Exam #3 return tomorrow 2-491

(only 6 people here)

Hermitian + Unitary Matrices

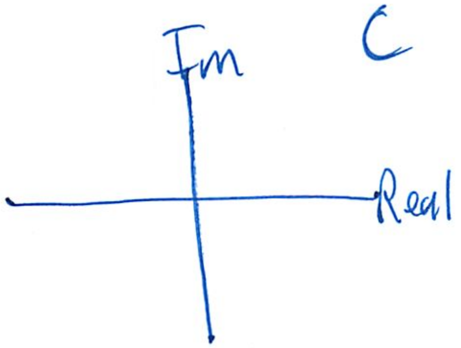
Was vector spaces over $\mathbb{R}$

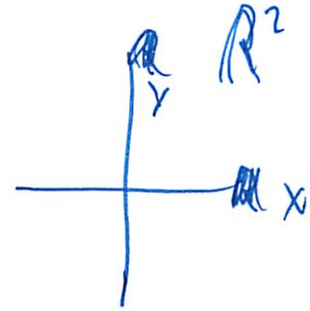
Now vector spaces over $\mathbb{C}$

$\mathbb{R}^n \to \mathbb{C}^n$

Diffs

1. Notion of transpose matrices change
   ↳ Become hermitian
2. Much harder to visualize





Things might seem unreasonable or false

$\mathbb{C}^2 \sim \mathbb{R}^4$

Each $\mathbb{C}$ has 2 dim — only for vis. $\mathbb{C}^2$ is still dim 2

②

example

a) What is the length of $V=\begin{pmatrix}1\\2\end{pmatrix}\in\mathbb{C}^2$?

$\sqrt{V^{\dagger}V}$

now $\sqrt{V^{\dagger}V}$

↑ take conjugate transpose

1. Transpose
2. switch signs of i so $i \to -i$
   $-i \to i$

For example

$\cancel{\sqrt{(1\ 2)(-i)}}$

$\sqrt{(1\ \ -i)\begin{pmatrix}1\\i\end{pmatrix}}$

↑note (+)

$= \sqrt{1-i^2}$

$= \sqrt{2}$

b) Is v ortogonal to $w=\begin{pmatrix}1\\-i\end{pmatrix}$?

↓

③

$W = V^H \cdot W = \begin{bmatrix} 1 & i \end{bmatrix} \begin{bmatrix} 1 \\ -i \end{bmatrix} = 1 + i^2$
$= 0$

So yes is orthogonal

* Always rembe to take conjugate

A square matrix is hemitan if $A^H = A$

Can skip many steps in Gram-schmidt
w/ e-vectors of Hermitan
Most are basically orthogonal already
Makes finding basis for the space easy

Say ① λ, β are 2 diff e-vales of A
$(A^H = A)$

② x, y are corresponding evectors

(4)

Prove: x, y are orthogonal

Soln $Ax = \lambda x$ $\quad\quad$ $Ay = \beta y$

Want to show orthogonal

$x \cdot y = x^H y = 0$

or $y^H x = 0$ } same thing

$y^H A x = y \lambda x$

$= \lambda y^H x$ ← can pull out λ since a #

$x^H A y = x^H \beta y$

$= \beta x^H y$

~~Should they be same? No~~

Take Hermitian of whole thing

$$\boxed{(AB)^H = B^H A^H}$$

$(x^H A y)^H = (x^H \beta y)^H$

$= (\beta x^H y)^H$

(5) $y^H A^H x = \beta y^H x$

$y^H A x$

equal to something earlier

So can compare two previous quantities

$\lambda y^H x = \beta y^H x$

So $y^H x = 0$

$(\lambda - \beta) y^H x = 0$

Review conj transpose / hermetian

$$\begin{bmatrix} 1+i & 2i \\ 0 & 2 \\ i & 3 \end{bmatrix}^H = \begin{bmatrix} 1-i & 0 & -i \\ -2i & 2 & 3 \end{bmatrix}$$

both are not =

So can't conclude vector is 0

Can cancel out

⑥

A is a <u>Unitary matrix</u> if it has orthonormal cols
L not orthonormal
Only when complex
Special term

$$A^H A = I$$

<u>Fact</u> λ of a Unitary matrix have length 1

$$Ux = \lambda x$$

$$\|Ux\|^2 = (Ux)^H (Ux)$$
$$= x^H U^H U x$$
$$= x^H I x$$
$$= \|x\|^2$$

$$\|Ux\|^2 = \|\lambda x\|^2$$
$$= |\lambda|^2 \|x\|^2$$

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Fourier Series

Why do we use? Motivation?

Like in calculus → Taylor series

Easiest to do calc in polynomials

Fav basis in Linear alg → orthonormal bases

Goal: Find orthonormal basis for the vector space of the function

~~Write each fn as~~

Can go down to just orthogonal

Vector space bit large → so restrict to fns defined $(0, 2\pi)$

But basis is set fns since in Vector space of fns

8

So when are 2 fns orthogonal → inner product

$$f \cdot g = \int_0^{2\pi} f(x)\, g(x)\, dx$$

When integrates to 0 → orthogonal

An orth basis for this space

$1, \cos x, \sin x, \cos 2x, \sin 2x$

Every func in space can be written as a linear combo of this bases

$$f(x) = a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + \dots$$

$$a_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos kx$$

$$b_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin kx$$

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(x)\, dx$$ ← avg of fn in the interval

Need orthogonal basis to find these coeffs

9

Discrete / FFT is much faster way to solve

$V_1, V_2, \dots V_n$ orth basis for V

$W = a_1 V_1 + \dots + a_n V_n$

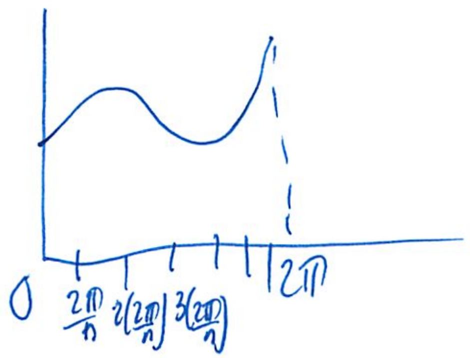
$W \cdot V_1 = (a_1 V_1 + a_2 V_2 + \dots + a_n V_n) \cdot V_1$

$= a_i \|V_i\|^2$

Instead of ∞ series

Use finite terms

└ sample only finite # of terms



↑ sample at certain pts

(10)

Want: A series $g(x) = \sum_{k=0}^{n-1} c_k e^{ikx}$

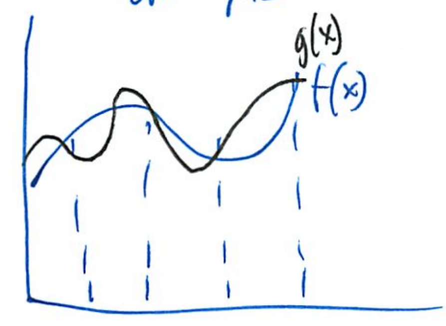
Such that $g(0) = f(0)$

$g\left(\frac{2\pi}{n}\right) = f\left(\frac{2\pi}{n}\right)$

$\vdots$

$g\left((n-1)\left(\frac{2\pi}{n}\right)\right) = f\left((n-1)\left(\frac{2\pi}{n}\right)\right)$

Has same intersections, but might be diff at pts



It turns out we need to solve for c

Discrete Fourier Trans.

$$\begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & w & w^2 & \cdots & w^{n-1} \\ 1 & w^2 & w^4 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & w^{n-1} & w^{2(n-1)} & \cdots & w^{(n-1)^2} \end{bmatrix}}_{F} \begin{bmatrix} c_0 \\ \vdots \\ c_{n-1} \end{bmatrix} \qquad w = e^{\frac{2\pi i}{n}}$$

Called discrete since only sampling discrete pts

(11)

It turns out this matrix F is easy

$\cancel{(F)^{-1} = (\ )F^H}$ $F^{-1} = \left(\frac{1}{n}\right) F^H$

? She is not sure of

But lots of multiplication

$y \cdot F^{-1} \sim O(n^2)$

So FFT

$$F_4 = \begin{bmatrix} I_2 & D_2 \\ I_2 & -D_2 \end{bmatrix} \begin{bmatrix} F_2 & \\ & F_2 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & 1 \\ & & 1 \end{bmatrix}$$

$D_2 = \text{diag}(1, w)$

So thats faster

Linear Transformation

map b/w 2 vector spaces

can expressed as a matrix

(like rotation + things)

(12)

A linear transformation $T : V \to W$ satisfies

a) $T(v_1 + v_2) = T(v_1) + T(v_2)$

b) $T(cv) = cT(v)$

c) $T(0) = 0$

(shift is not a linear transform)

Linear Transformations

Quiz 3 Me 65 ← almost the mode! Avg 72 box

Linear Transformations T gives matrix A

L can be functions or matrices

$T(c_1 v_1 + c_2 v_2 + \dots + c_n v_n)$

$= c_1 T(v_1) + c_2 T(v_2) + \dots + c_n T(v_n)$

rule/requirement of linearity

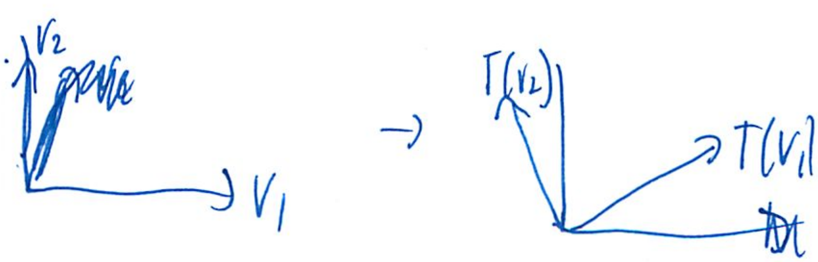
example $T(x) = Av$

Enough to know what $T(v_1), \dots, T(v_n)$

when inputs

$v_1, \dots v_n$ are a basis

Suppose T is a rotation



(2)

How to use transformations?

$c_1, \ldots, c_n$ are coordinates of v in the basis $v_1 \ldots v_n$

v in $\mathbb{R}^3$ basis v_1, v_2, v_3 in $\mathbb{R}^3$

$$v = c_1 v_1 + c_2 v_2 + c_3 v_3$$

$$\underbrace{\begin{bmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}}_{c} = \underbrace{\begin{bmatrix} v \end{bmatrix}}_{v}$$

basis so not in the same plane

$$c = B^{-1} v$$

note its the inverse matrix makes things more difficult

Change of basis stuff is a pain!

3

Applications

Choice of basis is critical

Image Processing

hair is the most difficult

black + white

every pixel $0 \to 255$

Pixel basis $\begin{bmatrix} 0 & 0 & 5 & 7 \\ - & - & - & \\ - & - & - & \\ & & & \end{bmatrix} \leftarrow$ Gray range

In the end we want to compress this stuff

Example of need to choose good basis

Must be able to invert, multiply quickly

Could make it a Fourier basis

(4)

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

↑ should be all 1s

Can capture large homogeneous areas at once

$$v_n = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ \vdots \end{bmatrix}$$ ← ossilating like crazy

but prob not desirable → noise

hope coeff small

remove when compress img

Want sins, cos

↳ for Fourier

very hard to beat for many purposes

Also wavelet basis

kinda like fourier

gives averages + differences

$$v = \begin{bmatrix} 5 \\ 7 \\ 0 \\ 2 \end{bmatrix} + c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + c_4 \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$

↑ the average $= 14/4$

(5)

True signal processing use more fine grained ~~filter~~ filters
└ diff class

How does this give a matrix A?

Construct A

1. We've chosen a basis

2. Write

$\downarrow$ inputs in $\mathbb{R}^n$ $\qquad \downarrow$ outputs in $\mathbb{R}^m$ $\leftarrow w_1, \dots, w_m$

$$Av_1 = a_{11} w_1 + a_{21} w_2 + \dots + a_{m1} w_m$$
$$Av_2 = a_{12} w_2 + a_{22} w_2 + \dots + a_{m2} w_m$$
$$\vdots$$
$$Av_n =$$

What if multiple transformations?

$$S(T(v)) \quad S \quad T \quad v$$

$k \times m \quad m \times n \quad n \times 1$

So $k \times 1$

$\hookrightarrow = z$ k dimensional space $\mathbb{R}^k$

(6)

$S(T(v))$

$c v_1 + c_2 v_2$

$c_1 T(v_1) + c_2 T(v_2)$

$c_1 S(T(v_1)) + c_2 S(T(v_2))$

Is represented by some matrix A

What is the input basis for S?

The Ws

So use Ws as output of T and input of S

Output of $S \to y$

$T(x) = Ax$

matrix that matches T

$S(w) = Bw$

$S(T(x)) = BA$

so thats why multiplied this way

(7)

$(BA)v =$

Remaining lectures about using linear algebra

Some problems here are optional - will be out on this

* This is last thing ~~on~~ on final

18.06

5/11

Applications of Lin Algebra

- One off games

Two Person Games

like Poker
applied fin math
like game theory

have x, y
payoff matrix

$$\begin{array}{cc} & \\ X & \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \\ & Y \end{array}$$

On each turn X picks a row
Y picks a col

that gives payoff

(2)

Note not a fair game

y will choose col 1 in all cases

x will ~~choose~~ want col 2 - but y will pick 0

So can't pick col 2 - needs col 1

payoff always = 1

∟ 1 is somewhat a saddle pt

x wants to maximize

y wants minimal (inc negitive)

New game

$$x \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$$

y

not so simple!

if y is still row 1, x does row 2

but then y will pick other row

x will swap back

3

No ~~pure~~ best sol

(don't get turn over)

Some mixed strategy

- will play rows randomly

X $\begin{array}{ll} x & \text{row } 1 \\ 1-x & \text{row } 2 \end{array}$

frac of time to play here

Need to find x

The key part here

$$\begin{array}{c}\begin{array}{r} x \\ 1-x \end{array}\left[\begin{array}{cc} \quad & \quad \\ \quad & \quad \end{array}\right] \\ \qquad\quad y \quad 1-y \end{array}$$

lin algebra in here

$0 \le x \le 1$

$0 \le y \le 1$

(4) key is linear algebra
└ w/ using inequalites

But if $m \times n$ matrix?
└ like in Poker
any realistic game

The best x and y

When pick x, y who wins

$1x + 3(1-x)$ ← when player y chooses col)

$2x + 0(1-x)$

So =

X wants these =, don't give y a reason to go one way or other

$1x + 3(1-x) = 2x$

$3x = 4$

$\boxed{x = 3/4}$ ← so x picks row 1 $\frac{3}{4}$ of time

(5)

$y + 2(1-y) = 3y$

$2 = 4y$

$y = \frac{1}{2}$

So what is the EV?

$\frac{3}{2}$

(missed how he found)

$= A \cdot X^T \cdot y$ happens to be as well

Linear Programming

(filming for WP)

involves vector, matrix, inequalities

Say looking for min cost

min cost $\underbrace{c^T x}_{\text{total cost}} = c_1 x_1 + \cdots + c_n x_n$

price, amt

(6)

Constraints $x_1 \geq 0 \quad \ldots \quad x_n \geq 0$

(changes lin to not just lin
So put in this constraints

But some constraints needed to live

(This reminds me of HS - but never gave it name)

$$\underset{m \times n}{A}\, x = \underset{m \times 1}{b}$$

Can be many x that satisfy this
$m < n$

~~feasible~~

feasible - one that satisfy constraints

optimal - the best $= x^*$

The central problem of optimization
- finding min or max

7

$A = \begin{bmatrix} 1 & 2 \end{bmatrix}$

Linear Programming problem

Constraint $Ax = b$

$x_1 + 2x_2 = b_1$

2 unknowns

$x_1 \geq 0$

$x_2 \geq 0$

Cost → of $x_1 = 2$

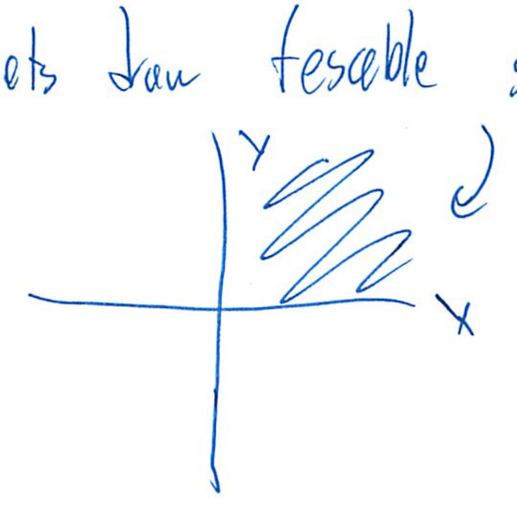
$x_2 = 5$

$= 2x_1 + 5x_2$

goal: min cost

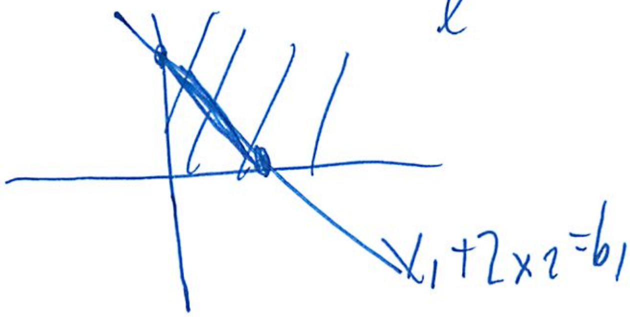
Whats the best combo of x, y

So lets draw fesable space



(8)

Then

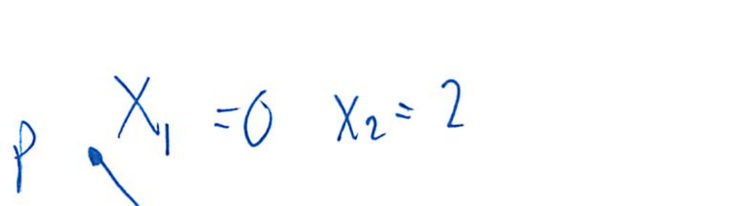


So ~~cannot~~ must be in shaded region and on line
— So anywhere on line seg

Practical problems have much larger dimensions

Who is the winner?

(good to find feasible then best)
(at least to think about)



Note winner will be one or the other

Not expecting a mixed sol

$C(x) = 2x_1 + 5x_2$

linear (straight line)

(9)

Function won't drop + come back up

$\text{Cost}(P) = 10$

$\text{Cost}(Q) = 8$

So Q is best choice

If had 100 options, but 60 constraints

40 vectors in W space

Lots of freedom

Picture in 100 D space

But best x is still at a corner

└ though exponential # of corners

Simplex method - still most popular method to solve

This was primal problem

(10)

2nd point: like game theory
lin algebra

diff player: the store
wants to $\uparrow$ profits

duality min cost = max ~~...~~ income to store
Same as 2 person game theory
example from start of lecture

Here store owner choosing c_s
(Interesting....) before buyer picking x_s

A, b, c

primal $\min c^T x$ (quantity) w/ $x \geq 0$ and $Ax = b$

dual $\max b^T y$ (prices) w/ $A^T y \leq c$

$y = \text{prices}$

$b^T y = (Ax)^T y = x^T A^T y \leq x^T c$

prove optimal income to store =
Optimal cost to buyer - duality theorem

(11)

Next examples more engineering focus

Grades

3 classes were hard

have not done #s

normally $1/3$ A

avg prob < 90

> $\frac{1}{3}$ get B

- below class avg

Some Cs

18.06 OH

5/9

Stuff not on final

FFT

don't know

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & m & m^2 & m^3 \\ 1 & m^2 & m^4 & m^6 \\ 1 & & & \\ 1 & & & \end{bmatrix} \quad m = w$$

$$\begin{bmatrix} I & d \\ I & -d \end{bmatrix} \begin{bmatrix} F & \\ & F \end{bmatrix} \begin{bmatrix} 1 & & & \\ & & 1 & \\ & 1 & & \\ & & & 1 \end{bmatrix}$$

Engineering

Lin algebra to model strings

Model displacements

②

Transforms

L is "choice of basis" in syllabus

Diff PIV on composition

Can choose basis that makes matrix simplistst

$c[\ ] + c[\ ]$

so easiest

If e-vectors - than the matrix is diagonal
as input + output

Also w/ SVD

u = input
v = output

Most confusing part: switching b/w abstract + concrete

If have matrix A

matrix function → $A \circ$ vector = vector
linear transformation

A is that matrix rel to the original basis

(3)

If want to write to another basis → matrix ~~will~~ will be diagonal

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T$

$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$A v_1 = \sigma_1 u_1$

$A v_2 = \sigma_2 u_2$

$A e_1 = a e_1 + c e_2$

$A e_2 = b e_1 + d e_2$

18.06 5/14

(5 min late)

More old finals to be posted

$$K = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 2 \end{bmatrix}$$

$n \times n$

pos def

pivots $\frac{2}{1}, \frac{3}{2}, \frac{4}{3}, \ldots$

det $2, 3, 4, \ldots$

evalues

evectors

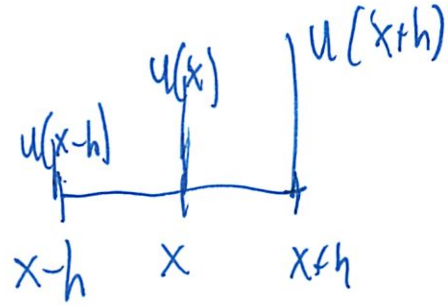
If you like lin algebra take 18.085
Comp. Sci. Engineering

Why is this matrix important

(2)

$$\frac{du}{dx} \sim ?$$

$$\frac{d^2u}{dx^2} \sim ?$$



How do you approx deriv ?
L what calculus is about

$$\frac{du}{dx} \sim \frac{u(x+h) - u(x)}{h}$$

is a better approx
Center difference

$$\sim \frac{u(x+h) - u(x-h)}{2h}$$ slope better

$$\frac{d^2u}{dx^2} \sim$$ 2nd difference
since 2nd derivative

$$\frac{u(x+h) - 2u(x) + u(x+h)}{h^2}$$

(3)

$$u_m = \sin k\pi x - k^2\pi^2 \sin k\pi x$$

So pos def case $\sim -\frac{d}{dx^2}$

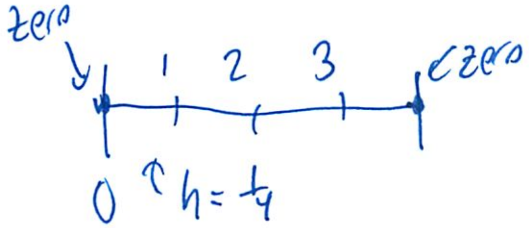
Now ready for evectors of K

$$V_k = (\sin k\pi h, \sin 2k\pi h, \dots, \sin Nk\pi h)$$

$h =$ step size

inverval $= [0, 1]$

↖chopped up into pieces of size h



$$h = \frac{1}{N+1}$$

$$V_k = \left(\sin \frac{k\pi}{N+1}, \sin \frac{2k\pi}{2N+1}, \dots, \sin \frac{Nk\pi}{N+1}\right)$$

↑vector of sins

↳ is an evector

(4)

Now use 18.06

Write as evalues + evectors

$\lambda_1, \ldots, \lambda_N$ $\quad V_1, \ldots, V_N$

real > 0

orthonormal $\rightarrow Q = [V_1 | \cdots V_n]$

sine matrix

$k =$ using ΛQ

$= Q \Lambda Q^T$

if try to break up to evectors

$$= \sum_{k=1}^{N} \lambda_k V_k V_k^T$$

$$= \begin{bmatrix} V_1 & \cdots & V_n \end{bmatrix} \begin{bmatrix} \lambda_1 V_1^T \\ \vdots \\ \lambda_N V_N^T \end{bmatrix}$$

multiply col by row

(5)

Get a rank 1 matrix

The whole matrix k

If multiply by v_1

Do we get $\lambda_1 v_1$?

$$\left(\sum_{k=1}^{N} \lambda_k v_k v_k^T\right) v_1 \stackrel{?}{=} \lambda_1 v_1$$

Vs are orthogonal

So 0 most of the time

Yes → Correct

Note for V_k didn't normalize

must divide by $\sqrt{\frac{2}{N}}$

⑥

What is $\sqrt{K}$?

$\cancel{\sqrt{K}}$

e^{-K}?

$e^{i\sqrt{K}}$?

$$\sqrt{K} = \sum_{k=1}^{N} \sqrt{\lambda_k}\, v_k v_k^T$$

what are λ_k?

$$\lambda_k = \cancel{2-2\cos k\theta}$$

$$2 - 2\cos k\theta$$

$$\theta = \frac{\theta}{N+1}$$

$\sqrt{2-2\cos k\theta}$ ← from trig?

$$\sqrt{2(1-\cos\theta)}$$

$$\sqrt{4\,\sin^2\frac{\theta}{2}}$$

$$2\sin\frac{\theta}{2}$$

$$\sum_{k=1}^{N} 2 \sin \frac{k\pi}{2(N+1)} \sin \frac{km\pi}{N+1} \sin \frac{kn\pi}{N+1}$$

$$\theta = \frac{\pi}{N+1}$$

$$\sum_{k=1}^{N} 2 \sin \frac{\theta}{2} \sin km\theta \sin kn\theta$$

What is $\sin \cdot \sin$?

$$\sin A \sin B = \frac{-\cos(A+B) + \cos(A-B)}{2}$$

$$\frac{1}{2} \sum \left(\sqrt{\lambda_k}\right) \left(\cos (m-n) \frac{k\pi}{N+1} - \cos (m+n) \frac{k\pi}{N+1} \right)$$

Final step: an approximation

↳ gets more accurate as $N \uparrow$

Replace messy sum w/ an integral

8

$$(\sqrt{K})_{mn} = C\int \left(2\sin\frac{k\theta}{2}\right)\left(\cos(m-n)k\theta - \cos(m+n)k\theta\right)$$

Rename $k\theta$ to ϕ

$$= C\int \left(2\sin\frac{\phi}{2}\right)\left(\cos(m-n)\phi - \cos(m+n)\phi\right)$$

Now solve / look it up

$$\sqrt{K} = \left(\frac{4}{\pi}\right) \begin{bmatrix} \ddots & 1 & -\frac{1}{3} & -\frac{1}{15} \\ -\frac{1}{3} & \ddots & \ddots & \ddots \\ -\frac{1}{15} & \ddots & \ddots & \ddots \end{bmatrix}$$

diagonals are $\frac{1}{1-4p^2}$

Can think of as ∞ matrix

Lactually easier

Main diag of $(\sqrt{K})^2 =$

$$= \left(\frac{4}{\pi}\right)^2 \left[1 + 2\frac{1}{9} + 2\left(\frac{1}{15}\right)^2 + \cdots\right]$$

(9)

Then put in calc

get $= 1.99 - \ldots$

what hoping for

main diag is 2

Not exactly a proof - but close

If want exponential e^{-k} instead?

e^{-k}

$$e^{-k} = \sum_{k=1}^{N} e^{-\lambda_k} v_k v_k^T$$

; replace

$$\frac{1}{2} \sum_{k=1}^{N} \left(e^{-\lambda_k}\right) \left(\cos \frac{(m-n)k\pi}{N+1} - \cos \frac{(m+n)k\pi}{N+1} \right)$$

Comes out the same

(10)

Why do we care?

$u_t = -k\, u^{\sqrt{k}}$ = Heat eq (? misread)

$u_{tt} = -k\, u$ = wave eqn

Last recitation

Linear Transform

(one of the most theoritacal)

You should do the optional HW

It will help on final

"If we pick a basis

any linear transform is a matrix"

"As soon as pick basis

$T: V \to W$

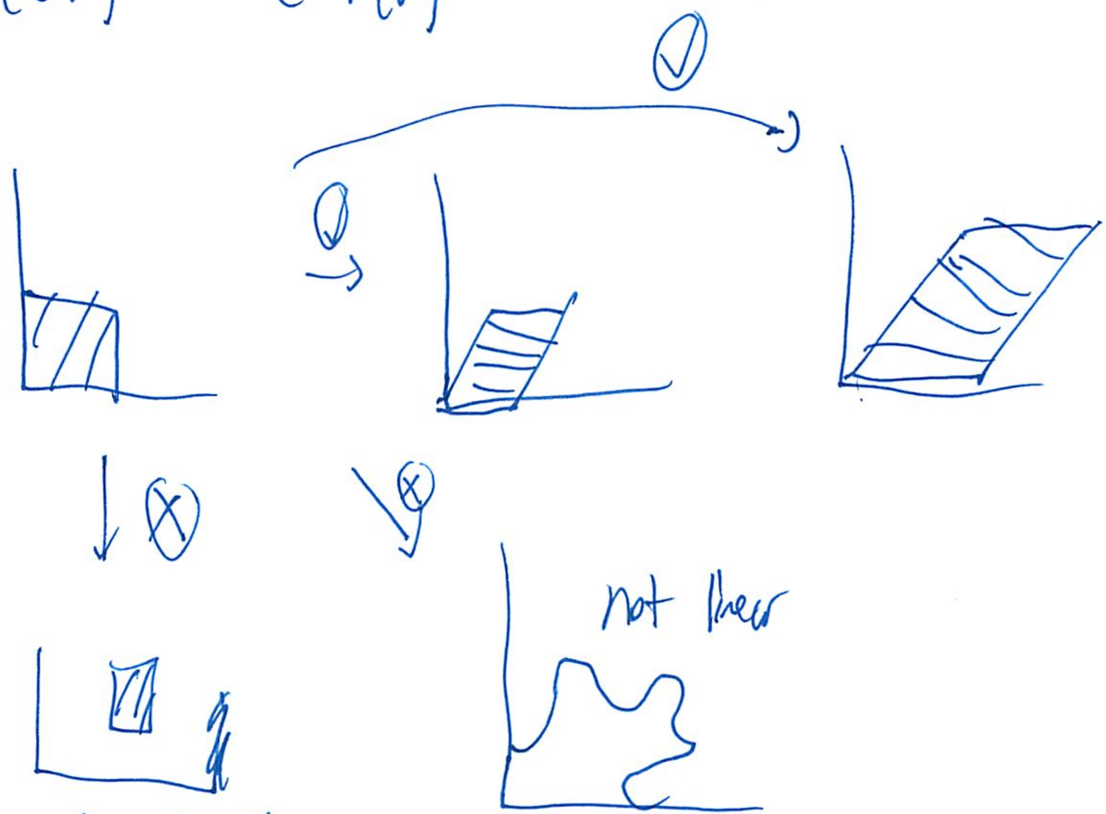
mapping one basis to another

(2)

1.) $T(v_1 + v_2) = T(v_1) + T(v_2)$

2.) $T(cv) = cT(v)$

ie



transformation

$\uparrow$ since $T(0) = T(0 \cdot v) = 0T(v) = 0$

$\boxed{T(0) = 0}$ ← should always hold

if it holds → not auto lin trans

3

Very important to realize if Lin Trans or not

a) $\mathbb{R}^2 \to \mathbb{R}$

$\begin{pmatrix} x \\ y \end{pmatrix} \to 3x - 2y + 1$

No

$f(0)$ is $0 + 0 + 1 = 1$

b) $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x & y \end{pmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

$f(0)$ is 0 ✓

but

$x^2 + 4xy + y^2$

Not linear!

$V_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \quad V_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$

Then $f\begin{pmatrix} x_1 + x_2 \\ \cancel{x_1} + y_2 \end{pmatrix} = f\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + f\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$

Won't work no matter what we choose

(4)

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$f\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = 1$$

$$f\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = 1$$

$$f\begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix} = f\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 6$$

$$1 \neq 6$$

Not trival to check

↳ but you can do it!

5

c) $g: V \to V$

V = vector space of twice diff. functions

$y \to y'' + 2y' + 5y$

Looks confusing

But $\checkmark$ is linear

$g(y_1 + y_2) = (y_1 + y_2)'' + 2(y_1 + y_2)' + 5(y_1 + y_2)$

$= y_1'' + y_2'' + 2y' + 2y_2' + 5y_1 + 5y_2$

$= (y_1'' + 2y_1' + 5y_1) + (y_2'' + 2y_2' + 5y_2)$

$= g(y_1) + g(y_2)$

Cond 1 $\checkmark$

⑥

Still need to check cond 2.

$$g(cy) = (cy)'' + 2(cy)' + 5(cy)$$

$$= cy'' + 2cy' + 5cy$$

$$= c(y'' + 2y' + 5y)$$

$$= c\,g(y)$$

✓ cond 2

So ✓ linear transform

---

d) $g: V \to V$ V = vector space

$V \to t^2 V$ t = constant

the ² doesn't do anything since

$$g(t_1 + t_2) = t^2 v_1 + t^2 v_2$$

$$= g(v_1) + g(v_2)$$

✓ cond 1

cond 2 just as easy to ✓

---

e) $h: \mathbb{C} \to \mathbb{C}$ ← one-dim vector spaces

$$x + iy \to x + y$$

Is this a linear transform?

$$h(i(x + iy)) =$$

$$= h(ix - y)$$

$$= -y + x$$

✓ cond 1 works

(g) $h(x+iy) = i(x+y)$

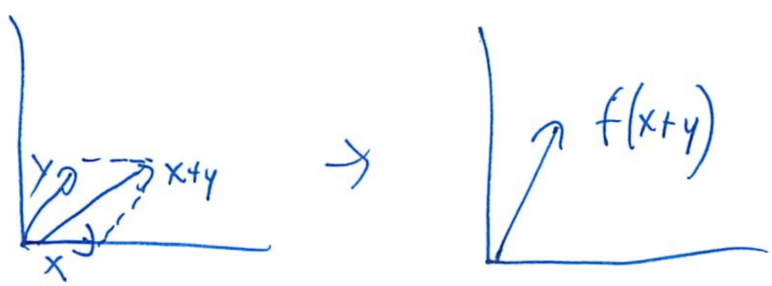
x usually diff

So no cond 2

(X) No

(f) $R^2 \to R^2$

Rotation by 30° CCW



Cond 1 : same as rotate 30° then add
✓ yes

Scale → rotate or rotate → scale same
✓ yes

(✓) So is lin trans.

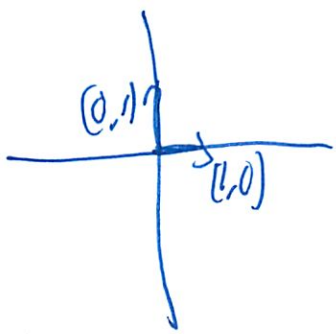
(9)

Pick a basis

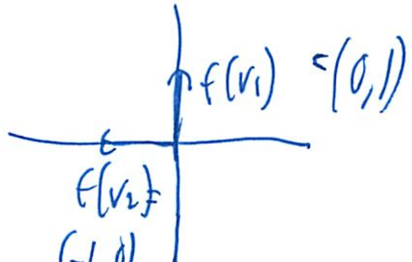
then write down a matrix representing the linear transformation

a) Rotation by 90° $\left(\frac{\pi}{2}\right)$ counterclock wise

Basis $= \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$



what happens when rotate 90°?



$v_1 \to \begin{pmatrix} 0 \\ 1 \end{pmatrix} = v_2$

$v_2 \to \begin{pmatrix} -1 \\ 0 \end{pmatrix} = -v_1$

(10)

So can represent lin transform as

$$V_1 = 0v_1 + 1v_2$$
$$V_2 = -1v_1 + 0v_2$$

So put that in a matrix

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

If choose a diff basis, matrix diff

$$\text{Basis}' = \quad v_1' = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$V_2' = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$B = ?$ Can ya come up w/ a basis for the lin trans?

$$V_1' = (\ ) = \_V_1 + \_V_2$$
$$V_2' = (\ ) = \_V_1 + \_V_2$$

(11)

$V_1' \Rightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -1\, v_1' + 1\, V_2'$

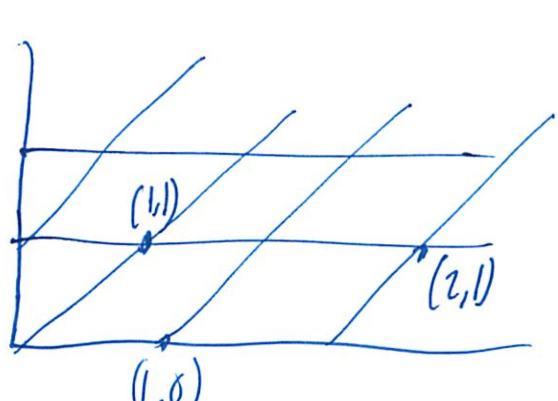
$V_2' \Rightarrow \begin{pmatrix} -1 \\ -1 \end{pmatrix} = -2 v_1' + 1\, V_2'$

$$B = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}$$

Notice 2 diff matrices depending on starting basis

It was still rotate by 90°

but now ~~the~~ no Cartisan grid, instead



(12)

Change of Basis Matrix

- important topic

Setup: input basis (v_1, v_2)

Output basis (v_1', v_2')

Goal Find a matrix M such that

$Mv_1 = v_1'$

$Mv_2 = v_2'$

Example a) Find M where

Input v_1, v_2: $\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}$

Output v_1', v_2': $\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}$

} trying to relate these 2 matrices

Write output in terms of input

(13)

$$V_1' = \begin{pmatrix}1\\0\end{pmatrix} = 1V_1 + 0V_2$$

$$V_2' = \begin{pmatrix}1\\1\end{pmatrix} = 1V_1 + 1V_2$$

$$M = \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}$$

← (I'm not sure this example is valid.)

b) Find M' where

Input $\begin{pmatrix}1\\0\end{pmatrix}$ $\begin{pmatrix}1\\1\end{pmatrix}$

Output $\begin{pmatrix}1\\0\end{pmatrix}$ $\begin{pmatrix}0\\1\end{pmatrix}$ ← standard basis

$$\begin{pmatrix}1\\0\end{pmatrix} = \underline{1}\,V_1 + \underline{0}\,V_2$$

$$\begin{pmatrix}0\\1\end{pmatrix} = \underline{-1}\,V_1 + \underline{1}\,V_2$$

$$A = \begin{bmatrix}1 & -1\\0 & 1\end{bmatrix}$$ ✓

(14)

Notice $MM' = I$

If it keeps all basis the same → identity

$A =$ in terms of $V_1 \ V_2$

$B =$ " " $V_1' \ V_2'$

$A = MBM^{-1}$ ← from similar matrices

$AM = BM$ They are the same linear transform

Another way to understand diagonalization

Suppose lin trans $\mathbb{R}^2 \to \mathbb{R}^2$

$V \to \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} V$

↑ vector ∘ matrix

note markov matrix

gives you the basis

(15)

1. Natural $\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}$ Could choose either bases

2. Eigen-bases $\lambda_1 = 1$
$\lambda_2 = \text{trace} - 1 = .5$

so $v_1' = \begin{pmatrix}3\\2\end{pmatrix}$
$v_2' = \begin{pmatrix}1\\-1\end{pmatrix}$

(1) $v_1 \to \begin{pmatrix}.8\\.2\end{pmatrix} = .8 v_1 + 0.2 v_2$

$v_2 \to \begin{pmatrix}.3\\.7\end{pmatrix} = .3 v_1 + .7 v_2$

$A = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix}$

(2) $v_1' \to \lambda_1 v_1' = \begin{pmatrix}3\\2\end{pmatrix}$
$v_2' \to .5 \lambda_2'$

$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ ← diagonal matrix

16

$V_1' = \begin{pmatrix} 3 \\ 2 \end{pmatrix} = 3v_1 + 2v_2$

$V_2' = \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 1v_1 - 1v_2$

$S = \begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix}$

$A = SDS^{-1}$

? diagonalizing a matrix

18.06
Sample Practice

5/16

Handed out a sample exam

1. Use elim

   L·U

   Find rank – the # of rows in U
   └ # of pivots

   Could go all the way RREF

   Then special sols

   ↑Can't just hardware tricks

---

2. Given basis for null space

   Rank? – 3 x 4
   – the nullspace

   Rank = ~~1~~ 3

   n = 4

   1-dim nullspace

   4 - 1 = 3

②

Complete Sol $\quad x = cz \quad$ any c

line

$$\begin{array}{cc} & \\ \text{RREF} = 3 & \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ & 4 \end{array}$$

(study)

free variable

Where 1

---

3. What are λ vales for proj

$\lambda = 1, 0$

But how many of each?

Can look at trace = 1

$\lambda = 1, 0, 0$

(study!)

What sub space?

Projects onto line through $(1, 2, -4)$

3

Apply P

$$= \frac{1}{21}\begin{bmatrix} -1 \\ -2 \\ 4 \end{bmatrix}$$

---

5. Matrix w/ orthonormal cols

doesn't have to be square

Want least squares sol.

$Qx = b$

$f = ($

$Q^TQ\hat{x} = Q^Tb$ for sure

$= Q^Tb$ just a multiplication

$Q^TQ = I$

4

4. $AB = 0$

$$\begin{bmatrix} - \\ - \\ - \end{bmatrix}\begin{bmatrix} | & | & | \end{bmatrix}$$

So this ~~tell~~ tells us $N(A) \cancel{\supseteq} C(B)$

A (cols of B) are 0

So Nullspace of A contains $C(B)$

$$N(A) \supseteq C(B)$$

$$N(B^T) \supseteq C(A^T)$$

$$\underset{\text{rank } r}{\overset{5 \times 7}{A}} \quad \underset{\text{rank } s}{\overset{7 \times 9}{B}} = 0$$

$$N(A) \supseteq C(B)$$

will be larger than

$$7 - r \geq s$$

(5)

$r + s \leq 7$

---

6. Gram Schmidt

$\llcorner$ Prof: Damn
might not be one of these on final

$a_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ $a_2 = \begin{bmatrix} 5 \\ 4 \\ -2 \end{bmatrix}$

$$q_1 = \frac{1}{3}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

$$q_2 = a_2 - \frac{a_2^T q_1}{q_1^T q_1} q_1$$

$$a_2 - \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \frac{\begin{bmatrix} 4 \\ 2 \\ -4 \end{bmatrix}}{\sqrt{36}} = q_2$$

$\llcorner$ should be orth to q_1
dot prod = 0

6

$$P = A(A^T A)^{-1} A^T$$
$$= QQ^T$$

---

7. If form basis

 L matrix invertable

b) Possible ranks 3

c) matrix for projection using orthonormal basis

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Pick the perfect basis so its easy
Prof ive didn't have much pratice on this

7

8. Determinant

$$\det \begin{bmatrix} 3 & 0 & 8 & 2 \\ 2 & 3 & 0 & 0 \\ 8 & 2 & 3 & 0 \\ 0 & 0 & 2 & 3 \end{bmatrix}$$

Cofactors

$$3 \cdot \det \begin{bmatrix} 3 & 0 & 0 \\ 2 & 3 & 0 \\ 0 & 2 & 3 \end{bmatrix} + 2 \cdot -\det \begin{bmatrix} 2 & 3 & 0 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= 3 \cdot 27 + 2 \cdot -8$$

$$= 81 - 16$$

$$= 65$$

Tricky part b) Can make bigger

$$\det \begin{bmatrix} 3 & 0 & 0 & 0 & 2 \\ 2 & 3 & 0 & 0 & 0 \\ 0 & 2 & 3 & 0 & 0 \\ 0 & 0 & 2 & 3 & 0 \\ 0 & 0 & 0 & 2 & 3 \end{bmatrix}$$

8

Not that ~~make~~ ~~easier~~ easy!

$$3 \cdot 3^4 + 2(16) + \cdots$$

---

9. 6 possible permutation matrices $P_1, \ldots, P_6$

a) det $= 1$ or -1

b) What could be the pivots?

1 or -1 (not allowing 0)

c) Trace? Could be 0 ✓

3 ← Identity matrix

2 × No if have 2, must have 3rd one

1

d) What could d be? $1, -1$

$$\begin{bmatrix} 1 & 7 & 8 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$ det of this?

(9)

$$\begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 1 & 0 & -\lambda \end{bmatrix}$$

$$-\lambda^3 + 1$$

$$\lambda = e^{\frac{2\pi i}{3}}, e^{-\frac{2\pi i}{3}}$$

Since orth all have abs value 1

But can have diff values around unit circle

10.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ & 2 & 1 & 1 \\ & & 3 & 1 \\ & & & 4 \end{bmatrix}$$

e vector matrix is also upper

$$\begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}$$

10

1999 Exam

1. ~~a~~ Find R

2. Gram schmid

3. Unplesent

-vector spaces w/ polynomial + functions

3c) $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$ $0, i, -i$

4. Projection

Pratice straigh line $\underbrace{\text{fitths}}_{\text{?? heard correctly}}$

5. λ values ~~(~~1, 1, 3~~)~~

Repeated eig values - sneaky

Could be 1 1 3

11

But also

$$\begin{array}{ccc} 1 & ? & \\ & 1 & \\ & & 3 \end{array}$$

Would not be diagonalizable

$\lambda = 1, 1, 3$

$A^2 - 3I$

$1, 1, 9$

$-2, -2, 6$

$\det = 24$

$A^T = 1, 1, 3$

$A^T - 2I$ invertable

Since 0 not an e-value

$-1, -1, 1$

(12)

7. Quadratic form - we didn't make a big deal of

---

6. $\begin{bmatrix} 3-\lambda & 4 \\ 1 & 0-\lambda \end{bmatrix}$

$\lambda^2 - 3\lambda - 4 = 0$

$(\lambda - 4)(\lambda + 1)$

$\lambda = 4, -1$

evectors $x_1 \quad x_2$

$u_n = A^n u_0$

$= c_1 \lambda_1^n x_1 + c_2 \lambda_2^n x_2$

$= \cancel{?} \ 0 + c_2 (-1)^n x_2$

13

Inital must be a multiple of c_2

Done

5/16 Practice Exam 2002

1 The 4 by 6 matrix A has all 2's below the diagonal and elsewhere all 1's:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 1 & 1 & 1 \end{bmatrix}$$

(a) By elimination factor A into L (4 by 4) times U (4 by 6).

(b) Find the rank of A and a basis for its nullspace (the special solutions would be good).

2 Suppose you know that the 3 by 4 matrix A has the vector $s = (2, 3, 1, 0)$ as a basis for its nullspace.

(a) What is the *rank* of A and the complete solution to $Ax = 0$?

(b) What is the exact row reduced echelon form R of A?

3 The following matrix is a *projection matrix*:

$$P = \frac{1}{21}\begin{bmatrix} 1 & 2 & -4 \\ 2 & 4 & -8 \\ -4 & -8 & 16 \end{bmatrix}.$$

(a) What subspace does P project onto?

(b) What is the *distance* from that subspace to $b = (1, 1, 1)$?

(c) What are the three eigenvalues of P? Is P diagonalizable?

5 Suppose the 4 by 2 matrix Q has orthonormal columns.

(a) Find the least squares solution $\widehat{\boldsymbol{x}}$ to $Q\boldsymbol{x} = \boldsymbol{b}$.

(b) Explain why QQ^{T} is not positive definite.

(c) What are the (nonzero) singular values of Q, and why?

4 (a) Suppose the product of A and B is the zero matrix: $AB = 0$. Then the __(1)__ space of A contains the __(2)__ space of B. Also the __(3)__ space of B contains the __(4)__ space of A. Those blank words are

(1)__________ (2)__________ (3)__________ (4)__________

(b) Suppose that matrix A is 5 by 7 with rank r, and B is 7 by 9 of rank s. What are the dimensions of spaces (1) and (2)? From the fact that space (1) contains space (2), what do you learn about $r + s$?

6 Let S be the subspace of $\mathbf{R}^3$ spanned by $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 5 \\ 4 \\ -2 \end{bmatrix}$.

(a) Find an orthonormal basis $\boldsymbol{q}_1, \boldsymbol{q}_2$ for S by Gram-Schmidt.

(b) Write down the 3 by 3 matrix P which projects vectors perpendicularly onto S.

(c) Show how the properties of P (what are they?) lead to the conclusion that $P\boldsymbol{b}$ is orthogonal to $\boldsymbol{b} - P\boldsymbol{b}$.

7 (a) If v_1, v_2, v_3 form a basis for $\mathbf{R}^3$ then the matrix with those three columns is ____.

(b) If v_1, v_2, v_3, v_4 span $\mathbf{R}^3$, give all possible ranks for the matrix with those four columns. __________.

(c) If q_1, q_2, q_3 form an orthonormal basis for $\mathbf{R}^3$, and T is the transformation that projects every vector v onto the plane of q_1 and q_2, what is the matrix for T in this basis? Explain.

8 Suppose the n by n matrix A_n has 3's along its main diagonal and 2's along the diagonal below and the $(1, n)$ position:

$$A_4 = \begin{bmatrix} 3 & 0 & 0 & 2 \\ 2 & 3 & 0 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 2 & 3 \end{bmatrix}.$$

Find by cofactors of row 1 or otherwise the determinant of A_4 and then the determinant of A_n for $n > 4$.

9 There are six 3 by 3 permutation matrices P.

(a) What numbers can be the *determinant* of P? What numbers can be *pivots*?

(b) What numbers can be the *trace* of P? What *four numbers* can be eigenvalues of P?

10 Suppose A is a 4 by 4 upper triangular matrix with $1, 2, 3, 4$ on its main diagonal. (You could put all 1's above the diagonal.)

(a) For $A - 3I$, which columns have pivots? Which components of the eigenvector x_3 (the special solution in the nullspace) are definitely zero?

(b) Using part (a), show that the eigenvector matrix S is also upper triangular.

1991
18.06 Final

1. Let $A = \begin{bmatrix} 1 & 2 & 1 & 7 \\ 2 & 4 & 1 & 10 \\ 3 & 6 & 1 & 13 \end{bmatrix}$.

(a) Find the *reduced* echelon form R of A.

(b) Find a matrix E which is a product of elementary matrices and such that $A = ER$.

(c) Give a basis for the null-space $\mathcal{N}(A)$.

(d) Give a basis for the column-space $\mathcal{R}(A)$.

(e) Give a factorization $A = LU$, where
L is a lower triangular matrix with 1's along the diagonal
U is an echelon matrix.

2. Write a_1, a_2, a_3 for the three columns of the matrix $A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 5 \\ 0 & 3 & 6 \end{bmatrix}$.

(a) What is the orthonormal basis determined by a_1, a_2, a_3 by the Gram-Schmidt process?

(b) Write down the factorization $A = QU$ where
Q has orthonormal columns,
U is square upper-triangular with positive entries along the main diagonal.

(c) What is the matrix of the projection to the span of the first two columns of A?

3. The parts of this problem are not related to one another.

(a) Using the inner product $(f,g) = \int_{-1}^{1} f(x)g(x)dx$, what is the polynomial of the form $mx + b$ which is nearest to x^3?

(b) What is the matrix, with respect to the basis $\{1, x, x^2\}$, of the operator on the vector space of polynomials of degree at most 2 which sends $f(x)$ to $xf'(x)$?

(c) Give an example of a 3-by-3 matrix with real coefficients and eigenvalues $0, i$, and $-i$ (where $i = \sqrt{-1}$).

(d) Write a list of all 4-by-4 Jordan matrices. (You need not include both of two similar Jordan matrices.) Then say how many linearly indendent eigenvectors each one has.

4. (a) Let $A = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $v = \begin{bmatrix} 0 \\ 1 \\ 8 \end{bmatrix}$. What is the point p of the column-space $\mathcal{R}(A)$ nearest to v?

(b) What are the coordinates of p with respect to the basis of $\mathcal{R}(A)$ given by the columns of A?

(c) For what values of b and m does $y = mx + b$ best fit the data given in the table below?

x	-1	0	1
y	0	1	8

5. I am thinking of a 3-by-3 matrix A with real entries and eigenvalues 1, 1, and 3. For each question below, give an answer and a reason for your answer; or else respond with "can't say" and give two such matrices giving different answers.

(a) Is A diagonalizable?

(b) What is $\det(A^2 - 3I)$?

(c) What is the rank of $A^T - 2I$?

(d) Is $(A - I)^2$ diagonalizable?

(e) Is A orthogonal?

(f) What is $\operatorname{tr}((A^T)^2)$?

6. Let $A = \begin{bmatrix} 3 & 4 \\ 1 & 0 \end{bmatrix}$.

(a) Find the eigenvalues of A.

(b) For each eigenvalue find a nonzero eigenvector, and write down a diagonalizing factorization $A = S\Lambda S^{-1}$. (You need not compute S^{-1}, but be explicit about your choice of S.)

(c) Consider the difference equation $u_n = Au_{n-1}$. For what initial vectors u_0 does the sequence u_n stay bounded?

(d) Write $u_n = \begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix}$, and express the difference equation $u_n = Au_{n-1}$ as a difference equation for the numbers a_n.

(e) For what initial values a_0, a_1, does the sequence a_n stay bounded?

7. Consider the quadratic form $q(x, y) = 9x^2 - 4xy + 6y^2$.

(a) Write down the associated symmetric matrix and find its eigenvalues.

(b) Find the eigenspaces, and sketch the level-set $q(x, y) = 10$.

8. Let A be a general skew-symmetric n-by-n matrix (so $A^T = -A$) with real entries.

(a) Explain why $\det A = 0$ if n is odd.

(b) Explain why all the eigenvalues of A are purely imaginary. [State a relevant theorem if you like; checking it (for a general such A, of course) will win you more credit.]

(c) Explain why $\det A \geq 0$ for all n.

18.06 Spring 2012 – Problem Set 10 (not handed in/not graded)

This short extra problem set is *not* to be handed in. The problems are meant to help you learn about linear transformations. The textbook problems are out of the 4th edition.

1. Do Problem 30 from Section 7.1.
2. Do Problem 31 from Section 7.1.
3. Do Problem 27 from Section 7.2.
4. Do Problem 32 from Section 7.2.

18.06 Wisdom. Get as ready as you can for the finals, by doing as many old exams as you have time for (found on the 18.06 website under "Past Courses"). Strive to master the concepts and techniques of Linear Algebra - you will need it, both here and beyond.

18.06 Spring 2012 – Problem Set 10 (not handed in/not graded)

This short extra problem set is *not* to be handed in. The problems are meant to help you learn about linear transformations. The textbook problems are out of the 4th edition.

1. Do Problem 30 from Section 7.1.

   Solution. The radius vectors of the circle are $(0,1)$ and $(0,1)$ and these vectors go to the vectors $(\sqrt{3},1)$ and $\left(-\frac{1}{2},\frac{\sqrt{3}}{2}\right)$ $\square$

2. Do Problem 31 from Section 7.1.

   Solution. A rectangular can be considered as a set

$$R = \{(x_0,y_0) + c(x_1,y_1) + d(x_2,y_2) : 0 \leq c,d \leq 1\}$$

   where(x_0,y_0), (x_1,y_2) and (x_2,y_2) are the vertices of the rectangular and (x_0,y_0) is in between the other two vertices (x_1,y_1) and (x_2,y_2). The image $T(R)$ of R is

$$T(R) = \{T(x_0,y_0) + cT(x_1,y_1) + dT(x_2,y_2) : 0 \leq c,d \leq 1\},$$

   which is a parallelogram or a line (with the assumption $T \neq 0$). The set $T(R)$ is a line iff T is not invertible.

   $\square$

3. Do Problem 27 from Section 7.2.

   Solution. There is a linear transformation S such that $S(w_i) = v_i$ and S is the inverse of T. Hence T is invertible. $\square$

4. Do Problem 32 from Section 7.2.

   Solution. False. If these n vectors are linearly dependant, there is a vector w which is not a linear combination of these n vectors and $T(w)$ is not determined by $T(v)$. $\square$

18.06 Wisdom. Get as ready as you can for the finals, by doing as many old exams as you have time for (found on the 18.06 website under "Past Courses"). Strive to master the concepts and techniques of Linear Algebra - you will need it, both here and beyond.