Massachusetts Institute of Technology
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Fall 2010)

Quiz 1 Announcements

need to make

Quiz 1: Closed-book, with one double-sided 8.5 x 11 formula sheet permitted. Please arrive early to find your seat before the prompt start at 7:30PM. Calculators will not be allowed (there won't be any hard numerical calculations in the quiz).

Date: Tuesday, October 12
Time: 7:30 - 9:00 PM
Location: 54-100
Content: Quiz 1 will cover the following class material (all boundaries are inclusive)
- Lectures 1 thru 7
- Textbook chapters 1 and 2
- Recitations 1 thru 7
- Tutorials 1 thru 3
- Problem Sets 1 thru 4

really glad we can have formula sheet!

no contineous - good!

Instructions: Multiple choice questions, if any, have a single correct answer, and no credit will be given for any incorrect answers. Other questions require fully-reasoned, convincing answers; partial credit is possible and reaching a correct conclusion does not guarantee full credit on these questions.

Practice Quizzes: Two past quizzes with full solutions are available on the OCW website (Spring05 & Spring06). An additional two quizzes have been posted on the course website (Spring09 & Fall09), which will be reviewed at the TA quiz 1 review session. Please note Quiz 1 coverage, course coverage, and course emphasis change each term. Hence past quizzes are not necessarily indicative of this term's quiz. Material presented in lecture, recitation, tutorial, and problem set exercises should be your primary source of preparation.
`http://ocw.mit.edu/OcwWeb/web/home/home/index.htm`
`http://stellar/S/course/6/fa10/6.041/materials.html`

Office Hours: The majority of the regular staff office hours are held on Monday, Tuesday, and Friday before the quiz date. Please check the course website for updates to times and any additional hours.

Optional 6.041/6.431 Quiz Review Session: There will be a two-hour 6.041/6.431 quiz review session administered by two TAs. The session will consist of two parts. In the first hour, a concise overview of the theory will be presented. In the second hour, selected problems from past quizzes will be solved. Though completely optional, the quiz review is a great opportunity to reinforce your understanding of the material and perhaps gain new insight. Details for the quiz review:

Date: Thursday, October 7
Time: 7:30 - 9:30 PM
Location: 32-141

Problems for the quiz review will be selected from the Spring 2009 and Fall 2009 Quiz 1 (each available on the course website under `Quiz Material`). We will review as many problems as time permits. Full solutions will be posted on-line following the review. We strongly recommend working through the problems before coming to the quiz review.

6.041 Quiz 1

10/14

$\cap$ = and $\subset$ subset of

$\cup$ = or ← don't forget remove overlap!

$A \subset C \rightarrow P(A) \le P(B)$

$$\boxed{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$$

$$P(A \cup B) \le P(A) + P(B)$$

$$\boxed{P(A \cup B \cup C) = P(A) + P(A^c \cap B) + P(A^c \cap B^c \cap C)}$$

Independent

$$P(A \cap B) = P(A)P(B)$$
$$P(A|B) = P(A)$$

conditionally independent seperate

$$\boxed{P(A \cap B | C) = P(A|C)P(B|C)}$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

↳ $P(A|B) \ge 0$

$P(\Omega|B) = 1$

$$\boxed{P(A_1 \cup A_2 \cup A_3 \ldots | B) = P(A_1|B) + P(A_2|B) + \ldots}$$

Counting Techniques n items

realign the universe

basic $n_1 n_2 n_3 \ldots n_m$

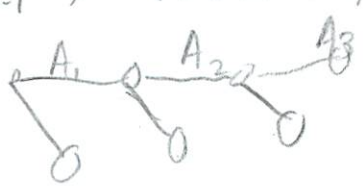
permentations $n(n-1) \ldots (n-k+1) = \frac{n!}{(n-k)!}$ ${}_nP_k$

order important k = seq lenght

Combonations $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

Multiplication Rule

$$\boxed{P\left(\bigcap_{i=1}^{n} A_i\right) = P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2)}$$



partitions

r-disjoint subsets

n_k elements in nth subset

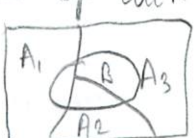
$$\binom{n}{n_1, n_2, \ldots n_r} = \binom{n}{n_1}\binom{n-n_1}{n_2}\cdots\binom{n-n_1-\ldots n_{r-1}}{n_r}$$

$$= \frac{n!}{n_1!(n-n_1)!} \cdot \frac{(n-n_1)!}{n_2!(n-n_1-n_2)!} \cdots \frac{(n-n_1-\ldots n_{r-1})!}{n_r!(n-n_1-\ldots n_{r-1})!}$$

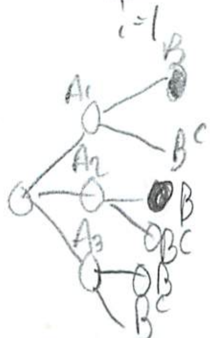
$$= \frac{n!}{n_1! \cdot \ldots \cdot n_r!}$$ multinomial formual

Total Probability Theorm

- add up each section



$$P(B) = \sum_{i=1}^{n} P(B \cap A_i) = \sum_{i=1}^{n} P(B|A_i)P(A_i)$$



Bayes Rule

likelihood, prior prob

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

↑ posterior prob

← normalization constant

Discrete Random Variables

$\{X = x\} = \{\omega \in \Omega \mid X(\omega) = x\} \subseteq \Omega$

$p_X(x) \ge 0$

$P(X \in S) = \sum_{x \in S} p_X(x)$

$Y = g(X)$ still works w/ everything

$E[X] = \sum_x x p_X(x)$

$E[aX + b] = aE[X] + b$ linearity of expectation

$E[g(X)] = \sum_x g(x) p_X(x)$

$\mathrm{Var}(X) = E[(X - E[X])^2] = E[X^2] - E[X]^2$

$\mathrm{var}(aX + b) = a^2 \mathrm{var}(X)$

joint

$p_{X,Y}(x,y) = P(\{X = x\} \cap \{Y = y\})$

$p_X(x) = \sum_y p_{X,Y}(x,y)$
$p_Y(y) = \sum_x p_{X,Y}(x,y)$) marginal

Total Expectation Theorm

$$E[X] = \sum_{i=1}^{n} P(A_i) E[X|A_i]$$

Independent

$p_{X|A} = p_X(x)$ $\forall x$ ← for all/given any

or

$p_{X,A}(x) = p_X(x)P(A)$ $\forall x$

$E[XY] = E[X]E[Y]$ inverse not true

$\mathrm{Var}(X + Y) = \mathrm{var}(X) + \mathrm{var}(Y)$ - constants don't matter

knowing info about 1 does not help you at all w/ other

$p_{X,Y}(x,y) = p_X(x)p_Y(y)$

$\binom{5}{1} = 5$

$\binom{5}{5} = 1$

$\frac{|P(A)|}{|P(\Omega)|}$ Discrete Uniform Law

Discrete Distributions

	X	PMF $p_X(k)$	$E[X]$	$var(X)$
Bernoulli	$\begin{cases} 1 & \text{sucess} \\ 0 & \text{failure} \end{cases}$	$\begin{cases} p & k=1 \\ 1-p & k=0 \end{cases}$	p	$p(1-p)$
Binomial	# of sucesses in n Bernoulli trials	$\binom{n}{k} p^k (1-p)^{n-k}$ $k = 0, 1, \ldots n$ ↑ # sucesses	np	$np(1-p)$
Geometric	# of trials till 1st sucess	$(1-p)^{k-1} p$ $k = 1, 2, 3$ ↑ # of trials	$\frac{1}{p}$	$\frac{1-p}{p^2}$ expected trials till sucess
Uniform	An integer in $[a, b]$	$\begin{cases} \frac{1}{b-a+1} & k = a, a+1, \ldots, b \\ 0 & \text{oterise} \end{cases}$	$\frac{a+b}{2}$	$\frac{(b-a)(b-a+2)}{12}$
Poisson		$e^{-\lambda} \frac{\lambda^k}{k!}$ $k = 0, 1, 2,$		

Bernoulli

- 2 outcomes
- like a coin flip
- used to construct more complex random variables

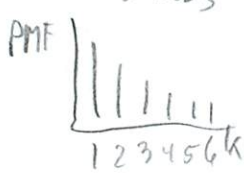
Binomial

- coin tossed n coins
- X = # sucesses in n trials

$n = 9$
$p = 1/2$

Geometric

- #(x) tosses needed till 1st sucess



Uniform

- rolling a fair 6-side die

$$p_X(k) = \begin{cases} 1/6 & \text{if } k = 1,2,3,4,5,6 \\ 0 & \text{otherwise} \end{cases}$$

Poisson

- like a binomial RV w/ very small p + large n
- good, quick approx for binomial

$$\lambda = \underset{\text{very large}}{n} \cdot \underset{\text{every small}}{p}$$

trials, eprob

10/11
Reading

Quiz I Review
Probabilistic Systems Analysis
6.041/6.431

Massachusetts Institute of Technology

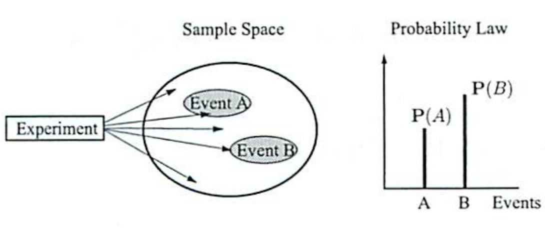
October 7, 2010

Quiz Information

- Closed-book with one double-sided 8.5 x 11 formula sheet allowed
- Date: Tuesday, October 12, 2010
- Time: 7:30 - 9:00 PM
- Location: 54-100
- Content: Chapters 1-2, Lecture 1-7, Recitations 1-7, Psets 1-4, Tutorials 1-3
- Show your reasoning when possible!

A Probabilistic Model:

- **Sample Space**: The set of all possible outcomes of an experiment.
- **Probability Law**: An assignment of a nonnegative number **P(E)** to each event E.



Probability Axioms

Given a sample space Ω:

1. **Nonnegativity:** $\mathbf{P}(A) \geq 0$ for each event A
2. **Additivity:** If A and B are disjoint events, then
$$\mathbf{P}(A \cup B) = P(A) + P(B)$$
If $A_1, A_2, \ldots,$ is a sequence of disjoint events,
$$\mathbf{P}(A_1 \cup A_2 \cup \cdots) = P(A_1) + P(A_2) + \cdots$$
3. **Normalization** $\mathbf{P}(\Omega) = 1$

$\cup$ = union or

$\cap$ = and

Properties of Probability Laws

part of

Given events A, B and C:

1. If $A \subset B$, then $\mathbf{P}(A) \leq \mathbf{P}(B)$
2. $\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B) - \mathbf{P}(A \cap B)$
3. $\mathbf{P}(A \cup B) \leq \mathbf{P}(A) + \mathbf{P}(B)$
4. $\mathbf{P}(A \cup B \cup C) = \mathbf{P}(A) + \mathbf{P}(A^c \cap B) + \mathbf{P}(A^c \cap B^c \cap C)$

I draw out graphically if don't understand

Discrete Models

- **Discrete Probability Law:** If Ω is finite, then each event $A \subseteq \Omega$ can be expressed as

$$A = \{s_1, s_2, \ldots, s_n\} \qquad s_i \in \Omega$$

Therefore the probability of the event A is given as

$$\mathbf{P}(A) = \mathbf{P}(s_1) + \mathbf{P}(s_2) + \cdots + \mathbf{P}(s_n)$$

- **Discrete Uniform Probability Law:** If all outcomes are equally likely,

$$\mathbf{P}(A) = \frac{|A|}{|\Omega|}$$

Conditional Probability

- Given an event B with $\mathbf{P}(B) > 0$, the conditional probability of an event $A \subseteq \Omega$ is given as

$$\mathbf{P}(A|B) = \frac{\mathbf{P}(A \cap B)}{\mathbf{P}(B)}$$

- $\mathbf{P}(A|B)$ is a valid probability law on Ω, satisfying
  1. $\mathbf{P}(A|B) \geq 0$
  2. $\mathbf{P}(\Omega|B) = 1$
  3. $\mathbf{P}(A_1 \cup A_2 \cup \cdots |B) = \mathbf{P}(A_1|B) + \mathbf{P}(A_2|B) + \cdots$, where $\{A_i\}_i$ is a set of disjoint events
- $\mathbf{P}(A|B)$ can also be viewed as a probability law on the restricted universe B.

I finally know what this symbol means

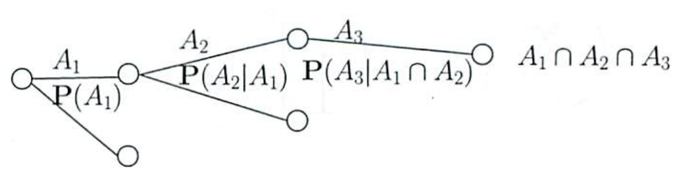
Multiplication Rule

- Let $A_1, \ldots, A_n$ be a set of events such that

$$\mathbf{P}\left(\bigcap_{i=1}^{n-1} A_i\right) > 0.$$

Then the joint probability of all events is

$$\mathbf{P}\left(\bigcap_{i=1}^{n} A_i\right) = \mathbf{P}(A_1)\mathbf{P}(A_2|A_1)\mathbf{P}(A_3|A_1 \cap A_2) \cdots \mathbf{P}(A_n|\bigcap_{i=1}^{n-1} A_i)$$



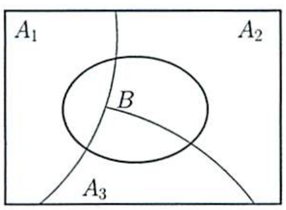
Oh - is this just multiply down the tree?

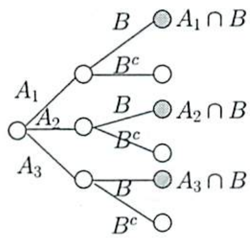
- makes much more sense now!

Total Probability Theorem

Let $A_1, \ldots, A_n$ be disjoint events that partition Ω. If $\mathbf{P}(A_i) > 0$ for each i, then for any event B,

$$\mathbf{P}(B) = \sum_{i=1}^{n} \mathbf{P}(B \cap A_i) = \sum_{i=1}^{n} \mathbf{P}(B|A_i)\mathbf{P}(A_i)$$



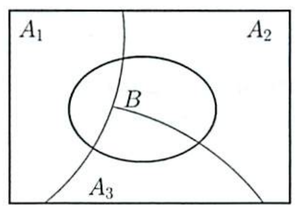


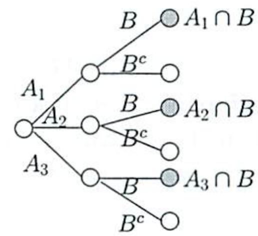
- we want P(B)
- So add up each section

Bayes Rule

Given a finite partition $A_1, \ldots, A_n$ of Ω with $\mathbf{P}(A_i) > 0$, then for each event B with $\mathbf{P}(B) > 0$

$$\mathbf{P}(A_i|B) = \frac{\mathbf{P}(B|A_i)\mathbf{P}(A_i)}{\mathbf{P}(B)} = \frac{\mathbf{P}(B|A_i)\mathbf{P}(A_i)}{\sum_{i=1}^{n} \mathbf{P}(B|A_i)\mathbf{P}(A_i)}$$





Relationship b/w 2 conditional prob that are reverse of each other

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$P(A|B)\,P(B) = P(A \cap B) = P(B|A)\,P(A)$$

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

Independence of Events

- Events A and B are **independent** if and only if

$$\mathbf{P}(A \cap B) = \mathbf{P}(A)\mathbf{P}(B)$$

or

$$\mathbf{P}(A|B) = \mathbf{P}(A) \quad \text{if } \mathbf{P}(B) > 0$$

- Events A and B are **conditionally independent** given an event C if and only if

$$\mathbf{P}(A \cap B|C) = \mathbf{P}(A|C)\mathbf{P}(B|C)$$

or

$$\mathbf{P}(A|B \cap C) = \mathbf{P}(A|C) \quad \text{if } \mathbf{P}(B \cap C) > 0$$

- Independence $\not\Leftrightarrow$ Conditional Independence.

Independence of a Set of Events

- The events $A_1, \ldots, A_n$ are **pairwise independent** if for each $i \neq j$

$$\mathbf{P}(A_i \cap A_j) = \mathbf{P}(A_i)\mathbf{P}(A_j)$$

- The events $A_1, \ldots, A_n$ are **independent** if

and ↓ multiply ↓

$$\mathbf{P}\left(\bigcap_{i \in S} A_i\right) = \prod_{i \in S} \mathbf{P}(A_i) \quad \forall\, S \subseteq \{1, 2, \ldots, n\}$$

- Pairwise independence $\not\Rightarrow$ independence, but independence $\Rightarrow$ pairwise independence.

Counting Techniques

- **Basic Counting Principle:** For an m-stage process with n_i choices at stage i,

$$\# \text{ Choices} = n_1 n_2 \cdots n_m$$

- **Permutations:** k-length sequences drawn from n distinct items without replacement (order is important):

$$\# \text{ Sequences} = n(n-1)\cdots(n-k+1) = \frac{n!}{(n-k)!}$$

- **Combinations:** Sets with k elements drawn from n distinct items (order within sets is not important):

$$\# \text{ Sets} = \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Counting Techniques-contd

- **Partitions:** The number of ways to partition an n-element set into r disjoint subsets, with n_k elements in the k^{th} subset:

$$\binom{n}{n_1, n_2, \ldots, n_r} = \binom{n}{n_1}\binom{n-n_1}{n_2}\cdots\binom{n-n_1-\cdots-n_r-1}{n_r}$$
$$= \frac{n!}{n_1! n_2!, \cdots, n_r!}$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$
$$\sum_{i=1}^{r} n_i = n$$

Chap 2

Discrete Random Variables

- A **random variable** is a real-valued function defined on the sample space:

$$X : \Omega \to R$$

just some # outcome

- The notation $\{X = x\}$ denotes an event:

set of events

$$\{X = x\} = \{\omega \in \Omega | X(\omega) = x\} \subseteq \Omega$$

actual event

- The **probability mass function (PMF)** for the random variable X assigns a probability to each event $\{X = x\}$:

$$p_X(x) = \mathbf{P}(\{X = x\}) = \mathbf{P}(\{\omega \in \Omega | X(\omega) = x\})$$

never saw this - just more useless stuff I think

PMF Properties

- Let X be a random variable and S a countable subset of the real line
- The axioms of probability hold:
  1. $p_X(x) \geq 0$
  2. $\mathbf{P}(X \in S) = \sum_{x \in S} p_X(x)$
  3. $\sum_x p_X(x) = 1$
- If g is a real-valued function, then $Y = g(X)$ is a random variable:

$$\omega \xrightarrow{X} X(\omega) \xrightarrow{g} g(X(\omega)) = Y(\omega)$$

with PMF

$$p_Y(y) = \sum_{x|g(x)=y} P_X(x)$$

Expectation

Given a random variable X with PMF $p_X(x)$:

- $\mathbf{E}[X] = \sum_x x p_X(x)$
- Given a derived random variable $Y = g(X)$:

$$\mathbf{E}[g(X)] = \sum_x g(x) p_X(x) = \sum_y y p_Y(y) = E[Y]$$

$$\mathbf{E}[X^n] = \sum_x x^n p_X(x)$$

- **Linearity** of Expectation: $\mathbf{E}[aX + b] = a\mathbf{E}[X] + b$.

Variance

The expected value of a derived random variable $g(X)$ is

$$\mathbf{E}[g(X)] = \sum_x g(x) p_X(x)$$

The variance of X is calculated as

- $var(X) = \mathbf{E}[(X - \mathbf{E}[X])^2] = \sum_x (x - \mathbf{E}[X])^2 p_X(x)$
- $var(X) = \mathbf{E}[X^2] - \mathbf{E}[X]^2$
- $var(aX + b) = a^2 var(X)$

Note that $var(x) \geq 0$

Multiple Random Variables

Let X and Y denote random variables defined on a sample space Ω.

- The **joint PMF** of X and Y is denoted by

$$p_{X,Y}(x, y) = \mathbf{P}(\{X = x\} \cap \{Y = y\})$$

- The **marginal PMFs** of X and Y are given respectively as

$$p_X(x) = \sum_y p_{X,Y}(x, y)$$

$$p_Y(y) = \sum_x p_{X,Y}(x, y)$$

Functions of Multiple Random Variables

Let $Z = g(X, Y)$ be a function of two random variables

- **PMF:**

$$p_Z(z) = \sum_{(x,y)|g(x,y)=z} p_{X,Y}(x, y)$$

"what is this again?"

- **Expectation:**

$$\mathbf{E}[Z] = \sum_{x,y} g(x, y) p_{X,Y}(x, y)$$

- **Linearity:** Suppose $g(X, Y) = aX + bY + c$.

$$\mathbf{E}[g(X, Y)] = a\mathbf{E}[X] + b\mathbf{E}[Y] + c$$

Conditioned Random Variables

- Conditioning X on an event A with $\mathbf{P}(A) > 0$ results in the PMF:

$$p_{X|A}(x) = \mathbf{P}(\{X = x\}|A) = \frac{\mathbf{P}(\{X = x\} \cap A)}{\mathbf{P}(A)}$$

- Conditioning X on the event $Y = y$ with $\mathbf{P}_Y(y) > 0$ results in the PMF:

$$p_{X|Y}(x|y) = \frac{\mathbf{P}(\{X = x\} \cap \{Y = y\})}{\mathbf{P}(\{Y = y\})} = \frac{p_{X,Y}(x,y)}{p_Y(y)}$$

Same as always
- and I do it graphically
- select only that row

Conditioned RV - contd

- Multiplication Rule: go along the branch

$$p_{X,Y}(x,y) = p_{X|Y}(x|y)p_Y(y)$$

- Total Probability Theorem:

$$p_X(x) = \sum_{i=1}^{n} \mathbf{P}(A_i)p_{X|A_i}(x)$$

$$p_X(x) = \sum_{y} p_{X|Y}(x|y)p_Y(y)$$

add up each part

Conditional Expectation

Let X and Y be random variables on a sample space Ω.

- Given an event A with $\mathbf{P}(A) > 0$

$$\mathbf{E}[X|A] = \sum_{x} x p_{X|A}(x)$$

- If $P_Y(y) > 0$, then

$$\mathbf{E}[X|\{Y = y\}] = \sum_{x} x p_{X|Y}(x|y)$$

- **Total Expectation Theorem:** Let $A_1, \ldots, A_n$ be a partition of Ω. If $\mathbf{P}(A_i) > 0 \; \forall i$, then

$$\mathbf{E}[X] = \sum_{i=1}^{n} \mathbf{P}(A_i)\mathbf{E}[X|A_i]$$

what is this again?
- just new universe
- and expectation

Independence

Let X and Y be random variables defined on Ω and let A be an event with $\mathbf{P}(A) > 0$.

- X is independent of A if either of the following hold:

$$p_{X|A}(x) = p_X(x) \; \forall x$$
$$p_{X,A}(x) = p_X(x)\mathbf{P}(A) \; \forall x$$

- X and Y are independent if either of the following hold:

$$p_{X|Y}(x|y) = p_X(x) \; \forall x \forall y$$
$$p_{X,Y}(x,y) = p_X(x)p_Y(y) \; \forall x \forall y$$

what is upside down $\forall$?
- universal quantification
- given any
- for all

Independence

If X and Y are independent, then the following hold:

- If g and h are real-valued functions, then $g(X)$ and $h(Y)$ are independent.
- $\mathbf{E}[XY] = \mathbf{E}[X]\mathbf{E}[Y]$ (inverse is not true)
- $var(X + Y) = var(X) + var(Y)$

Given independent random variables $X_1, \ldots, X_n$,

$$var(X_1+X_2+\cdots+X_n) = var(X_1)+var(X_2)+\cdots+var(X_n)$$

Some Discrete Distributions

	X	$p_X(k)$	$\mathbf{E}[X]$	$var(X)$
Bernoulli	$\begin{cases} 1 & \text{success} \\ 0 & \text{failure} \end{cases}$	$\begin{cases} p & k=1 \\ 1-p & k=0 \end{cases}$	p	$p(1-p)$
Binomial	Number of successes in n Bernoulli trials	$\binom{n}{k}p^k(1-p)^{n-k}$, $k=0,1,\ldots,n$	np	np(1-p)
Geometric	Number of trials until first success	$(1-p)^{k-1}p$, $k=1,2,\ldots$	$\frac{1}{p}$	$\frac{1-p}{p^2}$
Uniform	An integer in the interval [a,b]	$\begin{cases} \frac{1}{b-a+1} & k=a,\ldots,b \\ 0 & \text{otherwise} \end{cases}$	$\frac{a+b}{2}$	$\frac{(b-a)(b-a+2)}{12}$

thankfully we don't need to do proofs
and I understood most of the HW
will hopefully do better

Practice

10/11/10

6.041/6.431 Spring 2009 Quiz 1
Wednesday, March 11, 7:30 - 9:30 PM.

Name: ____________________

Recitation Instructor: ____________________

TA: ____________________

Question	Part	Score	Out of
0			3
1	all		40
2	a		5
	b		5
	c		6
	d		6
3	a		5
	b		6
	c		6
	d		6
	e		6
	f		6
	g		10
6.041 Total			100
6.431 Total			110

- Write your solutions in this quiz packet, only solutions in the quiz packet will be graded.
- Question one, multiple choice questions, will receive no partial credit. Partial credit for question two and three will be awarded.
- You are allowed one two-sided 8.5 by 11 formula sheet plus a calculator.
- You have 120 minutes to complete the quiz.
- Be neat! You will not get credit if we can't read it.
- Graded quizzes will be returned in recitation on Tuesday 3/17.

Problem 0: (3 pts) Write your name, your <u>assigned</u> recitation instructor's name, and <u>assigned</u> TA's name on the cover of the quiz booklet. The Instructor/TA pairing is listed below.

Recitation Instructor	TA	Recitation Time
Vivek Goyal	Natasa Blitvic	10 & 11 AM
Michael Collins	Danielle Hinton	10 & 11 AM
Shivani Agarwal	Stavros Valavanis	12 & 1 PM
Dimitri Bertsekas (6.431)	Aman Chawla (6.431)	1 & 2 PM

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Spring 2009)

Question 1
Multiple Choice Questions: **CLEARLY** circle the appropriate choice. Scratch paper is available if needed, though **NO** partial credit will be given for the Multiple Choice.

a. Which of the following statements is NOT true? Oh there are concept qus

(i) If $A \subset B$, then $\mathbf{P}(A) \le \mathbf{P}(B)$.
(ii) If $\mathbf{P}(B) > 0$, then $\mathbf{P}(A|B) \ge \mathbf{P}(A)$. ← smaller domain
(iii) $\mathbf{P}(A \cap B) \ge \mathbf{P}(A) + \mathbf{P}(B) - 1$. Oh wow thought was true
(iv) $\mathbf{P}(A \cap B^c) = \mathbf{P}(A \cup B) - \mathbf{P}(B)$.

- if disjoint then $P(A|B) = 0 < P(A)$

think all possible situations!

b. We throw n identical balls into m urns at random, where each urn is equally likely and each throw is independent of any other throw. What is the probability that the i-th urn is empty?

(i) $\left(1 - \frac{1}{m}\right)^n$ partitions
(ii) $\left(1 - \frac{1}{n}\right)^m$
(iii) $\binom{m}{n}\left(1 - \frac{1}{n}\right)^m$?, no clue
(iv) $\binom{n}{m}\left(\frac{1}{m}\right)^n$ guessing

ith ball in ith urn is $\frac{1}{m}$
so not going in $(1-\frac{1}{m})$
all throws independent $(1-\frac{1}{m})^n$

c. We toss two fair coins simultaneously and independently. If the outcomes of the two coins are the same, we win; otherwise, we lose. Let A be the event that the first coin comes up heads, B be the event that the second coin comes up heads, and C be the event that we win. Which of the following statements is false?

wait a sec answer sheet has diff choices and asks which is true - think I was right

(i) Events A and B are independent.
(ii) Events A and C are *not* independent.
(iii) Events A and B are *not* conditionally independent given C.
(iv) The probability of winning is 1/2.

H T H T

$P(A) = 2/4$
$P(B) = 2/4$
$P(C) = 2/4$
$P(A \cap C) = 1/4$ and $P(A), P(C)$ so ind

knowing that you won ~~adds~~ doesn't add. info

d. For a biased coin, the probability of "heads" is 1/3. Let h be the number of heads in five independent coin tosses. What is the probability $\mathbf{P}$(first toss is a head | $h = 1$ or $h = 5$)?

~~geometric~~ ? no simple

(i) $\frac{\frac{1}{3}(\frac{2}{3})^4}{5\frac{1}{3}(\frac{2}{3})^4+(\frac{1}{3})^5}$
(ii) $\frac{\frac{1}{3}(\frac{2}{3})^4}{\frac{1}{3}(\frac{2}{3})^4+(\frac{1}{3})^5}$
(iii) $\frac{\frac{1}{3}(\frac{2}{3})^4+(\frac{1}{3})^5}{5\frac{1}{3}(\frac{2}{3})^4+(\frac{1}{3})^5}$ guessish ✓
(iv) $\frac{1}{5}$

$\frac{P(A \cap B)}{P(B)}$ $P(A) = \frac{1}{3}$ $P(B) = \binom{5}{1}\frac{1}{3}\left(\frac{2}{3}\right)^4 + \binom{5}{5}\left(\frac{1}{3}\right)^5$

? both of these are special but what?

Come back - kinda hard for multiple choice!

Separate sheet →

e. A well-shuffled deck of 52 cards is dealt evenly to two players (26 cards each). What is the probability that player 1 gets all the aces?

(i) $\frac{\binom{48}{22}}{\binom{52}{26}}$

(ii) $\frac{4\binom{48}{22}}{\binom{52}{26}}$

(iii) $\frac{48!}{22!}\frac{52!}{26!}$

(iv) $\frac{4!\binom{48}{22}}{\binom{52}{26}}$

discrete uniform law ($\frac{P(\text{1 gets ace})}{P(\text{all possible outcomes})}$)

$\frac{\text{4 aces}}{\text{26 cards}}$ $\binom{26}{4}$ diff

$\binom{52}{26}$ → # hands a player can have

← but why the $\binom{48}{22}$, total guess

$\frac{26\times 25\times 24\times 23}{52\times 51\times 50\times 49}$

Given 4 aces + 22 other cards
Why are we assuming this is given?
$\binom{4}{4}=1$ ← assuming handled it out first

$\binom{4}{4}\cdot\binom{48}{24}$
counting principle

f. Suppose X, Y and Z are three independent discrete random variables. Then, X and $Y + Z$ are

(i) always ✓
(ii) sometimes
(iii) never

independent.

g. To obtain a driving licence, Mina needs to pass her driving test. Every time Mina takes a driving test, with probability 1/2, she will clear the test independent of her past. Mina failed her first test. Given this, let Y be the additional number of tests Mina takes before obtaining a licence. Then,

(i) $E[Y] = 1$.
(ii) $E[Y] = 2$.
(iii) $E[Y] = 0$.

~mean
– but each test does not matter

$1\cdot\frac{1}{2} + 0\cdot\frac{1}{2} = \frac{1}{2}$ $\quad \frac{1}{p} = \frac{1}{1/2} = 2$ ← but why divided by?

Geometric
↳ duh trials to 1st success

h. Consider two random variables X and Y, each taking values in $\{1, 2, 3\}$. Let their joint PMF be such that for any $1 \le x, y \le 3$,

$$P_{X,Y}(x,y) = \begin{cases} 0 & \text{if } (x,y) \in \{(1,3),(2,1),(3,2)\} \\ \textit{strictly positive} & \text{otherwise.} \end{cases}$$

Then,

(i) X and Y can be independent or dependent depending upon the *strictly positive* values.
(ii) X and Y are always independent.
(iii) X and Y can never be independent.

Y			
3	X		
2			X
1		X	
	1	2	3 X

← X are evenly distributed

1d

A = 1st toss head

$$P(A \mid \{H=1\} \text{ or } \{H=5\}) = \frac{P(A \cap (\{H=1\} \cup \{H=5\}))}{P(\{H=1\} \cup \{H=5\})}$$

$$= \frac{P(A \cap \{H=1\}) + P(A \cap \{H=5\})}{\ldots}$$

$$= P(H=5) + P(A \cap H=1)$$

know A must have happened

but this was a more complex representation — was it just a simplification? →

$$\frac{\left(\frac{1}{3}\right)^{5} + \frac{1}{3}\left(\frac{2}{3}\right)^{4}}{\binom{5}{1}\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^{4} + \binom{5}{5}\left(\frac{1}{3}\right)^{5}}$$

← don't care about order

← was correct here

$\binom{5}{1} = 5$

$\binom{5}{5} = 1$

1h

Here is that independent thing that confused me

if we know $X=1$ then we know Y can not be 3

$$P_{Y|X}(y|x) \neq P_Y(y)$$

I investage on a later sheet

i. Suppose you play a *matching coins* game with your friend as follows. Both you and your friend have a coin. Each time, you two reveal a side (i.e. H or T) of your coin to each other simultaneously. If the sides match, you WIN a 1 from your friend and if sides do not match then you lose a 1 to your friend. Your friend has a complicated (unknown) strategy in selecting the sides over time. You decide to go with the following simple strategy. Every time, you will toss your unbiased coin independently of everything else, and you will reveal its outcome to your friend (of course, your friend does not know the outcome of your random toss until you reveal it). Then,

(i) On average, you will lose money to your smart friend.
(ii) On average, you will neither lose nor win. That is, your average gain/loss is 0. guess-ish
(iii) On average, you will make money from your friend.

p(1-p) always [illegible] that his will match yours

j. Let $X_i, 1 \leq i \leq 4$ be independent Bernoulli random variable each with mean $p = 0.1$. Let $X = \sum_{i=1}^{4} X_i$. That is, X is a Binomial random variable with parameters $n = 4$ and $p = 0.1$. Then,

(i) $E[X_1|X = 2] = 0.1$.
(ii) $E[X_1|X = 2] = 0.5$.
(iii) $E[X_1|X = 2] = 0.25$.

draw 4 random variables?

X is sum
but if sum = 2 min it can be is 4?

Bernolli = 1 or 0
p = .1
add up 4 ind. tosses

Guessing

So if $\Sigma = 2$ then 1st toss must be 1 or 0 (no info)
Find p that 1st one was one

$$P(X_1 = 1 \mid X = 2) = .5$$

$$= \frac{P(X_1 = 1 \cap X = 2)}{P(X = 2)}$$

$$\frac{p \cdot \binom{3}{1} p (1-p)^2}{\binom{4}{2} p^2 (1-p)^2}$$

← 1 head in last 3 tosses for having 2 heads total

← 2 heads in 4 tosses simply 4 heads total

$$\frac{\binom{3}{1}}{\binom{4}{2}} = .5$$

$$E[\,] = 1 \cdot .5 + 0 \cdot P(X_1 = 0 \mid X = 2)$$
$$= .5$$

Still don't understand what they are asking

Shows its ind., right?

Question 2: $4+3+2=9$

Alice and Bob both need to buy a bicycle. The bike store has a stock of four green, three yellow, and two red bikes. Alice randomly picks one of the bikes and buys it. Immediately after, Bob does the same. The sale price of the green, yellow, and red bikes are \$300, \$200 and \$100, respectively.

Let A be the event that Alice bought a green bike, and B be the event that Bob bought a green bike.

a. What is $\mathbf{P}(A)$? What is $\mathbf{P}(A|B)$?

$P(A) = \frac{4}{9}$ ✓ $\quad \frac{P(A \cap B)}{P(B)} = \frac{\frac{4}{9} \cdot \frac{3}{8}}{\frac{4}{9}} = \frac{3}{8}$ ✓

b. Are A and B independent events? Justify your answer.

No. If A^c, then $P(B) = \frac{4}{8}$

If A then $P(B) = \frac{3}{8}$

$P(A) \neq P(A|B)$ formally

c. What is the probability that at least one of them bought a green bike?

$P(A \cup B) = \frac{4}{9} + \frac{4}{9} = \frac{8}{9}$

$= P(A) + P(B) - P(A \cap B)$

must remove overlap!

$= \frac{4}{9} + \frac{4}{9} - P(A|B) \cdot P(B)$

$= \frac{4}{9} + \frac{4}{9} - \frac{3}{8} \cdot \frac{4}{9}$

$= \frac{13}{18} = .722$

d. What is the probability that Alice and Bob bought bicycles of different colors?

all diff colors

$P(A^c \cap B^c) = (1-\frac{4}{9}) \cdot (1-\frac{4}{9})$

T. 4/9 still?

$P(\{G,G\}) = \frac{4}{9} \cdot \frac{3}{8}$ $\quad \frac{5}{9} \cdot \frac{5}{9}$ $\quad$ P(diff color) = 1 − P(same color)

$P(\{Y,Y\}) = \frac{3}{9} \cdot \frac{2}{8}$ $\quad 1 - (\frac{12}{72} + \frac{6}{72} + \frac{2}{72}) = \frac{13}{18}$

$P(\{R,R\}) = \frac{2}{9} \cdot \frac{1}{8}$

add each color match $= .722$

e. Given that Bob bought a green bike, what is the expected value of the amount of money spent by Alice?

$E[X \mid B]$ $\quad P(B) = \frac{4}{9}$ G given

1 less green

$\frac{3}{8} \cdot 300 + \frac{3}{8} \cdot 200 + \frac{2}{8} \cdot 100$

$\frac{900}{8} + \frac{600}{8} + \frac{200}{8} = \frac{1700}{8} \approx \$212.5$ ✓

f. Let G be the number of green bikes that remain on the store after Alice and Bob's visit. Compute $\mathbf{P}(B|G=3)$.

we know 1 bike is sold

$P(A \cap B^c) + P(A^c \cap B)$ divide + conqur

$\frac{4}{9} \cdot (1-\frac{3}{8}) + \frac{4}{9} \cdot (1-\frac{3}{8})$

$\frac{4}{9} \cdot \frac{5}{8} + \frac{4}{9} \cdot \frac{5}{8}$

$\frac{2 \cdot 4 \cdot 5}{9 \cdot 8} = \frac{40}{72} = \frac{20}{36} = \frac{10}{18} = \frac{5}{9}$

oh prob 1 bike sold is to Bob

− by symatry $\frac{1}{2}$

− since we knew they sold 1

− don't concern w/ prob of that

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Question 3:

Magic Games Inc. is a store that sells all sorts of fun games. One of its popular products is its magic 4-sided dice. The dice come in pairs; each die can be fair or crooked, and the dice in any pair can function independently or, in some cases, can have magnets inside them that cause them to behave in unpredictable ways when rolled together. Wow

Xavier and Yvonne together buy a pair of dice from this store. Each of them picks a die in the pair; one of them then rolls the two dice together. Let X be the outcome of Xavier's die and Y the outcome of Yvonne's die. The joint PMF of X and Y, $p_{X,Y}(x,y)$, is given by the following figure:

Y \ X	1	2	3	4
4	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{1}{10}$
3	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{1}{10}$	$\frac{1}{20}$
2	$\frac{1}{20}$	$\frac{1}{10}$	$\frac{1}{20}$	$\frac{1}{20}$
1	$\frac{1}{10}$	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{1}{20}$

(a) Find the PMF of the outcome of Xavier's die, $p_X(x)$.

$$\sum_{y=1}^{4} p_{X,Y}(x,y) \quad \begin{cases} 1/4 & x=1 \\ 1/4 & x=2 \\ 1/4 & x=3 \\ 1/4 & x=4 \end{cases}$$

say $\begin{cases} 1/4 & \text{if } x=1,2,3,4 \\ 0 & \text{oterwise} \end{cases}$

(b) Find the PMF of the outcome of Yvonne's die, $p_Y(y)$.

$$\begin{cases} 1/4 & Y=1 \\ 1/4 & Y=2 \\ 1/4 & Y=3 \\ 1/4 & Y=4 \end{cases}$$

(c) Are X and Y independent?

~~Yes - prob of X, Y are always~~ 1/4
– no matter outcome of oter die

No $p_X(x) \neq p_{X|Y}(x|2)$

So when is it ind?

* if knowing info about 1 does not help you w/ oter

Still don't really get it
where is a chart like this that is inp.

3c Independent - see tutorial 3

Y \ X	1	2	3
3	1/20	1/20	2/20
2	2/20	0	4/20
1	3/20	1/20	6/20

$X = 3$ is 2x as likly as $X = 1$
no matter what y is

(conditioned on $X \neq 2$)

- So this is something very difficult to ~~prove~~ achive - very rare case
- and likly on 2

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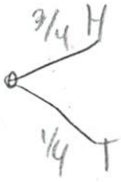
Zach and Wendy are intrigued by Xavier and Yvonne's dice; they visit the store and buy a pair of dice of their own. Again, each of them picks a die in the pair; one of them then rolls the two dice together. Let Z be the outcome of Zach's die and W the outcome of Wendy's die. The joint PMF of Z and W, $p_{Z,W}(z,w)$, is given by the following figure:

W \ Z	1	2	3	4
4	$\frac{1}{12}$	$\frac{1}{24}$	$\frac{1}{12}$	$\frac{1}{24}$
3	$\frac{1}{12}$	$\frac{1}{24}$	$\frac{1}{12}$	$\frac{1}{24}$
2	$\frac{1}{12}$	$\frac{1}{24}$	$\frac{1}{12}$	$\frac{1}{24}$
1	$\frac{1}{12}$	$\frac{1}{24}$	$\frac{1}{12}$	$\frac{1}{24}$

The store also sells a variety of magic coins, some fair and some crooked. Alice buys a coin that on each toss comes up heads with probability 3/4.

(d) Wondering whether to buy some dice as well, Alice decides to try out her friends' dice first. She does the following. First, she tosses her coin. If the coin comes up heads, she borrows Xavier and Yvonne's dice pair and rolls the two dice; if the coin comes up tails, she borrows Zach and Wendy's dice pair and rolls those instead. What is the probability that she rolls a double, i.e., that both dice in the pair she rolls show the same number?

$P(D|A)\,P(A) + P(D|A^c)\,P(A^c)$



$\frac{3}{4}\cdot\frac{1}{10} + \cancel{\frac{3}{4}\cdot\frac{1}{10} + \frac{3}{4}\cdot\frac{1}{10} + \frac{3}{4}\cdot\frac{1}{10}} +$

$\frac{1}{4}\cdot\frac{1}{12} + \cancel{\frac{1}{4}\cdot\frac{1}{24} + \frac{1}{4}\cdot\frac{1}{12} + \frac{1}{4}\cdot\frac{1}{24}}$ $+\frac{1}{24}$

$.3 + \frac{1}{16} = \cancel{.3833}$

correct

~~why only once?~~

no I had it right just did math wrong

$\frac{58}{160} = .3625$

So I got $4\cdot\frac{1}{4}\cdot\frac{1}{12} = \frac{1}{12}$

they have $\frac{1}{4}\cdot\frac{1}{4}$

Oh I see some are $\frac{1}{24}$ – copy error

10/12

(e) Alice has still not made up her mind about the dice. She tries another experiment. First, she tosses her coin. If the coin comes up heads, she takes Xavier and Yvonne's dice pair and rolls the dice repeatedly until she gets a double; if the coin comes up tails, she does the same with Zach and Wendy's dice. What is the expected number of times she will need to roll the dice pair she chooses? (Assume that if a given pair of dice is rolled repeatedly, the outcomes of the different rolls are independent.)

3/4 H → ~~tan~~ geometric

1/4 T → geometric

1 = double

$\frac{1}{10}+\frac{1}{10}+\frac{1}{10}+\frac{1}{10} = \frac{4}{10} = p^x$

$\frac{1}{12}+\frac{1}{24}+\frac{1}{12}+\frac{1}{24} = \frac{6}{24} = p^z$

$\cancel{\frac{3}{4}\left(\frac{4}{10}\right)+\frac{1}{4}\left(\frac{6}{24}\right)}$ – that was last problem

$\frac{3}{4}\cancel{(1-p^x)^{k-1}}$

expected value $= \frac{1}{p}$

damn – I did the var – so stupid

$\frac{3}{4}\left(\frac{\cancel{1-p^x}}{\cancel{p^2}p^x}\right) + \frac{1}{4}\left(\frac{\cancel{1-p^z}}{\cancel{p}p^z}\right)$

$\frac{3}{4}\cdot\cancel{3.75}\ \frac{1}{.4} + \frac{1}{4}\cdot\cancel{12}\ \frac{1}{.25}$

$\frac{23}{8}$ ~~5.81~~ 2.875 – seems weird

(f) Alice is bored with the dice and decides to experiment with her coin instead. She tosses the coin until she has seen a total of 11 heads. Let R be the number of tails she sees. Find $\mathbf{E}[R]$. (Assume independent tosses.)

$p[H] = \frac{3}{4}$

$p[T] = \frac{1}{4}$

R^c = # heads

$R = 1 - R^c$

$E[R] = 1 - E[R^c]$

Binomial? Geometric?

:(totally confused

T = time /tosses/ until sees 11 heads (each $p = \frac{3}{4}$)

tails $= T - 11$

$E[R] = E[T] - 11$

$= 11 \cdot \frac{1}{3/4} - 11$ ← geometric

$14.666 - 11$

$= \frac{11}{12} = .9167$ ← that's not $\frac{11}{12}$

:(solution wrong?

(g) Alice is still playing with her coin. Let A be the event that the second head she sees occurs on the 7th coin toss, and let S be the position of the first head. Find the conditional PMF of S given the event A, $p_{S|A}(s)$.

Want a PMF

A = 2nd head is on 7th toss
S = position 1st head

|_ _ H _ _ _| |H|

1 head
6 tosses

$\binom{6}{1} p = 6p(1-p)^5$

$6\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^5$

$= .00439 = \frac{9}{2048}$

count the 7th head

$\binom{6}{1}(1-p)^5 p^2$ $p = 3/4$

$S = s$ with $A \Rightarrow P(S=s \cap A)$

$= (1-p)^5 p^2$ for all values of S

$\frac{P(S=s \cap A)}{P(A)}$ = uniform dist.

$\begin{cases} \frac{1}{6} & s = 1,2,\ldots 6 \\ 0 & \text{otherwise} \end{cases}$

read carefully!

(h) Alice's friend Bob buys a coin from the same store that turns out to be fair, i.e., that on any toss comes up heads with probability 1/2. He tosses the coin repeatedly until he has seen either a total of 11 heads or a total of 11 tails. Let U be the number of times he will need to toss the coin. Find the PMF of U, $p_U(u)$. (Assume independent tosses.)

Want PMF again

- this was like that 1st one
- how do you do this formally, guessing $\sim \frac{1}{2}$

Intuition

Bob must toss a coin at least 11 times and at most 21 times

Intersection U tosses and 11 are heads is sum of $\binom{u-1}{10}$ sequences end w/ a head

dont forget this is true

- prob of each seq is $\left(\frac{1}{2}\right)^u$

If we consider any sequence w/ u tosses it is =ly likely to have 11 heads u − 11 tails, u − 11 heads 11 tails) P same

either w/ w/ H+T

$p_U(u) = \begin{cases} 2\binom{u-1}{10}\left(\frac{1}{2}\right)^u & u = 11, \ldots 21 \\ 0 & \text{otherwise} \end{cases}$

don't really get still − challenge: if $p \neq \frac{1}{2}$

Redo Practice 10/12

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Question 1
Multiple Choice Questions: **CLEARLY** circle the appropriate choice. Scratch paper is available if needed, though **NO** partial credit will be given for the Multiple Choice.

a. Which of the following statements is NOT true?

(i) If $A \subset B$, then $\mathbf{P}(A) \leq \mathbf{P}(B)$.

(ii) If $\mathbf{P}(B) > 0$, then $\mathbf{P}(A|B) \geq \mathbf{P}(A)$.

(iii) $\mathbf{P}(A \cap B) \geq \mathbf{P}(A) + \mathbf{P}(B) - 1$.

(iv) $\mathbf{P}(A \cap B^c) = \mathbf{P}(A \cup B) - \mathbf{P}(B)$.

b. We throw n identical balls into m urns at random, where each urn is equally likely and each throw is independent of any other throw. What is the probability that the i-th urn is empty?

(i) $\left(1 - \frac{1}{m}\right)^n$

(ii) $\left(1 - \frac{1}{n}\right)^m$

(iii) $\binom{m}{n}\left(1 - \frac{1}{n}\right)^m$

(iv) $\binom{n}{m}\left(\frac{1}{m}\right)^n$

c. We toss two fair coins simultaneously and independently. If the outcomes of the two coins are the same, we win; otherwise, we lose. Let A be the event that the first coin comes up heads, B be the event that the second coin comes up heads, and C be the event that we win. Which of the following statements is false?

(i) Events A and B are independent.

(ii) Events A and C are *not* independent.

(iii) Events A and B are *not* conditionally independent given C.

(iv) The probability of winning is 1/2.

d. For a biased coin, the probability of "heads" is 1/3. Let h be the number of heads in five independent coin tosses. What is the probability $\mathbf{P}(\text{first toss is a head} \mid h = 1 \text{ or } h = 5)$?

(i) $\frac{\frac{1}{3}(\frac{2}{3})^4}{5\frac{1}{3}(\frac{2}{3})^4+(\frac{1}{3})^5}$

(ii) $\frac{\frac{1}{3}(\frac{2}{3})^4}{\frac{1}{3}(\frac{2}{3})^4+(\frac{1}{3})^5}$

(iii) $\frac{\frac{1}{3}(\frac{2}{3})^4+(\frac{1}{3})^5}{5\frac{1}{3}(\frac{2}{3})^4+(\frac{1}{3})^5}$

(iv) $\frac{1}{5}$

e. A well-shuffled deck of 52 cards is dealt evenly to two players (26 cards each). What is the probability that player 1 gets all the aces?

(i) $\dfrac{\binom{48}{22}}{\binom{52}{26}}$

(ii) $\dfrac{4\binom{48}{22}}{\binom{52}{26}}$

(iii) $\frac{48!}{22!}\frac{52!}{26!}$

(iv) $\dfrac{4!\binom{48}{22}}{\binom{52}{26}}$

f. Suppose X, Y and Z are three independent discrete random variables. Then, X and $Y + Z$ are

(i) always
(ii) sometimes
(iii) never

independent.

g. To obtain a driving licence, Mina needs to pass her driving test. Every time Mina takes a driving test, with probability 1/2, she will clear the test independent of her past. Mina failed her first test. Given this, let Y be the additional number of tests Mina takes before obtaining a licence. Then,

(i) $E[Y] = 1$.
(ii) $E[Y] = 2$.
(iii) $E[Y] = 0$.

h. Consider two random variables X and Y, each taking values in $\{1, 2, 3\}$. Let their joint PMF be such that for any $1 \le x, y \le 3$,

$$P_{X,Y}(x, y) = \begin{cases} 0 & \text{if } (x, y) \in \{(1,3),(2,1),(3,2)\} \\ \textit{strictly positive} & \text{otherwise.} \end{cases}$$

Then,

(i) X and Y can be independent or dependent depending upon the *strictly positive* values.
(ii) X and Y are always independent.
(iii) X and Y can never be independent.

i. Suppose you play a *matching coins* game with your friend as follows. Both you and your friend have a coin. Each time, you two reveal a side (i.e. H or T) of your coin to each other simultaneously. If the sides match, you WIN a 1 from your friend and if sides do not match then you lose a 1 to your friend. Your friend has a complicated (unknown) strategy in selecting the sides over time. You decide to go with the following simple strategy. Every time, you will toss your unbiased coin independently of everything else, and you will reveal its outcome to your friend (of course, your friend does not know the outcome of your random toss until you reveal it). Then,

(i) On average, you will lose money to your smart friend.

(ii) On average, you will neither lose nor win. That is, your average gain/loss is 0.

(iii) On average, you will make money from your friend.

j. Let $X_i, 1 \leq i \leq 4$ be independent Bernoulli random variable each with mean $p = 0.1$. Let $X = \sum_{i=1}^{4} X_i$. That is, X is a Binomial random variable with parameters $n = 4$ and $p = 0.1$. Then,

(i) $E[X_1|X = 2] = 0.1$.

(ii) $E[X_1|X = 2] = 0.5$.

(iii) $E[X_1|X = 2] = 0.25$.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Spring 2009)

Question 2: $4+3+2=9$

Alice and Bob both need to buy a bicycle. The bike store has a stock of four green, three yellow, and two red bikes. Alice randomly picks one of the bikes and buys it. Immediately after, Bob does the same. The sale price of the green, yellow, and red bikes are \$300, \$200 and \$100, respectively.

Let A be the event that Alice bought a green bike, and B be the event that Bob bought a green bike.

a. What is $\mathbf{P}(A)$? What is $\mathbf{P}(A|B)$?

$P(A) = \frac{4}{9}$ $\quad P(A|B) = \frac{3}{8}$

? one green gone

b. Are A and B independent events? Justify your answer.

No $\quad P(A) \neq P(A|B)$

$\frac{4}{9} \neq \frac{3}{8}$

Bob's choice of bikes influences Alice

c. What is the probability that at least one of them bought a green bike?

$P(A \cup B) = \frac{4}{9} + \frac{4}{9} = \frac{8}{9}$

darn forgot the overlap again!

$\frac{4}{9} + \frac{4}{9} - P(A \cap B)$

$\frac{4}{9} + \frac{4}{9} - \left(\frac{4}{9} \cdot \frac{3}{8}\right)$ $\quad \leftarrow P(A)P(B|A) = \frac{P(A)P(B \cap A)}{P(A)}$

$\frac{8}{9} - \frac{1}{6} = \frac{13}{18} = .7222$ ✓ now right!

d. What is the probability that Alice and Bob bought bicycles of different colors? all colors

$P(\{G,G\}) = \frac{4}{9} \cdot \frac{3}{8} = \frac{1}{6}$

$P\{Y,Y\} \quad \frac{3}{9} \cdot \frac{2}{8} = \frac{1}{12}$

$P(RR \quad \frac{2}{9} \cdot \frac{1}{8} = \frac{1}{36}$

$1 - \left(\frac{1}{6} + \frac{1}{12} + \frac{1}{36}\right)$

$1 - \frac{5}{18}$

$\frac{13}{18} \quad .722$

e. Given that Bob bought a green bike, what is the expected value of the amount of money spent by Alice?

$E[X \mid B]$

So Alice $\frac{3}{8}$

$\frac{3}{8} \cdot 300 + \frac{3}{8} \cdot 200 + \frac{2}{8} \cdot 100$

$\frac{900}{8} + \frac{600}{8} + \frac{200}{8}$

$\frac{1700}{8} = \$212.5$

f. Let G be the number of green bikes that remain on the store after Alice and Bob's visit. Compute $\mathbf{P}(B|G=3)$.

We know 1 green sold
But to who A or B?

– even split $\frac{1}{2}$

– ignored "math way"

"math way"!

$A \setminus B$ = elements of A not in B

$$\frac{P(B \setminus A)}{P(\{A \setminus B\} \cup \{B \setminus A\})} = \frac{20/72}{40/72} = \frac{1}{2}$$

Question 3:

Magic Games Inc. is a store that sells all sorts of fun games. One of its popular products is its magic 4-sided dice. The dice come in pairs; each die can be fair or crooked, and the dice in any pair can function independently or, in some cases, can have magnets inside them that cause them to behave in unpredictable ways when rolled together.

Xavier and Yvonne together buy a pair of dice from this store. Each of them picks a die in the pair; one of them then rolls the two dice together. Let X be the outcome of Xavier's die and Y the outcome of Yvonne's die. The joint PMF of X and Y, $p_{X,Y}(x, y)$, is given by the following figure:

Y \ X	1	2	3	4
4	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{1}{10}$
3	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{1}{10}$	$\frac{1}{20}$
2	$\frac{1}{20}$	$\frac{1}{10}$	$\frac{1}{20}$	$\frac{1}{20}$
1	$\frac{1}{10}$	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{1}{20}$

(a) Find the PMF of the outcome of Xavier's die, $p_X(x)$.

(b) Find the PMF of the outcome of Yvonne's die, $p_Y(y)$.

(c) Are X and Y independent?

Zach and Wendy are intrigued by Xavier and Yvonne's dice; they visit the store and buy a pair of dice of their own. Again, each of them picks a die in the pair; one of them then rolls the two dice together. Let Z be the outcome of Zach's die and W the outcome of Wendy's die. The joint PMF of Z and W, $p_{Z,W}(z,w)$, is given by the following figure:

W \ Z	1	2	3	4
4	$\frac{1}{12}$	$\frac{1}{24}$	$\frac{1}{12}$	$\frac{1}{24}$
3	$\frac{1}{12}$	$\frac{1}{24}$	$\frac{1}{12}$	$\frac{1}{24}$
2	$\frac{1}{12}$	$\frac{1}{24}$	$\frac{1}{12}$	$\frac{1}{24}$
1	$\frac{1}{12}$	$\frac{1}{24}$	$\frac{1}{12}$	$\frac{1}{24}$

The store also sells a variety of magic coins, some fair and some crooked. Alice buys a coin that on each toss comes up heads with probability 3/4.

(d) Wondering whether to buy some dice as well, Alice decides to try out her friends' dice first. She does the following. First, she tosses her coin. If the coin comes up heads, she borrows Xavier and Yvonne's dice pair and rolls the two dice; if the coin comes up tails, she borrows Zach and Wendy's dice pair and rolls those instead. What is the probability that she rolls a double, i.e., that both dice in the pair she rolls show the same number?

(e) Alice has still not made up her mind about the dice. She tries another experiment. First, she tosses her coin. If the coin comes up heads, she takes Xavier and Yvonne's dice pair and rolls the dice repeatedly until she gets a double; if the coin comes up tails, she does the same with Zach and Wendy's dice. What is the expected number of times she will need to roll the dice pair she chooses? (Assume that if a given pair of dice is rolled repeatedly, the outcomes of the different rolls are independent.)

(f) Alice is bored with the dice and decides to experiment with her coin instead. She tosses the coin until she has seen a total of 11 heads. Let R be the number of tails she sees. Find $\mathbf{E}[R]$. (Assume independent tosses.)

(g) Alice is still playing with her coin. Let A be the event that the second head she sees occurs on the 7th coin toss, and let S be the position of the first head. Find the conditional PMF of S given the event A, $p_{S|A}(s)$.

(h) Alice's friend Bob buys a coin from the same store that turns out to be fair, i.e., that on any toss comes up heads with probability 1/2. He tosses the coin repeatedly until he has seen either a total of 11 heads or a total of 11 tails. Let U be the number of times he will need to toss the coin. Find the PMF of U, $p_U(u)$. (Assume independent tosses.)

6.041/6.431 Spring 2009 Quiz 1
Wednesday, March 11, 7:30 - 9:30 PM.
SOLUTIONS

Question	Part	Score	Out of
0			2
1	all		40
2	a		5
	b		2
	c		5
	d		5
	e		5
	f		5
3	a		2
	b		2
	c		2
	d		5
	e		5
	f		5
	g		5
	h		5
Total			100

Name: ____________________

Recitation Instructor: ____________________

- Write your solutions in this quiz packet, only solutions in the quiz packet will be graded.
- Question one, multiple choice questions, will receive no partial credit. Partial credit for question two and three will be awarded.
- You are allowed one two-sided 8.5 by 11 formula sheet plus a calculator.
- You have 120 minutes to complete the quiz.
- Be neat! You will not get credit if we can't read it.
- Graded quizzes will be returned in recitation on Tuesday 3/17.

Problem 0: (2 pts) Write your name, and your assigned recitation instructor's name, on the cover of the quiz booklet. The Instructors are listed below.

Recitation Instructor	Recitation Time
Devavrat Shah	10 & 11 AM
Shivani Agarwal	11AM & 12PM
Asu Ozdaglar	12 & 1 PM
Pablo Parrilo (6.431)	10 & 11AM

Question 1
Multiple Choice Questions: **CLEARLY** circle the appropriate choice. Scratch paper is available if needed, though **NO** partial credit will be given for the Multiple Choice. **Each multiple choice question is worth 4 points.**

a. Which of the following statements is NOT true?

(i) If $A \subset B$, then $\mathbf{P}(A) \leq \mathbf{P}(B)$.

(ii) $\boxed{\text{If } \mathbf{P}(B) > 0, \text{ then } \mathbf{P}(A|B) \geq \mathbf{P}(A).}$

(iii) $\mathbf{P}(A \cap B) \geq \mathbf{P}(A) + \mathbf{P}(B) - 1$.

(iv) $\mathbf{P}(A \cap B^c) = \mathbf{P}(A \cup B) - \mathbf{P}(B)$.

Solution: A counterexample: if we have two events A, B such that $P(B) > 0$ and $P(A) > 0$, but $A \cap B = \phi$, then $P(A|B) = 0$, but $P(A) > P(A|B)$. It's easy to come up with examples like this: for example, take any sample space with event A such that $P(A) > 0$, and $P(A^c > 0)$, it follows that $P(A|A^c) = 0$, but $P(A) > 0$.

b. We throw n identical balls into m urns at random, where each urn is equally likely and each throw is independent of any other throw. What is the probability that the i-th urn is empty?

(i) $\boxed{\left(1 - \frac{1}{m}\right)^n}$

(ii) $\left(1 - \frac{1}{n}\right)^m$

(iii) $\binom{m}{n}\left(1 - \frac{1}{n}\right)^m$

(iv) $\binom{n}{m}\left(\frac{1}{m}\right)^n$

Solution: The probability of the jth ball going into the ith urn is $1/m$. Hence, the probability of the jth ball not going into the ith urn is $(1 - 1/m)$. Since all throws are independent from one another, we can multiply these probabilities: the probability of all n balls not going into the ith urn, i.e. it is empty, is $\left(1 - \frac{1}{m}\right)^n$.

c. We toss two fair coins simultaneously and independently. If the outcomes of the two coin tosses are the same, we win; otherwise, we lose. Let A be the event that the first coin comes up heads, B be the event that the second coin comes up heads, and C be the event that we win. Which of the following statements is true?

(i) Events A and B are not independent.

(ii) $\boxed{\text{Events } A \text{ and } C \text{ are independent.}}$

(iii) Events A and B are conditionally independent given C.

(iv) The probability of winning is $3/4$.

Solution: The sample space in this case is $\Omega = \{(H,H),(H,T),(T,H),(T,T)\}$. The probability law is a uniform distribution over this space. We have $A = \{(H,H),(H,T)\}$, $B = \{(H,H),(T,H)\}$, and $C = \{(H,H),(T,T)\}$. By the discrete uniform law, $P(A) = P(B) =$

$P(C) = 1/2$. We also have $P(A \cap C) = 1/4$, hence $P(A \cap C) = P(A)P(C)$, and the two events are independent. Intuitively, knowing that you won adds no information about whether your coin turned up heads or not: stating this formally, we have $P(A|C) = P(A)$.

d. For a biased coin, the probability of "heads" is 1/3. Let H be the number of heads in five independent coin tosses. What is the probability $\mathbf{P}(\text{first toss is a head} \mid H = 1 \text{ or } H = 5)$?

(i) $\frac{\frac{1}{3}(\frac{2}{3})^4}{5\frac{1}{3}(\frac{2}{3})^4+(\frac{1}{3})^5}$

(ii) $\frac{\frac{1}{3}(\frac{2}{3})^4}{\frac{1}{3}(\frac{2}{3})^4+(\frac{1}{3})^5}$

(iii) $\boxed{\frac{\frac{1}{3}(\frac{2}{3})^4+(\frac{1}{3})^5}{5\frac{1}{3}(\frac{2}{3})^4+(\frac{1}{3})^5}}$

(iv) $\frac{1}{5}$

Solution: Let A be the event that the first toss is a head.

$$\begin{aligned}
P(A|\{H=1\} \text{ or } \{H=5\}) &= \frac{P(A \cap (\{H=1\} \cup \{H=5\}))}{P(\{H=1\} \cup \{H=5\})} \\
&= \frac{P((A \cap \{H=1\}) \cup (A \cap \{H=5\})}{P(\{H=1\} \cup \{H=5\})} \\
&= \frac{P(\{H=5\}) + P(A \cap \{H=1\})}{P(\{H=1\}) + P(\{H=5\})} \\
&= \frac{(1/3)^5 + (1/3)(2/3)^4}{\binom{5}{1}(1/3)(2/3)^4 + \binom{5}{5}(1/3)^5}.
\end{aligned}$$

e. A well-shuffled deck of 52 cards is dealt evenly to two players (26 cards each). What is the probability that player 1 gets all the aces?

(i) $\boxed{\frac{\binom{48}{22}}{\binom{52}{26}} = \frac{26\times25\times24\times23}{52\times51\times50\times49}}$

(ii) $\frac{4\binom{48}{22}}{\binom{52}{26}} = 4 \times \frac{26\times25\times24\times23}{52\times51\times50\times49}$

(iii) $\frac{48!}{22!}\frac{52!}{26!}$

(iv) $\frac{4!\binom{48}{22}}{\binom{52}{26}} = 4! \times \frac{26\times25\times24\times23}{52\times51\times50\times49}$

Solution: Let A be the event that player 1 gets all aces. By the disrete uniform law,

$$P(A) = \frac{|A|}{|\Omega|}. \tag{1}$$

$|\Omega| = \binom{52}{26}$ is the number of hands (26 cards from 52) player 1 can have. Additionally, once we have given player 1 all aces, then they must be given an additional 22 cards from the remaining 48 cards in the deck. Hence,

$$P(A) = \frac{\binom{48}{22}}{\binom{52}{26}}$$

f. Suppose X, Y and Z are three independent discrete random variables. Then, X and $Y + Z$ are

(i) $\boxed{\text{always}}$

(ii) sometimes

(iii) never

independent.

Solution: Since X is independent of Y and Z, X is independent of $g(Y, Z)$ for any function $g(Y, Z)$, including $g(Y, Z) = Y + Z$ (see page 114 of the book).

g. To obtain a driving licence, Mina needs to pass her driving test. Every time Mina takes a driving test, with probability $1/2$, she will clear the test independent of her past. Mina failed her first test. Given this, let Y be the additional number of tests Mina takes before obtaining a licence. Then,

(i) $E[Y] = 1$.

(ii) $\boxed{E[Y] = 2.}$

(iii) $E[Y] = 0$.

Solution: Y is defined as the number of additional tests Mina takes, so this is independent of the fact that she failed her first test. Y is a geometric RV with $p = 1/2$. Hence, $E[Y] = 1/p = 2$.

h. Consider two random variables X and Y, each taking values in $\{1, 2, 3\}$. Let their joint PMF be such that for any $1 \leq x, y \leq 3$, $P_{X,Y}(x, y) = 0$ if $(x, y) \in \{(1, 3), (2, 1), (3, 2)\}$, and $P_{X,Y}(x, y) > 0$ if $(x, y) \in \{(1, 1), (1, 2), (2, 2), (2, 3), (3, 1), (3.3)\}$. Then,

(i) X and Y can be independent or dependent depending upon the values of $P_{X,Y}(x, y)$ for $(x, y) \in \{(1, 1), (1, 2), (2, 2), (2, 3), (3, 1), (3.3)\}$.

(ii) X and Y are always independent.

(iii) $\boxed{X \text{ and } Y \text{ can never be independent.}}$

Solution: If, for example, we are given information that $X = 1$, we know that Y can never take value 3. However, without this information about X the probability $p_Y(3)$ is strictly positive and so $p_{Y|X}(y|x) \neq p_Y(y)$, for $x = 1$ and $y = 3$, i.e. X and Y can never be independent.

i. Suppose you play a *matching coins* game with your friend as follows. Both you and your friend each have your own coin. Each time, the two of you reveal a side (i.e. H or T) of your coin to each other simultaneously. If the sides match, you WIN \$1 from your friend and if sides do not match then you lose \$1 to your friend. Your friend has a complicated (unknown) strategy in selecting the sides over time. You decide to go with the following simple strategy. Every time, you will toss your unbiased coin independently of everything else, and you will reveal its outcome to your friend (of course, your friend does not know the outcome of your random toss until you reveal it). Then,

(i) On average, you will lose money to your smart friend.

(ii) $\boxed{\text{On average, you will neither lose nor win. That is, your average gain/loss is 0.}}$

(iii) On average, you will make money from your friend.

Solution: Let X_i be a random variable denoting your winnings at the i'th round of the game, i.e., $X_i = 1$ if you win, $X_i = -1$ if you lose. At each round your friend chooses either heads or tails, using some strategy that you don't know about. The key property is that *for any choice that you friend makes, we have* $p_{X_i}(1) = p_{X_i}(-1) = 0.5$: i.e., we always have a 0.5 probability that our coin toss will match the choice made by our friend. It can be verified that $\mathbf{E}[X_i] = 0$, and hence your average gain/loss is 0.

j. Let $X_i, 1 \leq i \leq 4$ be independent Bernoulli random variables each with mean $p = 0.1$. Let $X = \sum_{i=1}^{4} X_i$. Then,

(i) $E[X_1|X = 2] = 0.1$.

(ii) $\boxed{E[X_1|X = 2] = 0.5.}$

(iii) $E[X_1|X = 2] = 0.25$.

Solution: We have $P(X_1 = 1|X = 2) = 0.5$, because

$$\begin{aligned} P(X_1 = 1|X = 2) &= \frac{P(X_1 = 1 \cap X = 2)}{P(X = 2)} \\ &= \frac{p \times \binom{3}{1} p(1-p)^2}{\binom{4}{2} p^2 (1-p)^2} \\ &= \frac{\binom{3}{1}}{\binom{4}{2}} = 0.5 \end{aligned}$$

(Note that $\binom{4}{2} p^2(1-p)^2$ is the probability of seeing 2 heads out of 4 tosses, and $\binom{3}{1} p(1-p)^2$ is the probability of seeing 1 head in the last 3 tosses.)

Hence,

$$\mathbf{E}[X_1|X = 2] = 1 \times P(X_1 = 1|X = 2) + 0 \times P(X_1 = 0|X = 2) = 0.5$$

Question 2:

Alice and Bob both need to buy a bicycle. The bike store has a stock of four green, three yellow, and two red bikes. Alice randomly picks one of the bikes and buys it. Immediately after, Bob does the same. The sale price of the green, yellow, and red bikes are \$300, \$200 and \$100, respectively.

Let A be the event that Alice bought a green bike, and B be the event that Bob bought a green bike.

a. (5 points) What is $\mathbf{P}(A)$? What is $\mathbf{P}(A|B)$?

Solution: We have $\mathbf{P}(A) = 4/9$ (4 green bikes out of 9), and $\mathbf{P}(A|B) = 3/8$ (since we know that Bob has a green bike, Alice can have one of 3 green bikes out of the remaining 8).

b. (2 points) Are A and B independent events? Justify your answer.

Solution: Since $\mathbf{P}(A) \neq \mathbf{P}(A|B)$, the events are *not* independent. Informally, since there is a fixed quantity of green bikes, if Alice buys one, then the chances that Bob buys one too are slightly decreased.

c. (5 points) What is the probability that at least one of them bought a green bike?

Solution: The requested probability is $\mathbf{P}(A \cup B)$. We have

$$\begin{aligned}\mathbf{P}(A \cup B) &= \mathbf{P}(A) + \mathbf{P}(B) - \mathbf{P}(A \cap B) \\ &= \mathbf{P}(A) + \mathbf{P}(B) - \mathbf{P}(A|B) \cdot \mathbf{P}(B) \\ &= \frac{4}{9} + \frac{4}{9} - \frac{3}{8} \cdot \frac{4}{9} = \frac{13}{18} = 0.722.\end{aligned}$$

d. (5 points) What is the probability that Alice and Bob bought bicycles of different colors?

Solution: Let's compute first the probability that Alice and Bob bought bikes of the same color. We have

$$\mathbf{P}(\{G,G\}) = \frac{4}{9}\cdot\frac{3}{8}, \qquad \mathbf{P}(\{Y,Y\}) = \frac{3}{9}\cdot\frac{2}{8}, \qquad \mathbf{P}(\{R,R\}) = \frac{2}{9}\cdot\frac{1}{8}.$$

Therefore, the probability of buying bikes of different color is

$$\mathbf{P}(\text{different color}) = 1 - \mathbf{P}(\text{same color}) = 1 - \left(\frac{12}{72} + \frac{6}{72} + \frac{2}{72}\right) = \frac{13}{18} = 0.722.$$

e. (5 points) Given that Bob bought a green bike, what is the expected value of the amount of money spent by Alice?

Solution: If Bob bought a green bike, then the conditional probabilities of Alice buying a green, yellow, or red bike are $\frac{3}{8}$, $\frac{3}{8}$ and $\frac{2}{8}$, respectively. The expected amount of money spent by Alice is therefore

$$\$300\cdot\frac{3}{8} + \$200\cdot\frac{3}{8} + \$100\cdot\frac{2}{8} = \$212.50.$$

f. (5 points) Let G be the number of green bikes that remain in the store after Alice and Bob's visit. Compute $\mathbf{P}(B|G=3)$.

Solution: If $G=3$, then exactly one green bike was bought. By symmetry, there is equal chance that Alice or Bob bought it, thus $\mathbf{P}(B|G=3) = \frac{1}{2}$. Alternatively, define $A \setminus B$ as the elements of A that are not in B. We have:

$$\begin{aligned}\mathbf{P}(B|G=3) &= \mathbf{P}(B|\{A\setminus B\}\cup\{B\setminus A\}) \\ &= \frac{\mathbf{P}(B\setminus A)}{\mathbf{P}(\{A\setminus B\}\cup\{B\setminus A\})} = \frac{20/72}{40/72} = \frac{1}{2}.\end{aligned}$$

Question 3:

Magic Games Inc. is a store that sells all sorts of fun games. One of its popular products is its magic 4-sided dice. The dice come in pairs; each die can be fair or crooked, and the dice in any pair can function independently or, in some cases, can have magnets inside them that cause them to behave in unpredictable ways when rolled together.

Xavier and Yvonne together buy a pair of dice from this store. Each of them picks a die in the pair; one of them then rolls the two dice together. Let X be the outcome of Xavier's die and Y the outcome of Yvonne's die. The joint PMF of X and Y, $p_{X,Y}(x,y)$, is given by the following figure:

Y \ X	1	2	3	4
4	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{2}{20}$
3	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{2}{20}$	$\frac{1}{20}$
2	$\frac{1}{20}$	$\frac{2}{20}$	$\frac{1}{20}$	$\frac{1}{20}$
1	$\frac{2}{20}$	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{1}{20}$

(a) (2 points) Find the PMF of the outcome of Xavier's die, $p_X(x)$.

Solution:

$$p_X(x) = \sum_{y=1}^{4} p_{X,Y}(x,y) = \begin{cases} \frac{1}{4} & x = 1,2,3,4 \\ 0 & o.w. \end{cases}$$

(b) (2 points) Find the PMF of the outcome of Yvonne's die, $p_Y(y)$.

Solution:

$$p_Y(y) = \sum_{x=1}^{4} p_{X,Y}(x,y) = \begin{cases} \frac{1}{4} & y = 1,2,3,4 \\ 0 & o.w. \end{cases}$$

(c) (2 points) Are X and Y independent?

Solution: No. One of many counter examples: $p_X(x)$ does not equal $p_{X|Y}(x|2)$.

Zach and Wendy are intrigued by Xavier and Yvonne's dice. They visit the store and buy a pair of dice of their own. Again, each of them picks a die in the pair; one of them then rolls the two dice together. Let Z be the outcome of Zach's die and W the outcome of Wendy's die. The joint PMF of Z and W, $p_{Z,W}(z,w)$, is given by the following figure:

W \ Z	1	2	3	4
4	$\frac{2}{24}$	$\frac{1}{24}$	$\frac{2}{24}$	$\frac{1}{24}$
3	$\frac{2}{24}$	$\frac{1}{24}$	$\frac{2}{24}$	$\frac{1}{24}$
2	$\frac{2}{24}$	$\frac{1}{24}$	$\frac{2}{24}$	$\frac{1}{24}$
1	$\frac{2}{24}$	$\frac{1}{24}$	$\frac{2}{24}$	$\frac{1}{24}$

The store also sells a variety of magic coins, some fair and some crooked. Alice buys a coin that on each toss comes up heads with probability 3/4.

(d) (5 points) Wondering whether to buy some dice as well, Alice decides to try out her friends' dice first. She does the following. First, she tosses her coin. If the coin comes up heads, she borrows Xavier and Yvonne's dice pair and rolls the two dice. If the coin comes up tails, she borrows Zach and Wendy's dice pair and rolls those instead. What is the probability that she rolls a double, i.e., that both dice in the pair she rolls show the same number?

Solution: Let event D be the set of all doubles, and let event A be the event that Alice's coin toss results in heads. Using the law of total probability:

$$\begin{aligned} P(D) &= P(D|A)P(A) + P(D|A^c)P(A^c) \\ &= \frac{3}{4} \times \frac{4}{10} + \frac{1}{4} \times \frac{1}{4} \\ &= \frac{58}{160} = .3625 \end{aligned}$$

(e) (5 points) Alice has still not made up her mind about the dice. She tries another experiment. First, she tosses her coin. If the coin comes up heads, she takes Xavier and Yvonne's dice pair and rolls the dice repeatedly until she gets a double; if the coin comes up tails, she does the same with Zach and Wendy's dice. What is the expected number of times she will need to roll the dice pair she chooses? (Assume that if a given pair of dice is rolled repeatedly, the outcomes of the different rolls are independent.)

Solution: Let random variable N be the number of rolls until doubles is rolled. The distribution on N condition on the set of dice being rolled is a geometric random variable. Using the total expectation theorem, the expected value of N is:

$$\begin{aligned} E[N] &= E[N|A]P(A) + E[N|A^c]P(A^c) \\ &= \frac{3}{4} \times \frac{1}{\frac{4}{10}} + \frac{1}{4} \times \frac{1}{\frac{1}{4}} \\ &= \frac{23}{8} = 2.875 \end{aligned}$$

(f) (5 points) Alice is bored with the dice and decides to experiment with her coin instead. She tosses the coin until she has seen a total of 11 heads. Let R be the number of tails she sees. Find $\mathbf{E}[R]$. (Assume independent tosses.)

Solution: The time T until Alice sees a total of 11 heads is the sum of 11 independent and identically distributed geometric random variables with parameter $p = \frac{3}{4}$. Random variable R, the number of tails she sees, is $T - 11$. Thus:

$$\begin{aligned} E[R] &= E[T] - 11 \\ &= 11 \times \frac{1}{\frac{3}{4}} - 11 \\ &= \frac{11}{12} = .9167 \end{aligned}$$

(g) (5 points) Alice tries another experiment with her coin. Let A be the event that the second head she sees occurs on the 7th coin toss, and let S be the position of the first head. Find the conditional PMF of S given the event A, $p_{S|A}(s)$.

Solution: The probability of event A can be found by choosing one of the first 6 outcomes to be a head, the others tails, and then the outcome of the 7th toss to be head, which is $\binom{6}{1}(1-p)^5p^2$, where $p = \frac{3}{4}$. The intersection of $S = s$ with event A, $P(S = s \cap A)$, is an event with probability $(1-p)^5p^2$ for all values of s ($s = 1, \ldots, 6$). Consequently, $p_{S|A}(s) = \frac{P(S=s\cap A)}{P(A)}$ is a uniform distribution over the range of s ($s = 1, \ldots, 6$).

$$p_{S|A}(s) = \begin{cases} \frac{1}{6} & s = 1, \ldots, 6 \\ 0 & o.w. \end{cases}$$

(h) (5 points) Alice's friend Bob buys a coin from the same store that turns out to be fair, i.e., that on any toss comes up heads with probability 1/2. He tosses the coin repeatedly until he has seen either a total of 11 heads or a total of 11 tails. Let U be the number of times he will need to toss the coin. Find the PMF of U, $p_U(u)$. (Assume independent tosses.)

Solution: Bob must toss a coin at least 11 times and at most 21 times in order to have either 11 heads or 11 tails. The intersection of Bob requiring u tosses and 11 of those tosses being heads, is the sum of probability the $\binom{u-1}{10}$ sequences that conclude with a head and have a total of 11 heads. The probability of each of those sequences is $\left(\frac{1}{2}\right)^u$. If we consider any sequence in Bob's experiment with u tosses, since the coin is fair, that sequence is equally likely to have 11 heads and $u-11$ tails or 11 tails and $u-11$ heads. Consequently, the intersection of Bob requiring u tosses and 11 of those tosses being tails is identical to the probability that the sequence had 11 heads. Summing these two mutually exclusive probabilities which total $p_U(u)$:

$$p_U(u) = \begin{cases} 2\binom{u-1}{10}\left(\frac{1}{2}\right)^u & u = 11, \ldots, 21 \\ 0 & o.w. \end{cases}$$

Practice 10/12

6.041/6.431 Fall 2009 Quiz 1
Tuesday, October 13, 12:05 - 12:55 PM.

DO NOT TURN THIS PAGE OVER UNTIL YOU ARE TOLD TO DO SO

Name: ______________________

Recitation Instructor: ______________________

TA: ______________________

Question	Score	Out of
A		2
B.1		10
B.2 (a)		10
B.2 (b i)		12
B.2 (b ii)		12
B.2 (c)		10
B.3 (a)		10
B.3 (b)		12
B.3 (c)		12
B.3 (d i)		5
B.3 (d ii)		5
Your Grade		100

- This quiz has 2 problems, worth a total of 100 points.
- You may tear apart pages 3 and 4, as per your convenience.
- Write your solutions in this quiz booklet, only solutions in this quiz booklet will be graded. Be neat! You will not get credit if we can't read it.
- You are allowed one two-sided, handwritten, 8.5 by 11 formula sheet. Calculators are not allowed.
- Parts B.2 and B.3 can be done independently.

actually useful info here!

- You may give an answer in the form of an arithmetic expression (sums, products, ratios, factorials) of <u>numbers</u> that could be evaluated using a calculator. Expressions like $\binom{8}{3}$ or $\sum_{k=0}^{5}(1/2)^k$ are also fine.
- You have 50 minutes to complete the quiz.
- Graded quizzes will be returned in recitation on Thursday 10/15.

half as short

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Fall 2009)

Problem A: (2 points) Write your name, your assigned recitation instructor's name, and assigned TA's name on the cover of the quiz booklet. The Instructor/TA pairing is listed below.

Recitation Instructor	TA	Recitation Time
Jeffrey Shapiro	Jimmy Li	10 & 11 AM
Danielle Hinton	Uzoma Orji	1 & 2 PM
William Richoux	Ulric Ferner	2 & 3 PM
John Wyatt (6.431)	Aliaa Atwi	11 & 12 PM

Hint table

Summary of Results for Special Random Variables

Discrete Uniform over $[a, b]$:

$$p_X(k) = \begin{cases} \dfrac{1}{b-a+1}, & \text{if } k = a, a+1, \ldots, b, \\ 0, & \text{otherwise,} \end{cases}$$

$$\mathbf{E}[X] = \frac{a+b}{2}, \qquad \text{var}(X) = \frac{(b-a)(b-a+2)}{12}.$$

Bernoulli with Parameter p: (Describes the success or failure in a single trial.)

$$p_X(k) = \begin{cases} p, & \text{if } k = 1, \\ 1-p, & \text{if } k = 0, \end{cases}$$

$$\mathbf{E}[X] = p, \qquad \text{var}(X) = p(1-p).$$

Binomial with Parameters p and n: (Describes the number of successes in n independent Bernoulli trials.)

$$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}, \qquad k = 0, 1, \ldots, n,$$

$$\mathbf{E}[X] = np, \qquad \text{var}(X) = np(1-p).$$

Geometric with Parameter p: (Describes the number of trials until the first success, in a sequence of independent Bernoulli trials.)

$$p_X(k) = (1-p)^{k-1} p, \qquad k = 1, 2, \ldots,$$

$$\mathbf{E}[X] = \frac{1}{p}, \qquad \text{var}(X) = \frac{1-p}{p^2}.$$

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Problem B: (98 points) As a way to practice his probability skills, Bob goes apple picking. The orchard he goes to grows two varieties of apples: gala and honey crisp.

The **proportion of the gala apples** in the orchard is p $(0 < p < 1)$, the **proportion of the honey crisp apples** is $1 - p$. The number of apples in the orchard is so large that you can assume that picking a few apples does not change the proportion of the two varieties.

Independent of all other apples, the probability that a randomly picked **gala apple is ripe** is g and the probability that a randomly picked **honey crisp apple is ripe** is h.

1. **(10 points)** Suppose that Bob picks an apple at random (uniformly) and eats it. Find the probability that it was a **ripe gala** apple.

   Note: Parts 2 and 3 below can be done independently.

2. Suppose that Bob picks n apples at random (independently and uniformly).

   (a) **(10 points)** Find the probability that exactly k of those are **gala apples.**

   (b) Suppose that there are **exactly k gala apples** among the n apples Bob picked. Caleb comes by and gives Bob a **ripe gala** apple to add to his bounty. Bob then picks an apple at random from the $n + 1$ apples and eats it.

   (i) **(12 points)** What is the probability that it was a **ripe apple?**

   (ii) **(12 points)** What is the probability that it was a **gala apple if it was ripe?**

   (c) **(10 points)** Let $n = 20$, and suppose that Bob picked exactly 10 gala apples. What is the probability that the first 10 apples that Bob picked were all gala?

3. Next, Bob tries a different strategy. He starts with a tree of the **gala** variety and picks apples at random from that tree. Once Bob picks an apple off the tree, he carefully examines it to make sure it is ripe. Once he comes across an apple that is not ripe, he moves to **another gala tree.** He does this until he encounters an unripe apple on that second tree. Assume that each tree has a very large, essentially infinite, number of apples.

   (a) **(10 points)** Let X_i be the **number of apples** Bob picks off the ith tree, $(i = 1, 2)$. Write down the PMF, expectation, and variance of X_i.

   (b) **(12 points)** For $i = 1, 2$, let Y_i be the **total** number of **ripe apples** Bob picked from the first i trees. Find the expectation and the variance of Y_2. (Note that $Y_1 = X_1 - 1$ and $Y_2 = (X_1 - 1) + (X_2 - 1)$.)

   (c) **(12 points)** Find the joint PMF of Y_1 and Y_2.

   (d) In the following, answer just "yes" or "no." (Explanations will not be taken into account in grading.)

   (i) **(5 points)** Are X_1 and Y_2 independent?

   (ii) **(5 points)** Are X_2 and Y_1 independent?

Each question is repeated in the following pages. Please write your answer on the appropriate page.

Why?

pick ripe - Picking apples does not change other's prob.

	pick	ripe
G = Gala	p	g
H = Honey	$1-p$	h

Write out

$g = P(\text{ripe} \mid \text{gala})$

$h = P(\text{ripe} \mid \text{honey})$

1. **(10 points)** Suppose that Bob picks an apple at random (uniformly) and eats it. Find the probability that it was a **ripe gala** apple.

$P(\text{Pick gala} \cap \text{Was ripe})$

$= P(\text{Pick gala})\, P(\text{was ripe} \mid \text{pick gala})$ ← total probability theorm

$= p \cdot g$ ✓

Note: Parts 2 and 3 can be done independently.

2. Suppose that Bob picks n apples at random (independently and uniformly).

   (a) **(10 points)** Find the probability that exactly k of those are **gala apples.**

k sucesses → Binomial

$$p_K(k) = \begin{cases} \binom{n}{k} p^k (1-p)^{n-k} & 0 \le k \le n \\ 0 & \text{otherwise} \end{cases}$$

Was close
- just needed to add a bounding constraint although obvious - k must be less than 0

* must always define k *

(b) Suppose that there are **exactly** k **gala apples** among the n apples Bob picked. Caleb comes by and gives Bob a **ripe gala** apple to add to his bounty. Bob then picks an apple at random from the $n + 1$ apples and eats it.

(i) **(12 points)** What is the probability that it was a **ripe apple?**

* decompose correctly

has k + 1

Split it up
- he picks out of basket
- p that is ripe

P(ripe gala) + P(ripe honey)

$p^{k+1} \cdot g + p^{n-k} \cdot h$

P(ripe apple | Calab's ripe gala) P(Calab's ripe gala) + P(ripe apple | k gala) P(k gala) + P(ripe apple | n-k honey crisp) P(honey crisp)

? dit unsure lots of space

what is this property again? P(A|B)P(B)
total prob. theory - add up each section

$= 1 \cdot \frac{1}{n+1} + g \cdot \frac{k}{n+1} + h \cdot \frac{n-k}{n+1}$

know is ripe; prob that was gala; P ripe; prob was honey crisp

$= \frac{1 + gk + h(n-k)}{n+1}$

So this makes a lot of sense. What will make me think of it on the exam???

(ii) **(12 points)** What is the probability that it was a **gala apple if it was ripe?**

$$P(\text{Gala} \mid \text{Ripe}) = \frac{P(\text{Gala} \cap \text{Ripe})}{P(\text{Ripe})}$$

$$= \frac{p^{k+1} \cdot g}{g^{k+1} + h}$$

Confused

= as well

Also realize Bayers ✓

$$P(\text{Gala} \mid \text{Ripe}) = \frac{P(\text{Ripe} \mid \text{gala})\, P(A)}{P(B)}$$

$P(\text{Ripe} \cap \text{Gala}) =$

$P(\text{ripe} \cap \text{gala} \mid \text{Calab's ripe Gala})\, P(\text{Calab's ripe gala}) +$

$P(\text{ripe} \cap \text{gala} \mid k \text{ gala})\, P(k \text{ gala}) +$

$P(\text{ripe} \cap \text{Gala} \mid n-k \text{ honey crisp})\, P(n-k \text{ honey crisp})$

$$= 1 \cdot \frac{1}{n+1} + g\frac{k}{n+1} + 0\,\frac{n-k}{n+1}$$

$$= \frac{1+gk}{n+1}$$

$$= \frac{g \cdot p}{P(\text{ripe})}$$

← This is the hard part still

but solved it on [illegible] before! -did not realize it!

now plug in

$$\frac{\frac{1+gk}{n+1}}{\frac{1+gk+h(n-k)}{n+1}} = \frac{1+gk}{1+gk+h(n-k)}$$

(c) **(10 points)** Let $n = 20$, and suppose that Bob picked exactly 10 gala apples. What is the probability that the first 10 apples that Bob picked were all gala?

G G G G G G G G G G H H H H H H H H H H

is the only seq

$\binom{20}{20} p^{10}(1-p)^{10}$

No! $\boxed{\text{Note the conditionality!}}$

Do the full process each time - no shortcuts!

$p^{10}(1-p)^{10}$

A = 1st 10 are gala

B = know 10 have been picked from 20

$\boxed{P(A|B) = \frac{P(A \cap B)}{P(B)}}$

$P(B) = \binom{20}{10} p^{10}(1-p)^{10}$

$P(A \cap B)$ → seperate into disjoint events

- ok its exactly what I answered

But $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{p^{10}(1-p)^{10}}{\binom{20}{10} p^{10}(1-p^{10})}$

$= \frac{1}{\binom{20}{10}}$

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3. Next, Bob tries a different strategy. He starts with a tree of the gala variety and picks apples at random from that tree. Once Bob picks an apple off the tree, he carefully examines it to make sure it is ripe. Once he comes across an apple that is not ripe, he moves to **another gala tree.** He does this until he encounters an unripe apple on that second tree. Assume that each tree has a very large, essentially infinite, number of apples.

   & independent ✓

   (a) **(10 points)** Let X_i be the **number of apples** Bob picks off the ith tree, $(i = 1, 2)$. Write down the PMF, expectation, and variance of X_i.

Geometric, but reversing failure/sucess $k = 1, 2, 3, \ldots$

PMF $(g)^{k-1}(g-1)$ $(1-(1-g))^{k-1}(1-g)$ but we want k??

what I had, just flip this (to get sign right)

$E[X] = \dfrac{1}{1-g}$ ✓

$\text{Var} = \dfrac{1-(1-g)}{(1-g)^2} = \dfrac{\cancel{2-p}}{\cancel{p^2-2p+1}}$ $\dfrac{g}{(1-g)^2}$ close - but that flip

   (b) **(12 points)** For $i = 1, 2$, let Y_i be the **total** number of **ripe apples** Bob picked from the first i trees. Find the expectation and the variance of Y_2. (Note that $Y_1 = X_1 - 1$ and $Y_2 = (X_1 - 1) + (X_2 - 1)$.)

$i = 1, 2,$ ✗

why -1?

oh he discards the ~~rotten~~ apple of non-ripe

$Y_i = \sum_{i=1}^{i} (X_i - 1)$

do expectation normal way

$E[X] = \text{value} \cdot \text{prob}$

$(X_i - 1) \quad g^{k-1}(g-1)$

just a function of expectations

$E[Y_2] = E[(X_1 - 1) + (X_2 - 1)]$

I was thinking this but did not put

$= E[X_1] + E[X_2] - 2$

$= \cancel{\dfrac{g}{(1-g)^2} + \dfrac{g}{(1-g)^2} - 2}$

$= \dfrac{1}{g-1} + \dfrac{1}{g-1} - 2$

$= \dfrac{2g}{1-g}$

$\text{Var}(X+Y) = \text{var}(X) + \text{var}(Y)$

$= \dfrac{g}{(1-g)^2} + \dfrac{g}{(1-g)^2} - \cancel{2}$

$= \boxed{\dfrac{2g}{(1-g)^2}} - \cancel{2}$

adding + subtracting constaints does not matter

now I rember - but how supposed to know

$= \cancel{2\left(\dfrac{g}{(1-g)^2} - 1\right)}$

(c) **(12 points)** Find the joint PMF of Y_1 and Y_2.

Table? or what? Well what is PMF of one part?

No

~~Ans~~ Let $k \ge 0$ $l \ge k$

The event $Y_1 = k$ is identical to $X_1 = k+1$
$Y_2 = l$ $X_2 = l - k + 1$

Since X_1, X_2 are independent

$$(1-g)\, g^{k+1-1} (1-g)\, g^{l-k+1-1}$$

$$(1-g)^2 g^l \qquad 0 \le k \le l$$

$$p_{Y_1,Y_2}(k,l) = \begin{cases} g^l (1-g)^2 & k = 0, 1, \dots l \text{ and } l = k, k+1, \dots \\ 0 & \text{otherwise} \end{cases}$$

- this is p of l failures (picking ripe apples) and 2 sucesses (unripe)
- PMF does not depend on k - but sample space depends on $k \to l \ge k$
(Bob can't pick more ripe apples in 1st tree than in 1st 2 combined)

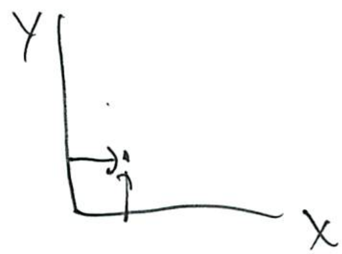
Joint PMF Review

10/12

$p_{X,Y}$
↳ pair of possible values

$$p_{X,Y}(x,y) = P(X=x, Y=y)$$
$$\hookrightarrow = P(\{X=x\} \cap \{Y=y\})$$

- determines probability of event which can be Specified by random variables X, Y
  - So it is that table thing



can sum up ↑ all collumns or rows ⟶
(marginal PMFs)

$$E[g(X,Y)] = \sum_x \sum_y g(x,y)\, p_{X,Y}(x,y)$$

$$E[aX+bY+c] = aE[X] + bE[Y] + c$$

↳ so nice shortcut on some problems

(64)

Back to c

But how does this relate to c?

~~is~~ its not P_{Y_1, Y_2}?

- yes it is

but how are they related?

I don't really get what they give here

$Y_1 = k \qquad X_1 = k+1$

$Y_2 = l \qquad X_2 = l - k + 1$

- So $X_1 = Y_1 + 1$

$Y_1 = X_1 - 1$

$X_2 = Y_2 - Y_1 + 1$

$X_2 = Y_2 - (X_1 - 1) + 1$

$X_2 = Y_2 - X_1 + 2$

$Y_2 = X_2 + X_1 - 2$ ← oh yeah it does work out!

But where does this idea of $k \to Y_1$, $l \to Y_2$ come from?

What sample problem was like that??

(2)

We know X_1, X_2 are ind.

But how does that help?

Multiply PMFs

$$X = g^{k-1}(1-g) \cdot \cancel{g^{k-1}(1-g)}$$

but we want

$$Y_1 \cdot Y_2 \rightarrow g^{\overbrace{k}^{k+1}-1}(1-g) \cdot (1-g)\, g^{\overbrace{k}^{l-k+1}-1}$$

$$g^{k+1-1}(1-g) \cdot (1-g)\, g^{l-k+1-1}$$

ok

but how did we know to sub in in that form

and remember what X is $\rightarrow$ the # of apples picked

Simplify

$$g^{k}(1-g) \cdot (1-g)\, g^{l-k}$$

$$(1-g)^2\, g^{l-k+k}$$

Make sure that am able to recognize this step! (the algebra)

$$(1-g)^2\, g^{l}$$

~~And then define what l~~

(3)

Then just define valid values of l and k

$k = 0, 1, \ldots$

$l = k, k+1, \ldots$ ← How do you know this is what it is?

well x_2

$l = k+1$

$x_2 = k+1 - k + 1 = 2$

↑ k must drop out!

(d) In the following, answer just "yes" or "no." (Explanations will not be taken into account in grading.)

(i) **(5 points)** Are X_1 and Y_2 independent?

No - Y_2 is sum of $X_1 - 1 + X_2 - 2$
right here

(ii) **(5 points)** Are X_2 and Y_1 independent?

Yes
$Y_1 = X_1 - 1$
no X_2

So that was a major failure

- How will I do better later today

Been studying 3 hrs now

- feels like more!
- but I do feel I can do much more

1 hr now till recitation

- which is review

then gym

Then ~~then~~ 3 hrs before exam

Looking back all except joint PMF are make sense

and I can do

- But how to know how to do from scratch?

Massachusetts Institute of Technology

Department of Electrical Engineering & Computer Science

6.041/6.431: Probabilistic Systems Analysis

(Fall 2009)

Quiz 1 Solutions:
October 13, 2009

1. **(10 points)** We start first by listing the following probabilities:

$$\begin{aligned} \mathbf{P}(\text{gala}) &= p \\ \mathbf{P}(\text{honey crisp}) &= 1-p \\ \mathbf{P}(\text{ripe} \mid \text{gala}) &= g \\ \mathbf{P}(\text{ripe} \mid \text{honey crisp}) &= h. \end{aligned}$$

The probability that Bob ate a ripe gala is:

$$\begin{aligned} \mathbf{P}(\text{ripe gala}) &= \mathbf{P}(\text{gala})\mathbf{P}(\text{ripe} \mid \text{gala}) \\ &= pg. \end{aligned}$$

2. (a) **(10 points)** If K is the number of gala apples selected from the n apples, then K is a binomial random variable. The probability that $K = k$ is:

$$p_K(k) = \begin{cases} \binom{n}{k} p^k (1-p)^{n-k}, & 0 \le k \le n \\ 0, & \text{otherwise.} \end{cases}$$

(b) i. **(12 points)** Among the $n+1$ apples, his bounty includes: 1 ripe gala from Caleb, k gala, $n-k$ honey crisp. We can compute the probability of eating a ripe apple conditioned on Bob selecting an apple from each of the three disjoint cases listed above. The total probability theorem allows us to then find the probability of Bob eating a ripe apple.

$$\begin{aligned} \mathbf{P}(\text{ripe apple}) &= \mathbf{P}(\text{ripe apple} \mid \text{Caleb's ripe gala})\mathbf{P}(\text{Caleb's ripe gala}) \\ &\quad + \mathbf{P}(\text{ripe apple} \mid k \text{ gala})\mathbf{P}(k \text{ gala}) \\ &\quad + \mathbf{P}(\text{ripe apple} \mid n-k \text{ honey crisp})\mathbf{P}(n-k \text{ honey crisp}) \\ &= 1 \cdot \frac{1}{n+1} + g \cdot \frac{k}{n+1} + h \cdot \frac{n-k}{n+1} \\ &= \frac{1 + gk + h(n-k)}{n+1}. \end{aligned}$$

ii. **(12 points)** Using Bayes' Rule:

$$\mathbf{P}(\text{gala} \mid \text{ripe apple}) = \frac{\mathbf{P}(\text{ripe} \cap \text{gala})}{\mathbf{P}(\text{ripe apple})}.$$

We can compute the $\mathbf{P}(\text{ripe} \cap \text{gala})$ as such:

$$\begin{aligned}
\mathbf{P}(\text{ripe} \cap \text{gala}) &= \mathbf{P}(\text{ripe} \cap \text{gala} \mid \text{Caleb's ripe gala})\mathbf{P}(\text{Caleb's ripe gala}) \\
&\quad + \mathbf{P}(\text{ripe} \cap \text{gala} \mid k \text{ gala})\mathbf{P}(k \text{ gala}) \\
&\quad + \mathbf{P}(\text{ripe} \cap \text{gala} \mid n-k \text{ honey crisp})\mathbf{P}(n-k \text{ honey crisp}) \\
&= 1 \cdot \frac{1}{n+1} + g \cdot \frac{k}{n+1} + 0 \cdot \frac{n-k}{n+1} \\
&= \frac{1+gk}{n+1}.
\end{aligned}$$

Combining this result with that of (i),

$$\mathbf{P}(\text{gala} \mid \text{ripe apple}) = \frac{\frac{1+gk}{n+1}}{\frac{1+gk+h(n-k)}{n+1}} = \frac{1+gk}{1+gk+h(n-k)}.$$

(c) **(10 points)** Let A be the event that the first 10 apples picked were all gala and let B be the event that exactly 10 gala apples were picked out of the 20 apples.

$$\mathbf{P}(A \mid B) = \frac{\mathbf{P}(A \cap B)}{\mathbf{P}(B)}.$$

The probability of the event B can be computed by using the result in (a), where $n = 20$ and $k = 10$. $\mathbf{P}(B) = \binom{20}{10}p^{10}(1-p)^{10}$. $\mathbf{P}(A \cap B)$ can be computed by seperating the event of A and B into disjoint events. The event $\{A \cap B\} = \{$1st 10 apples picked are gala and the last 10 apples are honey crisp$\}$. $\mathbf{P}(A \cap B) = p^{10}(1-p)^{10}$. Therefore,

$$\mathbf{P}(A \mid B) = \frac{p^{10}(1-p)^{10}}{\binom{20}{10}p^{10}(1-p)^{10}} = \frac{1}{\binom{20}{10}}.$$

3. (a) **(10 points)** Bob picks apples from each tree until he finds one that is not ripe, and so if we think of the event "picking an unripe apple" as a "success", then X_i is the number of apples picked (i.e. trials) until we get our first success. Therefore X_i is a geometrically distributed random variable with probability of success $1-g$.

Since each tree has a random collection of apples, the apples Bob picks from the first tree are independent of the apples he picks from the second tree, and therefore X_1 and X_2 are independent and identically distributed random variables.

The PMF of X_i is

$$\begin{aligned}
p_{X_1}(k) = p_{X_2}(k) &= (1-(1-g))^{k-1}(1-g) \\
&= \begin{cases} g^{k-1}(1-g), & k = 1, 2, 3, \ldots, \\ 0, & \text{otherwise}, \end{cases}
\end{aligned}$$

the expectation of X_i is

$$\mathbf{E}[X_1] = \mathbf{E}[X_2] = \frac{1}{1-g},$$

and the variance of X_i is

$$\begin{aligned}
\text{var}(X_1) = \text{var}(X_2) &= \frac{1-(1-g)}{(1-g)^2} \\
&= \frac{g}{(1-g)^2}.
\end{aligned}$$

(b) **(12 points)** Given that $Y_2 = (X_1 - 1) + (X_2 - 1)$, we can use the linearity property of expectations to find the expectation of Y_2.

$$\begin{aligned}
\mathbf{E}[Y_2] &= \mathbf{E}[(X_1 - 1) + (X_2 - 1)] \\
&= \mathbf{E}[X_1] + \mathbf{E}[X_2] - 2 \\
&= \frac{2}{1-g} - 2 \\
&= \frac{2g}{1-g}.
\end{aligned}$$

Since adding or subtracting a constant from a random variable has no effect on its variance, and since X_1 and X_2 are independent, we have

$$\begin{aligned}
\mathrm{var}(Y_2) &= \mathrm{var}((X_1 - 1) + (X_2 - 1)) \\
&= \mathrm{var}(X_1 + X_2) \\
&= \mathrm{var}(X_1) + \mathrm{var}(X_2) \\
&= \frac{2g}{(1-g)^2}.
\end{aligned}$$

(c) **(12 points)**

Let $k \geq 0$ and $\ell \geq k$. The event "$Y_1 = k$ and $Y_2 = \ell$" is identical to the event "$X_1 = k + 1$ and $X_2 = \ell - k + 1$". Since X_1 and X_2 are independent, the desired probability is $(1-g)g^{k+1-1}(1-g)g^{\ell-k+1-1} = (1-g)^2 g^\ell$, when $0 \leq k \leq \ell$, and zero otherwise.

$$p_{Y_1,Y_2}(k,\ell) = \begin{cases} g^\ell(1-g)^2, & k = 0, 1, \ldots \ell, \text{ and } \ell = k, k+1, \ldots, \\ 0, & \text{otherwise.} \end{cases}$$

Note that we can interpret this result as the probability of ℓ failures (i.e. picking ℓ ripe apples) and two successes (i.e. the two unripe apples that cause Bob to stop at each tree). Note also that although the expression for the joint PMF doesn't depend on k, the sample space does depend on k because $\ell \geq k$: Bob cannot pick more ripe apples in the first tree than he does in the first two trees combined.

(d) (i) **(5 points)** No. X_1 and Y_2 are not independent because $Y_2 \geq X_1 - 1$ and so knowing X_1 gives us information about Y_2. Specifically, if we know that Bob picked 10 total apples from the first tree (i.e. $X_1 = 10$), then we also know that Bob must have picked at least 9 ripe apples from the first two trees combined (i.e. $Y_2 \geq 9$).

(ii) **(5 points)** Yes. X_2 relates only to the second tree and Y_1 relates only to the first tree, and since the picking of apples is independent across trees, the two random variables are independent.