18.06

4/13

Exam 2

$\text{avg} = 75$ harder than he meant

$\sigma = 18$ me = 56

Diff eq $\frac{du}{dt} = Au$ $u(t) = e^{At} u(0)$

$e^{At} = I + At + \frac{(At)^2}{2!} + \ldots$ has eigenvalues

$1 + \lambda t + \frac{(\lambda t)^2}{2!} + \ldots$

(same evectors as A)

$A x_k = \lambda_k x_k \Rightarrow AS = S\Lambda$

$S = \begin{bmatrix} x_1 \\ \vdots \end{bmatrix} - \begin{bmatrix} x_n \\ \vdots \end{bmatrix}$ $A = S\Lambda S^{-1}$

[diagonalization - the building block of everything

$$A^k = S\Lambda^k S^{-1}$$

$$A^k u_0 = S\Lambda^k S^{-1} u_0$$

$$= c_1 \lambda_1^k x_k + \cdots + c_n \lambda_n^k x_n$$

See the point of eigenvalues + eigenvectors

Not A^k ← discrete steps

But e^{At} ← continuous

↑ the exponential

(wrote down on page 1)

$$e^{At} x_1 = e^{\lambda_1 t} x_1$$

$$e^{At} x_n = e^{\lambda_n t} x_n$$

$$\sum_0^\infty \frac{x^n}{n!} = e^x$$ "exponential series"

$$\sum_0^\infty x^n = 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$$ "geometric series"

$X = i\theta \rightarrow \cos\theta + i \sin\theta$ ← Euler

Can let x be a matrix
└ Series still converges!

e^A has a zero eigenvalue when?
└ never! - its always invetable

$(e^{At})^{-1} = e^{-At}$

$\boxed{\uparrow \frac{1}{e^x} = e^{-x}}$

as long as only one matrix A
if 2 matrices → not same eigenvectors

$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

need its eigenvalues
anti symetric ente

$A^T \overset{\checkmark}{=} -A$ ↓ so λ are pure imaginary

$$\boxed{\begin{array}{l}\text{Symmetric} \\ \downarrow \\ A^T = A \quad \searrow \lambda \text{ are all real}\end{array}}$$

So eigenvalues are

$$A - \lambda I = \begin{bmatrix} -\lambda & 1 \\ -1 & -\lambda \end{bmatrix}$$

$$= \lambda^2 + 1 = 0$$

$$\lambda = i, -i$$

Then $x_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}$

$$x_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

(I cant do this stuff in my head!)

$$\frac{du}{dt} = Au \qquad \frac{du_1}{dt} = 1 \cdot u_2$$

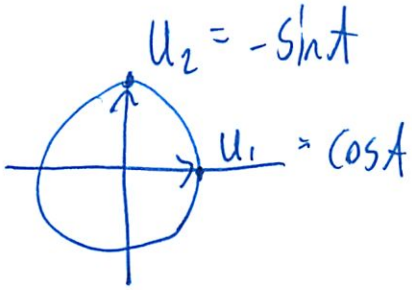
$$\frac{du_2}{dt} = -1 \cdot u_1$$

$$u_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

↑ start out

Physics' fav example— simple motion

⑤



$$u_1(t) = \cos t$$
$$u_2(t) = -\sin t$$

What is e^{At}

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = A^2 \cdot A^2$$

every 4 terms back to the identity

$$e^{At} = \begin{bmatrix} 1 - \frac{1t^2}{2!} + \frac{1t^4}{4!} - \frac{t^6}{6!} - - - - - & \_\_\_\_ \\ \_\_\_\_ & \_\_\_\_ \end{bmatrix}$$

↑ straight factorial ↑ see a pattern ↑ 2x2 matrix

$$\begin{bmatrix} \sim\text{---} & t - \frac{t^3}{3!} + \text{----} \\ \text{---} & \text{---} \end{bmatrix}$$

So $= \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

(looks familiar)

{Rare case when we can put in directly)

~~OM~~ Rotation matrix

Orthogonal

not sym or anti sym

If A = anti sym → e^{At} = orthogonal

Since if $\underset{\text{anti sym}}{A^T = -A}$ than e^{At} is orthogonal

$$e^{-At} = (e^{At})^T$$

important fact for a lot of purposes

⑦

Look at $(e^{At})^T = \left(I + At + \frac{A^2t^2}{2} + \cdots\right)^T$

$$= \left(I - At + \frac{A^2t^2}{2} + \cdots\right)$$

$$= e^{-At}$$

Checked " is orth
since transpose = inverse

For other diff eqn - prob don't want to use power series

e^{At} $u(0)$ =

↓

Combo of eigenvectors

$c_1 x_1 + \cdots + c_n x_n$

$$u(t) = e^{At} u(0) = c_1 e^{\lambda_1 t} x_1 + \cdots + c_n e^{\lambda_n t} x_n$$

↓

$c_1 x_1 + \cdots + c_n x_n$

If 2nd order

$$u'' - 2u' + u = 0$$

Want $\frac{du}{dt} = Au$

$$u = \begin{bmatrix} y \\ y' \end{bmatrix}$$

$$\frac{du}{dt} = \begin{bmatrix} y' \\ y'' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} y \\ y'' \end{bmatrix}$$

$$\begin{matrix} \uparrow & \uparrow \\ -y & y'' \end{matrix}$$

phase plane

└ funct and its velocity

What are the λs?

$$\begin{bmatrix} 0-\lambda & 1 \\ -1 & 2-\lambda \end{bmatrix}$$

$$\lambda^2 - 2\lambda + 1 = 0$$

↑ notice same as original problem!

double root / eigenvalue $\lambda = 1, 1$

Now eigenvector

$$\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$$

Whats in Nullspace?

Only 1D

So only 1 eigenvector

$$C_1 e^t x + C_2 t e^t$$

↑since repeat

Diff eq by linear algebra

Prof — not a whole lot of brilliant ideas

Next weeks hw: Diff eq + Markov

Shorter than usual

18.06 Spring 2012 – Problem Set 7

This problem set is due Thursday, April 19th, 2012 at 4pm (hand in to Room 2-106). The textbook problems are out of the 4th edition. For computational problems, please include a printout of the code with the problem set (for MATLAB in particular, `diary('filename')` will start a transcript session, `diary off` will end one.)

Every problem is worth 10 points.

1. Do Problem 2 from Section 8.3.
2. Do Problem 7 from Section 8.3 (do also the "challenge problem" part).
3. Do Problems 9 from Section 8.3.
4. Do Problems 12 from Section 8.3.
5. Do Problems 4 from Section 6.3.
6. Do Problems 5 from Section 6.3.
7. Do Problem 12 from Section 6.3.
8. Do Problem 24 from Section 6.3.
9. Do Problem 26 from Section 6.3.
10. Do Problem 30 from Section 6.3.

18.06 Wisdom. No 18.06 recitations on Tue April 17th - enjoy your Patriot's Day vacation!

~~2 challenge~~

~~$S\Lambda S^{-1}$ is what?~~

~~#5~~

~~#6 kinda~~

~~#7~~

~~#8 explain~~

~~#10~~

~~#9 explain~~

Michael Plasmeier

18.06 P-Set 7

94/100

4/17

8.3 #2 Diagonlize Markov to $A = S \Lambda S^{-1}$ by finding its other eigenvector

10/10

$$A = \begin{bmatrix} .9 & .15 \\ .1 & .85 \end{bmatrix}$$

So first find λ

~~$(.9-\lambda)(.85-\lambda) - .15 \cdot .1$~~

~~$.765 + \lambda^2 - .85\lambda - .9\lambda - .015$~~

~~$\lambda^2 - 1.75\lambda - .75 = 0$~~

~~$\lambda = 2.1, -.35$~~

I think I did that wrong

Matlab → $\lambda = 1, .75$ and was given

just use ↗

Eigenvectors

$\underline{\lambda = 1}$ $\begin{bmatrix} -.1 & .15 \\ .1 & -.15 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{matrix} 0 \\ 0 \end{matrix}$

②

~~$-.1x_1 + .15x_2 = 0$~~

~~$.1x_1 + -.15x_2 = 0$~~

Again Matlab

$$S = \begin{bmatrix} .8321 & -.7071 \\ .5547 & .7071 \end{bmatrix}$$

$$S^{-1} = \begin{bmatrix} .7211 & .7211 \\ -.5657 & .8485 \end{bmatrix}$$

$$A = S \underset{\text{matlab}}{\Lambda} S^{-1}$$

$$= \begin{bmatrix} .9 & .15 \\ .1 & .85 \end{bmatrix}$$ ✓ works

Limit $A^k = S \Lambda^k S^{-1}$

$$\begin{bmatrix} 1^k & \\ & .75^k \end{bmatrix} \text{ approaches } \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

So $S \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} S^{-1} \rightarrow \begin{bmatrix} .6 & .6 \\ .4 & .4 \end{bmatrix}$ "steady state"

③

8.3 #7 Find eigenvalues, vectors of A

9/10

$\lambda = 1, .5$

$$S = \begin{bmatrix} .8231 & -.7071 \\ .5547 & .7071 \end{bmatrix}$$

(same as last problem)

A^k is $S\Lambda^k S^{-1}$

$$S \begin{bmatrix} 1^k & 0 \\ 0 & .5^k \end{bmatrix} S^{-1}$$

$$S \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} S^{-1}$$ ← same exact as last problem

$$\begin{bmatrix} .6 & .6 \\ .4 & .4 \end{bmatrix} = \lim_{k \to \infty} A^k = A^{\infty}$$

④ Challenge

What Markov matrices produce this steady state

{Add cols to 1

~~Any one with~~

So $\lambda = 1, .75$ were this

$1, .5$

But I don't see the pattern

Book

$$\begin{bmatrix} .6 + .4a & .6 - .6a \\ .4 - .4a & .4 + .6a \end{bmatrix} \quad \begin{matrix} a \le 1 \\ .4 + .6a \ge 0 \end{matrix}$$

¿¡ But what is this ¿

OH! Here we know $A^{\infty} = \begin{bmatrix} .6 & .6 \\ .4 & .4 \end{bmatrix}$

and $\lambda = 1, < 1$

{unknown also call a

know eigenvector for 1

but not other eigenvector

Find evector in terms of a

also restrictions that $\sum col = 1$

Write 2nd evector in terms of B, C

Need to use the restrictions

(4b).

So $\begin{bmatrix} .8-a & .3 \\ .2 & .7-a \end{bmatrix} \begin{bmatrix} b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

~~$(.8-a)b + .3c = 0$~~

~~$.2b + (.7-a)c = 0$~~

~~but $.8-a+.2 = 1$ ← seems trivial~~

~~$.3 + .7-a = 1$~~

Want A in general

$\begin{pmatrix} .6 & b \\ .4 & c \end{pmatrix} \begin{pmatrix} 1 & \\ & a \end{pmatrix} \begin{pmatrix} .6 & b \\ .4 & c \end{pmatrix}^{-1} = \begin{bmatrix} .6 & .4 \\ .6 & .4 \end{bmatrix}$

but very complex to find

$Ax = \lambda x$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} .6 \\ .4 \end{pmatrix} = \begin{pmatrix} .6 \\ .4 \end{pmatrix}$

$a + c = 1$

$b + d = 1$

$.6a + .4b = .6$

$.6c + .4d = .4$ ✓

Solve from there. Also all ans ⊕ $a, b, c, d \geq 0$

9c)

Own)

$c = 1-a$

$d = 1-b$

$.6(1-a) + .4(1-b) = .4$

$.6 - .6a + .4 - .4b = .4$

$\frac{.6a}{.6} = \frac{.6 - .4b}{.6}$

$a = 1 - \frac{2}{3}b$

$.6 - .6(1 - \frac{2}{3}b) + .4 - .4b = .4$

$.6 - .6 - .4b + .4b - .4b = .4$

$-.4b = .4$

$b = -1$

$a = 1 - \frac{2}{3}(-1)$

$= 5/3$

$c = -2/3$

$d = 2$

b, c, d should be in terms of a.

?but that is not a function of something ...

(4d)

And violating $a,b,c,d > 0$

{but how would that work

Doesn't seem to be lattitude

And should be a function of ~~[illegible]~~ a,b

- not an exact ans
- I screwed this up

⑤ .

8.3 #9 Prove that the square of a Markov is still a Markov

10/10

S, Use $S\Lambda^2 S^{-1}$

like last time $\begin{bmatrix}1^2 & \\ & .75^2\end{bmatrix} = \begin{bmatrix}1 & \\ & .5625\end{bmatrix}$

So $S\Lambda^2 S^{-1} = \begin{bmatrix}.8250 & .5625 \\ .1250 & .7375\end{bmatrix}$

So confirmed, but not proved

We know $\lambda_1 = 1$

$\lambda_2 < 1$

So $\begin{bmatrix}1 & 0 \\ 0 & <1\end{bmatrix}$

When multiplying this

$$\begin{bmatrix}S_1 & S_3 \\ S_2 & S_4\end{bmatrix}\begin{bmatrix}1 & 0 \\ 0 & <1\end{bmatrix} = \begin{bmatrix}S_1 & S_3 \cdot <1 \\ S_2 & S_4 \cdot <1\end{bmatrix}$$

S_1 and S_2 are as they are

Then times S^{-1} more of the same

$$\begin{bmatrix} S_1 & S_3 \cdot c1 \\ S_2 & S_4 \cdot c1 \end{bmatrix} \begin{bmatrix} S_1^{-1} & S_3^{-1} \\ S_2^{-1} & S_4^{-1} \end{bmatrix}$$

Maintains the markov property

How do you know that the columns add to 1?

Book M^2 is still non neg

$$[1 \dots 1] M = [1 \dots 1]$$

So multiply on the right by M to find

$$[1 \dots 1] M^2 = [1 \dots 1]$$

So basically multiplying by a Markov dos nothing
A bit cleare that what I showed...

~~Question 4?~~

⑦

8.3 # 12 A Markov diff eq is not

8/10

$$\frac{du}{dt} = Au$$

but is

$$\frac{du}{dt} = (A - I)u$$

diagonal ⊖

rest A−I is ⊕

$\sum$ cols $= 0$

Find the eigenvalues of $B = A - I = \begin{bmatrix} -.2 & .3 \\ .2 & -.3 \end{bmatrix}$

$\lambda = 0, -.5$

Why does A−I have $\lambda = 0$?

So $(-.2-\lambda)(-.3-\lambda) - .2 \cdot .3$

$.06 + .2\lambda + .3\lambda + \lambda^2 - .06$

$\lambda^2 + .5\lambda = 0$

$\lambda = 0, -.5$ ✓

⑧

So it balances out?

Since this was $\begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix}$

The -1 on the diagonal made it

$$\begin{bmatrix} -.2 & .3 \\ .2 & -.3 \end{bmatrix}$$

Basically $x, -x$ $y, -y$

Then multipling together has $(-y)(-x) - (x)(y)$
$= 0$

When $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ multiply x_1 and x_2

what is steady state vs $t \to \infty$?

So $e^{\lambda_i t}$ are diff eq solns?

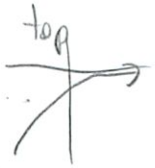
x_i are eigen vectors

? which are $\begin{bmatrix} .8321 & -.7071 \\ .5547 & .7071 \end{bmatrix}$ ← look familiar!

So what is the steady state sol'n?

So $e^{0t}\begin{bmatrix} .8321 \\ .5547 \end{bmatrix} = 1$

$e^{-.5t}\begin{bmatrix} -.7071 \\ .7071 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$



⑨

6.3#4 A door is opened

10/10

$v(0) = 30$ people

$w(0) = 10$ people

$$\frac{dv}{dt} = w - v$$

$$\frac{dw}{dt} = v - w$$

$$v + w = \text{fixed} = 40$$

$$\frac{du}{dt} = Au$$

So eigens help find steady state

$$u = e^{\lambda t} x$$

So A is

$$\begin{array}{cc} v & \\ w & \end{array}\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

✓ matches Book

But where was this described in the textbook?

Ah

$$\frac{d\begin{bmatrix} v \\ w \end{bmatrix}}{dt} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} v \\ w \end{bmatrix} = \begin{matrix} -v + w \\ v - w \end{matrix}$$

⑩

$\lambda = -2, 0$

$$S = \begin{bmatrix} .7071 & .7071 \\ -.7071 & .7071 \end{bmatrix}$$

?What is the significance of this #? $\frac{1}{\sqrt{2}}$

So $t=1$ is inital time? A^1 is just A

Or is inital $t=0$ - usually this way

So book shows starting from 0

$e^{-2(1)} = .135$

$e^{0(1)} = 1$

?How do we use that

Plug into original

v) $20 + .135(10) = 21.35$

w) $20 - .135(10) = 18.65$

So how does $S\Lambda S^{-1}$ stack up?

$$\begin{bmatrix}-1 & 1\\ 1 & -1\end{bmatrix}$$

just gives back A'

but wasn't this supposed to be the state?

Or only the steady state?

Or only Markov? ← no, don't think so

$$A = S\Lambda S^{-1}$$

$$A^2 = S\Lambda^2 S^{-1}$$ ← what does A^2 even mean? Here, you're just solving Diff. eqs in matrix form. See sol'n.

$$A^k = S\Lambda^k S^{-1}$$

$$A^\infty = S\Lambda^\infty S^{-1}$$ ← steady state

So try again w/ $t=\infty$

$$e^{0(\infty)} = 1$$

$$e^{-2(\infty)} = \infty$$? wait what

Try other way $\Lambda^\infty = \begin{bmatrix}\infty & 0\\ 0 & 0\end{bmatrix}$

$$S\Lambda^\infty S = \begin{bmatrix}\infty & -\infty\\ -\infty & \infty\end{bmatrix}$$

⊗ Not right either

②

Its A^{∞} is not steady state!

Only w/ $\lambda = 1$ I believe

(Chap 8.3)

So that was my issue

So stick to $u = e^{\lambda t} x$

But $e^{-2(\infty)} = 0$

$e^{0(\text{inf})} = 1$

$$u(t) = e^{\lambda_1 t} x_1 + e^{\lambda_2 t} x_2$$

no need to orthonormalize, although this is fine.

$$= Ce^{-2t} \begin{bmatrix} .7071 \\ -.7071 \end{bmatrix} + De^{0\cdot} \begin{bmatrix} .7071 \\ .7071 \end{bmatrix}$$

Ah, right need to calebrate on start

$$\begin{bmatrix} 30 \\ 10 \end{bmatrix} = C \begin{bmatrix} .7071 \\ -.7071 \end{bmatrix} + D \begin{bmatrix} .7071 \\ .7071 \end{bmatrix}$$

$C = \cancel{14.14} = 10\sqrt{2}$

$D = \cancel{28.28} = 20\sqrt{2}$

exact ans better

So then time ∞

$$\begin{bmatrix} ? \\ ? \end{bmatrix} = 10\sqrt{2} \cdot 0 \cdot \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} + 20\sqrt{2} \cdot 1 \cdot \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 20 \\ 20 \end{bmatrix}$$ ✓

⑬

6.3 #5 Reverse difusion

10/10

$$\frac{dv}{dt} = v - w$$

$$\frac{dw}{dt} = w - v$$

So $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

$\lambda = 0, 2$

$S = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

$\begin{bmatrix} 30 \\ 10 \end{bmatrix} = C \begin{bmatrix} -1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} + D \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

$C = -20\sqrt{2}$

$D = -10\sqrt{2}$

$u = -20\sqrt{2} \cdot 1 \cdot \begin{bmatrix} -1/\sqrt{2} \\ +1/\sqrt{2} \end{bmatrix} + -10\sqrt{2} \begin{bmatrix} -1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} e^{2\infty}$

$= \begin{bmatrix} \text{infinity} \\ -\text{infinity} \end{bmatrix} + \begin{bmatrix} -10 \\ -10 \end{bmatrix} = \begin{bmatrix} \text{inf} \\ -\text{inf} \end{bmatrix} \rightarrow \begin{bmatrix} 40 \\ 0 \end{bmatrix}$

? but how does that work w/ limits

(136)

OH) Write general sol as fn of t

$$\begin{bmatrix} v(t) \\ w(t) \end{bmatrix} = -20\sqrt{2}\, e^{0t} \begin{bmatrix} -1/\sqrt{2} \\ +1/\sqrt{2} \end{bmatrix} - 10\sqrt{2}\, e^{2t} \begin{bmatrix} -1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

$v + w$ constant

$$\lim_{t \to \infty} = \begin{bmatrix} \infty \\ -\infty \end{bmatrix}$$

Then apply limit

$$\begin{bmatrix} 40 \\ 0 \end{bmatrix}$$

✓ So I had understood it ☺

At some t reach 0, 40

Past that point, the intepretation of v, w as people in room breaks down

(like w/ partial/fraction of people)

⑭

6.3 #12 Subsitute $y = e^{\lambda t}$ into

10/10

$$y'' = 6y' - 9y$$

to show $\lambda = 3$ is a repeated root.

$$(e^{\lambda t})'' = 6(e^{\lambda t})' - 9(e^{\lambda t})$$

? is that what they meant?

OH! Do it

$$\lambda^2 e^{\lambda t} = 6\lambda e^{\lambda t} - 9e^{\lambda t}$$

non-zero #
can divide all by $e^{\lambda t}$

$$\lambda^2 = 6\lambda - 9$$

quadratic eq as λ

$$\lambda^2 - 6\lambda + 9 = 0$$

$$\lambda = +3, +3$$

① Answered 1st q.

⑮

For new matrix

$$\frac{d}{dt}\begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -9 & 6 \end{bmatrix}\begin{bmatrix} y \\ y' \end{bmatrix}$$

Show $\lambda = 3, 3$

So $\det(A - \lambda I)$ = what we found earlier

So it's the same

Only 1 line of eigenvectors

Since put in same λ, will get the same result

Or say nullspace of $A - 3I$ is 1-dimensonal

Show 2nd sol is $y = t e^{3t}$

- Saw in 18.03 we did this
- to avoid repeated root

Plug in again + show it works

⑯

$$(te^{3t})'' = 6(te^{3t})' - 9(te^{3t})$$

~~$9te^{3t} = 18te^{3t} - 9te^{3t}$~~

$\frac{d\,te^{3t}}{dt}$ ← product rule

$$3(3t+2)e^{3t} = 6(3t+1)e^{3t} - 9te^{3t}$$

$$(9t+6)e^{3t} = (18t+6)e^{3t} - 9te^{3t}$$

divide out e^{3t}

$$9t+6 = 18t+6-9t$$

$$9t+6 = 9t+6$$

✓ matches

⑰ ·

6.3 # 24 Write $A = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}$ as $S\Lambda S^{-1}$

10/10

$\lambda = 1, 3$

$S = \begin{bmatrix} 1 & .4472 \\ 0 & .8944 \end{bmatrix}$ $S^{-1} = \begin{bmatrix} 1 & -.5 \\ 0 & 1.11 \end{bmatrix}$

$$\begin{bmatrix} 1 & .4472 \\ 0 & .8944 \end{bmatrix} \begin{bmatrix} 1 & \\ & 3 \end{bmatrix} \begin{bmatrix} 1 & -.5 \\ 0 & 1.11 \end{bmatrix}$$

$$e^{\Lambda t} = \begin{bmatrix} e^{t} & \\ & e^{3t} \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} e^{t} & .5e^{3t} - .5e^{t} \\ 0 & e^{3t} \end{bmatrix}$$

So at $t = 0$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\frac{d e^{At}}{dt} = \begin{bmatrix} e^{t} & 1.5e^{3t} - .5e^{t} \\ 0 & 3e^{3t} \end{bmatrix}$$

At $t = 0$

$$= \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}$$

? what is that supposed to match w/ again ?

L OH] Change is A itself

Book] Much simpler $S = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ $S^{-1} = \begin{bmatrix} 1 & -1/2 \\ 0 & 1/2 \end{bmatrix}$

Why was mine so complex ?

And no explination

OH] Remember eigenvector is everything along the line

Matlab wants to have it normalized 

but $\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ is just as good

---

OH] $\frac{de^{At}}{dt} \to \frac{de^{A0}}{dt} = A$

we showed this

p 314 $\frac{de^{At}}{dt} = Ae^{At}$

problem is verifying this for specific instance ↓

(so w/ $t = 0$, e^{At} is I, so get A back

☺ Understand

6.3 #26 Give 2 reasons why e^{At} is never singular

?hmm

10/10 a) Write its inverse

?how supposed to find?

?Gauss-Jordan

book) $(e^{At})^{-1} = e^{-At}$

?how get that

OH) Proof of this is outside scope of course

$e^{x+y} = e^x e^y$ for #s

$(e^x)^{-1} = e^{-x}$

But does this generalize to matrices?

Not completely, but

$e^{A+B} = e^A e^B$? No, but if $AB = BA$, then yes

commute

So $tA, -tA$ commute ✓

So $e^{tA} + e^{-tA} = e^{tA-tA} = e^0 = 1$

Use power series to prove relation

⑩

b) Write down its eigenvalues

If $Ax = \lambda x$ Then $e^{At} x = \_\_\_ x$

? $e^{\lambda t}$

@ Back

Now the main qui Why is e^{At} never singular

Singlar = non invertable

It is invertable → so not singuler

Since $e^{At} = e^{-At}$

But still reed to show that

~ did in OH, see prev page

㉑

6.3 #30 Another good idea for $y'' = -y$ is

7/10 the trapezodial method (half forward / half back)

(iwtfi)

Best way to keep (Y_n, Z_n) on a circle

$$\begin{bmatrix} 1 & -\Delta t/2 \\ \Delta t/2 & 1 \end{bmatrix} \begin{bmatrix} Y_{n+1} \\ Z_{n+1} \end{bmatrix} =$$

$$\begin{bmatrix} 1 & \Delta t/2 \\ -\Delta t/2 & 1 \end{bmatrix} \begin{bmatrix} Y_n \\ Z_n \end{bmatrix}$$

a) Invert to write as $U_{n+1} = AU_n$

OH| Gives you new info

Rules for trapizodal method → outside scope of course

Go sentance by sintace

Find inverse Gauss Jordan or det

Symbol is parameter

Should be simple

$$\frac{1}{\det} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

㉒ .

Own

$$\det = (1) - \left(\frac{\Delta t}{2}\right)\left(-\frac{\Delta t}{2}\right)$$

$$= 1 + \frac{\Delta t^2}{4}$$

$$\text{left}^{-1} = \frac{1}{1+\Delta \frac{t}{4}} \begin{bmatrix} 1 & \Delta t/2 \\ -\Delta \frac{t}{2} & 1 \end{bmatrix}$$

Same w/o $\frac{1}{\det}$

Write as $U_{n+1} = A U_n$

$$U_{n+1} = \underbrace{\begin{bmatrix} 1 & +\Delta t/2 \\ -\frac{\Delta t}{2} & 1 \end{bmatrix}}_{?A} \begin{bmatrix} Y_n \\ Z_n \end{bmatrix}$$

? wait wasn't that done for us?
How does inverse fit in here?

(23).

OH) Then show A is orthogonal

$A^T A = I$ ← means orthogonal

(Just ignore the extra info + do the steps)

if input = norm 1 → then output is norm 1

norm $A \circ V$ = norm V

Own) ; So $A = \begin{bmatrix} 1 & \Delta t/2 \\ -\frac{\Delta t}{2} & 1 \end{bmatrix}$

transpose of this is

$$A^T = \begin{bmatrix} 1 & -\frac{\Delta t}{2} \\ \frac{\Delta t}{2} & 1 \end{bmatrix}$$

Multiply $A^T A = I$

$$\begin{bmatrix} 1 & -\Delta t/2 \\ \frac{\Delta t}{2} & 1 \end{bmatrix} \begin{bmatrix} 1 & \Delta t/2 \\ -\frac{\Delta t}{2} & 1 \end{bmatrix} = \begin{bmatrix} 1 + \frac{\Delta t^2}{4} & \frac{\Delta t}{2} - \frac{\Delta t}{2} \\ \frac{\Delta t}{2} - \frac{\Delta t}{2} & \frac{\Delta t^2}{4} + 1 \end{bmatrix}$$

$\frac{d}{dt} e^{At}$?

= not really I (I at $t=0$).

- did I multiply wrong?

㉔

These points U_n never leave the cicle

$$A = (I - B)^{-1}(I + B) \text{ is always orthogonal}$$

if $B^T = -B$

b) MATLAB

$U_0 = (1;0)$ to U_{32} w/ $\Delta t = \frac{2\pi}{32}$

; So how do we do enter values

$$U_1 = \begin{bmatrix} 1 \\ -.0982 \end{bmatrix}$$

$$U_2 = \begin{bmatrix} .9904 \\ -.1963 \end{bmatrix}$$

$$U_3 = \begin{bmatrix} .9711 \\ -.2936 \end{bmatrix}$$

etc on print out

does not seem to be a circle ...

back to $\begin{bmatrix} -1.16 \\ -.0117 \end{bmatrix} = U_{32}$

```
>> A = [1 (2*pi)/64; -(2*pi)/64 1]

A =

    1.0000    0.0982
   -0.0982    1.0000

>> U = [1; 0]

U =

     1
     0
```

u_0

```
>> U = A*U

U =

    1.0000
   -0.0982
```

u_1

```
>> U = A*U

U =

    0.9904
   -0.1963
```

2

```
>> U = A*U

U =

    0.9711
   -0.2936
```

3

```
>> U = A*U

U =

    0.9423
   -0.3889
```

4

```
>> U = A*U

U =

    0.9041
   -0.4814
```

5

```
>> U = A*U

U =
```

```
    0.8568          6
   -0.5702

>> U = A*U

U =                 7

    0.8008
   -0.6543

>> U = A*U

U =

    0.7366          8
   -0.7329

>> U = A*U

U =
                    9
    0.6647
   -0.8052

>> U = A*U

U =                 10

    0.5856
   -0.8705

>> U = A*U

U =                 11

    0.5001
   -0.9280

>> U = A*U

U =
                    12
    0.4090
   -0.9771

>> U = A*U

U =
                    13
    0.3131
   -1.0172
```

```
>> U = A*U

U =

    0.2132      14
   -1.0480

>> U = A*U

U =

    0.1104      15
   -1.0689

>> U = A*U

U =

    0.0054      16
   -1.0797

>> U = A*U

U =
                17
   -0.1006
   -1.0803

>> U = A*U

U =

   -0.2066      18
   -1.0704

>> U = A*U

U =

   -0.3117      19
   -1.0501

>> U = A*U

U =

   -0.4148      20
   -1.0195

>> U = A*U
```

```
U =

   -0.5149   21
   -0.9788

>> U = A*U

U =

   -0.6110   22
   -0.9282

>> U = A*U

U =

   -0.7021   23
   -0.8682

>> U = A*U

U =

   -0.7874   24
   -0.7993

>> U = A*U

U =

   -0.8658   25
   -0.7220

>> U = A*U

U =

   -0.9367   26
   -0.6370

>> U = A*U

U =

   -0.9993   27
   -0.5450

>> U = A*U

U =              28

   -1.0528
```

```
   -0.4469

>> U = A*U

U =

   -1.0967    29
   -0.3436

>> U = A*U

U =

   -1.1304     30
   -0.2359

>> U = A*U

U =                31

   -1.1536
   -0.1249

>> U = A*U
                    32
U =

   -1.1658
   -0.0117

>>
```

18.06 Spring 2012 – Problem Set 7

This problem set is due Thursday, April 19th, 2012 at 4pm (hand in to Room 2-106). The textbook problems are out of the 4th edition. For computational problems, please include a printout of the code with the problem set (for MATLAB in particular, `diary('filename')` will start a transcript session, `diary off` will end one.)

Every problem is worth 10 points.

1. Do Problem 2 from Section 8.3.

   Solution. Since $\begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ are the eigenvector vectors for the eigenvalues 1 and 0.75, respectively,

   $$S = \begin{bmatrix} 0.6 & -1 \\ 0.4 & 1 \end{bmatrix}.$$

   A^k approches to

   $$S \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} S^{-1} = \begin{bmatrix} 0.6 & 0.6 \\ 0.4 & 0.4 \end{bmatrix}.$$

   □

2. Do Problem 7 from Section 8.3 (do also the "challenge problem" part).

   Solution. The eigenvalues are 1 and 0.5, and the eigenvectors are

   $$\begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

   Since

   $$A^k = S \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}^k S^{-1},$$

   for

   $$S = \begin{bmatrix} 0.6 & 1 \\ 0.4 & -1 \end{bmatrix},$$

   $$A^\infty = S \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} S^{-1} = \begin{bmatrix} 0.6 & 0.6 \\ 0.4 & 0.4 \end{bmatrix}.$$

   Challenge problem Let $A = \begin{bmatrix} a & b \\ 1-a & 1-b \end{bmatrix}, 0 \leq a, b \leq 1$, be a Markov Matrix with steady state $\begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}$. Then

   $$A \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}.$$

Hence $0.6a + 0.4b = 0.6$. In other words,

$$A = \begin{bmatrix} 0.6 + 0.4x & 0.4 - 0.4x \\ 0.6 - 0.6x & 0.4 + 0.6x \end{bmatrix}$$

for some $-\frac{2}{3} \leq x \leq 1$.

$\square$

3. Do Problem 9 from Section 8.3.

Solution. If every entry of A is nonnegative, every entry of A^2 is also nonnegative. Since, for any $j = 1, \cdots, n$, $\sum_i (A)_{ij} = 1$,

$$\sum_i (A^2)_{ij} = \sum_{i,k} a_{ik}a_{kj} = \sum_k \sum_i a_{ik}a_{kj} = \left(\sum_k \left(\sum_i a_{ik}\right) a_{kj}\right) = \sum_k a_{kj} = 1.$$

$\square$

4. Do Problem 12 from Section 8.3.

Solution. The eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = -0.5$. We have a steady state for the Markov matrix A. For the steady state v, $(A - I)v = 0 = 0v$. So $A - I$ have $\lambda = 0$. If $u_t = e^{\lambda_1 t}c_1x_1 + e^{\lambda_2 t}c_2x_2$ for the initial value $u_0 = c_1x_1 + c_2x_2$, u_t converges to c_1x_1 as $t \to \infty$. $\square$

5. Do Problem 4 from Section 6.3.

Solution. $v + w$ is constant if and only if $\frac{d(v+w)}{dt} = 0$.

$$\frac{d(v + w)}{dt} = \frac{dv}{dt} + \frac{dw}{dt} = (w - v) + (v - w) = 0,$$

so $v + w$ is constant.

Let $u = \begin{bmatrix} v \\ w \end{bmatrix}$. Then

$$\frac{du}{dt} = \begin{bmatrix} \frac{dv}{dt} \\ \frac{dw}{dt} \end{bmatrix} = \begin{bmatrix} w - v \\ v - w \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix}.$$

The eigenvalue of $A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$ is given by solving $\det(A - \lambda I) = 0$. $\det(A - \lambda I) = (-1-\lambda)^2 - 1 = 1 + 2\lambda + \lambda^2 - 1 = \lambda(\lambda + 2)$ so the eigenvalues are $\lambda_1 = 0, \lambda_2 = -2$. We then observe that the corresponding eigenvectors are $x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, respectively.

Then the pure exponential solutions are given by

$$u_1(t) = e^{\lambda_1 t}x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$u_2(t) = e^{\lambda_2 t}x_2 = e^{-2t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

So the complete solutions are given by

$$u(t) = C\begin{bmatrix} 1 \\ 1 \end{bmatrix} + De^{-2t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} C + De^{-2t} \\ C - De^{-2t} \end{bmatrix}.$$

From the initial condition that $u(0) = \begin{bmatrix} v(0) \\ w(0) \end{bmatrix} = \begin{bmatrix} 30 \\ 10 \end{bmatrix}$, we get $C = 20$, $D = 10$.

That is,

$$v(t) = 20 + 10e^{-2t}$$
$$w(t) = 20 - 10e^{-2t}.$$

So $v(1) = 20 + 10e^{-2}, w(1) = 20 - 10e^{-2}, v(\infty) = w(\infty) = 20$.

□

6. Do Problem 5 from Section 6.3.

Solution. Now we have

$$\frac{du}{dt} = -Au = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} u.$$

The eigenvalues of $-A$ are given by -1 times the eigenvalues of A, so now we have $\lambda_1 = 0, \lambda_2 = 2$. The corresponding eigenvectors are the same as those of A, namely $x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Then the pure exponential solutions are given by

$$u_1(t) = e^{\lambda_1 t}x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$u_2(t) = e^{\lambda_2 t}x_2 = e^{2t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

So the complete solutions are given by

$$u(t) = C\begin{bmatrix} 1 \\ 1 \end{bmatrix} + De^{2t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} C + De^{2t} \\ C - De^{2t} \end{bmatrix}.$$

From the initial conditions $u(0) = \begin{bmatrix} v(0) \\ w(0) \end{bmatrix} = \begin{bmatrix} 30 \\ 10 \end{bmatrix}$, we get $C = 20$, $D = 10$, and $v(t) = 20 + 10e^{2t}$. So as $t \to \infty$, $v \to \infty$.

□

7. Do Problem 12 from Section 6.3.

Solution. Substituting $y = e^{\lambda t}$ into $y'' = 6y' - 9y$ gives

$$\lambda^2 e^{\lambda t} = 6\lambda e^{\lambda t} - 9e^{\lambda t},$$

so $e^{\lambda t}(\lambda - 3)^2 = 0$, which means $\lambda = 3$ is a repeated root.

In terms of the matrix equation, since the matrix has trace 6 and determinant 9, its only eigenvalue is 3, with one independent eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

To show that $y = te^{3t}$ is the second solution, just substitute this into the original differential equation. Since we have:

$$\begin{aligned} y' &= e^{3t} + 3te^{3t} \\ y'' &= 3e^{3t} + (3e^{3t} + 9te^{3t}) = 6e^{3t} + 9te^{3t}. \end{aligned}$$

Also,

$$6y' - 9y = 6e^{3t} + 18te^{3t} - 9te^{3t} = 6e^{3t} + 9te^{3t},$$

so we see that $y'' = 6y' - 9y$ when $y = te^{3t}$.

□

8. Do Problem 24 from Section 6.3.

Solution. A is an upper-triangular matrix, so we can read off its eigenvalues as the diagonal entries: 1, 3. By inspection we see that $(1, 0)$ is an eigenvector with eigenvalue 1. To find an eigenvector with eigenvalue 3 we observe

$$A - 3I = \left(\begin{smallmatrix} -2 & 1 \\ 0 & 0 \end{smallmatrix}\right),$$

and so $(1, 2)$ is in its nullspace. Thus

$$S = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 2 \end{smallmatrix}\right)$$

and

$$A = S\Lambda S^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix}.$$

Thus

$$e^{At} = Se^{\Lambda t}S^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} e^t & 0 \\ 0 & e^{3t} \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^t & \frac{1}{2}(e^{3t} - e^t) \\ 0 & e^{3t} \end{pmatrix}.$$

When $t = 0$ this is $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, as expected. Differentiating with respect to t, we get

$$\begin{pmatrix} e^t & \frac{1}{2}(3e^{3t} - e^t) \\ 0 & 3e^{3t} \end{pmatrix};$$

at $t = 0$ this is $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 3 \end{smallmatrix}\right) = A$.

□

9. Do Problem 26 from Section 6.3.

   Solution. e^{At} is nonsingular because

   (a) its inverse is given by e^{-At},

   (b) its eigenvalues are $e^{\lambda t}$ where λ is an eigenvalue of A — thus 0 is never an eigenvalue of e^{At}.

□

10. Do Problem 30 from Section 6.3.

   Solution. (a) $\begin{pmatrix} 1 & -\Delta t/2 \\ \Delta t/2 & 1 \end{pmatrix}^{-1} = \frac{1}{1+(\Delta t)^2/4}\begin{pmatrix} 1 & \Delta t/2 \\ -\Delta t/2 & 1 \end{pmatrix}$, so if $\mathbf{U}_n = (Y_n, Z_n)$ we have

$$\mathbf{U}_{n+1} = \frac{1}{1+(\Delta t)^2/4}\begin{pmatrix} 1 & \Delta t/2 \\ -\Delta t/2 & 1 \end{pmatrix}\begin{pmatrix} 1 & \Delta t/2 \\ -\Delta t/2 & 1 \end{pmatrix}\mathbf{U}_n = A\mathbf{U}_n$$

where

$$A = \frac{1}{1+(\Delta t)^2/4}\begin{pmatrix} 1-(\Delta t)^2/4 & \Delta t \\ -\Delta t & 1-(\Delta t)^2/4 \end{pmatrix}.$$

Then

$$\begin{aligned} A^{\mathrm{T}}A &= \frac{1}{(1+(\Delta t)^2/4)^2}\begin{pmatrix} 1-(\Delta t)^2/4 & -\Delta t \\ \Delta t & 1-(\Delta t)^2/4 \end{pmatrix}\begin{pmatrix} 1-(\Delta t)^2/4 & \Delta t \\ -\Delta t & 1-(\Delta t)^2/4 \end{pmatrix} \\ &= \frac{1}{(1+(\Delta t)^2/4)^2}\begin{pmatrix} (1-(\Delta t)^2/4)^2+(\Delta t)^2 & 0 \\ 0 & (1-(\Delta t)^2/4)^2+(\Delta t)^2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

If $B^T = -B$ and $A = (I-B)^{-1}(I+B)$ then $A^{\mathrm{T}}A = (I+B^{\mathrm{T}})(I-B^{\mathrm{T}})^{-1}(I-B)^{-1}(I+B) = (I-B)(I+B)^{-1}(I-B)^{-1}(I+B)$. But notice that $(I+B)(I-B) = I-B^2 = (I-B)(I+B)$, hence this equals $(I-B)(I-B)^{-1}(I+B)^{-1}(I+B) = I$. Similarly $AA^{\mathrm{T}} = (I-B)^{-1}(I+B)(I-B)(I+B)^{-1} = (I-B)^{-1}(I-B)(I+B)(I+B)^{-1} = I$, so A is indeed orthogonal.

   (b) If $\Delta t = 2\pi/32$ then using Matlab to compute A^{32} gives

$$\begin{pmatrix} 0.9998 & -0.0201 \\ 0.0201 & 0.9998 \end{pmatrix},$$

which is close to the identity, but there is clearly a potentially significant error.

□

OH

4/18

Only when $\lambda = 1$ is A^k steady state

18.06 4/18

Symmetric matrices

$$A = A^T = A^* = A'$$ ✓ Symbols for transpose in other subjects

Key facts

1. All eigenvalues are REAL
2. Eigenvectors are orthagonal

Can choose $S = Q$ eigen~~matrix~~ ~~[illegible]~~ vector matrix

$$A = S\Lambda S^{-1}$$

$$\downarrow$$

$$A = Q\Lambda Q^T$$

Examples

$A = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} = A^T$ $\lambda = 1, -1$

$A = \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix} = -A^T$ $\lambda = i, -i$ $S = \begin{bmatrix}1 & 1\\ i & -i\end{bmatrix}$ not sym

$A = \begin{bmatrix}2 & 1\\ -1 & 2\end{bmatrix}$ ~~[illegible]~~ $\lambda = 2+i \;\; 2-i$ not sym

②

Dealing w/ complex #s
L will do later ← in Chap 20 but discussing later today

(¿How does this match ???]

Fact

$$Q^T Q = I$$

$$Q^T = Q^{-1}$$

↓ so

$$A = Q \Lambda Q^T$$ ← One of the great theorms of lin algebra

Q = orthonormal

If $A^T \rightarrow Q \Lambda Q^T$ (¿ was heard correctly)

Since $\Lambda^T = \Lambda$

③

Examples (explained)

simplest possible →

$$\llcorner \; A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \rightarrow \text{Synetric so } \underset{\uparrow \text{real}}{\lambda = 1, -1}$$

$$\text{Then } Q = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \leftarrow \text{orthagonal}$$

↑ Since orthonormal

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \leftarrow \text{not sym} \qquad \lambda = i, -i \rightarrow S = \overset{\downarrow \text{orthagonal}}{\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}}$$

$$+2I \hookrightarrow \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} \text{ so add } 2 \text{ as well} \qquad 2+i, 2-i \text{ same } S$$

↑ note take dot product differently when complex

False reasoning for real A

$$Ax = \lambda x \qquad \leftarrow \text{ok}$$

take dot product (regular) w/ A^T

$$x^T A x = \lambda x^T x \qquad \leftarrow \text{ok}$$

④

False concludes eigenvalues are always real

Divide by one side / rearrange

$$\lambda = \frac{X^T A x}{X^T x}$$

$\nwarrow \|x\|^2$

Everything looks real

x looks real

? but this is false - λ does not have to be real

$$S = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}$$

appears dot product $\|x\| =$ $X^T x = \begin{bmatrix} 1 & i \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix} = 0$ ← but wrong way

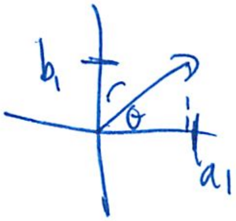
Must do something more to get lenght

$\|x\|$ is $\sqrt{2}$

Complex x in $C^n = \begin{bmatrix} a_1 + ib_1 \\ a_2 + ib_2 \\ \vdots \\ a_n + ib_n \end{bmatrix}$

So if $n=1$

└ size of $a_1 + ib_1$ is
(length)



r lenght of this vector

$r = |a_i + ib_i|$

$= \sqrt{a_1^2 + b_1^2}$

So $||x||$ is not $x^T x$

but $\sqrt{a_1^2 + b_1^2 + a_2^2 + b_2^2 + \dots + a_n^2 + b_n^2}$

Can only be 0 if everything is 0

⑥

So want a complex dot product

$$r^2 = a^2 + b^2$$

$$(a+ib)(a-ib)$$ ← complex conjugate

So $\overline{X}^T X$

$\uparrow$
x conjugate → in book uses X^H $\boxed{\|x\|^2 = x^H x}$

$$\begin{bmatrix} a_1 - ib_1 & \cdots \end{bmatrix} \begin{bmatrix} a_1 + ib_1 \\ \vdots \end{bmatrix} = \begin{bmatrix} a_1^2 + b_1^2 + \\ a_2^2 + b_2^2 + \\ \vdots \end{bmatrix}$$

* Don't forget to take the conjugate

~~If allowing A complex~~

$$A = A^H = A^* = A' = \overline{A}^T$$

$\uparrow \quad \uparrow$
symbols don't change
MATLAB auto uses A^H instead of A^T

↓ key facts to remember

So can't just have $S = Q$

So instead for symetric complex

$$A = A^H = \bar{A}^T$$

called "Hermitian"

if orthagonal for symetric complex

$Q^{-1} \rightarrow Q^H$

complex conj and transpose

↑ : Herd right

$$Q^H Q = I$$

called "unitary"

So all λ are real and eigenvectors orthagonal

↖ w/ $Q^H Q$ not $Q^T Q$

Since still symetric

$$\boxed{A = Q \Lambda Q^H}$$

Then since λ is real, can go backwards

$$A = A^H \iff A = Q \Lambda Q^H$$

[illegible]

⑧

Take conjugate transpose

⋮ (missed)

Spectral theorm
aka the principal axis theorm

Connects w/ an ellipsoid

3x3 symmetric matrix A

$$\begin{bmatrix} x & y & z \end{bmatrix} A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 1$$

all lie on an elipsoid

3 principal directions
- every ellipsoid can be lined up (by Q) to the principal axes

⑨

perp axis
(evector)

lenghts (λ)

$ax^2 + 2bxy + cy^2 = 1$

$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$

$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} =$

$= ax^2 + 2bxy + cy^2 = 1$

linear algebra is giving the key facts of geometry

n dim ⇒ no problem

Done everything except proved the key facts

Fact 1 |

can be complex

Given $A = A^H \to$ real λ's

Proof | $Ax = \lambda x$

$$x^H A x = \lambda \underbrace{x^H x}$$

x^H not x^T since | it might be complex

Is real

≠0 unless $x=0$

this is a ⊕ real #

$\underbrace{x^H A x}$

is also real

big step

is a #

[ ][ ][ ]

~~that~~ to prove real → take complex conj; (aka Hermetian

$(x^H A x)^H \to$ get same thing

$$= x^H A x$$

now

Fri About positive, definite matrices and examples theirof

⑪

Can verify

$$\bar{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_n(t) \end{bmatrix}$$

$$\bar{u}(t) = A\, u(t)$$

$$u(t) = e^{At} \cdot \bar{u}(0)$$

(think back to basic cases

$$A = S \Lambda S^{-1}$$

$$e^{At} = I + (At) + \frac{(At)^2}{2!} + \dots$$

$$= I + S(\Lambda t) S^{-1} + \frac{S(\Lambda^2 t) S^{-1}}{2!} + \dots$$

factor out S, S^{-1}

$$I = S S^{-1}$$

$$= S\left(I + \lambda t + \frac{(\lambda t)^2}{2!} + \cdots\right) S^{-1}$$

$$= S\begin{pmatrix} 1 + \lambda_1 t + \frac{(\lambda_1 t)^2}{2!} + \cdots & 0 & & 0 \\ 0 & 1 + \lambda_2 t + \cdots & & \\ \vdots & 0 & \ddots & \\ 0 & & & 1 + \lambda_n t + \frac{(\lambda_n t)^2}{2!} + \cdots \end{pmatrix}$$

$$= \begin{pmatrix} e^{\lambda_1 t} & & & \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ & & & e^{\lambda_n t} \end{pmatrix}$$

18.06

4/20

Building on Symmetric $A = A^T$

Today Positive Definite

1. all pivots > 0

2. all eigenvalues $\lambda > 0$

5 diff ways to identify

3. All "upper left" det are ⊕

$\begin{bmatrix} 5 & 4 \\ 4 & 3 \end{bmatrix}$ all pos - good, but not ⊕ def.

$\begin{array}{cc} ⊕ & ⊕ \\ 5 & 4 \\ 4 & 3 \end{array}$

det=5

det=-1

~~No~~

(No) - not ~~pa~~ ⊕ def

$\begin{bmatrix} 5 & 4 \\ 0 & -\frac{1}{5} \end{bmatrix}$ (No)

$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ det=2 det=3 $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

(Yes)

also w/ elim $\begin{bmatrix} ② ⊕ & 1 \\ 0 & (1.5) ⊕ \end{bmatrix}$ (Yes)

②

Also $\lambda = 3, 1$ Yes

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \begin{matrix} a > 0 \\ ac > b^2 \end{matrix}$$

(is it all 3 must be true, or any 1)

The more ⊕ #s are → the more ⊕ it pushes stuff

??

But note b^2

On diagional → big + pos

off diagonal → Small

$$\begin{bmatrix} 5 & -4 \\ -4 & 3 \end{bmatrix}$$ still no

⊕ det byitself is not enough

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$ det ⊕ but 1x1 det = ⊖ No

& is actually neg definite

③

$\begin{bmatrix} 2 & 1 \\ 1 & c \end{bmatrix}$ For which c is this ⊕ definite

✓ $\begin{matrix} \boxed{2} & 1 \\ 1 & c \end{matrix}$

↳ $\boxed{c > \frac{1}{2} \text{ for ⊕ def}}$

$\det = 2c - 1$

but what if $c = \frac{1}{2}$

$\det = 0$ ⊗ Not pos definite

Singular

$\lambda = 0, 2\frac{1}{2}$

But it is ⊕ semi definite

1. all $\lambda \geq 0$
2. all pivots ≥ 0

This has all the basic ideas of every chap

④

⊕ def rules cont → 4. A can be factored into $A = L L^T$

$$LL^T = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} =$$

?L ?L^T

Claim → will get ⊕ def matrix

How science uses it

$$= \begin{bmatrix} 1 & 1 \\ 1 & 5 \end{bmatrix}$$ Yes is ⊕ definite

L can't be singular

↳ would be semi-definite } (was that a rule?)

[cols must be ind

5. Def: $A = A^T$ is pos def <u>if</u>

key idea → $x^T A x > 0$ for all $x \neq 0$

$1\times n \quad n\times n \quad n\times 1$ = number

⑤

L can represent the energy in a system
<u>The</u> natural quantity

Ex $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$

Using #5

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = (x_1\ x_2) \begin{bmatrix} ax_1 + bx_2 \\ bx_1 + cx_2 \end{bmatrix}$$

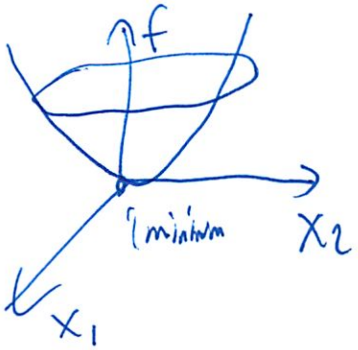
$$= ax_1^2 + 2b \cdot \underset{\uparrow}{x_1 x_2} + cx_2^2$$

cross term

dangerous the one that can go ⊖

$$= f(x_1, x_2)$$

Should be > for all x , $x \neq 0$



$f > 0,$
$\hookrightarrow$ A is pos def
$\hookrightarrow$ graph is a bowl

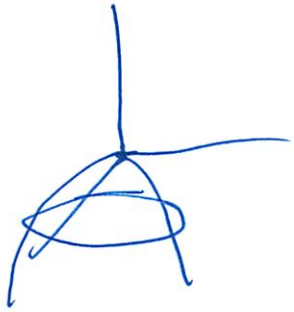
ex $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$f = x_1^2 + x_2^2$

What does the picture look like?

$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

$f = -x_1^2 - x^2$



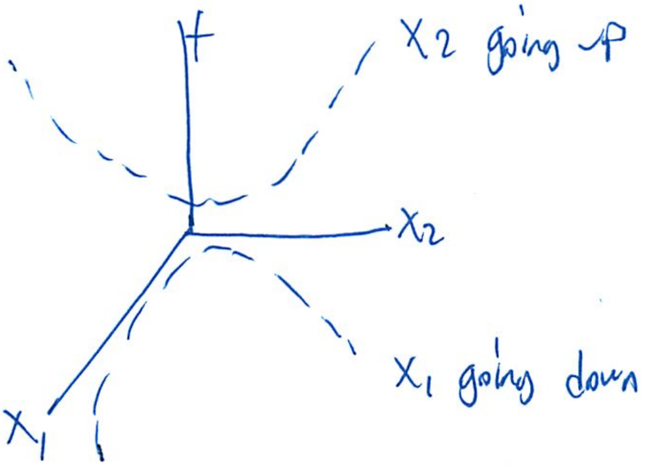
negitie

⑦

Saddle point

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$f = -x_1^2 + x_2^2$$



↱ like Rocky Mts pass

x_2 is mts range
min for mtns

x_1 is your path
max for you

$$\begin{bmatrix} 5 & -4 \\ -4 & 3 \end{bmatrix}$$

__Not__ ⊕ def
but saddle point

Connects w/ minimum
Lin calculus ⤷ next pg

⑧

$f(x)$ min $f'=0$ $f''>0$

max $f'=0$ $f''<0$

$$f(x,y) \quad \begin{matrix} \text{min} \rightarrow \\ \text{max} \rightarrow \end{matrix} \begin{pmatrix} df/dx = 0 \\ df/dy = 0 \end{pmatrix} \quad \left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) > \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$$

saddle

(confused on what's what)

$$A = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} \leftarrow \text{has to be } \oplus \text{ definite}$$

? test is same as $ac-b^2$

$$\frac{\partial f}{\partial x^2} > 0 \qquad \leftrightarrows 2 \text{ part test}$$

⑨

<u>Proving</u>

if 5 → then 2 ($\lambda > 0$)

Suppose $Ax = \lambda x$

then $\underbrace{x^T A x}_{\text{condition } 5} = \lambda \underbrace{x^T x}_{\text{always } +}$

$\underbrace{\quad}$

So λ ⊕

$$\lambda = \frac{x^T A x}{x^T x}$$

But 2 → 5

Start w/ λ ⊕

Prove $x^T A x > 0$

$$\underbrace{\lambda}_{\oplus} \underbrace{x^T x}_{\oplus} = \underbrace{x^T A x}$$

So must be ⊕

but <u>only</u> for the eigenvectors!

have not gotten them all!

18.06 Lecture

4/23

Matrices in Engineering

(watching online)

Matrix in Engineering

∟ but I don't see that online!

Oh well - guess it's not online ...

Chap 8.1 book

K = symetric (often ⊕ definite)

$K = A^T A$

$K = A^T C A$

Since $\frac{1}{2} u^T K u$ is energy (never negitive)

K = stiffness matrix

$K^{-1} f$ is response to forces

Direct linear → matrix eqns

Indirect continuous → diff eq'n

$Ku = f$

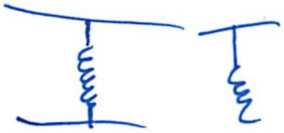
$M\frac{d^2u}{dt^2} + Ku = f$

Then $Kx = \lambda Mx$

(whats the point of this chap ??)

A Line of Springs

fixed-fixed → K_0 $A_0^T C_0 A_0$



fixed-free → K_1 $A_1^T C_1 A_1$

Free-free → $K_{singular}$

$u = (u_1, u_2, u_3)$ = movement of the masses up + down

$y = (y_1, y_2, y_3, y_4)$ or (y_1, y_2, y_3) = tension in the springs

③

u = Movements of n masses

e = Elongations of m springs

y = Internal forces in m springs

f = External forces

$$\begin{tikzcd}
\boxed{u} \arrow[d, "A"] & \boxed{f} \\
\boxed{e} \arrow[r, "C"] & \boxed{y} \arrow[u, "A^T"]
\end{tikzcd}$$

(a bunch more eq)

Removing eqns - matrices smaller

Then when 2x2 one spring is free

Can do circular

Can convert to continuous

$A^T C A u(x) = f(x)$ is

$$-\frac{d}{dx}\left(c(x)\frac{du}{dx}\right) = f(x)$$

(I hope we don't actually have to know)

18.06

4/23

Last time notes from Dan

$$R^2 = A \to R = Q\sqrt{\Lambda}\,Q^T$$

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

$$S = Q = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$R = Q\begin{bmatrix} 2 & 0 \\ 0 & \sqrt{2} \end{bmatrix} Q^T$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & \sqrt{2} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Big Fact If $A = R^T R$ and R has ind cols show $\bar{x}^T A \bar{x} > 0$ for $\bar{x} \neq 0$ (ie A pos def.)

$$X^T A x = X^T R^T R x = (x^T R^T)(Rx)$$

$$= (Rx^T)(Rx) \geq 0$$

($Rx = 0$ b/c R has ind cols)

②

Recall

A^{H} (hermiton) $= A^{T}$

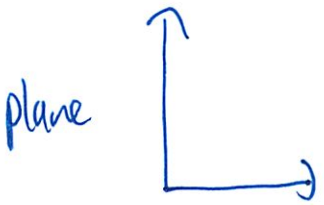
$X^{H} A^{H} x > 0 \rightarrow$ Pos def

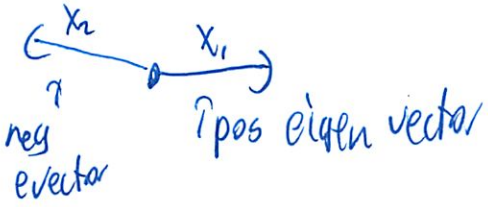
OH

4/25

Wed - Symmetric (6.4)
Fri - Pos definite (6.5)
Mon - App to Engineering (8.1)
Today - Similar

6.5 #2 → A_1 has one $\oplus$, one $\ominus$ λ





$\lambda \approx$ -0.little
12.little

~~pos~~ pos eigenvector is w/ eig. value that is pos

A · Symmetric so evectors are orthogonal

②

$x_n^T (A x_n)$

Dot prod $x_1 \cdot Ax_1$ is ⊕ since point in same dir
$x_2 \cdot Ax_2$ ⊖ diff
↑
eigenvector

So x_2 is a good canidate for a vector
such that $xAx^T < 0$
since x_2 is an eigenvector for a ⊖
eigenvalue

18.06 Spring 2012 – Problem Set 8

This problem set is due Thursday, April 26th, 2012 at 4pm (hand in to Room 2-106). The textbook problems are out of the 4th edition. For computational problems, please include a printout of the code with the problem set (for MATLAB in particular, `diary('filename')` will start a transcript session, `diary off` will end one.)

Every problem is worth 10 points.

1. Do Problem 3 from Section 6.4.
2. Do Problem 22 from Section 6.4 (now a paradox for you).
3. Do Problem 23 from Section 6.4.
4. Do Problem 30 from Section 6.4.
5. Do Problem 2 & 14 from Section 6.5.
6. Do Problem 28 from Section 6.5.
7. Do Problem 31 from Section 6.5.
8. Do Problem 35 from Section 6.5.
9. Do Problem 7 from Section 8.1.
10. Do Problem 11 from Section 8.1.

18.06 Wisdom.

- Your exams so far make up 30% of your final grade. If you wish to maximize your performance in 18.06, it is all about practice (without this, you won't know what you don't know either). Make a weekly study plan involving doing as many old exams as you can fit in your schedule (find them on the 18.06 web). Organize your work. Be systematic. Learn to check your results. Keep a list of all questions (and confusing concepts) you encounter on your journey, and ask your TAs to explain all the hows and whys. We are here for the same!
- Also try out the great Math Learning Center, that you can visit and ask the instructors your questions in a friendly atmosphere, and at evening hours too: *Room 2-102, Mon-Thu, 3:00–5:00pm and 7:30–9:30pm.*

Michael Plasmeier P-Set 8 86/100 4/23

1. 6.4 #3 Find the eigenvalues and the unit eigenvectors of $\begin{bmatrix} 2 & 2 & 2 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}$

8/10

Matlab

$\lambda = -2, 0, 4$ ✓ ↙ You should do this by hand! ⟨-2⟩

$$V = \begin{bmatrix} -1/\sqrt{3} & 0 & \sqrt{\frac{2}{3}} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix}$$

Book/ Unit vectors

$$\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} / \sqrt{2} \qquad \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} / \sqrt{6} \qquad \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} / \sqrt{3}$$

Same as what we had....

②

6.4 #22 ① If $AA^T = A^TA$ then A and A^T

10/10 share the same eigenvectors. (true)

② A and A^T always share the same eigenvalues,

③ Find the flaw; They have the same S and Λ

So $A = A^T$

OH | Presents w/ false argument → asks you to find mistake

① Remember if commute → share same eigenvector

└ from end of 6.2

② From $\det(A - \lambda I) = 0$

for A^T $\det(A^T - \lambda I) = 0$

Since how doing det does not change if you transpose it

Since $\det(A) = \det(A^T)$

③ Have same evalues

But only same evectors ~~in that form~~? Same

Hint; Order of eigenvalues and eigenvectors diff

for A and A^T – can't sub one in for the other

③

6.3 #23 Which of these Classes of Matrices does this belong to

9/10

$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$	$B = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$	
invertable ✓	invertable X	
Orthagonal ✓	orthgonal X	
projection X	projection ✓	$P^2 = P$ <u>and</u> $P = P^T$
permutable ✓	permutable X	
diagonizable ✓	diagonzable ✓	
Markov ✓	Markov ✓	

What factorizations are possible?

LU X	LU X	lower-upper
QR ✓	QR X	gram-schmidt
$S\Lambda S^{-1}$ ✓	$S\Lambda S^{-1}$ ✓	has an inverse
$Q\Lambda Q^{-1}$ ✓	$Q\Lambda Q^{-1}$ ✓	symetric

is ~~this wrong in the ans~~

~~prob not~~

③b

OH) Can do $S\Lambda S^{-1}$ whenever you have a complete set of eigenvalues and eigenvectors → can do

Does not matter if invertible

④

6.4 # 30 If λ_{max} has the largest eigenvalue of symmetric matrix A, no diagonal entry can be larger than λ_{max}. What is the 1st entry a_{11} of $A = Q \Lambda Q^T$. Show why $a_{11} \le \lambda_{max}$

10/10

In book)

$$\begin{bmatrix} q_{11} & \cdots & q_{1n} \end{bmatrix} \begin{bmatrix} \lambda_1 \bar{q}_{11} \\ \vdots \\ \lambda_n \bar{q}_{1n} \end{bmatrix} \le \lambda_{max} \left(|q_{11}|^2 + \cdots |q_{n1}|^2 \right) = \lambda_{max}$$

Why is this λ max?

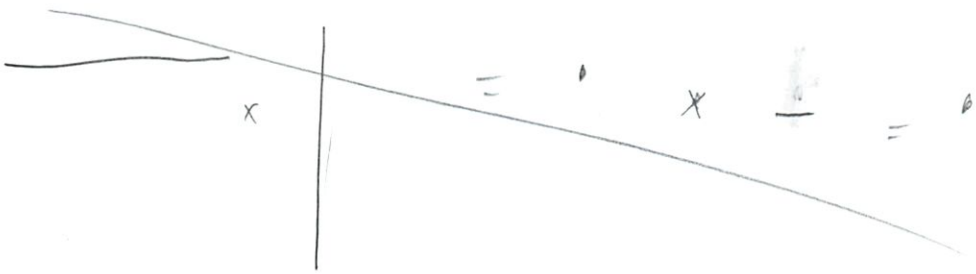
Oh!) If diagonal matrix, diagonals are λs
But not ness diagonal matrix here
↳ *any* symmetric matrix A

4b

Q = orthogonal

L esentially eigenvectors orthogonal

Have matrix as product of 3 matrices, so have an expression for it



1st grap ① $q_{11} \cdot \Lambda_{11} + q_{12} \cdot \underset{?0}{\lambda_{21}} + q_{1n} \cdot \underset{?0}{\lambda_{n1}}$

② $q_{12} \cdot \lambda_{22}$.

③ $q_{13} \cdot \lambda_{33}$

$a_{11} = q_{11}(q_{11}\Lambda_{11}) + q_{12}(q_{12}\lambda_{22}) + q_{13}(q_{13}\lambda_{33})$

$= q_{11}^2 \lambda_{11} + q_{12}^2 \lambda_{22} + q_{13}^2 \lambda_{33} + \ldots$

(4c)

Q is orthogonal - so rows + cols are orthonormal

↑ word for matrices ↑ word applies to rows + cols

So $q_{ij} < 1$

Also $\sum q_{ij}^2 = 1$

So $a_{11} \leq \lambda_{max}$

↑ if replace all coeff w/ largest one (λ_{max})

So $1 \cdot \lambda_{max}$ at most

⑤

6.5 #2. Which of A_1, A_2, A_3, A_4 has 2 positive eigenvalues?

10/10

$$A_1 = \begin{bmatrix} 5 & 6 \\ 6 & 7 \end{bmatrix} \quad A_2 = \begin{bmatrix} -1 & -2 \\ -2 & -5 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 1 & 10 \\ 10 & 100 \end{bmatrix} \quad A_4 = \begin{bmatrix} 1 & 10 \\ 10 & 101 \end{bmatrix}$$

So that means which is positive definate

$a > 0$ $\quad$ $ac - b^2 > 0$

A_1 $\quad$ $A_1 \rightarrow 35 - 36$ ⊗

~~A_2~~ $\quad$ $A_3 \rightarrow 100 - 100$ ⊗ $\boxed{\text{Semi}}$

A_3 $\quad$ $A_4 \rightarrow 101 - 100$ ✓

A_4

A[illegible] and A_4

Find an x such that $x^T A_1 x < 0$ So a fails the test

? which test is this

Oh $x^T A x > 0$ for every non 0 x

$$ax^2 + 2bxy + cy^2 < 0$$

?to fail

$$x = \begin{bmatrix} x \\ y \end{bmatrix}$$

⑥

$5x^2 + 2(6)xy + 7y^2 < 0$

$5x^2 + 12xy + 7y^2 < 0$

? if both −, the vale will be neg

Wolfram Alpha $x < 0$

$-\frac{5x}{7} < y < -x$

So One is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$5(1)^2 + 12(1)(-1) + 7(-1)^2$

$5 - 12 + 7 = 0$

which is not > 0

└ like semi definite or something like that

⑦

6.5 #14 If A is pos def → then A^{-1} is pos definite

Best proof: eigenvalues A^{-1} are pos since

└ Saw on last qu A, A^T are same eigenvalues

Here A, A^T eigenvalues are pos iff pivots are pos

But A^{-1} and A^T?

We know $Q^{-1} = Q^T$ w/ orthonormal

Book)

$$\lambda(A^{-1}) = \frac{1}{\lambda^{\text{of}}(A)}$$

↳ Oh duh!

2nd proof (for 2×2)

entries of A^{-1} $= \frac{1}{ac-b^2}\begin{bmatrix} c & -b \\ -b & a \end{bmatrix}$

passes the det test

↳So eigenvalues pos if $a > 0$
$ac - b^2 > 0$ □

⑧

6.5 # 28 Without multiplying $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$

10/10

$$\hookrightarrow \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

find the

a) det of A

?? is this $Q \Lambda Q^{-1}$?

$^{-1}$ is Q^T

?but how do you reconstruct fact?

(?and what does this do w/ ⊕ det?)

So $\det(A - \lambda I) = 0$

Ooh

Help | Duh product of eigenvalues

$2 \cdot 5 = 10$?remember these rules

⑨ ..

b) eigenvalues

$\lambda = 2, 5$ ← easy

c) eigenvectors

$$x = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}, \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

d) why its pos def
↳ all eigenvalues are ⊕

a) So could use these for det?
↳ but you found it 1st ...

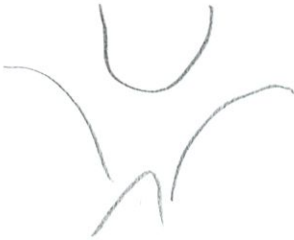
⑩

6.5 # 31 Which values of c give a bowl and which a saddle pt for

10/10

$$z = 4x^2 + 12xy + cy^2$$

So saddle pt



← awful pic

transitions at 9

< 9 Saddle pt

> 9 bowl

$= 9$ trough

? How do you tell?

Help Oh do pos def

det > 0 means local min → ie bowl

So positive def if > 9 / bowl

6.5 #35 Suppose C is pos definite (so $y^TCy > 0$ when $y \neq 0$) and A has ind cols (so $Ax \neq 0$ when $x \neq 0$)

10/10

Apply x^TA^TCAx to show A^TCA is ⊕ definite

So $x^TAx > 0$ for every non zero vector x

? what is there to show?

Also need the other facts

- all pivots ⊕

We know this applies to C
But how to apply to A?

Help) Just plug in
Get the definition
A^TCA is pos def?

⑫

8.1 #7 For 5 springs and 4 masses w/ both ends fixed, what are A, C, k ?

7/10

$e_1 = u_1$
$e_2 = u_2 - u_1$
$e_3 = u_3 - u_2$
$e_4 = u_4 - u_3$
$e_5 = -u_4$

$e = Au$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

from book
$e = Av$

x

$$C = \begin{bmatrix} c_1 & & & & \\ & c_2 & & & \\ & & 1 & & \\ & & & 1\,c_5 & \\ & & & & 1 \end{bmatrix}$$

$y = Ce$

⑬

force balance

~~$f_1 = y_1 - y_2$~~

~~$f_2 = y_2 - y_3$~~

~~$f_3 = y_3 - y_4$~~

~~$f_4 = y_4 - y_5$~~

Don't need

$$k = A^T C A$$

$$k_0 = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

⑭ ·

8.1 # 11 (MATLAB) Chem e has 1st deriv

2/10

$\frac{du}{dx}$ from fluid velocity

$\frac{d^2u}{dx^2}$ from diffusion

Replace $\frac{du}{dx}$ by a <u>forward diff</u>

then a <u>centered diff</u>

then a <u>backwards diff</u>

$\Delta x = \frac{1}{8}$

Graph sols of $-\frac{d^2u}{dx^2} + 10\frac{du}{dx} = 1$

$u(0) = u(1) = 0$

?, What is a forward difference

Mathworld $\Delta a_n = a_{n+1} - a_n$

Central $\delta(f_n) = \delta_n = f_{n+1/2} - f_{n-1/2}$

backward $\nabla_p - f_p - f_{p-1}$

⑮

I don't get this problem at all...

Book

E = diag(ones(6,1) 1)

k = 64 · 2(eye(7) − E − E')

D = 80 · (E − eye(7))

(k+D)\ones(7,1) ← forward

(k − D)\ones(7,1) ← backward

(k + D/2 − D'/2)\ones(7,1) ← cenered

What are E, k, D? Seem so random!

What is \ ?

Yeah I don't get this at all

```
>> E = diag(ones(6,1),1)

E =

     0     1     0     0     0     0     0
     0     0     1     0     0     0     0
     0     0     0     1     0     0     0
     0     0     0     0     1     0     0
     0     0     0     0     0     1     0
     0     0     0     0     0     0     1
     0     0     0     0     0     0     0


>> K = 64*2*(eye(7)-E-E')

K =

   128  -128     0     0     0     0     0
  -128   128  -128     0     0     0     0
     0  -128   128  -128     0     0     0
     0     0  -128   128  -128     0     0
     0     0     0  -128   128  -128     0
     0     0     0     0  -128   128  -128
     0     0     0     0     0  -128   128

>> D=80*(E-eye(7))

D =

   -80    80     0     0     0     0     0
     0   -80    80     0     0     0     0
     0     0   -80    80     0     0     0
     0     0     0   -80    80     0     0
     0     0     0     0   -80    80     0
     0     0     0     0     0   -80    80
     0     0     0     0     0     0   -80

>> (K+D)\ones(7,1)

ans =

   -0.0202
   -0.0410
   -0.0080
    0.0806
    0.0811
   -0.1547
   -0.3918

>> (K-D)\ones(7,1)

ans =
```

```
    0.0391
    0.0343
    0.0054
   -0.0205
   -0.0287
   -0.0208
   -0.0080

>> (K+D/2-D'/2)\ones(7,1)

ans =

    0.0012
   -0.0096
   -0.0276
   -0.0332
   -0.0069
    0.0420
    0.0629
```

18.06 Spring 2012 – Problem Set 8

This problem set is due Thursday, April 26th, 2012 at 4pm (hand in to Room 2-106). The textbook problems are out of the 4th edition. For computational problems, please include a printout of the code with the problem set (for MATLAB in particular, `diary('filename')` will start a transcript session, `diary off` will end one.)

Every problem is worth 10 points.

1. Do Problem 3 from Section 6.4.

   Solution. Solve $det(A-\lambda I)=0$ to get the eigenvalues $0, -2, 4$. Then find unit vectors in the nullspaces of the matrices $A-0I, A+2I, A-4I$ to get the unit eigenvectors $\frac{1}{\sqrt{2}}[0,1,-1]^T$, $\frac{1}{\sqrt{3}}[1,-1,-1]^T$, and $\frac{1}{\sqrt{6}}[2,1,1]^T$. □

2. Do Problem 22 from Section 6.4 (now a paradox for you).

   Solution. The flaw is that the correspondence between eigenvalues and eigenvectors need not be the same for A and A^T. For example, the matrix

   $$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

   has eigenvalues i and $-i$ with corresponding eigenvectors $[1,i]^T$ and $[i,1]^T$, while the transpose

   $$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

   has eigenvalues i and $-i$ with corresponding eigenvectors $[i,1]^T$ and $[1,i]^T$. □

3. Do Problem 23 from Section 6.4.

   Solution. A is invertible, orthogonal, permutation, diagonalizable, Markov. A has possible factorizations QR, $S\Lambda S^{-1}$ and $Q\Lambda Q^T$.

   B is projection, diagonalizable and Markov. B has possible factorizations LU, QR, $S\Lambda S^{-1}$ and $Q\Lambda Q^T$; note that the matrices U and R have only one non-zero diagonal entry. □

4. Do Problem 30 from Section 6.4.

   Solution. From the decomposition $A = Q\Lambda Q^T$ we have $a_{11} = \sum_i \lambda_i q_{1i}^2$. From here, we get $a_{11} \leq \sum_i \lambda_{max} q_{1i}^2 = \lambda_{max} \sum_i q_{1i}^2 = \lambda_{max}$ □

5. Do Problem 2 & 14 from Section 6.5.

Solution. **Problem 2** Only A_4 has two positive eigenvalues. The vector $x = [-7, 6]^T$ has $x^T Ax = -7 < 0$.

Problem 14 The eigenvalues of A^{-1} are positive because they are inverses of the eigenvalues of A. For the 2 by 2 case, one can also check directly that the entries of A^{-1} pass the determinant tests. ☐

6. Do Problem 28 from Section 6.5.

   Solution. Use the product formula for determinants to get:

   $$det(A) = 1 \times 10 \times 1 = 10.$$

   We recognize that A is factorised in the form $Q\Lambda Q^T$. Thus it has eigenvalues $\lambda_1 = 2$, $\lambda_2 = 5$, and corresponding eigenvectors $x_1 = (\cos\theta, \sin\theta)^T$ and $x_2 = (-\sin\theta, \cos\theta)$. Both eigenvalues are positive, so the matrix is positive definite. ☐

7. Do Problem 31 from Section 6.5.

   Solution. Complete the square to get

   $$z = 4x^2 + 12xy + cy^2 = (2x + 3y)^2 + (c - 9)y^2.$$

   Thus for $c > 9$ the graph of z is a bowl, and for $c < 9$ the graph has a saddle point. When $c = 9$, the graph, $z = (2x + 3y)^2$, is a "trough" staying at zero along the line $2x + 3y = 0$. ☐

8. Do Problem 35 from Section 6.5.

   Solution. "Factor" the tranposes to get

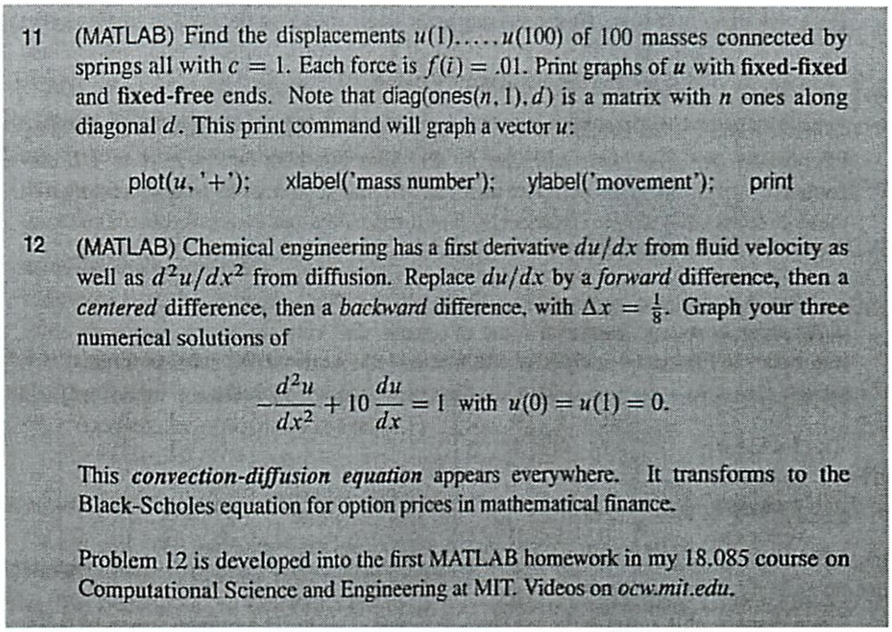
   $$x^T A^T CAx = (Ax)^T C(Ax).$$

   Since C is assumed positive definite, this energy can drop to zero only when $Ax = 0$. Since A is assumed to have independent columns, $Ax = 0$ only happens when $x = 0$. Thus $A^T CA$ has positive energy and is positive definite. ☐

9. Do Problem 7 from Section 8.1.

   Solution. (To first of the above) For 5 springs and 4 masses, the 5 by 4 matrix A has two non-zero diagonals: all $a_{ii} = 1$ and $a_{a+1,i} = -1$. With $C = \mathtt{diag}(c_1, c_2, c_3, c_4, c_5)$, we get $K = A^T CA$, symmetric tridiagonal with diagonal entries $K_{ii} = c_i + c_{i+1}$ and off-diagonals $K_{i+1,i} = -c_{i+1}$. With $C = I$ this K is the $-1, 2, -1$ matrix, and $K(2, 3, 3, 2) = (1, 1, 1, 1)$ solves $Ku = \mathtt{ones}(4, 1)$. (K^{-1} will solve $Ku = \mathtt{ones}(4)$.) ☐

10. Do Problem 11 from Section 8.1. There was a book version/numbering issue. Here are the two "problems 8.1.11" in question:



11 (MATLAB) Find the displacements $u(1), \dots, u(100)$ of 100 masses connected by springs all with $c = 1$. Each force is $f(i) = .01$. Print graphs of u with **fixed-fixed** and **fixed-free** ends. Note that diag(ones(n, 1), d) is a matrix with n ones along diagonal d. This print command will graph a vector u:

plot(u, '+'); xlabel('mass number'); ylabel('movement'); print

12 (MATLAB) Chemical engineering has a first derivative du/dx from fluid velocity as well as d^2u/dx^2 from diffusion. Replace du/dx by a *forward* difference, then a *centered* difference, then a *backward* difference, with $\Delta x = \frac{1}{8}$. Graph your three numerical solutions of

$$-\frac{d^2u}{dx^2} + 10\frac{du}{dx} = 1 \text{ with } u(0) = u(1) = 0.$$

This ***convection-diffusion equation*** appears everywhere. It transforms to the Black-Scholes equation for option prices in mathematical finance.

Problem 12 is developed into the first MATLAB homework in my 18.085 course on Computational Science and Engineering at MIT. Videos on *ocw.mit.edu*.

Solution. To "11" in the above snippet: The two graphs of 100 points are "discrete parabolas" starting at $(0, 0)$: symmetric around 50 in the fixed-fixed case, ending with slope zero in the fixed-free case.

To "12" in the above snippet: Forward/backward/centered for du/dx has a big effect because that term has the large coefficient. MATLAB:

```
E = diag(ones(6,1);
K = 64 * (2 * eye(7) - E - E');
D = 80 * (E - eye(7));
K + D\ones(7,1); % forward
(K - D')\ones(7,1); % backward
(K + D/2 - D'/2)\ones(7,1); % centered is usually the best : more accurate
```

□

18.06 Wisdom.

- Your exams so far make up 30% of your final grade. If you wish to maximize your performance in 18.06, it is all about practice (without this, you won't know what you don't know either). Make a weekly study plan involving doing as many old exams as you can fit in your schedule (find them on the 18.06 web). Organize your work. Be systematic. Learn to check your results. Keep a list of all questions (and confusing concepts) you encounter on your journey, and ask your TAs to explain all the hows and whys. We are here for the same!
- Also try out the great Math Learning Center, that you can visit and ask the instructors your questions in a friendly atmosphere, and at evening hours too: *Room 2-102, Mon-Thu, 3:00–5:00pm and 7:30–9:30pm.*

18.06 Recitation

4/24

Avg 6 pts later

Do your own HW - don't just copy from book

Diagonization

Input: Square matrix A

Output: Factorization $S \Lambda S^{-1}$

$$\underset{\text{invertable}}{S}\;\underset{\text{diagonal}}{\Lambda}\;S^{-1} \qquad \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

But not always possible

a) Diagonlize $\begin{bmatrix} 3 & 0 & -2 \\ 0 & 8 & 2 \\ 0 & -3 & 1 \end{bmatrix}$

1. Find λs
2. Find eigenvectors

②

1. $\underline{\lambda}$ $\det(A-\lambda I)=0$ solve for λ

$$A-\lambda I = \begin{bmatrix} 3-\lambda & 0 & -2 \\ 0 & 8-\lambda & 2 \\ 0 & -3 & 1-\lambda \end{bmatrix}$$

find det

$$\left[\begin{array}{cccccc} 3-\lambda & 0 & -2 & 3-\lambda & 0 \\ 0 & 8-\lambda & 2 & 0 & 8-\lambda \\ 0 & -3 & 1-\lambda & 0 & -3 \\ \ominus & \ominus & \ominus\ \oplus & \oplus & \oplus \end{array}\right]$$

$$= (3-\lambda)(8-\lambda\,(1-\lambda) - (3-\lambda)(2)(-3)$$

$$= (3-\lambda)\left[(8-\lambda)(1-\lambda) - (2)(-3)\right]$$

$$= (3-\lambda)(8-9\lambda+\lambda^2+6]$$

$$= (3-\lambda)(14-9\lambda+\lambda^2)$$

$$= (3-\lambda)(\lambda-2)(\lambda-7)$$

$\lambda_1 = 2$ ← all distict

$\lambda_2 = 3$ so able to find S

$\lambda_3 = 7$

Q: How to be sure not linear combn of

$$Ax_1 = \lambda_1 x_1$$

if $x_1 = x_2$

then $\lambda_1 \overset{\text{must}}{=} \lambda_2$

Contidiction ■

Eigenvector x_1 corresponding to $\lambda_1 = 2$

x_1 satisfies $Ax_1 = \lambda_1 x_1$

$$\Updownarrow$$

$$(A - \lambda_1 I)x_1 = 0$$

④

$$A - \lambda_1 I = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 6 & 2 \\ 0 & -3 & -1 \end{bmatrix}$$

To find $N(A - \lambda_1 I)$ row ops!

$$\downarrow$$

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 6 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

So nullspace $\ni$ $x_1 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$$a - 2c = 0$$
$$6b + 2c = 0$$

$b = 1$

$c = -3$

$a = -6$

⑤

Do the rest

$$x_2 = \begin{bmatrix}1\\0\\0\end{bmatrix}$$

$$\lambda_3 = \begin{bmatrix}-1\\-4\\2\end{bmatrix}$$

$$\underset{n\times n}{\underset{\uparrow}{A}} = \underset{n\times n}{\underset{\uparrow}{S}} \Lambda S^{-1}$$

~~$[S|I]$~~

$$S = \begin{bmatrix}-6 & 1 & -1\\ 1 & 0 & -4\\ -3 & 0 & 2\end{bmatrix}$$

$$\lambda = \begin{bmatrix}2 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & 7\end{bmatrix}$$

$$S^{-1} = \begin{bmatrix}0 & -1/5 & -2/5\\ 1 & -3/2 & -5/2\\ 0 & -3/10 & -1/10\end{bmatrix}$$

← I missed how we got that by hand

Ached; Gauss Jordan

$[S | I]$

⑥

b) Diagonalize $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$$\det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{bmatrix}$$

$$= (1-\lambda)^2$$

$$\lambda = 1$$

eigenvector

vector x s.t.

$$Ax = x \Leftrightarrow (A-I)x = 0$$

$$A - I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Not diagontizable

Generally not true that repeated eigenvalues means not diagonzizable ↓

⑦ $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

$$\det(A - \lambda I) = \det\begin{pmatrix} 2-\lambda & 0 \\ 0 & 2-\lambda \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in N(A - 2I)$$

↑ are in nullspace

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

↑ is diagonizable

Why do we care about diagonalizable matices?

c) Compute A^2, A^3, A^4 .....

$$A = S \Lambda S^{-1}$$

$$A^2 = S \Lambda S^{-1} S \Lambda S^{-1}$$

$$= S \Lambda I \Lambda S^{-1}$$

$$= S \Lambda^2 S^{-1}$$

$A3 = S\Lambda^3 S^{-1}$

$$\Lambda^n = \begin{bmatrix} 2^n & 0 & 0 \\ 0 & 3^n & 0 \\ 0 & 0 & 7^n \end{bmatrix}$$

So only 3 matrix multiplications

(This was all over the last p-set)
(will be on next exam)

d) Markov matrices

$\sum$ cols $= 1$

all entries > 0

$$A = \begin{bmatrix} .9 & .5 \\ .1 & .5 \end{bmatrix}$$

Fun fact $\lambda = 1$ is always an eigenvalue
for a Markov matrix

Trace $A = \lambda_1 + \lambda_2$

$1.4 = 1 + \lambda_2$ $\qquad$ $\lambda_2 = .4$

⑨

$$\lambda_1 = 1 \qquad x_1 = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

$$A - I = \begin{pmatrix} -.1 & .5 \\ .1 & -.5 \end{pmatrix}$$

$$\lambda = .4$$

$$A - .4I = \begin{pmatrix} .5 & .5 \\ .1 & .1 \end{pmatrix}$$

$$x_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$S = \begin{bmatrix} 5 & -1 \\ 1 & 1 \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & .4 \end{bmatrix}$$

Steady State

$$u_\infty = A u_\infty =$$

$$= A^2 u_\infty$$

$$\cdots$$

$$= A^\infty u_\infty$$

Steady state vector is the eigenvector corresponding to 1

$$u_\infty = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

Solving Diff Eq

Think back to 18.03

What is the sol'n to $x'(t) = a\,x(t)$?

guess $x = Ce^{\lambda t}$

$$C\lambda e^{\lambda t} = Ca\,e^{\lambda t}$$

$$\uparrow$$

$$\lambda = a$$

$$x(t) = x(0)\,e^{at}$$

Generalized: What is the sol to

$$\bar{u}'(t) = a\,\bar{u}(t)$$

$$\bar{u}(t) = \vec{u}(0) \cdot e^{at}$$

18.06

4/25

Similar Matrices

"Similar" matrices

↳ Same eigenvalues

$$\underline{B} = M^{-1} \underline{A} M$$

example

A is similar to Λ; choose $M = S$ = e vectors

(missed first few min of talk)

More detail needed if repeated λs

Was once climax of linear algebra

↳ "Jordan Form"

↳ what to do to diagonize if not full set eigenvectors

If n ind eigenvectors then $S^{-1} A S = \Lambda$

↑ Jordan Form

But what if repeated λ, 1 eigenvector

$$A = \begin{bmatrix} 2 & 1 \\ 4 & -2 \end{bmatrix}$$

$\lambda = 0, 0$

└ Not similar to $\Lambda = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

But how close can we get?

$J = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ ← put a 1 here

↑ will be trianglar

So this is similar

$J = M^{-1} A M$

↑ for some M

3×3 Jordan

Suppose $\lambda = 0, 0, 0$

we found a family of matrices

└ everyone in the family similar

$J = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ or, $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ or $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

↑ loner - no one similar

↑ rank 1

↑ rank 2

diff families

③

To be similar, A, B have same rank

Lwhen you multiply by an invetable matrix
you don't change the rank

So transfer it over

$MBM^{-1} = A$

rank A $\leq$ rank B

When multiple might get smaller

but when multiplying by investable ∍ no change

Why do B and A have the same eigenvalue?

Suppose $Bx = \lambda x$

Then $M^{-1}AMx = \lambda x$

But show A has eigenvalue same

Note eigenvalues diff

Mutiply both sides by M

$A(Mx) = \lambda(Mx)$

↑ what happened for eigenvector

④

Proof 2

know $\det(B - \lambda I) = 0$

know $\det(M^{-1}AM - \lambda I) = 0$

Show $\det(A - \lambda I) = 0$

So

$$\det(M^{-1}AM - \lambda I) = 0$$

$$\det(M^{-1}AM - \lambda M^{-1}M) = 0$$

$$\det(M^{-1}(A - \lambda I)M) = 0$$

$$\det(M^{-1}) \overset{\text{product rule}}{\cdot} \det(A - \lambda I) \cdot \det(M) = 0$$

$$1$$

$$\det(A - \lambda I) = 0$$

He likes the previous proof - since talks about eigenvectors as well

⑤

$$\begin{bmatrix}0&1&&\\1&0&1&\\&1&0&1\\&&1&0\end{bmatrix}$$ Sym + real

$A^T = A$

$$\begin{bmatrix}0&i&&\\-i&0&i&\\&-i&0&i\\&&-i&0\end{bmatrix}$$

$\bar{A}^T = A^H = A$

Hermetian + complex

$$= i\begin{bmatrix}0&1&&\\-1&0&1&\\&-1&0&1\\&&-1&0\end{bmatrix}$$

Same eigenvalues

Discussion on MITx

- I suggested Tutor
- Kahn Academy videos
    - ~~more~~ student : more brief concepts
- Student : Diff levels of difficulty

⑥

40 or 50 × 15 min lectures

Or a 5 min review of key ideas

1 of concept, 1 of solving problem

Prof: khan academy guy is friendly

The writing isn't great

Should I get out of the picture?

18.06

4/27

Singular Value Decomposition

(from 2005 OCW video - where it was out of order)

$$A = U \Sigma V^T$$

Σ diagonal

U, V orthogonal

"final and best factorization of a matrix"

diagonal → only diagonal is non zero

$$\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$$

will include sym pos def

∟ eig vectors are orthogonal

$S \to Q$

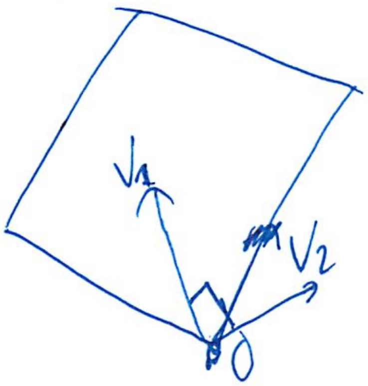
$\lambda \to \lambda$

Then only 1 U and V scene

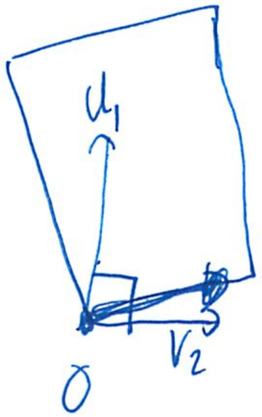
②

But usually not orthogonal

$\mathbb{R}^n$ row space



$\mathbb{R}^m$ col space



$$u_1 = A v_1$$

Looking for orthogonal basis in row space that gets knocked into orthogonal basis in col space

Gram Schmidt says how to get orthogonal basis in row space

$u_1 = v_1$
$u_2 = v_2 - \text{proj}(v_2)$
⋮ etc

③

If I take any orth basis and muliply by A

Lwon't be orth on other side

But we want to find those here that are orth on other side

$$A\begin{bmatrix} v_1 & v_2 & \cdots & v_r \end{bmatrix} = \begin{bmatrix} u_1 & \cdots & u_r \end{bmatrix}\begin{bmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \ddots \end{bmatrix}$$

↑ basis vector in row space

↑ basis vector col space

↑ multiplicative factor

Goal:

$$\sigma_1 u_1 = A v_1$$

$$\sigma_2 u_2 = A v_2$$

(think of what you want and express it as a matrix multiplication)

$$AV = U\Sigma$$

↑ goal

find orthonormal basis in row and col space

so like diagonalize matrix (to Σ)

take $A = \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix}$ invertable ↖ rank 2 ← this stuff should be natural!

④

want) V_1, V_2 in row space $\mathbb{R}^2$

U_1, U_2 in col space $\mathbb{R}^2$

$\sigma_1 > 0 \qquad \sigma_2 > 0$

words about null space

have a basis for null space

$V(r+1)$

$\searrow Vm$

take orthonormal basis (?? confused)

just get 0 on the diag matrix

$$A \begin{bmatrix} V_1 & V_2 & \cdots & V_r & V_{r+n} & V_n \end{bmatrix} = \begin{bmatrix} U_1 & U_2 & \cdots & U_r & U_n \end{bmatrix} \begin{bmatrix} \sigma & & & \\ & \sigma & & \\ & & 0 & \\ & & & 0 \end{bmatrix}$$

⑤

We need to get the matrix in shape

$\llcorner$ not sym so cant use e-vectors (not orth)

$$\begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix}$$

Some how must get the orthonormal V_1 V_2 U_1, U_2 to get it to work

$$A V_1 = \sigma_1 U_1$$

$$A V_2 = \sigma_2 U_2$$

$AV = U\Sigma$ ← matrices that get us there

Multiply by V^{-1}

$$A = U \Sigma V^{-1}$$

↑square orthogonal so

$$V^{-1} = V^T$$

$$= U \Sigma V^T$$

ⓔ

We have 2 orthogonal matrices here

Don't want to use both at once

Want to make the Us disappear → only left w/ Vs

So its the same combo as usual

$A^T A$ - pos def or at least pos semi-def

$$A^T A = V \Sigma^T U^T U \Sigma V^T$$

looks worse - but collapses

$$= V \underset{\text{diagonal}}{\Sigma^T I \Sigma} V^T$$

$$= V \Sigma^2 V^T$$

So this tells us the Vs

How find the Us?

↳ opposite order

②

V_s = e vectors A^TA
U_s = || AA^T

Σ = + square roots of $\sqrt{\sigma^2}$ so σ

Step 1. Compute A^TA
↳ want the evectors

$$A^TA = \begin{bmatrix} 4 & -3 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix} = \begin{bmatrix} 25 & 7 \\ 7 & 25 \end{bmatrix}$$

So what are the e vectors of this

$\lambda = 32, 18$ ← $\sqrt{\ }$ is sigma

$$x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Normalize

$$\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

⑧

$$\begin{bmatrix}4 & 4\\ -3 & 3\end{bmatrix} = \underset{\substack{?\\ \text{don't}\\ \text{know yet}}}{U} \underset{\Sigma}{\begin{bmatrix}\sqrt{32} & \\ & \sqrt{18}\end{bmatrix}} \underset{V^T}{\begin{bmatrix}1/\sqrt{2} & 1/\sqrt{2}\\ 1/\sqrt{2} & -1/\sqrt{2}\end{bmatrix}}$$

$$\Sigma = \begin{bmatrix}\sqrt{\lambda_1} & \\ & \sqrt{\lambda_2}\end{bmatrix}$$

So could figure out U from pieces

Something w/ inverse?

$U =$ ~~$A\Sigma^{-1}V^{T-1}$~~

$A\,\Sigma^{-1}\,V^{T-1}$ correct

but finding inverse is hard

Find U_1, U_2

$$AA^T = A\,\Sigma\,V^T V\,\Sigma^T U^T$$

$$= A\,\Sigma\,I\,\Sigma^T U^T$$

$$= A\,\Sigma^2\,U^T$$

? sym semi def matrix

⑨

Its'' e vectors go into U

$$A A^{T} = \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} 4 & -3 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 32 & 0 \\ 0 & 18 \end{bmatrix}$$

lucky
Came out diagonal

$\lambda = 32, 18$

$x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Not an accident that e-values same

AB / BA same e-values

<u>But</u> e-vectors different

So

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{32} & \\ & \sqrt{18} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

best orthonal matrix $= I$

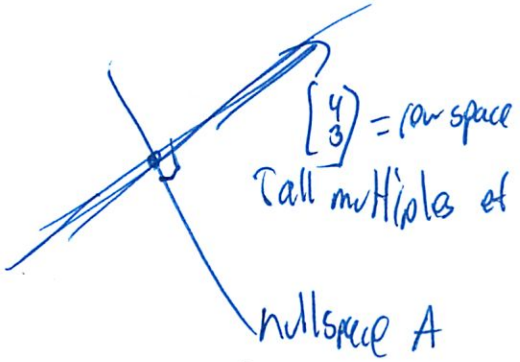
If multiply it back - it should check out / except he messed up a sign

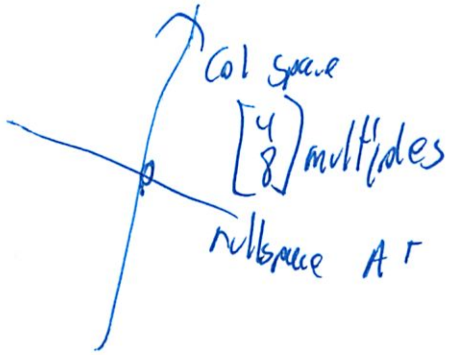
⑩

ex 2

$\begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix}$ ← rank 1

1D row and col space





↑So what $\begin{pmatrix} x \\ y \end{pmatrix}$ does it $= \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Choosing orthogonal basis are no problem

Make orthonormal

Only One V_1 to Choose

$$V_1 = \begin{bmatrix} .8 \\ .6 \end{bmatrix} \qquad U_1 = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$$

$$V_2 = \begin{bmatrix} .6 \\ -.8 \end{bmatrix} \qquad U_2 = \begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix}$$

So

$$\begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix} = \underset{U}{\begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & -1/\sqrt{5} \end{bmatrix}} \underset{\Sigma}{\begin{bmatrix} \sqrt{125} & 0 \\ 0 & 0 \end{bmatrix}} \underset{V^T}{\begin{bmatrix} .8 & .6 \\ .6 & -.8 \end{bmatrix}}$$

↑ $A^T A - \begin{pmatrix} 4 & 3 \\ 8 & 6 \end{pmatrix} \begin{pmatrix} 4 & 8 \\ 3 & 6 \end{pmatrix} = \begin{bmatrix} 80 & 60 \\ 60 & 45 \end{bmatrix}$

$\lambda = 0, 125$

⑪ We are choosing the right basis for the 4 subspaces of linear algebra

$v_1, \dots v_r$ is the right orthonormal basis for row space

$u_1 \dots u_r$ " " col space

$v_{r+1}, \dots v_n$ " " nullspace

$u_{r+1} \dots u_n$ Left nullspace

So these are the right bases make the matrix diagonal

$$A v_i = \sigma_i u_i$$

18.06 4/30

(Still need to watch last time)
(3 min late)

Orthogonal bases

Fourier series (continuous)

$$F(x) = \sum_{0}^{\infty} a_k \cos kx + \sum_{1}^{\infty} b_k \sin kx$$

Discrete fourier transform (DFT)
FAST fourier transform (FFT)

$$= \sum_{k=-\infty}^{\infty} c_k e^{ikx}$$

e^{ikx} (all int k)

are orth funs on $(0, 2\pi)$

So this lin algebra carries over to diff eq,
Continuous fourier transform

②

Orthogonal

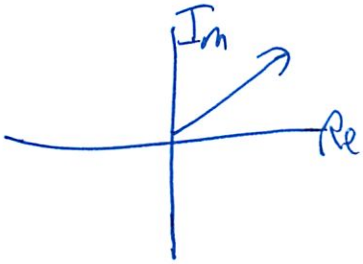
$$\bar{X}^T y = 0$$

$$X^H y = 0$$

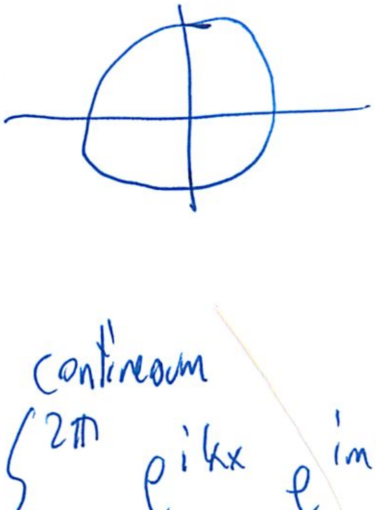
Complex function

Test for orthogonality $X_1^T Y_1 + X_2^T Y_2 + \cdots = 0$

$e^{ikx} = \cos kx + i \sin kx$ ← Euler



is on unit circle



~~e^{imx}~~

Whole continuum

$$\int_0^{2\pi} e^{ikx} e^{imx} dx$$

$= 0 \rightarrow$ orthogonal

But didn't take the complex conjugate of one of the factors

$$\int_0^{2\pi} e^{i(m-k)x}\,dx =$$

$$= \frac{e^{i(m-k)x}}{i(m-k)}\Bigg|_0^{2\pi} = 0$$ Good

Hard to visualize function space

But we have a basis

Same pattern as lin algebra - but ∞ly many functions

Finite Cases

Suppose have

$$F = c_1 v_1 + c_2 v_2 + \dots + c_n v_n$$

↑ Chap 4 - projections

Suppose Vs are orthogonal

c_1: Component in the v_1 direction

$$V_1^T F = C_1 V_1^T V_1$$

$$C_1 = \frac{\bar{V}_1^T F}{\bar{V}_1^T V_1}$$

↑ complex conj

What if job is find c_1?

Dot both sides w/ 1st function

└ integral $V_1 = e^{ix}$

$$\int_0^{2\pi} e^{-ix} F(x)\,dx = \int_0^{2\pi} C_1 e^{-ix} e^{ix}$$

$$= 2\pi c_1$$

$$\frac{1}{2\pi} \quad " \quad = C_1$$

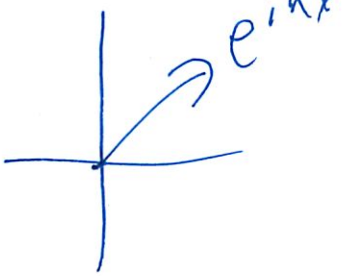
$$C_k = \frac{\int_0^{2\pi} e^{-ikx} F(x)}{2\pi}\,dx$$

(looks familiar)

⑤

At this point : most classes integration practice

For space



For all matrices → can do functions

Like Gram Schmidt

$1, x^2, x^3, \ldots$

$$1^T x = \int_{-1}^{1} 1 \cdot x \cdot dx = 0$$

area to left balances right = 0

$$1^T x^2 = \int_{-1}^{1} 1 \cdot x^2 \cdot dx$$

$\neq 0$ not orthogonal

but a basis

so gram schmidt comes it

⑥

Industries use diff eq
- like oil industry

So have functions

But to compute → go to vectors

FFT

Always expand vector F
like e^{ikx}, but vectors
Compute coeffs $c_1, \dots, c_n$

For all cs
Old way is n^2 multiplies
FFT $n \log n$ " ← "Holy Grail of algorithms"

$n = 2^{10} = 1024$

$n^2 = 2^{20} \approx 1$ mill

$n \log_2 n = 1024 \cdot 10 = 10{,}000$ } 100 to 1 for a simple math idea

Why are sin, cos so important?
Why is Fourier important?

e^{ikx} is an eigenvector for derivitive

$$\underset{\uparrow\text{operator}}{\frac{d}{dx}} e^{ikx} = \underbrace{ik}_{\lambda}\,\underbrace{e^{ikx}}_{\text{eigen function}}$$

but cave at imaginary
so skew sym. matrix

$$\frac{d^2}{dx^2} e^{ikx} = \underbrace{-k^2}_{\lambda}\,\underbrace{e^{ikx}}_{\text{eigenvector}}$$

Real eigenvalues
Orthogonal eigenvectors

Or switch sign

$$-\frac{d^2}{dx^2} e^{ikx} = k^2 e^{ikx}$$

pos semi def

Have eigenfunctions

Why Fourier important ?

$\llcorner$ since eigenvectors, eigenfunctions of $\frac{d}{dy}$

Can work on any linear eqs

Can't solve non-linear eqns

DFT

basis vectors $\begin{array}{c} k= \\ \\ \\ \\ \\ \end{array}\begin{array}{cccc} 0 & 1 & 2 & 3 \\ \end{array}$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix} \leftarrow \text{the 4 basis vectors} = F_4$$

~Complex

- orthogonal

$1, e^{ix}, e^{2ix}, e^{3ix}$

at $x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$

equal spacing → sampling

⑨

What would be the 6 Fourier vectors in 6 dimensions

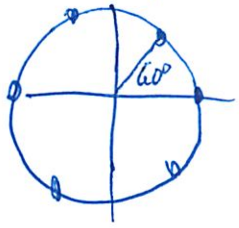
$N=6$

$$F_6 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & w & w^2 & w^3 & w^4 & w^5 \\ 1 & w^2 & w^4 & w^6 & w^8 & w^{10} \\ 1 & w^3 & w^6 & \vdots & \vdots & \vdots \\ 1 & w^4 & w^8 & \vdots & \vdots & \vdots \\ 1 & w^5 & w^{10} & & & \end{bmatrix}$$

← cols are orthogonal

$e^{i\theta}$ ← sample

space at ← 60°



but want in radians $\frac{\pi}{3}$

$\frac{\pi}{3}, \frac{2\pi}{3}, \frac{3\pi}{3}$

$w = e^{i\frac{\pi}{3}}, e^{i\frac{2\pi}{3}}, e^{i\pi}$

Complex, but nice pattern

18.06 Recitation

5/1

(2 min late)

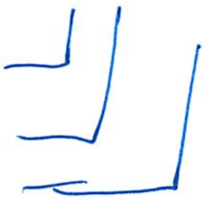
Exam Review Sun 1pm

Pos Def Matrices

When is a sq & symmetric matrix A pos def

A is already sym → so can diagonalize

1. All pos e-values
2. All pos pivots
3. Pos det on upper left



4. For any vector $x \neq 0$

   $x^T A x > 0$

5. $A = R^T R$ for some sq, invertable matrix R

   $R = QD$

   (see next)

$A = Q\Lambda Q^T$ if A sym

Eigenvalues are all real

various facts

$$\begin{bmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{bmatrix} = \begin{bmatrix}\sqrt{\lambda_1} & & \\ & \ddots & \\ & & \sqrt{\lambda_n}\end{bmatrix}^2 = D^2$$

$$A = QD^2Q^T$$
$$= QD(QD)^T$$

---

ex which are pos def?

a) $\begin{bmatrix}3 & 4\\ 4 & 6\end{bmatrix}$ yes by criteria 3

b) $\begin{bmatrix}2 & 1\\ 3 & 2\end{bmatrix}$ No not sym

c) $\begin{bmatrix}-1 & 2 & 7\\ 2 & 3 & 5\\ 7 & 5 & 2\end{bmatrix}$ No - by criteria 3

$-1 \leftarrow$ negative

$\begin{bmatrix} 2 & 0 \\ 0 & 7 \end{bmatrix}$ yes criteria 1

e vales ⊕

So small matrix → criteria 3

large matrix do the elim

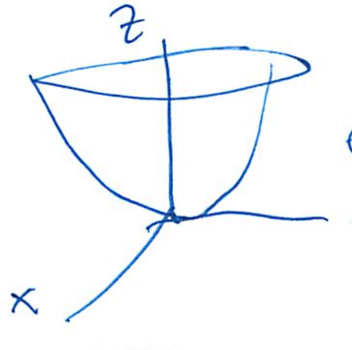
Ex Describe graph of $z = ax^2 + bxy + cy^2$

$$\begin{pmatrix} x & y \end{pmatrix} \underbrace{\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \end{pmatrix} = ax^2 + bxy + cy^2$$

Why is this useful?

If A is pos def, criteria 4 tells us

$z > 0$ for all $\begin{pmatrix} x \\ y \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

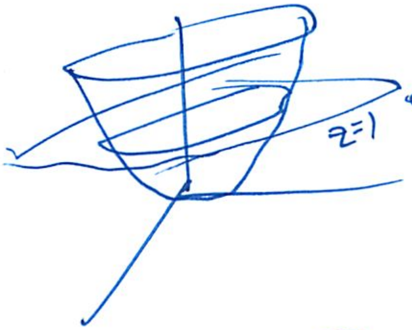


← will curve upwards

look like a bowl

④

If look at a slice



← is a circle

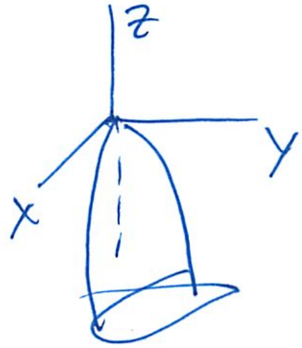
$ax^2+bxy+cy^2=1$

If A is pos def than this is an ellipse

A is neg def if $-A$ is pos def

↑she thinks this is true

Energy test is pretty clear → just reverses inequality



Could have mixed evalues – one ⊕
– one ⊖

then A is "indefinite"

You get a "saddle"

⇐ worst saddle ever!

If you horiz slice

↳ get a hyperbole

Jordan Forms

2 square matrices A and B are similar if there exists some invertable matrix M such that $A=M^{-1}BM$

↳ we see this all the time
Conj operation

Note: If A,B are similar, they have the same e-value

But: having same e-value does not guarentee they are similar

ⓔ

If e-vales same ∃ e vectors are not ness the same!

If $Ax = \lambda x$, eigenvector of B is $M^{-1}x$

Check $B(M^{-1}x) = (MAM^{-1})(Mx)$

$$= \underbrace{MAx}_{\lambda x},$$

Still does not tell us if similar

Put into Jordan Form

Will tell us if similar

A matrix is in Jordan form if we can write

$$A = \begin{bmatrix} J_1 & & 0 \\ & J_2 & \\ 0 & & \ddots \ J_n \end{bmatrix}$$

$$J_i = \begin{bmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{bmatrix}$$ ← Jordan blocks

λs are eigenvalues of A

ex/ $A = \begin{bmatrix} 5 & 4 & 2 & 1 \\ 0 & 1 & -1 & -1 \\ -1 & -1 & 3 & 0 \\ 1 & 1 & -1 & 2 \end{bmatrix}$ what is the Jordan form?

1. Find eigenvalues

$$\det(A - \lambda I) = 0$$

$$(\lambda - 1)(\lambda - 2)(\lambda - 4)^2$$

So we can kinda start Jordan form

Jordan form is 4×4

E values on diagonal

↙ Jordan block

$$\begin{bmatrix} \boxed{1} & & & \\ & \boxed{2} & & \\ & & 4 & \\ 0 & & & 4 \end{bmatrix}$$

↖ could be 1 or 2 blocks

Look at SDS^{-1}

↖ only if enough e-vectors

no duplicates

⑧

Since $\overset{4,4}{\text{same}}$ e-vector so 1 block since nullsp$(A-\lambda I)=0$

In large matrices, same evalue can have diff e-vectors
└since is dim $=2$

$$\begin{bmatrix} \boxed{1} & & \\ & \boxed{2} & \\ & & \boxed{4 \quad 4} \end{bmatrix}$$

We have a mystery matrix A
↑square

$$\det(A-\lambda I) = (\lambda-2)^3(\lambda-3)$$

What are the possible Jordan forms?

We know dim $=4$ since 4 evalues

$A = 4\times 4$

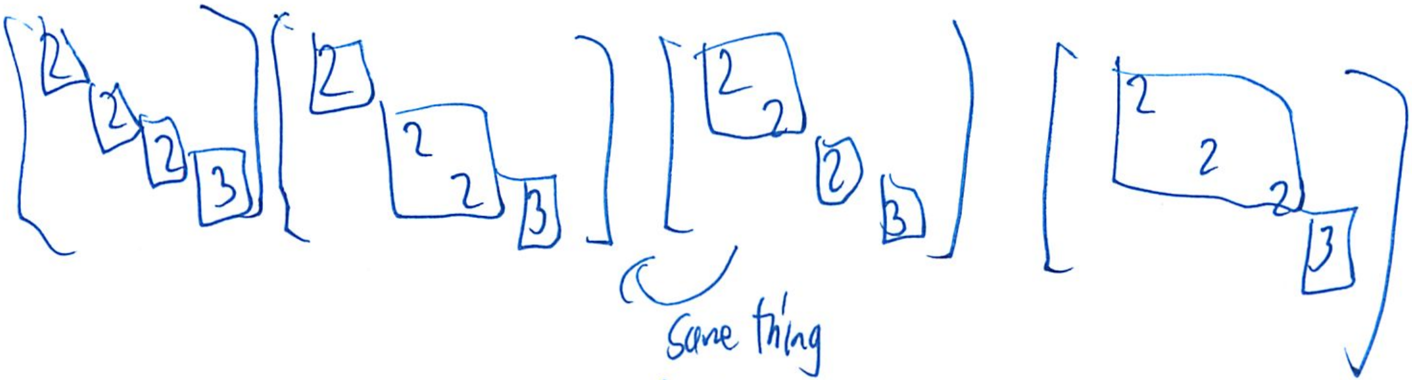
So Jordan form is 4×4

$$\begin{bmatrix} 2 & & & \\ & 2 & & \\ & & 2 & \\ & & & 3 \end{bmatrix}$$

not as useful to write $\begin{bmatrix} 2 & & & \\ & 3 & & \\ & & 2 & \\ & & & 2 \end{bmatrix}$ but still works

⑨

What are the Jordan blocks?



same thing
just multiply
by invertable

Singular Value Decomposition

$A =$ $m \times n$ matrix

We can make square matrices out of A

A^TA or AA^T

$n \times n$ $\quad$ $m \times m$

$u_i =$ eigenvectors

$V_i =$ eigenvectors

⑩

$$Av_i = \sigma_i u_i$$

So what effect does this make?

$$A\begin{bmatrix} | & & | \\ v_1 & \cdots & v_r \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ u_1 & \cdots & u_r \\ | & & | \end{bmatrix} \cdot \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \end{bmatrix}$$

$r = \text{rank } A$

We want A in terms of

Then want to move a part

but can't

So complete a matrix

$$A\underset{n\times n}{\overset{V}{\begin{bmatrix} | & & | & | & & | \\ v_1 & \cdots & v_r & v_{r+1} & \cdots & v_n \\ | & & | & | & & | \end{bmatrix}}}$$

$$= \underset{m\times m}{\overset{U}{\begin{bmatrix} | & & | & | & & | \\ u_1 & \cdots & u_r & u_{r+1} & \cdots & u_m \\ | & & | & | & & | \end{bmatrix}}} \underset{m\times n}{\overset{\Sigma}{\begin{bmatrix} \sigma_1 & & & & & \\ & \ddots & & & & \\ & & \sigma_r & & & 0 \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{bmatrix}}}$$

⑪

$V_{r+1} \dots V_n$ is an orthonormal basis for $N(A)$

$U_{r+1} \dots U_m$ is " " $N(A^T)$

Since cols orthonormal → can move w/ just multiplying by V^T (gives identity on LHS)

$$A = U \Sigma V^T$$

Ex) Find SVD of

$$A = \begin{bmatrix} 3 & -1 & 1 \\ 1 & 3 & 1 \end{bmatrix} = \underset{m}{2} \times \underset{n}{3}$$

1. V is 3×3

   U is 2×2

   Σ is 2×3

⑫

$$A^T A = \begin{bmatrix} 10 & -6 & 2 \\ -6 & 10 & 2 \\ 2 & 2 & 2 \end{bmatrix}$$

eigenvalues $= 0, \ 6, \ 16$

e-vectors $= \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

These are not V_i

Need to normalize!

$$\frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

So

$$V = \begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \\ 1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \\ -2/\sqrt{6} & 1/\sqrt{3} & 0 \end{bmatrix}$$

⑬

$Av_1 = \sigma v u_1$

$(8) = Av_1$ so throw out one

(will always happen somewhere)

$\frac{1}{\sqrt{3}}\begin{pmatrix}3\\3\end{pmatrix} = Av_2 = \sigma_2 u_2$

$u_2 = \begin{pmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{pmatrix}$ ← length 1 along $\frac{1}{\sqrt{3}}\begin{pmatrix}3\\3\end{pmatrix}$

~~But~~ $3/\sqrt{3}$ is $\sigma_2 \frac{1}{\sqrt{2}}$

So $\sigma_2 = \sqrt{6}$

And finally

$\frac{1}{\sqrt{2}}\begin{pmatrix}4\\-4\end{pmatrix} Av_3 = \sigma_3 u_3$

$u_3 = \begin{pmatrix}1/\sqrt{2}\\-1/\sqrt{2}\end{pmatrix}$ ← unit vector along $\begin{pmatrix}4\\-4\end{pmatrix}$

$\sigma_3 = 4$

(iii)

So can now write the SVD

SVD

$$A = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} \sqrt{6} & 0 & 0 \\ 0 & 4 & 0 \end{pmatrix} V$$

18.06 5/2

(20 min fire alarm)

Today FFT

Linear Transforms

Fri Exam review

Diagonization $S\Lambda S^{-1}$

A^n e^{At}

A^{+1}

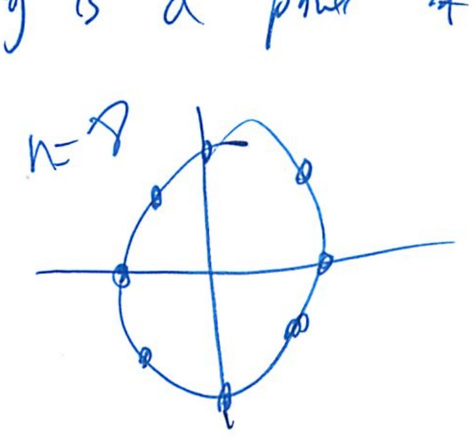
$A^T = A = Q\Lambda Q^T$

Pos def

Similar

---

everything is a power of $e^{2\pi i/8}$



$$F_8 = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & W & & W^7 \\ 1 & W^2 & & W^{14} \\ 1 & W^3 & & \vdots \\ 1 & W^4 & & \\ 1 & W^5 & & \vdots \\ 1 & W^6 & & \\ 1 & W^7 & & W^{49} \end{bmatrix}$$

⓪ If add the 8 #s it cancels

$$1 + w + w^2 + \dots + w^7 = 0$$

$$\frac{w^8 - 1}{w - 1}$$

$$w^8 = w^0 = 1$$

[illegible]

$$w^{(j-1)(k-1)}$$

8x8 [illegible] is 64 multiplications

FFT cuts this way down

<u>Key to the FFT</u>

The 8x8 connects closely to 2 copies of the 4x4

$$F_8 = \overset{\text{Simple}}{\begin{bmatrix} 1 \ddots 1 & 1 \ddots w^3 \\ 1 \ddots 1 & -1 \ddots w^3 \end{bmatrix}} \overset{\text{2 copies}}{\left[\begin{array}{c|c} F_4 & 0 \\ \hline 0 & F_4 \end{array}\right]} \underset{\text{permutation}}{\overset{P}{\left[\begin{array}{cccc} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \\ \hline 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{array}\right]}}$$

③

The 64 entries has been reduced to 32 non-zero entries

$$Pv = \begin{bmatrix} v_{even} \\ v_{odd} \end{bmatrix} = \left[\begin{array}{c} v_0 \\ v_2 \\ v_4 \\ v_6 \\ \hline v_1 \\ v_3 \\ v_5 \\ v_7 \end{array}\right]$$

Now recurse on the 4 into 2 and 2

So the Fs go to 1

So $\log_2 n$ steps n times $= n \log_2 n$

Book has the algebra to make it work

MIT is releasing an even faster FFT

↳ a fast, sparse FFT

Number theory people like 7

└ prime its stuff

Linear Transformation

Used to be on day 1

Matrix secondary for pure linear algebraists

$$\underset{\text{out}}{y} = T(\underset{\text{in}}{x})$$

linear functions - very restricted

$$T(cx+dy) = cT(x) + dT(y)$$

Best example $T(x) = Ax$

$$y = Ax$$

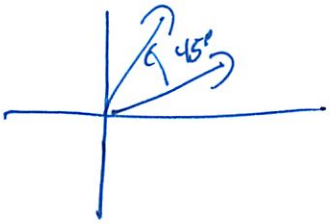
Is this a linear transformation?

⑤

1. T = rotate all vectors by 45%

(x, y) plane

linear? $T(x) = y$



Yes

What matrix does it?

- called a rotational matrix

$$A = \begin{bmatrix} \cos 45 & \sin 45 \\ -\sin 45 & \cos 45 \end{bmatrix}$$

2. Shift $T(x) = x + x_{fixed}$

Not linear

$T(0)$ has to come out 0

would kill the shift idea

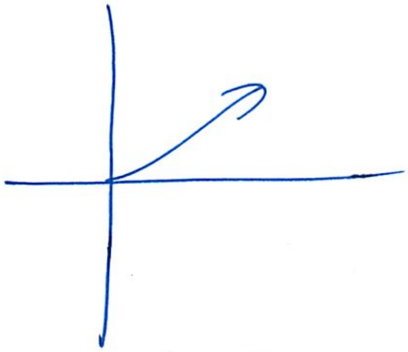
its always 0

In linear algebra: $\overset{\text{linear}}{\text{means}}$ no constant term

3. $\underline{\mathbb{R}^2}$

yes $\quad A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

4. <u>Shear</u>



$A = \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix}$

Linear transformations come from matrices

$$\underset{\mathbb{R}^m}{Y} = \underset{\uparrow}{T}(x)_{\in \mathbb{R}^n}$$

is a $m \times n$ matrix that does it

<u>only</u> example

⑦

To find A from T

↳ must choose Coords

L basically choosing a basis

Vectors you write are a combo of the bases

$\begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}$ is $3v_1 + 1v_2 + 4v_3$

$\uparrow$ basis vector

$T(cx+dy) = cT(x) + dT(y)$

$T \to A_{m \times n}$ depending on basis

If choose $\overset{ind}{e}$ vectors as basis

L would come out diagonal

"The right basis"

5/2

Review

Wed - Similar
Fri - SVD
M - Fourier, FFT, Complex

Can't even remember what day is what
This month in 18.06 went so fast

Similar matrices

diaagonization is not possible for every A
  ↳ if too few e-vectors

S is still best choice

but M is any invertable matrix

So $M^{-1}AM$ - might be diagonal

E-values same - no matter the M

$$A \underset{\text{similar}}{\approx} M^{-1}AM$$

I made up that symbol

If $B = M^{-1}AM$ then $A = MBM^{-1}$
  ↳ if A similar to B
    B is similar to A

②

If B = diagonizable its Λ
and M is S

But again here M is all invertable

---

* All matrices w/ same e-values are similar to each other *

Jordan Form

(poorly explained in the book)

aka Jordan normal / Jordan canonical form

- upper triangular matrix representing the operator on some basis

- any non-zero non-diagonal entries must = 1
  be above main diagonal
  and be identical diagonal entries on left + below

$$J = \begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_n \end{bmatrix} \quad J_i = \begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix}$$

each Jordan block

③

Jordan form is as close as we can get diagonalizing the matrices when only 1 evector

(This is a special case (should be clearer)

Not changed by m	Changed by m
e values	evectors
trace + det	null space
rank	col space
# ind evectors	row space
Jordan form	left nullspace
	Singular values

Every similar matrix $B = M^{-1}JM$

J^T is similar to J

④

Jordan Form

For every A, choose M so that $M^{-1}AM$ is as diagonal as possible

When A has full set evectors
↳ put into col of M
↳ ie $M = S$
$S \Lambda S^{-1}$

But otherwise → only s ind e vectors
↳ s blocks of J
The block handles one e vector of A
↳ again 1×1 blocks is Λ

This is the big theorm on matrix similarity

For each J solve $\frac{du}{dt} = Ju$

$$M J M^{-1} M J M^{-1} = M J I J M^{-1}$$
$$= M J^2 M^{-1}$$

⑤

[¿So this is like an estimation? - ask in OH...)

OH

5/2

Similar Matrices

Similarity ~~tt~~ transitive

$$A \leftrightarrow B$$

$$\begin{tikzcd} A \arrow[r, leftrightarrow] \arrow[rr, bend right, leftrightarrow, "\text{then}"'] & B \arrow[r, leftrightarrow] & C \end{tikzcd}$$

"equivilence relation"

Can start w/ single matrix

Ask what are all the matrices that are similar

Get a whole class of them that are similar

Or can be ~~sep~~ seperate

②

So we can divide into classes

- describe by its elements which are diagonal

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$ ← same e-values

- can always switch the order

So diagonal matrix represent similar graphs

Jordan Form

if just diagonal matrix → each equiv class not all will be included

by including the Jordan Form blocks, we include all of them

③

6.7 #3

How would you describe the rank 1

In middle matrix - only one non zero

$\sigma_1 \cdot$ 1st col u

Now if multiply by V^T

...

Cols are σ_1^2

(1)

OH2

5/2

Jordan Form Finding

Too theoritical - but can't do in 18.06

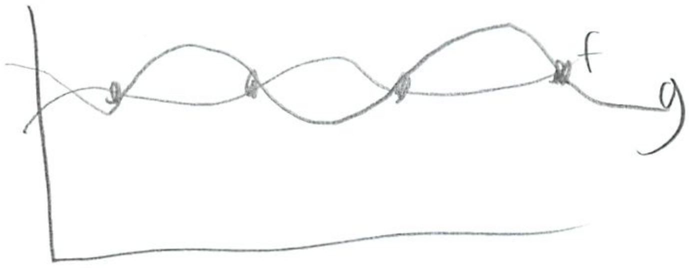
- know diagonals

- can guess blocks
- but can't define for sure in 18.06

10.1 - 15 - E is jst sym

②

So trying to find a g that meets the same pts



If know fourier matrix, you can solve for c

The coefficents of the unknown

SVD Reading

5/2

Solves 3 things

1. λ vectors are not orthogonal
2. Might not be enough e-vectors
3. Might not be square

* So singular vector fixes this

└ two of them U, V

Review FFT

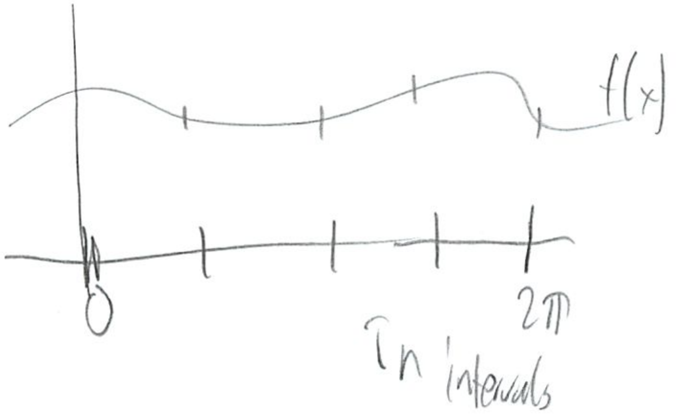
5/2

F^H = F conjugate transpose

Have fn

Evaluate at intervals

know the value of f(x) at $0, \frac{2\pi}{n}, 2\left(\frac{2\pi}{n}\right), 3\left(\frac{2\pi}{n}\right), \cdots (n-1)\left(\frac{2\pi}{n}\right)$



Look at these n values

Find some series

$$\sum_{k=1}^{n-1} c_k e^{ikx}$$

s.t. $f\left(\frac{2\pi}{n}\right) = q\left(\frac{2\pi}{n}\right)$

Chap 8.5 Reading

5/2

Fourier Series

From finite dimensions to infinite

∞ many components

1. V could be $(1, \frac{1}{2}, \frac{1}{4}, \dots)$

2. V ~~also~~ could be a function $(\sin(x))$

Fourier series connects them

Dot product of ∞ vectors is ∞ series

Does the series converge?

∞ lenght vectors → no

So only finite lenght vectors allowed

$\|v\|^2 = v \cdot v = v_1^2 + \dots$ must be finite

↳ called in "*Hilbert space*"

So $v=(1, \frac{1}{2}, \frac{1}{4} \dots)$ is included

Since lenght is

$$v \cdot v = 1 + \frac{1}{4} + \frac{1}{16} + \dots$$

$$= \frac{1}{1-\frac{1}{4}} = \frac{4}{3}$$

↑ I was never good at this!

Schwarz inequality $|v \cdot w| \le \|v\| \|w\|$

↑ ratio $= \cos\theta$

functions

$f(x)$ defined for $0 \le x \le 2\pi$

$$(f, g) = \int_0^{2\pi} f(x)\, g(x)\, dx$$

$$\|f\|^2 = \int_0^{2\pi} (f(x))^2\, dx$$

So $f(x) = \sin x$ comes w/ inner product of itself

$$\int_0^{2\pi} (\sin x)^2 dx = \pi$$

$\sin x$, $\cos x$ are orthagonal

③

Fourier Series

$$y(x) = a_0 + a_1 \cos x_1 + b_1 \sin x_a + a_2 \cos 2x$$
$$+ b_2 \sin 2x + \dots$$

Orthogonal basis

periodic → 2π

so just look $0 \to 2\pi$

even though ∞ series

δ is a small spike

if divide by lenght → orthonormal

(this is very familer from last semester

↳ do I remember it though ...)

④

Fourier Coefficents

$f(x) = a + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + \ldots$

Coeff a_1 $\int_0^{2\pi} f(x) \cos x \, dx =$

$$\int_0^{2\pi} a_1 \cos^2 x \, dx = \pi a_1$$

$$a_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos kx \, dx$$

$$b_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin kx \, dx$$

Linear algebra

∞ dim similar to n dim case

$$b = c_1 v_1 + c_2 v_2 + \ldots + c_n v_n$$

Multiply both sides by v_1^T

Use orthagonility so $v_1^T v_2 = 0$

c_1 term $v_1^T b = c_1 v_1^T v_1 + 0 + \ldots 0$

⑤

So $c_1 = \frac{v_1^T b}{v_1^T v_1}$

* Coeffs are easy to find for orthogonal basis

Eq for c's

$$c_1 v_1 + \dots + c_n v_n = b$$

or $$\begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = b$$

Reading
10.3
FFT

5/2

Multiply quickly by F F^{-1}

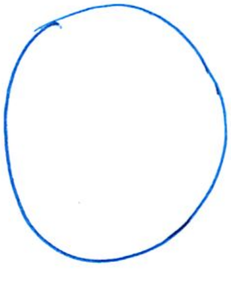
↑ Fourier

Represent f as a sum of harmonics

$c_k e^{ikx}$

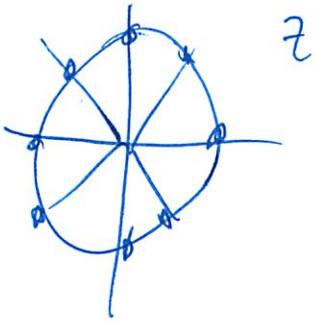
Look at through freq space as coeffs c_k
instead of physical space $f(x)$

Roots of unity



$z^n = 1$

Sols z are the nth roots of unity



$z^8 = 1$

$\frac{1}{8}(360) = 45$

②

Complex # $w = e^{i\theta} = e^{j2\pi/8}$

↑still confuses me somewhat...

Fourier matrix

$n = 4$

1s

$$F = \begin{bmatrix} 1 & 1 & \overset{\text{add } 1}{\curvearrowright} 1 & 1 \\ 1 & w & w^2 & w^3 \\ 1 & w^2 & w^4 & w^6 \\ 1 & w^3 & w^6 & w^9 \end{bmatrix}$$

double

1s ↓

Symetric <u>not</u> hermetian

<u>Unitary</u> $\left(\frac{1}{2}F^H\right)\left(\frac{1}{2}F\right) = I$

Cols of F give $F^H F = 4I$

inverse is $\frac{1}{4}F^H$ which is $F^{-1} = \frac{1}{4}\bar{F}$

③

Then

$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_4 \end{bmatrix} = Fc = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & w & w^2 & w^3 \\ 1 & w^2 & w^4 & w^6 \\ 1 & w^3 & w^6 & w^9 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

↑outputs ↑inputs

$$y_0 = c_0 + c_1 + c_2 + c_3$$

$$y_1 = c_0 + c_1 e^{i2\pi/4} + c_2 e^{i4\pi/4} + c_3 e^{i6\pi/4}$$

$$= c_0 + c_1 w + c_2 \underset{\uparrow \text{raise rotation}}{w^2} + c_3 w^3$$

<u>finite</u> fourier series

4 terms, evaluated at 4 points

<u>FFT Partial</u>

Matrix multiplication goes faster w/ zeros

Connect F_n w/ the half sized $F_{n/2}$

Or two copies of this

④

$$F_4 = \begin{bmatrix} F_2 & \\ & F_2 \end{bmatrix} \neq \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & i^2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & i^2 \end{bmatrix}$$

Need some factors

$$F_4 = \begin{bmatrix} 1 & & 1 & \\ & 1 & & i \\ 1 & & -1 & \\ & 1 & & -i \end{bmatrix} \begin{bmatrix} 1 & 1 & & \\ 1 & i^2 & & \\ & & 1 & 1 \\ & & 1 & i^2 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & & 1 & \\ & 1 & & \\ & & & 1 \end{bmatrix}$$

↑ random...

$$F_{1024} = \begin{bmatrix} I_{512} & D_{512} \\ I_{512} & -D_{512} \end{bmatrix} \begin{bmatrix} F_{512} & \\ & F_{512} \end{bmatrix} \begin{bmatrix} \text{even odd} \\ \text{permutation} \end{bmatrix}$$

↑ identity ↑ diagonal

Full FFT by Recursion

keep going to $F_{n/4}$

etc (recursion)

18.06 Spring 2012 – Problem Set 9

This problem set is due Thursday, May 3rd, 2012 at 4pm (hand in to Room 2-106). The textbook problems are out of the 4th edition. For computational problems, please include a printout of the code with the problem set (for MATLAB in particular, `diary('filename')` will start a transcript session, `diary off` will end one.)

Every problem is worth 10 points.

1. Do Problems 5 & 11 from Section 6.6.
2. Do Problems 17 & 19 from Section 6.6.
3. Do Problem 22 from Section 6.6.
4. Do Problems 3 & 6 from Section 6.7.
5. Do Problems 9 & 10 from Section 6.7.
6. Do Problem 14 from Section 6.7.
7. Do Problems 1 & 4 from Section 8.5.
8. Do Problems 3 & 19 from Section 10.2.
9. Do Problem 7 from Section 10.3.
10. Do Problem 15 from Section 10.3.

18.06 Wisdom. To reiterate: You should get as ready as you can for Exam 3, and for the finals, by doing as many old exams as you have time for (found on the 18.06 website under "Past Courses"). Get into a good rhythm with these rehearsals to stay fully tuned in on Linear Algebra, and to master the concepts - you will need it, both here and beyond.

Michael Plasmeier 18.06 P-set 9 91/100 5/3

6.6 #5 Similar Matrices

10/10 Which of the 6 are similar

↳ same e-value in any order

Find λ in Matlab for speed + correctness

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \lambda = 1, 1$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \lambda = -1, 1$$

$$\left.\begin{aligned} &\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad \lambda = 1, 0 \\ &\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \quad \lambda = 1, 0 \end{aligned}\right] \text{similar}$$

$$\left.\begin{aligned} &\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \quad \lambda = 0, 1 \\ &\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \quad \lambda = 0, 1 \end{aligned}\right] \text{similar}$$

Similar order does not matter!

Prob also same pattern I forgot

②

6.6 #11 Solve $\frac{du}{dt} = Ju$ for J in #10 $J = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$

Stating from $u(0) = (5, 2)$

Remember $t e^{\lambda t}$

About Jordan Form ...

So J is as close to diagonal as can get

Can't solve from changing variables

$$\frac{du}{dt} = Ju \quad \text{is} \quad \frac{dx}{dt} = \lambda x + y$$

$$\frac{dy}{dt} = \lambda y$$

Then work backward -> triangular

$$y = y(0) e^{\lambda t}$$

$$x = (x(0) + t y(0)) e^{\lambda t}$$

So I see the pattern -> but why?

(2b)

Then fill in start point

$$y = 2e^{\lambda t}$$

$$x = (5 + 2t)\, e^{\lambda t}$$

? is that it ?

↓ this form

$$u = \begin{bmatrix} 5 \\ 2 \end{bmatrix} e^{\lambda t} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} t e^{\lambda t}$$

③

6.6 #17 True or False

10/10 a) A sym matrix can't be similar to a non sym matrix

I think need full set of evalues for sym matrix – so can't be similar

True

Book's Diagonizing a non sym $\Rightarrow A = S\Lambda S^{-1}$

Then Λ is non sym and similar (False)

(confused by what is what when its symmetric)

So can diag w/ out sym → but if sym its $Q\Lambda Q^T$ – and can handle repeated evalues

b) An invatible matrix can't be similar to a singular matrix.

True. Singular means one of the eigenvalues is 0

No e-vale can be 0 in an invertable

So can't be similar ✓ True

④

c) A can't be similar to $-A$ unless $A=0$

¿ Is this something about pos or neg definite?

Eigenvalues should be $-$, would not be the same True

Book) $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ are similar So False

└ So only needed to find 1 counter example...

d) A can't be similar to $A+I$

So what are λ of I

↳ 1, 1, 1, ...

But how do λ change when add

Add cellwise ~ So everything one more

So will be different

✓ True

6.6 #19 If A is 6x4 and B is 4x6

AB and BA have diff sizes

but w/ blocks

$$M^{-1}FM = \begin{bmatrix} I & -A \\ 0 & I \end{bmatrix} \begin{bmatrix} AB & 0 \\ A & 0 \end{bmatrix} \begin{bmatrix} I & A \\ 0 & I \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ B & BA \end{bmatrix} = G$$

\# zeros?

a) What are the size of the 4 blocks

? what blocks → Jordan blocks

So $M^{-1}AM = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_s \end{bmatrix}$

Jordan block

note long diregession

One Jordan block per unique e-vector/e-value

So do we know which blocks are here?

kinda saw in recitation

⑥

Key Fact: Note same λ does not mean similar

We don't have specific #s

Otherwise could draw blocks

Then look at SDS^{-1}

└ I don't get how this is working w/ splitting up into blocks

If same e-vector ??

Back to q.

Still no clue what it's asking

Book | Diagonal blocks 6 x 6
4 x 6

AB has same E values as BA plus 6-4 zeros

6b

What is F? - just letter gien to something

OH)

$$M = \begin{bmatrix} I & A \\ 0 & I \end{bmatrix}$$

$$F = \begin{bmatrix} AB & 0 \\ B & 0 \end{bmatrix}$$

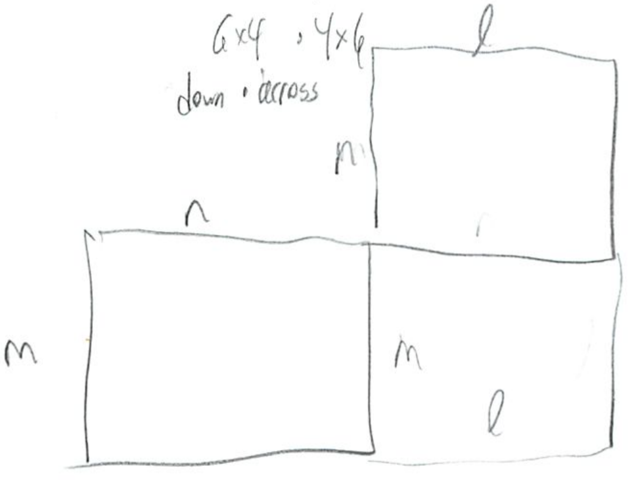
0 is not just 0
L is a matrix 0

So a asks what is size of each

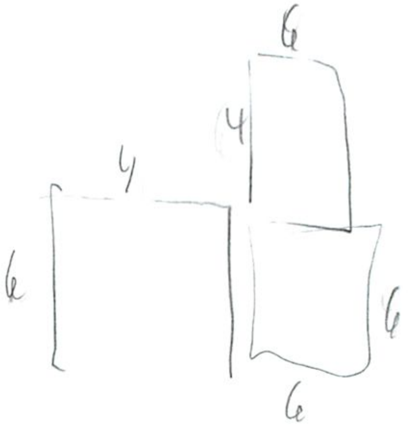
So upper right is 6x4
lower left is 4x6

$A \cdot B$ is

$6 \times 4 \cdot 4 \times 6$
down · accross

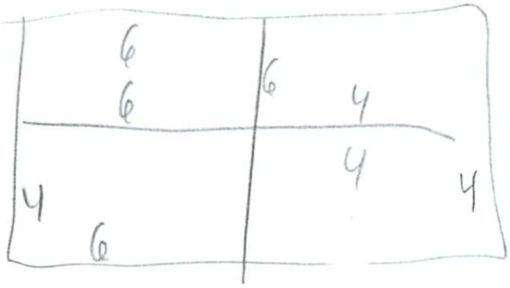


So here



(60)

60



⑦

b) +1 This eq is $M^{-1}FM = G$ so F, G have same 10 e-values ↑since similar

F has 6 evalues of AB plus 4 zeros

G has 4 evalues of BA plus 6 zeros

AB has same evalues as BA plus ___ 0s

So AB is size 6 x 6

BA is 4 x 4

So 6 - 4 = 2 ✓

Gilb #22 If an $n \times n$ matrix A has all eigenvalues $\lambda = 0$ prove that A^n = zero matrix,

10/10

(Maybe prove that J^n = zero matrix, by direct manipulation,
Or use Cayley Hamilton theory)

So $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\lambda = 0, 0, 0$

We want $B = M^{-1} A M$ to get B
which is similar to A
which is nearly as diag as possible

$\llcorner$ similar is same evalues

⑨

$$M^{-1}AM = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_s \end{bmatrix} = J$$

$$^{r}\begin{bmatrix} 0 & 1 & 0 \\ & 0 & 1 \\ & & 0 \end{bmatrix}_{r}$$ 3 λ repeated = 1 block

$$J^n = \begin{bmatrix} \text{not just} & 0^n & 1^n & \\ & & 0^n & 1^n \\ & & & 0^n \end{bmatrix}$$

but it shows its the same

And since $J = B$ and is similar to A

A^n is the same

[? seems too easy ...

Book | $A = MJM^{-1}$

$A^n = MJ^nM^{-1} = 0$

Each J^k has 1s on the kth diagonal

$\det(A - \lambda I) = \lambda^n$ So $J^n = 0$ by C-H Theorem

Kinda what I did - but I don't get

⑩

6.7 #3 For this $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ and all rank 1 matrices, why is σ_1^2 the sum of all a_{ij}^2 ?

10/10

So SVD is

$$A = U \Sigma V^T = u_1 \sigma_1 v_1^T + \cdots + u_r \sigma_r v_r^T$$

A is diagonalized

$$A v_1 = \sigma_1 u_1$$
$$A v_2 - \sigma_2 u_2$$

OH] $A = \begin{bmatrix} | & | \\ u_1 & u_2 \\ | & | \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} — v_1 — \\ — v_2 — \end{bmatrix}$

u, v not orthonormal ↓ when multiply get 1
L can't just multiply

Multiply both sides by A^T

$$AA^T = \begin{bmatrix} | & | \\ u_1 & u_2 \\ | & | \end{bmatrix} \begin{pmatrix} \sigma_1 & 0 \\ 0 & 0 \end{pmatrix} \underbrace{\begin{bmatrix} - v_1 - \\ - v_2 - \end{bmatrix} A^T}_{A \begin{pmatrix} | & | \\ v_1 & v_2 \\ | & | \end{pmatrix}^T}$$

since $A^T B^T = (BA)^T$

ⓦ ∴

$$\left(\overbrace{\begin{bmatrix}\sigma & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}| & |\\ u_1 & u_2\\ | & |\end{bmatrix}}\right)^T$$

the important line

By def SVD

$\sigma_1 u_1 = Av_1$

$\sigma_2 u_2 = Av_2$

$$\begin{pmatrix}\sigma_1 & \\ & \sigma_2\end{pmatrix}\begin{pmatrix}| & |\\ u_1 & u_2\\ | & |\end{pmatrix} = A\begin{pmatrix}| & |\\ v_1 & v_2\\ | & |\end{pmatrix}$$

vectors by convention are long

$$= \begin{bmatrix}| & |\\ u_1 & u_2\\ | & |\end{bmatrix}\begin{bmatrix}\sigma_1^2 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}| & |\\ u_1 & u_2\\ | & |\end{bmatrix}^T$$

orthonormal so $Q^T = Q^{-1}$

$AA^{-1} = I$

$$= \begin{bmatrix}\sigma_1^2 & 0\\ 0 & 0\end{bmatrix}$$ magically vanishes

Since

$$\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}$$

$$= \begin{bmatrix}a_{11}^2 + a_{12}^2 & *\\ * & a_{21}^2 + a_{22}^2\end{bmatrix}$$

$$= \begin{pmatrix}\sigma_1^2 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

$$= \begin{pmatrix}\sigma_1^2 & 0\\ 0 & 0\end{pmatrix}$$

can just add

(TA confused) ☹

⑫

6.7 #6 Compute A^TA and AA^T for this A

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Multiply the matrices $U\Sigma V^T$ to recova A

Σ has same shape as A.

Step 1. Compute A^TA

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

2. Find $\lambda = 0, 1, 3$

3. Find eig vectors $= \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{6} \\ \sqrt{2/3} \\ 1/\sqrt{6} \end{bmatrix}$

4. Fill in what we know

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = U \cdot \underbrace{\begin{bmatrix} \sqrt{0} & & \\ & \sqrt{1} & \\ & & \sqrt{3} \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ -1/\sqrt{3} & 0 & \sqrt{2/3} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix}}_{V^T}$$

5. Calculate AA^T

$$\begin{bmatrix}1&1&0\\0&1&1\end{bmatrix}\begin{bmatrix}1&0\\1&1\\0&1\end{bmatrix}=\begin{bmatrix}2&1\\1&2\end{bmatrix}$$

6. $\lambda = 1, 3$ ← will be the same, but dim diff...

7. $X = \begin{bmatrix}-1/\sqrt{2} & 1/\sqrt{2}\\ 1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}$

8. Fill it in

$$A = \underbrace{\begin{bmatrix}-1/\sqrt{2} & 1/\sqrt{2}\\ 1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}\begin{bmatrix}\sqrt{0} & & \\ & \sqrt{1} & \\ & & \sqrt{3}\end{bmatrix}}_{\text{dim error}}\begin{bmatrix}1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6}\\ -1/\sqrt{3} & 0 & \sqrt{2/3}\\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6}\end{bmatrix}^T$$

hmm - how to avoid

$$\boxed{\begin{aligned}&\text{dim}\quad (x \times \underbrace{y)\cdot(y}_{\text{same}} \cdot z)\\ &\qquad = (x \cdot z)\end{aligned}}$$

⑭

¿So elim the 0 eig vector

$$A = \begin{bmatrix} -1/\sqrt{2} & \sqrt{3}/2 \\ 1/\sqrt{2} & \sqrt{3}/2 \end{bmatrix} \begin{bmatrix} V^T \end{bmatrix}$$

but now 2x2 3x3 ...

kill the $\lambda=0$ col ?

$$V^T = \begin{bmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{6} & \sqrt{2/3} & 1/\sqrt{6} \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$ ✓ worked out

basically don't include $\lambda=0$, right?

Σ I got was $\begin{bmatrix} \sqrt{1} & \\ & \sqrt{3} \end{bmatrix}$

why is it the same shape as A?

or $\begin{bmatrix} 0 & \sqrt{1} & 0 \\ 0 & 0 & \sqrt{3} \end{bmatrix}$?

& diff from example I found online

There were no non-square examples in the book

Oh add 0s at the end

$$\begin{bmatrix} \sqrt{1} & 0 & 0 \\ 0 & \sqrt{3} & 0 \end{bmatrix}$$

Example on WP

$$A = \begin{bmatrix} & 5 \\ 4 & \end{bmatrix}$$

$$U = \begin{bmatrix} & 4 \\ 4 & \end{bmatrix} \quad \Sigma = \begin{bmatrix} & 5 \\ 4 & \end{bmatrix} \quad V^T = \begin{bmatrix} & 5 \\ 5 & \end{bmatrix}$$

So thats how it is supposed to work out

Order matters!

⑮6

6.7 #9 Suppose $u_1, \ldots u_n$ and $v_1 \ldots v_n$ are orthonormal bases for R^n

10/10

Construct the matrix A that transforms each v_j into u_j to give

$$Av_1 = u_1$$

$$\vdots$$

$$Av_n = u_n$$

So u, vs are orthonormal basises

normally

$$\left[\begin{array}{l} Av_1 = \sigma_1 u_1 \\ Av_2 = \sigma_2 u_2 \\ \vdots \\ Av_r = \sigma_r v_r \end{array}\right.$$

So here A is something that includes σ and real A

$$A = \text{real } A \cdot \sigma^{-1}$$

⑯

Book | $A = UV^T$ since all $\sigma = 1$

So $\Sigma = I$

So why $\sigma = 1$

Oh the orthonormal is not always there?

Orthogonal gives $V^T V = I$

$U^T U = I$

I kinda had it

Still not good w/ all these rules...

Final A is usually $U \Sigma V^T$

in this case Σ is identity

Since all multipliers are given as 1

*I didn't realize this was given and part of problem

⑰

whats so special about rank 1 ?

6.7 # 10 Construct a matrix w/ rank 1 that has

$$Av = 12u$$

for $v = \frac{1}{2}(1, 1, 1, 1)$ and

$u = \frac{1}{3}(2, 2, 1)$

Only 1 singular

So find A

$$\underset{4 \cdot 1}{A} \underset{1 \cdot 4}{\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}} = 12 \begin{bmatrix} \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

$?\sigma_1?$

$$\begin{bmatrix} 24 \\ \hline 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{aligned} \tfrac{1}{2} &= 12 \cdot \tfrac{2}{3} \\ \tfrac{1}{2} &= 0 \cdot \tfrac{2}{3} \end{aligned}$$

? is that the right approach ?

or any multiple

$$\begin{bmatrix} 24 \\ 12 \\ 6 \\ 3 \end{bmatrix}$$

This feels weirdly wrong ...

17b

Friend) Similar to 9

$$A = \begin{pmatrix} 2/3 \\ 2/3 \\ 1/3 \end{pmatrix} (12) \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix} =$$

↑ must deal w/ the 12

$$= \begin{bmatrix} 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 \\ 2 & 2 & 2 & 2 \end{bmatrix}$$

⑱

6.7 #14 Suppose A is invertable w/ $\sigma_1 > \sigma_2 > 0$

10/10

Change A by as small a matrix as possible to produce a singular matrix A_0

Hint : U and V don't change

From $A = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} \sigma_1 & \\ & \sigma_2 \end{bmatrix} \begin{bmatrix} V_1 & V_2 \end{bmatrix}^T$

So make A from investable to singular

Can't change u or V

So have to change σs

So change one of them

$$\begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & \overset{1}{\cancel{V}_1} \\ & V_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} u_1\sigma_1 & u_2\sigma_2 \end{bmatrix}$$

$$u_1 \sigma_1 V_1 + u_2 \sigma_2 V_2 = \text{final}$$

; So change either

⑭

Book | Change σ_2

? Why is σ_2 better than σ_1 ?

Friend | Since were told $\sigma_1 > \sigma_2$

Not changing matrix by as much

(stupid qu)

㉔

Fourier series

8.5 #1 Integrate

10/10

$$2\cos jx \cos kx = \cos(j+k)x + \cos(j-k)x$$

to show $\cos jx \perp \cos kx$

Book what is this trying to show

$$\int_0^{2\pi} \cos((j+k)\,x)\,dx =$$

$$= \left[\frac{\sin((j+k)x)}{j+k}\right]_0^{2\pi}$$

$$= 0$$

$$\int_0^{2\pi} \cos((j-k)\,x)dx) = 0$$

Notice $j-k \neq 0$ in denom

If $j=k \to \int_0^{2\pi} \cos^2 j\,x\,dx = \pi$

resonance

(21b)

Final)

Given identity

Itegrate it

See 0

Since have 2 sin functions added together

eval at $0, 2\pi$ so will be 0

b) Should be factor of 2

But get π

$\int 2\cos^2 jx \rightarrow$ Comes out 2π

8.5 #4 1st 3 Lagrende polynomials are $1, x, x^2-\frac{1}{3}$

Choose c so x^3-cx is orthagonal

$$\int_{-1}^{1} (x)(x^3-cx)\,dx =$$

$$= \int_{-1}^{1} (x^4-cx^2)\,dx$$

$$= \frac{1}{5}x^5 - \frac{c}{3}x^3 \Big|_{-1}^{1}$$

$$= 2\left(\frac{1}{5}-\frac{c}{3}\right) = 0$$

$$\frac{1}{5} = \frac{c}{3}$$

$$c = 3/5$$

Odd fun is just 0

Look for even function

Thats the even fn

㉒

10.2 #3 Solve $Az=0$ to find a vector in nullspace of A in #2

10/10

$$A = \begin{bmatrix} i & 1 & i \\ 1 & i & i \end{bmatrix}$$

Show z is $\perp$ to cols of A^H

$$\begin{bmatrix} i & 1 & i \\ 1 & i & i \end{bmatrix} z = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$i z_1 + z_2 + i z_3 = 0$$

$$z_1 + i z_2 + i z_3 = 0$$

So $z_3 = -i$

$z_1 = -1$

$z_2 = i$

$$-i + i + +1 = 0$$

$$-1 + -1 + +1 = -1$$

Not it - is anything possible?

㉓

I should have solved more robustly

Book 1 Ah if you do mix

$$z = \begin{bmatrix} 1+i \\ 1+i \\ -2 \end{bmatrix}$$

$$i(1+i) + (1+i) + i(-2) = 0$$
$$i - 1 + 1 + i - 2i = 0 \ ✓$$

$$(i+1) + i(1+i) + i(-2) = 0$$
$$i + 1 + i - 1 - 2i = 0 \ ✓$$

Show z is $\perp$ to col of A^H

So $A = \begin{bmatrix} i & 1 & i \\ 1 & i & i \end{bmatrix}$ $\quad z = \begin{bmatrix} 1+i \\ 1+i \\ -2 \end{bmatrix}$

Show orthogonal by dot product is 0
Just basically did that

㉘

Show $\underline{z}$ is <u>not</u> $\perp$ to cols of A^+

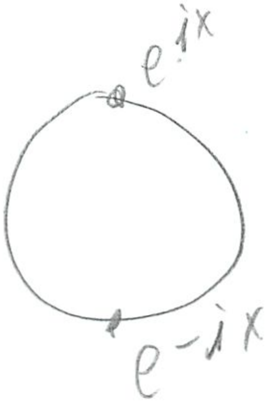
$$A^+ = \begin{bmatrix} i & 1 \\ 1 & i \\ i & i \end{bmatrix} \quad z = \begin{bmatrix} 1+i \\ 1+i \\ -2 \end{bmatrix}$$

Can't multiply nicely → dimension

So good col space is not $C(A^+)$ but $C((A^H))$

㉕

10.2 #19 The function e^{-ix} and e^{ix} are orthogonal on interval $0 \le x \le 2\pi$ since inner product is $\int_0^{2\pi}$ ______ $= 0$



; across from circle

but how is that orthogonal

well start at 0

so equal distances away

; How would you write this

2e

Book) Vs are cols of a unitary matrix U

So $U^H = U^{-1}$

Then $z = UU^H z$ = multiply by cols

$$= v(v_1^H z) + \dots + v_n(v_n^H z)$$

a typical othonormal expansion

¿But how is this $\int_0^{2\pi}$?

Friend) inner product $= \int_0^{2\pi} e^{-ix} \cdot e^{ix} \, dx$

$$= \int_0^{2\pi} e^{2ix} \, dx$$

$$= \frac{1}{2i} \left[e^{2ix} \right]_0^{2\pi}$$

$= 0$ ☺

㉗

10.3 #7 Put vector $c = [1,0,1,0)$ to find

$$y = Fc$$

1/10

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & w & w^2 & w^3 \\ 1 & w^2 & w^4 & w^6 \\ 1 & w^3 & w^6 & w^9 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$y_0 = 1 + 1$

$y_1 = 1 + w^2$

$y_2 = 1 + w^4$

$y_3 = 1 + w^6$

like that sin/cos thing

$c = (0,1,0,1)$

$y_0 = 1 + 1$

$y_1 = w + w^3$

$y_2 = w^2 + w^6$

$y_3 = w^3 + w^9$

?

28

10.3 #15 Fast convolution To multiply $C \circ x$

10/10

Can multiply $F(E(F^{-1}x))$ instead

Direct way uses n^2 sep multiplications

Knowing E and F, the 2nd way uses
$n \lg_2 n + n$ multiplications

How many from E, F, F^{-1}?

(6.006 qu)

└ not really

Don't understand process well enough

(29)

Book Diagonal $\underline{E}$ $= n$

F, F^{-1} need $\frac{1}{2} n \lg_2 n$ for each FFT

✓

18.06 Spring 2012 – Problem Set 9

This problem set is due Thursday, May 3rd, 2012 at 4pm (hand in to Room 2-106). The textbook problems are out of the 4th edition. For computational problems, please include a printout of the code with the problem set (for MATLAB in particular, `diary('filename')` will start a transcript session, `diary off` will end one.)

Every problem is worth 10 points.

1. Do Problems 5 & 11 from Section 6.6.

   Solution. **Problem 5.** $\begin{bmatrix}1&1\\0&0\end{bmatrix}$, $\begin{bmatrix}0&0\\1&1\end{bmatrix}$, $\begin{bmatrix}1&0\\1&0\end{bmatrix}$ and $\begin{bmatrix}0&1\\0&1\end{bmatrix}$ are similar (they all have eigenvalues 1 and 0).

   $\begin{bmatrix}1&0\\0&1\end{bmatrix}$ is by itself (eigenvalue 1), and $\begin{bmatrix}0&1\\1&0\end{bmatrix}$ is also by itself, with eigenvalues 1 and -1.

   Problem 11. $\mathbf{u}(0) = \begin{bmatrix}5\\2\end{bmatrix} = \begin{bmatrix}v(0)\\w(0)\end{bmatrix}$. The equation

   $$\frac{d\mathbf{u}}{dt} = \begin{bmatrix}\lambda&1\\0&\lambda\end{bmatrix}\mathbf{u}$$

   has

   $$\frac{dv}{dt} = \lambda v + w \qquad \text{and} \qquad \frac{dw}{dt} = \lambda w.$$

   Then $w(t) = 2e^{\lambda t}$ and $v(t)$ must include $2te^{\lambda t}$ (this comes from the repeated λ). To match $v(0) = 5$, the solution is $v(t) = 2te^{\lambda t} + 5e^{\lambda t}$. ☐

2. Do Problems 17 & 19 from Section 6.6.

   Solution. **Problem 17**

   (a) *False:* Diagonalize a nonsymmetric matrix $A = S\Lambda S^{-1}$. Then Λ is symmetric and similar.

   (b) *True:* A singular matrix has $\lambda = 0$.

   (c) *False:* $\begin{bmatrix}0&1\\-1&0\end{bmatrix}$ and $\begin{bmatrix}0&-1\\1&0\end{bmatrix}$ are similar (they have $\lambda = \pm 1$).

   (d) *True:* Adding I increases all eigenvalues by 1.

   Problem 19 Diagonal blocks 6 by 6, 4 by 4; AB has the same eigenvalues as BA plus $6 - 4$ zeroes. ☐

3. Do Problem 22 from Section 6.6.

   Solution. Let $J = M^{-1}AM$ be the Jordan form of A. Since J is a strictly upper triangular matrix, $J^n = 0$. Hence $A^n = (M^{-1}JM)^n = M^{-1}J^nM = 0$. ☐

4. Do Problems 3 & 6 from Section 6.7.

Solution. **Problem 3** There is a orthogonal matrix V such that $VA^TAV^T = (VA^T)\cdot (VA^T)^T = \begin{bmatrix} \sigma^2 & 0 \\ 0 & 0 \end{bmatrix}$. Let $a_1 = [a_{11}a_{12}]$ and $a_2 = [a_{21}a_{22}]$. Since orthogonal matrices preserve length of vectors,

$$\sigma^2 + 0^2 = ||Va_1^T||^2 = ||a_1^T||^2 = a_{11}^2 + a_{12}^2$$

and

$$0^2 + 0^2 = ||Va_2^T||^2 = ||a_2^T||^2 = a_{21}^2 + a_{22}^2.$$

Hence $\sum a_{ij}^2 = \sigma^2$.

Problem 6

The eigenvalues of $AA^T = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ are 3 and 1. The corresponding normal eigenvectors are $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$ and $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$. Hence we have $U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$.

The eigenvalues of $A^TA = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ are 3, 1 and 0. The corresponding normal eigenvectors are $\begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}$, $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$, and $\begin{bmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$. Hence we have $V = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & 0 & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{bmatrix}$.

Hence

$$A = U \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} V^T.$$

$\square$

5. Do Problems 9 & 10 from Section 6.7.

Solution. **Problem 9** Since A is a orthogonal matrix, $AA^T = A^TA = I$. The only eigenvalue of I is 1, so $A = UIV^T = UV^T$.

Problem 10 Note that $A = U\Sigma V^T$ where U and V are orthogonal matrices with first columns u and v, respectively, and $\Sigma = \begin{bmatrix} 12 & \\ & \end{bmatrix}$. Our matrix $A = 12uv^T$ and the only singular value if $\sigma_1 = 12$.

$\square$

6. Do Problem 14 from Section 6.7.

Solution. Let $A = U\Sigma V^T = Udiag(\sigma_1, \cdots, \sigma_n)V^T$ with $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n > 0$. The closet singular matrix is $A_0 = U\Sigma_0 V^T = Udiag(\sigma_1, \cdots, \sigma_{n-1}, 0)V^T$. $\square$

7. Do Problems 1 & 4 from Section 8.5.

Solution. **Problem 1** If $j \neq k$,the integral of RHS is

$$\int_0^{2\pi} \cos(j+k)x + \cos(j-k)x dx = \frac{\sin(j+k)x}{j+k} + \frac{\sin(j-k)x}{j-k}\Big|_{x=0}^{2\pi} = 0.$$

Hence $(\cos jx, \cos kx) = 0$. If $j = k$,

$$\int_0^{2\pi} \cos(j+k)x + \cos(j-k)x dx = \frac{\sin(j+k)x}{j+k} + 1\Big|_{x=0}^{2\pi} = 2\pi.$$

So $(\cos jx, \cos kx) = \pi$.

Problem 4 For any c, $x^3 - cx$ is orthogonal to 1 and $x^2 - \frac{1}{3}$, and

$$\int_{-1} 1x \cdot (x^3 - cx) dx = \frac{2}{5} - 2\frac{c}{3}.$$

If we let $c = \frac{3}{5}$, then $x^3 - cx$ is orthogonal to every other functions.

$\square$

8. Do Problems 3 & 19 from Section 10.2.

Solution. **Problem 3** $z = C[-1-i, -1-i, 2]^T$ for some constant C. Since $z^H A^H = 0$, $z\overline{a_i} = 0$ for any column a_i of A. However, $z \cdot [-i, 1, -i] = -1 - i$. Hence columns of A^T are not orthogonal to z.

Problem 19 Note that $e^{-ix} = \cos x - i \sin x$ and $e^{ix} = \cos x + i \sin x$. Their inner product is

$$\int_0^{2\pi} e^{-ix}\overline{e^{ix}} dx = \int_0^{2\pi} \cos^2 x - \sin^2 x - 2i \cos x \sin x dx = \int_0^{2\pi} \cos 2x - i \sin 2x dx = 0.$$

$\square$

9. Do Problem 7 from Section 10.3.

Solution.

$$c = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \to \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \to \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \to \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \end{bmatrix} = Fc.$$

Also,

$$C = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \to \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \to \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \end{bmatrix} \to \begin{bmatrix} 2 \\ 0 \\ -2 \\ 0 \end{bmatrix} = FC.$$

$\square$

10. Do Problem 15 from Section 10.3.

Solution. Diagonal E needs n multiplications, while the Fourier matrix F and F^{-1} need $\frac{1}{2} n \log_2 n$ multiplications each by the **FFT**. The total is much less than the ordinary n^2 for C times x. ☐

18.06 Wisdom. To reiterate: You should get as ready as you can for Exam 3, and for the finals, by doing as many old exams as you have time for (found on the 18.06 website under "Past Courses"). Get into a good rhythm with these rehearsals to stay fully tuned in on Linear Algebra, and to master the concepts - you will need it, both here and beyond.

18.06

5/4/12

(1 min late)

Find M if Possible $\quad$ NOT possible

$$M\begin{bmatrix}1&1&1\\1&2&2\\1&2&2\end{bmatrix}M^{-1}=\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}$$

If same λ's do I have to find M?

└ No repeated λs → they will be similar

└ means will be diagonizable

for sure Λ will be ok

Only if not → need to do Jordan Form

details of Jordan form not on there

②

$\det(A - \lambda I)$ and then find roots

↳ NOT good for finding λ on large matrices

do repeated MAM^{-1}

λ get better + better

more + more triangular

$A = QR$

└ Look at $A_2 = RQ$

Note QR is similar to RQ

? more nearly triangular

$$\underline{M}\, QR\, \underline{M^{-1}} = RQ$$

$$Q^{-1} Q R Q = RQ$$

③

$$A = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$$

$$A^{100} = c_1$$

1. Find λ

$$\begin{vmatrix} 4-\lambda & 3 \\ 1 & 2-\lambda \end{vmatrix} = \lambda^2 - 6\lambda + 5 = 0$$

$$(\lambda - 5)(\lambda - 1)$$

$$x = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Are x orth? No, since original matrix is <u>not</u> symmetric

$$\frac{du}{dt} = Au$$

$$u(0) = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$u = c_1 e^{5t} \begin{bmatrix} 3 \\ 1 \end{bmatrix} + c_2 e^{t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

④

At $t=0$

$$\begin{bmatrix}4\\0\end{bmatrix} = c_1\underset{\substack{\downarrow\\1}}{\begin{bmatrix}3\\1\end{bmatrix}} + c_2\underset{\substack{\downarrow\\1}}{\begin{bmatrix}1\\-1\end{bmatrix}}$$

Repeated λ

Look for e vectors

ie $\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}$ $\quad \lambda = \underset{\uparrow \text{since singular}}{0,0},3$

$$x = \underset{\uparrow \text{since symmetric (missed)}}{\begin{bmatrix}\;\\\;\\\;\end{bmatrix} \;\; \begin{bmatrix}1\\1\\1\end{bmatrix}}$$

$\begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix}$ $\quad \lambda = 1,1,1$

matrix singular

$$A = 1I = \begin{bmatrix}0&1&1\\0&0&1\\0&0&0\end{bmatrix}$$

⑤

$$\frac{du}{dt} = Au$$

$$\frac{d}{dt}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$\lambda_1 = 0$ ← know its singular $(\lambda = 0)$

$\lambda_2 = \frac{1}{2}$

$\lambda_3 = 1$

$$X = \overset{\lambda_1}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}, \overset{\lambda_2}{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}} \overset{\lambda_3}{\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}}$$

Could you reconstruct the matrix from this? Yes

$A = S\Lambda S^{-1}$ ←

Symmetric

↳ Since the 3 x are orthogonal

⑥

Markov matrix? No

The largest $\lambda = 1$ is a good sign

But has $\ominus$ in the x_3

λ for orthogonal Q is that when you multiply by a #, lenght does not change

$$\| Qx \|^2 = \| x \|^2$$

So all $\lambda = 1$

$$(Qx)^T (Qx) = x^T x$$

Since $x^T Q^T Q x =$

$$= x^T I x$$
$$= x^T x$$

[illegible]

Projection matrix? No

λ can only be $\underset{\text{perp to plane?}}{0}$ or $\underset{\text{? projection onto}}{1}$

⑦

Pos def

$$A = \begin{bmatrix} 1 & b & 0 \\ b & 4 & 2 \\ 0 & 2 & 4 \end{bmatrix}$$

If that is ⊕ def, what #s can b be?

If $b = 0$, Yes

$b = 10$ No

Largest b?

do $\left.\begin{matrix} 1 & b \\ b & 4 \end{matrix}\right|$ test

if $b = 2$, this is on border line (semi def)

But does this cause the $\underline{3\times3}|$ test to fail?

Yes, ⊖ det $= -4$

So not semi def

invertable or singular ? (did he answer?)

⑧ Requirements

$4-b^2 > 0$

$16-4-4b^2 > 0$

$12 > 4b^2$

Fourier is not on exam

Complex is not on exam, on final

This exam's basically all the ~~exam~~ LA stuff

inc SVD

SVD

$AV = U\Sigma$

$A = U\Sigma V^T$

Try computing

$A = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$ does not need to be square

rank = 1

⑨

$$A^TA = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \qquad \underbrace{\underset{1\cdot 4 \quad 4\cdot 1}{AA^T}}_{1\cdot 1} = [4]$$

Want orthogonal x, but first need λ

$$\lambda = [4, 0, 0, 0] \qquad \lambda = 4$$

note similar λ is on purpose

$$[1\,1\,1\,1] = \overset{U}{[\,1\,]}\;\overset{\Sigma}{[\,2\,0\,0\,0\,]}\;\overset{V^T}{\begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ \text{—} & \text{—} & \text{—} & \text{—} \\ \text{—} & \text{—} & \text{—} & \text{—} \\ \text{—} & \text{—} & \text{—} & \text{—} \end{bmatrix}}$$

← row sp

← ~~[illegible]~~

← null space 4 dimensions get multiplied by 0 anyway

↑ stuff in col space dim=1

1x4

4 x 4

Singular value = $\sqrt{4} = 2$

$$\boxed{Av = \sigma u}$$

~ just 1

Orthogonal bases for all the spaces

Prof: Be able to do SVD for a 2x2

⑩

$$A = \begin{bmatrix} 1/2 & 3 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$A^{k \to \infty}$

What happens?

Diagonalize

$\lambda = 1/2, 1, 1$

$$\begin{bmatrix} (1/2)^k & & \\ & 1^k & \\ & & 1^k \end{bmatrix} \overset{\text{in limit}}{=} \begin{bmatrix} 0 & & \\ & 1 & \\ & & 1 \end{bmatrix}$$

Exam 11 AM Walker Mon

18.06
Review Session

5/6

1-6 8.1 8.2 8.3 8.5

Fourier - Prof said no

But the early part is on there

Syllabus

Diff Eq

Symmetric Matrix

Pos Def Matrix

Engineering

Similar Matrices

SVD

Fourier Series, ~~FFT~~

Not FFT

Linear Transformations

②

Practice exam

1. $A = \begin{bmatrix} .7 & .4 \\ .3 & .6 \end{bmatrix}$

a) Find Λ and S

$$S^{-1}AS = \Lambda$$

b) Find A^k

c) Find lim as $k \to \infty$ of $u_k = A^k u_0$

if $u_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

So I rememebe this qu

Can I find λ, x by hand?

$\det(A - I\lambda) = 0$

$(.7-\lambda)(.6-\lambda) - .3 \cdot .4$

$.42 - .7\lambda - .6\lambda + \lambda^2 - .12 = 0$

$\lambda^2 - 1.3\lambda + .30 = 0$

③

$$b^{2} \pm \sqrt{4ac}$$

$$\frac{-b \pm \sqrt{b^2-4ac}}{2a}$$ Remember!

$$\frac{1.3 \pm \sqrt{1.3^2 - 4 \cdot 1 \cdot .30}}{2 \cdot 1}$$

(small #s!)

$$\begin{array}{r} 1.3 \\ \underline{1.3} \\ 39 \\ \underline{130} \\ 1.69 \end{array}$$

$$\frac{1.3 \pm \sqrt{1.69 - 1.2}}{2}$$

$$1.3 \pm \sqrt{.49}$$

$$\frac{1.3 \pm .07}{2} \qquad \frac{1.37}{2} \qquad \frac{1.23}{2}$$

$$X = \begin{bmatrix} .7 - \frac{1.37}{2} & .4 \\ .3 & .6 - \frac{1.37}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

⑧ ¿Am I doing this right → too much algebra

then $\begin{bmatrix} x_1 & x_1 \\ x_2 & x_2 \end{bmatrix}\begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix}\begin{bmatrix} & \\ & \end{bmatrix}$ ← need to find inv

what are the rules here?

If sym → then $Q^{-1} = Q^T$?

But not sym, so find w/ Gauss Jordan?

└ Need to review

Multiply $[A \; I]$ by A^{-1} to get $(I \; A^{-1})$

basically $\left[\begin{array}{c|cc} A & 1 & \\ & & 1 \end{array}\right]$ ← do to both sides

elim, then RREF

get $\left[\begin{array}{cc|c} 1 & & A^{-1} \\ & 1 & \end{array}\right]$

⑤

A^k is just multiply Λ^k

Take the lim to ∞

I don't ~~know~~ know what that $u_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ means

Something diff eq possibly?

Answer

1.a. Lets find Λ

$$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

Notice A = markov matrix Ohh...

$\lambda = 1$

$a = .3$

Since $\sum \lambda = \text{trace}(A)$

$= \text{Sum of diagonal}$

$1 + \lambda_2 = .7 + .6$

(5b)

$$\prod \lambda = \det(A)$$

To find S

$\underline{\lambda = 1}$ $N(A - \lambda I) = N(A - I)$

$$= N\begin{pmatrix} -.3 & .4 \\ .3 & -.4 \end{pmatrix}$$

$$x_1 = \begin{bmatrix} 4 \\ 3 \end{bmatrix} \qquad \leftarrow \text{ie } \begin{bmatrix} -.3 & .4 \\ .3 & -.4 \end{bmatrix}\begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\underline{\lambda = .3}$ $N(A - .3\,I)$

$$= N\begin{pmatrix} .4 & .4 \\ .3 & .3 \end{pmatrix}$$

$$x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

(5c)

Put in order of e-values

$$S = \begin{bmatrix} 4 & 1 \\ 3 & -1 \end{bmatrix} \quad \Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

~~Now~~

If wanted S^{-1}

- could do Gauss Jordan
- or the short way

$$\frac{1}{ac-bd}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

But don't have to find here

b) $A = S\Lambda S^{-1}$

$A^k = (S\Lambda S^{-1})(S\Lambda S^{-1})$

$= S\Lambda^k S^{-1}$

↑ still need to multiply by S S^{-1}

5d

$$\begin{bmatrix} 4 + 3(0.3)^k & 4 - 4(0.3)^k \\ 3 - 3(.3)^k & 3 + 4(.3)^k \end{bmatrix}$$

c) $$\lim_{k \to \infty} A^k \underset{\substack{\uparrow \\ \text{starting} \\ \text{point}}}{u_0} = \frac{1}{7}\begin{bmatrix} 4 & 4 \\ 3 & 3 \end{bmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

since $(.3)^k = 0$

$$= \frac{1}{7}\begin{pmatrix} 8 \\ 6 \end{pmatrix}$$

---

⑥

2. Suppose A is an investable 3x3 matix

a) Show how to prove that $x^T A^T A x$ is always ⊕

if x is not 0

Why fail if A not $^{-1}$?

So this chap/section I'm particularly bad at.

So $A^TA = I$?

Well $A^{-1}A$

If A = Orthonormal cols then $A^TA = I$

↑Q called

$$\left[\qquad\right]$$

↑each col is lenght 1

$\sqrt{\Sigma(\,)^2}$

When dot product any 2 cols = 0

⑦

Nothing special w/ A sym

~~AA^{T} $A^{T}A$~~

~~AB^{-1} $B^{-1}A$~~

$(AB)^{-1} = B^{-1}A^{-1}$ always true

$$\underset{I}{\overset{\shortparallel}{AA^{-1}}} = \underset{I}{\overset{\shortparallel}{A^{T}A}} \text{ if diagonal}$$

$\leftarrow$ in all cases

Or if A $1[\quad 3 \quad]$

$$3\left|\quad\right]^{1} \; 1\left[\quad 3 \quad\right] = 3\times 3$$

Oh A is 3x3

So $\underset{3\times 3}{A^{T}}\underset{3\times 3}{A} = 3\times 3$

⑧

6) prove A^TA is simlar to AA^T

Does it follow same e-value + evectors?

This is what I was confused on

Similar $B = M^{-1}AM$ is similar to M

↑ Any invertable matrix

Don't have to find Jordan Form

↳ if same, then simlar

Can guess M – no robust way to find

In order to be similar must have same λ
but having same λ → does not mean similar
but will be if both have full set e-vectors
↳ ie diagonlizable

⑨ similar def

A, B are n × n matices (must be square)

A, B are <u>similar</u> if there is some n×n matrix M

Such that $A = M^{-1}BM$

If A and B are similar, they have same evalue.

Having the same evalues does not gurentee similarity

eg $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ are not similar

---

Gussing M is usally pretty hard

└ won't likly need to find

A, B are similar if same Jordan form

⑩

So basically show

$$A^TA = M^{-1}AA^TM$$

? what next then?

Reversable

$$MAM^{-1} = \cancel{MM^{-1}}B\cancel{MM^{T}}$$

Answer

$$(AB)^T = B^TA^T$$

So $x^TA^TAx = (Ax)^TAx$

$= ||Ax||^2 > 0$

Only when $Ax = 0$

A is invertable

↓

$Ax = 0$ only when $x = 0$

Otherwise this quant is always positive

⑪

~~So A^TA = dot product of vector~~

Vector^T Vector = dot product of vector

= lenght of vector

b) $$A^TA = \underbrace{A^{-1}A}_{I}A^TA$$

Can multiply by those because by def of similar

$$A^TA = IA^TA$$

So similar

So then it follows (since similar) same e-value but not same e-vectors

⑫

c) If SVD is writen $A = U \Sigma V^T$

What is $A^T A$ reduced to simplest form?

(didn't think about)

$$A^T A = (U \Sigma V^T)^T (U \Sigma V^T)$$

Using $(AB)^T = B^T A^T$

$$(ABC)^T = C^T B^T A^T$$

$$= (V^T)^T \Sigma^T \underbrace{U^T U}_{I} \Sigma V^T$$

$$= V \Sigma^T \Sigma V^T$$

?not square so

not Σ^2

Since not $n \times n$

also not orthonormal cols

Thats it

(13)

~~1~~3. Suppose a 3x3 real sym matrix A has e-values 4, 4, 0

a) Find the det from evalues $(2A-I)^{-1}$

??? why 2A

So $A = S \Lambda S^{-1}$

$$\begin{bmatrix} 4 & & \\ & 4 & \\ & & 0 \end{bmatrix}$$

? This means its likely that other form I forget the name of. SVD

$A = U \Sigma V^T$

But really $\det(A) = \prod \lambda$

$= 4 \cdot 4 \cdot 0 = 0$?

But they want 2A?

lii

b) T or F A has 3 ind ev ectors
and can be diagonizable

false - λ repeats
λ 0

which is it?
or both

c) T or F x^TAx is never negitive

¿didn't we just show that's always tre?

d) T or F $\frac{1}{4}A$ is Markov

False -> Markov is diagonlizable - don't think so...
it has only 1 and < 1
Just guessing F - feel pretty sure

(15)

e) If A has orthonormal e vectors

q_1 q_2 q_3 w/ $\lambda = 4, 4, 0$

Find a formula for A in terms of q_1 q_2 q_3

using diagonlization

What does diagonlization mean again?

$$S^{-1}AS = \Lambda = \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix}$$

That's what I thought ...

But does this have to be SVD

$$\begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix} \begin{bmatrix} 4 & & \\ & 4 & \\ & & 0 \end{bmatrix} \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}^T$$

That's σ not λ

$$\Sigma = \begin{bmatrix} \sigma_1^2 & & \\ & \sigma_2^2 & \\ & & \ddots \end{bmatrix}$$

(15b)

Answer a) Need to find the e-values of $\left(\overset{2A-I}{}\right)^{-1}$

M has evalue λ

↳ so M^{-1} has evalue $\frac{1}{\lambda}$

So if can find e values of ↳ $2A-I$

Since $Mx = \lambda x$

$Ix = x$

$(M-I)x = (\lambda-1)x$

$M-I$ has e-value $\lambda-1$

A has e-values $4, 4, 0$

$2A$ has evalues $8, 8, 0$

$2A-I$ has $7, 7, -1$

$(2A-I)^{-1}$ has $\frac{1}{7}\ \frac{1}{7}, -1$

det is Π so $\frac{1}{7} \cdot \frac{1}{7} \cdot -1 = -\frac{1}{49}$

15C

b) We do have enough info

If 3x3 real sym matrix ∃ can always diag

↑ remember
were told

True

c) True A is pos semi def

This is one of the tests

If diag. Then all e-vect ind

Try to find e-vectors to see if diag.

0 e-value means not inv

d) Is there an A such that $\frac{1}{4}A$ is Markov

↳ since if there is → not enough info

Is not enough info since

True if $\frac{1}{4}A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & .5 & .5 \\ 0 & .5 & .5 \end{bmatrix}$

(15d)

False $\frac{1}{4}A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

So diagionlize to check

TA: kinda guess

Markov is a strong condition

e) How to diag a?

But it gives the vectors

So just write the formula

$$A = \underset{\uparrow \text{ were told orthonormal}}{\overset{Q}{\begin{bmatrix} | & | & | \\ q_1 & q_2 & q_3 \\ | & | & | \end{bmatrix}}} \underset{\uparrow \text{ order must match}}{\overset{\lambda}{\begin{bmatrix} 4 & & \\ & 4 & \\ & & 0 \end{bmatrix}}} \overset{Q^T}{\begin{bmatrix} - & q_1 & - \\ - & q_2 & - \\ - & q_3 & - \end{bmatrix}}$$

Multiply it out

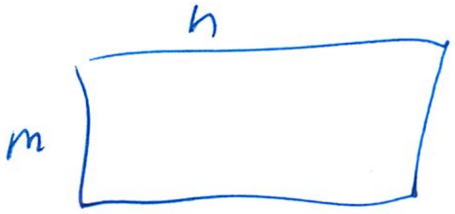
$$= 4q_1q_1^T + 4q_2q_2^T$$

Might have had to orthonormalize w/ Gram Schmidt

(16)

SVD Primer

$A = m \times n$ matrix
rank r



Look at $AA^T_{m\times m}$ orthonormal evectors $u_1, \dots u_r$
$A^TA_{n\times n}$ " " $v_1, \dots v_r$

$AV = U\Sigma$

are square so can find λ, x
same rank r

To find σ_i, solve $Av_i = \sigma u_i$ → Throw out those that correspond to evalue 0

Must normalize ← or σ screwed up
← study

So use Gram-Schmidt

Already orthogonalize - unless the λ same → must orthonormalize
if λ diff

$V = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_R \\ | & & | \end{bmatrix}$ ← want to be square

~~so enlarge w/ 0s~~ $v_{r+1} \cdots v_n$

~~So add~~ enlarge w/ orthonormal basis for $N(A)$ until square

~~$U = [u_1 \cdots u_k]$~~

16b

$$U = \left[\begin{array}{cccccc} | & & | & | & & | \\ u_1 & \cdots & u_r & \underbrace{u_{r+1} \quad \cdots \quad u_n}_{\text{Orthonormal basis for } N(A^T)} \\ | & & | & | & & | \end{array}\right]$$

? enlarge w/

⑰

4a Find the evales of $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

Then evales $A = \begin{bmatrix} a & 1 \\ 1 & a \end{bmatrix}$ $R = \begin{bmatrix} b & -1 \\ 1 & b \end{bmatrix}$

Looks like pos def ...

a, b real

a) Under what condition on "a" does all sols of $\frac{du}{dt} = A u$ approach 0 as $t \to \infty$

Answers

A has e-values $1, -1$ $(a+1, a-1)$

R: $i, -i$

$(b+1, b-i)$

b) $a < -1$

c) $b < 0$

d) $a > 1$

⑱

This is a diff eq problem

When solving diff eq w/ bounds → look at λ values

So λ is ⊖ when want sol stable

λ ⊕ instable

c) Since complex λ - want real parts negitive

Review for Exam 3

5/6

6.2 Diagonalizing

$S^{1}AS = \Lambda$

So Λ = e values λ

S = e vectors

$\prod \lambda = \det$

$\sum \lambda$ = trace = $\sum$ diagonals entries

$S^2 = S\Lambda S^{-1} S \Lambda S^{-1} =$

$= S \Lambda^2 S^{-1}$

$u_{k+1} = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix} u_k$

$AB = BA$ → means share same e-vector

∴ always share same evalue

don't think so since not auto true

②

6.3 | Application to Diff eq

$\frac{du}{dt} = Au$ w/ $u(0)$ at $t=0$

$u = e^{\lambda t}x$ when $Ax = \lambda t$

~~Then solve for~~

put togeter Σ solutions

$u(t) = Ce^{t}\begin{bmatrix}1\\1\end{bmatrix} + De^{-t}\begin{bmatrix}1\\-1\end{bmatrix} =$

? specific sol

Solve for C, D w/ $u(0)$

Should practice this

$\frac{du}{dt} = Au = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}u$ $\quad u(0) = \begin{pmatrix}y\\z\end{pmatrix}$

So this is

$\frac{d}{dt}\begin{bmatrix}y\\z\end{bmatrix} = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}y\\z\end{bmatrix}$

③

Find λ

$$\det(A - \lambda I) = 0$$

$\boxed{-1}$ yes $\neq a$??

well want eigenvectors

$\lambda = -1$

$$\begin{bmatrix} -1 & 1 \\ \cancel{0}\,1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-x + y = 0$

$x + -y = 0$

$x = y \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Somehow

$\underline{\lambda = 1}$?markov

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x + y = 0$

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

⑨

So

$$e^{1t}\begin{bmatrix}1\\1\end{bmatrix} + e^{-1t}\begin{bmatrix}1\\-1\end{bmatrix}$$

$$\begin{bmatrix}4\\2\end{bmatrix}0 = C e^{t}\begin{bmatrix}1\\1\end{bmatrix} + De^{-1t}\begin{bmatrix}1\\-\end{bmatrix}$$

$$e^0 = 1$$

$$\begin{bmatrix}4\\2\end{bmatrix} = C\begin{bmatrix}1\\1\end{bmatrix} + D\begin{bmatrix}1\\-1\end{bmatrix}$$

? This was given

$C + D = 4$

$C - D = 2$

$C = 2 + D$

$2 + D + D = 4$

$2D = 2$

$D = 1$ ✓

$C = 3$

$$u(t) = 3e^{1t}\begin{bmatrix}1\\1\end{bmatrix} + 1e^{-t}\begin{bmatrix}1\\-1\end{bmatrix}$$

Ok that should be it

had a

Second order

$$d \begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} & \\ & \end{bmatrix} \begin{bmatrix} y \\ y' \end{bmatrix}$$

Matrix exponential
skipping

$\boxed{\text{lec 4 Symmetric}}$ $P = P^T$

e-vectors of $P \overset{\text{and}}{\#} P^T$ are ⊥ unit vectors

most important matrices

~~$Ax = \lambda x$~~

$SS^T = I$ $S^{-1} = S^T$ ~~right~~ yes only when orthogonal

$Q^T = Q^{-1}$

Only real evalues
e-vectors can be choosen orthonormal

? Since symmetric

⑥

$A = Q \Lambda Q^{-1} = Q \Lambda Q^T$

(Remember what stuff is what!)

E values can be complex

└ but not if sym!

Prod of pivots = determine

~~$A = A^T$~~

pos eig vales = # pos pivots

All sym matrices are diag.

Always enough eig vectors

(I never look in worked examples section...)

(Despite me relying on this...)

6.5 Pos Definite Matrices

Sym matrices that have ⊕ evals makes it truly special (but not rare)

1. all pivots > 0
2. all $\lambda > 0$

} but these statements are equivilant!

3. All upper left det are ⊕

$$\begin{array}{cc} & \oplus \quad [\text{illegible}] \\ \text{det}=5 & \begin{matrix} 5 & \cancel{3}4 \\ 4 & 3 \end{matrix} \\ \text{det}=-1 & \end{array}$$

No

So for $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $a > 0$ and $ac - b^2 > 0$

4. A can be factored into $A = LL^T$

lower triangular

Can't be singular

Cols must be ind

⑧

$$5.\ \overbrace{A = A^T}^{\text{this is sym}}\ \ \underline{\text{if}}\ \ x^TAx > 0$$

for all $x \neq 0$

(The normal def

(pos def must be sym?
Don't think so - WP
But in this chap they need to be)

also note
$x = \begin{bmatrix} x \\ y \end{bmatrix}$
$x^TAx =$
$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} =$
$ax^2 + 2bxy + cy^2 > 0$

Ⓐ Just look at $x^TAx > 0$
means pos def

x^TAx is the <u>energy</u> in a system
Where the other rules are from

Ⓐ If A, B ⊕ sym ⊕ def, $A + B$ will be

When it has one of the propert's it has them all

Also $A = R^T R$ for matrix R w/ ind col's

└ If the cols of R are ind, $A = R^T R$ is ⊕ def

Pos Semi Def

When $x^T A x = 0$

When $\lambda = 0$

$Ax = \lambda x$

$x^T A x = \lambda x^T x$

$x^T A x = \lambda \|x\|^2$

So ⊕ x ⊕ = ⊕

A factors in $R^T R$ w/ dep Cols in R

Le.le Similar Matrices

$B = M^{-1} A M$ is similar to M

└ any invertable matrix

(must be square I'm pretty sure)

Can't actually find, but can guess...

⑩

What is $X^T X$

So for sym/ orthonormal $X^T = X^{-1}$

So $X^{-1} X = I$

but normally ...

(TA question but don't have ...

$||v||$ = lenght $= \sqrt{v^2}$

dot product $v \cdot v = v_1 u_1 + v_2 u_2$

So is $X^T x$ the length ?

$$X = 1\left[\ \overset{5}{\quad}\ \right]$$

$$5\left[\ \overset{1}{\ }\ \right] 1\left[\ \overset{5}{\quad}\ \right] = 5 \times 5 \left[\quad\right]$$

lenght squared ?

$A^T A$ is Square, sym, pos semi def

$(AB)^T = B^T A^T$

⑪

i And last time we say $(AB)^{-1} = B^{-1}A^{-1}$ ✓

$||v|| = \sqrt{v_1^2 + v_2^2 + v_3^2}$

The dot product of a vector is the lenght squared

$v = [1, 2, 3]$

$||v||^2 = \begin{bmatrix}1\\2\\3\end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix} = 1 + 4 + 9 = 14$

So it must be $X = X^T$ is $||X||^2$

Plus here $A = A^T$ Sym

Since x = vector!

⑫

Similar

evalues don't change for similar

evectors do

det, trace, rank, Jordan form same/unchanged

evectors, col, row, null space changed

6.7 SVD

$$Av_1 = \sigma_1 u_1$$
$$Av_2 = \sigma_2 u_2$$
$$Av_3 = \sigma_3 u_3$$
$$\vdots$$

w/ $AA^T = u_s$
$A^TA = V_s$

$$A = U \Sigma V^T$$

[illegible]

Be sure to actually try one...

$\begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$ So

↓

⑬

$AA^{\dagger}$

$$\begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 4 \\ 4 & 2 \end{bmatrix}$$ ✓

$(8-\lambda)(2-\lambda) - 16 = 0$

$16 - 10\lambda + \lambda^2 - 16 = 0$

$\lambda^2 - 10\lambda = 0$

$$\frac{+10 \pm \sqrt{100 - 4 \cdot 1 \cdot 0}}{2}$$

$$\frac{10 \pm 10}{2} \quad \lambda = 0, 10$$ ✓

$\underline{\lambda = 0}$

$$\begin{bmatrix} 8 & 4 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$8x + 4y = 0$

~~$4x + 2y =$~~ 0

; unspecified or multiple ...

$8x = -4y$

$2x = -y$

$$\begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

⑭

$\lambda = 10$) $\begin{bmatrix} -2 & 4 \\ 4 & -8 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$-2x + 4y = 0$

$4y = 2x$

$2y = x$

$4x = +8y$ ↕ ~~i Contradict~~

$x = +2y$

$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$WA = \begin{bmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{bmatrix} \quad \begin{bmatrix} -2/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix}$

So ? what is lenght of $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$\sqrt{1^2 + 2^2} = \sqrt{5}$ so they did it right

but my signs are diff

~~And why did they conflict above?~~

But they should work $2(-1) = (-2)$ ✓

⑮

So $U = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$

Now V is $A^T A$

$$\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \cancel{3}5 & \cancel{3}5 \\ \cancel{3}5 & \cancel{3}5 \end{bmatrix}$$ Can't add

$$\det (\cancel{3}5-\lambda)(\cancel{3}5-\lambda) - \cancel{9}25$$

$$\cancel{9}25 - \cancel{6}10\lambda + \lambda^2 - \cancel{9}25 = 0$$

$$\lambda^2 - \cancel{6}10 = 0$$

(doing this the long way:)

$$\cancel{\frac{6 \pm \sqrt{36}}{2}}$$

$$\lambda = 0, \cancel{6}\, 10$$

$\underline{\lambda = 0}$ $$\begin{bmatrix} \cancel{3}5 & \cancel{3}5 \\ \cancel{3}5 & \cancel{3}5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

⑯

$\boxed{\lambda = \cancel{7}6}$

$$\begin{bmatrix} -\cancel{3}^{5} & \cancel{3}^{5} \\ \cancel{3}\,5 & -\cancel{3}\,5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-3x + 3y = 0$

$3x = 3y$

$x = y$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$V = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

So σ is

Oh wait λ should be same...

0, 10

~~$\sqrt{0}$ $\sqrt{10}$~~

$\sigma = 0, \sqrt{10}$

But Σ is σ^2 so $\begin{bmatrix} 0 & \\ & 10 \end{bmatrix}$

Don't forget to normalize!!

↳ orthonormal

⑰

$A = U \Sigma V^T$

$$\begin{bmatrix}2&2\\1&1\end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix}2&1\\1&-2\end{bmatrix}\begin{bmatrix}\sqrt{10}&\\&0\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1&1\\1&-1\end{bmatrix}$$

↑ note order

I think you can also swap order
but ~~[illegible]~~ non zero values usually first

And A does not have to be square
had some trouble on p-set
(which I now turned in)
(but should try ..)

$A = \begin{bmatrix}2&1\end{bmatrix}$

So $AA^T = \begin{bmatrix}2&1\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}5\end{bmatrix}$

$\lambda = 5$

~~5Ix~~ ~~[illegible]~~ but how

$\underline{\begin{bmatrix}5-5\end{bmatrix}\begin{bmatrix}x\end{bmatrix} = 0}$?

Wolfram alpha $x = 1$? can be any value ?

⑱

$A^T A$

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$$

$$(4-\lambda)(1-\lambda) - 4$$

$$4 - 5\lambda + \lambda^2 - 4 = 0$$

$$\lambda^2 - 5\lambda$$

$$\frac{5 \pm 5}{2}$$

$$\lambda = 0, 5$$

(; yeah adds 0

$\underline{\lambda = 0)}$

$$\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$4x + 2y = 0$$

$$2x = -y$$

$$\begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$\underline{\lambda = 5|}$

$$\begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-x + 2y = 0$$

$$2y = x$$

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

So then what

$$\begin{bmatrix}1\end{bmatrix}\begin{bmatrix}\sqrt{5} & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}2 & 1\\ 1 & -2\end{bmatrix}$$

Try it

$$\begin{bmatrix}1\cdot 1 \cdot 2 \cdot 2\end{bmatrix} \otimes \text{ error}$$

(Wish I had my p-set where I figured this out...)

Or add 0s?

$$\underset{2\times 2}{\begin{bmatrix}1 & \\ & 0\end{bmatrix}}\underset{2\times 2}{\begin{bmatrix}\sqrt{5} & \\ & 0\end{bmatrix}} \ ✓ \text{ works } = \begin{bmatrix}\sqrt{5} & \\ & 0\end{bmatrix}$$

Then 2×2 × 2×2 works

$$\begin{bmatrix}2\sqrt{5} & \sqrt{5}\\ 0 & 0\end{bmatrix}$$

Oh never <u>normalized</u>!

$$\sqrt{2^2+1^2} = \sqrt{5}$$

$$\frac{1}{\sqrt{5}}\begin{bmatrix}2\sqrt{5} & \sqrt{5}\end{bmatrix} = \begin{pmatrix}2 & 1\end{pmatrix} \ ✓$$

Basically it appears just add 0s

8.1 Matrices in Eng

I hope he doesn't ask anything here...
Absent that day and no video online...

8.2 Graphs in Networks

(practice exam didn't cover this much) ...

incidence matrix
note every vector in nullspace is $\perp$ to every vector in row space

---

8.3 Markov Matrices

Cols add to 1
all values pos or 0

Converges to same u_∞ no matter u_0
└ Thats why I was confused in the review ~~class~~ session

So λ is 1 and <1

(look at trace)

$u_0 = \lambda^{1k} \cdot (\text{coeff}) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \text{other}$

$u_k = x_1 + c_2(\lambda_2)^k x_2 + c_n(\lambda_n)^k x_n$

Where does coeff come from?

Oh from start u_0 - like C +D in diff eq

I believe

Can use w/ consumption matrix

8.5 Fourier Series

From finite dimensions to ∞ dim

Vector becomes a function $f(x)$

$||v||$ must be finite

Is hermetian on it?

not in index! - 10.2 not on sheet

㉒

Schwartz ineq $|v \cdot w| \le \|v\| \, \|w\|$

inner product

$$(f,g) = \int_0^{2\pi} f(x)\, g(x) \, dx$$

length squared

$$\|f\|^2 = \int_0^{2\pi} (f(x))^2 \, dx$$

Fourier series

$$y(x) = a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \ldots$$

Fourier coeffs

$$a_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos kx \, dx$$

$$b_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin kx \, dx$$

Will we have to do anything here?

(23)

Inverse Practice Gauss Jordan

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

Convert AI to IA^{-1}

$$l = -\frac{1}{2} \quad \begin{bmatrix} 2 & -1 & 0 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{bmatrix}$$

$-1-(-\frac{1}{2})2$

$2-(-\frac{1}{2})-1$

$$\begin{bmatrix} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1\frac{1}{2} & -1 & -\frac{1}{2} & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & \end{bmatrix}$$

$l = \frac{0}{2}$

$l = -\frac{1}{1.5} = -\frac{2}{3}$

$$\begin{bmatrix} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1\frac{1}{2} & -1 & +1/2 & 1 & 0 \\ 0 & 0 & 4/3 & +1/3 & +2/3 & 1 \end{bmatrix}$$

$-1-(-\frac{2}{3})(1.5)$

$2-(-\frac{2}{3})(-1)$

$0-(+\frac{1}{2})(-2/3)$

↑ Why did I screw up this side?

$0-(-\frac{1}{2})(1)$

Oh only ·2 ☹

Then other one is wrong

Now REF

↳ want 0s above bivots

rows added to rows above them

So ~~row~~ row 2 = ~~3/4 row 2~~ row 2 + $\frac{3}{4}$ row 3

$$\left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & 0 & \frac{3}{4} & \frac{3}{2} & \frac{3}{4} \\ 0 & 0 & 4/3 & 1/3 & 2/3 & 1 \end{array}\right]$$

$-1+\frac{3}{4}\left(\frac{4}{3}\right)$

$\frac{1}{2}+\frac{3}{4}\left(\frac{1}{3}\right)$

$\frac{1}{2}+\frac{3}{12} - \frac{9}{12} = \frac{3}{4}$

$1+\frac{3}{4}\left(\frac{2}{3}\right)$

$= 1+\frac{6}{12}$

Now row 1 = row 1 + $\frac{2}{3}$ row 2 figured it out

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 0 & \frac{3}{2} & \cancel{0}\,1 & \frac{1}{2} \\ 0 & \frac{3}{2} & 0 & 3/4 & 3/2 & 3/4 \\ 0 & 0 & 4/3 & 1/3 & 2/3 & 1 \end{array}\right]$$

Close

$1+\frac{2}{3}\left(\frac{3}{4}\right)$

$1+\frac{6}{12}$

$\frac{2}{3}\left(\frac{3}{4}\right) = \frac{6}{12}$

Now Gram Schmidt

to make orthonormal vectors

for $A = QR$

messed this up on prev exam...
R is weird/useless

$$\begin{bmatrix} | & | & | \\ a & b & c \\ | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | \\ q_1 & q_2 & q_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} q_1^T a & q_1^T b & q_1^T c \\ & q_2^T b & q_2^T c \\ & & q_3^T c \end{bmatrix}$$

$$\overset{\swarrow \text{orthogonal}}{A} = \overset{\swarrow \text{original}}{a}$$

$$B = b - \frac{A^T b}{A^T A} A$$

lots of painful multiplying

$$C = c - \frac{A^T c}{A^T A} A - \frac{A^T c}{A^T A} B \quad \begin{matrix} \leftarrow \text{one looking at} \\ \leftarrow \text{look at old ones} \end{matrix}$$

㉖

Try it $a=\begin{bmatrix}1\\-1\\0\end{bmatrix}$ $b=\begin{bmatrix}2\\0\\-2\end{bmatrix}$ $c=\begin{bmatrix}3\\-3\\3\end{bmatrix}$

$$A=a=\begin{bmatrix}1\\-1\\0\end{bmatrix}$$

$$B= b-\frac{A^Tb}{A^TA}A$$

$$=\begin{bmatrix}2\\0\\-2\end{bmatrix}-\frac{{}^{1}\begin{bmatrix}1 & -1 & 0\end{bmatrix}_{3}\ {}^{3}\begin{bmatrix}2\\0\\-2\end{bmatrix}}{\begin{bmatrix}1 & -1 & 0\end{bmatrix}\begin{bmatrix}1\\-1\\0\end{bmatrix}}\begin{bmatrix}1\\-1\\0\end{bmatrix}$$

$$=\begin{bmatrix}2\\0\\-2\end{bmatrix}-\frac{\begin{bmatrix}2+0+0\end{bmatrix}}{\begin{bmatrix}1+1+0\end{bmatrix}}\begin{bmatrix}1\\-1\\0\end{bmatrix}$$

$$=\begin{bmatrix}2\\0\\-2\end{bmatrix}-\begin{bmatrix}1\\-1\\0\end{bmatrix}=\begin{bmatrix}1\\+1\\-2\end{bmatrix}\quad\begin{bmatrix}1\\1\\-2\end{bmatrix}$$

~~WTF~~ Figured it out

copy error, algebra error

㉗

$$C = C - \frac{B^T C}{B^T B} B - \frac{A^T C}{A^T A} A$$ ✓

$$= \begin{bmatrix}3\\-3\\3\end{bmatrix} - \frac{\begin{bmatrix}1 & 1 & -2\end{bmatrix}\begin{bmatrix}3\\-3\\3\end{bmatrix}}{\begin{bmatrix}1 & 1 & 2\end{bmatrix}\begin{bmatrix}1\\1\\2\end{bmatrix}} \begin{bmatrix}1\\1\\2\end{bmatrix} - \frac{A^T c}{A^T A} A$$

$$= \begin{bmatrix}3\\-3\\3\end{bmatrix} - \frac{\begin{bmatrix}3 - 3 - 6\end{bmatrix}}{\begin{bmatrix}1 + 1 + 4\end{bmatrix}} \begin{bmatrix}1\\1\\2\end{bmatrix} - \frac{A^T c}{A^T A} A$$

$$= \begin{bmatrix}3\\-3\\+3\end{bmatrix} - \frac{-6}{6}\begin{bmatrix}1\\1\\2\end{bmatrix} = \begin{bmatrix}4\\-2\\5\end{bmatrix} - \cancel{A}c$$ etc

㉘

$$= \begin{bmatrix} 4 \\ -2 \\ 5 \end{bmatrix} - \frac{\begin{bmatrix} 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ -3 \\ 3 \end{bmatrix}}{\begin{bmatrix} \cancel{4\,2\,h\,2} \\ 1 \;\; -1 \;\; 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ -2 \\ 5 \end{bmatrix} - \frac{\begin{bmatrix} 3 + 3 + 0 \end{bmatrix}}{\begin{bmatrix} 1 + 1 + 0 \end{bmatrix}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ -2 \\ 5 \end{bmatrix} - \frac{6}{2} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Some algebra error somewhere

Don't forget to normalize

$$q_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad q_2 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \quad q_3 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

5/7

$(AB)^{-1} = B^{-1}A^{-1}$

$AA^{-1} = A^{-1}A$ if diagonalizable?

Sym?

Sym always means diag...

M has	e value	λ
M^{-1}		$\frac{1}{\lambda}$
$2M$		2λ
$M-I$		$\lambda-1$

Don't enlarge w/ 0s for SVD U and V

instead insert orthonormal basis

for $N(A)$ for V ← n-r cols

$N(A^T)$ U ← m-r cols

Quick Flip though...

18.06 Quiz 3 Professor Strang May 6, 2011

Your PRINTED name is ____________________ **1.**

Your Recitation Instructor (and time) is ____________________ **2.**

Instructors: (Hezari)(Pires)(Sheridan)(Yoo) **3.**

Please show enough work so we can see your method and give due credit.

1. (a) Find two eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}.$$

$\lambda = 2, 5$

$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$(2-\lambda)(5-\lambda) - 3 \cdot 0$

$10 - 7\lambda + \lambda^2 = 0$

(too small space)

(b) Express any vector $u_0 = \begin{bmatrix} a \\ b \end{bmatrix}$ as a combination of the eigenvectors.

? exponential

$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = S^{-1} u_0 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a-b \\ b \end{bmatrix}$ ✓

?did we ever learn?

(c) What is the solution $u(t)$ to $\frac{du}{dt} = Au$ starting from $u(0) = u_0$?

$e^{\lambda t} \begin{bmatrix} x \end{bmatrix}$

? evector?

yes

$u(t) = c\, e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2 =$

(d) Find a formula $u_k =$ ____________ for the solution to $u_{k+1} = Au_k$ which starts from that vector u_0. Set $k = -1$ to find $A^{-1} u_0$.

Hmm is the same

despite its inverse

2. This problem is about the matrix

$$A = \begin{bmatrix} \sqrt{2} & 1 \\ 0 & \sqrt{2} \end{bmatrix}.$$

(a) Find all eigenvectors of A. Exactly why is it impossible to diagonalize A in the form $A = S\Lambda S^{-1}$?

λ repeat → no ← r0, e-vectors are the same
non real λ

(b) Find the matrices U, Σ, V^T in the Singular Value Decomposition $A = U\,\Sigma\,V^T$. Tell me *two orthogonal vectors* v_1, v_2 in the plane so that Av_1 and Av_2 are also orthogonal.

Did before - -

(c) Find a matrix B that is similar to A (but different from A). Show that A and B meet the requirement to be similar *(what is it?)*.

λ guess show $B = M^{-1}AM$

F $M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

3. Suppose A is a real m by n matrix.

(a) Prove that the symmetric matrix A^TA has the property $x^T(A^TA)x \geq 0$ for every vector x in R^n. Explain each step in your reason.

Pos de

$(x^T A^T)(Ax)$

$(Ax)^T(Ax)$

will be ≥ 0

but not for same reason as before?

(b) According to part (a), the matrix A^TA is positive semidefinite at least — and possibly positive definite. Under what condition on A is A^TA positive definite?

(c) If $m < n$ prove that A^TA is *not* positive definite.

18.06 Quiz 3 Professor Strang May 6, 2011

Your PRINTED name is ____________________ 1.

Your Recitation Instructor (and time) is ____________________ 2.

Instructors: (Hezari)(Pires)(Sheridan)(Yoo) 3.

Please show enough work so we can see your method and give due credit.

1. (a) Find two eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}.$$

$p(\lambda) = \det(A - \lambda I) = \lambda^2 - 7\lambda + 10 = (\lambda-2)(\lambda-5) = 0 \Rightarrow \lambda_1 = 2$, $\lambda_2 = 5$

$N(A - 2I) = N\left(\begin{bmatrix} 0 & 3 \\ 0 & 3 \end{bmatrix}\right) = \text{span}\left\{\underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{x_1}\right\}.$

$N(A - 5I) = N\left(\begin{bmatrix} -3 & 3 \\ 0 & 0 \end{bmatrix}\right) = \text{span}\left\{\underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{x_2}\right\}.$

(b) Express any vector $u_0 = \begin{bmatrix} a \\ b \end{bmatrix}$ as a combination of the eigenvectors.

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = S^{-1} u_0 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a-b \\ b \end{bmatrix}$$

So $u_0 = c_1 x_1 + c_2 x_2 = (a-b)\begin{bmatrix} 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$

(c) What is the solution $u(t)$ to $\frac{du}{dt} = Au$ starting from $u(0) = u_0$?

$$u(t) = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2 = (a-b)e^{2t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + b e^{5t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

(d) Find a formula u_k = __________ for the solution to $u_{k+1} = Au_k$ which starts from that vector u_0. Set $k = -1$ to find $A^{-1}u_0$.

$$u_k = A^k u_0 = c_1 \lambda_1^k x_1 + c_2 \lambda_2^k x_2 = (a-b)2^k\begin{bmatrix} 1 \\ 0 \end{bmatrix} + b\,5^k\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

$$u_{-1} = (a-b)\,2^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + b\,5^{-1}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{a-b}{2} + \frac{b}{5} \\ \frac{b}{5} \end{bmatrix}$$

2. This problem is about the matrix

$$A = \begin{bmatrix} \sqrt{2} & 1 \\ 0 & \sqrt{2} \end{bmatrix}.$$

(a) Find all eigenvectors of A. Exactly why is it impossible to diagonalize A in the form $A = S\Lambda S^{-1}$? $p(\lambda) = (\lambda - \sqrt{2})(\lambda - \sqrt{2}) = 0 \Rightarrow \lambda_1 = \lambda_2 = \sqrt{2}$ repeated eigenvalues

$N(A - \sqrt{2} I) = N\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = \text{Span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$

There are not enough independent eigenvectors to form an invertible matrix S with eigenvectors as its columns.

(b) Find the matrices U, Σ, V^T in the Singular Value Decomposition $A = U\Sigma V^T$. Tell me *two orthogonal vectors* v_1, v_2 in the plane so that Av_1 and Av_2 are also orthogonal. $B = A^TA = \begin{bmatrix} \sqrt{2} & 0 \\ 1 & \sqrt{2} \end{bmatrix}\begin{bmatrix} \sqrt{2} & 1 \\ 0 & \sqrt{2} \end{bmatrix} = \begin{bmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 3 \end{bmatrix}$

eigenvalues of B $\rightarrow p_B(\lambda) = \lambda^2 - 5\lambda + 4 = (\lambda-1)(\lambda-4) = 0 \Rightarrow \lambda_1 = 4,\ \lambda_2 = 1$ for B.

$\Rightarrow \sigma_1 = \sqrt{\lambda_1} = 2,\ \sigma_2 = \sqrt{\lambda_2} = 1. \Rightarrow \Sigma = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}.$

eigenvectors of B $\rightarrow \begin{cases} N(B - 4I) = N\begin{bmatrix} -2 & \sqrt{2} \\ \sqrt{2} & -1 \end{bmatrix} = \text{Span}\left\{\begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix}\right\} \xRightarrow[\text{normalize}]{} v_1 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix} \\ N(B - I) = N\begin{bmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 2 \end{bmatrix} = \text{Span}\left\{\begin{bmatrix} \sqrt{2} \\ -1 \end{bmatrix}\right\} \Rightarrow v_2 = \frac{1}{\sqrt{3}}\begin{bmatrix} \sqrt{2} \\ -1 \end{bmatrix}. \end{cases}$

$u_1 = \frac{Av_1}{\sigma_1} = \frac{1}{\sqrt{3}}\begin{bmatrix} \sqrt{2} \\ 1 \end{bmatrix}.$

$u_2 = \frac{Av_2}{\sigma_2} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ -\sqrt{2} \end{bmatrix}.$ $A = \begin{bmatrix} u_1 \mid u_2 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} v_1 \mid v_2 \end{bmatrix}^T.$

$\boxed{v_1 \perp v_2 \text{ and } Av_1 \perp Av_2 \text{ because } u_1 \perp u_2.}$

(c) Find a matrix B that is similar to A (but different from A).

Show that A and B meet the requirement to be similar *(what is it?)*.

we say $B \underset{\text{similar}}{\sim} A$ if $B = MAM^{-1}$ for some invertible M.

choose for example $M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. (You can choose any M you like!)

Then $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \sqrt{2} & 1 \\ 0 & \sqrt{2} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1} = \cdots = \begin{bmatrix} \sqrt{2} & 0 \\ 1 & \sqrt{2} \end{bmatrix}$

and $B \neq A$ but $B \sim A$!

3. Suppose A is a real m by n matrix.

(a) Prove that the symmetric matrix A^TA has the property $x^T(A^TA)x \geq 0$ for every vector x in R^n. Explain each step in your reason.

$x^T (A^T A) x = (x^T A^T) A x = (Ax)^T Ax = (Ax) \cdot (Ax) \geqslant 0.$

(b) According to part (a), the matrix A^TA is positive semidefinite at least — and possibly positive definite. Under what condition on A is A^TA positive definite?

we want to see under what condition

$x^T (A^TA) x = 0$ implies $x = 0$.

So let $x^T A^T A x = 0$. By (a) we get $(Ax) \cdot (Ax) = 0$.

So $Ax = 0$. Hence to get $x = 0$ from $Ax = 0$,

we need $N(A) = \{0\}$ or A must have independent columns.

(c) If $m < n$ prove that A^TA is *not* positive definite.

we use (b) and show that if $m < n$ then $N(A) \neq \{0\}$.

OK! we know that $\dim N(A) = n - r$ where $r = \text{rank}(A)$.

But $r \leq m$. So

$\dim N(A) = n - r \geqslant n - m$. Since $m < n$, $n - m > 0$

therefore: $N(A) \neq \{0\}$.

18.06 Quiz 3 May 8, 2010 Professor Strang

Your PRINTED name is: ____ **1.**

Your recitation number is ______________________________ **2.**

3.

1. **(40 points)** Suppose u is a unit vector in R^n, so $u^T u = 1$. This problem is about the n by n symmetric matrix $H = I - 2u\,u^T$.

   (a) Show directly that $H^2 = I$. Since $H = H^T$, we now know that H is not only symmetric but also ________________________.

   (b) One eigenvector of H is u itself. Find the corresponding eigenvalue.

   (c) If v is any vector perpendicular to u, show that v is an eigenvector of H and **find the eigenvalue.** With all these eigenvectors v, that eigenvalue must be repeated how many times? Is H **diagonalizable**? Why or why not?

   (d) Find the diagonal entries H_{11} and H_{ii} in terms of $u_1, \ldots, u_n$. Add up $H_{11} + \ldots + H_{nn}$ and separately add up the eigenvalues of H.

2. **(30 points)** Suppose A is a positive definite symmetric n by n matrix.

(a) How do you know that A^{-1} is also positive definite? (We know A^{-1} is symmetric. I just had an e-mail from the International Monetary Fund with this question.)

(b) Suppose Q is any **orthogonal** n by n matrix. How do you know that $QAQ^T = QAQ^{-1}$ is positive definite? Write down which test you are using.

(c) Show that the block matrix

$$B = \begin{bmatrix} A & A \\ A & A \end{bmatrix}$$

is positive **semidefinite.** How do you know B is not positive definite?

3. **(30 points)** This question is about the matrix

$$A = \begin{bmatrix} 0 & -1 \\ 4 & 0 \end{bmatrix}$$

.

(a) Find its eigenvalues and eigenvectors.

Write the vector $u(0) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ as a combination of those eigenvectors.

(b) Solve the equation $\frac{du}{dt} = Au$ starting with the same vector $u(0)$ at time $t = 0$.

In other words: the solution $u(t)$ is what combination of the eigenvectors of A?

(c) Find the 3 matrices in the Singular Value Decomposition $A = U\,\Sigma\,V^T$ in two steps.

–First, compute V and Σ using the matrix $A^T A$.

–Second, find the (orthonormal) columns of U.

18.06 Quiz 3 Solutions May 8, 2010 Professor Strang

Your PRINTED name is: ______________________ 1.

Your recitation number is ______________ 2.

3.

1. **(40 points)** Suppose u is a unit vector in R^n, so $u^Tu = 1$. This problem is about the n by n symmetric matrix $H = I - 2uu^T$.

(a) Show directly that $H^2 = I$. Since $H = H^T$, we now know that H is not only symmetric but also ______________.

$\boxed{\text{Solution}}$ Explicitly, we find $H^2 = (I - 2uu^T)^2 = I^2 - 4uu^T + 4uuu^Tuu^T$ (2 points): since $u^Tu = 1$, $H^2 = I$ (3 points). Since $H = H^T$, we also have $H^TH = 1$, implying that H is an orthogonal (or unitary) matrix.

(b) One eigenvector of H is u itself. Find the corresponding eigenvalue.

$\boxed{\text{Solution}}$ Since $Hu = (I - 2uu^T)u = u - 2uu^Tu = u - 2u = -u$, $\lambda = -1$.

(c) If v is any vector perpendicular to u, show that v is an eigenvector of H and **find the eigenvalue.** With all these eigenvectors v, that eigenvalue must be repeated how many times? Is H **diagonalizable**? Why or why not?

$\boxed{\text{Solution}}$ For any vector v orthogonal to u (i.e. $u^Tv = 0$), we have $Hv = (I - 2uu^T)v = v - 2uu^Tv = v$, so the associated λ is 1. The orthogonal complement to the space spanned by u has dimension $n-1$, so there is a basis of $(n-1)$ orthonormal eigenvectors with this eigenvalue. Adding in the eigenvector u, we find that H is diagonalizable.

(d) Find the diagonal entries H_{11} and H_{ii} in terms of $u_1, \ldots, u_n$. Add up $H_{11} + \ldots + H_{nn}$ and separately add up the eigenvalues of H.

$\boxed{\text{Solution}}$ Since ith diagonal entry of uu^T is u_i^2, the i diagonal entry of H is $H_{ii} = 1 - 2u_i^2$ (3 points). Summing these together gives $\sum_{i=1}^n H_{ii} = n - 2\sum_{i=1}^n u_i^2 = n - 2$ (3 points). Adding up the eigenvalues of H also gives $\sum \lambda_i = (1) - 1 + (n-1)(1) = n - 2$ (4 points).

2. **(30 points)** Suppose A is a positive definite symmetric n by n matrix.

(a) How do you know that A^{-1} is also positive definite? (We know A^{-1} is symmetric. I just had an e-mail from the International Monetary Fund with this question.)

$\boxed{\text{Solution}}$ Since a matrix is positive-definite if and only if all its eigenvalues are positive (5 points), and since the eigenvalues of A^{-1} are simply the inverses of the eigenvalues of A, A^{-1} is also positive definite (the inverse of a positive number is positive) (5 points).

(b) Suppose Q is any **orthogonal** n by n matrix. How do you know that $QAQ^T = QAQ^{-1}$ is positive definite? Write down which test you are using.

$\boxed{\text{Solution}}$ Using the energy text ($x^TAx > 0$ for nonzero x), we find that $x^TQAQ^Tx = (Q^Tx)^TA(Q^Tx) > 0$ for all nonzero x as well (since Q is invertible). Using the positive eigenvalue test, since A is similar to QAQ^{-1} and similar matrices have the same eigenvalues, QAQ^{-1} also has all positive eigenvalues. (5 points for test, 5 points for application)

(c) Show that the block matrix

$$B = \begin{bmatrix} A & A \\ A & A \end{bmatrix}$$

is positive **semidefinite**. How do you know B is not positive definite?

$\boxed{\text{Solution}}$ First, since B is singular, it cannot be positive definite (it has eigenvalues of 0). However, the pivots of B are the pivots of A in the first n rows followed by 0s in the remaining rows, so by the pivot test, B is still semi-definite. Similarly, the first n upper-left determinants of B are the same as those of A, while the remaining ones are 0s, giving another proof. Finally, given a nonzero vector

$$u = \begin{bmatrix} x \\ y \end{bmatrix}$$

where x and y are vectors in $\mathbf{R}^n$, one has $u^TBu = (x+y)^TA(x+y)$ which is nonnegative (and zero when $x + y = 0$).

3. **(30 points)** This question is about the matrix

$$A = \begin{bmatrix} 0 & -1 \\ 4 & 0 \end{bmatrix}$$

(a) Find its eigenvalues and eigenvectors.

Write the vector $u(0) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ as a combination of those eigenvectors.

$\boxed{\text{Solution}}$ Since $\det(A - \lambda I) = \lambda^2 + 4$, the eigenvalues are $2i, -2i$ (4 points). Two associated eigenvectors are $[1 \quad -2i]^T, [1 \quad 2i]^T$, though there are many other choices (4 points). $u(0)$ is just the sum of these two vectors (2 points).

(b) Solve the equation $\frac{du}{dt} = Au$ starting with the same vector $u(0)$ at time $t = 0$.

In other words: the solution $u(t)$ is what combination of the eigenvectors of A?

$\boxed{\text{Solution}}$ One simply adds in factors of $e^{\lambda t}$ to each term, giving

$$u(t) = e^{2it} \begin{bmatrix} 1 \\ -2i \end{bmatrix} + e^{-2it} \begin{bmatrix} 1 \\ 2i \end{bmatrix}$$

(c) Find the 3 matrices in the Singular Value Decomposition $A = U \Sigma V^T$ in two steps.

–First, compute V and Σ using the matrix $A^T A$.

–Second, find the (orthonormal) columns of U.

$\boxed{\text{Solution}}$ Note that $A^T A = V \Sigma^T U^T U \Sigma V^T = V \Sigma^2 V^T$, so the diagonal entries of Σ are simply the positive roots of the eigenvalues of

$$A^T A = \begin{bmatrix} 0 & 4 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} 16 & 0 \\ 0 & 1 \end{bmatrix}$$

i.e. $\sigma_1 = 4, \sigma_2 = 1$. Since $A^T A$ is already diagonal, V is the identity matrix. The columns of U should satisfy $Au_1 = \sigma_1 v_1, Au_2 = \sigma_2 v_2$: by inspection, one obtains

$$u_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, u_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, U = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

18.06 Professor Johnson Quiz 3 May 1, 2009

Your PRINTED name is: ____________________

Please circle your recitation:

(R01)	M2	2-314	Qian Lin
(R02)	M3	2-314	Qian Lin
(R03)	T11	2-251	Martina Balagovic
(R04)	T11	2-229	Inna Zakharevich
(R05)	T12	2-251	Martina Balagovic
(R06)	T12	2-090	Ben Harris
(R07)	T1	2-284	Roman Bezrukavnikov
(R08)	T1	2-310	Nick Rozenblyum
(R09)	T2	2-284	Roman Bezrukavnikov

Grading

1 ______

2 ______

3 ______

4 ______

Total: ______

1 (20 pts.) For each part, give **as much information as possible** about the **eigenvalues** of the matrix A described in that part. (Each part describes a *different* matrix A. A may be complex.)

(a) The recurrence $\mathbf{u}_{k+1} = A\mathbf{u}_k$ has a solution where $\|\mathbf{u}_k\| \to 0$ as $k \to \infty$ for one initial vector $\mathbf{u}_0$, but also has a solution with $\|\mathbf{u}_k\| \to \infty$ as $k \to \infty$ for a *different* choice of the initial vector $\mathbf{u}_0$.

(b) The equation $(A^2 - 4I)\mathbf{x} = \mathbf{b}$ has no solution for some right-hand side $\mathbf{b}$.

(c) $A = e^{B^T B}$ for some real matrix B with full column rank.

(d) $A = B^T B$ for a 4×3 real matrix B, and the matrix BB^T has eigenvalues $\lambda = 3, 2, 1, 0$. (Hint: think about the SVD of B.)

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2 (20 pts.) You are given the matrix

$$A = \begin{pmatrix} 0.5 & 0.2 & 0.2 \\ 0.1 & 0.5 & 0.5 \\ 0.4 & 0.3 & 0.3 \end{pmatrix}.$$

(i) What are the eigenvalues of A? [*Hint:* Very little calculation required! You should be able to see two eigenvalues by inspection of the form of A, and the third by an easy calculation. You *shouldn't* need to compute $\det(A - \lambda I)$ unless you really want to do it the hard way.]

(ii) The vector $\mathbf{u}(t)$ solves the system

$$\frac{d\mathbf{u}}{dt} = A\mathbf{u}$$

for some initial condition $\mathbf{u}(0)$. If you are told that $\mathbf{u}(t)$ approaches some constant vector as $t \to \infty$, give as much true information as possible regarding the initial condition $\mathbf{u}(0)$.

[*Note:* be sure you understand that this is *not the same thing* as solving the recurrence $\mathbf{u}_{k+1} = A\mathbf{u}_k$! Imagine how you would find $\mathbf{u}(t)$ if you knew what $\mathbf{u}(0)$ was.]

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3 (10 pts.) The 3×3 matrix A has three independent eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ with corresponding eigenvalues λ_1, λ_2, and λ_3 (that is, $A\mathbf{v}_i = \lambda_i \mathbf{v}_i$ for $i = 1, 2, 3$).

If

$$\mathbf{b} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$$

for some coefficients c_1, c_2, and c_3, then write (in terms of λ_i, c_i, and $\mathbf{v}_i$) a formula for the solution $\mathbf{x}$ of

$$A^2\mathbf{x} + 2A\mathbf{x} - 3I\mathbf{x} = \mathbf{b}$$

(you can assume that a solution exists for any $\mathbf{b}$).

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4 (15 pts.) A is a 3×3 real-symmetric matrix. Two of its eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = -1$ with eigenvectors $\mathbf{v}_1 = (1, 1, 1)$ and $\mathbf{v}_2 = (1, -1, 0)$, respectively. The third eigenvalue is $\lambda_3 = 0$.

(I) Give an eigenvector $\mathbf{v}_3$ for the eigenvalue λ_3. (*Hint:* what must be true of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$?)

(II) Using your result from (I), write the matrix e^A as the product of three matrices, and explicitly give the three matrices. (You need not work out the arithmetic, but your answer should contain no matrix inverses or matrix exponentials. *If you find yourself doing a lot of arithmetic, you are forgetting a useful property of this matrix!*)

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18.06 Spring 2009 Exam 3 Practice

General comments

Exam 2 covers the first 31 lectures of 18.06, mainly focusing on lectures 19–31 (eigenproblems). The topics covered are (very briefly summarized):

1. All of the topics from exams 1 and 2, although of course these are not the focus of the exam.

2. Determinants: their properties, how to compute them (simple formulas for 2×2 and 3×3, usually by elimination for matrices $> 3 \times 3$), their relationship to linear equations (zero determinant = singular), their use for eigenvalue problems.

3. Eigenvalues and eigenvectors: their definition $\boxed{A\vec{x} = \lambda\vec{x}}$, their properties, the fact that for an eigenvector the matrix (or *any function of the matrix*) acts just like a number. Computing λ from the characteristic polynomial $\det(A - \lambda I)$ and $\vec{x}$ from $N(A - \lambda I)$; zero eigenvalues $\lambda = 0$ just correspond to $N(A)$. Understand (from the definition) why, if A has an eigenvalue λ, then A^k has an eigenvalue λ^k, αA has an eigenvalue $\alpha\lambda$, and $A + \beta I$ has an eigenvalue $\lambda + \beta$, all with the *same* eigenvector.

4. Diagonalization $A = S\Lambda S^{-1}$: where it comes from, its use in understanding properties of matrices and eigenvalues. **The basic idea that, to solve a problem involving A, you first expand your vector in the basis of the eigenvectors (S), then for each eigenvector you treat A as just a number λ, then at the end you add up the solutions.**

5. Similar matrices: A and $B = MAM^{-1}$ have the same eigenvalues for any invertible matrix M, and if $A\vec{x} = \lambda\vec{x}$ then $B\vec{y} = \lambda\vec{y}$ for $\vec{y} = M\vec{x}$. Similar matrices have the same trace (sum of the eigenvalues) and determinant (product of the eigenvalues).

6. Using eigenvalues/eigenvectors to solve problems involving matrix powers, such as linear recurrences (e.g. Fibonacci). Multiplying by A many times tends towards the eigenvector for the largest $|\lambda|$. Markov matrices: what the defining properties are, and the consequences (a steady state with $\lambda = 1$, all other solutions decay away, the sum of the components of the vector is conserved, a unique steady state if all entries of the matrix are > 0). $A^n = S\Lambda^n S^{-1}$.

7. Using eigenvalues/eigenvectors to solve linear systems of differential equations $\frac{d\vec{u}}{dt} = A\vec{u}$ with initial conditions $\vec{u}(0)$. Practical scheme: expand $\vec{u}(0)$ in eigenvector basis and multiply each term by $e^{\lambda t}$. Formal solution: $e^{At}\vec{u}(0)$, and the meaning of the matrix exponential $e^A = Se^{\Lambda}S^{-1}$ and how to compute it and manipulate it.

8. If $A = A^T$ (real-symmetric), then the eigenvalues are real and the eigenvectors are orthogonal (or can be chosen orthogonal), and A is diagonalizable as $A = Q\Lambda Q^T$ for an orthogonal Q. If $A = B^T B$ where B has full column rank, then A is positive definite: all $\lambda > 0$ and all pivots > 0 and $\vec{y}^T A\vec{y} > 0$ for any $\vec{y} \neq 0$; connection to minimization problems (like least-squares).

9. Complex matrices, for which we replace $\vec{x}^T$ and A^T by and $\vec{x}^H = \overline{\vec{x}^T}$ $A^H = \overline{A^T}$ (and why). What to do if you get a complex λ: consequences for matrix powers (recurrence relations) and differential equations are oscillating solutions, using $e^{i\theta} = \cos\theta + i\sin\theta$.

10. Defective matrices and generalized eigenvectors: what to do if A is not diagonalizable, especially for a practical problem like $A^k\vec{u}$ or $e^{At}\vec{u}$. (Note that real-symmetric, real-orthogonal, Hermitian, and unitary matrices are never defective, nor are $n \times n$ matrices with n distinct eigenvalues.)

11. Singular value decomposition $A = U\Sigma V^T$ and their relationship to eigenvectors/eigenvalues of A^TA and AA^T.

The central concept from this part of the course is highlighted in boldface above. Once you have an eigenvector, *any* operation involving the matrix just becomes that operation with the single number λ. And single numbers are easy to handle. So, we try to find the eigenvectors and then express every vector in that basis (aside from rare defective cases), at which point problems become easy (ideally)! Also, you should be able to recognize and reason about how and why special forms of the matrix A (symmetric, Markov, singular, etcetera) give you additional information about the eigenvectors and eigenvalues.

Defective matrices and SVDs will see at most limited coverage on the exam, perhaps one part of a problem each, at most.

Some practice problems

The 18.06 web site has exams from previous terms that you can download, with solutions. I've listed a few practice exam problems that I like below, but there are plenty more to choose from. The exam will consist of 3 or 4 questions (perhaps with several parts each), and you will have one hour. You can find the solutions to these problems on the 18.06 web site (in the section for old exams/psets).

1. (Fall 2002 exam 3.) **(a)** What are the eigenvalues of the 5×5 matrix $A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$? Please look at A, not at $\det(A - \lambda I)$. **(b)** Solve $\frac{d\vec{u}}{dt} = A\vec{u}$ starting from $\vec{u}(0) = (0,1,1,1,2)^T$. (First split $\vec{u}(0)$ as the sum of two eigenvectors of A.) **(c)** Using part (a), what are the *eigenvalues* and *trace* and *determinant* of the matrix B which is the same as A except that it has zeros on its diagonal?
2. (Fall 2002 exam 3.) **(a)** if A is similar to B show that e^A is similar to e^B. (Hint: first write down the definitions of "similar" and e^A.) **(b)** If A has 3 eigenvalues $\lambda = 0,2,4$, find the eigenvalues of e^A. **(c)** Explain this connection with determinants: $\det(e^A) = e^{\text{trace of } A}$.
3. (Fall 2002 exam 3.) Companies in the US, Asia, and Europe have assets of \$12 trillion. At the start, \$6 trillion are in the US and \$6 trillion are in Europe. Each year, half the US money stays home, 1/4 each goes to Asia and Europe. For Asia and Europe, half stays home and half is sent to the US, hence
$$\begin{pmatrix} \text{US} \\ \text{Asia} \\ \text{Europe} \end{pmatrix}_{\text{year } k+1} = \begin{pmatrix} .5 & .5 & .5 \\ .25 & .5 & 0 \\ .25 & 0 & .5 \end{pmatrix} \begin{pmatrix} \text{US} \\ \text{Asia} \\ \text{Europe} \end{pmatrix}_{\text{year } k+1}.$$
(a) The eigenvalues and eigenvectors of this *singular* matrix A are what? **(b)** The limiting distribution of the \$12 trillion after many many years is US=?, Asia=?, Europe=?
4. (Fall 2002 exam 2.) If you know that $\det A = 6$, what is $\det B$ for B given by:
$$A = \begin{pmatrix} \text{row 1} \\ \text{row 2} \\ \text{row 3} \end{pmatrix} \qquad B = \begin{pmatrix} \text{row 3 + row 2 + row 1} \\ \text{row 2 + row 1} \\ \text{row 1} \end{pmatrix}$$
5. (Spring 2004 exam 3.) For the symmetric matrix $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{pmatrix}$, you are given that one of the eigenvalues is $\lambda = 1$ with a line of eigenvectors $\vec{x} = (c,c,0)$. **(a)** That line is the nullspace of what matrix constructed from A? **(b)** Find (in any way) the other two eigenvalues of A and two corresponding eigenvectors. **(c)** The diagonalization $A = S\Lambda S^{-1}$ has an especially nice form because $A = A^T$. Write all entries in the nice symmetric diagonalization of A. **(d)** Give a reason why e^A *is* or *is not* a symmetric positive-definite matrix.

6. (Spring 2004 exam 3.) **(a)** Find the eigenvalues and eigenvectors (depending on c) of $A = \begin{pmatrix} 0.3 & c \\ 0.7 & 1-c \end{pmatrix}$. For which c is the matrix A *not diagonalizable*?[1] **(b)** What is the largest range of (real) values of c so that A^n approaches a limiting matrix A^∞ as $n \to \infty$? **(c)** What is that limit of A^n (still depending on c)?

7. (Spring 2005 exam 3.) **(a)** Find all the eigenvalues and all the eigenvectors of the following A. *It is a symmetric Markov matrix with a repeated eigenvalue.*

$$A = \begin{pmatrix} 2/4 & 1/4 & 1/4 \\ 1/4 & 2/4 & 1/4 \\ 1/4 & 1/4 & 2/4 \end{pmatrix}.$$

(b) Find the limit of A^k as $k \to \infty$. **(c)** Choose any positive numbers r, s, and t so that $A - rI$ is positive-definite, $A - sI$ is indefinite, and $A - tI$ is negative definite. **(d)** Suppose that this $A = B^T B$. What are the singular values σ_i in the SVD of B?

8. (Spring 2005 exam 3.) **(a)** Complete this 2×2 matrix A, depending on the real number a, so that its eigenvalues are $\lambda = 1$ and $\lambda = -1$. $A = \begin{pmatrix} a & 1 \\ ? & ? \end{pmatrix}$. **(b)** How do you know that A has two independent eigenvectors? **(c)** Which choices of a give orthogonal eigenvectors and which don't?

9. (Spring 2005 exam 3.) Suppose that the 3×3 matrix A has 3 independent eigenvectors $\vec{x}_{1,2,3}$ and corresponding eigenvalues $\lambda_{1,2,3}$. (The λ's might not be different.) **(a)** Describe the general form of every solution $\vec{u}(t)$to the differential equation $\frac{d\vec{u}}{dt} = A\vec{u}$ in terms of the λ's and $\vec{x}$'s. (The answer $e^{At}\vec{u}(0)$ is not sufficient.) **(b)** Starting from any vector $\vec{u}_0$, suppose $\vec{u}_{k+1} = A\vec{u}_k$. What are the conditions on the $\vec{x}$'s and λ's to guarantee that $\vec{u}_k \to 0$ as $k \to \infty$? Why?

10. (Fall 2005 exam 3.) This 4×4 matrix H is a special matrix called a "Hadamard matrix:"

$$H = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

It has two key properties: $H^T = H$, and $H^2 = 4I$. **(a)** Figure out the eigenvalues of H and explain your reasoning. **(b)** Figure out H^{-1} and $\det H$. Explain your reasoning. **(c)** This matrix S contains three eigenvectors of H. Find a 4-th eigenvector $\vec{x}_4$ and explain your reasoning.

$$S = \begin{pmatrix} 1 & 1 & 0 & ? \\ 1 & 0 & -1 & ? \\ 1 & 0 & 1 & ? \\ -1 & 1 & 0 & ? \end{pmatrix}.$$

e**(d)** Find the solution to $d\vec{u}/dt = H\vec{u}$ given that $\vec{u}(0)$ is the 3rd column of S.

11. (Fall 2005 exam 3.) Suppose A is a 3×3 symmetric matrix with eigenvalues $2, 5, 7$ and corresponding eigenvectors $\vec{x}_1$, $\vec{x}_2$, and $\vec{x}_3$. **(a)** Suppose $\vec{x}$ is a linear combination $\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2 + c_3\vec{x}_3$. Find $A\vec{x}$. Now find $\vec{x}^T A\vec{x}$ using the symmetry of A. Explain why $\vec{x}^T A\vec{x} > 0$ unless $\vec{x} = 0$.

12. (Fall 2006 exam 3.) **(a)** Find all three eigenvalues of A, and an eigenvector matrix S. $A = \begin{pmatrix} -1 & 2 & 4 \\ 0 & 0 & 5 \\ 0 & 0 & 1 \end{pmatrix}$. **(b)** Explain why $A^{1001} = A$. Is $A^{1000} = I$? **(c)** The matrix $A^T A$ for this A is $A^T A = \begin{pmatrix} 1 & -2 & -4 \\ -2 & 4 & 8 \\ -4 & 8 & 42 \end{pmatrix}$.How many

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[1] The solutions are a little tricky. A is *not* a Markov matrix because c may be < 0. However, its columns sum to 1, and that was enough to give us an eigenvalue $\lambda = 1$ in our analysis of Markov matrices. In the non-diagonalizable case, the solution's formula for part (b) is incorrect. As we know from lecture 30, for a repeated eigenvalue $\lambda = 1$ that is defective, there is a term in A^n that goes as 1^n and another term that goes as $n\,1^{n-1}$. Since the latter blows up, the defective case does not have a finite A^∞ limit.

eigenvalues of A^TA are positive? zero? negative? Does A^TA have the same eigenvectors as A? (Don't compute anything, but explain your answers.)

13. (Fall 2006 exam 3.) Suppose the $n \times n$ matrix A has n orthonormal eigenvectors $\vec{q}_1, \ldots, \vec{q}_n$ and n positive eigenvalues $\lambda_1, \ldots, \lambda_n$. That is, $A\vec{q}_j = \lambda_j \vec{q}_j$. **(a)** What are the eigenvalues and eigenvectors of A^{-1}? **(b)** Any vector $\vec{b}$ can be written as a combination of the eigenvectors $\vec{b} = c_1\vec{q}_1 + c_2\vec{q}_2 + \cdots + c_n\vec{q}_n$ for some coefficients c_j. What is a quick formula for c_1 using the orthogonality of the $\vec{q}$'s? **(c)** The solution to $A\vec{x} = \vec{b}$ is also a combination of the eigenvectors $A^{-1}\vec{b} = d_1\vec{q}_1 + d_2\vec{q}_2 + \cdots + d_n\vec{q}_n$. What is a quick formula for d_1. (You can write it in terms of the c's even if you didn't answer part b.)

14. (Fall 2007 exam 3.) Suppose that we form a sequence of real numbers f_k defined by the recurrence relation $f_{k+1} = f_k - f_{k-1} + f_{k-2}$, starting with the initial numbers $f_0 = 2$, $f_1 = 1$, and $f_2 = 2$. **(a)** Define a 3-component vector $\vec{g}_k = (f_k, f_{k-1}, f_{k-2})$ and a 3×3 matrix A so that $\vec{g}_{k+1} = A\vec{g}_k$. **(b)** If you constructed A correctly, the three eigenvalues should be 1 and $\pm i$, and the latter two eigenvectors should be $(-1, \pm i, 1)$. Check that you have these $\pm i$ eigenvalues and eigenvectors, and find the $\lambda = 1$ eigenvector. **(c)** Give an explicit formula for f_k for any k (formulas involving A^k are not acceptable; elementary arithmetic and powers of complex numbers only). **(d)** Is there any choice of initial conditions (f_0, f_1, and f_2) that will make $|f_k|$ diverge as $k \to \infty$? Explain.

15. (Fall 2007 exam 3.) Some 3×3 real matrix A has eigenvalues $\lambda_1 = 0$, $\lambda_2 = 1$, and $\lambda_3 = 2$, with corresponding eigenvectors $\vec{x}_1 = (1,0,0)$, $\vec{x}_2 = (0,1,2)$, and $\vec{x}_3 = (0,1,1)$. **(a)** Give a basis for the nullspace $N(A)$, the column space $C(A)$, and the row space $C(A^T)$. **(b)** Find *all* solutions (the complete solution) $\vec{x}$ to $A\vec{x} = \vec{x}_2 - 3\vec{x}_3$. **(c)** Is A real-symmetic, orthogonal, Markov, or none of the above?

18.06 Quiz 3 Solution

Hold on Friday, 1 May 2009 at 11am in Walker Gym.
Total: 65 points.

Problem 1:

For each part, give **as much information as possible** about the **eigenvalues** of the matrix A described in that part. (Each part describes a *different* matrix A. A may be complex.)

(a) The recurrence $\mathbf{u}_{k+1} = A\mathbf{u}_k$ has a solution where $\|\mathbf{u}_k\| \to 0$ as $k \to \infty$ for one initial vector $\mathbf{u}_0$, but also has a solution with $\|\mathbf{u}_k\| \to \infty$ as $k \to \infty$ for a *different* choice of the initial vector $\mathbf{u}_0$.

(b) The equation $(A^2 - 4I)\mathbf{x} = \mathbf{b}$ has no solution for some right-hand side $\mathbf{b}$.

(c) $A = e^{B^TB}$ for some real matrix B with full column rank.

(d) $A = B^TB$ for a 4×3 real matrix B, and the matrix BB^T has eigenvalues $\lambda = 3, 2, 1, 0$. (Hint: think about the SVD of B.)

$\boxed{\text{Solution}}$ (20 points = 5+5+5+5)

(a) (There was a bug in this problem: in the first condition, we should have required the initial vector $\mathbf{u}_0$ to be nonzero.) The first condition implies that A has an eigenvalue with absolute value $|\lambda| < 1$. The second condition implies that either A has an eigenvalue with absolute value $|\lambda| > 1$, or A is defective for 2 eigenvalues λ with $|\lambda| = 1$.

(b) The condition says that $A^2 - 4I$ is singular. But we know that, if $\lambda_1, \dots, \lambda_n$ are eigenvalues of A, then the eigenvalues of $A^2 - 4I$ are $\lambda_1^2 - 4, \dots, \lambda_n^2 - 4$. The condition $A^2 - 4I$ being singular says that one of $\lambda_i^2 - 4$ is zero, and hence $\lambda_i = 2$ or -2. That is to say A has an eigenvalue 2 or -2.

(c) Since B has full column rank, the eigenvalues of B^TB are postive real numbers λ_i. Hence, we know $A = e^{B^TB}$ has eigenvalues e^{λ_i}; they are real numbers bigger than 1.

(d) Since BB^T and B^TB have the same set of *nonzero* eigenvalues. So B^TB must have eigenvalues $3, 2, 1$. Moreover, since B is a 4×3 matrix, B^TB is a 3×3 matrix. Hence, $3, 2, 1$ are exactly all the eigenvalues.

Problem 2: You are given the matrix

$$A = \begin{pmatrix} 0.5 & 0.2 & 0.2 \\ 0.1 & 0.5 & 0.5 \\ 0.4 & 0.3 & 0.3 \end{pmatrix}.$$

(i) What are the eigenvalues of A? [*Hint:* Very little calculation required! You should be able to see two eigenvalues by inspection of the form of A, and the third by an easy calculation. You *shouldn't* need to compute $\det(A - \lambda I)$ unless you really want to do it the hard way.]

(ii) The vector $\mathbf{u}(t)$ solves the system

$$\frac{d\mathbf{u}}{dt} = A\mathbf{u}$$

for some initial condition $\mathbf{u}(0)$. If you are told that $\mathbf{u}(t)$ approaches some constant vector as $t \to \infty$, give as much true information as possible regarding the initial condition $\mathbf{u}(0)$.

[*Note:* be sure you understand that this is *not the same thing* as solving the recurrence $\mathbf{u}_{k+1} = A\mathbf{u}_k$! Imagine how you would find $\mathbf{u}(t)$ if you knew what $\mathbf{u}(0)$ was.]

$\boxed{\text{Solution}}$ (20 points = 10+10)

(i) First, the last two columns of A are the same. Hence A is singular and it must have an eigenvalue $\lambda_1 = 0$. Also, we observe that A is a Markov matrix. This means that $\lambda_2 = 1$ is an eigenvalue of A. Finally, we know the trace of A is the sum of its three eigenvalues. So, $\mathrm{Tr}(A) = 0.5 + 0.5 + 0.3 = 1.3$ and the last eigenvalue is $\lambda_3 = 1.3 - 1 - 0 = 0.3$.

(ii) We can write $\mathbf{u}(0) = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$ using three eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, which correspond to $\lambda_1 = 0$, $\lambda_2 = 1$, and $\lambda_3 = 0.3$, respectively. We know that this system has solution $\mathbf{u}(t) = c_1\mathbf{v}_1 + c_2e^t\mathbf{v}_2 + c_3e^{0.3t}\mathbf{v}_3$. So, if either one of c_2 and c_3 is nonzero, the system would blow up as $t \to \infty$. Therefore, the only possibility for $\mathbf{u}(t)$ to approaches some constant is to have $c_2 = c_3 = 0$, that is to say that $\mathbf{u}(0)$ is a multiple of the eigenvector $\mathbf{v}_1 = (0, -1, 1)^T$. In this case, $\mathbf{u}(t) = \mathbf{u}(0) = c_1\mathbf{v}_1$ is a constant.

Problem 3:

The 3 × 3 matrix A has three independent eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ with corresponding eigenvalues λ_1, λ_2, and λ_3 (that is, $A\mathbf{v}_i = \lambda_i \mathbf{v}_i$ for $i = 1, 2, 3$).

If

$$\mathbf{b} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$$

for some coefficients c_1, c_2, and c_3, then write (in terms of λ_i, c_i, and $\mathbf{v}_i$) a formula for the solution $\mathbf{x}$ of

$$A^2\mathbf{x} + 2A\mathbf{x} - 3I\mathbf{x} = \mathbf{b}$$

(you can assume that a solution exists for any $\mathbf{b}$).

$\boxed{\text{Solution}}$ (10 points)

Using the eigenvalues $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, we have

$$\begin{aligned}\mathbf{x} &= (A^2 + 2A - 3I)^{-1}\mathbf{b} \\ &= (A^2 + 2A - 3I)^{-1}(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3) \\ &= \frac{c_1}{\lambda_1^2 + 2\lambda_1 - 3}\mathbf{v}_1 + \frac{c_2}{\lambda_2^2 + 2\lambda_2 - 3}\mathbf{v}_2 + \frac{c_3}{\lambda_3^2 + 2\lambda_3 - 3}\mathbf{v}_3.\end{aligned}$$

Problem 4: A is a 3 × 3 real-symmetric matrix. Two of its eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = -1$ with eigenvectors $\mathbf{v}_1 = (1,1,1)^{\mathrm{T}}$ and $\mathbf{v}_2 = (1,-1,0)^{\mathrm{T}}$, respectively. The third eigenvalue is $\lambda_3 = 0$.

(I) Give an eigenvector $\mathbf{v}_3$ for the eigenvalue λ_3. (*Hint:* what must be true of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$?)

(II) Using your result from (I), write the matrix e^A as the product of three matrices, and explicitly give the three matrices. (You need not work out the arithmetic, but your answer should contain no matrix inverses or matrix exponentials. *If you find yourself doing a lot of arithmetic, you are forgetting a useful property of this matrix!*)

$\boxed{\text{Solution}}$ (15 points = 7+8)

(I) For a real-symmetric matrix, its eigenvectors are orthogonal to each other. So, by inspection, in order for $\mathbf{v}_3$ to be perpendicular to $\mathbf{v}_2$, we need its first two components same. Hence, we should take $\mathbf{v}_3$ to be $(1,1,-2)^{\mathrm{T}}$. To easy the second part, we can normalize the eigenvectors

$$\begin{aligned}\mathbf{q}_1 &= \mathbf{v}_1/\|\mathbf{v}_1\| = (1,1,1)^{\mathrm{T}}/\sqrt{3}, \\ \mathbf{q}_2 &= \mathbf{v}_2/\|\mathbf{v}_2\| = (1,-1,0)^{\mathrm{T}}/\sqrt{2}, \\ \mathbf{q}_3 &= \mathbf{v}_3/\|\mathbf{v}_3\| = (1,1,-2)^{\mathrm{T}}/\sqrt{6}.\end{aligned}$$

Alternatively, we can use Gram-Schmidt to find (a multiple of) $\mathbf{v}_3$ as follows. We start with $\mathbf{v} = (1,0,0)$,

$$\mathbf{v}_3 = \mathbf{v} - (\mathbf{v}\cdot\mathbf{q}_1)\mathbf{q}_1 - (\mathbf{v}\cdot\mathbf{q}_2)\mathbf{q}_2 = (1,0,0)^{\mathrm{T}} - \frac{1}{3}(1,1,1)^{\mathrm{T}} - \frac{1}{2}(1,-1,0)^{\mathrm{T}} = (\frac{1}{6}, \frac{1}{6}, -\frac{1}{3})^{\mathrm{T}}.$$

(II) We can write

$$\begin{aligned}A &= S\Lambda S^{-1} = Q\Lambda Q^{\mathrm{T}} \\ &= \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{6} & 1/\sqrt{6} & -2/\sqrt{6} \end{pmatrix}.\end{aligned}$$

Hence,

$$\begin{aligned}e^A &= Qe^{\Lambda}Q^{\mathrm{T}} \\ &= \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \end{pmatrix} \begin{pmatrix} e & 0 & 0 \\ 0 & 1/e & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{6} & 1/\sqrt{6} & -2/\sqrt{6} \end{pmatrix}.\end{aligned}$$

18.06 Professor Strang Quiz 3 May 7th, 2012

Your PRINTED name is: Michael Plasmeier

Grading

1 20

2 18

3 27

65

Please circle your recitation:

r01	T 11	4-159	Ailsa Keating	`ailsa`
r02	T 11	36-153	Rune Haugseng	`haugseng`
[✓ r03]	T 12	4-159	Jennifer Park	`jmypark`
r04	T 12	36-153	Rune Haugseng	`haugseng`
r05	T 1	4-153	Dimiter Ostrev	`ostrev`
r06	T 1	4-159	Uhi Rinn Suh	`ursuh`
r07	T 1	66-144	Ailsa Keating	`ailsa`
r08	T 2	66-144	Niels Martin Moller	`moller`
r09	T 2	4-153	Dimiter Ostrev	`ostrev`
r10	ESG		Gabrielle Stoy	`gstoy`

1 (33 pts.)

Suppose an $n \times n$ matrix A has n independent eigenvectors $x_1, \ldots, x_n$. Then you could write the solution to $\frac{du}{dt} = Au$ in three ways: Diff eq

$$u(t) = e^{At}u(0), \quad \text{or}$$
$$u(t) = Se^{\Lambda t}S^{-1}u(0), \quad \text{or}$$
$$u(t) = c_1e^{\lambda_1 t}x_1 + \ldots + c_ne^{\lambda_n t}x_n.$$

Here, $S = [x_1 \,|\, x_2 \,|\, \ldots \,|\, x_n]$. ← evectors

(a) From the definition of the exponential of a matrix, show why e^{At} is the same as $Se^{\Lambda t}S^{-1}$.

2

Well $A = S\Lambda S^{-1}$

$e^{At} = Se^{\Lambda t}S^{-1}$ explain this step.

Since $A \to e^{At}$ through some process
$\Lambda \to e^{\Lambda t}$

(b) How do you find $c_1, \ldots, c_n$ from $u(0)$ and S? ← $t = 0$

5

You set $u(0) = c_1 e^{\lambda_1 0}[x_1] + c_2 e^{\lambda_2 0}[x_2] + \ldots$

$$\underset{\text{given}}{u(0)} = \underset{\text{find}}{C_1}\,\underset{\text{given}}{X_1} + \underset{\text{find}}{C_2}\,\underset{\text{given}}{X_2} + \ldots$$

n eq, n unknowns (since x_i is $n \times 1$ size)

explicit formula?

a) didn't look at

(c) For this specific equation, write $u(t)$ in any one of the three forms, using *numbers* not symbols: You can choose which form.

13

$$\frac{du}{dt} = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \text{ starting from } u(0) = \begin{bmatrix} 4 \\ 3 \end{bmatrix}.$$

$$(1-\lambda)(4-\lambda)+2$$
$$4-5\lambda+\lambda^2+2=0$$
$$\lambda^2-5\lambda+6$$
$$\frac{5 \pm \sqrt{25-4\cdot 6}}{2}$$
$$\lambda = \frac{5 \pm 1}{2}$$
$$=3,2$$

$\lambda=3$
$$\begin{bmatrix} -2 & 2 \\ -1 & 1 \end{bmatrix}$$
$$-2x+2y=0$$
$$x=y$$
$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\lambda=2$
$$\begin{bmatrix} -1 & 2 \\ -1 & 2 \end{bmatrix}$$
$$-x+2y=0$$
$$2y=x$$
$$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

~~$\frac{du_1}{dt} = u_1 + 2u_2$~~

~~$\frac{du_2}{dt} = -u_1 + 4u_2$~~

$$u = c_1 e^{3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{2t}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} 4 \\ 3 \end{bmatrix} = c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$4 = c_1 + 2c_2$$
$$3 = c_1 + c_2$$
$$c_1 = 3 - c_2$$

$$4 = (3-c_2)+2c_2$$
$$= 3 + c_2$$
$$1 = c_2$$

$$c_1 = 2$$

$$\rightarrow u = 1e^{3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 2e^{2t}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

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2 (30 pts.)

This question is about the real matrix

$$A = \begin{bmatrix} 1 & c \\ 1 & -1 \end{bmatrix}, \quad \text{for} \quad c \in \mathbb{R}.$$

(a) - Find the eigenvalues of A, depending on c.

- For which values of c does A have real eigenvalues? Sym means real e values

$c=1$ certainly

$(1-\lambda)(-1-\lambda) - c$

$-1 + 0\lambda + \lambda^2 - c = 0$

$\lambda^2 - (1+c) = 0$ $\quad$ c is constant, real #

$$\lambda = \frac{0 \pm \sqrt{0^2 - 4 \cdot 1 \cdot (1+c)}}{2} = \frac{\pm\sqrt{4(1+c)}}{2}$$

So when is $-4(1+c) \geq 0$

$1+c \geq 0$

$c \geq -1$

When $c = -1$ $\lambda = 0$ which is real

8/8

$\underline{I^{-1}}$

$$\left[\begin{array}{ccc|ccc} 1 & & & 1 & & \\ & 1 & & & 1 & \\ & & 1 & & & 1 \end{array}\right]$$

$I = I^{-1}$

$\ell = \frac{1}{0}$

Swap

$$\left[\begin{array}{cc|cc} 0 & 1 & 1 & \\ 1 & 0 & & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

? Same

(b) - For one particular value of c, convince me that A is similar to both the matrix

$$B = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix},$$

$$A = \begin{bmatrix} 1 & c \\ 1 & -1 \end{bmatrix}$$

and to the matrix

$$C = \begin{bmatrix} 2 & 2 \\ 0 & -2 \end{bmatrix}.$$

- Don't forget to say which value c this happens for.

Similar $B = M^{-1} A M$ is similar to A

Were not taught a robust way to compare

What if $M = I$

Distinct e-vals ⇒ can diagonalize

~~$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} ? \begin{bmatrix} 1 & c \\ 1 & -1 \end{bmatrix}$ No~~

~~$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ No~~

You can find a M

Since λ values must be the same not sufficient

So $(2-\lambda)(-2-\lambda) = 0$

$-4 + 0\lambda + \lambda^2$

$\lambda^2 - 4$

$\frac{-b \pm \sqrt{0^2 - 4 \cdot 1 \cdot (-4)}}{2} = \pm 2$

So when $\pm 2 = \frac{\pm\sqrt{4(1+c)}}{2}$

→

+)

$4 = \sqrt{4(1+c)}$

$16 = 4(1+c)$

$4 = 1+c$

$3 = c$

-)

$-4 = -\sqrt{4(1+c)}$

same

Other

$(2-\lambda)(-2-\lambda) - 2 \cdot 0$ = same λ as before

When $c=3$

5/11

(c) For one particular value of c, convince me that A cannot be diagonalized. It is not similar to a diagonal matrix Λ, when c has that value.

- Which value c?
- Why not?

A can't be diag when repeated eig values - repeated rows cols

Like $c = -1$

$$\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

$$(1-\lambda)(-1-\lambda) + 1$$

$$-1 + 0\lambda + \lambda^2 + 1$$

$$\lambda^2 = 0$$

$$\lambda = 0, 0$$

repeated λ

Not similar to Λ Why?

5/11

18/30

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3 (37 pts.)

Oh A is not this directly $\begin{bmatrix}\lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \\ & & & \lambda_n\end{bmatrix}$ small to large

(a) Suppose A is an $n \times n$ symmetric matrix with eigenvalues $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$.

10

- What is the largest number real number c that can be subtracted from the diagonal entries of A, so that $A - cI$ is positive semidefinite?

- Why?

All eig values must be >0 for pos def

≥ 0 pos semi def

So largest c is

$\lambda_1 - c = 0$

↑ smallest λ

$\lambda_1 = c$

A-cI symm.? −3

(b) Suppose B is a matrix with independent columns.

$B = \begin{bmatrix} \quad \end{bmatrix}$ ind cols

- What is the nullspace $N(B)$?
- Show that $A = B^T B$ is positive definite. Start by saying what that means about $x^T Ax$.

Nullspace of B is

$$\begin{bmatrix} B \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The x, y such that $B = 0$

$A = B^T B$ is pos definite

We know $x^T Ax > 0$ for pos def

$Ax = \lambda x^T$

$A x^T A x = \lambda x^T A x$

positive on both sides

$x^T x = ||x||^2$ which is pos

$\lambda =$ is pos, since pos def

2

So $B^T B$

(can t rember th proof :)

15

(c) This matrix A has rank $r = 1$:

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}.$$

SVD

Similar to book! almost exactly same!

- Find its largest singular value σ from A^TA.
- From its column space and row space, respectively, find unit vectors u and v so that

$$Av = \sigma u, \quad \text{and} \quad A = u\sigma v^T.$$

- From the nullspaces of A and A^T put numbers into the full SVD (Singular Value Decomposition) of A:

$$A = \begin{bmatrix} | & & | \\ u & & \cdots \\ | & & | \end{bmatrix} \begin{bmatrix} \sigma & & 0 \\ & & \\ 0 & & \cdots \end{bmatrix} \begin{bmatrix} | & & | \\ v & & \cdots \\ | & & | \end{bmatrix}^T.$$

$\underline{AA^T}$

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1\cdot1+1\cdot1 & 1\cdot2+1\cdot2 \\ 2\cdot1+2\cdot1 & 2\cdot2+2\cdot2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix}$$

$(2-\lambda)(8-\lambda)-16$

$16-10\lambda+\lambda^2-16=0$

$\lambda^2-10\lambda=0$

$\lambda=0,10$

$\underline{\lambda=0}$ $\begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$2x+4y=0$

$x=-2y$

$\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$\underline{A^TA}$

$$\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 1\cdot1+2\cdot2 & 1\cdot1+2\cdot2 \\ 1\cdot1+2\cdot2 & 1\cdot1+2\cdot2 \end{bmatrix} = \begin{bmatrix} 5 & 5 \\ 5 & 5 \end{bmatrix}$$

$(5-\lambda)(5-\lambda)-25.$

$25-10\lambda+\lambda^2-25$

$\lambda^2-10\lambda=0$

$\lambda=0,10$

$\underline{\lambda=0}$ $\begin{bmatrix} 5 & 5 \\ 5 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$5x+5y=0$

$x=-y$

$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\boxed{\lambda=10}$ $\begin{bmatrix} -8 & 4 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$4x = 2y$

$2x = y$

$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\frac{1}{\sqrt{5}}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$

$\boxed{\lambda=10}$ $\begin{bmatrix} -5 & 5 \\ 5 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$5x = 5y$

$x = y$

$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

$$A = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} \sqrt{10} & \\ & 0 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^T$$

largest singular value $\sigma = \sqrt{10}$

~~normalize~~

~~no null space needed~~ - square or we did

√ makes sens

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18.06

Debreif

5/7

Think figured out all but the 2 proof qu
    Should be better at those
    If had more time to study
    well for final . . .

Figured out a mistake on one of the pratical qu fairly quickly at end
    L very stressful
    but think I pulled it together rather well

18.06 Professor Strang Quiz 3 – Solutions May 7th, 2012

Grading

1

2

3

Your PRINTED name is: ____________________

Please circle your recitation:

r01	T 11	4-159	Ailsa Keating	`ailsa`
r02	T 11	36-153	Rune Haugseng	`haugseng`
r03	T 12	4-159	Jennifer Park	`jmypark`
r04	T 12	36-153	Rune Haugseng	`haugseng`
r05	T 1	4-153	Dimiter Ostrev	`ostrev`
r06	T 1	4-159	Uhi Rinn Suh	`ursuh`
r07	T 1	66-144	Ailsa Keating	`ailsa`
r08	T 2	66-144	Niels Martin Moller	`moller`
r09	T 2	4-153	Dimiter Ostrev	`ostrev`
r10	ESG		Gabrielle Stoy	`gstoy`

1 (33 pts.)

Suppose an $n \times n$ matrix A has n independent eigenvectors $x_1, \dots, x_n$. Then you could write the solution to $\frac{du}{dt} = Au$ in three ways:

$$\begin{aligned} u(t) &= e^{At}u(0), \quad \text{or} \\ u(t) &= Se^{\Lambda t}S^{-1}u(0), \quad \text{or} \\ u(t) &= c_1 e^{\lambda_1 t}x_1 + \ldots + c_n e^{\lambda_n t}x_n. \end{aligned}$$

Here, $S = [x_1 \mid x_2 \mid \dots \mid x_n]$.

(a) From the definition of the exponential of a matrix, show why e^{At} is the same as $Se^{\Lambda t}S^{-1}$.

Solution. Recall that $A = S\Lambda S^{-1}$, and $A^k t^k = S\Lambda^k t^k S^{-1}$. Then, definition of the exponential:

$$\boxed{\exp(At) = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!} = S\left(\sum_{k=0}^{\infty} \frac{\Lambda^k t^k}{k!}\right)S^{-1} = Se^{\Lambda t}S^{-1}.}$$

☐

(b) How do you find $c_1, \dots, c_n$ from $u(0)$ and S?

Solution. Since $e^0 = 1$, we see that

$$u(0) = c_1 x_1 + \ldots + c_n x_n = S\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix},$$

where we used the definition of the matrix product. Thus the answer is:

$$\boxed{\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = S^{-1}u(0).}$$

☐

(c) For this specific equation, write $u(t)$ in any one of the (added: latter two of the) three forms, using *numbers* not symbols: You can choose which form.

$$\frac{du}{dt} = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix} u, \quad \text{starting from} \quad u(0) = \begin{bmatrix} 4 \\ 3 \end{bmatrix}.$$

Solution. We diagonalize A and get:

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}.$$

Thus $c = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, so for the second form

$$\boxed{u(t) = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{3t} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix},}$$

while in the third form:

$$\boxed{u(t) = e^{2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 2e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.}$$

☐

2 (30 pts.)

This question is about the real matrix

$$A = \begin{bmatrix} 1 & c \\ 1 & -1 \end{bmatrix}, \quad \text{for} \quad c \in \mathbb{R}.$$

(a) - Find the eigenvalues of A, depending on c.

- For which values of c does A have real eigenvalues?

Solution. Since $0 = \mathrm{tr}A = \lambda_1 + \lambda_2$, we see that $\lambda_2 = -\lambda_1$.

Also, $-1 - c = \det A = -\lambda_1^2$. Thus,

$$\boxed{\lambda = \pm\sqrt{1+c}.}$$

Therefore,

$$\boxed{\text{the eigenvalues are real precisely when } c \geq -1.}$$

$\square$

(b) - For one particular value of c, convince me that A is similar to both the matrix

$$B = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix},$$

and to the matrix

$$C = \begin{bmatrix} 2 & 2 \\ 0 & -2 \end{bmatrix}.$$

- Don't forget to say which value c this happens for.

Solution. If two matrices are similar, then they do have the same eigenvalues (those are $2, -2$ for both B and C). Here we must therefore have $0 = \text{tr}A$ and $-1 - c = \det A = -4$. We see that this happens precisely when $c = 3$, where we check that indeed the eigenvalues are $2, -2$. However, this does not guarantee that they are similar - and hence is not convincing.

Convincing: The eigenvalues $2, -2$ are different, so both A, B and C are diagonalizable, with the same diagonal matrix (for example to $\Lambda = B$!). Therefore A, B and C are all similar when $c = 3$. ☐

(c) For one particular value of c, convince me that A cannot be diagonalized. It is not similar to a diagonal matrix Λ, when c has that value.

- Which value c?
- Why not?

Solution. As we saw above, $\text{tr}A = 0$, so regardless of c the eigenvalues come in pairs $\lambda_2 = -\lambda_1$. This means that whenever $\lambda_1 \neq 0$, we have two different eigenvalues, and hence A is diagonalizable (not what we're after).

Thus we need $\lambda_1 = \lambda_2 = 0$, a repeated eigenvalue, which happens when $c = -1$ (so $\det A = 0$) as the only suspect – does it work?

$$\boxed{\text{Convincing: For } c = -1, \text{ we have } N(A - 0 \cdot I) = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}}$$

With only a 1-dimensional space of eigenvectors for the matrix, we are convinced that A is not diagonalizable for $c = -1$. ☐

3 (37 pts.)

(a) Suppose A is an $n \times n$ symmetric matrix with eigenvalues $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$.

- What is the largest number real number c that can be subtracted from the diagonal entries of A, so that $A - cI$ is positive semidefinite?

- Why?

Solution. - We first realize that: If A is symmetric, then $A - cI$ is also symmetric, since in general $(A + B)^T = A^T + B^T$ (simple, but very important to check!).

- Then we realize that the eigenvalues of $A - cI$ are $\lambda_1 - c \leq \lambda_2 - c \leq \ldots \leq \lambda_n - c$.

Therefore:

$$\boxed{c = \lambda_1 \text{ is the largest that can ensure positive semidefiniteness (and it does).}}$$

$\square$

(b) Suppose B is a matrix with independent columns.

- What is the nullspace $N(B)$?

- Show that $A = B^T B$ is positive definite. Start by saying what that means about $x^T Ax$.

Solution. - Then $Bx = 0$ only has the zero solution, so $N(B) = \{0\}$.

- Again, we start by observing that $A^T = A$ is symmetric. Then we recall what positive definite means (the "energy" test):

$$\boxed{x^T Ax > 0 \quad \text{whenever} \quad x \neq 0.}$$

Thus, we see here (by definition the inner product property of the transpose of a matrix):

$$\boxed{x^T Ax = x^T(B^T(Bx)) = (Bx)^T(Bx) = \|Bx\|^2 \geq 0.}$$

So $A = B^T B$ is positive semidefinite. But finally, the equality $\|Bx\|^2 = 0$, only happens when $Bx = 0$ which by $N(B) = \{0\}$ means $x = 0$. ☐

(c) This matrix A has rank $r = 1$:

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}.$$

- Find its largest singular value σ from $A^T A$.

- From its column space and row space, respectively, find unit vectors u and v so that

$$Av = \sigma u, \quad \text{and} \quad A = u\sigma v^T.$$

- From the nullspaces of A and A^T put numbers into the full SVD (Singular Value Decomposition) of A:

$$A = \begin{bmatrix} | & & | \\ u & & \cdots \\ | & & | \end{bmatrix} \begin{bmatrix} \sigma & & 0 \\ & & \\ 0 & & \cdots \end{bmatrix} \begin{bmatrix} | & & | \\ v & & \cdots \\ | & & | \end{bmatrix}^T.$$

Solution. We compute:

$$A^T A = \begin{bmatrix} 5 & 5 \\ 5 & 5 \end{bmatrix}.$$

Thus the two eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = 10$, and $\sigma = \sqrt{10}$. For v, we find a vector in $N(A^T A - 10I)$, and normalize to unit length:

$$v = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}.$$

Then we find u using

$$u = \frac{Av}{\sigma} = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}.$$

Since we have the orthogonal sums of subspaces $\mathbb{R}^2 = \mathbb{R}^m = c(A) \oplus N(A^T)$ and also $\mathbb{R}^2 = \mathbb{R}^n = c(A^T) \oplus N(A)$, we need to find one unit vector from each of $N(A)$ and $N(A^T)$ and augment to v and u, respectively:

$$v_2 = \begin{bmatrix} 1\sqrt{2} \\ -1\sqrt{2} \end{bmatrix} \in N(A),$$

$$u_2 = \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} \in N(A^T),$$

Thus, we finally see the full SVD:

$$\boxed{A = U\Sigma V^T = \begin{bmatrix} 1/\sqrt{5} & -2\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} \begin{bmatrix} \sqrt{10} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}^T.}$$

We remember, as a final check, to verify that the square matrices U and V both contain orthonormal bases of $\mathbb{R}^2$ as they should:

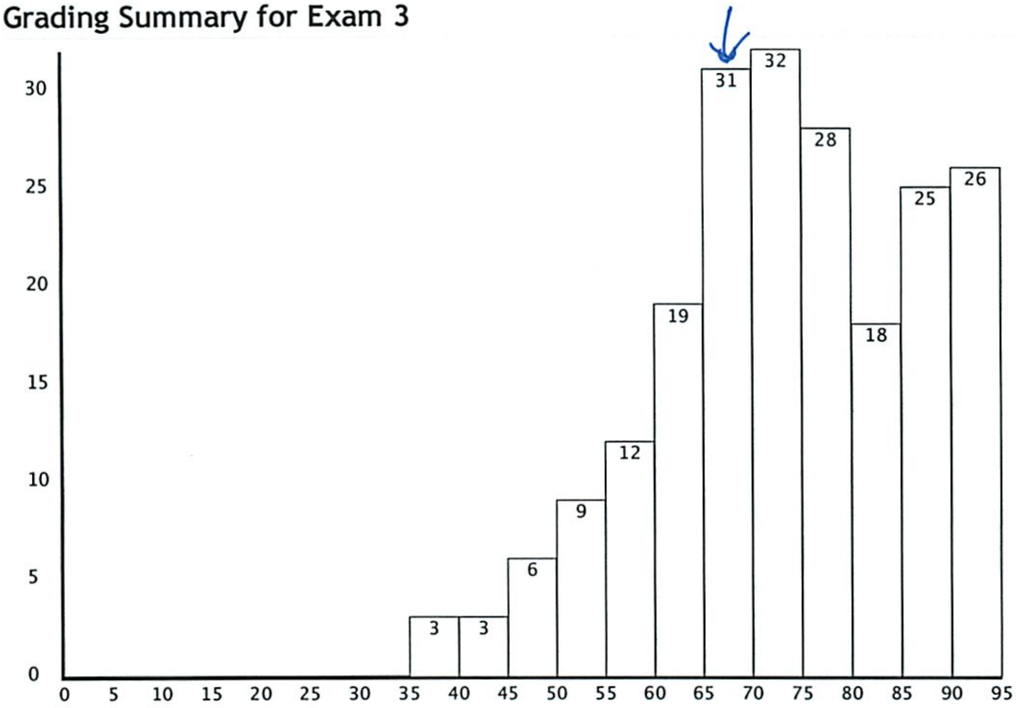
$$UU^T = I_2,$$
$$VV^T = I_2.$$

☐

18.06 Linear Algebra

Dashboard | Students | Assignments

Grading Summary for Exam 3



Number of Scores:	212
Average:	72.96
Standard Deviation:	13.43

18.06 Linear Algebra

Grade Report

Grade Report for Michael E. Plasmeier

Assignment/Exam Name	Graph	Due Date	Points	Max Pts	Weight
Homework 1	📊	02.16.2012	83.00	100.00	1.50%
Homework 2	📊	02.23.2012	65.00	100.00	1.50%
Homework 3	📊	03.01.2012	77.00	100.00	1.50%
Homework 4	📊	03.08.2012	77.00	100.00	1.50%
Exam 1	📊	03.09.2012	69.00	100.00	15.00%
Homework 5	📊	03.22.2012	83.00	100.00	1.50%
Homework 6	📊	04.05.2012	85.00	100.00	1.50%
Exam 2	📊	04.11.2012	56.00	100.00	15.00%
Homework 7	📊	04.19.2012	94.00	100.00	1.50%
Homework 8	📊	04.26.2012	86.00	100.00	1.50%
Homework 9	📊	05.03.2012	91.00	100.00	1.50%
Exam 3	📊	05.08.2012	65.00	100.00	15.00%

CUMULATIVE SCORE 39.62 / 58.50 = 67.72%

Instructor's Comments