $\mathbb{R}^m$

Assume a_j are lin. ind.

* Find the combo $p = \hat{x}_1 a_1 + \ldots + \hat{x}_n a_n$ closest to a given vector b. *

15

We are projecting each b in $\mathbb{R}^m$ onto the subspaces spanned by the a's to get p.

We are looking for a particular combo $p = A\hat{x}$ that is closest to b.

$\hat{x}$ is the best choice - closest vector in the col space

---

Geometry!

error vector $b - A\hat{x}$ is $\perp$ to the subspace

$$a_1^T(b - A\hat{x}) = 0 \quad \vdots \quad a_n^T(b - A\hat{x}) = 0 \quad = \begin{bmatrix} a_1^T & - \\ \vdots & \\ a_n^T & - \end{bmatrix} \begin{bmatrix} b - A\hat{x} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

So combo $p = \hat{x}_1 a_1 + \dots + \hat{x}_n a_n = A\hat{x}$ that is closest to b comes from

$$\underbrace{A^T}_{\text{col space}} \underbrace{(b - A\hat{x})}_{\text{error vector}} = 0 \quad \text{- or -} \quad A^T(A\hat{x}) = A^T b$$

are perpendicular

so is nullspace (A^T)

(16)

The symmetric matrix $A^T A$ is n by n

It is invertable if the a's are ind.

The sol to $\hat{x} = (A^T A)^{-1} A^T b$

The projection of b onto the subspace is p

$$p = A\hat{x} = A(A^T A)^{-1} A^T b$$

This formula shows the $n \times n$ proj matrix $p = Pb$

$$P = A(A^T A)^{-1} A^T$$

Note Can't split $(A^T A)^{-1}$ since ~~...~~ A is not invertable

---

$A^T A$ is invertable iff A has lin. ind cols

Square, symetric, invertable

$$\underset{A^T}{(n \text{ by } m)} \cdot \underset{A}{(m \text{ by } n)}$$

So $(A^T A) = (n \text{ by } n)$

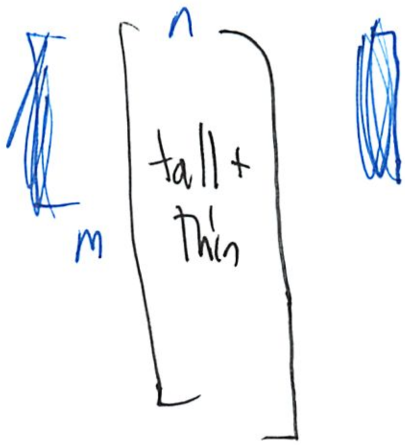
(Hard to remember all the thing that are invertable!)

3/20

4.3 Least Sqs Approximation

Often $Ax = b$ has no solution

- Since more eqs than unknowns
  more rows than cols

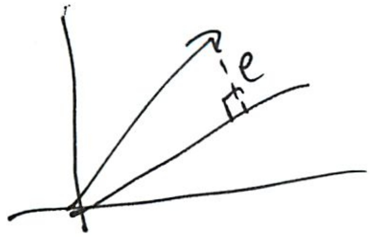


$m > n$

- So b is outside col space
- Elim reaches a block + is impossible to finish
  - I never tried going that far...

- $e = b - Ax$ is 0 when exact sol
  $\neq 0$ when no exact sol

Then $\hat{x}$ is least sq sol when lenght of e is as close as possible



So basically measure that?

Prev section emphized p – this emphasizes $\hat{x}$

$$p = A\hat{x}$$

Fund eq is still $A^TA\hat{x} = A^Tb$

$$\boxed{\text{Basically when } Ax = b \text{ w/ no sols multiply by } A^T \text{ and solve } A^TA\hat{x} = A^Tb}$$

? what does that mean?

Example

$\quad\quad\quad\quad\quad\quad\quad\quad\downarrow t \ \downarrow b$

Fit line to $(0,6), (1,0), (2,0)$

$\uparrow b = C + Dt$

$t=0$	$\rightarrow$	$C + D\cdot 0 = 6$
$t=1$		$C + D\cdot 1 = 0$
$t=2$		$C + D\cdot 2 = 0$

$\uparrow t \quad \uparrow b$

(3)

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \quad x = \begin{bmatrix} C \\ D \end{bmatrix} \quad b = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}$$

$Ax = b$ not solvable

(same #s in last section!)

└ example 3

found $A^T A = \begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix}$

$A^T b = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$

Now solve $A^T A \hat{x} = A^T b$ to find $\hat{x}$?

$$\begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$$

gives

$$\hat{x} = \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$$ ← i guess + check

Try

$$3\hat{x}_1 + 3\hat{x}_2 = 6$$
$$3\hat{x}_1 + 5\hat{x}_2 = 0$$

2 eq 2 unknown

no g + ✓

(4)

$$3\hat{x}_1 = 6 - 3\hat{x}_2$$

$$\hat{x}_1 = 2 - \hat{x}_2$$

$$3(2 - \hat{x}_2) + 5\hat{x}_2 = 0$$

$$6 - 3\hat{x}_2 + 5\hat{x}_2 = 0$$

$$2\hat{x}_2 = -6$$

$$\hat{x}_2 = -3$$

$$3\hat{x}_1 = 6 - 3(-3)$$

$$3\hat{x}_1 = 15$$

$$\hat{x}_1 = 5$$

Now back to the new method

~~Still correct that 5 = 2~~

Still Correct that $5 - 3t$ is best line

but what if 100 points?

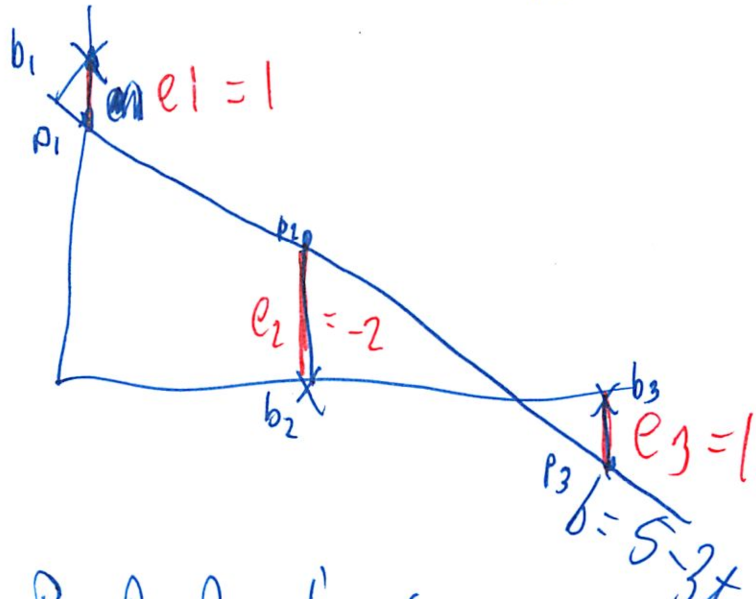
Or the points don't make a straight line?

didn't we do something like this in ~~elementary~~ middle school?

(5)

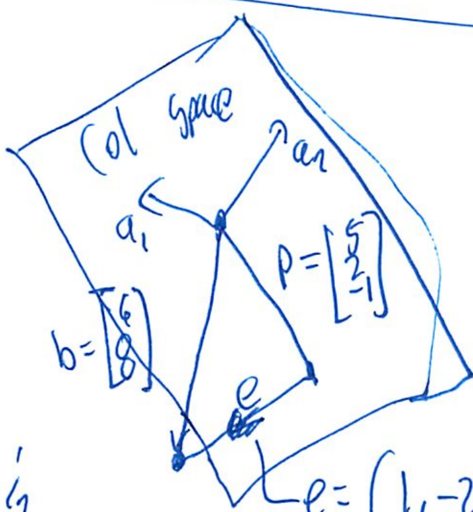
Want to minimize error

X = selected pt



minimize
$$E = e_1^2 + e_2^2 + e_3^2$$

p_1, p_2, p_3 lie in a line
Since are in col space



$\hat{b}$ is __not__ in col space

$e = (1, -2, 1)$

$e = b - A\hat{x}$

← 3D space: b, p, e space

projection
$$p = 5a_1 - 3a_2$$

(b)

Algebra

Every vector b splits in 2 parts

↳ part in col space → p

→ perp. part in $N(A^T)$ is e

$$Ax = b = p + e \quad \text{impossible}$$

↓ instead

$$A\hat{x} = p \quad \text{solvable}$$

So we want to minimize e

$$\|Ax - b\|^2 = \|Ax - p\|^2 + \|e\|^2$$

$$c^2 = b^2 + a^2$$

So:

7

Vector $Ax - p$ in col space is $\perp$ to e in left null space

We reduce $Ax - p$ to 0 by choosing x to be $\hat{x}$

This leaves smallest possible error $e = (e_1, e_2, e_3)$

↳ Makes $E = ||Ax - b||^2$ as small as possible

Calculus

$$E = e_1^2 + e_2^2 + e_3^2$$

$$= ||Ax - b||^2$$

$$= (C + D \cdot 0 - 6)^2 + (C + D \cdot 1)^2 + (C + D \cdot 2)^2$$

↑ ↑ unknowns

2 partial derivs

d treated as constant → $\dfrac{\partial E}{\partial C} = 2(C + D \cdot 0 - 6) + 2(C + D \cdot 1) + 2(C + D \cdot 2) = 0$

c constant → $\dfrac{\partial E}{\partial D} = 2(C + D \cdot 0 - 6)(0) + 2(C + D \cdot 1)(1) + 2(C + D \cdot 2)(2) = 0$

↑ extra factors from chain rule

8

C factors are 1, 1, 1

Now cancle 2 from every term

$$3C + 3D = 6$$
$$3C + 5D = 0$$

$$\downarrow$$

$$\begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix} = A^T A = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$$

¿wait what happened there?

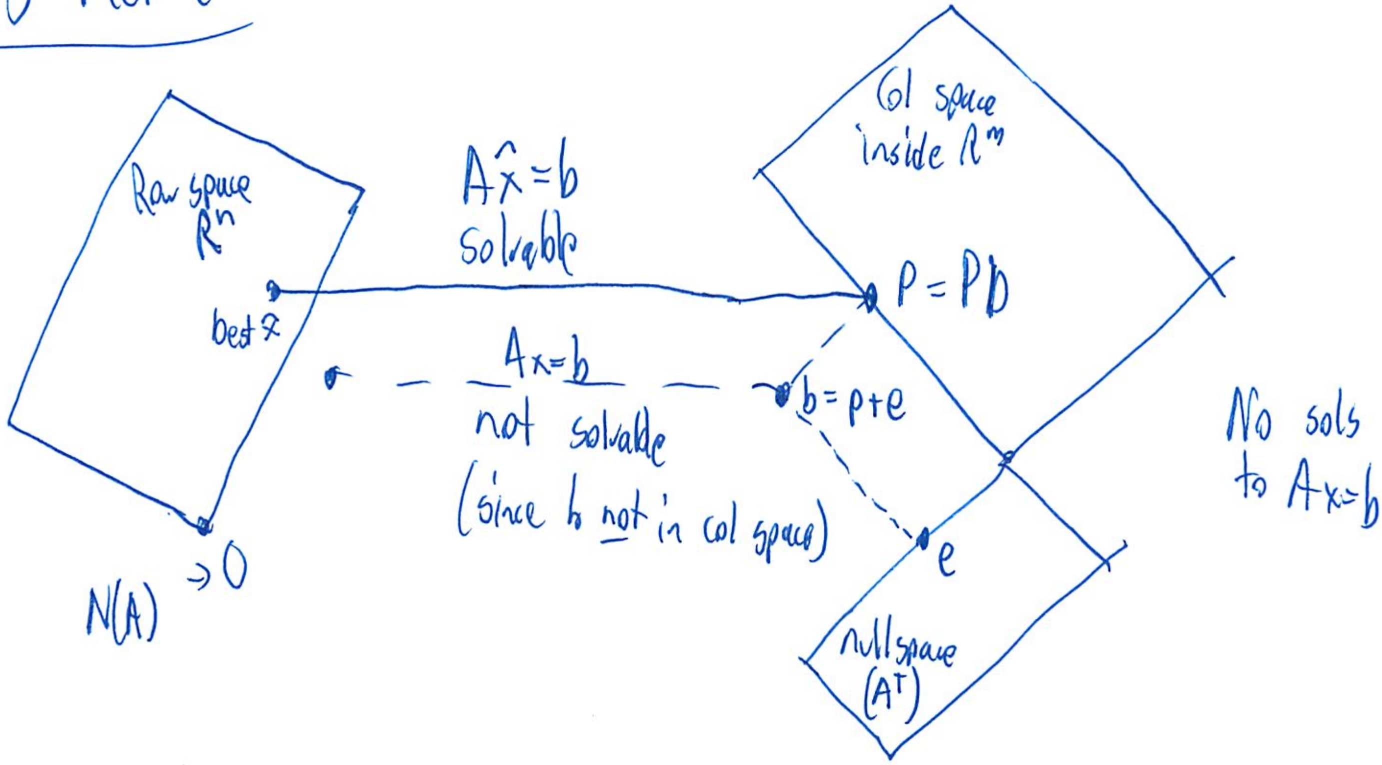
$$\boxed{\text{The Partial Derivs of } \|Ax - b\|^2 \text{ are } 0 \text{ when } A^T A \hat{x} = A^T b}$$

At $t = 0, 1, 2$ the line goes through $p = 5, 2, -1$ (not $b = (6, 0, 0)$)

The errors are $1, -2, 1 = e$

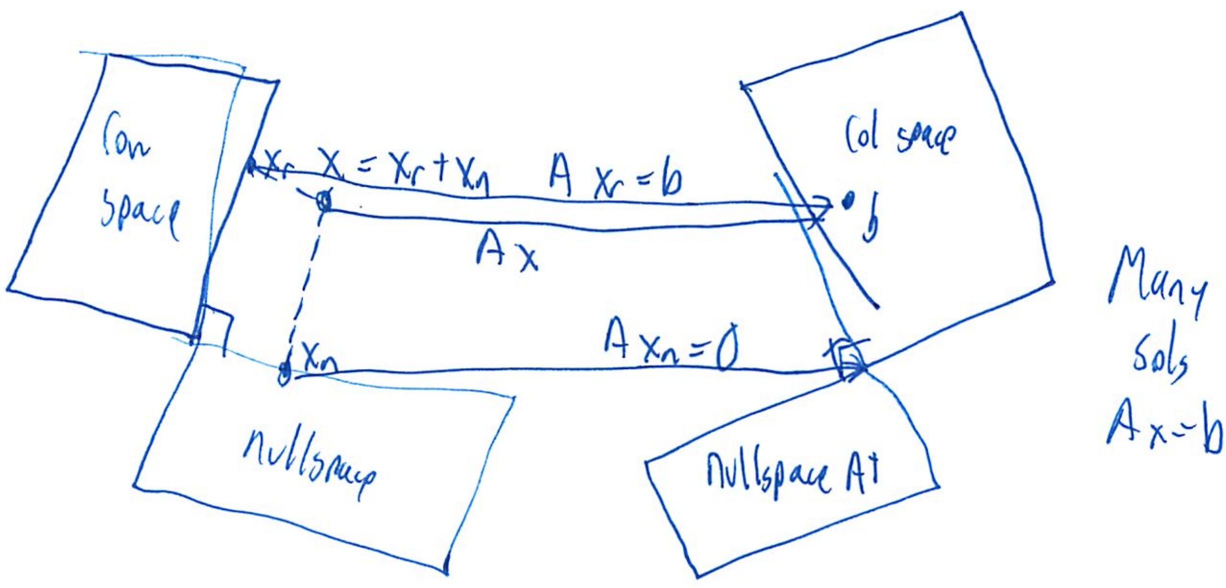
(9)

The Big Picture



? p is closest to b (ahh!)

This unlike earlier Figure 4.3 in section 4.1



(10)

* Instead of splitting up x, we are splitting up b *
" " $Ax=b$ we solve $A\hat{x}=p$
$e=b-p$ is unavoidable

$N(A)$ is very small → just one point
Only sol to $Ax=0$ is $x=0$
So A^TA is invertable

The eqn $A^TA\hat{x}=A^Tb$ fully determines best vector $\hat{x}$

(I think I never take the time to look at this in depth

Chap 7: complete pic x splits to $x_r + x_n$
b " " $p+e$

Remember Chap 3.4 The Complete sol to $Ax=b$

$$X = X_p + X_n$$

? X to get it to b

Every x splits into
a row space component x_p +
null space component x_n

$Ax_r = Ax$
$Ax_n = 0$

When full col rank
full row rank → $Ax=b$ undedetermined (~~inf~~ many sols)

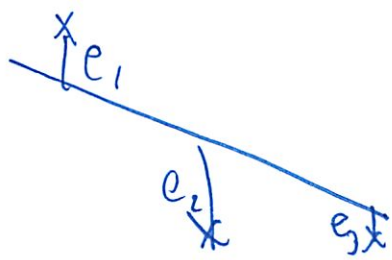
(still under here in book – remove my notes..)

(11)

Fitting a Straight Line

Clearest app of least squares app

$m \geq 2$ points



minimize $E = e_1^2 + e_2^2 + e_3^2 + \dots + e_m^2$

$$Ax = b = \begin{matrix} C + Dt_1 = b_1 \\ \vdots \\ C + Dt_m = b_m \end{matrix} \quad \text{w/ } A = \begin{bmatrix} 1 & t_1 \\ 1 & t_2 \\ \vdots & \vdots \\ 1 & t_m \end{bmatrix}$$

(looks familiar)

Col space is so thin that b is outside of it

~~When b happens to lie in the Col~~

(12)

Solve $A^T A \hat{x} = A^T b$ for $\hat{x} = (C, D)$

$e_i = b_i - C - Dt_i$

So $A^T A = \begin{bmatrix} 1 & \cdots & 1 \\ t_1 & \cdots & t_m \end{bmatrix} \begin{bmatrix} 1 & t_1 \\ \vdots & \vdots \\ 1 & t_m \end{bmatrix} = \begin{bmatrix} m & \sum t_i \\ \sum t_i & \sum t_i^2 \end{bmatrix}$

nice trick :)

On the RHS

$A^T b = \begin{bmatrix} 1 & \cdots & 1 \\ t_1 & \cdots & t_m \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} = \begin{bmatrix} \sum b_i \\ \sum t_i b_i \end{bmatrix}$

So

line $C + Dt$ minimizes $e_1^2 + \cdots + e_m^2 = \|Ax - b\|^2$

where $A^T A \hat{x} = A^T b$

$$\boxed{\begin{bmatrix} m & \sum t_i \\ \sum t_i & \sum t_i^2 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} \sum b_i \\ \sum t_i b_i \end{bmatrix}}$$

(13)

Fitting by ~~[illegible]~~ Parabola

Not all things straight lines
Fit stuff by parabola

$b = C + Dt + Et^2$ throw a ball

height at time t

$m > 3$ points

$C + Dt_1 + Et_1^2 = b_1$

$\vdots$

$C + Dt_m + Et_m^2 = b_m$

$\rightarrow A = \begin{bmatrix} 1 & t_1 & t_1^2 \\ \vdots & \vdots & \vdots \\ 1 & t_m & t_m^2 \end{bmatrix}$

Choose $\hat{x} = (C, D, E)$ to satisfy

$$A^T A \hat{x} = A^T b$$

(need to practice this!)

(14)

4.4 Orthogonal Bases + Gram-Schmidt

- See how orthogonality makes it ~~###~~ easy to find $\hat{x}$ and p and P.
- Dot products are 0 - since A^TA is diagonal
- Constructs orthogonal vector
  - will be the cols of a new matrix Q
- Orthogonical dot product $= 0$

  aka $q_i^T q_j = 0$ if $i \neq j$
- Normal ~~m~~ lenghts $= 1 = ||q_i||$
- Q = matrix w/ orthonormal cols
- $Q^TQ = I$

$$\underbrace{\begin{bmatrix} - q_1^T - \\ - q_2^T - \\ - q_3^T - \end{bmatrix}}_{Q^T} \overbrace{\begin{bmatrix} | & | & | \\ q_1 & q_2 & q_3 \\ | & | & | \end{bmatrix}}^{Q} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} = I$$

(15)

Example 1 Rotation

$$Q = \begin{bmatrix} \cos\theta, & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

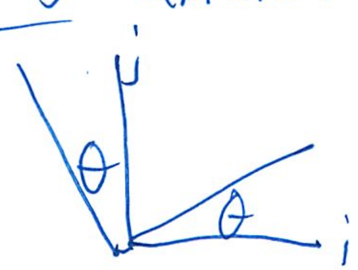
$$Q^T = \begin{bmatrix} \cos\theta, & \sin\theta \\ -\sin\theta, & \cos\theta \end{bmatrix}$$

Example 2 Permutation

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{since } P^{-1} = P^T$$

Orthogonal (not orthonormal) - still diagonal but not identity

Example 3 Rotation



(don't really get this ...)

(16)

Projections Using Orthogonal Bases : Q Replaces A

$A^T A \xrightarrow{\text{entries}} a_i^T a_j$

$Q^T Q \rightarrow q_i^T q_j$

$= I$

Least squares sol of $Qx = b$

is $\hat{x} = Q^T b$

Projection matrix $\rightarrow P = QQ^T$

Nothing to invert!

$$p = \begin{bmatrix} | & | & | \\ q_1 & q_2 & q_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} q_1^T b \\ \vdots \\ q_3^T b \end{bmatrix} =$$

$$= q_1(q_1^T b) + \dots + q_n(q_n^T b)$$

When Q = square $\rightarrow$ whole subspace is square

$b = p$

(17)

Gram Schmidt Process

Orthogonal is good

Makes inverse easy

But how to create orthonormal vectors?

(went over in Recitation)

---

lowercase ↳ input

uppercase ↳ output

Have 3 ind vectors a, b, c

$$q_1 = \frac{A}{\|A\|} \quad q_2 = \frac{B}{\|B\|} \quad q_3 = \frac{C}{\|C\|}$$

1. Choose $A = a$

2. Find B.

$$B = b - \frac{A^T b}{A^T A} A$$

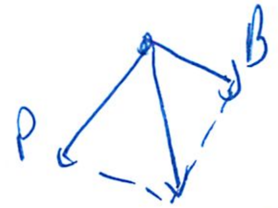
B is the error vectore e

3. Find C

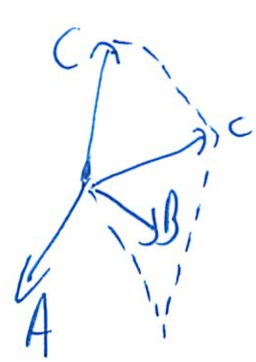
$$C = c - \frac{A^T c}{A^T A} A - \frac{B^T c}{B^T B} B$$

? subtract out components in those directions

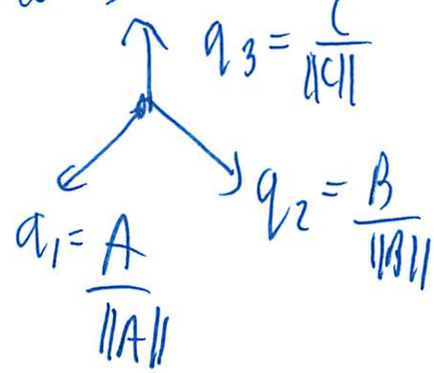
(18)



$A = a\ b$



So unit vectors



So we did the math in today's recitation

Factorization $A = QR$

$$A = QR$$

↑ triangular

$$\begin{bmatrix} a & b & c \end{bmatrix} = \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix} \begin{bmatrix} q_1^T a & q_1^T b & q_1^T c \\ & q_2^T b & q_2^T c \\ & & q_3^T c \end{bmatrix}$$

basically Gram-Schmidt in a nutshell

What is $q_i^T a$?

I can't seem to pick it out now...

Dot product?

Least Squares

$$R^T R \hat{x} = R^T Q^T b$$

or

$$R \hat{x} = Q^T b$$

or

$$\hat{x} = R^{-1} Q^T b$$

So when $Ax = b$ is impossible, we solve $R\hat{x} = Q^T b$ by back substitution (fast)

18.06 Spring 2012 – Problem Set 5

This problem set is due Thursday, March 22nd, 2012 at 4pm (hand in to Room 2-106). The textbook problems are out of the 4th edition. For computational problems, please include a printout of the code with the problem set (for MATLAB in particular, `diary('filename')` will start a transcript session, `diary off` will end one.)

Every problem is worth 10 points.

1. Do Problems 11 & 16 from Section 4.2.
2. Do Problem 27 from Section 4.2.
3. Do Problem 30 from Section 4.2.
4. Do Problem 31 from Section 4.2.
5. Do Problems 1 & 3 from Section 4.3.
6. Do Problem 26 from Section 4.3.
7. Do Problems 4 & 14 from Section 4.4.
8. Do Problem 7 from Section 4.4.
9. Do Problems 18 & 24 from Section 4.4.
10. In this exercise, we use MATLAB for learning about Gram-Schmidt.

    Suppose the 4×3 matrix A has 1's when $i = j$ and -1's when $i = j + 1$ and otherwise $A_{ij} = 0$. From the MATLAB command

    $$[\mathtt{Q}, \mathtt{R}] = \mathtt{qr(A)},$$

    find the Gram-Schmidt orthonormal basis for the column space $C(A)$. Renormalize those basis vectors to contain nice fractions by dividing by the diagonal entries of R. Now the vectors are orthogonal, not orthonormal.

    Guess what the pattern you see here is, and use this to find a 4th vector in $\mathbb{R}^4$ orthogonal to these three columns of Q (and of A).

18.06 Wisdom. On problem sets and exams, always try to think of ways to check your results. It both assists your own learning, when you reflect on and relate to what you've worked through, plus you avoid loosing points!

Michael Plasmeier   18.06 PS5   83/100   3/21

11, Project b onto col space of A by solving
$$A^T A \hat{x} = A^T b \quad \text{and} \quad p = A\hat{x}$$

10/10

a) $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ $b = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \hat{x} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 1 \cdot 1 + 1 \cdot 0 + 0 \cdot 0 & 1 \cdot 1 + 1 \cdot 1 + 0 \cdot 0 \\ 1 \cdot 1 + 0 \cdot 0 + 0 \cdot 0 & 1 \cdot 1 + 0 \cdot 1 + 0 \cdot 0 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 1 \cdot 2 + 1 \cdot 3 + 0 \cdot 4 \\ 1 \cdot 2 + 0 \cdot 3 + 0 \cdot 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \hat{x} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

$$x_1 + 2x_2 = 5$$
$$x_1 + x_2 = 2$$

2

$x_1 = 2 - x_2$

$(2 - x_2) + 2x_2 = 5$

$2 - x_2 + 2x_2 = 5$

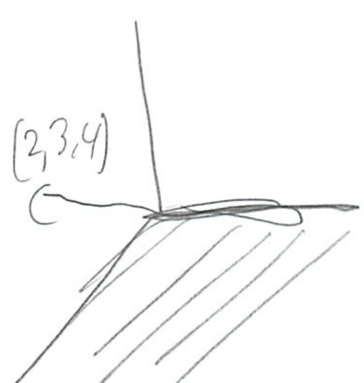
$2 + x_2 = 5$

$x_2 = 3$

$x_1 = 2 - 3 = -1$

$\hat{x} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

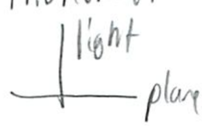
OH! $\mathbb{R}^3$

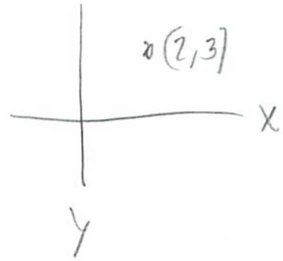


So if project b onto xy plane

break into x, y, z component

Shine light orthonormal onto plane





So projection $= (2, 3, 0)$

$p =$

(3)

Remember $p = A\hat{x}$

$$\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \cdot -1 + 1 \cdot 3 \\ 0 \cdot -1 + 1 \cdot 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$$ ✓

$e = b - p$

$$= \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}$$ ✓

✓

Verify $A^T e = 0$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \cdot 0 + 1 \cdot 0 + 0 \cdot 4 \\ 1 \cdot 0 + 1 \cdot 0 + 0 \cdot 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$ ✓

(4)

b) $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$ $b = \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix}$

So if shine light on $\begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix}$ is $\begin{bmatrix} 4 \\ 4 \\ 0 \end{bmatrix}$

? works when projecting onto x, y plane

which was since we ~~are~~ were projecting onto col

Space which was $\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$. the the trick

prob won't work.

So old fashioned way (not so old-fashioned!)

A has 2 cols

[when $A^T A$ is not invertable, pick A

that represents / is a basis for the space]

$$p = A \frac{A^T B}{A^T A}$$

(5)

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \frac{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix}}{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \frac{\begin{bmatrix} 4+4+6 \\ 4+4+0 \end{bmatrix}}{\begin{bmatrix} 1+1+0 & 1+1+0 \\ 1+1+0 & 1+1+1 \end{bmatrix}}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \frac{\begin{bmatrix} 14 \\ 8 \end{bmatrix}}{\begin{bmatrix} 2 & 2 \\ 2 & 3 \end{bmatrix}}$$

So can't divide maties

Insted $p = A\hat{x} = A(A^TA)^{-1}A^Tb$

(what was shown in recitation

~~But this is not invertable!~~ Is invertably

(6) to find $(A^TA)^{-1}$

$$\left[\begin{array}{cc|cc} 2 & 2 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{array}\right]$$

$\downarrow$ eliminate to LHS being identity
RHS is $(A^TA)^{-1}$

row2 = row2 − 1row1

$$\left[\begin{array}{cc|cc} 2 & 2 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{array}\right]$$

$\downarrow$ row1 = $\frac{1}{2}$ row1

$$\left[\begin{array}{cc|cc} 1 & 1 & 1/2 & 0 \\ 0 & 1 & -1 & 1 \end{array}\right]$$

$\downarrow$ row1 = row1 − row2

$$\left[\begin{array}{cc|cc} 1 & 0 & 3/2 & -1 \\ 0 & 1 & -1 & 1 \end{array}\right]$$

$(A^TA)^{-1}$

7

$$A \quad (A^TA)^{-1}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3/2 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 3/2 + -1 & -1+1 \\ 3/2 + -1 & -1+1 \\ 0 + -1 & 0+1 \end{bmatrix} = \overset{A(A^TA)^{-1}}{\begin{bmatrix} 1/2 & 0 \\ 1/2 & 0 \\ -1 & 1 \end{bmatrix}}$$

$$\overset{A(A^TA)^{-1}}{\begin{bmatrix} 1/2 & 0 \\ 1/2 & 0 \\ -1 & 1 \end{bmatrix}} \overset{A^T}{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}} = \begin{bmatrix} 1/2 \cdot 1 - 0 & 1/2 - 0 & 1/2 \\ 1/2 - 0 & 1/2 - 0 & 1/2 \\ -1+1 & -1+1 & -1 \end{bmatrix} = \overset{A(A^TA)^{-1}A^T}{\begin{bmatrix} +1/2 & +1/2 & +1/2 \\ +1/2 & +1/2 & +1/2 \\ 0 & 0 & -1 \end{bmatrix}}$$

$$\overset{A(A^TA)^{-1}A^T}{\begin{bmatrix} +1/2 & +1/2 & +1/2 \\ +1/2 & +1/2 & +1/2 \\ 0 & 0 & -1 \end{bmatrix}} \overset{b}{\begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix}} = \begin{bmatrix} +1/2 \cdot 4 + +1/2 \cdot 4 + +1/2\, 6 \\ \\ -6 \end{bmatrix} = \overset{p}{\begin{bmatrix} +14 \\ +14 \\ -6 \end{bmatrix}}$$

Now other way, ~~just for fun~~ Since its shorter!

$\boxed{\text{shold be } \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix}}$

$$A^T A \hat{x} = A^T b$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \hat{x} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix}$$

8

$$A^T A = \begin{bmatrix} 1+1+0 & 1+1+1 \\ 1+1+0 & 1+1+0 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 2 & 2 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 4+4+6 \\ 4+4+0 \end{bmatrix} = \begin{bmatrix} 14 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 \\ 2 & 2 \end{bmatrix} \hat{x} = \begin{bmatrix} 14 \\ 8 \end{bmatrix}$$

$$2x_1 + 3x_2 = 14$$
$$2x_1 + 2x_2 = 8$$

$$2x_1 = 8 - 2x_2$$
$$x_1 = 4 - x_2$$

$$2(4 - x_2) + 2x_2 = 8$$
$$8 - 2x_2 + 2x_2 = 8$$
$$0 = 0$$

woops

(9)

$$2(4-x_2)+3x_2=14$$

$$8-2x_2+3x_2=14$$

$$x_2=6$$

$$x_1=4-x_2$$

$$=4-6$$

$$x_1=-2$$

$$\begin{bmatrix}-2\\6\end{bmatrix}$$

$$p=A\hat{x}=\begin{bmatrix}1&1\\1&1\\0&1\end{bmatrix}\begin{bmatrix}-2\\6\end{bmatrix}$$

$$=\begin{bmatrix}-2+6\\-2+6\\0+6\end{bmatrix}=\begin{bmatrix}4\\4\\6\end{bmatrix}$$ (1) correct ✓

~~thats b, not p~~ same

~~What went wrong?~~ but did 1st method wrong

this is obvious easier here - always?

10

$$e = b - p = \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix} - \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0$$

4.2 #16 What linear combo $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ is closest to $\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$

¿So this is find line b/w pts?

So is it $A = \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ -1 & 1 \end{bmatrix}$ $b = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$

¿Could do 4.3 Least Squares Method or way off ??

$$A^T A \hat{x} = A^T b$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ -1 & 1 \end{bmatrix} \hat{x} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

(11)

$$A^T A = \begin{bmatrix} 1\cdot 1 + 0\cdot 2 + 1\cdot -1 & 1\cdot 1 + 0\cdot 0 + 1\cdot 1 \\ 1\cdot 1 + 2\cdot 2 + -1\cdot -1 & 1\cdot 1 + 2\cdot 0 + -1\cdot 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 2 \\ 6 & 0 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 1\cdot 2 + 0\cdot 1 + 1\cdot 1 \\ 1\cdot 2 + 2\cdot 1 + -1\cdot 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 2 \\ 6 & 0 \end{bmatrix} \hat{x} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

$$2\hat{x}_2 = 3$$

$$6\hat{x}_1 = 3$$

$$\hat{x}_1 = \frac{1}{2} \qquad \hat{x} = \begin{bmatrix} 1/2 \\ 3/2 \end{bmatrix}$$

$$\hat{x}_2 = \frac{3}{2}$$

(12)

$$P\ \frac{1}{2}\begin{bmatrix}1\\2\\-1\end{bmatrix}+\frac{3}{2}\begin{bmatrix}1\\0\\1\end{bmatrix}=\begin{bmatrix}1/2\\1\\-1/2\end{bmatrix}+\begin{bmatrix}3/2\\0\\3/2\end{bmatrix}=\begin{bmatrix}2\\1\\1\end{bmatrix}$$

~~No wrong approach...~~

~~No - checked back of book - did work wrong~~

One little mistake and you're screwed
algebra

So b is in plane

$Pb = b$

? what does this mean?

$p = b$

Is that correct

$p = A\hat{x}$

could do to check

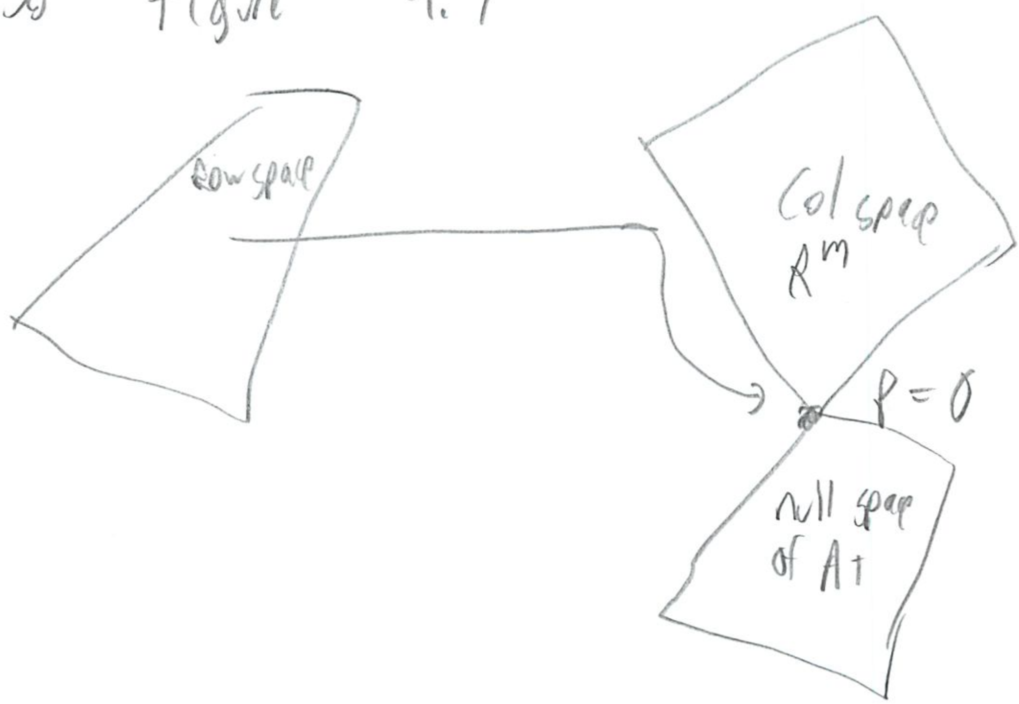
I guess fits exactly - so thats why

(13)

4.2 # 27 If $A^T A x = 0$ then $Ax = 0$

1/10 New proof The vector Ax is in the nullspace of ____ . Ax is always in the col space of ___ . To be in both $\perp$ spaces, Ax must be 0.

So Figure 4.7



(Answer not in back) ? I'm confused!.

(14)

4.2 #30

a) Find Projection Matrix P_C

10/10

$$A = \begin{bmatrix} 3 & 6 & 6 \\ 4 & 8 & 8 \end{bmatrix}$$

so only 1 ind col

$$P = A(A^T A)^{-1} A^T$$

only invertable if lin ind. cols

So $A = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ ✓

So can use special 1 col formula

$$P = \frac{a a^T}{a^T a}$$

$$\frac{\begin{bmatrix} 3 \\ 4 \end{bmatrix} \begin{bmatrix} 3 & 4 \end{bmatrix}}{\begin{bmatrix} 3 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix}} = \frac{\begin{bmatrix} 3 \cdot 3 & 3 \cdot 4 \\ 4 \cdot 3 & 4 \cdot 4 \end{bmatrix}}{[3 \cdot 3 + 4 \cdot 4]} = \frac{\begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix}}{25}$$ ✓

(15)

b) Find the 3x3 projection matrix P_R onto row space of A

Trow space

$$\begin{bmatrix} 3 & 4 \\ 6 & 8 \\ 6 & 8 \end{bmatrix} \rightarrow \begin{bmatrix} 3 \\ 6 \\ 6 \end{bmatrix}$$

can have fraction linear cols

$$\frac{\begin{bmatrix} 3 \\ 6 \\ 6 \end{bmatrix}\begin{bmatrix} 3 & 6 & 6 \end{bmatrix}}{\begin{bmatrix} 3 & 6 & 6 \end{bmatrix}\begin{bmatrix} 3 \\ 6 \\ 6 \end{bmatrix}} = \frac{\begin{bmatrix} 3\cdot 3 & 3\cdot 6 & 3\cdot 6 \\ 6\cdot 3 & 6\cdot 6 & 6\cdot 6 \\ 6\cdot 3 & 6\cdot 6 & 6\cdot 6 \end{bmatrix}}{\begin{bmatrix} 3\cdot 3 + 6\cdot 6 + 6\cdot 6 \end{bmatrix}}$$

$\cancel{0}2$

$$= \frac{\begin{bmatrix} 9 & 18 & 18 \\ 18 & 36 & 36 \\ 18 & 36 & 36 \end{bmatrix}}{81}$$

(16)

Ahha could have $\begin{bmatrix}1\\2\\2\end{bmatrix}$ as basis — duh

Can divide everything by 9

$$P_R = \frac{\begin{bmatrix}1 & 2 & 2\\2 & 4 & 4\\2 & 4 & 4\end{bmatrix}}{9}$$

$$B = \begin{bmatrix}9/25 & 12/25\\12/25 & 16/25\end{bmatrix}\begin{bmatrix}3 & 6 & 6\\4 & 8 & 8\end{bmatrix}\begin{bmatrix}1/9 & 2/9 & 2/9\\2/9 & 4/9 & 4/9\\2/9 & 4/9 & 4/9\end{bmatrix}$$

$$= \begin{bmatrix}3 & 6 & 6\\4 & 8 & 8\end{bmatrix} = A$$

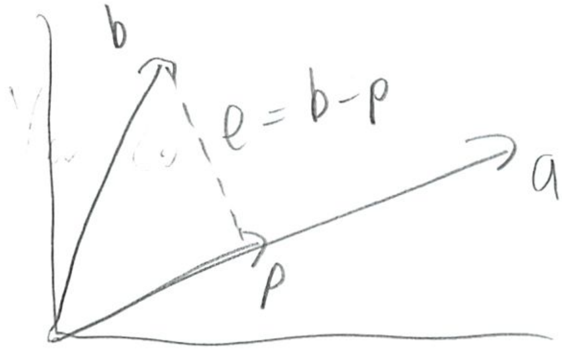
since $P_C A = A$

$A P_R = A$

(17)

4.2#31 In R^m suppose I gives you b and p and n linearly ind vectors $a_1, \ldots, a_n$. How would you test to see if p is the proj onto the subspace spanned by the as?

10/10



You could verify $e = b - p$

~~So this is since we have I~~

~~wait is this Identity or the author?~~

~~Vectors p and e should add to b = (1,1,1)~~

is orthogonal / perpendicular to a

didn't read whole sol

(18)

4.3 #1 With $b = 0, 8, 8, 20$ at $t = 0, 1, 3, 4$

10/10

Write down the 4 eqs

$p = 1, 5, 13, 17$

Find $A\hat{x} = p$

(I think I got this one - spent a lot of time reading)

$C + D0 = 0$

$C + D(1) = 8$

$C + D(3) = 8$

$C + D(4) = 20$

$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}$ ✓ $x = \begin{bmatrix} C \\ D \end{bmatrix}$ $b = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$

always 1s

(19)

Now this

$$A^T A \hat{x} = A^T b$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \hat{x} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 1+1+1+1 & 0+1+3+4 \\ 0+1+3+4 & 0+1+9+12 \end{bmatrix} = \begin{bmatrix} 4 & 8 \\ 8 & 22 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 0+8+8+20 \\ 0+8+24+80 \end{bmatrix} = \begin{bmatrix} 36 \\ 122 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 8 \\ 8 & 22 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 36 \\ 122 \end{bmatrix}$$

$$4C + 8D = 36$$

$$8C + 22D = 112$$

$$C = 1$$

$$D = 4 \qquad \hat{x} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

✓

(20)

Now find p_s

$1 + 4(0) = 1$

$1 + 4(1) = 5$

$1 + 4(3) = 13$

$1 + 4(4) = 17$

$$p = A\hat{x} = \begin{bmatrix} 1 \\ 5 \\ 13 \\ 17 \end{bmatrix}$$

2oh cool

So errors

$0 - 1 = -1$

$8 - 5 = 3$

$8 - 13 = -5$

$20 - 17 = 3$

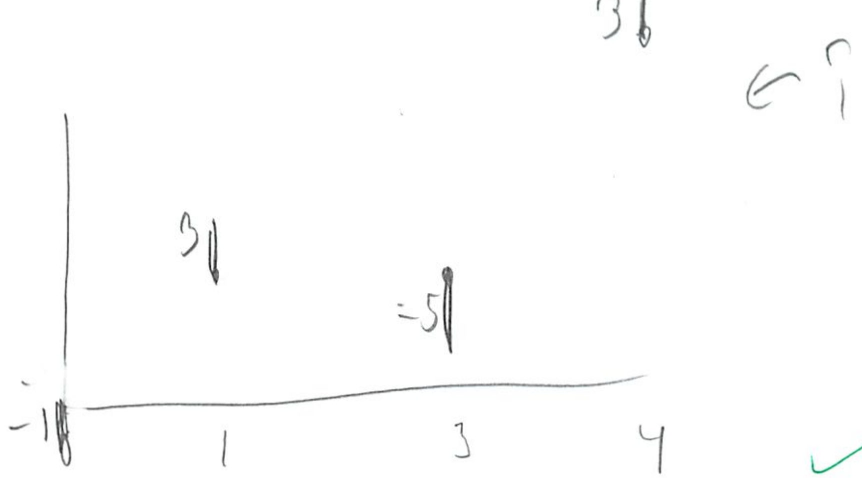
Sum $(-1)^2 + 3^2 + (-5)^2 + (3)^2 = 1 + 9 + 25 + 9 = 44$

(0)

(21)

$b = (0, 8, 8, 20)$

4.3 #3 Check that $e = b - p = (-1, 3, -5, 3)$
is $\perp$ to both Cols of A



Cols of A

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}$$

i is + still the same
e is $\perp$ in col space

(22)

Dot prods w/ col of A

$$\begin{bmatrix} -1 & 3 & -5 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = -1 + 3 + -5 + 3 = 0$$ ✓

$$\begin{bmatrix} -1 & 3 & -5 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 3 \\ 4 \end{bmatrix} = 0 + 3 - 15 + 12 = 0$$ ✓

Then find $\|e\|$

$$\sqrt{(-1)^2 + (3)^2 + (-5)^2 + (3)^2}$$

$$\sqrt{1 + 9 + 25 + 9}$$

$$\sqrt{44}$$ ✓ same as before

✓

(23)

4.3 #26 Find a plane ✓ one more dim that gives the best fit to the 4 values $b = (0,1,3,4)$ at corners $(1,0)$ $(0,1)$ $(-1,0)$ $(0,-1)$ of a square

2/10

$C + Dx + Ey = b$ at those 4 pts are $Ax = b$

w/ $x = \begin{pmatrix} C \\ D \\ E \end{pmatrix}$ What is A?

~~so don't need to know about pts~~ the x y values

$$C + D1 + E0 = 0$$
$$C + D0 + E1 = 1$$
$$C + D{-1} + E0 = 3$$
$$C + D0 + E{-1} = 4$$

$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$ $b = \begin{bmatrix} 0 \\ 1 \\ 3 \\ 4 \end{bmatrix}$

$Ax = b$ is not solvable!

24

$$\begin{bmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0+1+0-1 & 0+0+0+0 & 0+1+0+1 \\ 1+0-1+0 & 1+0+1+0 & 0+0+0+0 \\ 1+1+1+1 & 1+0-1+0 & 0+1+0+-1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 4 & 0 & 0 \end{bmatrix}$$

$C = 2$ $\quad \hat{x} = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}$ X

$D = 2$

$E = 4$

(need to see - what to use inputs w/ that)

So $\underset{C}{2}(\ \ ) + \underset{D}{2}(\underset{X}{0}) + \underset{E}{4}(\underset{Y}{0}) = ? = 2$

0

Avg of bs $\dfrac{0+1+3+4}{4} = 2$

(25)

4.4 #4 Give example

a) A matrix Q that has orthonormal cols but $QQ^T \neq I$

10/10

$$Q = \begin{bmatrix} 2/3 & 2/3 \\ 2/3 & -1/3 \\ 2/3 & -2/3 \end{bmatrix} \quad Q^T = \begin{bmatrix} 2/3 & 2/3 & 1/3 \\ 2/3 & -1/3 & -2/3 \end{bmatrix}$$

$$Q \cdot Q^T = \begin{bmatrix} 8/9 & 2/9 & -2/9 \\ 2/9 & 5/9 & 4/9 \\ -2/9 & 4/9 & 5/9 \end{bmatrix} \neq I$$

* not square

b) Two orthogonal vectors that are not lin ind.

$$\vec{a} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \vec{b} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

orthogonal → dot product 0

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$ ✓

lin ind →

$$0 \cdot \vec{a} + 1 \cdot \vec{b} = 0$$

defies lin ind def

(26)

c) Orthonormal basis for R^3 including $q_1 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$

{orthogonal <u>and</u>

sums add to 1

$q_1 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$ what is orthogonal?

make it normal

$q_2 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix}$

$q_3 = q_1 \times q_2$

$$\begin{vmatrix} i & j & k \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{vmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ -2/\sqrt{6} \end{bmatrix}$$

$\|q_3\| = 0$

all are orthogonal

$q_1^T q_2 = 0$

$q_2^T q_3 = 0$

$q_3^T q_1 = 0$

(27)

4.4 #14

Complete the Gram-Schmidt process in Problem 13 by computing $q_1 = \frac{a}{\|a\|}$

$q_2 = \frac{B}{\|B\|}$

and factoring into QR

$$\begin{vmatrix} 1 & 4 \\ 1 & 0 \end{vmatrix} = \begin{bmatrix} q_1 & q_2 \end{bmatrix} \begin{bmatrix} \|a\| & ? \\ 0 & \|B\| \end{bmatrix}$$

R is told to us

for 3x3 $\begin{bmatrix} q_1^T a & q_1^T b & q_1^T c \\ & q_2^T b & q_2^T c \\ & & q_3^T c \end{bmatrix}$

So 2x2 $\begin{bmatrix} & q_1^T b \\ & \end{bmatrix}$

Now find q_1 and q_2

$q_1 = \frac{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}{\sqrt{2}}$ $q_2 = \frac{\begin{bmatrix} 4 \\ 0 \end{bmatrix}}{4}$

Are not orthogolal to each other

2) b

Must do Gram Schmidt process

1. A you keep $= \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

2. $B = b - \dfrac{A^T b}{A^T A} A$

only 1 col

$$B = \begin{bmatrix} 4 \\ 0 \end{bmatrix} - \frac{\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \end{bmatrix}}{\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 4 \\ 0 \end{bmatrix} - \frac{\begin{bmatrix} 4 + 0 \end{bmatrix}}{\begin{bmatrix} 1 \cdot 1 + 1 \cdot 1 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 4 \\ 0 \end{bmatrix} - \frac{4}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ 0 \end{bmatrix} - \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

(28)

Normalize to get to q

- Sum of squares adds to 1

$$q_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$q_2 = \begin{bmatrix} 2/\sqrt{8} \\ -2/\sqrt{8} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

$$\|A\| = \sqrt{2}$$

$$\|B\| = \sqrt{8}$$

So

$$QR = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{2} & q_1^T b \\ 0 & \sqrt{8} \end{bmatrix}$$

✓

$$q_1^T b = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 4 \\ 0 \end{bmatrix} = 4/\sqrt{2} = 2\sqrt{2}$$

why didn't I realize that!

$$\sqrt{8} = 2\sqrt{2}$$

✓ Matches

(29)

4.4 #7

If Q has orthogonal cols, what is the least squares sol $\hat{x}$ → $Qx = b$

10/10

Oh simple since you did the work

$$\hat{x} = R^{-1} Q^T b$$

¿Is that it? standard

or ↳ $Q^T Q \hat{x} = Q^T b$

$I \hat{x} = Q^T b$

$\hat{x} = Q^T b$

4.4 #18

Find orthogonal vectors A, B, C for

10/10

$a = (1, -1, 0, 0)$

$b = (0, 1, -1, 0)$

$c = (0, 0, 1, -1)$

$d = (1, 1, 1, 1)$

¿just do Gram-Schmidt?

30

1. $A = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$

do one col at a time

2. $B = b - \dfrac{A^T b}{A^T A} A$

$$= \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} - \frac{\begin{pmatrix} 1 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}}{\begin{pmatrix} 1 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} - \frac{(0 - 1 + 0 + 0)}{(1 + 1 + 0 + 0)} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1/2 \\ 1/2 \\ -1 \\ 0 \end{pmatrix}$$

sign error corrected

(31)

Now for c

$$C = c - \frac{A^{\dagger} c}{A^{\dagger} A} A - \frac{B^{\dagger} c}{B^{\dagger} B} B$$

$$= \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} - \frac{\begin{pmatrix} 1 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}}{\begin{pmatrix} 1 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} - \text{rest}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} - \cancel{\frac{(0)}{2} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}} - \frac{\begin{pmatrix} 1/2 & +1/2 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}}{\begin{pmatrix} 1/2 & 1/2 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1/2 \\ 1/2 \\ -1 \\ 0 \end{pmatrix}} \begin{pmatrix} 1/2 \\ 1/2 \\ 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} - \frac{-1}{\frac{1}{4} + \frac{1}{4} + 1} \begin{pmatrix} 1/2 \\ 1/2 \\ 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} + \frac{2}{3} \begin{pmatrix} 1/2 \\ 1/2 \\ 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 1/3 \\ 1/3 \\ 2/3 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 1/3 \\ 2/3 \\ -1 \end{pmatrix} \rightarrow \text{Correction} \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \\ -1 \end{pmatrix}$$

(32)

4.4 #24 a) Find a basis for the subspace S in R^4 spanned by sols of

$$X_1 + X_2 + X_3 - X_4 = 0$$

$$x_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad x_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \quad x_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

b) Find a basis for the orthogonal complement $S^\perp$

$$b = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$ normal to hyper plan

c) Find b_1 in S and b_2 in $S^\perp$ so that

$b_1 + b_2 = b = (1,1,1,1)$

$$b_1 \text{ in } S = \frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} + \frac{1}{8}\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 3/2 \end{pmatrix}$$

$$b_2 \text{ in } S^\perp = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ -1/2 \end{bmatrix}$$

23

$$\vec{b}_1 + \vec{b}_2 = \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 3/2 \end{pmatrix} + \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ -1/2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$ 1 ✓

(34)

10. Matlab Gram-Schmidt

10/10

4 x 3 matrix

$$A = \begin{array}{c c} & \begin{array}{ccc} 0 & 1\ j & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ i\ 2 \\ 3 \end{array} & \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \end{array}$$

$[Q, R] = qr(A)$

$Q \cdot diag(R)^{-1}$

Augment w/ $\begin{matrix} 0 \\ 0 \\ 0 \\ 1 \end{matrix}$

```
>> A=[1 0 0; -1 1 0; 0 -1 1; 0 0 -1]

A =

     1     0     0
    -1     1     0
     0    -1     1
     0     0    -1

>> [Q, R] = qr(A)

Q =

    0.7071    0.4082    0.2887   -0.5000
   -0.7071    0.4082    0.2887   -0.5000
         0   -0.8165    0.2887   -0.5000
         0         0   -0.8660   -0.5000


R =

    1.4142   -0.7071         0
         0    1.2247   -0.8165
         0         0    1.1547
         0         0         0


>> o= [0 0 0 1]

o =

     0     0     0     1

>> r= [R o']

r =

    1.4142   -0.7071         0         0
         0    1.2247   -0.8165         0
         0         0    1.1547         0
         0         0         0    1.0000


//I did not know how to do this in MatLab!
>> r = [1.4142 0 0 0; 0 1.2247 0 0; 0 0 1.1547 0; 0 0 0 1]

r =

    1.4142         0         0         0
         0    1.2247         0         0
         0         0    1.1547         0
         0         0         0    1.0000
```

```
>> Q/r

ans =

    0.5000    0.3333    0.2500   -0.5000
   -0.5000    0.3333    0.2500   -0.5000
         0   -0.6667    0.2500   -0.5000
         0         0   -0.7500   -0.5000
```

18.06 Spring 2012 – Problem Set 5

This problem set is due Thursday, March 22nd, 2012 at 4pm (hand in to Room 2-106). The textbook problems are out of the 4th edition. For computational problems, please include a printout of the code with the problem set (for MATLAB in particular, `diary('filename')` will start a transcript session, `diary off` will end one.)

Every problem is worth 10 points.

1. Do Problems 11 & 16 from Section 4.2.

Solution.

Problem 11

(a)

$$AA^T = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}.$$

$$\hat{x} = (A^TA)^{-1}A^Tb = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}.$$

Hence

$$p = A\hat{x} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, e = \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}.$$

(b)

$$AA^T = \begin{bmatrix} 2 & 2 \\ 2 & 3 \end{bmatrix}.$$

$$\hat{x} = (A^TA)^{-1}A^Tb = \frac{1}{2}\begin{bmatrix} 3 & -2 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \end{bmatrix}.$$

Hence

$$p = A\hat{x} = \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix}, e = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Problem 16

Linear combination of $(1, 2, -1)$ and $(1, 0, 1)$ which is closest to b is the projection of b onto the plane spanned by $(1, 2, -1)$ and $(1, 0, 1)$.

projection of b onto the plane

$$= \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ -1 & 1 \end{bmatrix}\left(\begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & 0 \\ -1 & 1 \end{bmatrix}\right)^{-1}\begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}.$$

Hence $(2, 1, 1)$ itself is on the plane and $\frac{1}{2}(1, 2, -1) + \frac{3}{2}(1, 0, 1) = (2, 1, 1)$. □

2. Do Problem 27 from Section 4.2.

   Solution. The vector Ax is in the nullspace of A^T. Ax is always in the column space of A. To be in both of those perpendicular spaces, Ax must be zero. $\square$

3. Do Problem 30 from Section 4.2.

   Solution. (a) Column space of A is spanned by one vector $a = [3,4]^T$. The projection matrix
$$P_c = \frac{aa^T}{a^Ta} = \frac{1}{25}\begin{bmatrix} 9 & 12 \\ 12 & 25 \end{bmatrix}.$$

   (b) The row space of A is spanned by one vector $b = [1,2,2]^T$. The projection matrix
$$P_r = \frac{bb^T}{b^Tb} = \frac{1}{9}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 2 & 4 & 4 \end{bmatrix}.$$

   $B = P_cAP_r = A$. Since P_c projects each columns of A to themselves, $P_cA = A$. Similarly, $P_rA^T = A^T$ and $A = (P_rA^T)^T = AP_r^T = AP_r$. Hence $P_cAP_r = A$.

   $\square$

4. Do Problem 31 from Section 4.2.

   Solution. p is the projection of b onto the subspace spanned by $a_1, \cdots, a_n$ if and only if $b - p$ is perpendicular to the vectors $a_1, \cdots, a_n$. $\square$

5. Do Problems 1 & 3 from Section 4.3.

   Solution. **Problem 1**

   We have
$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}.$$
$$\begin{bmatrix} 4 & 8 \\ 8 & 26 \end{bmatrix}\hat{x} = \begin{bmatrix} 36 \\ 112 \end{bmatrix}$$
   by substituting this matrix to the normal equation $A^TA\hat{x} = A^Tv$. Hence
$$\hat{x} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}, A\hat{x} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 13 \\ 17 \end{bmatrix}, e = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ -5 \\ 3 \end{bmatrix}.$$
   The minimum value $E = e_1^2 + e_2^2 + e_3^2 + e_4^2 = 44$.

Problem 3 Since $eA = 0$, e is perpendicular to both columns of A. The shortest distance $||e||$ from b to the column space of A is $\sqrt{E}$ for the E in Problem 1. Hence it is $\sqrt{44}$.

□

6. Do Problem 26 from Section 4.3.

Solution. We would like to minimize the error in the equation of the form $Ax = b$ given by

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} C \\ D \\ E \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 3 \\ 4 \end{pmatrix}.$$

Although this equation is not solvable, we solve a new equation of the form $A\hat{x} = p$, where p is the projection of b to $C(A)$. Recall that $p = A(A^TA)^{-1}A^Tb = (1/2, 1/2, 7/2, 7/2)$. Solving $A\hat{x} = p$ then yields $\hat{x} = (2, -1, -3/2)$.

Then our plane of best fit is given by the equation $2 - x - \frac{3}{2}y = 0$, and at the centre of the square (i.e. $x = y = 0$), the plane passes through $(0, 0, 2)$. 2 is the average of $0, 1, 3, 4$.

□

7. Do Problems 4 & 14 from Section 4.4.

Solution.

Problem 4

(a) Take $Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then $QQ^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \neq I$.

(b) $(0, 0)$ and $(1, 0)$ are orthogonal since their dot product is zero. But they are not independent.

(c) One possible set of orthonormal basis including q_1 is

$$\begin{aligned} q_1 &= (1, 1, 1)/\sqrt{3} \\ q_2 &= (1, -1, 0)/\sqrt{2} \\ q_3 &= (1, 1, -2)/\sqrt{6}. \end{aligned}$$

Problem 14

$B = b - xa$ should be orthogonal to a, with $x = \frac{a^Tb}{a^Ta} = 2$, so $B = (2, -2)$. Then $q_1 = \frac{1}{\sqrt{2}}(1, 1)$ and $q_2 = \frac{1}{2\sqrt{2}}(2, -2) = (1/\sqrt{2}, -1/\sqrt{2})$. Then

$$QR = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} \sqrt{2} & 2\sqrt{2} \\ 0 & 2\sqrt{2} \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 1 & 0 \end{pmatrix}.$$

□

8. Do Problem 7 from Section 4.4.

   Solution. We normally solve $A^T A\hat{x} = A^T b$ to get the least squares solution $\hat{x}$. Now, if Q is an orthonormal matrix, we have $Q^T Q = I$, so that we just get $\hat{x} = Q^T b$. □

9. Do Problems 18 & 24 from Section 4.4.

   Solution. **Problem** 18

   Note that we are only looking for orthogonal vectors, not orthonormal ones. So we have

   $$A = a = (1, -1, 0, 0),$$
   $$B = b - \text{projection of } b \text{ onto } A = (1/2, 1/2, -1, 0),$$
   $$C = c - \text{projection of } c \text{ onto } A - \text{projection of } c \text{ onto } B = (1/3, 1/3, 1/3, -1).$$

   Problem 24

   (a) A possible basis is given by $(1, -1, 0, 0), (1, 0, -1, 0), (1, 0, 0, 1)$.

   (b) Since S contains solutions to $(1, 1, 1, -1)^T x = 0$, a basis of S is given by $(1, 1, 1, -1)$.

   (c) Split $(1, 1, 1, 1) = b_1 + b_2$ by projection onto $S^\perp$ and S: $b_2 = 1/2(1, 1, 1, -1)$ and $b_1 = 1/2(1, 1, 1, 3)$.

   □

10. In this exercise, we use MATLAB for learning about Gram-Schmidt.

    Suppose the 4×3 matrix A has 1's when $i = j$ and -1's when $i = j + 1$ and otherwise $A_{ij} = 0$. From the MATLAB command

    $$[\texttt{Q}, \texttt{R}] = \texttt{qr(A)},$$

    find the Gram-Schmidt orthonormal basis for the column space $C(A)$. Renormalize those basis vectors to contain nice fractions by dividing by the diagonal entries of R. Now the vectors are orthogonal, not orthonormal.

    Guess what the pattern you see here is, and use this to find a 4th vector in $\mathbb{R}^4$ orthogonal to these three columns of Q (and of A).

    Solution. `>>A = [1 0 0; -1 1 0; 0 -1 1; 0 0 1]`

```
A =

 1  0  0
-1  1  0
```

```
 0 -1  1
 0  0 -1

>>[Q,R] = qr(A)

Q =

-0.7071 -0.4082 -0.2887 0.5000
 0.7071 -0.4082 -0.2887 0.5000
      0  0.8165 -0.2887 0.5000
      0       0  0.8660 0.5000

R =

-1.4142  0.7071       0
      0 -1.2247  0.8165
      0       0 -1.1547
      0       0       0
```

So an orthonormal basis of $C(A)$ is given by $(-0.7071, 0.7071, 0, 0)$,$(-0.4082, -0.4082, 0.8165, 0)$, and $(-0.2887, -0.2887, -0.2887, 0.8660)$.

From $R = \begin{pmatrix} -1.4142 & 0 & 0 & 0 \\ 0 & -1.2247 & 0 & 0 \\ 0 & -1.1547 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1/2 & 0 \\ 0 & 1 & 2/3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, the orthogonal matrix is given by $QD = \begin{pmatrix} 1 & 1/2 & 1/3 & 1/2 \\ -1 & 1/2 & 1/3 & 1/2 \\ 0 & -1 & 1/3 & 1/2 \\ 0 & 0 & -1 & 1/2 \end{pmatrix}$. where D is the 4×4 diagonal matrix with the diagonal entries being $(-1.4142, -1.2247, -1.1547, 1)$.

If we were to find the fourth vector orthogonal to all the columns of Q, we would guess $(1/4, 1/4, 1/4, 1/4)$. One can check that this is indeed perpendicular to all the columns.

Remark 0.1. When we use the qr command, MATLAB always returns a square matrix Q with orthonormal columns (in our case, we only needed three orthonormal vectors in $\mathbb{R}^4$, but MATLAB finds a fourth vector that is orthonormal to the first three, and returns it as the last column of Q). In order to avoid this issue, you could have used the command `[Q,R] = qr(A,0)`. Then it would return:

```
Q =

   -0.7071   -0.4082   -0.2887
    0.7071   -0.4082   -0.2887
         0    0.8165   -0.2887
```

```
         0         0   -0.8660


R =

   -1.4142    0.7071         0
         0   -1.2247    0.8165
         0         0   -1.1547
```

Sorry about the confusion!

$\square$

18.06 Wisdom. On problem sets and exams, always try to think of ways to check your results. It both assists your own learning, when you reflect on and relate to what you've worked through, plus you avoid loosing points!

18.06 Lecture

3/21

(missed the lecture before - still need to watch on YouTube)

Before Board

1. Big Formula det $A = \sum$ $n!$ terms
2. Cofactors (dets of size $n-1$)
3. Formula of A^{-1} // Cramer's Rule for $A^{-1}b$
4. Volume of box

Today: Cover determinants

Then: Eigenvalues, Eigenvectors

are a connection to

Last time: Long list of formulas for the determinants

Big formula: $n \times n$ matrix

produce 1 # (det) by some formula

he did 2×2 and 3×3

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

(2)

$$\underbrace{a}_{\text{factor}} \quad \underbrace{\begin{matrix} e & f \\ h & i \end{matrix}}_{\text{cofactor}}$$

next Can take f from row 2, col 3

then must take h last

$$\det A = \sum_{n! \text{ terms}} a_1$$

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$A_{11} \quad A_{23} \quad A_{32}$

all the permutations of $\{1, \dots, n\}$ col #s

(3)

$$\det A = \sum_{n!\text{ terms}} a_{1\alpha}\, a_{2\beta} \cdots a_{3\omega}$$

Missing 1 key part of det

$\pm$

Which one? <u>Sign of permutation</u>

perm is 1, 3, 2 odd $\ominus$

lex is 1, 2, 3 even $\oplus$

<u>4x4</u>

$$A = P = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Only 1 choice in the 1st row

1 2nd row

⋮ etc ⋮

(4)

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

makes no difference
Since must still each row, col once

← previous. Will see det is ± 1, which one?

(+)

$\boxed{\text{in } 3\times 3 \quad [\;\;] \text{ is } (-)}$

but here
4, 3, 2, 1
reverse permutation
even for 4×4

do exchanges to get in right order

<u>heap sort</u>

4	3	2	1		
1	3	2	4		4 and 1
1	2	3	4	(2)	2 and 3
	?even			even	

5

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & 0 & 0 \\ d & e & f \\ g & h & i \end{vmatrix} + \begin{vmatrix} 0 & b & 0 \\ \text{---} & \text{---} & \text{---} \\ \text{---} & \text{---} & \text{---} \end{vmatrix} + \begin{vmatrix} 0 & c & 0 \\ \text{---} & \text{---} & \text{---} \\ \text{---} & \text{---} & \text{---} \end{vmatrix}$$

copy same rows

now same w/ 2nd row

$$\begin{vmatrix} a & 0 & 0 \\ d & 0 & 0 \\ g & h & i \end{vmatrix} \quad \begin{vmatrix} a & 0 & 0 \\ 0 & e & 0 \\ g & h & i \end{vmatrix} \quad \begin{vmatrix} a & 0 & 0 \\ 0 & 0 & f \\ g & h & i \end{vmatrix}$$

now have 9

det = 0
(automatically)
singular

Want to know all possible basis for $C(A)$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} x \\ y \\ 0 \\ 0 \end{bmatrix}$$

all of col space $C(A)$

(4)

2 bases could be

$$\begin{bmatrix} a \\ c \\ g \\ 0 \end{bmatrix} \begin{bmatrix} b \\ d \\ g \\ 0 \end{bmatrix}$$ Must be ind

When are those 2 vectors ind?

$ad \neq bc$

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

~~det ≠ 0~~

det = 0

24 terms in det

└ ±1

└ 1 − 1 + 1 − 1... etc

Since

- singular
- Cols are very dep
- ~~ar~~

7

Kick it up a notch

$$A = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}$$

This is not 0

Has some structure. What does it mean?

$$\begin{bmatrix} 2 & 1 & 1 & 1 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}$$

# of terms is not that bad

$$\begin{aligned} &+2 \cdot 1 \cdot 1 \cdot 1 \\ = &-1 \cdot -1 \cdot 1 \cdot 1 \\ &-1 \cdot 1 \cdot -1 \cdot 1 \\ &\;\;1 \cdot 1 \cdot 1 \cdot -1 \end{aligned} \quad = 5$$

Will revisit when we do eigenvalues

$$\text{eigenvalues} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 5 \end{bmatrix}$$

8

$$A = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}$$

$$\underbrace{a(ei - fh)}_{\text{Cofactor of } a} + b\underbrace{(-di - fg)}_{\ominus \text{ comes in here}} + c(dh - eg)$$

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$ ← little code.
Take 2×2 det.
Give it that sign

You could create det by starting w/ 1×1 matrix

$$\det [17] = 17$$

Can define det building up out of cofactors

(9)

$$\det A = a_{11}\underbrace{A_{11}}_{\text{Cofactor}} + a_{12}A_{12} + \cdots + a_{1n}A_{1n}$$ Cofactors

Some exercise in cofactors in HW

$$A^{-1} = \frac{C^T}{\det(A)}$$ ← cofactor ← denominator ← Prof i neat formula

For example - tell me the 1 entry of the inverse

know invertable

$\det \neq 0$
$= 5$

So Cofactor of 2 $= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\det(C) = 1$

So $A^{-1} = 1\ 1/5$

(10)

$$AC^T = \begin{array}{cc} A & C^T \\ \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ & \vdots & \\ & \vdots & \end{bmatrix} & \begin{bmatrix} A_{11} & \\ A_{12} & \cdots \\ A_{13} & \end{bmatrix} \end{array}$$

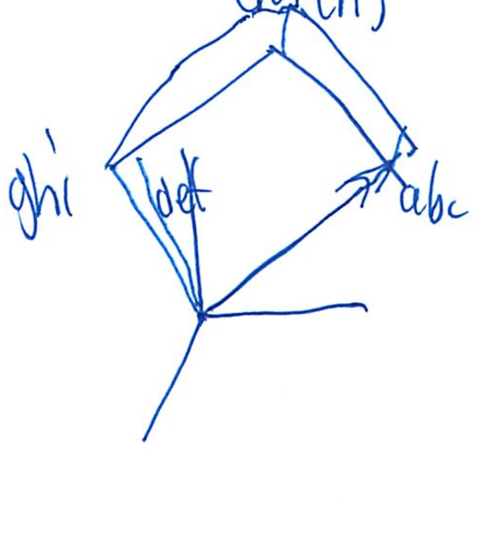
cofactors of row 1 go into col 1

$$= \begin{bmatrix} \det(A) & 0 & 0 \\ 0 & \det(A) & 0 \\ 0 & 0 & \ddots \end{bmatrix} = \det(A) \text{ on diagonals, } 0 \text{ elsewhere}$$

$$= \det(A)\, I$$

So divide by $\det(A)$

$$\frac{AC^T}{\det(A)} = A^{-1}$$ proved



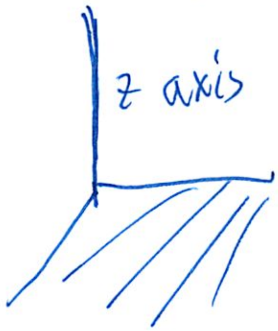
Out of det comes central formula for geometry

18.06 OH

3/21

Orthogonal

R^3



2 subspaces

xy plane

$v \in V$

$w \in W$

$v \cdot w = 0$ ← if orthogonal

2 vector spaces are orthogonal complements

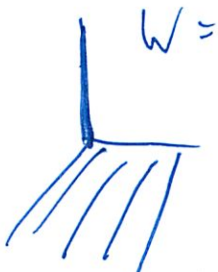
X is an, vector in R^3

V and W are orthogonal complements

if we can find $v \in V$ and $w \in W$

$X = V + W$

So



$W = \{0, 0, z\}$

$V = \{(a, b, 0)\}$

②

So take random vector ~~a~~ so add to $\{1,2,5\}$

So $\{1,2,0\} + \{0,0,5\}$

~~a~~ Can do for any a, b, z

So any vector in $\mathbb{R}^3$ could be found

So $\{a,b,0\}$ and $\{0,0,z\}$ are orth. complements

So if $v =$ ~~a~~ x axis are orthogonal

So w/ $(a,0,0)$, $(0,0,b)$

So pic - row space and null space are orth. complements

(3)

So if had $\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

$R(A) \subseteq R^2$

$N(A) \subseteq R^2$ ← since what we get when multiply by

Are orthogonal – complements

Row space could be any dimension
– not just a plane

Can write <u>any</u> x as something from row space (x_r) and something from null space (x_n)

Always true – no matter col or row rank

4

When not invertable

4.2 #38 $\mathbb{R}^2$

Col space - span of all cols

Find the projection matrix

Take line that is ~~[illegible]~~ inside this - project onto space

Simple 2D formula

18.06

3/2/13

Last lecture on determinants

Cofactors

inverse $A^{-1} = \frac{C^T}{\det A}$ Cramers Rule

Volume

Use Cofactors

$$\det \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} = D_4$$

$-1, 1, 1$

$$[1] \quad D_1 = 1$$

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = D_2 = 2$$

2

$$\begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix} D_3 = 3$$

So now what about D_4

$1 \cdot D_3 + D_2$

$D_n = 1 \cdot D_{n-1} + D_{n-2}$ (fibinachi?)

$$\left[\begin{array}{c|ccc} 1 & 1 & 0 & — \\ \hline -1 & & & \\ 0 & & D_{n-1} & \\ 1 & & & \end{array}\right]$$

So D_1 D_2 $\rightarrow$ "almost fibinicchi"

$\overset{\shortparallel}{1}$ $\overset{\shortparallel}{2}$ 5 8 13 $\dashrightarrow$ a one missing

This was using cofactors

in a tridiagonal matrix

(3)

$Ax = b$

$x = A^{-1} b$ ← Cramers Rule − for $Ax = b$

$\downarrow$

So $A^{-1} = \dfrac{C}{\det(A)}$

$$x_1 = \frac{\det[b \; a_2 \cdots a_n]}{\det(A)}$$

$$\begin{array}{c} \qquad\quad\;\; \downarrow \text{col } 2 \\ x_2 = \dfrac{\det[a_1 \; b \; a_3 \cdots a_n]}{\det(A)} \end{array}$$

Suppose $Ax = b$
2nd col of A

$$x = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} \begin{array}{l} \leftarrow x_1 \\ \leftarrow x_2 \\ \\ \\ \end{array}$$

If b is a_2 (2nd col)
↳ 1st two cols are the same

(4)

If $b = a_2$ then $x_2 = 1$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}$$

$$x_1 = \frac{df - bg}{ad - bc}$$

$$\begin{bmatrix} f & d \\ g & d \end{bmatrix} \quad \begin{bmatrix} a & f \\ c & g \end{bmatrix}$$

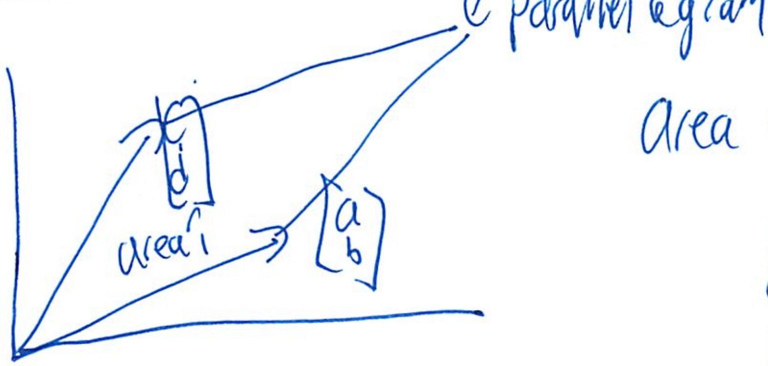
fg in col 1

keep rest of matrix

$$x_2 = \frac{ag - cf}{ad - bc}$$

(5)

Volume



area = base · height

base = $\sqrt{a^2+b^2}$
height = ?

wrong way to do it

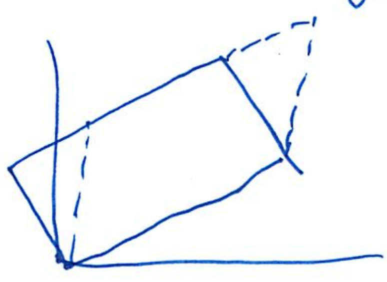
better way?

area = $\begin{bmatrix} a & c \\ b & d \end{bmatrix}$ = ad − bc

Why?

- cross product

- Does a rectage have same area for parallel a gram



Gram Schmidt!

$\begin{bmatrix} a & c \\ b & d \end{bmatrix} \to \begin{bmatrix} a & c-ra \\ b & d-rb \end{bmatrix}$

⑥

Doing col ops

– talked earlier about row ops

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c-ra & d-ra \end{bmatrix}$$

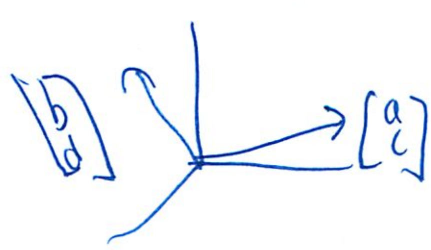
doing row op

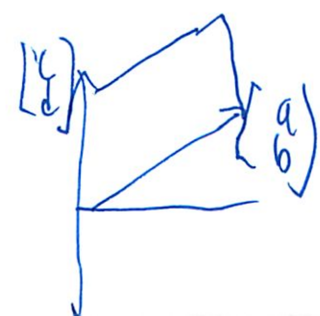
$$= \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a & b \\ -ra & -rb \end{bmatrix}$$

det 0

Gram-Schmidt takes box and straights it out to a cube

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \det \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$





(7) Edges from rows or cols

Where we are going: Eigenvalues

Uses deteminants

~~Eigenvector =~~ Ax in same dir as x

$\uparrow$ eigenvector of A

$$Ax = \lambda x$$

$\uparrow$ eigen vector $\quad$ $\uparrow$ eigenvalue

$A^2 x$ is in the same direction

$$A^2x = A(Ax) = A(\lambda x) = \lambda^2 x$$

If can find special dir - it makes the problem 1 directional

8

Each one — ~~sss~~ special property

Step 1 Solve $A x = \lambda x$

find

Step 2 ~~A (B~~ $(A - \lambda I) x = 0$

it just right shift $+ \lambda$

will be singular

then x = nullspace

So look for λ 1st

must be singular

So ~~xxxx~~

$\det(A - \lambda I) = 0$

18.06 Lecture

4/2

λ Eigenvalues / Eigenvectors x

Ax is in the same direction as x

? only special vectors

2nd half of linear algebra

end of 2nd quiz

projection

least sqs

det

eigenvalues

Row ops mess up eigenvalues

elim

So need something else

Eigenvectors — a vector — multiply by A, does not change direction

$$Ax = \lambda x$$

some multiple

normally n eigenvectors (directions) for an $n \times n$ matrix

(2)

Normal situation

have enough eigenvectors

~~n~~ n ind eigenvectors

---

big eq $\boxed{Ax = \lambda x}$

- not linear
- Solving is harder than elim
- Can't get exact ans in finite # of steps
- Matlab: eig(A)

Easy facts:

$A^2 x =$

eigenvector is same for $A, A^2, A^{-1}, A^p, A + 16I$

any power

random #

$= A(Ax)$

$= A(\lambda x)$

$= \lambda(Ax)$

$= \lambda^2 x$

(3)

Start $Ax = \lambda x$

multiply both sides by A^{-1}

$x = \lambda A^{-1} x$

divide both sides by λ

$\frac{1}{\lambda} x = A^{-1} x$

So A^2, A^{-1}, A^p, $A + 16I$

$\downarrow$ $\downarrow$ $\downarrow$ $\downarrow$

λ^2 $\frac{1}{\lambda}$ λ^p $\lambda + 16$

$$(A + 16I)x = Ax + 16x$$
$$= \lambda x + 16x$$

$A + B$

$By = \omega x$

diff letter

Can't easily answer - its not $\lambda + \omega$

they are in diff dir -> diff eigenvectors

(4)

it would have to be $AB = BA$
to have same eigenvectors

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

two eigenvectors

Symmetric - so will get 2 eigenvectors (no special case)

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

first
? eigenvector

$$\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

~~eigenvector~~

$$= 4 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= 4 x_1$$

? $\lambda_1 = 4$

(5)

If did $\begin{bmatrix} 10 \\ 10 \end{bmatrix}$

$$= \begin{bmatrix} 40 \\ 40 \end{bmatrix}$$

$$= 4 \begin{bmatrix} 10 \\ 10 \end{bmatrix}$$

$$= 4 x_1$$

So a line of eigenvectors

Matlab would do unit vectors

$$\pm \begin{bmatrix} 1 \\ 1 \end{bmatrix} / \sqrt{2}$$

But not the 0 vector

$$A x_2 = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} -$$

? Sym matrix perp

$$= \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

⑥

$$= 2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$\lambda_2 = 2$

Sum of eigenvectors

$$\lambda_1 + \lambda_2 = 6$$

$= \text{Sum of diagonal} = \text{trace}$

$= 6$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ eigenvalues?

$$x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ w/ } \lambda_1 = 1$$

$$x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{ w/ } \lambda_2 = -1$$

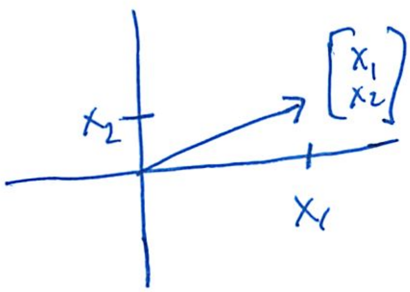
notice from above shifted by 3!

7

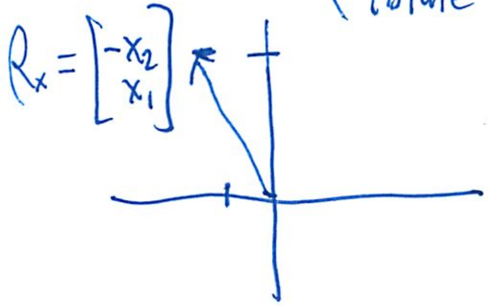
$$R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

? 90° rotation

$$R\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}$$



rotate by 90°



eigenvector / eigenvalue <u>can be</u> complex $(a+bi)$

$$Rx = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} -i \\ 1 \end{bmatrix} = -i\begin{bmatrix} 1 \\ i \end{bmatrix}$$

(8)

In $\mathbb{C}^2$ space — not $\mathbb{R}^2$

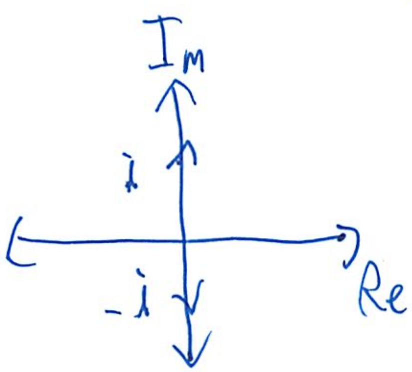
Need a 2nd one

└ get a 2nd one for free → "complex conjugate"

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} i \\ 1 \end{bmatrix} = +i \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

So $\lambda = -i, i$

└ complex conj pair



What if we changed the sign?

$$B = \begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix}$$

(9)

Remember the 3 just shifts the values by 3

~~$B = R$~~

$$C = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Close to R

90° rotation the other way

$C = -R$

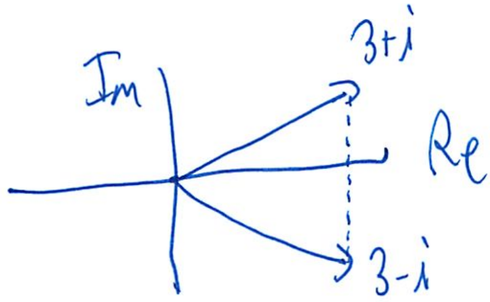
$$B = 3I + C$$
$$= 3I - R$$

$$\lambda(B) = \begin{cases} 3-(-i) & = 3+i \\ 3-i & = 3-i \end{cases}$$

shift by 3

reverse signs

10



Can also easy check w/ det
multiply eigenvalues to get det

$\in$ 2 easy check
- trace $+$ Σ
- det $\times$ Π

$$(3+i)(3-i) = 10$$

Those 2 were <u>orthogonal</u> matrices

Orthogonal = Q

$\hookrightarrow Q^T Q = I$

aka orthonormal cols

ex $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$,

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$\lambda = 1, -1 \quad i, -i$

$e^{i\theta}, e^{-i\theta}$ respectfully

(11)

$\boxed{\text{Always } |\lambda| = 1}$

$c + is$, $c - is$

$e^{i\theta} = c + is$

$e^{-i\theta} = c - is$

We're looking at every matrix - asking eigenvector
eigenvalue

~~To prove~~ $\cancel{Qx = \lambda x}$

To prove
$|\lambda| = 1$

Use $Q^T Q = I$

$\overline{Q}\,\overline{x} = \overline{\lambda}\,\overline{x}$

(12)

$$(a-ib)(x-iy) = \overline{(a+ib)(x+iy)}$$

(1) $Qx = \lambda x$

(2) $\bar{x}^T Q^T = \bar{\lambda} \, x^{-T}$

└ transpose and take complex conj

$$(\bar{x}^T Q^T)(Qx) = (\bar{\lambda} \bar{x}^T)(\lambda x)$$

$$\bar{x}^T x = \bar{\lambda} \lambda \, \bar{x}^T x$$

on both sides

So $\bar{\lambda} \lambda \overset{\text{must}}{=} 1$

$= |\lambda|^2$

18.06 Det Reading

4/3

(going to move fast + focus on what don't know)

det = 0 when no inverse

$\det(A^{-1}) = \frac{1}{\det(A)}$

~~Product of 2 pivo~~

3 ways

1. Multiple n pivots in U (Pivot formula)
2. Add up n! terms ("big" formula)
3. Combine n smaller det (Cofactor formula)
   - add accross the top ← where I was confused in recitation

---

Sign changes when 2 rows col exchanged

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ ← 1 row ex from I

(2)

$A - \lambda I = 0$, eigven value

$\det(AB) = \det(A)\det(B)$

~~$\det(A+B) = \det(A) + \det(B)$~~

$\det(A^T) = \det A$

---

the multiply and linearity stuff saw in video

and all the rules

5.2 Permutations + Cofactors

The 3 ways to find

(saw them all before)

Cofactors ↓not this one

$\det A = a_{11}(a_{22}a_{33} - a_{23}a_{32}) + a_{12}(\overset{\text{will be} \ominus}{a_{23}a_{31}} - a_{21}a_{33}) + a_{13}\underbrace{\underbrace{(a_{21}a_{32} - a_{22}a_{31})}_{M_{ij}}}_{-1\ C_{ij}}$

③

5.3 Cramer's Rule, Inverses, and Volumes

Solves $Ax = b$ by algebra, not elim

We also invert A

Cramers' Rule Solves $Ax = b$

Somehow get x_1 (I'm so confused here!)

$$\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \begin{bmatrix} x_1 & 0 & 0 \\ x_2 & 1 & 0 \\ x_3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{bmatrix} = B_1$$

↑ Replace 1st col of I w/ x

↑ get B

Multiply a col at a time

Take det of the 3 matrices

$$(\det A)(x_1) = \det B_1$$

$$\boxed{x_1 = \frac{\det B_1}{\det A}}$$

I think this is how

Now changing a col of A gives B_1

(above - right?)

4

To find x_2, put the vector x in the 2nd col of I

$$\begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \begin{bmatrix} 1 & x_1 & 0 \\ 0 & x_2 & 0 \\ 0 & x_3 & 1 \end{bmatrix} = \begin{bmatrix} a_1 & b & a_3 \end{bmatrix} = B_2$$

$$(\det A)(x_2) = \det B_2$$

$$\boxed{x_2 = \frac{\det B_2}{\det A}}$$

etc

(I think this is what we did at end of recitation)

Example

$$3x_1 + 4x_2 = 2$$
$$5x_1 + 6x_2 = 4$$

$$\underset{A}{\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}} \underset{x}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} = \underset{b}{\begin{bmatrix} 2 \\ 4 \end{bmatrix}}$$

$$A \quad x \quad = \quad b$$

Goal: Solve for x_1, ~~&~~ x_2

(5)

$\det A = \begin{vmatrix} 3 & 4 \\ 5 & 6 \end{vmatrix} = 18 - 20 = -2$

$\det B_1 = \begin{vmatrix} 2 & 4 \\ 4 & 6 \end{vmatrix} = 12 - 16 = -4$

ahh I see
That fancy $\begin{matrix} x_1 & 0 \\ x_2 & 1 \end{matrix}$ thing is just to produce this B_1, B_2 thing...
Basically just put B in for A col by col and find det

$\det B_2 = \begin{vmatrix} 3 & 2 \\ 5 & 4 \end{vmatrix} = 12 - 10 = 2$

So $x_1 = \frac{-4}{-2} = 2 \qquad x_2 = \frac{2}{-2} = -1$

Then verify $\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

6

To find A^{-1}

$$(A^{-1})_{ij} = \frac{C_{ji}}{\det(A)}$$ ← note ji

$$A^{-1} = \frac{C^T}{\det A}$$ ← remember cross out row, col compute det, then sign quilt

---

Area of triangle

half of a 3 by 3 det

Parallelogram area

is a 2 by 2 det

Cross product

result is a vector

$U \times V$

2 by 2 cofactors

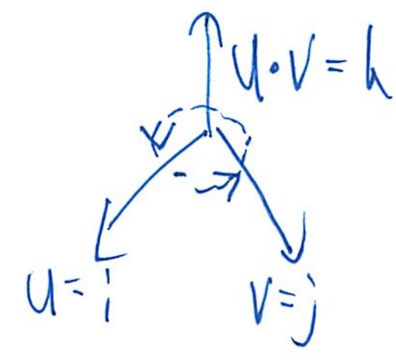
7

$$u \times v = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= (u_2 v_3 - u_3 v_2)i + (u_3 v_1 - u_1 v_3)j + (u_1 v_2 - u_2 v_1)k$$

Vector is $\perp$ to u and v

This is this right hand rule



lenght $\|u\| \|v\| |\sin\theta|$

Triple product

add a scalar

$$(u \cdot v) \cdot w = \begin{vmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

18.06 OH

4/4

Row Ops

Adding a multiple of one row to another

elim = a specific order or row ops

Not just U - That is only 1 case

Instead a diff obvious
- always works
- but may be hard

He sees 2 steps

So use $t\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} ta & tb \\ c & d \end{vmatrix}$

$$\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$$

Holds for any row

since could switch

Look at rules 4 and 6

②

Modify matrix to meet those cases

So from 2nd row subtract 1st
3rd 1st

$$\begin{bmatrix} 101 & 201 & 301 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}$$

Subtract 3rd row from 2nd

$$\begin{bmatrix} 101 & 201 & 301 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

2 row = so det A ~~...~~ $= 0$

Don't know why wrong on Matlab

3

b) $\det \begin{bmatrix} 1 & t & t^2 \\ t & 1 & t \\ t^2 & t & 1 \end{bmatrix}$

(TA was having trouble pattern finding)

Just do systematic elim

And get formula

And simplify / factor

18. Just elim again

18.06 Spring 2012 – Problem Set 6

This problem set is due Thursday, April 5th, 2012 at 4pm (hand in to Room 2-106). The textbook problems are out of the 4th edition. For computational problems, please include a printout of the code with the problem set (for MATLAB in particular, `diary('filename')` will start a transcript session, `diary off` will end one.)

Every problem is worth 10 points.

1. Do Problems 9 & 15 from Section 5.1.
2. Do Problems 18 & 22 from Section 5.1.
3. Do Problems 8 & 9 from Section 5.2.
4. Do Problem 20 from Section 5.2.
5. Do Problem 29 from Section 5.2.
6. Do Problem 34 from Section 5.2.
7. Do Problems 4 & 8 from Section 5.3.
8. Do Problems 20 & 25 from Section 5.3.
9. Do Problem 1 from Section 6.1.
10. Use MATLAB to "prove" all the facts you remember (or may not remember?) about determinants. First define the following matrices to test on (copy paste into MATLAB - retyping is time- and patience-consuming):

```
%Two random 4 x 4 matrices:
A = rand(4,4);
B = rand(4,4);

%An elementary subtraction of rows (by left-multiplying. Of columns if you right-)
E = [1 -3  0  0;
     0  1  0  0;
     0  0  1  0;
     0  0  0  1];

%An "odd" permutation:
P_odd = [1  0  0  0;
         0  1  0  0;
         0  0  0  1;
         0  0  1  0];

%An "even" permutation:
P_even = [0  1  0  0;
```

```
          1  0  0  0;
          0  0  0  1;
          0  0  1  0];

%Another 4 x 4...almost, the 1st row is missing:
C = rand(3,4);

%Two random row vectors
a1 = rand(1,4);
a2 = rand (1,4);

%Two matrices having the a_1, a_2 as 1st rows
D1 = [a1;
      C ]

D2 = [a2;
      C ]
%Matrix with sum as 1st row
D = [a1 + a2;
        C  ]
```

Now, using these matrices do the following tests. We've slipped in a couple of *false* ones - to make it more exciting (take a guess before you hit < `enter` >).

(a) `det(D1)` + `det(D2)` = `det(D)`

(b) `det(A)` + `det(B)` = `det(A + B)`

(c) `det(10 * A)` = `10 * det(A)`

(d) `det(E * A)` = `det(A)` = `det(A * E)`

(e) `det(P_odd * A)` = `-det(A)`
`det(P_even * A)` = `det(A)`

Which ones in (a)-(e) are correct, and which are false?

18.06 Wisdom. Enjoy your spring break!

②

5.1 #15. Use row ops to simplify and compute det

; So elim method ;

$$\det\begin{bmatrix} 101 & 201 & 301 \\ 102 & 202 & 302 \\ 103 & 203 & 303 \end{bmatrix}$$

$l = \frac{102}{101}$

$$\begin{bmatrix} 101 & 201 & 301 \\ 0 & 202-\frac{102}{101}\cdot 201 & 302-\frac{102}{101}\cdot 301 \\ 0 & 203-\frac{103}{101}\cdot 201 & 303-\frac{103}{101}\cdot 301 \end{bmatrix}$$

$102 - \frac{102}{101}\cdot 101$

$l = \frac{103}{101}$

$$\begin{bmatrix} 101 & 201 & 301 \\ 0 & -\frac{100}{101} & -\frac{200}{101} \\ 0 & 0 & 15.684 \end{bmatrix}$$

$$l = \frac{203 - \frac{103}{101}\cdot 201}{202 - \frac{102}{101}\cdot 201} = 2$$

$$101 \cdot -\frac{100}{101} \cdot 15.684 = -1568$$

Wolfram Alpha: 40,000

- must have made an algebra error somewhere

Book says 0 - say what!

Michael Plasmeier   P-Set 6   85/100   9/3

5.1 #9 Do these matrices have det 0, 1, 2, or 3?

10/10

? Should we just calculate?

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

2 row swaps +1

Then multiply pivots

$$B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Can't easily do pivots
But can do cofactors

det = 2

$$C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$? Can use the big formula?

det = 0

②

5.1 #15 Use row ops to simplify + compute

det

can be elim

or can be to a certain trick

$$\det \begin{bmatrix} 101 & 201 & 301 \\ 102 & 202 & 302 \\ 103 & 203 & 303 \end{bmatrix}$$

So from 2nd row subtract 1st
3rd row " 1st

$$\begin{bmatrix} 101 & 201 & 301 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}$$

Subtract 3rd row from 2nd

$$\begin{bmatrix} 101 & 201 & 301 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

2 rows = so det A = 0

(3)

$$\det \begin{bmatrix} 1 & t & t^2 \\ t & 1 & t \\ t^2 & t & 1 \end{bmatrix}$$

Or are we supposed to cale 3
L don't see how it applies

?? How are we being asked to solve this?
? elim

$l_{21} = t \qquad l_{31} = t^2$

$$\begin{bmatrix} 1 & t & t^2 \\ 0 & 1-t & t-t^3 \\ 0 & t-t^2 & -t^4 \end{bmatrix} \qquad l_{32} = \frac{t-t^2}{1-t^2} = \frac{t}{t+1}$$

$$\begin{bmatrix} 1 & t & t^2 \\ 0 & 1-t & t-t^3 \\ 0 & 0 & t^2(-t+t-1) \end{bmatrix}$$

$$-t^4 - \frac{t}{t+1}(t-t^3) = t^2(-t^2+t-1)$$

$$1 \cdot (1-t)(t^2(-t^2+t-1))$$
$$= t^5 - 2t^4 + t^2$$

Book: $1-2t^2+t^4$
$= (1-t^2)^2$

must have done something wrong

(4)

5.1 # 18 Use row ops to show 3x3 Vandermode det is

8/10

$$\det\begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix} = (b-a)(c-a)(c-b)$$

This isn't elim again is it?
All ans should come at the same ...

$l_{21} = 1 \quad l_{31} = 1$

$$\begin{bmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a \\ 0 & c-a & c^2-a \end{bmatrix} \qquad l_{32} = \frac{c-a}{b-a}$$

$$\begin{bmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a \\ 0 & 0 & \end{bmatrix}$$

$$? \ (c^2-a) \cdot \frac{c-a}{b-a}(b^2-a)$$

$$= \frac{(b^2-a)(c-a)(c^2-a)}{(b-a)}$$

$$1 \cdot \cancel{(b-a)} \cdot \frac{(b^2-a)(c-a)(c^2-a)}{\cancel{(b-a)}}$$

(5)

Do row-by-row factoring

$$= (b-a)(c-a) \det \begin{bmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 1 & c+b \end{bmatrix}$$

Use 1st col as cofactor col

$$= (b-a)(c-a)(+1) \det \begin{bmatrix} 1 & b+a \\ 1 & c+a \end{bmatrix}$$

$$= (b-a)(c-a)(c+a-b-a)$$

$$= (b-a)(c-a)(c-b)$$

I must be screwing stuff up since it seems diff ways = diff ans

I think dividing by b-a is a bad idea, since we could have a=b.

6

5.1 # 22 From $ad - bc$ find the det of A and $A - I$ and $A - \lambda I$

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\det = 4 - 1 = 3$$

$$A' = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$\det = \frac{1}{\det A} = \frac{1}{3}$$

or $\begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$ ← apply to everything

$$\frac{2}{3} \cdot \frac{2}{3} - \left(-\frac{1}{3} \cdot -\frac{1}{3}\right)$$

$$\frac{4}{9} - \frac{1}{9} = 1/3$$

$$A - \lambda I = \begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix}$$

7

$(2-\lambda)(2-\lambda) - (1)(1)$

$4 - 4\lambda + \lambda^2 - 1$

$\lambda^2 - 4\lambda + 3$

Why didn't you finish your calculations? :(

-2

8

5.2 #8 If det A is not 0, at least one of

8/10

the n! terms in formula 8 is not 0

Deduce from the big formula that some ordering of the rows of A leaves no zeros on the diagonal.

Formula 8: The big formula

$\sum (\det P)\, a_{1\alpha}\, a_{2\beta} \cdots a_{n\omega}$

Sum over all n! col permutations

Not 0 means invertable

~~All the rows are ind~~

There are non 0 pivots

One of the n! Permutations is I

This will be non zero result. 0

(9)

5.2 #9 Show that 4 is the largest determinant for a 3×3 matrix of 1s and -1s

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

? 5+ and 4 −

But that gets proccessed

Its the cofactors ← (is that the best way to look at it)

So 3! = 6 permutations

↳ or 6 terms

Last 3 are odd (1 ex) (+)

First 3 even (0 or 2 ex) (−)

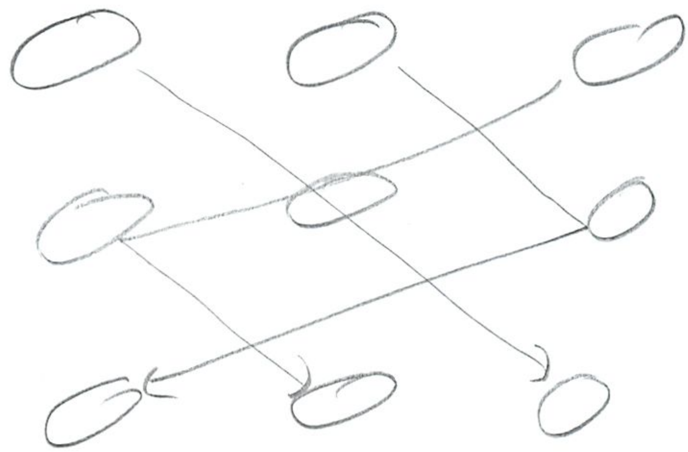
So for max all the even are (−)

odd are (+)

I don't get book sol either

7b

w/ friends!

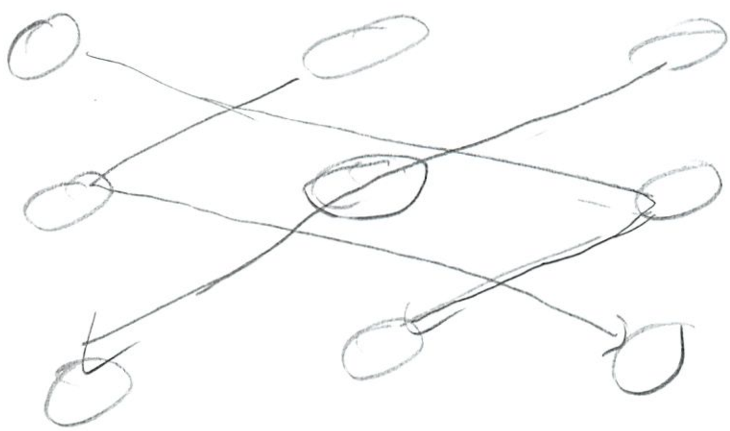


even

need an = #
of -1 on each
ind paths

need even # of - ones

odd - ~~?~~ trying to get -1s



odd

There are 6 permutations valued either ⊕ or ⊖
In order for the determinant to be > 4
each of those permutations must yield a +1.

(9c).

Assume for the sake of contridiction that this is true.
Look at the even and odd permutations seperatly.

Even	Odd
All permutations are ind	All permutations are ind

Need even # of -1s to get $+1$ in all even permutations
odd # of -1 to get $+1$s odd

But Can't have it both ways.
Best you can do is 5, 1
L can do example
if I diagonal is 1s, rest is -1
Then get
$5-1=4$

Can't have even # and an odd # of -1 entries in matrix at the same time!

Not completely rigorous. See sol'n.

(10).

5.2 #20

10/10 Find G_2, G_3 and then by row ops G_4

Can you predict G_n?

¿What are G

It seems 0 on diag, 1s elsewhere

From ans it seems they want the det

$\det G_2 = -1$

$G_3 = 2$ ← matlab

$G_4 = -3$

Have to do some elim thing

Just look at pattern?

$(n-1) \times (-1)^{n-1}$

(1) Thats it

(11)

5.2 #29 About the Big Formula

8/10

For E_4 in #15, 5 of the $4! = 25$ terms are non 0. Find those 5 terms to show $E_4 = -1$

$$E_4 = \begin{vmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{vmatrix}$$

I actually do

I guess you can see

$$\begin{vmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{vmatrix} \begin{vmatrix} & 1 & & \\ 1 & & & \\ & & 1 & \\ & & & 1 \end{vmatrix} \begin{vmatrix} 1 & & & \\ & & 1 & \\ & 1 & & \\ & & & 1 \end{vmatrix}$$

$$\begin{vmatrix} 1 & & & \\ & 1 & & \\ & & & 1 \\ & & 1 & \end{vmatrix} \begin{vmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{vmatrix} \begin{vmatrix} & 1 & & \\ 1 & & & \\ & & 1 & \\ & & & 1 \end{vmatrix}$$

There are 6 permutations here?

(12)

5.2 #34 This problem shows in 2 ways that det $A = 0$

5/10

$$A = \begin{bmatrix} x & x & x & x & x \\ x & x & x & x & x \\ 0 & 0 & 0 & x & x \\ 0 & 0 & 0 & x & x \\ 0 & 0 & 0 & x & x \end{bmatrix} \quad x = \text{any value}$$

Not that = rows - x can be any value

Bottom 3 rows are dep since the last 3 rows are all in the same plane

? a 3 vectors in 4th, 5th plane

basically 2 variables 3 equations

Part (b)?

(13)

5.3 #4 Quick proof of Cramer's Rule

10/10

Det is a linear combo of col 1.

It is 0 if 2 cols are =

When $b = Ax = x_1 a_1 + x_2 a_2 + x_3 a_3$

goes into 1st col of A, the det of B_1 is

$$|b \ a_2 \ a_3| = |x_1 a_1 + x_2 a_2 + x_3 a_3 \quad a_2 \ a_3|$$
$$= x_1 |a_1 \ a_2 \ a_3|$$
$$= x_1 \det A$$

a) What formula from x_1 froms from left = right?

It is $x_1 = \dfrac{\det B_1}{\det A}$

✓ Same as back of book

(14)

b) What step leads to middle eq?

There's 4!

So we are going

$$|x_1 a_1 + x_2 a_2 + x_3 a_3 \quad a_2 \quad a_3| \to x_1 |a_1 \ a_2 \ a_3|$$

Linear det

So is that

$$|x_1 a_1 \quad a_2 \quad a_3| +$$

$$|x_2 a_2 \quad a_2 \quad a_3| +$$

$$|x_3 a_3 \quad a_2 \quad a_3|$$

Factor out c, is this legal

$$x_1 |a_1 \ a_2 \ a_3| +$$

$$x_2 |a_2 \ a_2 \ a_3| +$$

$$x_3 |a_3 \ a_2 \ a_3| \quad 0$$

Q5)

$x_2 x_2$ and x_3 a_3

Then repeated cols, so

$$x_1 \, |a_1 \; a_2 \; a_3|$$

which is

$$x_1 \det A$$

← ??

??Since we can transpose??

repeated cols ⇒ lin. dep cols.

✓

(16)

5.3 #8 Find the cofactors of A and multiply AC^T to find det A

$$A = \begin{bmatrix} 1 & 1 & 4 \\ 1 & 2 & 2 \\ 1 & 2 & 5 \end{bmatrix}$$

cofactor is cross out row col and find det of that

$$\begin{bmatrix} 6 & -3 & 0 \\ +3 & 1 & -1 \\ -6 & +2 & 1 \end{bmatrix}$$

remember sign quilt is included in cofactor

$$A^{-1} = \frac{C^T}{\det A}$$

So $\det A A^{-1} = C^T$

$$\det A = \frac{C^T}{A^{-1}}$$

$$= AC^T$$

matlab

$$\begin{bmatrix} 1 & 1 & 4 \\ 1 & 2 & 2 \\ 1 & 2 & 5 \end{bmatrix} \begin{bmatrix} 6 & 3 & -6 \\ -3 & 1 & 2 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = 3I$$

(17).

So this is $(\det A) I$ so $\det A = I$

If change 4 to 100 $\to$ det A unchanged

since cofactor is 0

18

5.3 # 20 Hadaman matrix H

10/10

Orthogonal rows
box is a hyper cube

$$|H| = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{vmatrix}$$

= Volume of hypercube in $\mathbb{R}^4$

Find the vol = find the det

L use matlab

det (H) = 16

↑ side lenght 2^4

S^n

(19)

5.3 # 25 An n-dimensional cube has how many corners?

2^n (✓)

Edges? $n \cdot 2^{n-1}$ (✓)

? each corner has n edge out
but each corner shares a corner

(n-1) dimension faces?

n faces meet at a corner

how many corners share a face? 2^{n-1}

So $2n$ faces (✓)

Since front + back to every dimension

The cube in $\mathbb{R}^n$ whose edges are the rows of $2I$ has volume ____.

2^n

? $2I$ matrix is n 2s

cube would be 2, 2, 2

Hypercube computers have parallel processors at corners w/ connections along the edges.

(20)

6.1 #1 Example at start of chap

$$A = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} .70 & .45 \\ .30 & .55 \end{bmatrix}$$

$$A^\infty = \begin{bmatrix} .6 & .6 \\ .4 & .4 \end{bmatrix}$$

Find eigenvalues.

$$\begin{bmatrix} .8-\lambda & .3 \\ .8 & .7-\lambda \end{bmatrix}$$

$$\lambda^2 - 1.5\lambda + .66 - .06 = 0$$

$$\lambda^2 - \frac{3}{2} + \frac{1}{2} = 0$$

Quadratic formula

$$\lambda = \frac{1.5 \pm \sqrt{(3/2)^2 - 4/2}}{2}$$

$$= 1, .5$$

Then $\lambda^2 = 1$ or $.25$

$$\lambda^\infty = 1 \text{ or } 0$$

(21)

a) Show from A how a cow ex can produce diff eigenvales

$$B = \begin{bmatrix} .2 & .7 \\ .8 & .3 \end{bmatrix}$$

$\det(B\lambda - I) = 0$

$\lambda^2 - .5\lambda + .06 - .56 = 0$

$\lambda^2 - \frac{1}{2}\lambda - \frac{1}{2} = 0$

$\lambda = 1$ or $-.5$

? diff since ⊖

b) Why is 0 eigenvalue changed by elim

$\lambda = 0 \rightarrow$ singular matrix

elim singular matrix is 0

So singular

(22)

10. Use MatLab to prove our facts

6/10 rref = elimination (and then some)

a) false x

b) false

c) false

d) false x

e) true, true

```
>> A = rand(4,4);
>> B = rand(4,4);
>> E = [1 -3 0 0;
0 1 0 0;
0 0 1 0;
0 0 0 1];
>> P_odd = [1 0 0 0;
0 1 0 0;
0 0 0 1;
0 0 1 0];
>> P_even = [0 1 0 0;
11 0 0 0;
0 0 0 1;
0 0 1 0];
>> P_even = [0 1 0 0;
1 0 0 0;
0 0 0 1;
0 0 1 0];
>> C = rand(3,4);
>> a1 = rand(1,4);
>> a2 = rand (1,4);
>> D1 = [a1;
C ]

D1 =

    0.1869    0.4898    0.4456    0.6463
    0.2769    0.8235    0.9502    0.3816
    0.0462    0.6948    0.0344    0.7655
    0.0971    0.3171    0.4387    0.7952

>> D2 = [a2;
C ]

D2 =

    0.7094    0.7547    0.2760    0.6797
    0.2769    0.8235    0.9502    0.3816
    0.0462    0.6948    0.0344    0.7655
    0.0971    0.3171    0.4387    0.7952

>> D = [a1 + a2;
C ]

D =

    0.8962    1.2445    0.7216    1.3260
    0.2769    0.8235    0.9502    0.3816
    0.0462    0.6948    0.0344    0.7655
    0.0971    0.3171    0.4387    0.7952
```

```
>> det(D1) + det(D2) == det(D)

ans =

     0

>> 1 == 1

ans =

     1

>> 1 == 2

ans =

     0

>> det(A) + det(B) == det(A + B)

ans =

     0

>> det(10*A) == 10*det(A)

ans =

     0

>> det(E*A) == det(A)

ans =

     0

>> det(A) == det(A*E)

ans =

     0

>> det(P_odd* A) == det(A)

ans =

     0

>> det(P_odd* A) == -det(A)

ans =
```

```
     1

>> det(P_even*A) == det(A)

ans =

     1
```

18.06 Spring 2012 – Problem Set 6

This problem set is due Thursday, April 5th, 2012 at 4pm (hand in to Room 2-106). The textbook problems are out of the 4th edition. For computational problems, please include a printout of the code with the problem set (for MATLAB in particular, `diary('filename')` will start a transcript session, `diary off` will end one.)

Every problem is worth 10 points.

1. Do Problems 9 & 15 from Section 5.1.

   Solution. **Problem 9**

   $\det(A) = 1$: exchange row 1 and row 3, and then row 1 and row 2.

   $\det(B) = 2$: subtract rows 1 and 2 from row 3 then columns 1 and 2 from column 3.

   $\det(C) = 0$: the rows are equal. (Note that $C = A + B$, but $\det(C) \neq \det(A) + \det(B)$.)

   Problem 15

   The first determinant is zero: subtract row 2 from row 3, and row 1 from row 2, to get a matrix with two equal rows.

   The second determinant is $(1 - t^2)^2 = 1 - 2t^2 + t^4$: subtract t times row 2 from row 1, and t times row 3 from row 2, to get a lower-triangular matrix. □

2. Do Problems 18 & 22 from Section 5.1.

   Solution. **Problem 18**

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix} = \begin{vmatrix} b-a & b^2-a^2 \\ c-a & c^2-a^2 \end{vmatrix}$$

   where to reach the 2×2 determinant, we eliminate a and a^2 in row 1 by column operations. Now factor out $b - a$ and $c - a$ from the 2×2 determinant:

$$(b-a)(c-a)\begin{vmatrix} 1 & b+a \\ 1 & c+a \end{vmatrix} = (b-a)(c-a)(c-b).$$

   Problem 22

   $\det(A) = 3$, $\det(A^{-1}) = 1/3$, $\det(A - \lambda I) = \lambda^2 - 4\lambda + 3 = (\lambda - 1)(\lambda - 3)$. The numbers $\lambda = 1$ and $\lambda = 3$ give $\det(A - \lambda I) = 0$. □

3. Do Problems 8 & 9 from Section 5.2.

Solution. **Problem 8**

Some term $a_{1i_1}a_{2i_2}\dots a_{ni_n}$ in the big formula is not zero. Move rows $1, 2, \dots, n$ into rows $i_1, i_2, \dots, i_n$. Then these non-zero a's will be on the main diagonal.

Problem 9

There are 6 terms in the big formula, all ± 1. Thus, the determinant must be an *even* integer.

To get $+1$ for all the three product terms corresponding to the even permutations, the matrix needs to have an *even* total number of -1 entries (this is easy to see in this 3×3 situation, where $3 \cdot 3!/2 = 3^2$ happens to hold, so each matrix entry shows up exactly once somewhere in the 3 product terms coming from the even permutations). On the other hand, to also get $+1$ for all the product terms corresponding to the odd permutations, the matrix would need to have an *odd* total number of -1 entries.

Thus at least one term in the big formula must be a -1, and the maximal determinant is $+4$. Namely, this is attained for example for the matrix:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}.$$

$\square$

4. Do Problem 20 from Section 5.2.

   Solution. $G_2 = -1$, $G_3 = 2$, and $G_4 = -3$. We guess that $G_n = (-1)^{n-1}(n-1)$. $\square$

5. Do Problem 29 from Section 5.2.

   Solution. There are five non-zero products, all ± 1. Here are the $(row, column)$ coordinates of the terms, and the signs:

$$\begin{array}{cl} + & (1,1)(2,2)(3,3)(4,4) \\ + & (1,2)(2,1)(3,4)(4,3) \\ - & (1,2)(2,1)(3,3)(4,4) \\ - & (1,1)(2,2)(3,4)(4,3) \\ - & (1,1)(2,3)(3,2)(4,4) \end{array}$$

   Total: -1. $\square$

6. Do Problem 34 from Section 5.2.

   Solution. (a) Consider the 3 by 5 matrix formed by the last three rows of A. It has only two non-zero columns, and so it has rank at most 2. Therefore, the last three rows of A are linearly dependent.

(b) Consider a term in the big formula for $\det A$; it is a product of entries of A, one entry in each row and column. Consider the entries coming from the last three rows of A; there are three of them, and at most two can be in the last two columns of A. Therefore, at least one entry falls in the 3 by 3 block of zeroes, and so the whole term of the big formula is 0.

□

7. Do Problems 4 & 8 from Section 5.3.

Solution. **Problem 4**

(a) We get the familiar formula $x_1 = |\mathbf{b}\ \mathbf{a}_2\ \mathbf{a}_3| / \det(A)$

(b) We use linearity of the determinant in the first column:

$$|x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3\ \mathbf{a}_2\ \mathbf{a}_3| = x_1|\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_3| + x_2|\mathbf{a}_2\ \mathbf{a}_2\ \mathbf{a}_3| + x_3|\mathbf{a}_3\ \mathbf{a}_2\ \mathbf{a}_3|$$

The last two terms are zero because two of the columns are the same.

Problem 8 The cofactor matrix is

$$C = \begin{bmatrix} 6 & -3 & 0 \\ 3 & 1 & -1 \\ -6 & 2 & 1 \end{bmatrix}$$

Then $AC^T = 3I$ so $\det(A) = 3$. If you change that 4 to 100, $\det(A)$ is unchanged because the corresponding cofactor is 0.

□

8. Do Problems 20 & 25 from Section 5.3.

Solution. **Problem 20** The rows of the Hadamard matrix generate a 4-dimensional hypercube with side length 2. Therefore $\det(H) = \pm 16$. It turns out that the rows of H are in such an order that in fact $\det(H) = 16$.

Problem 25 An n dimensional cube has 2^n corners, $n\,2^{n-1}$ edges and $2n$ $(n-1)$-dimensional faces. The cube in $\mathbb{R}^n$ generated by the rows of $2I$ has volume 2^n.

□

9. Do Problem 1 from Section 6.1.

Solution. The eigenvalues of A are 1 and 1/2; the eigenvalues of A^2 are 1 and 1/4; the eigenvalues of A^∞ are 1 and 0.

(a) If we swap the rows of A, the resulting matrix has eigenvalues 1 and -1/2

(b) A matrix has a zero eigenvalue if and only if it has a non-trivial nullspace. The nullspace of a matrix is not changed by the steps of elimination; therefore, a zero eigenvalue is not changed by the steps of elimination.

□

10. Use MATLAB to "prove" all the facts you remember (or may not remember?) about determinants. First define the following matrices to test on (copy paste into MATLAB - retyping is time- and patience-consuming):

```
%Two random 4 x 4 matrices:
A = rand(4,4);
B = rand(4,4);

%An elementary subtraction of rows (by left-multiplying. Of columns if you right-)
E = [1 -3  0  0;
     0  1  0  0;
     0  0  1  0;
     0  0  0  1];

%An "odd" permutation:
P_odd = [1  0  0  0;
         0  1  0  0;
         0  0  0  1;
         0  0  1  0];

%An "even" permutation:
P_even = [0  1  0  0;
          1  0  0  0;
          0  0  0  1;
          0  0  1  0];

%Another 4 x 4...almost, the 1st row is missing:
C = rand(3,4);

%Two random row vectors
a1 = rand(1,4);
a2 = rand (1,4);

%Two matrices having the a_1, a_2 as 1st rows
D1 = [a1;
      C ]

D2 = [a2;
      C ]
%Matrix with sum as 1st row
D = [a1 + a2;
        C  ]
```

Now, using these matrices do the following tests. We've slipped in a couple of *false* ones - to make it more exciting (take a guess before you hit $<$ `enter` $>$).

(a) $\texttt{det(D1)} + \texttt{det(D2)} = \texttt{det(D)}$

(b) $\texttt{det(A)} + \texttt{det(B)} = \texttt{det(A + B)}$

(c) $\texttt{det(10} * \texttt{A)} = \texttt{10} * \texttt{det(A)}$

(d) $\texttt{det(E} * \texttt{A)} = \texttt{det(A)} = \texttt{det(A} * \texttt{E)}$

(e) $\texttt{det(P\_odd} * \texttt{A)} = -\texttt{det(A)}$
$\texttt{det(P\_even} * \texttt{A)} = \texttt{det(A)}$

Which ones in (a)-(e) are correct, and which are false?

Solution. (a) True

(b) False

(c) False

(d) True

(e) True

□

18.06 Wisdom. Enjoyed your spring break? True!

18.06 Recitation

4/3

Exam Review Session next week

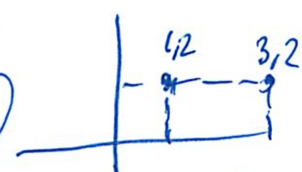
Determinants

Only take det for square matrices $n \times n$

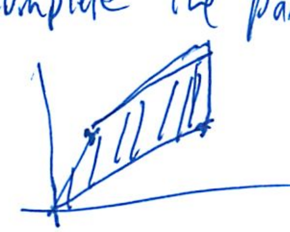
Properties

Determinates are signed volumes of the box made by the columns of the matrix

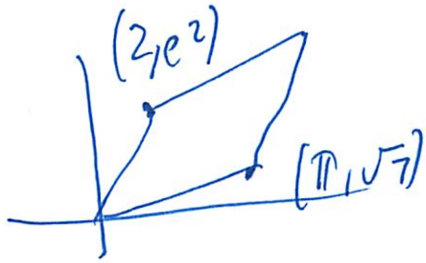
eg $\det\begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix} = \pm$ Volume of



complete the paralelogram



Backwards Given parallelogram
Want to compute the area



Write as matrix, take det

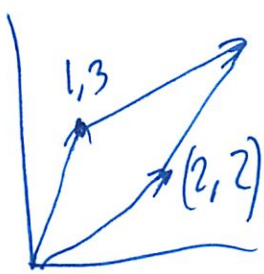
$$\text{abs}\left(\det\left(\begin{bmatrix} \pi & 2 \\ \sqrt{7} & e^2 \end{bmatrix}\right)\right)$$

②

$= |\pi e^2 - 2\sqrt{7}|$

$\det A = \det A^T$

- hard to prove w/ formula
- but clear w/ picture
  - just a rotation of parallelogram

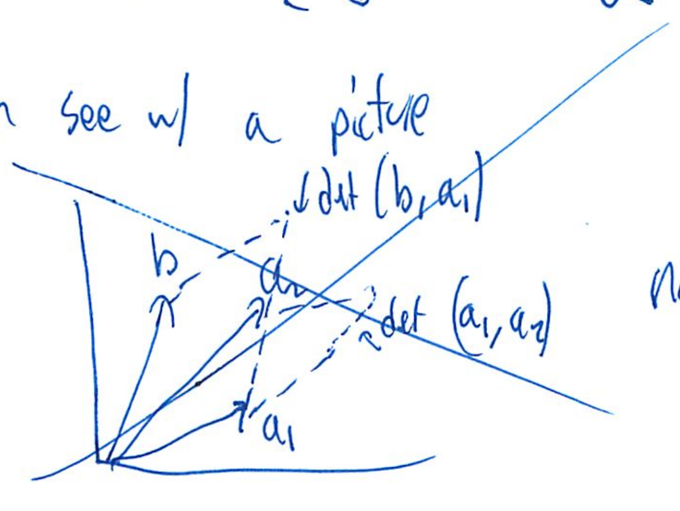


- det are linear in cols and rows

$$\det\begin{bmatrix} xa_1 + yb & a_2 & \cdots & a_n \end{bmatrix}$$

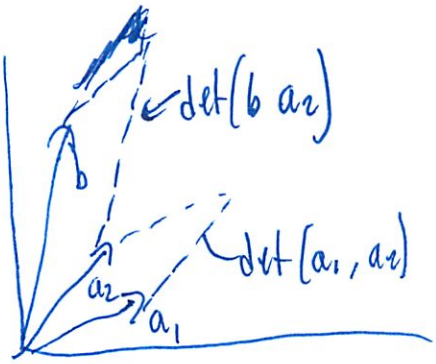
$$= x\det\begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix} + y\det\begin{bmatrix} b & a_2 & \cdots & a_n \end{bmatrix}$$

- can see w/ a picture



not this picture!

③



If you line them up along a_2

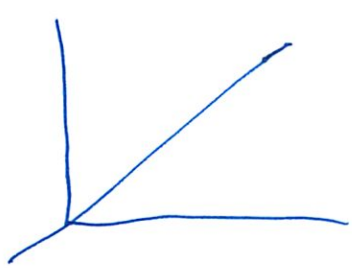
the parallelagrams

Can see that adding should give you

? ① heard wrong?

If A is not invertable

det A = 0



Det AB = det A ∘ det B

(4)

$, QQ^T = I$

If Q is orthogonal, $\det Q =$

$\det Q \cdot \det Q^T = \det QQ^T = \det I = 1$

$= (\det Q)^2$

So $\det Q = \pm 1$

$$\det \begin{pmatrix} x_1 & & * \; * \; * \; * \\ & \ddots & \\ 0 & & x_n \end{pmatrix} \leftarrow \text{anything}$$

$\uparrow 0$

$= x_1 \cdots x_n$

So might be eaiser to get matrix like this (U) w/ Gausian elim - to get the det

(5)

Ex $A = \begin{bmatrix} 2 & 6 & 2 \\ 1 & 4 & 2 \\ 5 & 4 & 0 \end{bmatrix}$

a) Compute using the "big formula"

- involves permutation
- write all 3x3 permutation matrices

$$3! = 3 \cdot 2 \cdot 1 = 6$$

$$\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & & 1 \\ & 1 & \end{bmatrix} \begin{bmatrix} & 1 & \\ 1 & & \\ & & 1 \end{bmatrix}$$

$$\begin{bmatrix} & 1 & \\ & & 1 \\ 1 & & \end{bmatrix} \begin{bmatrix} & & 1 \\ 1 & & \\ & 1 & \end{bmatrix} \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix}$$

Compute the signs

- how many rows swapped to I
- odd $\ominus$
- even $\oplus$

(e)

Respectfully $0 \oplus$ $1 \ominus$ $1 \ominus$

$2 \oplus$ $2 \oplus$ $1 \ominus$

Note can only swap adj $\quad$ 1 2 3 $\quad$ 1 2 3 4
- first and last are adj
- but not cons 2,4 when 4 cons

Now compute rest of the terms

$(2)(4)(0)$ $\quad$ $(2)(2)(9)$ $\quad$ $(6)(1)(0)$

$(6)(2)(5)$ $\quad$ $(2)(1)(9)$ $\quad$ $(5)(4)(2)$

Multiply them all together

$+0$ $\quad$ -36 $\quad$ -0

$+60$ $\quad$ $+18$ $\quad$ -40

Sum $= 2$

(7)

Trick

$$\begin{array}{ccccc} \multicolumn{3}{c}{\text{write}} & \multicolumn{2}{c}{\text{Copy 1st 2 col}} \\ 2 & 6 & 2 & 2 & 6 \\ 1 & 4 & 2 & 1 & 4 \\ 5 & 9 & 0 & 5 & 9 \\ \ominus & \ominus\ \ominus & \oplus & \oplus & \oplus \end{array}$$

Gets to complicated on 4x4

Cramer's Rule

b) Solve $Ax = \begin{pmatrix}1\\8\\0\end{pmatrix}$ using Cramers Rule

replace 1st col of A w/ b

$$x_1 = \frac{\det\begin{bmatrix}1&6&2\\8&4&2\\0&9&0\end{bmatrix}}{\det A} = -9$$

replace 2nd col of A w/ b

$$x_2 = \frac{\det\begin{bmatrix}2&1&2\\1&8&2\\5&0&0\end{bmatrix}}{\det A} = -5$$

$$x_3 = \frac{\det\begin{bmatrix} & & \end{bmatrix}}{\det A} = -\frac{11}{2}$$

⑧

c) What is the cofactor matrix C?

$$C = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

↑ 3x3 when matrix is 3x3

$$C_{ij} = (-1)^{i+j} M_{ij}$$

↑ Minors

delete first row and col

↑ minor

$$C = \begin{bmatrix} \det() & -\det() & \det() \\ -\det() & \det() & -\det() \\ \det() & -\det() & \det() \end{bmatrix}$$

← 9 diff det

↑ Signs make a checkerboard

⑨

$$= \begin{bmatrix} \det\begin{pmatrix} 4 & 2 \\ 9 & 0 \end{pmatrix} & -\det\begin{pmatrix} 1 & 2 \\ 5 & 0 \end{pmatrix} & \det\begin{pmatrix} 1 & 4 \\ 5 & 9 \end{pmatrix} \\ -\det\begin{pmatrix} 6 & 2 \\ 9 & 0 \end{pmatrix} & \det\begin{pmatrix} 2 & 2 \\ 5 & 0 \end{pmatrix} & -\det\begin{pmatrix} 2 & 6 \\ 5 & 9 \end{pmatrix} \\ \det\begin{pmatrix} 6 & 2 \\ 4 & 2 \end{pmatrix} & -\det\begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix} & \det\begin{pmatrix} 2 & 6 \\ 1 & 4 \end{pmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} -18 & 10 & -11 \\ 18 & -10 & 12 \\ 4 & -2 & 2 \end{bmatrix}$$

purpose of C is to find inverse quicker

alt to elim

d) What is A^{-1}?

$$A^{-1} = \frac{C^T}{\det A} = \underset{\det A}{\frac{1}{2}} C^T$$

e) What is the vol of the box whose side are the cols of A? 2

f) cons - no change

Can find det w/ minors - won't cover in recitation

(10)

Purposes of Eigenvalues

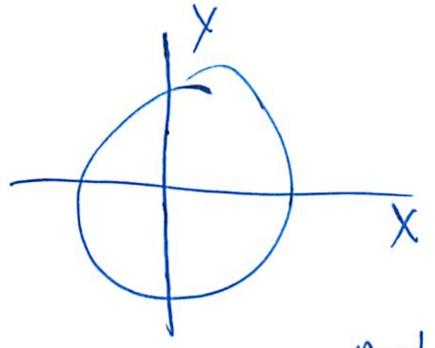
Def λ is an eigenvalue of A if there is some non-zero vector x such that

$$Ax = \lambda x$$ ← eigenvector

↑ eigen value

X can't be 0

Unit Circle



$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

matrix sends points on the circle somewhere else

What is the image of the unit circle under the function A?

(11)

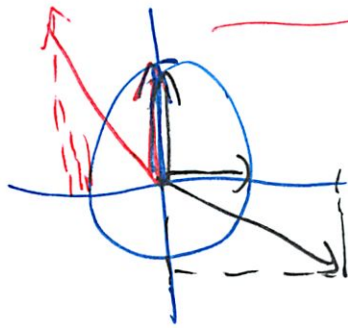
$A: \mathbb{R}^2 \to \mathbb{R}^2$

$x \mapsto Ax$

Take standard bases

$\boxed{A\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}2\\-1\end{pmatrix}}$

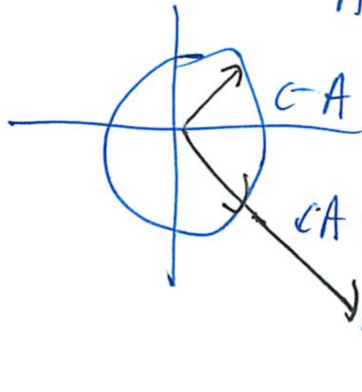
$\boxed{A\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}-1\\2\end{pmatrix}}$



Eigenvectors of A $\begin{pmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{pmatrix}$ $\lambda_1 = 1$

are ~~some~~ better bases to use $\begin{pmatrix}1/\sqrt{2}\\-1/\sqrt{2}\end{pmatrix}$ $\lambda_2 = 3$

$Ax_1 = \lambda_1 x_1 = x_1$

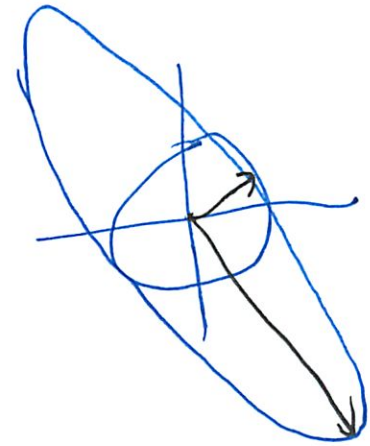


← A does not change this vector

← A stretches vector by a factor of 3

$Ax_2 = \lambda_2 x_2 = 3x_2$

12



Don't have to be orthonormal

$A x_k = \lambda_k x_k \quad k = 1, \ldots, n$

n ind. e.vectors x_k

n e.values – real or complex
may be repeated

In matrix form

$$\boxed{AS = S\Lambda}$$

e.vector matrix
e.value matrix

$$\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

simple to do when diagonal

key $A \begin{bmatrix} x_1 & x_2 & \cdots & x_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$

Just need to check it

(2)

$$\begin{bmatrix} Ax_1 & \cdots & Ax_n \end{bmatrix} = \begin{bmatrix} \lambda_1 x_1 & \cdots & \lambda_n x_n \end{bmatrix}$$

Cols match up

$$Ax_1 = \lambda_1 x_1$$
$$\vdots$$
$$Ax_n = \lambda_n x_n$$

$AS = S\Lambda$

L 2 ways to think about

$S^{-1}AS = \Lambda$ "diagonilze A"

$A = S \Lambda S^{-1}$ L like $A = LU$ "factorization of A"

e vector evalue

inverse of

ind rows → shows that S must be invertable

13

$y'' + by' + cy = 0$

constant coeff diff eq

remember from 18.03 (he hates the class)

look for exponential sol

$y = e^{\lambda t}$

plug it in

$(\lambda^2 + b\lambda + c) e^{\lambda t} = 0$

never 0

So quad. eq w/ 2 roots/exponents

Those give the 2 sol

$e^{\lambda_1 t}$ or

$e^{\lambda_2 t}$

Bad case

imaginary sol

but we can deal w/

(4)

Real problem is repeat

$\underline{t}\, e^{\lambda t}$

Suppose wanted 1000th power of the matrix A

exactly what evalues/evectors are perfect for!

Want A^{1000} to solve difference eq
not differential

$u_{k+1} = A u_k$
each step

Start w/ u_0 given

So solution $u_k = A^k u_0$

For differential eq $u(t) = e^{tA} u(0)$ ← analog

← digital

⑤

$$A^{1000} = (S\Lambda S^{-1})(S\Lambda S^{-1})\cdots(S\Lambda S^{-1})$$

cancles 999 times

$$= S\Lambda^{1000}S^{-1}$$

e values of A to 1000 th power

$$= S\begin{bmatrix}\lambda_1^{1000} & & \\ & \ddots & \\ & & \lambda_n^{1000}\end{bmatrix}S^{-1}$$

So must do the prep work to use this method finding S, Λ

Fib #5

$$F_{n+2} = F_{n+1} + F_n$$

$F_0 = 0$

$F_1 = 1$

$0, 1, 2, 3, 5, 8, 13, \ldots F_{1000}$

6

How do we find F_{1000}?

Diff eq was one step

This is a two step rule

Need a simple trick

$$u_n = \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix}$$

1st order system w/ vectors

$$\underbrace{\begin{bmatrix} F_{n+2} \\ F_{n+1} \end{bmatrix}}_{u_{n+1}} = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix}}_{u_n}$$

↑ turn 2nd order problem into 1st order

(7) $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

What are the powers of this matrix?

$$A^2 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

next pair of fib

$$A^3 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$$

1st job is to find eivalues / e vectors

(have not done this yet in 18.06)

$$Ax = \lambda x$$

Write as

$$(A - \lambda I)\, x = 0$$

$$\det(A - \lambda I) = 0$$

$$\det \begin{bmatrix} 1-\lambda & 1 \\ 1 & 0-\lambda \end{bmatrix} =$$

subtract $\lambda \cdot I$ (along diagonal)

8

(this looks familiar now)

$= \lambda^2 - \lambda - 1 = 0$

L is like $F_{n+2} - F_{n+1} - F_n = 0$

? fibinichis rule

Can use quadratic formula

$\lambda = \frac{+1 \pm \sqrt{1+4}}{2}$ $\qquad \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$= 1/2 \pm \sqrt{5}/2$

$= \frac{1 \pm 2.236}{2}$

$= \frac{3.236}{2}, \frac{-1.236}{2}$

$= \lambda_1 = 1.618, \lambda_2 = -.618$

How fast does ~~x~~ Fib #s grow?

Look at largest eigen value

1.618

The larger one dominates

(9)

$$A^{1000} = S \begin{bmatrix} 1.618^{1000} & \\ & \ddots \\ & & (-0.618)^{1000} \end{bmatrix} S^{-1}$$

By using S we get an exact formula

Notice biggest eigenvalue dominates

Trace $= \sum$ diagonals $\overset{\begin{bmatrix}1&1\\1&0\end{bmatrix}}{=} 1$

$= \sum \lambda = 1$ ✓

Det $-1 = \frac{-1+\sqrt{5}}{2} \cdot \frac{1+\sqrt{5}}{2}$

$= -\left(\frac{5-1}{4}\right)$

$= -1$ ✓

10

If

$u_{k+1} = A u_k$

$A^k \nearrow \infty$ Unstable

$A^k \searrow$ stable 0

<u>Theorm</u> $A^k \to 0$ as $k \to \infty$ <u>IF</u>

<u>all</u> $\lambda_i < 1$ stable

$A^k \to \infty$ as $k \to \infty$ <u>IF</u>

<u>any</u> $\lambda_i > 1$ Unstable

notice all and any

leaves a space in between <u>neutral</u>

ex Markoff Matrices

- max $\lambda = 1$
- can have some < 1

(11)

$$\frac{du}{dt} = Au \quad \text{then} \quad u(t) = e^{At} u(0)$$

Stable $|u(t)| \to 0$

$\boxed{\text{Powers}}$

NEEDs λ_i

$$\lambda = a + ib$$

$$e^{\lambda t} = e^{(a+ib)t}$$

$$= |e^{at}||e^{ibt}| = 1$$

a must be ⊖

The real part of λ

Next time: Markoff matrices

18.06 (4/6)

Last time: computed powers of a matrix

$$u_k = A^k u_0$$

$$= (S \Lambda S^{-1})^k u_0$$

↑ eigenvector matrix cols

↑ eigenvalue matrix

$$= S \Lambda^k S^{-1} u_0$$

diagonizable $=$ n ind. eigenvectors

Pieces are some multiple 1st eigenvector

$$u_k = c_1 (\lambda_1)^k x_1 \quad \leftarrow x_1 \text{ piece} +$$

$$c_2 (\lambda_2)^k x_2 \quad \leftarrow x_2 \text{ piece} +$$

$$\cdots +$$

$$c_n (\lambda_n)^k x_n$$

cs come from splitting u_0 into eigenvectors

②

know evectors $x_1, \ldots, x_n$ are a basis for $\mathbb{R}^n$ since ind.

c's from

$$u_0 = c_1 x_1 + \cdots + c_n x_n$$

$$u_0 = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = Sc$$

$$c = S^{-1} u_0$$

(I have no clue here...)

$$\underbrace{\begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}}_{S} \underbrace{\begin{bmatrix} \lambda_1^k & & \\ & \ddots & \\ & & \lambda_n^k \end{bmatrix}}_{\Lambda^k} \underbrace{\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}}_{S^{-1} u_0}$$

(3)

Last time if any $|\lambda_i| > 1$ $u_s \to \infty$

if all $|\lambda_i| < 1$ $v_s \to 0$

Today in between case

$\lambda_1 = 1$ or any λ - but not all

other $\lambda_i = < 1$

Somewhat stable case

$u_k = c_1 \lambda_1^k x_1 + \dots c_n \lambda_n^k x_n$

What happens to sol as go out in time $(k \uparrow)$

Other parts go to 0

$\lambda_1 \to 1$

So $\lim = c_1 x_1$

L steady state

L highly important

4

If $u_0 = c_1 x_1$

Then $u_1 = A u_0 = x_1$

↑ same!

$u_2 = c_1 x_1$

$u_{etc} = c_1 x_1$

↑ some multiple scaling

Markoff matrix: every col adds to 1

all entries > 0

keep total fixed - people not created / lost

But people move around b/w

"transition matrix"

$$\begin{bmatrix} u_{\text{inside MA}} \\ u_{\text{outside MA}} \end{bmatrix}_{t=0}$$

(5)

$$\begin{array}{c} \downarrow 80\% \text{ stay} \qquad \downarrow 10\% \text{ outsiders move in} \\ \begin{bmatrix} .8 & .1 \\ .2 & .9 \end{bmatrix} \begin{bmatrix} U_{\text{inside MA}} \\ U_{\text{outside MA}} \end{bmatrix}_{t=0} = \begin{bmatrix} U_{\text{inside MA}} \\ U_{\text{outside MA}} \end{bmatrix}_{t=1} \\ \uparrow 20\% \text{ leave MA} \qquad 90\% \text{ outsiders stay out} \end{array}$$

So at $t = 100$?

Find eigen value of

$$\begin{bmatrix} .8 & .1 \\ .2 & .9 \end{bmatrix}$$

$\lambda_1 = 1 \leftarrow$ by markoff rule

$\lambda_2 = .7 \leftarrow$ from finding trace

Check w/ det ✓

Or systematic

$$\det \begin{bmatrix} .8-\lambda & .1 \\ .2 & .9-\lambda \end{bmatrix}$$

$= \lambda^2 - 1.7\lambda + .7$ find roots

(6)

Find eigenvectors

$$A \qquad A - 1I \qquad A - .7I$$

$$\begin{bmatrix} .8 & .1 \\ .2 & .9 \end{bmatrix} \quad \begin{bmatrix} -.2 & .1 \\ .2 & -.1 \end{bmatrix} \quad \begin{bmatrix} .1 & .1 \\ .2 & .2 \end{bmatrix}$$

singular singular

$$x_1 = \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix} \qquad x_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

evector evector

So if we start w/ 200 ~~million~~ people in MA
100 in US

$$u_0 = \begin{bmatrix} 200 \\ 100 \end{bmatrix}$$

$$u_1 = \begin{bmatrix} .8 \cdot 200 + .1 \cdot 100 \\ .2 \cdot 200 + .9 \cdot 100 \end{bmatrix} = \begin{bmatrix} 170 \\ 130 \end{bmatrix}$$

note total still 300

u_2 = etc

$\boxed{\text{Note no births / deaths}}$

But u_{100}?

7

~~What~~ But what about ∞?

$$U_\infty = \begin{bmatrix} 100 \\ 200 \end{bmatrix}$$

eigenvector gives ans

(I think I remember this...)
from 18.03

If all 300 people start in MA

still $\begin{bmatrix} 100 \\ 200 \end{bmatrix}$, no change!

How do we know this fits our stable case

no markoff matrix, > 1
evalue

Another example

$$\begin{bmatrix} .6 & .3 & .0 \\ .4 & .3 & .1 \\ 0 & .4 & .9 \end{bmatrix}$$

kinda broke rule w/ 0
but still get
eigenvalue of 1

(8)

$$A - I = \begin{bmatrix} -.4 & .3 & 0 \\ .4 & -.7 & .1 \\ 0 & .4 & -.1 \end{bmatrix}$$

det = 0 matrix is singular

Cols dep

How can we see that?

Add cols get 0 ← must be singular

Before cols add to 1

Left nullspace

$$B^T y = \begin{bmatrix} -.4 & .4 & 0 \\ .3 & -.7 & .4 \\ 0 & .1 & -.1 \end{bmatrix} \begin{bmatrix} y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Show that rows are dep

Some combo of rows produces a 0 row

So row rank = 2

$$y = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

(9)

row space dim $= 2$
L rank

So tells us cols are dep ton

Not $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\lambda_{max} = 1$
other $|\lambda_i| < 1$

$A = \begin{bmatrix} a & b \\ & \end{bmatrix}$

$A = \begin{bmatrix} 1 & 0 \\ 0 & -1/2 \end{bmatrix}$

$A \begin{bmatrix} 1 & 0 \\ 17 & -1/2 \end{bmatrix}$

Tcomple so $\lambda_{max} = 1$
how get $\lambda = 1$?
make cols add to 1

$A = \begin{bmatrix} a & b \\ 1-a & 1-b \end{bmatrix}$

(10)

Plus other eigenvalue $\lambda = 1,\ a-b$

Markov if

1. Cols $\Sigma = 1$ ✓
2. $0 < A < 1$

   $0 < b < 1$

Course 6 classes based on this

Note some are the transpose

Rows $\Sigma = 0$

He does Σ cols $= 0$

(2 min late)

$$u = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \qquad \frac{du}{dt} = Au$$

$$\frac{du}{dt} = a\,u$$

↑ (1 by 1)

$$u(t) = e^{at} u(0)$$

Can write sol 3 ways

$$u(t) = e^{At}\, u(0)$$

← whats the exponential of a matrix

Use eigenvalues

$$= S e^{\Lambda t} S^{-1} u(0)$$

since $A^k = (S \Lambda S^{-1})^k = S \Lambda^k S^{-1}$

diagonalized matrix

← watch each eigenvector grow or decay

$$= c_1 e^{\lambda_1 t} x_1 + \dots + c_n e^{\lambda_n t} x_n$$ ← written out form

②

What is e^{At}? e^x?

← most important thing Calculus produces?

$$\frac{d e^x}{dx} = e^x$$

$$e^{x+y} = (e^x)(e^y)$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$$

∞ series

So then

e^{At} ← so can take time deriv

← x becomes a matrix

$$= I + \frac{At}{1!} + \frac{(At)^2}{2!} + \cdots + \frac{(At)^n}{n!}$$

∞ series

Then

$$\frac{d e^{At}}{dt} = A e^{At}$$

$$= 0 + A + A^2 t + \cdots + \frac{A^n t^{n-1}}{(n-1)!}$$

$\frac{2t}{2}$

③

~~Not True~~

Fallacy $e^{A+B} \neq e^{A}e^{B}$

matrices

$$e^{A+B} = I + (A+B) + \frac{(A+B)^2}{2} = A^2 + AB + BA + B^2$$

$$e^{A}e^{B} = \left(I + A + \frac{A^2}{2} + \cdots\right)\left(I + B + \frac{B^2}{2} + \cdots\right)$$

$$= A^2, AB, B^2$$

Never a BA so fails

Instead $e^{A(t_1 + t_2)} = e^{At_1} e^{At_2}$

But we haven't really shown how to actually do

41

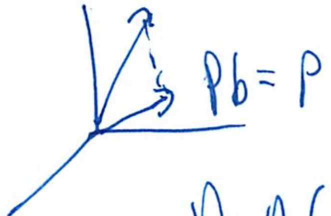
__Quiz__ Up to 6.1

- not this (6.3)
- or diagonization (6.2)
- but will have to find an eigenvalue (6.1)

Begining of course included

Which matrices are invertible? always

- only identity
- projection __no__



$P = A(A^TA)^{-1}A^T$

? cols of A are a basis for the subspace

eigenvectors of P

(5)

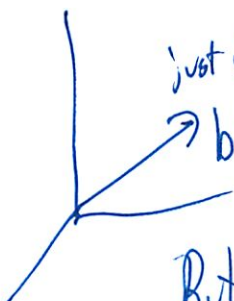
Questions are more understanding than calculating

eigenvectors of $P \Rightarrow$ all b in $C(A)$

$\lambda = 1$

same

just draw B

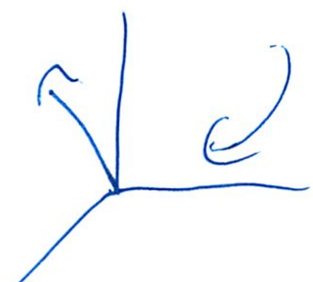


$b = P = Pb$

But not all λ are 1

others $\lambda = 0$

↳ B is $N(A^T)$



Permutation Matrix

- all are invertable $P = \begin{bmatrix} 1 & 1 \end{bmatrix}$
- its transpose $P^{-1} = P^T = \begin{bmatrix} 1 & & \\ & & 1 \\ & 1 & \end{bmatrix}$

(6)

Also all orthogonal matrices Q

$$Q^T Q = I$$

$$Q^{-1} = Q^T$$

No Markov this quiz

Determinants

1. Uses pivots
2. Uses cofactors - already includes the $\pm$
3. Big formula - $n!$ terms

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 3 & 0 \end{bmatrix} \quad C\cancel{A}_{11} = 1 \cdot \det \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 3 & 0 \end{bmatrix} = 0$$

same so 0

$$C\cancel{A}_{12} = \overset{\text{built in } \ominus}{-1} \det \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 3 & 0 \end{bmatrix} = -1 \cdot -3 = 3$$

7

$C_{13} = ---$

$C_{14} = ---$

So

$$\det A = a_{11} C_{11} + a_{12} C_{12} + a_{13} C_{13} + a_{14} C_{14}$$

$$= 0 + 2 \cdot 3 + \_\_ + \_\_$$

$$\text{So } A^{-1} = \frac{C^T}{\det A}$$ ← first row is first col in inverse

$$= \frac{\begin{bmatrix} 0 & - & - & - \\ 3 & - & - & - \\ - & - & - & - \\ - & - & - & - \end{bmatrix}}{\det A}$$

8

$$\begin{bmatrix} 3 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

? Cofactors are the way to go w/ this

$$\det \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 4 \end{bmatrix}$$ ← elim is good here

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \quad \det = 6$$

U = upper triangular

(9)

Gram Schmidt

Least Squares

← forget this

$Ax = b$

A is 50×3

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Most importan eq Chap 4

$$A^T A x = A^T b$$

if real sol

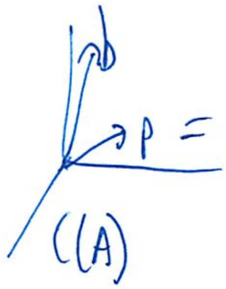
if invertable

$$\overline{A^T A}\,\hat{x} = A^T b$$

$\hat{x}$ = minimizes the squared error

$$\|b - Ax\|^2$$

10



$p = Pb$

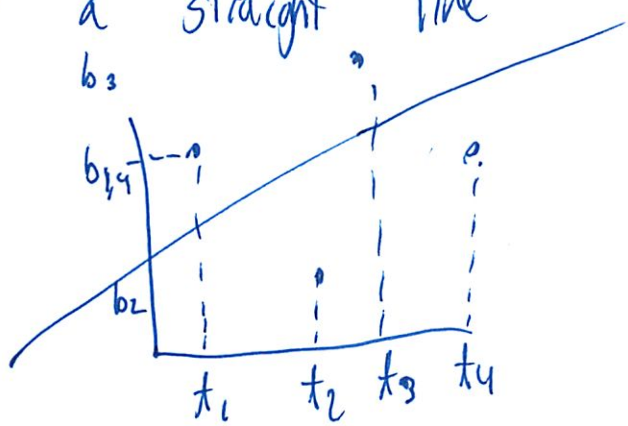
Tp is $\perp$ to error $\|e\|^2 = \|b - Ax\|^2$

$p = A\hat{x}$

↑as close as you can get to Vector V

So connect all 3 of the ideas

Fit a straight line



Can't solve the 4 eq exactly

But can get close!

$$A = \begin{bmatrix} 1 & t_1 \\ 1 & t_2 \\ 1 & t_3 \\ 1 & t_4 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

← exp set up

← the data

(ii)

Is A^TA invertable?

$\underset{n\times m}{A^T}\underset{m\times n}{A}$ is invertable iff

$\begin{bmatrix} \cancel{\ } A^T \end{bmatrix} \begin{bmatrix} A \\ \\ \\ \end{bmatrix} = m > n$ ← odds in your favor that invertable

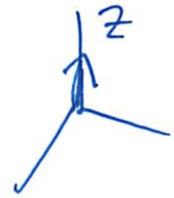
$\begin{bmatrix} B^T \\ \\ \\ \end{bmatrix} \begin{bmatrix} B \end{bmatrix} = m < n$

→ if A has ind cols

18.06 Recitation

4/10

Projection matrix is not invertable

But when project  it becomes 0

this is an eigenvalue

Spring 07

Problem 1

a) Compute $\det \begin{pmatrix} 1 & 1 & 2 \\ 1 & 3 & 1 \\ 4 & 1 & 1 \end{pmatrix}$

3 ways

- Big Formula
- Cofactor
- Elim

(2)

Quick way to do 3x3

- prob elim
- but also

$$\begin{bmatrix} 1 & 1 & 2 & 1 & 1 \\ 1 & 3 & 1 & 1 & 3 \\ 4 & 1 & 1 & 4 & 1 \end{bmatrix}$$

↑ ↑ repeat

Then multiply along diagonal, Minus for going by

$-24 \quad -1 \quad -1 \quad +3 \quad +4 \quad +2$

↑ Sum this $= -17$

(3)

Compute det of this 4x 4 matrix

$$\det\begin{pmatrix} 1 & 1 & 1 & 2 \\ 1 & 1 & 3 & 1 \\ 1 & 4 & 1 & 1 \\ 5 & 1 & 1 & 1 \end{pmatrix}$$

- elim prob could do, but lets skip
- Big formula too big → $4! = 24$
- Cofactor possible

better when a lot of 0s

do row ops

Subtract row 2 = row 2 - row 1

row 3 = row 3 - row 1

row 4 = ~~5~~ row 3 - row 1

$$\begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & 0 & 2 & -1 \\ 0 & 3 & 1 & -1 \\ 0 & -4 & -4 & -9 \end{pmatrix}$$

4

So go down the 1st row

$= 1 \cdot C_{11} + 0 \cdot C_{21} \cdot 0 \cdot C_{31} \cdot 0\, C_{41}$

Can do along any col or row

But the ones w/ a lot of 0s are nice

$$= 1 \cdot (-1)^{\overset{\text{row, col}}{1+1}} \cdot \det\begin{pmatrix} 0 & 2 & -1 \\ 3 & 1 & -1 \\ -4 & -4 & -9 \end{pmatrix}$$

Compute as before

$= 74$

Common mistake on P-Set

$$\det \frac{1}{3}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} =$$

$$\neq \frac{1}{3} \det\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \quad \text{WRONG!}$$

$$= \det\begin{pmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{pmatrix}$$

⑤ Since duplicative only take out of top row

Factor out constant row by row

$$= \frac{1}{3} \det \begin{pmatrix} 2 & -1 \\ -1/3 & 2/3 \end{pmatrix}$$

$$= \frac{1}{9} \det \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

take out constan to power of n here $n = 2$ $\left(\frac{1}{3}\right)^n$

" So I asked and all 4 of these formulas are =

- Still confused

c) What is (1,4) entry of B^{-1}

$$\left(B^{-1}\right)_{1,4} = \left(\frac{C^T}{\det B}\right)_{1,4} = \frac{1}{\det B} \left(C^T\right)_{1,4}$$

$$= \frac{1}{\det B} \left(C\right)_{4,1}$$

⑥

So $C_{4,1}$ is

$$= (-1)^{1+4} \det\begin{pmatrix} 1 & 1 & 2 \\ 1 & 3 & 1 \\ 4 & 1 & 1 \end{pmatrix}$$

$$= -1 \cdot -17$$

$$= +17$$

So $B^{-1} = \frac{17}{74}$

Projection question - Problem 2

a) Let $b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 5 \\ -1 & 0 & -2 \end{pmatrix}$

Find projection of b onto $C(A)$

~~AA~~ Projecting onto $\mathbb{R}^3$

? Are the cols ind

↳ No! $2\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + 5\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ -2 \end{pmatrix}$

So really $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \end{pmatrix}$ (new matrix we invented?)

Now projection formula

$$P = B(B^TB)^{-1}B^Tb$$

just do this

example of middle part

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} =$$

easy to invert
(need to study inverting)

$$(B^TB)^{-1} = \; = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}^{-1}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix}$$

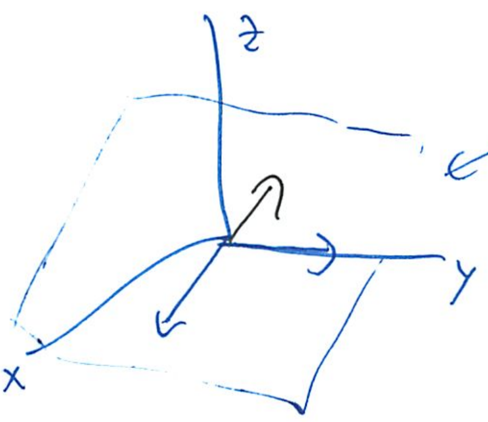
Then plug rest in

(8)

b) What is the Least Squares Approx

$$\underset{B}{\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \end{pmatrix}} x = \underset{b}{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}$$

When we look at col pic
are in $\mathbb{R}^3$ - 3 components



← They span a plane

Other vector outside the plane
Our 2 vectors can't combine for black vector
Since not on the plane
But can approximate

(9)

1.) What is the projection of $\begin{pmatrix}1\\1\\1\end{pmatrix}$ onto $C(B)=?$

Answer is $\begin{pmatrix}0\\1\\0\end{pmatrix}$ from part b)

So what are the system ~~of~~ of eqns that we need to solve?

$$\begin{pmatrix}1 & 0\\0 & 1\\-1 & 0\end{pmatrix} x = \begin{pmatrix}0\\1\\0\end{pmatrix}$$

$$B \quad x = b$$

First half of semester

Only need 2nd col

Since that is the projection

$$x = \begin{pmatrix}0\\1\end{pmatrix}$$

Note: Not always the second vector

Can solve by elimination

(10)

Roundabout way: If $Ax = b$ is not solvable multiply by A^T, and solve $A^TAx = A^Tb$

---

20) What is the projection of $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ onto col space of

$$\begin{pmatrix} 10000 & 0 & 2 \\ 0 & 1 & 5 \\ -1 & 0 & -2 \end{pmatrix}$$

Cols ind

So span R3

So we can just leave our $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ result

Since its in the col space of our 3x3 matrix

11

3. Gram Schmidt

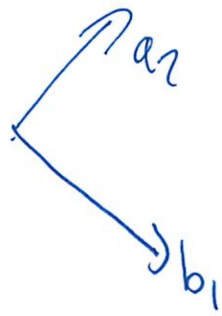
Have 4 vectors

$$a_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad a_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad a_3 = \begin{pmatrix} 0 \\ 0 \\ 2 \\ 0 \end{pmatrix} \quad a_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 2 \end{pmatrix}$$

Get orthogonal (not orthonormal) basis

1. Always start w/ a_1
   - Pick it to be your 1st basis
   - $b_1 = a_1$

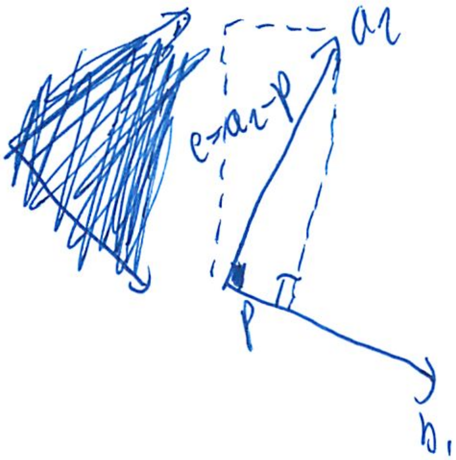
2. Find b_2, orthogonal to b_1



$$b_2 = a_2 - \frac{b_1^T a_2}{b_1^T b_1} b_1$$

12

Projecting a_2 to b_1



$$\frac{\begin{pmatrix} 1 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}}{\begin{pmatrix} 1 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \frac{2}{2} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

So $b_2 = a_2 -$ that $-$

$$b_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

(13)

take away projections of both parts

3. Now $b_3 = a_3 - \frac{b_2^T a_3}{b_2^T b_2} b_2 - \frac{b_1^T a_3}{b_1^T b_1} b_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

answer

4. $b_4 = a_4 - \frac{b_3^T a_4}{b_3^T b_3} b_3 - \frac{b_2^T a_4}{b_2^T b_2} b_2 -$

$\frac{b_1^T a_4}{b_1^T b_1} = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$

answer

If asks orthonormal, divide by lenght

$= \sqrt{\sum \text{components}^2}$

Problem 4

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$$

a) Show that the cols of A are orth. to each other

(14)

Check 1st vs 2nd $\perp$ = Orthogonal
" 3rd
" 4th

2nd vs 3rd
" 4th
3rd vs 4th

Take dot product

$$(1, 0, -1, 0) \cdot (0, 1, 0, -1) = 0$$

means $\perp$

Calc the determinant

$$\det A = \det \begin{pmatrix} & \end{pmatrix}$$

What is easiest way → elim to upper triangular form

$$= \det \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} = 4$$

← row 3 = row 2 + row 1
← row 4 row 2 + row 4

Can't swap rows!

25

c) Calc A^{-1}

Could do via cofactors

Or elim

- add identity to get augmented matrix

But can exploit that cols of A are othagonial

General facts: If col's of B are orthogonal
square

$$B^T B = \begin{pmatrix} x_1 & 0 & 0 & 0 \\ 0 & x_2 & 0 & 0 \\ 0 & 0 & x_3 & 0 \\ 0 & 0 & 0 & x_4 \end{pmatrix}$$

other stuff 0

Here nice all cols are the same length

$A^T A = 2I$

So $A^{-1} = \frac{1}{2} A^T$

um

(16)

Problem 5

Complete the 2x2 matrix A

So that its eigenvales are 1, -1

no matter what a is

$$A = \begin{pmatrix} a & 1 \\ ?_1 & ?_2 \end{pmatrix}$$

know trace $= \lambda_1 + \lambda_2 = 0$

$$a + ?_2 = 0$$

So $?_2 = -a$

know det $= \lambda_1 \lambda_2 = -1$

$$= a(-a) - ?_1 = -1$$

So $-a^2 + 1 = ?_1$

$$A = \begin{bmatrix} a & 1 \\ -a^2+1 & -a \end{bmatrix}$$

EXAM 2 REVIEW

JENNIFER PARK

This is an unofficial summary of topics we have covered so far in class, and what I think is important for the upcoming exam. Please do NOT take this as a complete list of things to study, but rather a rough guideline on the things that you absolutely must know for the exam. **The exam is on chapters 1-5, 6.1 and 8.2. Note that although this review sheet only covers chapters 4-5 and 6.1, the exam is cumulative!**

1. Definitions

Make sure you can define these things correctly for the exam:

- The **four fundamental subspaces** of a matrix is...
- Two vectors v and w are **orthogonal** if... Two subspaces V and W of $\mathbb{R}^n$ are **orthogonal** if...
- Of the four fundamental subspaces, which spaces are orthogonal?
- The orthogonal complement of a subspace V is...
- The **projection** of a vector b onto a subspace V is... When the subspace is 1-dimensional, the definition simplifies to...
- The vectors $v_1, \cdots v_n$ are **orthonormal** if...
- If P is a permutation matrix, the **sign** of P is...
- The (i, j) **cofactor** of a matrix A is given by...
- The **eigenvalue** of the matrix A is the constant λ such that...
- The **eigenvector** x of the matrix A is... (can x be zero?)
- The **incidence matrix** of a directed graph is...

2. Computations

Make sure you can do the following computations:

- Projection of a vector onto a subspace.
- Least squares approximation.
- Fitting a straight line? Parabola?
- Gram-Schmidt process. Can you get an orthogonal basis from a given basis? Orthonormal basis?
- What is the inverse of the matrix Q whose columns are orthonormal?
- $A = QR$ factorization.
- $\det A$ for square matrices A via: products of pivots, "big formula", cofactors.
- Solving a system of equations using Cramer's rule.
- Finding A^{-1} using cofactors.
- Computing the volume of a parallelopiped ("a box") using determinants.
- Finding eigenvalues and eigenvectors.

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Chap 6 Reading

4/8

Eigenvalues + EigenVectors

$Ax = b$ steady state

$\frac{du}{dt} = Au$ changes w/ time

- grow
- decay
- ossilate

Can't find w/ isolation

$\boxed{Ax = \lambda x}$

or to easily find A^{100}

eigenvectors Certain vectors x are in the same direction as Ax

just a # "λ" times the original x

→ eigenvalue

tells you if vector stretched or shrunk

e

$Ax = 0x$ means eigenvector x is in nullspace

$Ax = 1x$ means all vectors are eigenvectors of I

Find eigenvectors

$$A = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix}$$

$$\det \begin{bmatrix} .8-\lambda & .3 \\ .2 & .7-\lambda \end{bmatrix}$$

$$(.8-\lambda)(.7-\lambda) - .3 \cdot .2$$

$$\lambda^2 - \frac{3}{2}\lambda + \frac{1}{2}$$

$$(\lambda - 1)(\lambda - \tfrac{1}{2})$$

$$\lambda = 1 \quad \lambda = \tfrac{1}{2}$$

Since here $A - \lambda I$ becomes singular ($0 = \det$)

x_1 in null space of $A - I$

x_2 in null space of $A - \frac{1}{2} I$

(3)

So

$$Ax = \lambda x$$

x_1 $\quad \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} \begin{bmatrix} x_1 \end{bmatrix} = 1 \begin{bmatrix} x_1 \end{bmatrix}$

? where did the book get $x_1 = \begin{bmatrix} .6 \\ .4 \end{bmatrix}$

x_1 is an eigenvector

Note When A is squared, eigenvectors stay the same

eigenvalues are squared

A^2 eigen vector $= \begin{bmatrix} .6 \\ .4 \end{bmatrix}$

A^2 eigen value $= 1^2, \left(\frac{1}{2}\right)^2$

$\lambda = 1$ → $Ax_1 = x_1 = \begin{bmatrix} .6 \\ .4 \end{bmatrix}$

$Ax_2 = \lambda_2 x_2 = \begin{bmatrix} .5 \\ -.5 \end{bmatrix}$

$\lambda = .5$ → $x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\lambda = 1$ → $A^2 x_1 = (1)^2 x_1$

$\lambda = .25$ $A^2 x_2 = (.5)^2 x_2 = \begin{bmatrix} .25 \\ -.25 \end{bmatrix}$

4

All other vectors are combos of the 2 eigenvectors

$$\begin{bmatrix} .8 \\ .2 \end{bmatrix} = x_1 + .2x_2$$

↑ 1st col A

For A^2

$$\begin{bmatrix} .7 \\ .3 \end{bmatrix} = (1)^1 x_1 + \left(\tfrac{1}{2}\right)^1 .2\, x_2$$

A^{100}

$$\begin{bmatrix} .6 \\ .4 \end{bmatrix} = (1)^{99} x_1 + \underbrace{\left(\tfrac{1}{2}\right)^{99} .2\, x_2}_{\text{very small}}$$

Note this particular A is a <u>Markov matrix</u>

Since $\lambda_1 = 1$ ← steady part

$\lambda_2 = .5$ ← decaying part – almost disappears

$$x_1 = \begin{bmatrix} .6 \\ .4 \end{bmatrix} = \text{the steady state}$$

5

Projections ~~Reflections~~

$$P = \begin{bmatrix} .5 & .5 \\ .5 & .5 \end{bmatrix}$$

$\lambda = 1, 0$

$x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

I think I remember the linear combos thereof

$Px_1 = x_1$ steady state (fill the col space)

$Px_2 = 0$ nullspace (fill the), nullspace projected to 0

Since

1. Each col of P $\sum = 1$, so $\lambda_1 = 1$
2. P is singular, so $\lambda_2 = 0$
3. P is symetric, so λ_1, λ_2 are perpendicular

Projections $\lambda = 0, 1$

Permutations $|\lambda| = 1$

(6)

Reflections

$$R = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$\lambda = 1, -1$

$x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

signs reversed

$R(\text{reflection}) = 2P - I$

difference

$\lambda = 2(1) - 1 = 1$

$2(0) - 1 = -1$

x of $R^2 = \lambda^2$

$R^2 = I$

$1^2 = 1, \; (-1)^2 = 1$

(i)

Finding them

$Ax = \lambda x$

$(A - \lambda I)x = 0$

↑ eigen vectors make up the nullspace of $A - \lambda I$

↑ I could never come up w/ that lang...

If $(A - \lambda I)x = 0$ has a nonzero sol

$A - \lambda I$ is not invertible / singular

$\det(A - \lambda I)$ must $= 0$

λ is an eigenvalue of A, only if

For each λ, solve $(A - \lambda I)x = 0$

or $Ax = \lambda x$ to find eigenvector x

Example 2 Find Eigenvalues

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$ is singular

Singular $\rightarrow \lambda = 0$

Subtract λ on the diagonals

$A - \lambda I$

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{bmatrix}$$

Take the det

$$(1-\lambda)(4-\lambda) - 4$$

$$\lambda^2 - 5\lambda$$

Set = to 0 + factor

$$\lambda^2 - 5\lambda = 0$$

$$\lambda_1 = 0$$
$$\lambda_2 = 5$$

did this already

(9)

Find eigenvector

Solve $(A - \lambda I)x = 0$ for x

Sepertly for $\lambda_1 = 0$ and $\lambda_2 = 5$

$\lambda_1 = 0$

$$(A - 0I)x = \begin{bmatrix} 1-0 & 2 \\ 2 & 4-0 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$1y - 2z = 0$$

$$2y - 4z = 0$$

$$y = 2z$$

$$2(2z) - 4z = 0$$

$$0 = 0$$

$$2y = 4z$$

hmm - not helping

since 1 is a multiple of the other

Any multiple of $(b, -a)$ works

So $\begin{bmatrix} +2 \\ -1 \end{bmatrix}$

(10)

$\lambda_1 = 5$

$$\begin{bmatrix} 1-5 & 2 \\ 2 & 4-5 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-4y + 2z = 0$$

$$2y - 1z = 0$$

$$2y = z$$

$$-4y + 2(2y) = 0$$

multiple again

$(b, -a)$

$\cancel{\begin{bmatrix} -1 \\ -2 \end{bmatrix}}$ only the top row is a,b

$\begin{bmatrix} 2 \\ 4 \end{bmatrix} \to$ reduce $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

The eigen~~vectors~~ values can be =, so only 1 eigenvector

Without n eigenvectors in a n×n matrix, no basis

↳ can't diagonalize a matrix

(11)

Good News, Bad News

row ops/elim change λ

but diag of $U = \lambda$

$\prod \lambda = \det$

$\sum \lambda = \sum$ diagonals $=$ trace

Imaginary

eigenvalues may be $ai+b$

$$Q = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$\lambda = i, -i$

$$x = \begin{bmatrix} 1 \\ i \end{bmatrix}, \begin{bmatrix} i \\ 1 \end{bmatrix}$$

Note 1. $|\lambda| = 1$

2. Q is a skew-symmetric matrix

(12)

Diagonalizing a Matrix

Easy to multiply an eigenvector

Like a diagonal matrix

Matrix $A \longrightarrow \Lambda$ diagonal matrix

when use eigenvectors properly

$$S^{-1}AS = \Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} = \text{\underline{eigenvalue matrix}}$$

$$S = \text{\underline{eigenvector matrix}} = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}$$

Remember S^{-1} is like dividing by S

$$\begin{aligned} AS &= S\Lambda \\ &\downarrow S^{-1}AS = \Lambda \\ A &= S\Lambda S^{-1} \end{aligned}$$

must have n ind eigenvectors

(13)

$$A^2 = S\Lambda S^{-1} S\Lambda S^{-1}$$

since $S^{-1}S = I$

$$= S\Lambda^2 S^{-1}$$

this is easy

---

1. Remember no repeated eigenvectors / repeated rows
2. Can multiply by constants
3. Eigenvalues in S in same order as eigenvalues in Λ

<u>Not</u> about invertibility

Markov → steady state

$|\lambda| \leq 1$ → A^k goes to 0 in lim of k

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Fibinachi #s

0, 1, 1, 2, 3, 5, 8, 13

$F_{k+2} = F_{k+1} + F_k$

~~Call~~ Find F^{100}

Can do slow way

Or

$$U_k = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}$$

$$u_{k+1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} u_k$$

$$\begin{array}{l} F_{k+2} = F_{k+1} + F_k \\ F_{k+1} = F_{k+1} \end{array}$$

So every step multiplies by $A = \begin{bmatrix} 1 & 1 \\ \cancel{1}0 & 0 \end{bmatrix}$

$$u_{100} = A^{100} u_0$$

(15)

So find eigenvalues

$$\begin{bmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{bmatrix}$$

$$\lambda^2-\lambda-1$$

$$\lambda_1 = \frac{1+\sqrt{5}}{2} \approx 1.618$$

$$\lambda_2 = \frac{1-\sqrt{5}}{2} \approx -.618$$

eigenvectors

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\lambda_1-\lambda_2}\left(\begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix} - \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}\right)$$

$$u_0 = \frac{x_1 - x_2}{\lambda_1-\lambda_2}$$

$$u_{100} = \frac{(\lambda_1)^{100}x_1 - (\lambda_2)^{100}x_2}{\lambda_1-\lambda_2}$$

$$= 2.54 \cdot 10^{20}$$

16

Matrix Powers A^k

This factorization is for computing powers

1. Write u_0 as combo of eigenvectors

   $c_1 x_1 + \dots + c_n x_n$

   $c = S^{-1} u_0$

2. Multiply each eigenvector x_i by $(\lambda_i)^k$

   Now have $\Lambda^k S^{-1} u_0$

3. Add up the pieces $c_i(\lambda_i)^k x_i$ to find the sol $u_k = A^k u_0$

   $S \Lambda^k S^{-1} u_0$

So

$$u_{k+1} = A u_k$$

$$u_k = A^k u_0$$

$$= c_1(\lambda_1)^k x_1 + \dots + c_n(\lambda_n)^k x_n$$

(17)

AB and $A+B$

~~~~ A, B usually don't share same eigenvector

So eigenvalues $A+B \neq \lambda + \beta$

Only share eigenvectors if

$$AB = BA$$

6.3 Application to Diff Eq

$$\frac{du}{dt} = A\underline{u}$$

deriv of $e^{\lambda t} = \lambda e^{\lambda t}$

* Goal: Convert constant coeff diff eq into linear algebra *

$\frac{du}{dt} = \lambda t$ has the sols $u(t) = Ce^{\lambda t}$

$\underline{u}$ is a matrix | Choose $C = u(0)$? time 0
~~~~

(18)

Can check a sol

Plug $u(t) = e^{\lambda t}x$ into $\frac{du}{dt} = Au$

$Au = Ae^{\lambda t}\underline{x}$

$\lambda > 0$ grows

$\lambda < 0$ decays

(remembering this from last semester...)

(Do the rest later)

Math Review 4/10

Exam 2

Chap 1-5, 6.1, 8.2
(WTF?)

Previous 1-3,5

New: 4th Subspace

Orthagonality

Projections

Least Squares

Gram Smidt

Determinents

Cramers Rule

Eigenvalues + Vectors

Review Unit 1's notes

②

RREF is only 1s in the diagonal

$$\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$$

$$A \; I \rightarrow I \; A^{-1}$$

A^{-1}

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

← Cofactor Transpose

C^T

Gauss – Jordan elim

(3)

Gauss Jordan

$$
\begin{array}{cc}
A & I \\
\left[\begin{array}{ccc|ccc}
a_{11} & \cdots & a_{n1} & 1 & 0 & 0 \\
\vdots & \ddots & & 0 & 1 & 0 \\
a_{n1} & & a_{nn} & 0 & 0 & 1
\end{array}\right]
\end{array}
$$

elim till

$$
\begin{array}{cc}
I & A^{-1} \\
\left[\begin{array}{ccc|ccc}
1 & 0 & 0 & b_{11} & \cdots & b_{n1} \\
0 & 1 & 0 & & \ddots & \\
0 & 0 & 1 & b_{n1} & & b_{nn}
\end{array}\right]
\end{array}
$$

ahhh I think I get it

- basically want RREF

  └ what killed me on exam before

Invertable = Non 0 det

(4)

Col space and row space
L The space spanned by the col
Write a basis → pivot cols

$N(A) \subseteq \mathbb{R}^n$

Then all those sol types!

Null space The sol $Ax = 0$
easier to elim 1st ? x for where = 0

$Ax = b$ only solvable if b is in col space of A

Pivot variables and free variables
free choose 1 in each, 0 in rest of free
to get all the special sols
then add across

$$x = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix} = \begin{pmatrix} -x_2 - x_4 \\ x_2 \\ -x_4 \\ x_4 \end{pmatrix}$$

⑤

rank = # pivots

So complete sol

$$X = x_p + x_n$$

$A x_p = b$ $\qquad$ $A x_n = 0$

<u>Sizes</u> ~~[illegible]~~

<u>full col rank</u> $r = n$

$$\begin{array}{cc} & m \\ n & \begin{bmatrix} \text{tall} \\ \text{+ thin} \\ \\ \end{bmatrix} \end{array}$$

all cols pivots / no free

Nullspace only $x = 0$

$Ax = b$ 0 or 1 sol

<u>full col rank</u> $r = m$

$$\begin{array}{cc} & m \\ n & \begin{bmatrix} & \text{Short + wide} & \\ & & \end{bmatrix} \end{array}$$

All rows have pivots

$Ax = b$ sol for every b

$n - m$ special sols

∞ sols

(d)

width	height		$Ax=b$
$r=m$	$r=n$	square	1 sol
$r=m$	$r<n$	short + wide	∞ sol
$r<m$	$r=n$	tall + thin	0 or 1 sol
$r<m$	$r<n$	no full rank	0 or ∞ sol

I remember

$\begin{bmatrix} 1 \text{ sol} \end{bmatrix}$ $\begin{bmatrix} \cancel{\infty} \; \infty \end{bmatrix}$ $\begin{bmatrix} 0 \text{ or} \\ 1 \end{bmatrix}$ $\begin{bmatrix} 0 \text{ or} \\ \infty \end{bmatrix}$

no full rank

1   ∞ 0 or 1   0 or ∞

1   ∞ 0 or 1   0 or ∞

1   ∞ 0 or 1   0 or ∞

7

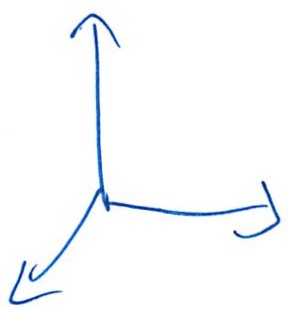
i saw before i

Lin ind

only $Ax=0$ is $x=0$

~~that~~ i thats tall + thin, right i

Not on a plane



Not

Span if lin combos fill the space

basis lin ind + fill the space

dim # of vectors in basis

I feel more, but not perfectly comfortable w/ def

Col space has dim r

ind cols = # ind rows r

8

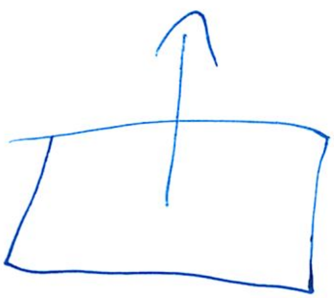
dim nullspace $= n - r$ $\rightarrow$ sum R^n ← # of cols

but for col space

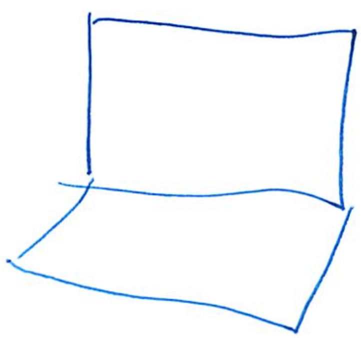
dim left nullspace $= m - r$ $\rightarrow$ sum R^m

---

4 Subspaces Orthagonal



orthagonal



not orthagonal

<u>Orthagonal complement</u> − every vector is $\perp$ to V

$V^{\perp}$

"

Orthagonal to all elements

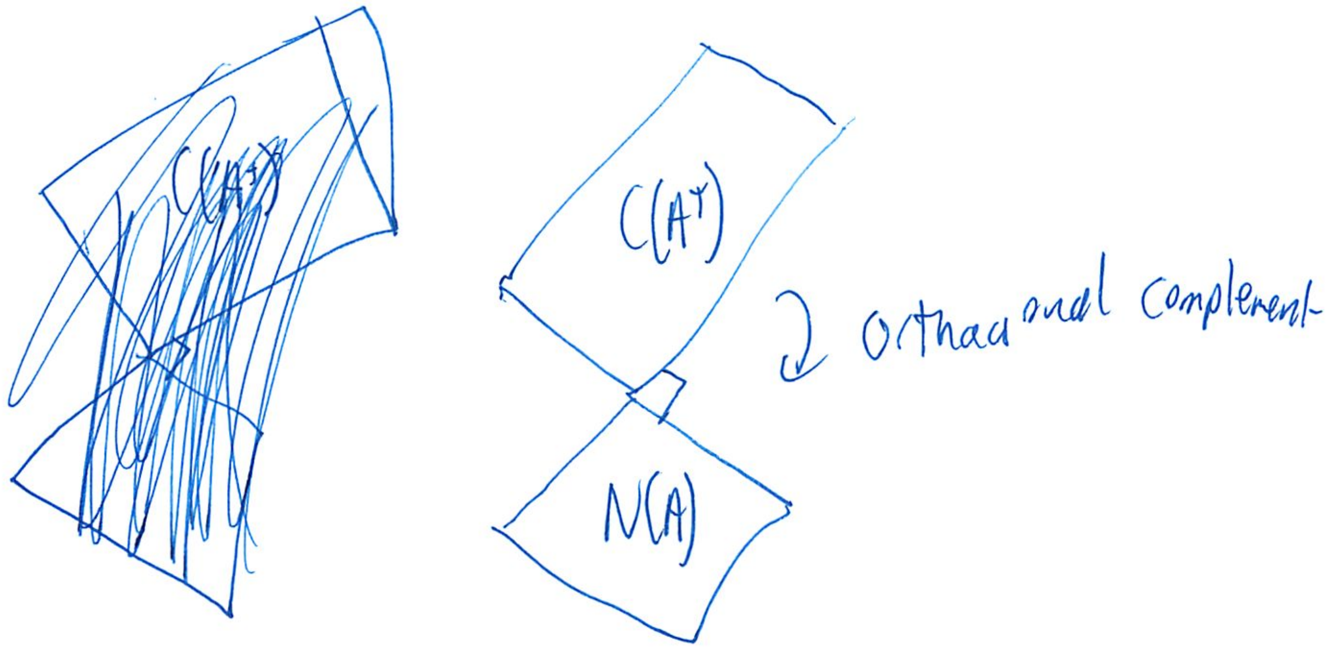
from online it seems you have 2 vectors

then the one orthagonal to those 2 $\uparrow_z$ $\swarrow_u$ $\rightarrow v$

9

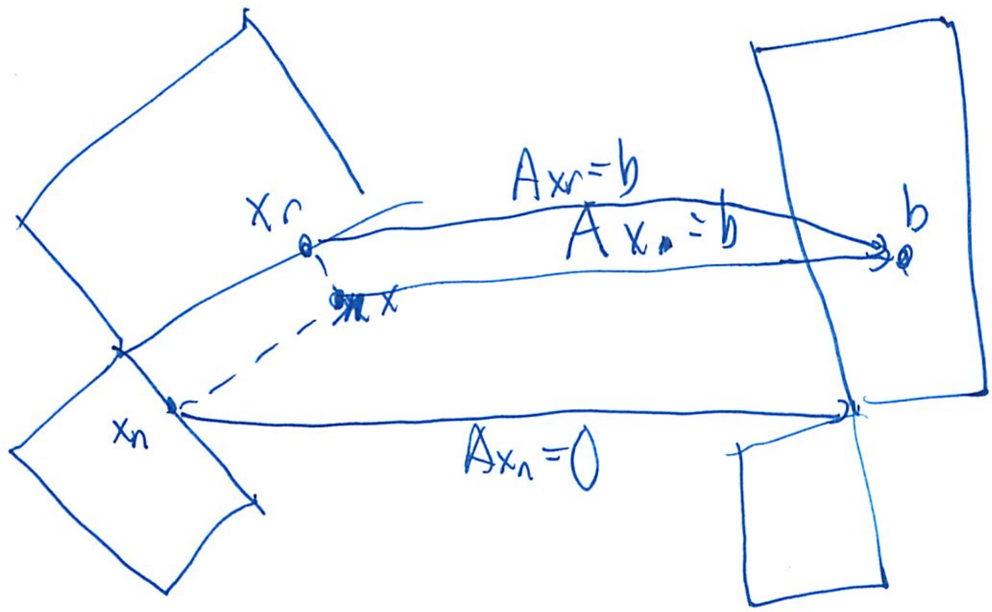
So $N(A)$ is $\perp$ of $C(A^T)$

nullspace — row space



Then also

$$X = X_r + X_n$$



I still don't get this pic!

(10)

Projections

← new stuff now

project b onto a line p

└ the projection $p = Pb$

So then $Pb = p$ to check

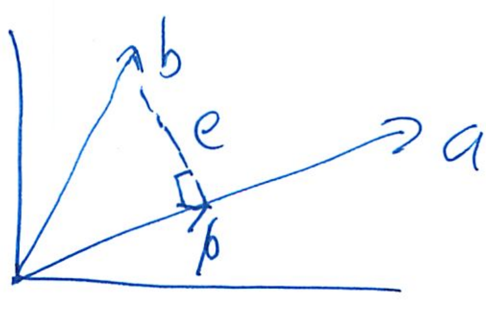
like a light shining from above

$p = \hat{x} a$

1. Find $\hat{x}$
2. Find vector p
3. Find matrix P

$e = b - \hat{x} a$

$\perp a$



(11)

$$p = \hat{x} a = \frac{a^T b}{a^T a} a$$

$$P = \frac{a a^T}{a^T a}$$

So for $a = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

$$P = \frac{\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \end{bmatrix}}{\begin{bmatrix} 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}} = \frac{\begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 2 & 4 & 4 \end{bmatrix}}{1+4+4}$$

$$p = Pb = \frac{1}{9} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 2 & 4 & 4 \end{bmatrix} \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}_{b} = \frac{1}{9} \begin{bmatrix} 5 \\ 10 \\ 10 \end{bmatrix}$$

(12)

Remember the formula!

$$P = \frac{a a^T}{a^T a}$$ ← remember large ← small / single #

$$\frac{a a^T}{a^T a}$$

$$p = a \hat{x} = a \frac{a^T b}{a^T a}$$

$$x = \frac{a^T b}{a^T a}$$

$$P = \frac{a a^T}{a^T a}$$

$$p = Pb$$

(13)

$$p = Pb$$

$$p = a\hat{x}$$

$$x = \frac{a^Tb}{a^Ta}$$

$$P = \frac{aa^T}{a^Ta}$$

before

$$\begin{bmatrix} 1 \end{bmatrix} \quad \begin{bmatrix} 0 \text{ or } 1 \\ \text{correct} \end{bmatrix} \quad \begin{bmatrix} \infty \end{bmatrix} \quad \begin{bmatrix} 0 \text{ or } \infty \\ \neq \end{bmatrix}$$

$$\begin{bmatrix} 1 \end{bmatrix} \quad \begin{bmatrix} 0 \text{ or } 1 \end{bmatrix} \quad \begin{bmatrix} \infty \end{bmatrix} \quad \begin{bmatrix} 0 \text{ or } \infty \\ \neq \end{bmatrix}$$

$$\begin{bmatrix} 1 \end{bmatrix} \quad \begin{bmatrix} 0 \text{ or } 1 \end{bmatrix} \quad \begin{bmatrix} \infty \end{bmatrix} \quad \begin{bmatrix} 0 \text{ or } \infty \\ \neq \end{bmatrix}$$

(14)

Can project onto a subspace

$$p = A\hat{x}$$

$$= A(A^TA)^{-1}A^Tb$$

$$P = A(A^TA)^{-1}A^T$$

remember these

$$P = A(A^TA)^{-1}A^T$$

$$P = \frac{aa^T}{a^Ta}$$

$$P = A(A^TA)^{-1}A^T$$

(15)

How to actually find

Find $\underbrace{A^TA}$, $\underbrace{A^Tb}$

Solve $A^TA\hat{x} = A^Tb$

Since $P = A(A^TA)^{-1}A^Tb$ ①

get $\hat{x} = \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix}$

$p = A\hat{x}$

$e = b - p$

$P^2 = P$

Could find $(A^TA)^{-1}$ if you ~~w~~ wanted
normal way ~~t~~ to find inverse
- as before in notes

Ahh remember its $C^T \Leftarrow$ transpose

(16)

But remember only invertable if ind cols

Least Squares Approx

$$A^T A \hat{x} = A^T b$$

That was same as before

but w/ very specific

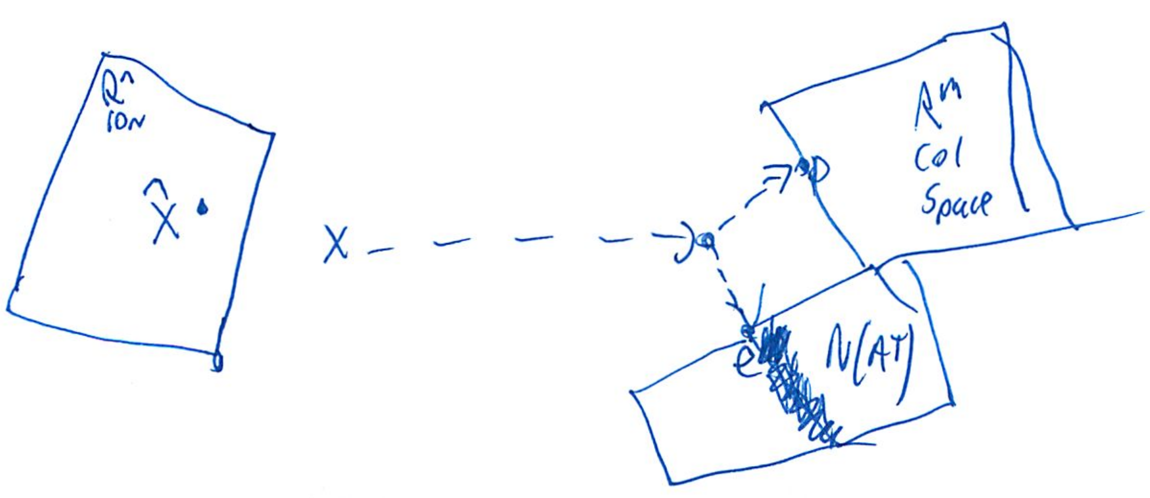
$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \quad x = \begin{bmatrix} C \\ D \end{bmatrix} \quad b = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}$$

always 1 | t (ind axis)

$y = C + Dt$

y position (dep axis)

So then same process



(17)

Will go to - but don't need to memorize

$$\begin{bmatrix} m & \sum t_i \\ \sum t_i & \sum t_i^2 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} \sum b_i \\ \sum t_i b_i \end{bmatrix}$$

Orthonormal

$$q_i^T q_j = \begin{cases} 0 & i \neq j \text{ ortogonal} \\ 1 & i = j \text{ unit vectors} \end{cases}$$

$Q^T Q = I$

? like always? $\cancel{A^T A = I}$

No thats $A^{-1} A = I$

Oh but $Q^T = Q^{-1}$ when Q is square

Can do projections w/ Orthogonal Bases

Q replaces A

$I \hat{x} = Q^T b$ $\quad p = Q \hat{x}$ $\quad P = Q I Q^T$

18

Gram Schmidt

$$A = a$$

$$B = b - \frac{A^T b}{A^T A} A$$

$$C = C - \frac{A^T c}{A^T A} A - \frac{B^T c}{B^T B} B$$

etc

Another formula to remember!

$$B = b - \frac{AB^T b}{A^T A} A$$

? don't foget

Then actually multiply

Lastly normalize

– divide by lenght = Σ values

19

~~And~~

Now we have the factorization $A = QR$

$R =$ triangular $= Q^T A$

$$\begin{bmatrix} q_1^T a & q_1^T b & q_1^T c \\ & q_2^T b & q_2^T c \\ & & q_3^T c \end{bmatrix}$$

Seems silly

Least squares

$$x = R^+ Q^T b$$

not an operational formula

20

Determinants

$$A^{-1} = \frac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$ad-bc$

memorize

flip a, d

change sign b, c

3 ways to find

Some properties

1. $\det I = 1$
2. Can swap any row + swap sign
3. Lin function of each row sep

$$t\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} ta & tb \\ c & d \end{vmatrix}$$

any row

I'm not good at this one

(21)

$$\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$$

Only 1 row changes at once

So for other rows → swap it to the top
do the ops
swap it back

$\det 2I \neq 2 \det I$

more like vol

(I still don't get how this lines up w/ class today)

So I think $t \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

means $t \cdot \det\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

not $\det\left(t\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)$

Oh I think I got it

$\det\left(t\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)$ is $\det\begin{pmatrix} ta & tb \\ tc & td \end{pmatrix}$ is $t \det\begin{pmatrix} a & b \\ tc & td \end{pmatrix}$

straight multiply

special rule 2

(22)

3. then $A^2 \det \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

(swap row, remove, swap back

Thats that vol A^n where $n = \#$ rows

Mathes class

4. two rows $= \Rightarrow \det = 0$

5. Subtract one row from another $\to$ det unchanged

$$\begin{vmatrix} a & b \\ c-la & d-lb \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

6. Row of 0s $\Rightarrow \det = 0$

7. Triangular $\to$ just diagonal entries
upper or lower

8. Singular $\to \det = 0$

9. $|AB| = |A|\,|B|$

So $|A|\,|A^{-1}| = I$

Rember both $AA^{-1} = I$
$A^{-1}A = I$

(23)

10. $|A^T| = |A|$

<u>Pivot method</u>

Elim

Multiply diagonal

<u>Big Formula</u>

$n!$ terms

One from each row and col

(Hopefully don't need odd/even BS)

Multiply within each term

Than add all the terms

Its important

No! ~~All even~~ Half the terms are $\ominus$

Odd # of row ex back to identity = $\ominus$

Even $\oplus$

even = 0, 2

odd = 1, 3

(24)

So

is

Cofactors

$$\det A = a_{11}\left(a_{22}a_{33} - a_{23}a_{32}\right) + a_{12}\left(a_{23}a_{31} - a_{21}a_{33}\right) + a_{13}\underbrace{\left(a_{21}a_{32} - a_{22}a_{31}\right)}_{\text{Cofactors}}$$

Choose an entry from each row + col

$$C_{ij} = (-1)^{i+j} \det M_{ij}$$

minor $\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$

? So cofactors can only be across top row?

Useful when many 0s

Can go far w/ this recursivly

Ahh so since $|A| = |A^T|$ can do 1st col

? Any other row?

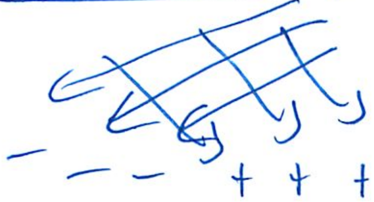
So yes I think you can do any col

or any row

Found in book

The checkerboard +, - matrix - ya only use 1 row

Quick way



? which method is this?

I think big formula

26

Cramers Rule

Solves $Ax = b$

I think the operational formulas

$$x_1 = \frac{\det B_1}{\det A} \qquad x_2 = \frac{\det B_2}{\det A} \qquad \text{etc}$$

ith col of
A replaced w/ b

$$B_1 = \begin{bmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{bmatrix}$$

Example Solve

$$3x_1 + 4x_2 = 2$$
$$5x_1 + 6x_2 = 4$$

Try on my own

$$A = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix} \qquad b = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

det = -2

$$B_1 = \begin{bmatrix} 2 & 4 \\ 4 & 6 \end{bmatrix} \qquad B_2 = \begin{bmatrix} 3 & 2 \\ 5 & 4 \end{bmatrix}$$

-4 $\qquad$ 2

$$x_1 = \frac{-4}{-2} = +2 \qquad x_2 = \frac{2}{-2} = -1$$

Can't do mental math

27

Another formula for A^{-1}

$$(A^{-1})_{ij} = \frac{C_{ji}}{\det A}$$ ← notice transpose

← Cofactor is minor w/ sign checkerboard

Area of triangle = half 3x3 det

Cross Product

$$u \times v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= (u_2 v_3 - u_3 v_2)\, i + \text{etc}$$

Right hand rule

(28)

Eigenvalues + Eigenvectors

East to find ~~A^{100}~~ A^{100}

$Ax = \lambda x$

$$\begin{bmatrix} .8-\lambda & .3 \\ .2 & .7-\lambda \end{bmatrix}$$

↑ on I diagonal

$\lambda^2 - \frac{3}{2}\lambda + \frac{1}{2}$

$(\lambda - 1)(\lambda - \frac{1}{2})$

↑ make sure I can actually still factor

$+ \quad \times$

Or can do $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Then plug in to $(A - \lambda I)x = 0$

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$ solve w/ elim

↑ I subtracted $\lambda = 0$ on diagonal

for each λ

29

Those are the eigenvectors

Then what do w/ those?

So we can seperate anything into eigenvectors

$\begin{bmatrix} .8 \\ .2 \end{bmatrix} = x_1 + .2\, x_2$

given start

Multiply each x_i by λ_i to be like 2

$= x_1 \lambda^{99} + .2\, \lambda^{99} x_2$

keep

$= x_1 (1)^{99} + .2 \left(\frac{1}{2}\right)^{99} x_2$

$= \begin{bmatrix} .6 \\ .4 \end{bmatrix} + \sim 0$

Which was x_1

So notice this was $\lambda = 1$ and < 1

Markov matrix

30

P has $\lambda = 1, 0$

R has $\lambda = 1, -1$

Q has $\lambda = i, -i$

Sum of diagonal = trace = sum of λ

det = product of λ

Why do I keep forgetthing these 2 facts

Q

"Formula Sheet"
Exam 2

$A^{-1} = \frac{C^T}{\det A} \rightarrow \frac{1}{ad-bc}\begin{bmatrix} +d & -b \\ -c & +a \end{bmatrix}$

RREF 1s on diag only

$[A \; I] = [I \; A^{-1}]$ "Gauss Jordan" augmented elim

$\begin{bmatrix} 1 \end{bmatrix}$ $\begin{bmatrix} 0 \text{ or } 1 \end{bmatrix}$ $\begin{bmatrix} \infty \end{bmatrix}$ $\begin{bmatrix} \infty \text{ or } 0 \\ \neq \end{bmatrix}$ no full row rank

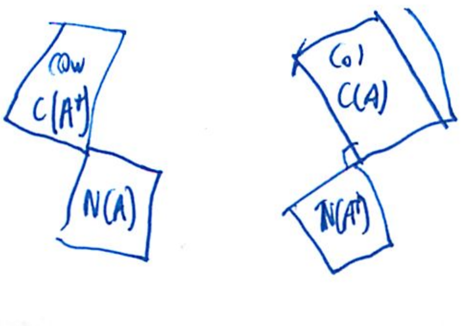
---

<u>Span</u> if lin combo fills the space

<u>basis</u> lin ind + fills the space

<u>dim</u> # of vectors in a basis

dim nullspace $= n - r$



$p = Pb = a\hat{x} = a\frac{a^Tb}{a^Ta}$

$P = \frac{aa^T}{a^Ta}$ ← one #

Projections

$A^TA\hat{x} = A^Tb$

$p = A\hat{x} = A(A^TA)^{-1}A^Tb$

$P = A(A^TA)^{-1}A^T$

$\hat{x} = \frac{a^Tb}{a^Ta}$

$P^2 = P$

$e = b - p$

Least Squares

$A^TA\hat{x} = A^Tb$

← do each separately, solve for $\hat{x}$

$A = \begin{bmatrix} 1 & * \\ 1 & * \\ 1 & * \end{bmatrix}$ ← ind $\quad x = \begin{bmatrix} C \\ D \end{bmatrix} \quad b = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}$ ← dep

---

$q_i^Tq_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$

$Q^TQ = I$

$AA^T = I$	$A^TA = I$

②

Gram Schmidt

$A = a$

$B = b - \frac{A^T b}{A^T A} A$

$C = c - \frac{A^T c}{A^T A} A - \frac{B^T c}{B^T B} B$

$A = QR$

Det

1. $\det I = 1$
2. Can swap any row + swap sign
3. Lin function of each row sep
   - any row
   - only 1 row at a time
   - $t \cdot \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \det\begin{pmatrix} ta & tb \\ c & d \end{pmatrix}$
4. two rows = → det = 0
5. Can subtract one row from another w/o changing det
6. Row of 0s → det = 0
7. Triangular → det = just diagonal entries upper or lower
8. Singular → det = 0
9. $|AB| = |A||B|$
10. $|A^T| = |A|$

Big Formula

Note even/odd # of switches

Cofactors

Any row or col

Have proper sign (checkerboard)

Cramer's Rule

Solves $Ax = b$

$x_1 = \frac{\det B_1}{\det A}$ $\quad x_2 = \frac{\det B_2}{\det A}$

A replaced w/ b for that row

Eigenvalues

$Ax = \lambda x$

Find $A - I\lambda$

Take det of

Set = 0

$\det(A - \lambda I) = 0$

Factor

Then find eigen vector

$(A - \lambda I)x = 0$

for each λ

P $\lambda = 1, 0$

R $\lambda = 1, -1$

Q $\lambda = i, -i$

Formula Practice

4/11

$[\ 1\ ]\quad[\ \infty\ ]\quad[\ 0 \text{ or } 1\ ]\quad\begin{bmatrix} 0 \text{ or } \infty \\ \neq \end{bmatrix}$ ✓

$$\left.\begin{aligned} p &= A\hat{x} \\ p &= Pb \end{aligned}\right\} = \text{✓}$$

$$\hat{x} = \cancel{\frac{aa^T}{a^Ta}a} = \cancel{x}\,\frac{a^Tb}{a^Ta}$$ close

$$= \cancel{a}\,\frac{a^Tb}{a^Ta}$$

↑ not strictly included in $\hat{x}$

$$\underbrace{A^TA}_{\text{2 groups}}\hat{x} = \underbrace{A^Tb}$$ ✓

$$P = A(A^TA)^{-1}A^T\cancel{b}$$

$$A\hat{x} = A(A^TA)^{-1}A^Tb$$

②

Then

$A = a$

$B = b - \frac{A^T b}{A^T A} A$ ✓

$P = A(A^TA)^{-1}A^T \cancel{b}$

$A^TA\,\hat{x} = A^Tb$

Think of what we need operationally

$\underset{\text{small}}{p} = A\hat{x} = A(A^TA)^{-1}A^T$

Operationally find A^TA A^Tb

find $\hat{x}$

find $A\hat{x} = p$

$P = A(A^TA)^{-1}A^T$

$\frac{1}{\det}$ cofactors

(3)

Cramers

$$X_1 = \frac{\det B_1}{\det A}$$ ← replace col 1 A w/ b

$$\det(A - I\lambda) = 0$$

set $(A - I\lambda)x = 0$
↑ solve

So back to proj

$\underbrace{A^T A \qquad\qquad A^T b}_{\text{find } \hat{x}}$

$p = A\hat{x}$

$P = A(A^T A)^{-1} A^T$ no b

$p = A(A^T A)^{-1} A^T b$

(4)

$$p = Pb = a\hat{x} = a \frac{a^T b}{a^T a}$$

$$P = \frac{a a^T}{a^T a}$$

$$p = \frac{a^T b}{a^T a} a$$

$$P = \frac{a a^T}{a^T a}$$

Hopefully I remember this

8.1 Preview

4/10

incidence matrix

$$\begin{matrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{matrix}$$

← ~~edges~~ vertices

$\Sigma =$ deg

↑ ~~vertices~~ edges

$\Sigma = 2$

Kirchoff : Flow in = flow out

18.06 Professor Johnson Quiz 2 April 1, 2009

4/10 Pratice

Your PRINTED name is: ____________________

Please circle your recitation:

(R01)	M2	2-314	Qian Lin
(R02)	M3	2-314	Qian Lin
(R03)	T11	2-251	Martina Balagovic
(R04)	T11	2-229	Inna Zakharevich
(R05)	T12	2-251	Martina Balagovic
(R06)	T12	2-090	Ben Harris
(R07)	T1	2-284	Roman Bezrukavnikov
(R08)	T1	2-310	Nick Rozenblyum
(R09)	T2	2-284	Roman Bezrukavnikov

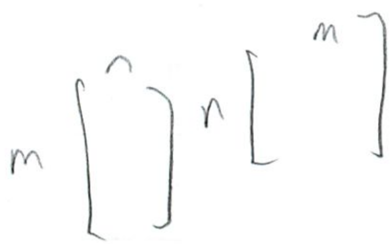
Grading	
1	
2	
3	
Total:	

1 (20 pts.) (a) If P is the projection matrix onto the *null* space of A, then $P\mathbf{y} - \mathbf{y}$, for any $\mathbf{y}$, is in the ___row___ space of A. ?? wtf

(b) If $A\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{x}$, then the closest vector to $\mathbf{b}$ in $N(A^T)$ is ___linear approx___ (best answer). 0

(c) If the *rows* of A (an $m \times n$ matrix) are independent, then the dimension of $N(A^TA)$ is ___$m \times n$___. $n-m$ Same

(d) If a matrix U has orthonormal *rows*, then $I =$ ___UU^T___ and the projection matrix onto the *row* space of U is ___U^TU___. (Your answers should be the simplest expressions involving U and U^T only.)



I didn't get any of those
Why?

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2 (30 pts.) The matrix

$$A = \begin{pmatrix} 1 & 2 & 1 & -7 \\ 2 & 4 & 1 & -5 \\ 1 & 2 & 2 & -16 \end{pmatrix}$$

is converted to row-reduced echelon form by the usual row-elimination steps, resulting in the matrix:

$$\begin{array}{cccc} i & d & i & d \end{array}$$

$$R = \begin{pmatrix} 1 & 2 & 0 & 2 \\ 0 & 0 & 1 & -9 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$ ↑ not left RREF

(♣) The *minimum* number of columns of A that form a *dependent* set of vectors is 2. The *maximum* number of columns of A that forms an *independent* set of vectors is 2. ✓

Col space 2 dimensional
so 2 cols in the basis

(♢) Give an *orthonormal* basis for the *row* space of A. (Careful: be sure you start with a basis for the row space, not containing any dependent vectors.) Your answer may contain square roots left as $\sqrt{\textit{some number}}$.

(♠) Given the vector $\mathbf{b} = \begin{pmatrix} 2 & 5 & -9 & 3 \end{pmatrix}^T$, compute the *closest* vector $\mathbf{p}$ to $\mathbf{b}$ in the *row space* $C(A^T)$? (Hint: less calculation is needed if you use your answer from ♢.)

(♡) In terms of your answer $\mathbf{p}$ to ♠ above, what is the closest vector to $\mathbf{b}$ in the *nullspace* $N(A)$? (No calculation required, and you need not have solved ♠: you can leave your answer in terms of $\mathbf{p}$ and $\mathbf{b}$.)

b-p Oh duh;

basis - #s add up to it

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$ normal already

$$b = \begin{bmatrix} 2 \\ 5 \\ -9 \\ 3 \end{bmatrix} =$$ Least squares?

2
4
9
4

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3 (20 pts.) You are told that the least-square linear fit to three points $(0, b_1)$, $(1, b_2)$, and $(2, b_3)$ is $C + Dt$ for $C = 1$ and $D = -2$. That is, the fit is $1 - 2t$.

In this question, you will work backwards from this fit to reason about the unknown values $\mathbf{b} = \begin{pmatrix} b_1 & b_2 & b_3 \end{pmatrix}^T$ at the coordinates $t = 0, 1, 2$.

(i) Write down the explicit equations that $\mathbf{b}$ must satisfy for $1 - 2t$ to be the least-square linear fit. (The points do *not* have to fall exactly on the line.)

(ii) If all the points fall *exactly* on the line $1 - 2t$, then $\mathbf{b} =$ __________. Check that this satisfies your equations in (i).

(iii) More generally, if all the points fall exactly on *any* straight line, then $\mathbf{b}$ is in the ______________ space of what matrix? (Write down the matrix.)

$$\underset{A}{\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Solve for

told $C = 1$
$D = -2$

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 + 0 = 1 \\ 1 - 2 = -1 \\ 1 - 4 = -3 \end{bmatrix}$$

☺ their way seemed a lot more complicated!

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18.06 Quiz 2 Solution

Hold on Wednesday, 1 April 2009 at 11am in Walker Gym.
Total: 70 points.

Problem 1:

(a) If P is the projection matrix onto the *null* space of A, then $P\mathbf{y} - \mathbf{y}$, for any $\mathbf{y}$, is in the __________ space of A.

(b) If $A\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{x}$, then the closest vector to $\mathbf{b}$ in $N(A^T)$ is __________ (best answer).

(c) If the *rows* of A (an $m \times n$ matrix) are independent, then the dimension of $N(A^T A)$ is __________.

(d) If a matrix U has orthonormal *rows*, then $I =$ __________ and the projection matrix onto the *row* space of U is __________. (Your answers should be the simplest expressions involving U and U^T only.)

$\boxed{\text{Solution}}$ (20 points = 5+5+5+5)
Answers: (a) row; (b) 0; (c) $n - m$; (d) UU^T, $U^T U$.

(a) Since $P\mathbf{y}$ is the projection to the nullspace of A, $P\mathbf{y} - \mathbf{y}$ is orthogonal to the null space; it then must lie in the row space of A.

(b) Since $A\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{x}$, $\mathbf{b}$ is in the column space $C(A)$ of A. We know that the left nullspace $N(A^T)$ is orthogonal to the column space. So the closest vector to $\mathbf{b}$ is 0.

(c) We derived in Problem 7 of Pset 4 that the nullspace of $A^T A$ is the same as the nullspace of A; the latter has dimension $n - m$ because the matrix A is of full row rank m.
Alternatively, we also derived the following in lecture, and it is in the text, and on the practice-exam handout: the ranks of A and $A^T A$ are the same, so both equal to m. Since A has full row rank and $A^T A$ has n columns, $N(A^T A)$ has dimension $n - m$.

(d) Note that $U^T = Q$, a matrix with orthonormal columns. We saw in class that $I = Q^T Q = UU^T$, and the projection matrix onto $C(Q) = C(U^T)$ is $QQ^T = U^T U$.

Problem 2: The matrix

$$A = \begin{pmatrix} 1 & 2 & 1 & -7 \\ 2 & 4 & 1 & -5 \\ 1 & 2 & 2 & -16 \end{pmatrix}$$

is converted to row-reduced echelon form by the usual row-elimination steps, resulting in the matrix:

$$R = \begin{pmatrix} 1 & 2 & 0 & 2 \\ 0 & 0 & 1 & -9 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(♣) The *minimum* number of columns of A that form a *dependent* set of vectors is __________. The *maximum* number of columns of A that forms an *independent* set of vectors is __________.

(♢) Give an *orthonormal* basis for the *row* space of A. (Careful: be sure you start with a basis for the row space, not containing any dependent vectors.) Your answer may contain square roots left as $\sqrt{\textit{some number}}$.

(♠) Given the vector $\mathbf{b} = \begin{pmatrix} 2 & 5 & -9 & 3 \end{pmatrix}^T$, compute the *closest* vector $\mathbf{p}$ to $\mathbf{b}$ in the *row space* $C(A^T)$? (Hint: less calculation is needed if you use your answer from ♢.)

(♡) In terms of your answer $\mathbf{p}$ to ♠ above, what is the closest vector to $\mathbf{b}$ in the *nullspace* $N(A)$? (No calculation required, and you need not have solved ♠: you can leave your answer in terms of $\mathbf{p}$ and $\mathbf{b}$.)

$\boxed{\text{Solution}}$ (30 points = 6+10+10+4)
Answers: (♣) 2, 2; (♢) see below; (♠) $\mathbf{p} = (2\ \ 4\ \ 0\ 4)^T$; (♡) $\mathbf{b} - \mathbf{p}$.

(♣) The key point of the problem is that the dependency of columns in R and A is the same. By inspection of R (or A), the first two columns are dependent, so that is the smallest dependent set. R has two pivots, so A is rank 2 and the column space is 2-dimensional, so 2 is the maximum number of independent columns. Equivalently, the maximum number of independent columns is the number of columns in any basis for $C(A)$, such as the 2 pivot columns.

(♢) Note that the (row-reduced) echelon form R has the same row space as A. We may therefore start Gram-Schmidt on the pivot rows of R, which form a basis for the row space of R and A.

$$\mathbf{q}_1 = \frac{\mathbf{a}_1}{\|\mathbf{a}_1\|} = (\tfrac{1}{3}\ \ \tfrac{2}{3}\ \ 0\ \ \tfrac{2}{3})^{\mathrm{T}}$$

$$\begin{aligned}\mathbf{q}_2 &= \frac{\mathbf{a}_2 - \mathbf{q}_1^{\mathrm{T}}\mathbf{a}_2\mathbf{q}_1}{\|\mathbf{a}_2 - \mathbf{q}_1^{\mathrm{T}}\mathbf{a}_2\mathbf{q}_1\|} = \frac{(0\ \ 0\ \ 1\ \ -9)^{\mathrm{T}} + 6\cdot(\frac{1}{3}\ \ \frac{2}{3}\ \ 0\ \ \frac{2}{3})^{\mathrm{T}}}{\|(0\ \ 0\ \ 1\ \ -9)^{\mathrm{T}} + 6\cdot(\frac{1}{3}\ \ \frac{2}{3}\ \ 0\ \ \frac{2}{3})^{\mathrm{T}}\|} \\ &= \frac{(2\ \ 4\ \ 1\ \ -5)^{\mathrm{T}}}{\|(2\ \ 4\ \ 1\ \ -5)^{\mathrm{T}}\|} = (2\ \ 4\ \ 1\ \ -5)^{\mathrm{T}}/\sqrt{46}.\end{aligned}$$

REMARK: One can also obtain an orthonormal basis by starting with 2 rows of A since in this case any 2 rows are independent and form a basis. But the pivot rows of R are a nicer basis (more zeros), and the calculations are therefore much simpler.

(♠) The closest vector $\mathbf{p}$ should be given by the projection to the row space. That is

$$\mathbf{p} = \mathbf{p}^{\mathrm{T}}\mathbf{q}_1\mathbf{q}_1 + \mathbf{p}^{\mathrm{T}}\mathbf{q}_2\mathbf{q}_2 = 6\cdot(\frac{1}{3}\ \ \frac{2}{3}\ \ 0\ \ \frac{2}{3})^{\mathrm{T}} + 0 = (2\ \ 4\ \ 0\ \ 4)^{\mathrm{T}}.$$

(♡) The closest vector $\mathbf{p}$ of $\mathbf{b}$ in the row space is exactly the projection in the row space. But the row space and the nullspace are orthogonal to each other. Then, $\mathbf{b}-\mathbf{p}$ is exactly the orthogonal projection in the nullspace $N(A)$; it is the closest vector to $\mathbf{b}$ in the nullspace.

Problem 3: You are told that the least-square linear fit to three points $(0, b_1)$, $(1, b_2)$, and $(2, b_3)$ is $C + Dt$ for $C = 1$ and $D = -2$. That is, the fit is $1 - 2t$.

In this question, you will work backwards from this fit to reason about the unknown values $\mathbf{b} = (b_1\ \ b_2\ \ b_3)^T$ at the coordinates $t = 0, 1, 2$.

(i) Write down the explicit equations that $\mathbf{b}$ must satisfy for $1 - 2t$ to be the least-square linear fit. (The points do *not* have to fall exactly on the line.)

(ii) If all the points fall *exactly* on the line $1 - 2t$, then $\mathbf{b} =$ __________. Check that this satisfies your equations in (i).

(iii) More generally, if all the points fall exactly on *any* straight line, then $\mathbf{b}$ is in the ______________ space of what matrix? (Write down the matrix.)

$\boxed{\text{Solution}}$ (20 points = 10+5+5)

Answers: (i) see below; (ii) $\mathbf{b} = (1\ \ -1\ \ -3)$; (iii) see below.

(i) The system that we would solve if the line passed exactly through all of the points is

$$\begin{pmatrix}1 & 0\\ 1 & 1\\ 1 & 2\end{pmatrix}\begin{pmatrix}C\\ D\end{pmatrix} = \begin{pmatrix}b_1\\ b_2\\ b_3\end{pmatrix}$$

However, since the line may not pass through all the points this system may have no solution, and instead we find the least-square solution by solving the normal equations:

$$\begin{pmatrix}1 & 1 & 1\\ 0 & 1 & 2\end{pmatrix}\begin{pmatrix}1 & 0\\ 1 & 1\\ 1 & 2\end{pmatrix}\begin{pmatrix}C\\ D\end{pmatrix} = \begin{pmatrix}1 & 1 & 1\\ 0 & 1 & 2\end{pmatrix}\begin{pmatrix}b_1\\ b_2\\ b_3\end{pmatrix}$$

That is

$$\begin{pmatrix}3 & 3\\ 3 & 5\end{pmatrix}\begin{pmatrix}C\\ D\end{pmatrix} = \begin{pmatrix}b_1 + b_2 + b_3\\ b_2 + 2b_3\end{pmatrix}$$

Since the least-square fit is $1-2t$, the above linear system has solution $(1\ \ -2)^{\mathrm{T}}$. Hence, b_1, b_2, b_3 should satisfy

$$\begin{aligned} b_1 + b_2 + b_3 &= -3 \\ b_2 + 2b_3 &= -7 \end{aligned}$$

(ii) If all the points fall exactly on the line $1 - 2t$,

$$b_1 = 1 - 2\cdot 0 = 1, \quad b_2 = 1 - 2\cdot 1 = -1, \quad b_3 = 1 - 2\cdot 2 = -3.$$

We plug the solution back in the relations above and check.

$$1 - 1 - 3 = -3, \quad -1 + 2 \times (-3) = -7.$$

(iii) If all points fall exactly on a straight line, the following system would have a solution.

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}.$$

In other words, **b** lies in the column space of the matrix

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

18.06 Spring 2009 Exam 2 Practice

General comments

Diff than here

Exam 2 covers the first 18 lectures of 18.06. It does *not* cover determinants (lectures 19 and 20). There will also be *no* questions on graphs and networks. The topics covered are (very briefly summarized):

1. All of the topics from exam 1.

2. Linear independence [key point: the columns of a matrix A are independent if $N(A) = \{0\}$], bases (an independent set of vectors that spans a space), and dimension of subspaces (the number of vectors in *any* basis).

3. The four fundamental subspaces (key points: their dimensions for a given rank r and $m \times n$ matrix A, their relationship to the solutions [if any] of $Ax = b$, their orthogonal complements, and how/why we can find bases for them via the elimination process).

4. What happens to the four subspaces as we do matrix operations, especially elimination steps and more generally how the subspaces of AB compare to those of A and B. The fact (important for projection and least-squares!) that A^TA has the same rank as A, the same null space as A, and the same column space as A^T, and why (we proved this in class and another way in homework).

5. Orthogonal complements $S^\perp$ for subspaces S, especially (but not only) the four fundamental subspaces.

6. Orthogonal projections: given a matrix A, the projection of b onto $C(A)$ is $p = A\hat{x}$ where $\hat{x}$ solves $A^TA\hat{x} = A^Tb$ [always solvable since $C(A^TA) = C(A^T)$]. If A has full column rank, then A^TA is invertible and we can write the projection matrix $P = A(A^TA)^{-1}A^T$ (so that $A\hat{x} = Pb$, but it is *much* quicker to solve $A^TA\hat{x} = A^Tb$ by elimination than to compute P in general). $e = b - A\hat{x}$ is in $C(A)^\perp = N(A^T)$, and $I - P$ is the projection matrix onto $N(A^T)$.

7. Least-squares: $\hat{x}$ minimizes $\|Ax - b\|^2$ over all x, and is the *least-squares* solution. That is, $p = A\hat{x}$ is the *closest* point to b in $C(A)$. Application to least-square curve fitting, minimizing the sum of the squares of the errors.

8. Orthonormal bases, forming the columns of a matrix Q with $Q^TQ = I$. The projection matrix onto $C(Q)$ is just QQ^T, and $\hat{x} = Q^Tb$. Obtaining Q from A (i.e., an orthonormal basis from any basis) by Gram-Schmidt, and the correspondence of this process to $A = QR$ factorization where $R = Q^TA$ is invertible and upper-triangular. Using $A = QR$ to solve equations (either $Ax = b$ or $A^TA\hat{x} = A^Tb$). Q is an *orthogonal matrix* only if it is square, in which case $Q^T = Q^{-1}$.

9. Dot products of functions, and hence Gram-Schmidt, orthonormal bases (e.g. Fourier series or orthogonal polynomials), orthogonal projection, and least-squares for functions.

As usual, the exam questions may turn these concepts around a bit, e.g. giving the answer and asking you to work backwards towards the question, or ask about the same concept in a slightly changed context. We want to know that you have really internalized these concepts, not just memorizing an algorithm but knowing *why* the method works and where it came from.

Some practice problems

The 18.06 web site has exams from previous terms that you can download, with solutions. I've listed a few practice exam problems that I like below, but there are plenty more to choose from. (Note: exam 2 in several previous terms asked about determinants; we *won't* have any determinant questions until exam 3.) The exam will consist of 3 or 4 questions (perhaps with several parts each), and you will have one hour. You can find the solutions to these problems on the 18.06 web site (in the section for old exams/psets). On the last page I give practice problems for orthogonal functions and orthogonal projections of functions.

1. (Fall 2002 exam 2.) **(a)** Choose c and the last column of Q so that you have an orthogonal matrix:

$$Q = c\begin{bmatrix} 1 & -1 & -1 & ? \\ -1 & 1 & -1 & ? \\ -1 & -1 & -1 & ? \\ -1 & -1 & 1 & ? \end{bmatrix}.$$

(b) Project $b = (1,1,1,1)^T$ onto the first column of Q. Then project b onto the plane spanned by the first two columns. **(c)** Suppose the last column of this matrix (where the ?'s are) were changed to $(1,1,1,1)^T$. Call this new matrix A. If Gram-Schmidt is applied to the 4 columns of A, what would be the 4 outputs q_1, q_2, q_3, q_4? (Don't do a lot of calculations...please!)

2. (Fall 2008 exam 2.) [The parts of this question are independent and can be done in any order.] **(a)** P is the projection matrix onto $C(A)$, where A has independent columns. Q is a square orthogonal matrix with the same number of rows as A. In its simplest form, in terms of P and Q, what is the projection matrix onto the column space of QA? **(b)** The vectors a, b, and c are independent. The matrix P is the projection matrix onto the span of a and b. Suppose we apply Gram-Schmidt onto the vectors a, b,and c to produce orthonormal vectors q_1,q_2,and q_3. Write the unit vector q_3 in simplest form in terms of P and c only. **(c)** The vectors a, b, and c are independent, and the matrix A has these three vectors as its columns. You are given the QR decomposition of A, where Q is orthogonal and R is 3×3 upper-triangular as usual. Write $\|c\|$ in terms of only the elements of R, in simplest form.

3. (Fall 2008 exam 2.) Suppose we have obtained from measurements n data points (t_i, b_i) and you are asked to find a best least-squares fit function of the form $y = C + Dt + E(1-t)$. Are C, D, and E uniquely determined? Write down a solvable system of equations that gives a solution to the least-squares problem.

4. (Fall 2008 exam 2.) **(a)** If A is invertible, must the column space of A^{-1} be the same as the column space of A? **(b)** If A is square, must the column space of A^2 be the same as the column space of A?

5. (Fall 2005 exam 1.) Suppose A is $m\times n$ with *linearly dependent columns*. Complete with as much true information as possible: **(a)** The rank of A is .....? **(b)** The nullspace of A contains .....? **(c)** The equation $A^T y = b$ has no solution for some right-hand sides b because ......? (more words needed)

6. (Fall 2005 exam 1.) Suppose A is the 3×4 matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \end{bmatrix}.$$

(a) A basis for $C(A)$ is .....? **(b)** For which vectors $b = (b_1, b_2, b_3)^T$does $Ax = b$ have a solution? (Give specific conditions on $b_{1,2,3}$.) **(c)** Explain why there is no 4×3 matrix B for which $AB = I$ (3×3). Give a good reason (the mere fact that A is rectangular is *not* sufficient).

7. (Spring 2005 exam 1.) Suppose the columns of a 7×4 matrix A are linearly independent. **(a)** After row operations reduce A to U or R, how many rows will be all zero (or is it impossible to tell)? **(b)** Assume that no row swaps were required for elimination. What is the row space of A? Explain why this equation will surely be solvable: $A^T y = (1,0,0,0)^T$.

8. (Fall 2005 exam 2.) The matrix Q has orthonormal columns q_1, q_2, q_3:

$$Q = \begin{bmatrix} 0.1 & 0.5 & a \\ 0.7 & 0.5 & b \\ 0.1 & -0.5 & c \\ 0.7 & -0.5 & d \end{bmatrix}.$$

(a) What equations must be satisfied by the numbers a, b, c, d? Is there a unique choice for those (real) numbers, apart from multiplying them all by -1? **(c)** Suppose Gram-Schmidt starts with those same first two columns and with the third column $a_3 = (1,1,1,1)^T$. What third column would it choose for q_3. (You can leave a square root as $\sqrt{\cdots}$ if you want to.)

9. (Fall 2005 exam 2.) Our measurements at times $t = 1,2,3$ are $b = 1,4,b_3$. We want to fit those points by the nearest line $C + Dt$, using least-squares. **(a)** Which value for b_3 will put the three points on a straight line? Give C and D for this line. Will least squares choose that line if the third measurement is $b_3 = 9$? (Yes or no.) **(b)** What is the lienar system $Ax = b$ that would be solved exactly for $x = (C,D)$ if the three points do lie on a line? Compute the projection matrix P onto the column space of A. You can use the 2×2 inverse formula $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. **(c)** What is the rank of that projection matrix P? How is the column space of P related to the column space of A? (You can answer this part without your answer from **b.**) **(d)** Suppose $b_3 = 1$. Write down the equation for the best least-squares solution $\hat{x}$, and show that the best straight line is horizontal in this case.

10. (Fall 2006 exam 2.) Suppose we take measurements at the 21 equally spaced times $t = -10, -9, \ldots, 9, 10$. All measurements are $b_i = 0$ except that $b_{11} = 1$ at the middle time $t = 0$. **(a)** Using least squares, what are the best C and D to fit those 21 points by a straight line $C + Dt$? **(b)** You are projecting the vector b onto what subspace? (*Give a basis.*) Find a nonzero vector perpendicular to that subspace.

11. (Fall 2006 exam 2.) The Gram-Schmidt methods produces orthonormal vectors q_1, q_2, q_3 from independent vectors a_1, a_2, a_3 in $\mathbb{R}^5$. Put those vectors into the columns of 5×3 matrices Q and A, respectively. **(a)** Give formulas using Q and A for the projection matrices P_Q and P_A onto the column spaces of Q and A, respectively. **(b)** Does P_Q equal P_A, and why or why not? What is $P_Q Q$? **(c)** Suppose a_4 is a new vector, and a_1, a_2, a_3, a_4 are independent. Which of the following (if any) is the new Gram-Schmidt vector q_4? **1:** $\frac{P_Q a_4}{\|P_Q a_4\|}$. **2:** $\frac{a_4 - \frac{a_4^T a_1}{a_1^T a_1} a_1 - \frac{a_4^T a_2}{a_2^T a_2} a_2 - \frac{a_4^T a_3}{a_3^T a_3} a_3}{\|\cdots\text{same vector}\cdots\|}$. **3.** $\frac{a_4 - P_A a_4}{\|a_4 - P_A a_4\|}$.

12. (Spring 2004 exam 2.) We are given two vectors a and b in $\mathbb{R}^4$, $a = (2,5,2,4)^T$ and $b = (1,2,1,0)^T$. **(a)** Find the projection p of the vector b onto the line through a. Check(!) that the error $e = b - p$ is perpendicular to....what? **(b)** The subspace S of all vectors in $\mathbb{R}^4$ that are perpendicular to this a is 3-dimensional. Compute the projection q of b onto this perpendicular subspace S. (It doesn't need a big computation!)

13. (Spring 2004 exam 2.) Suppose that q_1, q_2, and q_3 are 3 orthonormal vectors in $\mathbb{R}^n$. They go into the columns of an $n \times 3$ matrix Q. **(a)** What inequality ($\leq$ or $\geq$) do you know for n? Is there any condition on n required in order to have $Q^T Q = I$? Is there any condition on n required to have $QQ^T = I$? **(b)** Give a nice matrix formula involving b and Q for the projection p of a vector b onto the column space of Q. Complete the sentence: p is the closest vector .... **(c)** Suppose the projection of b onto that column space is $p = c_1 q_1 + c_2 q_2 + c_3 q_3$. Find a formula for c_1 that only involves b and q_1 (possibly using dot products).

14. (Spring 2005 exam 2.) If the output vectors from Gram-Schmidt are: $q_1 = (\cos\theta, \sin\theta)^T$ and $q_2 = (-\sin\theta, \cos\theta)$ for some θ, describe all possible input vectors a_1 and a_2.

15. (Spring 2005 exam 2.) If a and b are nonzero vectors in $\mathbb{R}^n$, what number x minimizes the squared length $\|b - xa\|^2$?

16. (Spring 2005 exam 2.) Find the projection p of the vector $b = (1,2,6)^T$ onto the plane $x + y + z = 0$ in $\mathbb{R}^3$. (You may want to first find a basis for this 2-dimensional subspace, perhaps even an orthogonal basis.)

17. (Spring 2005 exam 2.) You are given the matrix

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \\ 1 & 2 \end{bmatrix}.$$

Suppose P_1 is the projection matrix onto the 1-dimensional subspace spanned by the first column of A. Suppose P_2 is the projection matrix onto the 2-dimensional column space of A. After thinking a little, compute the product P_2P_1.

None of the first two exams in previous terms covered orthogonal functions—these are in the standard 18.06 syllabus, especially Fourier series, but previously weren't covered until later in the term, after eigenproblems. A couple of problems about orthogonal functions appeared on your last problem set, which you should review, and a couple more practice problems on this topic are:

1. Suppose you are given three functions $a_1(t)$, $a_2(t)$, and $b(t)$ for $0 \le t \le 1$. Define dot products of any two functions $f(t)$ and $g(t)$ by $f(t)\cdot g(t) = \int_0^1 f(t)g(t)dt$ (hence, $\|f(t)\| = \sqrt{\int_0^1 f(t)^2 dt}$). Suppose we want the "best-fit" function $p(t) = Ca_1(t) + Da_2(t)$ that minimizes $\|p(t) - b(t)\|$ over all possible C and D. Give an explicit formula for $p(t)$ in terms of some integrals and other expressions involving $a_1(t)$, $a_2(t)$, and $b(t)$ only.

2. The functions $q_1(t) = \sin(t)/\sqrt{\pi}$, $q_2(t) = \sin(2t)/\sqrt{\pi}$, and $q_3(t) = \cos(t)/\sqrt{\pi}$ are orthonormal if we define dot products of any two functions $f(t)$ and $g(t)$ by $f(t)\cdot g(t) = \int_0^{2\pi} f(t)g(t)dt$. **(a)** Write the function $b(t) = t$ as the sum of two functions, one in the span of q_1, q_2 and q_3 and one perpendicular to q_1, q_2 and q_3. You should write your answer explicitly in terms of integrals *etc.*, but you need not evaluate the integrals (this isn't 18.01). **(b)** If you were to do Gram-Schmidt on the set of four functions q_1, q_2, q_3, b, in that order, what would you get?

Solutions:

1. This is just a least-squares problem. There are a couple of ways to do this, but the way we learned in class is to first find an orthonormal basis by Gram-Schmidt: $q_1(t) = a_1/\|a_1\| = a_1(t)/\sqrt{\int_0^1 a_1(t)^2 dt}$, $q_2(t) = (a_2 - q_1[q_1\cdot a_2])/\|\cdots\| = [a_2 - q_1\int_0^1 q_1(t)a_2(t)dt]/\|\cdots\|$. Then $p(t) = q_1(q_1\cdot b) + q_2(q_2\cdot b) = q_1(t)\int_0^1 q_1(t')b(t')dt' + q_2(t')\int_0^1 q_2(t')b(t')dt'$.

2. **(a)** We are just writing $b(t) = p(t) + e(t)$, where $p(t)$ is the orthogonal projection and $e(t) = b(t) - p(t)$. Exactly as for vectors, we can write the orthogonal projection as:

$$p(t) = \sum_{i=1}^{3} q_i(q_i\cdot b) = \frac{\sin(t)}{\pi}\int_0^{2\pi}\sin(t')t'dt' + \frac{\sin(2t)}{\pi}\int_0^{2\pi}\sin(2t')t'dt' + \frac{\cos(t)}{\pi}\int_0^{2\pi}\cos(t')t'dt',$$

and thus $e(t) = t - p(t)$ is perpendicular to q_1, q_2, q_3. **(b)** q_1 to q_3 are already orthonormal, so they wouldn't be changed by Gram-Schmidt. When you do Gram-Schmidt on the last function $b(t)$, you would subtract off the projection and then normalize...but this is precisely the function $q_4(t) = e(t)/\|e(t)\| = e(t)/\sqrt{\int_0^{2\pi} e(t')^2 dt'}$.

The key point that I want you to understand is that you just do exactly the same steps as you would for vectors, and the "only" change is that the dot products become some kind of integral (depending on what the function dot product was chosen to be).

18.06 Quiz 2 April 7, 2010 Professor Strang

Your PRINTED name is: ______________________________ **1.**

Your recitation number or instructor is ______________________________ **2.**

3.

______

1. (30 points)

(a) Find the matrix P that projects every vector b in R^3 onto the line in the direction of $a = (2, 1, 3)$.

(b) What are the column space and nullspace of P? Describe them geometrically and also *give a basis for each space.*

(c) What are *all* the eigenvectors of P and their corresponding eigenvalues? (You can use the geometry of projections, not a messy calculation.) The diagonal entries of P add up to ______ .

2. (30 points)

(a) $p = A\widehat{x}$ is the vector in $C(A)$ nearest to a given vector b. If A has independent columns, what equation determines $\widehat{x}$? What are all the vectors perpendicular to the error $e = b - A\widehat{x}$? What goes wrong if the columns of A are dependent?

(b) Suppose $A = QR$ where Q has orthonormal columns and R is upper triangular invertible. Find $\widehat{x}$ and p in terms of Q and R and b (not A).

(c) (Separate question) If q_1 and q_2 are any orthonormal vectors in R^5, give a formula for the projection p of any vector b onto the plane spanned by q_1 and q_2 *(write p as a combination of q_1 and q_2).*

3. (40 points) This problem is about the n by n matrix A_n that has zeros on its main diagonal and all other entries equal to -1. In MATLAB $A_n = \text{eye}\,(n) - \text{ones}\,(n)$.

(a) Find the determinant of A_n. Here is a suggested approach:

Start by adding all rows (except the last) to the last row, and then factoring out a constant. (You could check $n = 3$ to have a start on part b.)

(b) For any invertible matrix A, the $(1,1)$ entry of A^{-1} is the ratio of ________________ .

So the $(1,1)$ entry of A_4^{-1} is ________________ .

(c) Find *two orthogonal eigenvectors* with $A_3\,x = x$. (So $\lambda = 1$ is a double eigenvalue.)

(d) What is the third eigenvalue of A_3 and a corresponding eigenvector?

18.06 Quiz 2 April 7, 2010 Professor Strang

Your PRINTED name is: ____________________ **1.**

Your recitation number or instructor is ____________________ **2.**

3.

1. (33 points)

(a) Find the matrix P that projects every vector b in R^3 onto the line in the direction of $a = (2, 1, 3)$.

$\boxed{\text{Solution}}$ The general formula for the orthogonal projection onto the column space of a matrix **A** is

$$P = \mathbf{A}(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T.$$

Here,

$$\mathbf{A} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \quad \text{so that} \quad \boxed{\mathbf{P} = \frac{1}{14}\begin{bmatrix} 4 & 2 & 6 \\ 2 & 1 & 3 \\ 6 & 3 & 9 \end{bmatrix}}$$

Remarks:

- Since we're projecting onto a one-dimensional space, $\mathbf{A}^T\mathbf{A}$ is just a number and we can write things like $P = (\mathbf{A}\mathbf{A}^T)/(\mathbf{A}^T\mathbf{A})$. *This won't work in general.*
- You don't have to know the formula to do this. The i^{th} column of **P** is, pretty much by definition, the projection of e_i ($e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$) onto the line in the direction of a. And this is something you should know how to do without a formula.

RUBRIC: There was some leniency for computational errors, but otherwise there weren't many opportunities for partial credit.

(b) What are the column space and nullspace of P? Describe them geometrically and also *give a basis for each space.*

$\boxed{\text{Solution}}$ The column space is the $\boxed{\text{line in } R^3 \text{ in the direction of } a = (2, 1, 3).}$ One basis for it is

$$\boxed{\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}}$$

and there's not really much choice in giving this basis (you can rescale by a non-zero constant).

The nullspace is the $\boxed{\text{plane in } R^3 \text{ that is perpendicular to } a = (2, 1, 3)}$ (i.e., $2x+y+z = 0$.) One basis for it is

$$\boxed{\begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}}$$

though there are a lot of different looking choices for it (any two vectors that are perpendicular to a and not in the same line will work).

RUBRIC: 6 points for giving a correct basis, and 4 points for giving the complete geometric description. Note that it is not correct to say e.g., $N(\mathbf{P}) = R^2$. It is correct to say that $N(\mathbf{P})$ is a (2-dimensional) plane in R^3, but this is not a complete geometric description unless you say (geometrically) *which* plane it is: the one perpendicular to a/to the line through a.

(c) What are *all* the eigenvectors of P and their corresponding eigenvalues? (You can use the geometry of projections, not a messy calculation.) The diagonal entries of P add up to ______ .

Solution The diagonal entries of P add up to $\boxed{1 = \text{the sum of the eigenvalues}}$

Since **P** is a projection, it's only possible eigenvalues are $\lambda = 0$ (with multiplicity equal to the dimension of the nullspace, here 2) and $\lambda = 1$ (with multiplicity equal to the dimension of the column space, here 1). So, a complete list of eigenvectors and eigenvalues is:

- $\lambda = 0$ with multiplicity 2. The eigenvectors for $\lambda = 0$ are precisely the vectors in the null space. That is, all linear combinations of $\begin{bmatrix} 3 & 0 & -2 \end{bmatrix}^T$ and $\begin{bmatrix} -1 & 2 & 0 \end{bmatrix}^T$.
- $\lambda = 1$ with multiplicity 1. The eigenvectors for $\lambda = 1$ are precisely the vectors in the column space. That is, all multiples of $\begin{bmatrix} 2 & 1 & 3 \end{bmatrix}^T$.

RUBRIC: 2 points for the sum of eigenvalues, 4 points for a full list (with multiplicities) of eigenvalues, and 4 points for a complete description of all eigenvectors. In light of the emphasized "*all*," you'd lose 1 point if you gave two eigenvectors for $\lambda = 0$ and didn't say that all (at least non-zero) linear combinations were also eigenvectors for $\lambda = 0$.

2. (34 points)

(a) $p = A\hat{x}$ is the vector in $C(A)$ nearest to a given vector b. If A has independent columns, what equation determines $\hat{x}$? What are all the vectors perpendicular to the error $e = b - A\hat{x}$? What goes wrong if the columns of A are dependent?

Solution $\hat{x}$ is determined by the equation $\hat{x} = (A^TA)^{-1}A^Tb$ (since A has independent columns, A^TA is invertible whether or not A is square). The vectors perpendicular to an arbitrary error vector are the elements of the column space of A. If the columns of A are dependent, A^TA is no longer invertible, and there is no unique nearest vector (i.e. there are multiple solutions).

RUBRIC: 4 points for the determining equation (1 point off for actually inverting A^TA or saying that it was invertible), 3 points for identifying the column space, and three points for identifying the multiple solutions (1 point off if you just say that A^TA is not invertible). Note that you cannot write $A^{-1}B$ as $\frac{B}{A}$: this only works for numbers because multiplication and division are commutative, which is not true for matrices.

(b) Suppose $A = QR$ where Q has orthonormal columns and R is upper triangular invertible. Find $\hat{x}$ and p in terms of Q and R and b (not A).

Solution Since $Q^TQ = I$ and R is invertible, we obtain

$$\begin{aligned} \hat{x} &= (A^TA)^{-1}A^Tb = ((QR)^T(QR))^{-1}(QR)^Tb \\ &= (R^TQ^TQR)^{-1}R^TQ^Tb = R^{-1}(R^T)^{-1}R^TQ^Tb = R^{-1}Q^Tb \\ p &= (QR)\hat{x} = QQ^Tb \end{aligned}$$

Note that QQ^T is not the identity matrix in general.

RUBIC: 6 points for finding $\hat{x}$, 4 points for p. One point off from each if the equations are not simplified, more points off for bad form, having variables other than Q, R and b, etc.

(c) If q_1 and q_2 are any orthonormal vectors in R^5, give a formula for the projection p of any vector b onto the plane spanned by q_1 and q_2 *(write p as a combination of q_1 and q_2)*.

Solution $p = (q_1^Tb)q_1 + (q_2^Tb)q^2$.

RUBRIC: little partial credit. If you identified the difference between b and p instead, you may have gotten some points.

3. (33 points) This problem is about the n by n matrix A_n that has zeros on its main diagonal and all other entries equal to -1. In MATLAB $A_n = \text{eye}(n) - \text{ones}(n)$.

(a) Find the determinant of A_n. Here is a suggested approach:

Start by adding all rows (except the last) to the last row, and then factoring out a constant. (You could check $n = 3$ to have a start on part b.)

$\boxed{\text{Solution}}$ Following the hint, add all of the rows to the last row (which does not change the determinant). Thus the matrix becomes

$$\begin{bmatrix} 0 & -1 & -1 & \cdots & -1 \\ -1 & 0 & -1 & \cdots & -1 \\ -1 & -1 & 0 & \cdots & -1 \\ \vdots & \vdots & \vdots & & \vdots \\ -(n-1) & -(n-1) & -(n-1) & \cdots & -(n-1) \end{bmatrix}.$$

Next, pull out the factor of $-(n-1)$ from the last row. As the determinant is linear in each row separately, we get

$$\begin{vmatrix} 0 & -1 & -1 & \cdots & -1 \\ -1 & 0 & -1 & \cdots & -1 \\ -1 & -1 & 0 & \cdots & -1 \\ \vdots & \vdots & \vdots & & \vdots \\ -(n-1) & -(n-1) & -(n-1) & \cdots & -(n-1) \end{vmatrix} = (1-n) \begin{vmatrix} 0 & -1 & -1 & \cdots & -1 \\ -1 & 0 & -1 & \cdots & -1 \\ -1 & -1 & 0 & \cdots & -1 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{vmatrix}.$$

Next, add the last row back to each of the other rows (which again keeps the determinant the same). So now we want to find

$$(1-n) \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{vmatrix}.$$

This matrix is lower triangular. So its determinant is the product of the entries on its diagonal. Thus the above quantity is $(1-n)$.

Alternately, one can find the determinant of the matrix by finding all its eigenvalues. As $A_n = I - \mathbf{ones}(n)$, we know that $N(A_n - I) = N(-\mathbf{ones}(n))$. The latter nullspace has dimension $n-1$. Thus 1 is an eigenvalue of multiplicity $n-1$, and the corresponding eigenvectors are all the nonzero vectors whose entries add up to 0.

In addition, all of the rows of A_n add up to $1-n$. So $1-n$ is an eigenvalue with eigenvector $(1, 1, \dots, 1)$. Thus we have found all of the eigenvectors and eigenvalues. The determinant is the product of the eigenvalues, so it is $1^{n-1} \cdot (1-n)$ or $1-n$.

RUBRIC: 2 points for following the hint, 2 points for pulling out the factor of $(1-n)$ correctly, 2 points for adding the last row to the other rows, 2 points for the correct answer.

(b) For any invertible matrix A, the $(1,1)$ entry of A^{-1} is the ratio of ____________ .

So the $(1,1)$ entry of A_4^{-1} is ____________ .

$\boxed{\text{Solution}}$ Cramer's rule gives $A^{-1} = \frac{1}{|A|}C^T$ where C is the cofactor matrix, whose (i,j) entry is $(-1)^{i+j}|M_{ij}|$ where M_{ij} is the submatrix obtained by deleting row i and column j of the (arbitrary) invertible matrix A. Thus the entry with $i = j = 1$ is $|M_{11}|/|A|$.

In the case where $A = A_n$, the submatrix M_{11} is A_{n-1}; so the desired formula is $|A_{n-1}|/|A_n|$. Now, $|A_n| = 1-n$ by part (a). So $|A_4| = -3$ and $|A_3| = -2$. Thus the $(1,1)$ entry of A_4^{-1} is $2/3$.

RUBRIC: 5 points for the correct ratio, 5 points for the correct application to the current problem. If the wrong ratio was given, then no credit was given for applying it.

(c) Find *two orthogonal eigenvectors* with $A_3 x = x$. (So $\lambda = 1$ is a double eigenvalue.)

$\boxed{\text{Solution}}$ In solution 2 of part (a) above, we saw that the eigenvectors are all the nonzero vectors whose entries add up to 0. Two obvious such vectors are $(1, -1, 0)$ and $(0, 1, -1)$, but there are many more linearly independent pairs.

However, $(1, -1, 0)$ and $(0, 1, -1)$ are not orthogonal! So we must find another pair. We can use the Gram–Schmidt process to get orthogonal vectors, or we can just try to guess two orthogonal vectors whose entries add up to 1. For example, $(1, -1, 0)$ and $(1, 1, -2)$ work. (Note that the vectors are not required to have unit length.)

RUBRIC: up to 5 points for a correct method, 2 points for finding linearly independent vectors, 3 points for orthogonality.

(d) What is the third eigenvalue of A_3 and a corresponding eigenvector?

$\boxed{\text{Solution}}$ In solution 2 of part (a) above, we saw that the third eigenvalue is -2 and a corresponding eigenvector is $(1, 1, 1)$.

Another way to proceed is to notice that the trace of A_3 is 0. However, the trace is the sum of the eigenvalues, and two of them are 1. So the third must be -2. Alternatively, in part (a), we saw that $|A_3| = -2$. However, the determinant is the product of the eigenvalues, and two of them are 1. So the third must be -2.

A third way to proceed is to find the characteristic polynomial of A_3, which is $\lambda^3 - 3\lambda + 2$. Since 1 is a double root, we can find the third root by dividing twice by $\lambda - 1$.

RUBRIC: 5 points for the eigenvalue, 5 points for a corresponding eigenvector.

18.06 Quiz 2 Professor Strang April 6, 2011

Your PRINTED name is ______________________________ **1.**

Your Recitation Instructor (and time) is ______________________________ **2.**

Instructors: (Pires)(Hezari)(Sheridan)(Yoo) **3.**

Please show enough work so we can see your method and give due credit.

1. **(8 pts. each)** Suppose a_1 and a_2 are orthogonal unit vectors in $\mathbf{R}^5$.

   (a) What are the requirements on a matrix P to be a projection matrix? Verify that $P = a_1 a_1^T + a_2 a_2^T$ satisfies those requirements.

   (b) If a_3 is in $\mathbf{R}^5$, what combination of a_1 and a_2 is closest to a_3?

   (c) Find a combination c of a_1, a_2, a_3 that is perpendicular to a_1 and a_2. *If possible, choose* $c \neq 0$. Describe all cases when $c = 0$ is the only possibility.

   (d) Show that a_1 and a_2 and c are eigenvectors of P (*if* $c \neq 0$) and find their eigenvalues.

2. (7 pts. each)

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 0 & 0 & 9 & 10 \\ 0 & 0 & 11 & 12 \end{bmatrix}.$$

(a) Find all nonzero terms in the big formula $\det A = \sum \pm a_{1\alpha}\, a_{2\beta}\, a_{3\gamma}\, a_{4\delta}$ and combine them to compute $\det A$.

(b) Find all the pivots of A.

(c) Find the cofactors C_{11}, C_{12}, C_{13}, C_{14} of row 1 of A.

(d) Find column 1 of A^{-1}.

3. **(8 pts. each)** Suppose A is a 2 by 2 matrix and $Ax = x$ and $Ay = -y$ ($x \neq 0$ and $y \neq 0$).

(a) (Reverse engineering) What is the polynomial $p(\lambda) = \det(A - \lambda I)$?

(b) If you know that the first column of A is $(2, 1)$, find the second column:

$$A = \begin{bmatrix} 2 & ? \\ 1 & ? \end{bmatrix}.$$

(c) For that matrix in part (b), find an invertible S and a diagonal matrix Λ so that $A = S\Lambda S^{-1}$.

(d) Compute A^{101}. (If you don't solve parts (b) -(c), use the description of A at the start. In all questions **show enough work** so we can see your method and give due credit.)

(e) If $Ax = x$ and $Ay = -y$ (with $x \neq 0$ and $y \neq 0$) prove that x and y are *independent.*

Start of a proof: Suppose $z = cx + dy = 0$. Then $Az =$ (follow from here.)

18.06 Quiz 2 — Professor Strang — April 6, 2011

Your PRINTED name is ______________________ 1.

Your Recitation Instructor (and time) is ______________________ 2.

Instructors: (Pires)(Hezari)(Sheridan)(Yoo) 3.

Please show enough work so we can see your method and give due credit.

1. (8 pts. each) Suppose a_1 and a_2 are orthogonal unit vectors in $\mathbf{R}^5$.

(a) What are the requirements on a matrix P to be a projection matrix? Verify that $P = a_1a_1^T + a_2a_2^T$ satisfies those requirements.

(b) If a_3 is in $\mathbf{R}^5$, what combination of a_1 and a_2 is closest to a_3?

(c) Find a combination c of a_1, a_2, a_3 that is perpendicular to a_1 and a_2. *If possible, choose* $c \neq 0$. Describe all cases when $c = 0$ is the only possibility.

(d) Show that a_1 and a_2 and c are eigenvectors of P (*if* $c \neq 0$) and find their eigenvalues.

1:

a: P is a projection (orthogonal projection!) if

$$\begin{cases} P^2 = P, \\ P^T = P. \end{cases}$$

We know check this for $P = a_1a_1^T + a_2a_2^T$.

- $P^2 = (a_1a_1^T + a_2a_2^T)(a_1a_1^T + a_2a_2^T) = a_1\underbrace{a_1^Ta_1}a_1^T + a_1\underbrace{a_1^Ta_2}a_2^T + a_2\underbrace{a_2^Ta_1}a_1^T + a_2\underbrace{a_2^Ta_2}a_2^T$.

  Since $a_1^Ta_2 = 0$, $a_2^Ta_1 = 0$, and $a_1^Ta_1 = a_2^Ta_2 = 1$, we get

  $P^2 = a_1a_1^T + a_2a_2^T = P$.

- $P^T = (a_1a_1^T + a_2a_2^T)^T = (a_1^T)^T a_1^T + (a_2^T)^T a_2^T = a_1a_1^T + a_2a_2^T = P$.

b: The closest combination is $Pa_3 = (a_1^Ta_3)a_1 + (a_2^Ta_3)a_2$.

c: c = error term $= a_3 - Pa_3 = a_3 - (a_1^Ta_3)a_1 - (a_2^Ta_3)a_2$.

$c = 0$ only if c is in the plane generated by a_1 and a_2.

d: Since P is the projection on the column space of $A = [\,a_1 \mid a_2\,]$, we have:

$Pa_1 = a_1 \Rightarrow \lambda_1 = 1$

$Pa_2 = a_2 \Rightarrow \lambda_2 = 1$

$Pc = 0 \Rightarrow \lambda_3 = 0$.

2. **(7 pts. each)**

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 0 & 0 & 9 & 10 \\ 0 & 0 & 11 & 12 \end{bmatrix}.$$

(a) Find all nonzero terms in the big formula $\det A = \sum \pm a_{1\alpha} a_{2\beta} a_{3\gamma} a_{4\delta}$ and combine them to compute $\det A$.

(b) Find all the pivots of A.

(c) Find the cofactors C_{11}, C_{12}, C_{13}, C_{14} of row 1 of A.

(d) Find column 1 of A^{-1}.

2:

a: $\det A = 1\,(6\cdot(9\cdot 12 - 10\cdot 11)) - 2\,(5\,(9\cdot 12 - 10\cdot 11)) = 8$

b: By row reduction:

A reduces to $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -16 \\ 0 & 0 & 9 & 10 \\ 0 & 0 & 0 & -\frac{12}{9} \end{bmatrix}.$

Hence the pivots are: $1, -4, 9, -\frac{12}{9}$.

c:

$C_{11} = \det \begin{bmatrix} 6 & 7 & 8 \\ 0 & 9 & 10 \\ 0 & 11 & 12 \end{bmatrix} = -12$

$C_{12} = -\det \begin{bmatrix} 5 & 7 & 8 \\ 0 & 9 & 10 \\ 0 & 11 & 12 \end{bmatrix} = 10$

$C_{13} = \det \begin{bmatrix} 5 & 6 & 8 \\ 0 & 0 & 10 \\ 0 & 0 & 12 \end{bmatrix} = 0$

$C_{14} = -\det \begin{bmatrix} 5 & 6 & 7 \\ 0 & 0 & 9 \\ 0 & 0 & 11 \end{bmatrix} = 0.$

d: From c we get

$(A^{-1})_{11} = -\frac{12}{8}$

$(A^{-1})_{21} = \frac{10}{8}$

$(A^{-1})_{31} = 0$

$(A^{-1})_{41} = 0.$

3. **(8 pts. each)** Suppose A is a 2 by 2 matrix and $Ax = x$ and $Ay = -y$ ($x \neq 0$ and $y \neq 0$).

(a) (Reverse engineering) What is the polynomial $p(\lambda) = \det(A - \lambda I)$?

(b) If you know that the first column of A is $(2, 1)$, find the second column:

$$A = \begin{bmatrix} 2 & ? \\ 1 & ? \end{bmatrix}.$$

(c) For that matrix in part (b), find an invertible S and a diagonal matrix Λ so that $A = S\Lambda S^{-1}$.

(d) Compute A^{101}. (If you don't solve parts (b) -(c), use the description of A at the start. In all questions **show enough work** so we can see your method and give due credit.)

(e) If $Ax = x$ and $Ay = -y$ (with $x \neq 0$ and $y \neq 0$) prove that x and y are *independent*.

Start of a proof: Suppose $z = cx + dy = 0$. Then $Az =$ (follow from here.)

3:

a: $p(\lambda) = (1-\lambda)(-1-\lambda) = \lambda^2 - 1$

b: We know that $\mathrm{Tr}A = 1 + (-1) = 0$.
On the other hand if we put $A = \begin{bmatrix} 2 & a_{12} \\ 1 & a_{22} \end{bmatrix}$
then $\mathrm{Tr}A = 2 + a_{22}$. Hence $a_{22} = -2$.
To find a_{12} we note that on one hand
$\det A = 1 \cdot (-1) = -1$ and on the other hand
$\det A = 2a_{22} - a_{12} = -4 - a_{12}$. Therefore $a_{12} = -3$.
So $A = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}$.

(c): It is easy to see that X an eigenvector of $\lambda_1 = 1$
is $X = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and for Y an eigenvector of $\lambda_2 = -1$
we have $Y = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. So we can choose $S = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$.

d: From c we have $A^{101} = S\Lambda^{101}S^{-1} = S\Lambda S^{-1} = A$.
Note that $\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and therefore $\Lambda^{101} = \Lambda$.

e: On one hand since $z = 0$ we have $Az = 0$.
On the other hand $Az = A(cx + dy) = cAX + dAY$
$= cX - dY$.
Therefore
$\begin{cases} z = cX + dY = 0 \\ Az = cX - dY = 0 \end{cases}$ $\xRightarrow{\text{since } X \neq 0,\ Y \neq 0}$ $c = d = 0 \Rightarrow X$ and Y are linearly independent.

Solutions

18.06 Professor Strang Quiz 2 April 11th, 2012

Your PRINTED name is: ____________________

Grading

1

2

3

Please circle your recitation:

r01	T 11	4-159	Ailsa Keating	ailsa
r02	T 11	36-153	Rune Haugseng	haugseng
r03	T 12	4-159	Jennifer Park	jmypark
r04	T 12	36-153	Rune Haugseng	haugseng
r05	T 1	4-153	Dimiter Ostrev	ostrev
r06	T 1	4-159	Uhi Rinn Suh	ursuh
r07	T 1	66-144	Ailsa Keating	ailsa
r08	T 2	66-144	Niels Martin Moller	moller
r09	T 2	4-153	Dimiter Ostrev	ostrev
r10	ESG		Gabrielle Stoy	gstoy

1 (40 pts.)

(a) Find the projection p of the vector b onto the plane of a_1 and a_2, when

$$b = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad a_1 = \begin{bmatrix} 1 \\ 7 \\ 1 \\ 7 \end{bmatrix}, \quad a_2 = \begin{bmatrix} -1 \\ 7 \\ 1 \\ -7 \end{bmatrix}.$$

Solution. Observe that $a_1^T a_2 = 0$. Thus

$$p = \frac{a_1^T b}{a_1^T a_1} a_1 + \frac{a_2^T b}{a_2^T a_2} a_2 = \frac{8}{100} a_1 - \frac{8}{100} a_2 = \begin{bmatrix} 4/25 \\ 0 \\ 0 \\ 28/25 \end{bmatrix}.$$

□

(b) What projection matrix P will produce the projection $p = Pb$ for every vector b in $\mathbb{R}^4$?

Solution. Let A be the 4×2 matrix with columns a_1, a_2. P is given by $P = A(A^T A)^{-1} A^T$. Notice that

$$A^T A = \begin{bmatrix} 100 & 0 \\ 0 & 100 \end{bmatrix}.$$

(a_1 and a_2 are orthogonal and of same length.)

Thus

$$P = \frac{1}{100} A A^T = \frac{1}{100} \begin{bmatrix} 2 & 0 & 0 & 14 \\ 0 & 98 & 14 & 0 \\ 0 & 14 & 2 & 0 \\ 14 & 0 & 0 & 98 \end{bmatrix}.$$

□

(c) What is the determinant of $I - P$? Explain your answer.

Solution. $I - P$ is the matrix of the projection to the orthgonal complement of $C(A)$, i.e. $N(A^T)$. In particular, $I - P$ has rank the dimension of $N(A^T)$, which is 3. Thus $I - P$ is singular, and $\det(I - P) = 0$. □

(d) <u>What are all nonzero eigenvectors of P with eigenvalue</u> $\lambda = 1$?

How is the number of independent eigenvectors with $\lambda = 0$ of a square matrix A connected to the rank of A?

(You could answer (c) and (d) even if you don't answer (b).)

Solution. The non-zero eigenvectors with eigenvalue $\lambda = 1$ are all the non-zero linear combinations of a_1 and a_2, i.e. all the non-zero vectors in the plane spanned by a_1 and a_2.

Suppose A is a $n \times n$ matrix, with rank r.

$$\begin{aligned}\#\text{ independent zero-eigenvectors of } A &= \#\text{ independent vectors in } N(A)\\ &= \text{dimension of } N(A) = n - r\end{aligned}$$

□

2 (30 pts.)

(a) Suppose the matrix A factors into $A = PLU$ with a permutation matrix P, and 1's on the diagonal of L (lower triangular) and pivots $d_1, \ldots, d_n$ on the diagonal of U (upper triangular).

What is the determinant of A? EXPLAIN WHAT RULES YOU ARE USING.

Solution. Use

$$\det(A) = \det(P) \cdot \det(L) \cdot \det(U)$$

where we make two uses of the rule $\det(MN) = \det(M)\det(N)$, for any two $n \times n$ matrices M and N. We will compute each of the determinants on the right-hand side.

The determinant of a triangular matrix is the product of its diagonal entries; this is true whether the matrix is upper or lower triangular. Thus

$$\det(L) = 1 \qquad \text{and} \qquad \det(U) = d_1 \cdot d_2 \cdot \ldots \cdot d_n.$$

The determinant changes sign whenever two rows are swapped. Thus

$$\det(P) = \begin{cases} +1 & \text{if } P \text{ is even (even \# of row exchanges)} \\ -1 & \text{if } P \text{ is odd (odd \# of row exchanges)} \end{cases}$$

and so

$$\det(A) = \pm d_1 \cdot d_2 \cdot \ldots \cdot d_n$$

where the sign depends on the parity of P.

□

(b) Suppose the first row of a new matrix A consists of the numbers $1, 2, 3, 4$. Suppose the cofactors C_{ij} of that first row are the numbers $2, 2, 2, 2$.

(Cofactors already include the $\pm$ signs.)

Which entries of A^{-1} does this tell you and what are those entries?

Solution. Using the cofactor expansion in the first row gives

$$\begin{aligned}\det(A) &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} + a_{14}C_{14} \\ &= 1 \times 2 + 2 \times 2 + 3 \times 2 + 4 \times 2 \\ &= 20\end{aligned}$$

As $A^{-1} = C^T / \det(A)$, where C is the cofactor matrix, this data gives us the entries of the first column of A^{-1}; they are all $2/20 = 1/10$. $\square$

(c) What is the determinant of the matrix $M(x)$? For which values of x is the determinant equal to zero?

$$M(x) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 2 & x \\ 1 & 1 & 4 & x^2 \\ 1 & -1 & 8 & x^3 \end{bmatrix}$$

Solution. **Solution no. 1.**

From, for instance, the 'Big Formula', we know that $\det(M)$ is a cubic polynomial in x. Say

$$\det(M) = ax^3 + bx^2 + cx + d.$$

We can calculate d by setting $x = 0$. Using the cofactor expansion in the last column, we get that

$$d = -\begin{vmatrix} 1 & -1 & 2 \\ 1 & 1 & 4 \\ 1 & -1 & 8 \end{vmatrix} = -\begin{vmatrix} 1 & -1 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 6 \end{vmatrix} = -12.$$

We will determine the other coefficients of $\det(M)$ by finding three roots for it. x is a root of $\det(M)$ if and only if $M(x)$ is a singular matrix. Now, notice that

$$\begin{aligned} (1,1,1) &= (x,x^2,x^3) \quad \text{for } x = 1 \\ (1,-1,1) &= (x,x^2,x^3) \quad \text{for } x = -1 \\ (2,4,8) &= (x,x^2,x^3) \quad \text{for } x = 2. \end{aligned}$$

Thus $M(x)$ is singular for $x = 1, -1$ and 2; moreover, this implies that

$$\det(M) = a(x-1)(x+1)(x-2).$$

As $d = 2a$, we must have $a = -6$. Thus

$$\det(M) = -6(x-1)(x+1)(x-2) = -6x^3 + 12x^2 + 6x - 12.$$

The values of x for which $M(x)$ is singular are $1, -1$ and 2.

Solution no. 2.

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 2 & x \\ 1 & 1 & 4 & x^2 \\ 1 & -1 & 8 & x^3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 1 & x-1 \\ 0 & 0 & 3 & x^2-1 \\ 0 & -2 & 7 & x^3-1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 1 & x-1 \\ 0 & 0 & 3 & x^2-1 \\ 0 & 0 & 6 & x^3-x \end{vmatrix}$$

$$= \begin{vmatrix} -2 & 1 & x-1 \\ 0 & 3 & x^2-1 \\ 0 & 6 & x^3-x \end{vmatrix} = -2 \begin{vmatrix} 3 & x^2-1 \\ 6 & x^3-x \end{vmatrix} = -6x^3 + 12x^2 + 6x - 12$$

In the first step, subtract the first row from the second, third and fourth rows. In the second step, subtract the second row from the fourth. For the third and fourth steps, use the cofactor expansion in the first column.

We factorize $\det(M)$ by guessing roots, trying small integers; we find that $1, -1$ and 2 are all roots, which gives

$$\det(M) = -6(x-1)(x+1)(x-2).$$

The values of x for which $M(x)$ is singular are $1, -1$ and 2. $\square$

3 (30 pts.)

(a) Starting from independent vectors a_1 and a_2, use Gram-Schmidt to find formulas for two orthonormal vectors q_1 and q_2 (combinations of a_1 and a_2):

Solution.

$$q_1 = \frac{a_1}{||a_1||}$$

$$q_2 = \frac{a_2 - (a_2^T q_1)q_1}{||a_2 - (a_2^T q_1)q_1||} = (a_2 - \frac{(a_2^T a_1)}{a_1^T a_1}a_1) \, / \, ||a_2 - \frac{(a_2^T a_1)}{a_1^T a_1}a_1||$$

□

(b) The connection between the matrices $A = [a_1 \; a_2]$ and $Q = [q_1 \; q_2]$ is often written $A = QR$. From your answer to Part (a), what are the entries in this matrix R?

Solution. Re-arranging the expressions above gives

$$a_1 = q_1 ||a_1||$$

$$a_2 = (a_2^T q_1)q_1 + ||a_2 - (a_2^T q_1)q_1||q_2$$

and thus

$$R = \begin{bmatrix} a_1^T q_1 & a_2^T q_1 \\ a_1^T q_2 & a_2^T q_2 \end{bmatrix} = \begin{bmatrix} ||a_1|| & a_2^T q_1 \\ 0 & ||a_2 - (a_2^T q_1)q_1|| \end{bmatrix}$$

□

(c) The least squares solution $\widehat{x}$ to the equation $Ax = b$ comes from solving what equation? If $A = QR$ as above, show that $R\widehat{x} = Q^T b$.

Solution. $\widehat{x}$ comes from solving $A^T A\widehat{x} = A^T b$.

Suppose we have $A = QR$. Notice that:

- $Q^T Q = I$, so $A^T A = (QR)^T QR = R^T Q^T QR = R^T R$.
- As a_1 and a_2 are independent, R is invertible. Thus R^T is also invertible.

Thus we have

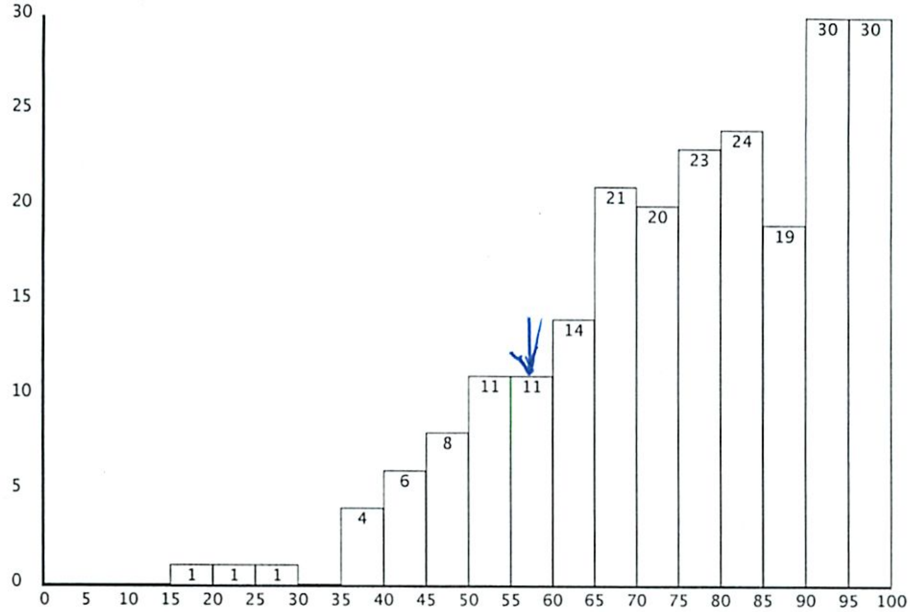
$$\begin{aligned} A^T A\widehat{x} &= A^T b \\ \Leftrightarrow \quad R^T R\widehat{x} &= R^T Q^T b \\ \Leftrightarrow \quad R\widehat{x} &= Q^T b. \end{aligned}$$

□

18.06 Linear Algebra

Dashboard | Students | Assignments

Grading Summary for Exam 2



Number of Scores:	224
Average:	75.36
Standard Deviation:	17.55

→ wish I was higher

Grade 56/100

18.06 Professor Strang Quiz 2 April 11th, 2012

Your PRINTED name is: Michael Plasmeier

Grading	
1	18
2	20
3	18
	56

Please circle your recitation:

r01	T 11	4-159	Ailsa Keating	ailsa
r02	T 11	36-153	Rune Haugseng	haugseng
r03	T 12	4-159	Jennifer Park	jmypark
r04	T 12	36-153	Rune Haugseng	haugseng
r05	T 1	4-153	Dimiter Ostrev	ostrev
r06	T 1	4-159	Uhi Rinn Suh	ursuh
r07	T 1	66-144	Ailsa Keating	ailsa
r08	T 2	66-144	Niels Martin Moller	moller
r09	T 2	4-153	Dimiter Ostrev	ostrev
r10	ESG		Gabrielle Stoy	gstoy

$P = A(A^TA)^{-1}A^T =$

$x_1 = \frac{\det B_1}{\det A}$ Cartier

$p = A\hat{x} = A(A^TA)^{-1}A^Tb = a\frac{a^Tb}{a^Ta}$

$B = b - \frac{A^Tb}{A^TA}$ G-S

$\frac{AA^T}{A^TA}$ = something

$$A^T b = \begin{bmatrix} -1 & 7 & 1 & -7 \\ 1 & 7 & 1 & 7 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1+0+0-7 \\ 1+0+0+7 \end{bmatrix}$$

$$= \begin{bmatrix} -8 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 100 \\ 100 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -8 \\ 8 \end{bmatrix}$$

$$100 x_2 = -8$$

$$x_2 = -\frac{8}{100} = -.08$$

$$100 x_1 = 8$$

$$x_1 = .08$$

$$p = a\hat{x} = \begin{bmatrix} 1 & -1 \\ 7 & 7 \\ 1 & 1 \\ 7 & -7 \end{bmatrix} \begin{bmatrix} -.08 \\ .08 \end{bmatrix} = \begin{bmatrix} -.08 - .08 \\ -.56 + .56 \\ -.08 + .08 \\ -.56 - .56 \end{bmatrix} = \begin{bmatrix} -.16 \\ 0 \\ 0 \\ -1.12 \end{bmatrix}$$

think that's it, but decimals

1 (40 pts.)

(a) Find the projection p of the vector b onto the plane of a_1 and a_2, when

matrix

$$b = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad a_1 = \begin{bmatrix} 1 \\ 7 \\ 1 \\ 7 \end{bmatrix}, \quad a_2 = \begin{bmatrix} -1 \\ 7 \\ 1 \\ -7 \end{bmatrix}. \qquad A = \begin{bmatrix} 1 & -1 \\ 7 & 7 \\ 1 & 1 \\ 7 & -7 \end{bmatrix}$$

$$p = a\hat{x} = a\frac{a^Tb}{a^Ta}$$

$$p = A(A^TA)^{-1}A^Tb$$

$$A^TA\hat{x} = A^Tb$$

$$A^TA = \begin{bmatrix} -1 & 7 & 1 & -7 \\ 1 & 7 & 1 & +7 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 7 & 7 \\ 1 & 1 \\ 7 & -7 \end{bmatrix} = \begin{bmatrix} -1+49+1-49 & 1+49+1+49 \\ 1+49+1+49 & -1+49+1-49 \end{bmatrix} = \begin{bmatrix} 0 & 100 \\ 100 & 0 \end{bmatrix}$$

8

prev page

(b) What projection matrix P will produce the projection $p = Pb$ for every vector b in $\mathbb{R}^4$?

~~a matrix that spans the space~~

$$P = A(A^TA)^{-1}A^T$$

10

So $(A^TA)^{-1}$: $\dfrac{\begin{bmatrix} 0 & -100 \\ -100 & 0 \end{bmatrix}}{-10{,}000}$ flip signs & swap

$$A(A^TA)^{-1} = \begin{bmatrix} 1 & -1 \\ 7 & 7 \\ 1 & 1 \\ 7 & -7 \end{bmatrix}\begin{bmatrix} 0 & 1/100 \\ 1/100 & 0 \end{bmatrix} = \begin{bmatrix} 0-\frac{1}{100} & 1/100+0 \\ 0+7/100 & 7/100+0 \\ 0+1/100 & 1/100+0 \\ 0-7/100 & 7/100+0 \end{bmatrix}$$

$$p = A(A^TA)^{-1}A^T =$$

$$\begin{bmatrix} -1/100 & 1/100 \\ 7/100 & 7/100 \\ 1/100 & 1/100 \\ -7/100 & 7/100 \end{bmatrix}\begin{bmatrix} -1 & 7 & 1 & -7 \\ 1 & 7 & 1 & 7 \end{bmatrix} = \begin{bmatrix} 1/100+1/100 & -7/100+7/100 & -1/100+1/100 \\ -7/100+7/100 & 49/100+49/100 & 7/100+7/100 \\ -1/100+1/100 & 7/100+7/100 & 1/100+1/100 \\ 7/100+7/100 & -49/100+49/100 & 49/100+49/100 \end{bmatrix}$$

$$\begin{bmatrix} -7/100 + 7/100 \\ -49/100 + 49/100 \\ -7/100 + 7/100 \\ 49/100 + 49/100 \end{bmatrix} = \underbrace{\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 98 & 14 & 0 \\ 0 & 14 & 2 & 0 \\ 14 & 0 & 98 & 98 \end{bmatrix}}_{100} = P$$

if correct a waste of time

I −
put ~~ones~~ on the diagonal

(c) What is the determinant of $I - P$? Explain your answer.

$$\det \begin{bmatrix} \frac{\begin{matrix} -1 & 0 & 0 & 0 \\ 0 & -97 & 14 & 0 \\ 0 & 14 & -1 & 0 \\ 14 & 0 & 98 & -97 \end{matrix}}{100} \end{bmatrix} \to \frac{-1}{100} \cdot \det \begin{bmatrix} \frac{\begin{matrix} -97 & 14 & 0 \\ 14 & -1 & 0 \\ 0 & 98 & -97 \end{matrix}}{100} \end{bmatrix}$$

0

Not triangular − but almost
− Cofactor?

Just apply the formula. There's prob. some sort of trick as well
Can't just add

(d) What are all nonzero eigenvectors of P with eigenvalue $\lambda = 1$? I'm guessing this should have been easier

How is the number of independent eigenvectors with $\lambda = 0$ of an $n \times n$ square matrix A connected to the rank of A? # eigenvectors = rank of matrix = # ind cols/rows

0

(You could answer (c) and (d) even if you don't answer (b).)

So $(P - \lambda I)x = 0$
(? reverse of above? − but this is the formula

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 97 & 14 & 0 \\ 0 & 14 & 1 & 0 \\ 14 & 0 & 98 & 97 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 = 0$
$97x_2 + 14x_3 = 0$
$14x_2 + 1x_3 = 0$
$14x_1 + 98x_3 + 97x_4 = 0$

$97x_2 = -14x_3$
$x_3 = -\frac{97}{14}x_2$
$14x_2 + -\frac{97}{14}x_2 = 0$

Again prob earlier error making this hard

Gold
$98x_3 + 97x_4 = 0$
$97x_4 = -98x_3$
$x_4 = -\frac{98}{97}x_3$

$14 - \frac{97}{14} x_2 = 0$
$x_2 = 0$
$x_3 = 0$

$x_4 = 0$
don't want that one!

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2 (30 pts.)

(a) Suppose the matrix A factors into $A = PLU$ with a permutation matrix P, and 1's on the diagonal of L (lower triangular) and pivots $d_1, \ldots, d_n$ on the diagonal of U (upper triangular).

What is the determinant of A? EXPLAIN WHAT RULES YOU ARE USING.

(rule) $|A| = |P||L||U|$

~~is always 1 (rule)~~ multiple diagonals $L = 1$'s (given) $U = d_1 \cdot d_2 \cdot \ldots \cdot d_n$ (rule for diagonals)

6

So $|A| = d_1 \cdot d_2 \cdot \ldots \cdot d_n$

(b) Suppose the first row of a new matrix A consists of the numbers 1, 2, 3, 4. Suppose the cofactors C_{ij} of that first row are the numbers 2, 2, 2, 2.

(Cofactors already include the $\pm$ signs.)

Which entries of A^{-1} does this tell you and what are those entries?

$$A = \begin{bmatrix} 1 & \cdots \\ 2 & \cdots \\ 3 & \cdots \\ 4 & \cdots \end{bmatrix} \qquad C = \begin{bmatrix} 2 & \cdots \\ 2 & \cdots \\ 2 & \cdots \\ 2 & \cdots \end{bmatrix}$$

7

So $A^{-1} = \dfrac{C^T}{\det(A)} = \dfrac{\begin{bmatrix} 2 & 2 & 2 & 2 \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}}{20}$

det A = sum of a row of cofactors · original values

$= 1 \cdot 2 + 2 \cdot 2 + 3 \cdot 2 + 4 \cdot 2 =$

$= 2 + 4 + 6 + 8 = 20$

(c) What is the determinant of the matrix $M(x)$? For which values of x is the determinant equal to zero?

$$M(x) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 2 & x \\ 1 & 1 & 4 & x^2 \\ 1 & -1 & 8 & x^3 \end{bmatrix}.$$

Elim

$l_{21} = 1$

$l_{31} = 1$

$l_{41} = 1$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 1 & x-1 \\ 0 & 0 & 3 & x^2-1 \\ 0 & -2 & 7 & x^3-1 \end{bmatrix}$$

row swap

$\ominus$

$l_{32} = \frac{-2}{-2} = 1$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 1 & x-1 \\ 0 & -2 & 7 & x^3-1 \\ 0 & 0 & 3 & x^2-1 \end{bmatrix}$$

$l_{43} = \frac{1}{2}$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 1 & x-1 \\ 0 & 0 & 6 & x^3-1-(x-1) \\ 0 & 0 & 3 & x^2-1 \end{bmatrix}$$

7

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 1 & x-1 \\ 0 & 0 & 6 & x^3-x \\ 0 & 0 & 0 & x^2-1-\frac{1}{2}(x^3-x) \end{bmatrix}$$

→ Diagonal - read down the diagonal

neg from before row swap

$-(1 \cdot -2 \cdot 6 \cdot x^2 - 1 - \frac{1}{2}(x^3-x))$

Set = to 0

$+12(x^2-1-\frac{x^3-x}{2}) = 0$

$\frac{\quad}{-12} \quad +12 \qquad \frac{\quad}{-12} \quad +12$

$x^2 - \frac{x^3-x}{2} = 12$

$2x^2 - x^3 - x = 24$

$x(-x^2+2x-1) = 24$

$x = 0$

$x = 1$

$x =$

can at of the to facto

$-x^2 + 2x - 25$

$\frac{-2 \pm \sqrt{4-2\cdot-1\cdot-25}}{2a}$

$x = \frac{+2 \pm \sqrt{-46}}{2}$

whatever that is

again prob algebra error

yes

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3 (30 pts.)

(a) Starting from independent vectors a_1 and a_2, use Gram-Schmidt to find formulas for two orthonormal vectors q_1 and q_2 (combinations of a_1 and a_2):

5/10

$$q_1 = a_1 / \|a_1\| \quad 0\%$$

$$q_2 = \frac{a_2 - \dfrac{q_1^T a_2}{q_1^T q_1}}{\| \cdots \|} \qquad = 1.$$

(b) The connection between the matrices $A = [a_1\ a_2]$ and $Q = [q_1\ q_2]$ is often written $A = QR$. From your answer to Part (a), what are the entries in this matrix R?

forgot exactly [illegible] this is

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \begin{bmatrix} q_1^T a_1 & q_1^T a_1 \\ & q_2^T a_2 \end{bmatrix} \quad 0\%$$

7/10

(c) The least squares solution $\widehat{x}$ to the equation $Ax = b$ comes from solving what equation? If $A = QR$ as above, show that $R\widehat{x} = Q^T b$.

6/10

$A^T A \hat{x} = A^T b$ ✓

$R = A^T A$?! ← doesn't really line up w/ last one

(what else)

$Q^T = A^T$ ← this either

So $A^T A \hat{x} = A^T b$

So something special going on here...

Orthonormals are stand ins

$\underline{Q^T Q = I} = A^T A$?! ✓

└ broke down to component bases and normalized

No.

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not as bad as I thought - took several iterations
lots of alg issues