Massachusetts Institute of Technology
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Fall 2010)

Quiz 2 Announcements

?, Confused

Quiz 2: Closed-book, with two handwritten double-sided 8.5 x 11 formula sheet permitted. Please arrive early to find your seat before the prompt start at 7:30PM. Calculators will not be allowed (there won't be any hard numerical calculations in the quiz).

guess two allowed so don't need to duplicate 1st one

> *Date:* Tuesday, November 2
> *Time:* 7:30 - 9:30 PM
> *Location:* 54-100
> *Content:* Quiz 2 will cover the following class material (all boundaries are inclusive)
> Lectures 1 thru 12
> Textbook chapters 1 and 4 (except section 4.4 on transforms)
> Recitations 1 thru 13
> Tutorials 1 thru 6
> Problem Sets 1 thru 6

Instructions: Multiple choice questions, if any, have a single correct answer, and no credit will be given for any incorrect answers. Other questions require fully-reasoned, convincing answers; partial credit is possible and reaching a correct conclusion does not guarantee full credit on these questions.

Practice Quizzes: Two past quizzes with full solutions are available on the OCW website (Spring05 & Spring06). An additional two quizzes have been posted on the course website (Spring08 & Fall09), which will be reviewed at the TA quiz 2 review session. Please note Quiz 2 coverage, course coverage, and course emphasis change each term. Hence past quizzes are not necessarily indicative of this term's quiz. Material presented in lecture, recitation, tutorial, and problem set exercises should be your primary source of preparation.
`http://ocw.mit.edu/OcwWeb/web/home/home/index.htm`
`http://stellar/S/course/6/fa10/6.041/materials.html`

Office Hours: The majority of the regular staff office hours are held on Monday, Tuesday, and Friday before the quiz date. Please check the course website for updates to times and any additional hours.

Optional 6.041/6.431 Quiz Review Session: There will be a two-hour 6.041/6.431 quiz review session administered by two TAs. The session will consist of two parts. In the first hour, a concise overview of the theory will be presented. In the second hour, selected problems from past quizzes will be solved. Though completely optional, the quiz review is a great opportunity to reinforce your understanding of the material and perhaps gain new insight. Details for the quiz review:

> *Date:* Thursday, October 28
> *Time:* 7:30 - 9:30 PM
> *Location:* 32-123

Problems for the quiz review will be selected from the Spring 2008 and Fall 2009 Quiz 2 (each available on the course website under `Quiz Material`). We will review as many problems as time permits. Full solutions will be posted on-line following the review. We strongly recommend working through the problems before coming to the quiz review.

6.041/6.431 Probabilistic Systems Analysis

Quiz II Review
Fall 2010

1 Probability Density Functions (PDF)

For a continuous RV X with PDF $f_X(x)$,

$$P(a \le X \le b) = \int_a^b f_X(x)dx$$
$$P(X \in A) = \int_A f_X(x)dx$$

Properties:

- Nonnegativity:

$$f_X(x) \ge 0 \ \forall x$$

- Normalization:

$$\int_{-\infty}^{\infty} f_X(x)dx = 1$$

2 PDF Interpretation

Caution: $f_X(x) \neq P(X = x)$

- if X is continuous, $P(X = x) = 0 \quad \forall x$!!
- $f_X(x)$ can be ≥ 1

Interpretation: "probability per unit length" for "small" lengths around x

$$P(x \le X \le x + \delta) \approx f_X(x)\delta$$

3 Mean and variance of a continuous RV

$$\begin{aligned}
E[X] &= \int_{-\infty}^{\infty} x f_X(x)dx \\
\mathrm{Var}(X) &= E\left[(X - E[X])^2\right] \\
&= \int_{-\infty}^{\infty} (x - E[X])^2 f_X(x)dx \\
&= E[X^2] - (E[X])^2 \ (\ge 0) \\
E[g(X)] &= \int_{-\infty}^{\infty} g(x) f_X(x)dx \\
E[aX + b] &= aE[X] + b \\
\mathrm{Var}(aX + b) &= a^2\mathrm{Var}(X)
\end{aligned}$$

10/27

4 Cumulative Distribution Functions

Definition:

$$F_X(x) = P(X \leq x)$$

monotonically increasing from 0 (at $-\infty$) to 1 (at $+\infty$).

- Continuous RV (CDF is continuous in x):

$$F_X(x) = P(X \leq x) = \int_{-\infty}^{x} f_X(t)dt$$

$$f_X(x) = \frac{dF_X}{dx}(x)$$

- Discrete RV (CDF is piecewise constant):

$$F_X(x) = P(X \leq x) = \sum_{k \leq x} p_X(k)$$

$$p_X(k) = F_X(k) - F_X(k-1)$$

5 Uniform Random Variable

If X is a uniform random variable over the interval [a,b]:

$$f_X(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

$$F_X(x) = \begin{cases} 0 & \text{if } x \leq a \\ \frac{x-a}{b-a} & \text{if } a \leq x \leq b \\ 1 & \text{otherwise } (x > b) \end{cases}$$

$$E[X] = \frac{b-a}{2}$$

$$\text{var}(X) = \frac{(b-a)^2}{12}$$

6 Exponential Random Variable

X is an exponential random variable with parameter λ:

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$F_X(x) = \begin{cases} 1 - e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$E[X] = \frac{1}{\lambda} \quad \text{var}(X) = \frac{1}{\lambda^2}$$

Memoryless Property: Given that $X > t$, $X - t$ is an exponential RV with parameter λ

7 Normal/Gaussian Random Variables

General normal RV: $N(\mu, \sigma^2)$:

$$\begin{aligned} f_X(x) &= \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2} \\ E[X] &= \mu, \quad \text{Var}(X) = \sigma^2 \end{aligned}$$

Property: If $X \sim N(\mu, \sigma^2)$ and $Y = aX + b$

then $Y \sim N(a\mu + b, a^2\sigma^2)$

8 Normal CDF

Standard Normal RV: $N(0,1)$
CDF of standard normal RV Y at y: $\Phi(y)$
- given in tables for $y \geq 0$
- for $y < 0$, use the result: $\Phi(y) = 1 - \Phi(-y)$
To evaluate CDF of a general standard normal, express it as a function of a standard normal:

$$X \sim N(\mu, \sigma^2) \Leftrightarrow \frac{X-\mu}{\sigma} \sim N(0,1)$$

$$P(X \leq x) = P\Big(\frac{X-\mu}{\sigma} \leq \frac{x-\mu}{\sigma}\Big) = \Phi\Big(\frac{x-\mu}{\sigma}\Big)$$

9 Joint PDF

Joint PDF of two continuous RV X and Y: $f_{X,Y}(x,y)$

$$P(A) = \int\int_A f_{X,Y}(x,y)dxdy$$

Marginal pdf: $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy$
$E[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)f_{X,Y}(x,y)dxdy$
Joint CDF: $F_{X,Y}(x,y) = P(X \leq x, Y \leq y)$

10 Independence

By definition,

$$X,\ Y \text{ independent} \Leftrightarrow f_{X,Y}(x,y) = f_X(x)f_Y(y) \quad \forall(x,y)$$

If X and Y are independent:

- $E[XY]$=$E[X]E[Y]$
- $g(X)$ and $h(Y)$ are independent
- $E[g(X)h(Y)] = E[g(X)]E[h(Y)]$

11 Conditioning on an event

Let X be a continuous RV and A be an event with $P(A) > 0$,

$$\begin{aligned} f_{X|A}(x) &= \begin{cases} \frac{f_X(x)}{P(X \in A)} & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases} \\ P(X \in B | X \in A) &= \int_B f_{X|A}(x)dx \\ E[X|A] &= \int_{-\infty}^{\infty} x f_{X|A}(x)dx \\ E[g(X)|A] &= \int_{-\infty}^{\infty} g(x) f_{X|A}(x)dx \end{aligned}$$

If $A_1, \ldots, A_n$ are disjoint events that form a partition of the sample space,

$$\begin{aligned} f_X(x) &= \sum_{i=1}^{n} P(A_i) f_{X|A_i}(x) \text{ (}\approx\text{ total probability theorem)} \\ E[X] &= \sum_{i=1}^{n} P(A_i) E[X|A_i] \text{ (total expectation theorem)} \\ E[g(X)] &= \sum_{i=1}^{n} P(A_i) E[g(X)|A_i] \end{aligned}$$

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12 Conditioning on a RV

X, Y continuous RV

$$\begin{aligned} f_{X|Y}(x|y) &= \frac{f_{X,Y}(x,y)}{f_Y(y)} \\ f_X(x) &= \int_{-\infty}^{\infty} f_Y(y) f_{X|Y}(x|y) dy \quad (\approx totalprobthm) \end{aligned}$$

Conditional Expectation:

$$\begin{aligned} E[X|Y=y] &= \int_{-\infty}^{\infty} x f_{X|Y}(x|y) dx \\ E[g(X)|Y=y] &= \int_{-\infty}^{\infty} g(X) f_{X|Y}(x|y) dx \\ E[g(X,Y)|Y=y] &= \int_{-\infty}^{\infty} g(x,y) f_{X|Y}(x|y) dx \end{aligned}$$

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Total Expectation Theorem:

$$\begin{aligned} E[X] &= \int_{-\infty}^{\infty} E[X|Y=y] f_Y(y) dy \\ E[g(X)] &= \int_{-\infty}^{\infty} E[g(X)|Y=y] f_Y(y) dy \\ E[g(X,Y)] &= \int_{-\infty}^{\infty} E[g(X,Y)|Y=y] f_Y(y) dy \end{aligned}$$

15

13 Continuous Bayes' Rule

X, Y continuous RV, N discrete RV, A an event.

$$\begin{aligned} f_{X|Y}(x|y) &= \frac{f_{Y|X}(y|x) f_X(x)}{f_Y(y)} = \frac{f_{Y|X}(y|x) f_X(x)}{\int_{-\infty}^{\infty} f_{Y|X}(y|t) f_X(t) dt} \\ P(A|Y=y) &= \frac{P(A) f_{Y|A}(y)}{f_Y(y)} = \frac{P(A) f_{Y|A}(y)}{f_{Y|A}(y)P(A) + f_{Y|A^c}(y)P(A^c)} \\ P(N=n|Y=y) &= \frac{p_N(n) f_{Y|N}(y|n)}{f_Y(y)} = \frac{p_N(n) f_{Y|N}(y|n)}{\sum_i p_N(i) f_{Y|N}(y|i)} \end{aligned}$$

16

14 Derived distributions

Def: PDF of a *function* of a RV X with known PDF: $Y = g(X)$.
Method:

- Get the CDF:

$$F_Y(y) = P(Y \leq y) = P(g(X) \leq y) = \int_{x|g(x)\leq y} f_X(x)dx$$

- Differentiate: $f_Y(y) = \frac{dF_Y}{dy}(y)$

Special case: if $Y = g(X) = aX + b$, $f_Y(y) = \frac{1}{|a|} f_X(\frac{x-b}{a})$

15 Convolution

$W = X + Y$, with X, Y independent.

- Discrete case:

$$p_W(w) = \sum_x p_X(x) p_Y(w-x)$$

- Continuous case:

$$f_W(w) = \int_{-\infty}^{\infty} f_X(x) f_Y(w-x)\, dx$$

Graphical Method:

- put the PMFs (or PDFs) on top of each other
- flip the PMF (or PDF) of Y
- shift the flipped PMF (or PDF) of Y by w
- cross-multiply and add (or evaluate the integral)

In particular, if X, Y are independent and normal, then $W = X + Y$ is normal.

16 Law of iterated expectations

$E[X|Y = y] = f(y)$ is a number.
$E[X|Y] = f(Y)$ is a random variable
(the expectation is taken with respect to X).
To compute $E[X|Y]$, first express $E[X|Y = y]$ as a function of y.

Law of iterated expectations:

$$E[X] = E[E[X|Y]]$$

(equality between two real numbers)

17 Law of Total Variance

$\mathrm{Var}(X|Y)$ is a random variable that is a function of Y (the variance is taken with respect to X).
To compute $\mathrm{Var}(X|Y)$, first express

$$\mathrm{Var}(X|Y=y) = E[(X - E[X|Y=y])^2|Y=y]$$

as a function of y.

Law of conditional variances:

$$\mathrm{Var}(X) = E[\mathrm{Var}(X|Y)] + \mathrm{Var}(E[X|Y])$$

(equality between two real numbers)

18 Sum of a random number of iid RVs

N discrete RV, X_i i.i.d and independent of N.
$Y = X_1 + \ldots + X_N$. Then:

$$\begin{aligned} E[Y] &= E[X]E[N] \\ \mathrm{Var}(Y) &= E[N]\mathrm{Var}(X) + (E[X])^2\mathrm{Var}(N) \end{aligned}$$

19 Covariance and Correlation

$$\begin{aligned} \mathrm{Cov}(X,Y) &= E[(X - E[X])(Y - E[Y])] \\ &= E[XY] - E[X]E[Y] \end{aligned}$$

- By definition, X, Y are uncorrelated $\Leftrightarrow \mathrm{Cov}(X,Y) = 0$.
- If X, Y independent $\Rightarrow$ X and Y are uncorrelated. (the converse is not true)
- In general, Var(X+Y)= Var(X)+ Var(Y)+ 2 Cov(X,Y)
- If X and Y are uncorrelated, Cov(X,Y)=0 and Var(X+Y)= Var(X)+Var(Y)

Correlation Coefficient: (dimensionless)

$$\rho = \frac{\mathrm{Cov}(X,Y)}{\sigma_X \sigma_Y} \quad \in [-1, 1]$$

$\rho = 0 \Leftrightarrow$ X and Y are uncorrelated.
$|\rho| = 1 \Leftrightarrow \quad X - E[X] = c[Y - E[Y]]$ (linearly related)

6.041 Review Session

10/27

Will cover contineous
- but of course include concepts from quiz 1

Lectures 1-12
Recitations 1-13
Tutorials 1-6
P-set 1-6

Some discrete + contineous are building blocks

discrete
- binomial
- geometric — see next pg
- Bernoulli - ~~0 or 1 success or failure~~ ~~$P_K(k) = \begin{cases} p & \text{if } k=1 \\ 1-p & k=0 \end{cases}$~~
- Uniform

Contineous
- Uniform
- exponential
- normal
- poisson - not on quiz 2

(2)

Discrete

Bernoulli (p)

$$P_X(k) = \begin{cases} p & k=1 \\ 1-p & k=0 \end{cases} \qquad E[X]=p \qquad var(X)=p(1-p)$$

Binomial (n,p)

$$X = X_1 + X_2 + X_3 + \dots + X_n$$

$$P_X(k) = \binom{n}{k} p^k (1-p)^{n-k} \qquad k=0,1,2\dots \qquad E[X]=np$$

$$var(X) = np(1-p)$$

just add up their variances

N = # of indep. Bernoulli trials to 1st success

Geometric (p)

$$P_X(k) = (1-p)^{k-1} p \qquad E[X]=\frac{1}{p} \qquad var(X) = \frac{1-p}{p^2}$$

$$k=1,2,3\dots$$

Uniform [a,b]

$$P_X(k) = \frac{1}{b-a+1} \qquad k=a, a+1, \dots b \qquad E[X] = \frac{b+a}{2} \qquad var(X) = \frac{(b-a)(b-a+2)}{12}$$

now I know what that means!

(3)

Contineous

Uniform (a, b)

$$f_x(x) = \frac{1}{b-a} \quad x \in [a,b] \qquad E[X] = \frac{a+b}{2} \qquad var(X) = \frac{(b-a)^2}{12}$$

exponential (λ)

$$f_x(x) = \lambda e^{-\lambda x} \quad x \geq 0 \qquad E[X] = 1/\lambda \qquad var(X) = 1/\lambda^2$$

normal (μ, σ^2)

$$f_x(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(x-\mu)^2}{2\sigma^2}} \qquad E[X] = \mu \qquad var(X) = \sigma^2$$

(4)

$$E[X] = \int_X x f_X(x)\,dx$$

$$Var(X) = E[X - E[X]] = \int (x - E[X])^2 f_X(x)\,dx$$

CDFs

$$P(X \leq x) = \int_{-\infty}^{x} f_X(x)\,dx$$

$$= \int_{-\infty}^{x} f_X(t)\,dt$$ dummy variable

← need a lot of work on these

$$P(a \leq X \leq b) = F_X(b) - F_X(a)$$

$\lim_{x \to -\infty} F_X(x) = 0$ left $F_X(x) \geq 0$

$\lim_{x \to \infty} F_X(x) = 1$ right Monotonically non decreasing

exponential CDF

$$F_X(x) = \begin{cases} 1 - e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

normal CDF

Use sheet w/ ~~#~~ Φ which is for Normal(0, 1)

$Y = \frac{X - \mu}{\sigma}$ So now $Y = N(0,1)$

(5)

joint PDF

$$f_{X,Y}(x,y)$$

$$P(x,y \in B) = \iint_B f_{X,Y}(x,y)\,dx\,dy$$

$$f_{X,Y}(x,y) = f_X(x) f_{Y|X}(y|x)$$

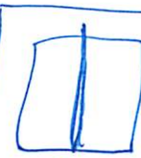
marginal pdf

integrate over all the possible values

$$f_X(x) = \int_Y f_{X,Y}(x,y)\,dy$$

renormalize?

- no - already normalized
- all the prob adds up to 1
- squishing it to that axis



?conditional

not marginal

conditioning

- can condition on a RV
- $f_{X|Y}(x|y) = \dfrac{f_{X,Y}(x,y)}{f_Y(y)}$ ← now must renormalize!

denominator does that

- condition on an event

$$f_{X|B}(x) = \frac{f_X(x)}{P(B)}$$ ← normalize w/ denom

$X \in B$ only for!

(e)

Memoryless property

- exponential

$X = \exp(\lambda)$

$f_X(x) = \lambda e^{-\lambda x} \quad x \geq 0$

$X \geq t$

So can use conditioning to prove

$$f_{X|X \geq t}(x) = \frac{\lambda e^{-\lambda x}}{e^{-\lambda t}}$$

← 1-CDF of exponential evaluated at t

$$= \lambda e^{-\lambda(x-t)}$$

still get exponential, but shifted

notice the exponential math

Bayes Rule

$$X \to \boxed{\text{measurement}} \to Y \to \boxed{} \to X$$

don't know what was sent

observe Y

know conditional pdf $f_{Y|X}(y|x)$

inference what x sent $f_{X|Y}(x|y)$

$$f_{X|Y}(x|y) = \frac{f_X(x)\, f_{Y|X}(y|x)}{\int_x f_X(x)\, f_{Y|X}(x)\,dx}$$

joint

need practice on actually doing these

if x is discrete

$$P_{X|Y}(x|y) = \frac{P_X(x)\, f_{Y|X}(x)}{\sum_x P_X(x)\, f_{Y|X}(y|x)}$$

(7)

Derived distributions

$Z = g(X)$

$f_Z(z) = ?$

First find CDF

$F_Z(z) = P(Z \leq z)$

$= P(g(X) \leq z)$

use fundemental theorm of calculus
differenitate

$$f_Z(z) = \frac{\partial F_Z(z)}{\partial z}$$

also if $Z = f(X, Y)$

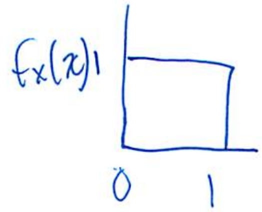
Convolution

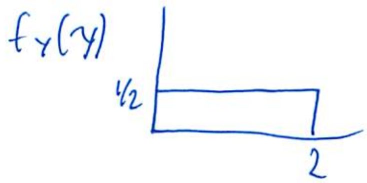
X, Y independant

$Z = X + Y$

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(t)\, f_Y(z-t)\, dt$$

flip + slide



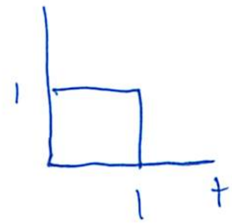


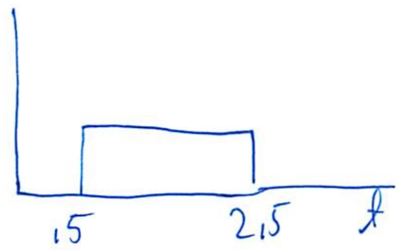
- how much sliding is there
- or was it just that one problem

$f_Z(2.5) = ?$

8

Flip + slide!





Multiply PDF
- only nonzero values
- so where <u>overlap</u>

$f_Z(2.5) = \int_{.5}^{1} 1 \cdot \frac{1}{2}\, dt$ ← only 1 region here
remember that other p-set qu

<u>Coverance</u> (X, Y)

$\text{Cov}(X, Y) = E[XY] - E[X]E[Y]$

$\text{Var}(X) = \text{Cov}(X, Y)$ ← not always I think

$\rho(X, Y)$ ← dimensionless correlation coefficent

$= \dfrac{\text{Cov}(X, Y)}{\sigma_X \sigma_Y}$

if X, Y are independent → covariance = 0

9

Law of iterated expectations

$$E[X] = E[E[X|Y]]$$

Law of total variance

$$var(X) = E[var(X|Y)] + var(E[X|Y])$$

Sum of R# of RV

$$Y = X_1 + \dots + X_n$$

X_i must be inp. of each other and N

$$E[Y] = E[E[Y|N]]$$

$$= E[X_i]\,E[N]$$

$var(Y)$ = add variances

$$= E[N\,var(X_i)] + var(N\,E[X_i])$$

$$= E[N]\,var(X_i) + (E[X_i])^2\,var(N)$$

TA: ~~$~~ considerably easier than 08

6.041 Fall 2009 Quiz 2
Tuesday, November 3, 7:30 - 9:30 PM.

DO NOT TURN THIS PAGE OVER UNTIL YOU ARE TOLD TO DO SO

Name: ____________________

Recitation Instructor: ____________________

TA: ____________________

Question	Score	Out of
1		2
2 (a)		7
2 (b)		7
2 (c)		7
2 (d)		7
2 (e)		7
2 (f)		7
3		10
4 (a i)		5
4 (a ii)		5
4 (b)		7
4 (c)		8
5 (a)		7
5 (b)		7
5 (c)		7
Your Grade		100

- This quiz has 5 problems, worth a total of 100 points.
- **When giving a formula for a PDF, make sure to specify the range over which the formula holds.**
- Please make sure to return the entire exam booklet intact.
- Write your solutions in this quiz booklet, only solutions in this quiz booklet will be graded. Be neat! You will not get credit if we can't read it.
- You are allowed 2 two-sided, handwritten, formulae sheets. Calculators not allowed.
- You may give an answer in the form of an arithmetic expression (sums, products, ratios, factorials) of numbers that could be evaluated using a calculator. Expressions like $\binom{8}{3}$ or $\sum_{k=0}^{5}(1/2)^k$ are also fine.
- You have 2 hrs. to complete the quiz.
- Graded quizzes will be returned in recitation on Thursday 11/5.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Fall 2009)

Problem 1: (2 points) Write your name, your assigned recitation instructor's name, and assigned TA's name on the cover of the quiz booklet. The Instructor/TA pairing is listed below.

Recitation Instructor	TA	Recitation Time
Jeffrey Shapiro	Jimmy Li	10 & 11 AM
Danielle Hinton	Uzoma Orji	1 & 2 PM
William Richoux	Ulric Ferner	2 & 3 PM
John Wyatt (6.431)	Aliaa Atwi	11 & 12 PM

	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767

The standard normal table. The entries in this table provide the numerical values of $\Phi(y) = \mathbf{P}(Y \leq y)$, where Y is a standard normal random variable, for y between 0 and 1.99. For example, to find $\Phi(1.71)$, we look at the row corresponding to 1.7 and the column corresponding to 0.01, so that $\Phi(1.71) = .9564$. When y is negative, the value of $\Phi(y)$ can be found using the formula $\Phi(y) = 1 - \Phi(-y)$.

Problem 2. (42 points)
The random variable X is exponential with parameter 1. Given the value x of X, the random variable Y is exponential with parameter equal to x (and mean $1/x$).

Note: Some useful integrals, for $\lambda > 0$:

$$\int_0^\infty xe^{-\lambda x}\,dx = \frac{1}{\lambda^2}, \qquad \int_0^\infty x^2 e^{-\lambda x}\,dx = \frac{2}{\lambda^3}.$$

(a) (7 points) Find the joint PDF of X and Y.

(b) (7 points) Find the marginal PDF of Y.

(c) (7 points) Find the conditional PDF of X, given that $Y = 2$.

(d) (7 points) Find the conditional expectation of X, given that $Y = 2$.

(e) (7 points) Find the conditional PDF of Y, given that $X = 2$ and $Y \geq 3$.

(f) (7 points) Find the PDF of e^{2X}.

Problem 3. (10 points)
For the following questions, mark the correct answer. If you get it right, you receive 5 points for that question. You receive no credit if you get it wrong. A justification is not required and will not be taken into account.

Let X and Y be continuous random variables. Let N be a discrete random variable.

(a) (5 points) The quantity $\mathbf{E}[X \mid Y]$ is always:

(i) A number.
(ii) A discrete random variable.
(iii) A continuous random variable.
(iv) Not enough information to choose between (i)-(iii).

(b) (5 points) The quantity $\mathbf{E}[\,\mathbf{E}[X \mid Y, N] \mid N]$ is always:

(i) A number.
(ii) A discrete random variable.
(iii) A continuous random variable.
(iv) Not enough information to choose between (i)-(iii).

Problem 4. (25 points)
The probability of obtaining heads in a single flip of a certain coin is itself a random variable, denoted by Q, which is uniformly distributed in $[0, 1]$. Let $X = 1$ if the coin flip results in heads, and $X = 0$ if the coin flip results in tails.

(a) (i) (5 points) Find the mean of X.
(ii) (5 points) Find the variance of X.

(b) (7 points) Find the covariance of X and Q.

(c) (8 points) Find the conditional PDF of Q given that $X = 1$.

Problem 5. (21 points)
Let X and Y be **independent continuous** random variables with marginal PDFs f_X and f_Y, and marginal CDFs F_X and F_Y, respectively. Let

$$S = \min\{X, Y\}, \qquad L = \max\{X, Y\}.$$

(a) (7 points) If X and Y are standard normal, find the probability that $S \geq 1$.

(b) (7 points) Fix some s and ℓ with $s \leq \ell$. Give a formula for

$$\mathbf{P}(s \leq S \text{ and } L \leq \ell)$$

involving F_X and F_Y, and no integrals.

(c) (7 points) Assume that $s \leq s + \delta \leq \ell$. Give a formula for

$$\mathbf{P}(s \leq S \leq s + \delta,\ \ell \leq L \leq \ell + \delta),$$

as an integral involving f_X and f_Y.

Each question is repeated in the following pages. Please write your answer on the appropriate page.

Problem 2. (42 points)
The random variable X is exponential with parameter 1. Given the value x of X, the random variable Y is exponential with parameter equal to x (and mean $1/x$).

Note: Some useful integrals, for $\lambda > 0$:

$$\int_0^\infty xe^{-\lambda x}\,dx = \frac{1}{\lambda^2}, \qquad \int_0^\infty x^2e^{-\lambda x}\,dx = \frac{2}{\lambda^3}.$$

(a) (7 points) Find the joint PDF of X and Y.

(b) (7 points) Find the marginal PDF of Y.

(c) (7 points) Find the conditional PDF of X, given that $Y = 2$.

(d) (7 points) Find the conditional expectation of X, given that $Y = 2$.

(e) (7 points) Find the conditional PDF of Y, given that $X = 2$ and $Y \geq 3$.

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(c) (7 points) Assume that $s \leq s + \delta \leq \ell$. Give a formula for

$$\mathbf{P}(s \leq S \leq s + \delta,\ \ell \leq L \leq \ell + \delta),$$

as an integral involving f_X and f_Y.

Quiz 2 Fall 04 in Review Session 10/27

$X \sim \exp(1)$ ← so $\lambda = 1$

Given $X = x$ $Y \sim \exp(x)$ $\lambda = x$ ← note, I prob would not have even read that

a) $f_{X,Y}(x,y)$

$$f_X(X) = \begin{cases} e^{-x} & X \geq 0 \\ 0 & \text{else} \end{cases}$$ ← always do other

$$f_{Y|X}(y|x) = \begin{cases} xe^{-xy} & y \geq 0 \\ 0 & \text{else} \end{cases}$$ never knew how to do this

$= f_X(x) f_{Y|X}(y|x)$ ← where is that formula from, but oh

$$= \begin{cases} e^{-x} x e^{-xy} & x \geq 0, y \geq 0 \\ 0 & \text{else} \end{cases}$$

b) $f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx$ just do formula

$= \int_0^{\infty} xe^{-(y+1)x}\,dx$ ↙ See exam for hints on integration

\# Small y constant w/ regard to variable of integration

$$= \begin{cases} \frac{1}{(y+1)^2} & y \geq 0, \\ 0 & \text{else} \end{cases}$$

(2)

c) $f_{X|Y}(x|2) = \begin{cases} \dfrac{f_{X,Y}(x,2)}{f_Y(2)} = \dfrac{xe^{-3x}}{1/3^2} & x \geq 0 \\ 0 & \text{else} \end{cases}$

fill y in as 2, combine

actually easy when know what to do

d) find conditional expectation

$E[X|Y=2]$

- integrate over conditional PDF from c

$= \int_0^\infty x f_{X|Y}(x|2)\, dx$

← just put it in + do the math!

$= \int_0^\infty 9x^2 e^{-3x}\, dx$

formula given for integration at top of exam page

$= 9 \frac{2}{3^3}$ ← can leave like that on exam

$= \frac{2}{3}$

e) $f_{Y|(X=2)\cap(Y\geq 3)}(y|2)$

↑ conditioned on this event

looks confusing

but exponential w/ param X

this is " " " X=2

conditioned that $Y \geq 3$

$Y \sim \exp(2)$

③

but memoryless, so remaining time is exponential

$(Y-3) \sim \exp(2)$

↑ so does not matter except for bounds

$$f_{Y|X \cap (Y \geq 3)}(y|2) = \begin{cases} 2e^{-2(y-3)} & y-3 \geq 0 \rightarrow y \geq 3 \\ 0 & \text{else} \end{cases}$$

Could also do ~~full out conditional right~~

$$\frac{f_{Y|X}(y|2)}{P(Y \geq 3 | X=2)} \quad y \geq 3$$

same exponential RV

$1 - F_{Y|X}(3|2)$

x is 2

$1-(1-e^{-2 \cdot 3})$

e^{-6}

simplifies

get same thing

conditioning on an event

$$\boxed{f_{X|A}(x) = \frac{f_X(x)}{P(A)} \quad x \in A}$$

conditioning everywhere and use Y

$$\boxed{f_{Y|X \cap A}(y|x) = \frac{f_{Y|X}(y|x)}{P(A|X=x)} \quad Y \in A}$$

added to formula sheet

f) PDF e^{-2x}

$Z = e^{-2x}$

Find $f_Z(z)$

2 step derived distribution method

$F_Z(z) = P(Z \leq z) = P(e^{2x} \leq z)$

↑ monotonically increasing

so when take log of both sides - don't need to flip sign

↓ review when to change side →

change side when

$x \div$ by $\ominus$# to both sides

by integer

- ie not $x^2 > x$
- need to test the values of x

4

$= P(2X \le \log(z))$

divide both sides by 2

$= P\left(X \le \frac{\log(z)}{2}\right)$ ← oh that is reduction process

where is X valid?

$= \int_{-\infty}^{\frac{1}{2}\log z} f_X(x)\,dx$ ↑ where that condition satisfied

$= \int_0^{1/2 \log z} e^{-x}\,dx$ which value of z is it true for?

$= 1 - e^{-1/2 \log z}$

↑ note $e^{\log}$ properties ~~will just leave~~

$x^5 \cdot x^7 = x^{12}$

$\frac{x^5}{x^7} = x^{-2}$

$(x^5)^7 = x^{30}$

$8 = 2^x$

$x = \log_2 8$

$8 = e^x$

$x = \ln 8$

$e^{\frac{1}{2}\log x} = x^{\frac{1}{2}}$

$e^{\log x} = x$

↓ boundries

$F_Z(z) = \begin{cases} 1 - z^{-1/2} & \frac{1}{2}\log z \ge 0 \\ & z \ge 1 \\ 0 & \text{else } z < 1 \end{cases}$ ↓ simplify

quiz (seems very hard!)

differentiate

$f_Z(z) = \frac{d}{dz} F_Z(z) = \begin{cases} \frac{1}{2} z^{-3/2} & z \ge 1 \\ 0 & \text{otherwise} \end{cases}$

~~← computer got something different!!~~

Copy error

~~Still no~~

duh differentiate! not integration!

(really review this problem!)

---

3. X, Y = contineous RV

N = discrete RV

E[X|Y] → always # or discrete RV or contineous RV or not enough info

~~note its not E[X|Y=y] ← #~~

~~So this is contineous RV~~

(5)

Oh different if independent or continuous RV

$\uparrow$ #

$\hookrightarrow$ non independent

So not enough info \ ~~called~~ called a discrete RV

b) $E[E[X|Y,N]|N]$

again depends if each is independent w/ it

$E[g(Y,N)|N]$

$\to h(N)$

$\to$ #

) both are ~~called~~ discrete RV

(lots of students were angry)

~~[illegible]~~ ~~the~~ clarification

$E[g(Y,N)|N=n]$

$= E[g(Y,n)]$ ← I think ; how convert to little n and then back ?

$= \int_{-\infty}^{\infty} g(y,n) f_{Y|N}(y|n) dy$

$= h(n)$

but no $=n$ given so use cap

$= h(N)$

(6)

$E[g(Y) | W]$

$= \int_{-\infty}^{\infty} g(y) f_Y(y) \, dy$

$= E[g(Y)] = a$ #

aka a discrete RV - will just guess on this qu

---

4. Prob of heads is Q

$Q \sim$ uniform $[0, 1]$

$X = \begin{cases} 1 & \text{Heads w/p } q \\ 0 & \text{Tails w/p } 1-q \end{cases}$ bernoulli

Find $E[X]$

$P_{X|Q}(k|q) = \begin{cases} q & \text{heads} & k=1 \\ 1-q & \text{tails} & k=0 \end{cases}$

$E[X|Q] = Q$

$E[X|Q=q] = q$

same form (new chap 4 notation) Study!

law of iterated expectations

$E[X] = E[E[X|Q]] = E[Q] = \frac{1}{2}$

why not $\frac{b-a}{2}$?
- it is
- I guess they did longer way

? why type of problem is it?

7

b. Find var(x)

- law of total var

$$= E[\underbrace{var(X|Q)}] + var(\underbrace{E[X|Q]}_{Q})$$

$$\boxed{\begin{aligned} &Var(X|Q=q) \\ &= q(1-q) \end{aligned}}$$

$$= Q(1-Q)$$

$$= E[Q(1-Q)] + var(Q)$$

can break it up

$$= E[Q] - E[Q^2] + var(Q)$$ ← easy to find, but can get rid of it

$$= E[Q] - E[Q^2] + (E[Q^2] - (E[Q])^2)$$

$$= E[Q] - (E[Q])^2$$

$$= \frac{1}{2} - \left(\frac{1}{2}\right)^2$$

$\frac{1}{b-a}$

$$= \frac{1}{4}$$

in solutions

X ~ Bernoulli p = ½

var(X) = p(1-p) = ¼

yeah is this much simpler than it seems

c) Cov(X, Q) what is this again

- p-set last qu

$$= E[XQ] - E[X]E[Q]$$

- but they did it a different way here
- and a different way on the p-set

law of iterated expectations

condition on rv Q

8

$E[XQ] = E[E[XQ \mid Q]]$

if was small q could take it out of expectation

$E[XQ \mid Q=q] = qE[X \mid Q=q]$

if don't specify small q, use Q

$= E[Q \underbrace{E[X \mid Q]}_{Q}]$

$= E[Q^2]$

$= Var(Q) + E[Q]$ to avoid integral (clever!)

$= \int_0^1 q^2 \, dq$

need to actually do to see

$= \frac{1}{3} q^3 \Big|_0^1$

so each coin flip has different probablity

$= \frac{1}{3}$

So back to problem

$Cov(X, Q) = \frac{1}{3} - \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{12}$

Q: is it coincidence same as var

- would have to think about it more
- later in course

(9)

these problems are ~~so~~ boring

c) Find Conditional PDF of Q given X=1

$$f_{Q|X}(q\mid 1) = \frac{f_Q(q)\, p_{X|Q}(1\mid q)}{p_X(1)}$$

X=

Bayes Rule

convert 1 to another

same as $E[X] = \frac{1}{2}$

$\int f_Q(q)\, \underbrace{p_X(1\mid q)}_{q}\, dq$ ← total prob theorm

= same as expected value
also not hard to do integral

fill in each piece

$$= \frac{1 \cdot q}{\frac{1}{2}} \quad q \in [0,1]$$

$$= \begin{cases} 2q & q \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

---

5. Different concept
did not pay much attension in class
find prob of sample space

X, Y ind.

$S = \min\{X, Y\}$ $L = \max\{X, Y\}$

a) $X, Y \sim N(0,1)$

$P(S \geq 1) = P(\min\{X, Y\} \geq 1)$

when both X and Y are greater than 1 ← interpert

$= P(X \geq 1 \cap Y \geq 1)$

$= P(X \geq 1)\, P(Y \geq 1)$ since ind.

(10)

X, Y are standard normal
-otherwise need to standardize

$= (1 - F_X(1))(1 - F_Y(1))$

↑ definition of CDF
(need to see!

weird, but know how to do!
- would not have gotten

$= (1 - \Phi(1))^2$

plug in by looking at table

$= (1 - .8413)^2$ ← can leave on exam

$= (.1587)^2$

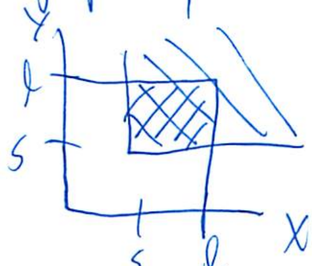
b) now a little trickier

$s \leq \ell$

$P(s \leq S \cap L \leq \ell)$

$= P(s \leq \min(X, Y) \cap \max(X, Y) \leq \ell)$

2 ways to do
-graphically



↑ wrong way?

$= s \leq \min(X, Y)$

$= \max(X, Y) \leq \ell$

↑ So box is intersect

$= P(s \leq X \leq \ell \cap s \leq Y \leq \ell)$ ↑ loss of fidelity?

X, Y indp

(11)

$= P(s \le X \le l)\, P(s \le Y \le l)$

write in terms of CDF

$=$

$(F_X(l) - F_X(s))(F_Y(l) - F_Y(s))$

$= P((X \ge s \cap Y \ge s) \cap (X \le l \cap Y \le l))$

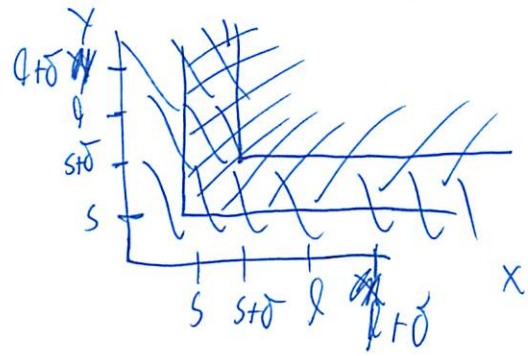
4 conditions

all intersects so can get rid or parentesee

-could have solved w/o looking at graph

c) $P(s \le S \le s+\delta \cap l \le L \le l+\delta)$

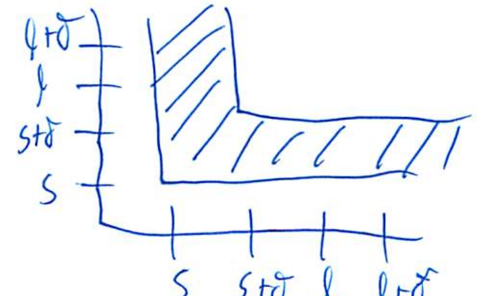
$= P(s \le \min(X,Y) \le s+\delta \cap l \le \max(X,Y) \le l+\delta)$



$\delta > 0$ given ~~assume~~

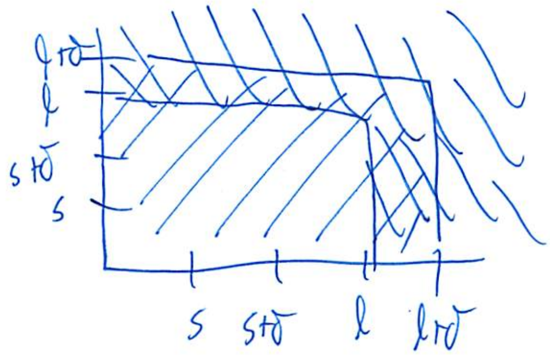
// $s \le \min(x,y)$

\\\ $\min(x,y) \le s+\delta$



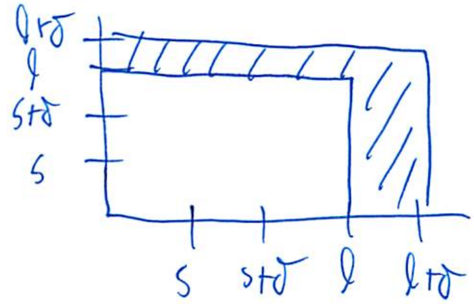
/// $s \le \min(x,y) \le s+\delta$

12



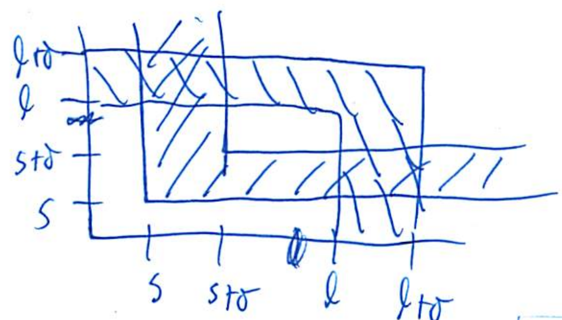
$\backslash\backslash\backslash = l \le \max(x,y)$

$/// = \max(x,y) \le l+\delta$



$/// = l \le \max(x,y) \le l+\delta$

So for both



So overlapping areas

- check that can't overlap

$$f_{X,Y}(x,y) = f_X(x) f_Y(y)$$

answer

$$\boxed{\int_{l}^{l+\delta}\int_{s}^{s+\delta} f_{X,Y}(x,y)\,dx\,dy + \int_{s}^{s+\delta}\int_{l}^{l+\delta} f_{X,Y}(x,y)\,dx\,dy}$$

would not have gotten what this means

prob the challenge problem

- we'll have something different

Massachusetts Institute of Technology

Department of Electrical Engineering & Computer Science

6.041/6.431: Probabilistic Systems Analysis

(Fall 2009)

Quiz 2 Solutions:
November 3, 2009

Problem 2. (49 points)

(a) (7 points)
We start by recognizing that $f_X(x) = e^{-x}$ for $x \geq 0$ and $f_{Y|X}(y \mid x) = xe^{-xy}$ for $y \geq 0$. Furthermore, $f_{X,Y}(x,y) = f_X(x) \cdot f_{Y|X}(y \mid x)$. Substituting for $f_x(x)$ and $f_{Y|X}(y \mid x)$ yields,

$$f_{X,Y}(x,y) = \begin{cases} xe^{-(1+y)x}, & x \geq 0, y \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

(b) (7 points)
The marginal PDF of Y can be found by integrating the joint PDF of X and Y.

$$\begin{aligned} f_Y(y) &= \int_X f_{X,Y}(x,y)dx \\ &= \int_0^\infty xe^{-(1+y)x}dx \end{aligned}$$

$$f_Y(y) = \begin{cases} \frac{1}{(1+y)^2}, & y \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

(c) (7 points)
We are asked to compute the PDF of the random variable X while conditioning on another random variable Y. The conditional PDF of X given that $Y = 2$ is

$$f_{X|Y}(x \mid 2) = \frac{f_{X,Y}(x,2)}{f_Y(2)} = \frac{xe^{-3x}}{\frac{1}{3^2}}$$

$$f_{X|Y}(x \mid 2) = \begin{cases} 9xe^{-3x}, & x \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

(d) (7 points)

$$\begin{aligned} \mathbf{E}[X \mid Y = 2] &= \int_X x \cdot f_{X|Y}(x \mid 2)dx \\ &= 9\int_0^\infty x^2 e^{-3x}dx \\ &= 9 \cdot \frac{2}{3^3} \\ &= \frac{2}{3}. \end{aligned}$$

(e) (7 points)
In the new universe in which $X = 2$, we are asked to compute the conditional PDF of Y given the event $Y \geq 3$.

$$f_{Y|X,Y\geq 3}(y \mid 2) = \frac{f_{Y|X}(y \mid 2)}{\mathbf{P}(Y \geq 3 \mid X = 2)}.$$

We first calculate the $\mathbf{P}(Y \geq 3 \mid X = 2)$.

$$\begin{aligned}
\mathbf{P}(Y \geq 3 \mid X = 2) &= \int_3^\infty f_{Y|X}(y \mid 2)dy \\
&= \int_3^\infty 2e^{-2y}dy \\
&= 1 - F_{Y|X}(3 \mid 2) \\
&= 1 - (1 - e^{-2\cdot 3}) \\
&= e^{-6},
\end{aligned}$$

where $F_{Y|X(3|2)}$ is the CDF of an exponential random variable with $\lambda = 2$ evaluated at $y = 3$. Substituting the values of $f_{Y|X}(y \mid 2)$ and $\mathbf{P}(Y \geq 3 \mid X = 2)$ yields

$$f_{Y|X,Y\geq 3}(y \mid 2) = \begin{cases} 2e^6e^{-2y}, & y \geq 3 \\ 0, & \text{otherwise.} \end{cases}$$

Alternatively, $f_{Y|X}(y \mid 2)$ is an exponential random variable with $\lambda = 2$. To compute the conditional PMF $f_{Y|X,Y\geq 3}(y \mid 2)$, we can apply the memorylessness property of an exponential variable. Therefore, this conditional PMF is also an exponential random variable with $\lambda = 2$, but it is shifted by 3.

(f) (7 points)
Let's define $Z = e^{2X}$. Since X is an exponential random variable that takes on non-negative values $(X \geq 0), Z \geq 1$. We find the PDF of Z by first computing its CDF.

$$\begin{aligned}
F_Z(z) &= \mathbf{P}(Z \leq z) \\
&= \mathbf{P}(e^{2X} \leq z) \\
&= \mathbf{P}(2X \leq \ln z) \\
&= \mathbf{P}(X \leq \frac{\ln z}{2}) \\
&= 1 - e^{-\frac{\ln z}{2}} \\
&= 1 - e^{\ln z^{-\frac{1}{2}}}
\end{aligned}$$

The CDF of Z is:

$$F_Z(z) = \begin{cases} 1 - z^{-\frac{1}{2}} & z \geq 1 \\ 0, & z < 1 \end{cases}$$

Differentiating the CDF of Z yields the PDF

$$f_Z(z) = \begin{cases} \frac{1}{2}z^{-\frac{3}{2}} & z \geq 1 \\ 0, & z < 1 \end{cases}$$

Alternatively, you can apply the PDF formula for a strictly monotonic function of a continuous random variable. Recall if $z = g(x)$ and $x = h(z)$, then

$$f_Z(z) = f_X(h(z)) \left| \frac{dh}{dy}(z) \right|.$$

In this problem, $z = e^{2x}$ and $x = \frac{1}{2}\ln z$. Note that $f_Z(z)$ is nonzero for $z > 1$. Since X is an exponential random variable with $\lambda = 1$, $f_X(x) = e^x$. Thus,

$$\begin{aligned} f_Z(z) &= e^{-\frac{1}{2}\ln z} \left| \frac{1}{2z} \right| \\ &= e^{\ln z^{-\frac{1}{2}}} \frac{1}{2z} \\ &= \frac{1}{2} z^{-\frac{3}{2}} \quad z \geq 1, \end{aligned}$$

where the second equality holds since the expression inside the absolute value is always positive for $z \geq 1$.

Problem 3. (10 points)

(a) (5 points) The quantity $\mathbf{E}[X \mid Y]$ is always:

(i) A number.

(ii) A discrete random variable.

(iii) A continuous random variable.

(iv) $\boxed{\text{Not enough information to choose between (i)-(iii).}}$

If X and Y are not independent, then $\mathbf{E}[X \mid Y]$ is a function of Y and is therefore a continuous random variable. However if X and Y are independent, then $\mathbf{E}[X \mid Y] = \mathbf{E}[X]$ which is a number.

(b) (5 points) The quantity $\mathbf{E}[\mathbf{E}[X \mid Y, N] \mid N]$ is always:

(i) A number.

(ii) $\boxed{\text{A discrete random variable.}}$

(iii) A continuous random variable.

(iv) Not enough information to choose between (i)-(iii).

If X, Y and N are not independent, then the inner expectation $G(Y, N) = \mathbf{E}[X \mid Y, N]$ is a function of Y and N. Furthermore $\mathbf{E}[G(Y, N) \mid N]$ is a function of N, a discrete random variable. If X, Y and N are independent, then the inner expectation $\mathbf{E}[X \mid Y, N] = \mathbf{E}[X]$, which is a number. The expectation of a number given N is still a number, which is a special case of a discrete random variable.

Problem 4. (25 points)

(a) (i) (5 points)
Using the Law of Iterated Expectations, we have

$$\mathbf{E}[X] = \mathbf{E}[\mathbf{E}[X \mid Q]] = \mathbf{E}[Q] = \frac{1}{2}.$$

(ii) (5 points)
X is a Bernoulli random variable with a mean $p = \frac{1}{2}$ and its variance is $\text{var}(X) = p(1-p) = 1/4$.

(b) (7 points)
We know that $\text{cov}(X, Q) = \mathbf{E}[XQ] - \mathbf{E}[X]\mathbf{E}[Q]$, so first let's calculate $\mathbf{E}[XQ]$:

$$\mathbf{E}[XQ] = \mathbf{E}[\mathbf{E}[XQ \mid Q]] = \mathbf{E}[Q\mathbf{E}[X \mid Q]] = \mathbf{E}[Q^2] = \frac{1}{3}.$$

Therefore, we have

$$\text{cov}(X, Q) = \frac{1}{3} - \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{12}.$$

(c) (8 points)
Using Bayes' Rule, we have

$$f_{Q|X}(q \mid 1) = \frac{f_Q(q)p_{X|Q}(1 \mid q)}{p_X(1)} = \frac{f_Q(q)\mathbf{P}(X = 1 \mid Q = q)}{\mathbf{P}(X = 1)}, \quad 0 \le q \le 1.$$

Additionally, we know that

$$\mathbf{P}(X = 1 \mid Q = q) = q,$$

and that for Bernoulli random variables

$$\mathbf{P}(X = 1) = \mathbf{E}[X] = \frac{1}{2}.$$

Thus, the conditional PDF of Q given $X = 1$ is

$$\begin{aligned} f_{Q|X}(q \mid 1) &= \frac{1 \cdot q}{1/2} \\ &= \begin{cases} 2q, & 0 \le q \le 1, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Problem 5. (21 points)

(a) (7 points)

$$\begin{aligned} \mathbf{P}(S \ge 1) &= \mathbf{P}(\min\{X, Y\} \ge 1) = \mathbf{P}(X \ge 1 \text{ and } Y \ge 1) = \mathbf{P}(X \ge 1)\mathbf{P}(Y \ge 1) \\ &= (1 - F_X(1))(1 - F_Y(1)) = (1 - \Phi(1))^2 \approx (1 - 0.8413)^2 \approx 0.0252. \end{aligned}$$

(b) (7 points)
Recalling Problem 2 of Problem Set 6, we have

$$\begin{aligned}
\mathbf{P}(s \le S \text{ and } L \le \ell) &= \mathbf{P}(s \le \min\{X,Y\} \text{ and } \max\{X,Y\} \le \ell) \\
&= \mathbf{P}(s \le X \text{ and } s \le Y \text{ and } X \le \ell \text{ and } Y \le \ell) \\
&= \mathbf{P}(s \le X \le \ell)\mathbf{P}(s \le Y \le \ell) \\
&= (F_X(\ell) - F_X(s))(F_Y(\ell)F_Y(s)).
\end{aligned}$$

(c) (7 points)
Given that $s \le s+\delta \le \ell$, the event $\{s \le S \le s+\delta, \ell \le L \le \ell+\delta\}$ is made up of the union of two disjoint possible events:

$$\{s \le X \le s+\delta, \ell \le Y \le \ell+\delta\} \cup \{s \le Y \le s+\delta, \ell \le X \le \ell+\delta\}.$$

In other words, either $S = X$ and $L = Y$, or $S = Y$ and $L = X$. Because the two events are disjoint, the probability of their union is equal to the sum of their individual probabilities.

Using also the independence of X and Y, we have

$$\begin{aligned}
\mathbf{P}(s \le S \le s+\delta, \ell \le L \le \ell+\delta) &= \mathbf{P}(s \le X \le s+\delta, \ell \le Y \le \ell+\delta) \\
&\quad + \mathbf{P}(s \le y \le s+\delta, \ell \le X \le \ell+\delta) \\
&= \mathbf{P}(s \le X \le s+\delta)\mathbf{P}(\ell \le Y \le \ell+\delta) \\
&\quad + \mathbf{P}(s \le y \le s+\delta)\mathbf{P}(\ell \le X \le \ell+\delta) \\
&= \int_s^{s+\delta} f_X(x)dx \int_\ell^{\ell+\delta} f_Y(y)dy \\
&\quad + \int_s^{s+\delta} f_Y(y)dy \int_\ell^{\ell+\delta} f_X(x)dx
\end{aligned}$$

Tutorial 7
October 28/29, 2010

1. Alice and Bob alternate playing at the casino table. (Alice starts and plays at odd times $i = 1, 3, \ldots$; Bob plays at even times $i = 2, 4, \ldots$.) At each time i, the net gain of whoever is playing is a random variable G_i with the following PMF:

$$p_G(g) = \begin{cases} \frac{1}{3} & g = -2, \\ \frac{1}{2} & g = 1, \\ \frac{1}{6} & g = 3, \\ 0 & \text{otherwise} \end{cases}$$

Assume that the net gains at different times are independent. We refer to an outcome of -2 as a "loss."

(a) They keep gambling until the first time where a loss by Bob immediately follows a loss by Alice. Write down the PMF of the total number of rounds played. (A round consists of two plays, one by Alice and then one by Bob.)

(b) Write down the PMF for Z, defined as the time at which Bob has his third loss.

(c) Let N be the number of rounds until each one of them has won at least once. Find $\mathbf{E}[N]$.

2. Problem 6.6, page 328 in text.

Sum of a geometric number of independent geometric random variables

Let $Y = X_1 + \cdots + X_N$, where the random variable X_i are geometric with parameter p, and N is geometric with parameter q. Assume that the random variables $N, X_1, X_2, \cdots$ are independent. Show that Y is geometric with parameter pq. *Hint:* Interpret the various random variables in terms of a split Bernoulli process.

3. A train bridge is constructed across a wide river. Trains arrive at the bridge according to a Poisson process of rate $\lambda = 3$ per day.

(a) If a train arrives on day 0, find the probability that there will be no trains on days 1, 2, and 3.

(b) Find the probability that the next train to arrive after the first train on day 0, takes more than 3 days to arrive.

(c) Find the probability that no trains arrive in the first 2 days, but 4 trains arrive on the 4^{th} day.

(d) Find the probability that it takes more than 2 days for the 5^{th} train to arrive at the bridge.

Tutorial 7

10/29

Bernoulli process

- seq of iid RV
- binary trials
- Bernoulli
- Binomial
- Geometric
- Pascal
- fresh start →

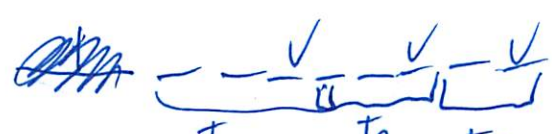
just go forward
past does not matter

Pascal RV

\# of trials/time till kth sucess

$k=1 \to$ geometric

very easy to reason about



$$Y_k = T_1 + T_2 + \dots + T_k$$

each T is geometric w/ param p

independent - so disjoined

$$E[Y_k] = k \cdot \frac{1}{p}$$

↑ linearity of expectations

(2)

$$Var(Y_k) = k \cdot \frac{1-p}{p^2}$$

$P_{Y_k}(y) =$ prob that kth sucess happens in trial y

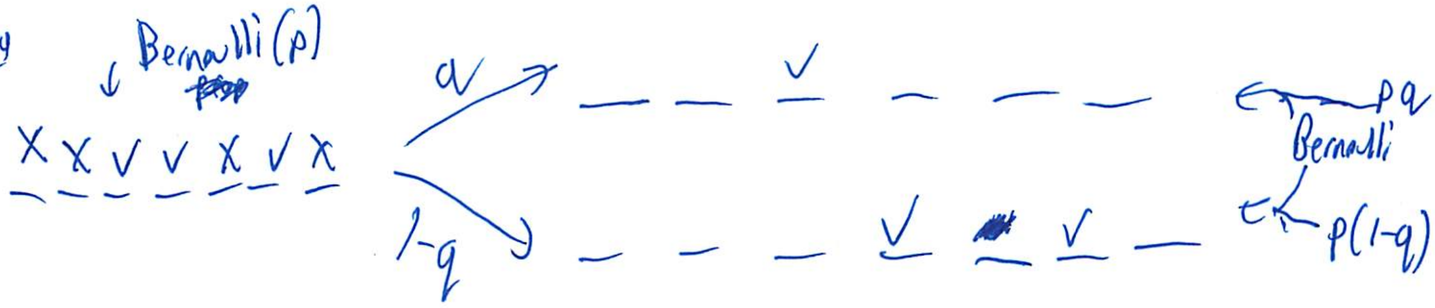
$$= p^k (1-p)^{y-k} \ldots$$

Now count sequences that have $k-1$ sucesses over $y-1$ slots

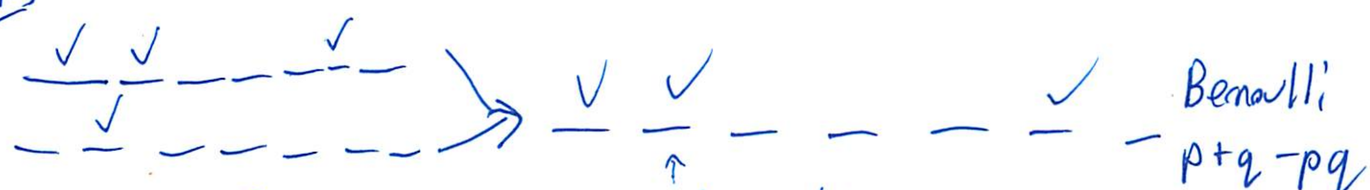
$$= \binom{y-1}{k-1} p^k (1-p)^{y-k} \qquad y = k, k+1, \ldots$$

Splitting + Merging Bernoulli Process

Splitting



Merging



This is dependent - if mosquito, know not a tick

* must be independant *

(5)

#1. Alice + Bob taking turns

$\uparrow$ odd turns $\uparrow$ even

$$\underbrace{\underset{1}{\underline{A}}\ \underset{2}{\underline{B}}}_{R_1}\ \underbrace{\underset{3}{\underline{A}}\ \underset{4}{\underline{B}}}_{R_2}\ \underbrace{\underset{5}{\underline{A}}\ \underset{6}{\underline{B}}}_{R_3}$$

$$P_G(g) = \begin{cases} 1/3 & g=-2 \\ 1/2 & g=1 \\ 1/6 & g=3 \\ 0 & \text{else} \end{cases}$$

a) What is the # of rounds till $\underline{L}\,\underline{L} = x$

$\uparrow$ is $g=-2$

Want PMF of X

$p_X(x) =$ geometric "sucess" we are calling $g=-2$

$P(\text{sucess}) = \left(\frac{1}{3}\right)^2 = \frac{1}{9}$

So geometric(1/9)

$$= \left(\frac{8}{9}\right)^{x-1}\left(\frac{1}{9}\right) \quad x = 1, 2, 3 \ldots$$

b) Z = time until Bob has 3rd loss

Pascal $\leftarrow$ but Pascal for concesitive #
but bob only plays even times
So multiply by 2

R = # of rounds till Bob's 3rd loss

$Z = 2R$

(4)

$$P_R(r) = \binom{r-1}{3-1}\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^{r-3} \qquad r = 3, 4, 5, \ldots$$

- must count all the ways can position

$$P_Z(z) = P_R\left(\frac{z}{2}\right)$$

$$= \binom{z/2 - 1}{3-1}\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^{z/2-3} \qquad z = 6, 8, 10, \ldots$$

c) $E[N] = ?$ $\quad N =$ # of rounds until each has won at least once

$= E[N \mid \text{Bob wins 1st}] \cdot P(\text{Bob wins 1st}) + E[N \mid \text{Alice wins 1st}) \cdot P(\text{Alice wins 1st}) + E[N \mid \text{Both win 1st}] \cdot P(\text{Both win 1st}) + E[N \mid \text{Both lose}]\, P(\text{both lose})$

↑ total expectation theorem

iterate over cases

? is this the best way to partition

A_1 = both win

A_2 = only alice wins

A_3 = " bob "

A_4 = no one wins

↓ (short hand!)

$$= \sum_i^4 E[N \mid A_i] \cdot P(A_i)$$

(5)

$$= 1 \cdot \left(\frac{2}{3}\right)^2 + \left(1 + \frac{1}{2/3}\right) \cdot \left(\frac{2}{3} \cdot \frac{1}{3}\right) + \underbrace{\left(1 + \frac{1}{2/3}\right) \cdot \left(\frac{2}{3} \cdot \frac{1}{3}\right)}_{\text{same}} + \left(1 + E[N]\right)\left(\frac{1}{3}\right)^2$$

have to wait till Bob wins

start fresh

✓ X | restart

just time until 1st sucess from Bob

Geometric $\left(\frac{2}{3}\right)$

grap $E[N]$ on 1 side

algebra ...

$E[N] = \frac{15}{8}$

Poisson Process

Can happen anywhere on line

- time homogeneity

$P(k, \tau)$ = prob of k arrivals in interval of lengh τ

each interval of same leanght has same probability p

- independence

- small interval probablity: τ is very small

$\lambda\tau$ ← prob of 1 arrival

$1 - \lambda\tau$ ← prob 0 arrivals

0 ← prob multiple arrivals

(5b

Can define all these variables for Poisson process.

Poisson RV

$$p(k,\tau) = e^{-\lambda\tau}\frac{(\lambda\tau)^k}{k!} \qquad k=0,1,2\ldots\ldots$$

$$E[X] = \lambda\tau \qquad \mathrm{var}(X) = \lambda\tau$$

exponential

time until 1st arrival

erlang

like Pascal, continuous time until kth arrival

$$Y_k = T_1 + T_2 + \ldots + T_k$$

↑ each exponential   all independent.

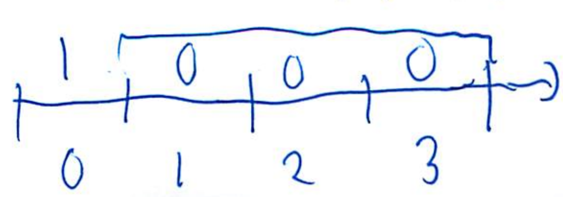
$$E[Y_k] = k\cdot\frac{1}{\lambda} \qquad \mathrm{var}(Y_k) = \frac{k}{\lambda^2}$$

#3. continuous time on a bridge

$\lambda = 3$ trains/day

On day 0 see 1 train

P(0 trains on days 1,2,3 | 1 train day 0)



(6)

Days are disjoint, so days independent
So day 0 does not matter

poisson RV

$= P(0,3)$

just
care
about 3 days

$$= e^{-3\cdot 3} \frac{(3,3)^0}{0!}$$

$$= e^{-3\cdot 3}$$

$$= e^{-9}$$

---

b) Find P that after 1st train arrives on day 3, the next train takes more than 3 days

= Same as part a!

$= e^{-9}$

---

c) Find P(no trains 0, 1 | 4 trains day 4)

multiply 2 since ind.

= P(no trains day 0,1) P(4 trains day 4)

(7)

$= P(0,2) \cdot P(4,1)$

= substitute same formula again

- - - -

---

d) Find p it takes more than 2 days for 5th train to arrive

= P(5th train arrives <u>after</u> day 2)

$= P(0,2) + P(1,2) + P(2,2) + P(3,2) + P(4,2)$

= substitute in

---

2. $Y = X_1 + X_2 + \ldots + X_N$

X_i s are iid

~ geometric

N is ind. of X

$N \sim geom(q)$

Show $Y \sim geom(pq)$

This is PMF

Derived dist - tedious

Transforms - not covering this semester

Random Sum - only gives E + var

Hint: interpret in split Bernoulli processes

(8)

Hard part is interpeting problem

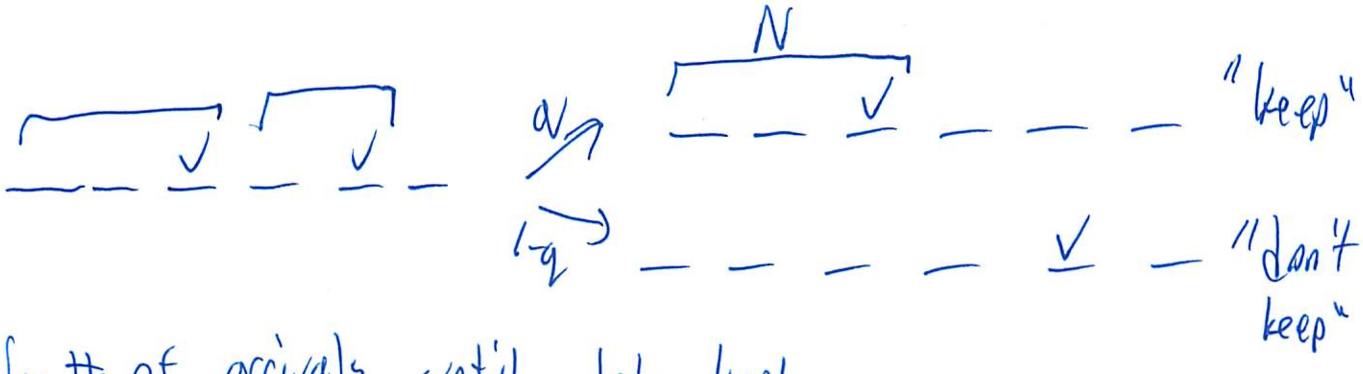
X_i = interarrival times

Interperate X_i as interarrival times in Bernouli p process

"keep" ~~an~~ an arrival w/ $p = q$

"Discard" " " " $p = 1-q$

) aka split



N = # of arrivals until 1st kept

$Y = X_1 + X_2 + \dots + X_n$ = total time until 1st arrival decide to keep

= time until 1st arrival in Bernoulli pq process

$\sim$ geometric (pq)

(go through book and try to solve - confusing at 1st)

6.041 Review

10/31

review CDF!
- note cdf tables for common
- but should be still able to do manually

do joint CDF
- and the other various joint qu

looked back at last test
- proof was disastroeos
- and interped one problem wrong

Practice 11/1/2010

6.041/6.431 Spring 2008 Quiz 2
Wednesday, April 16, 7:30 - 9:30 PM.

DO NOT TURN THIS PAGE OVER UNTIL YOU ARE TOLD TO DO SO

Name: ____________________

Recitation Instructor: ____________________

TA: ____________________

6.041/6.431: ____________________

Question	Part	Score	Out of
0			3
1	all		36
2	a		4
	b		5
	c		5
	d		8
	e		5
	f		6
3	a		4
	b		6
	c		6
	d		6
	e		6
Total			100

- Write your solutions in this quiz packet, only solutions in the quiz packet will be graded.
- Question one, multiple choice questions, will receive no partial credit. Partial credit for question two and three will be awarded.
- You are allowed 2 two-sided 8.5 by 11 formula sheet plus a calculator.
- You have 120 minutes to complete the quiz.
- Be neat! You will not get credit if we can't read it.
- We will send out an email with more information on how to obtain your quiz before drop date.
- **Good Luck!**

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Spring 2008)

Problem 0: (3 pts) Write your name, your assigned recitation instructor's name, and assigned TA's name on the cover of the quiz booklet. The Instructor/TA pairing is listed below. Also write the class you are registered for: 6.041 or 6.431.

Recitation Instructor	TA	Recitation Time
Vivek Goyal	Natasa Blitvic	10 & 11 AM
Michael Collins	Danielle Hinton	10 & 11 AM
Shivani Agarwal	Stavros Valavanis	12 & 1 PM
Dimitri Bertsekas (6.431)	Aman Chawla (6.431)	1 & 2 PM

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Spring 2008)

Question 1: Multiple choice questions. **CLEARLY** circle the best answer for each question below. Each question is worth 4 points each, with no partial credit given.

a. (4 pts) Let X_1, X_2, and X_3 be independent random variables with the continuous uniform distribution over $[0,1]$. Then $\mathbf{P}(X_1 < X_2 < X_3) =$

(i) 1/6
(ii) 1/3
(iii) 1/2
(iv) 1/4

b. (4 pts) Let X and Y be two continuous random variables. Then

(i) $\mathbf{E}[XY] = \mathbf{E}[X]\mathbf{E}[Y]$
(ii) $\mathbf{E}[X^2 + Y^2] = \mathbf{E}[X^2] + \mathbf{E}[Y^2]$
(iii) $f_{X+Y}(x+y) = f_X(x)f_Y(y)$
(iv) $\text{var}(X+Y) = \text{var}(X) + \text{var}(Y)$

c. (4 pts) Suppose X is uniformly distributed over $[0,4]$ and Y is uniformly distributed over $[0,1]$. Assume X and Y are independent. Let $Z = X + Y$. Then

(i) $f_Z(4.5) = 0$
(ii) $f_Z(4.5) = 1/8$
(iii) $f_Z(4.5) = 1/4$
(iv) $f_Z(4.5) = 1/2$

d. (4 pts) For the random variables defined in part (c), $\mathbf{P}(\max(X,Y) > 3)$ is equal to

(i) 0
(ii) 9/4
(iii) 3/4
(iv) 1/4

e. (4 pts) Consider the following variant of the hat problem from lecture: N people put their hats in a closet at the start of a party, where each hat is uniquely identified. At the end of the party each person randomly selects a hat from the closet. Suppose N is a Poisson random variable with parameter λ. If X is the number of people who pick their own hats, then $\mathbf{E}[X]$ is equal to

(i) λ
(ii) $1/\lambda^2$
(iii) $1/\lambda$
(iv) 1

f. (4 pts) Suppose X and Y are Poisson random variables with parameters λ_1 and λ_2 respectively, where X and Y are independent. Define $W = X + Y$, then

(i) W is Poisson with parameter $\min(\lambda_1, \lambda_2)$

(ii) W is Poisson with parameter $\lambda_1 + \lambda_2$

(iii) W may not be Poisson but has mean equal to $\min(\lambda_1, \lambda_2)$

(iv) W may not be Poisson but has mean equal to $\lambda_1 + \lambda_2$

g. (4 pts) Let X be a random variable whose transform is given by $M_X(s) = (0.4 + 0.6e^s)^{50}$. Then

(i) $\mathbf{P}(X = 0) = \mathbf{P}(X = 50)$

(ii) $\mathbf{P}(X = 51) > 0$

(iii) $\mathbf{P}(X = 0) = (0.4)^{50}$

(iv) $\mathbf{P}(X = 50) = 0.6$

h. (4 pts) Let X_i, $i = 1, 2, \ldots$ be independent random variables all distributed according to the pdf $f_X(x) = x/8$ for $0 \le x \le 4$. Let $S = \frac{1}{100}\sum_{i=1}^{100} X_i$. Then $\mathbf{P}(S > 3)$ is approximately equal to

(i) $1 - \Phi(5)$

(ii) $\Phi(5)$

(iii) $1 - \Phi\left(\frac{5}{\sqrt{2}}\right)$

(iv) $\Phi\left(\frac{5}{\sqrt{2}}\right)$

i. (4 pts) Let X_i, $i = 1, 2, \ldots$ be independent random variables all distributed according to the pdf $f_X(x) = 1$, $0 \le x \le 1$. Define $Y_n = X_1 X_2 X_3 \ldots X_n$, for some integer n. Then $\text{var}(Y_n)$ is equal to

(i) $\frac{n}{12}$

(ii) $\frac{1}{3^n} - \frac{1}{4^n}$

(iii) $\frac{1}{12^n}$

(iv) $\frac{1}{12}$

Question 2: Each Mac book has a lifetime that is exponentially distributed with parameter λ. The lifetime of Mac books are independent of each other. Suppose you have two Mac books, which you begin using at the same time. Define T_1 as the time of the first ~~laptop~~ failure and T_2 as the time of the second ~~laptop~~ failure.

a. (4 pts) Compute $f_{T_1}(t_1)$

b. (5 pts) Let $X = T_2 - T_1$. Compute $f_{X|T_1}(x|t_1)$

c. (5 pts) Is X independent of T_1? Give a mathematical justification for your answer.

d. (8 pts) Compute $f_{T_2}(t_2)$ and $\mathbf{E}[T_2]$

e. (5 pts) Now suppose you have 100 Mac books, and let Y be the time of the first laptop failure. Find the best answer for $\mathbf{P}(Y < 0.01)$

Your friend, Charlie, loves Mac books so much he buys S new Mac books every day! On any given day S is equally likely to be 4 or 8, and all days are independent from each other. Let S_{100} be the number of Mac books Charlie buys over the next 100 days.

f. (6 pts) Find the best approximation for $\mathbf{P}(S_{100} \leq 608)$. Express your final answer in terms of $\Phi(\cdot)$, the CDF of the standard normal.

Question 3: Saif is a well intentioned though slightly indecisive fellow. Every morning he flips a coin to decide where to go. If the coin is heads he drives to the mall, if it comes up tails he volunteers at the local shelter. Saif's coin is not necessarily fair, rather it possesses a probability of heads equal to q. We do not know q, but we do know it is well-modeled by a random variable Q where the density of Q is

$$f_Q(q) = \begin{cases} 2q & \text{for } 0 \leq q \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Assume conditioned on Q each coin flip is independent. Note parts a, b, c, and $\{d, e\}$ may be answered independent of each other.

a. (4 pts) What's the probability that Saif goes to the local shelter if he flips the coin once?

In an attempt to promote virtuous behavior, Saif's father offers to pay him \$4 every day he volunteers at the local shelter. Define X as Saif's payout if he flips the coin every morning for the next 30 days.

b. (6 pts) Find $\text{var}(X)$

Let event B be that Saif goes to the local shelter at least once in k days.

c. (6 pts) Find the conditional density of Q given B, $f_{Q|B}(q)$

While shopping at the mall, Saif gets a call from his sister Mais. They agree to meet at the Coco Cabana Court yard at exactly 1:30PM. Unfortunately Mais arrives Z minutes late, where Z is a continuous uniform random variable from zero to 10 minutes. Saif is furious that Mais has kept him waiting, and demands Mais pay him R dollars where $R = \exp(Z + 2)$.

d. (6 pts) Find Saif's expected payout, $\mathbf{E}[R]$

e. (6 pts) Find the density of Saif's payout, $f_R(r)$

Spring 08 Quiz 2 11/1
+ Studying

a) X_1

Oh this is that weird CDF qu from the book
I remember 1/6 from looking at last night
But I don't get it

✓ simple version X_1 X_2

$P(X_1 < X_2) + P(X_2 > 1) + P(X_1 = X_2) = 1$

? prob this occures ← Oh that makes more sense
- why didn't I see this before!

? so those 3 possibilities are complete + disjoint

$P(X_1 < X_2) = P(X_2 > X_1)$ by symtirry

$P(X_1 = X_2) = 0$ since contineous

so $P(X_1 < X_2) = 1/2$

---

now back to our problem

$P(X_1 < X_2 < X_3) + P(X_1 < X_3 < X_2)$ ... 6 of these $= 1$

all = each other by symmatry so $= \frac{1}{6}$

~~book~~

(2)

Book does something different

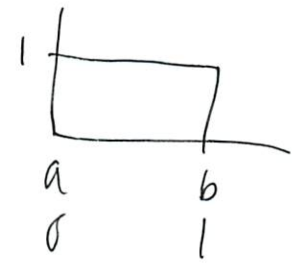
-oh no book problem is different -so where do I remember this?

-should do a CDF review from lecture

-hmmm don't see it

-well just $\int_{-\infty}^{x} f_X(t)\,dt$

So preove uniform



$$\int_{-\infty}^{x} 1\,dt$$

$$t \Big|_{-\infty}^{x}$$

Or $\dfrac{x-a}{b-a} = \dfrac{x-0}{b-1} = x$

Since uniform

$$\begin{cases} x & a \le x \le b \\ 0 & x < a \\ 1 & b > x \end{cases}$$

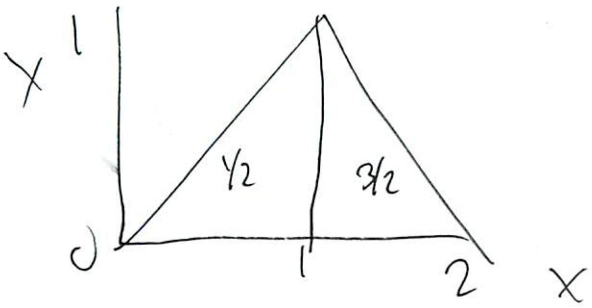
important to include

Oh that was not that hard

look back at that p-set qu

③

Pset 5 #3 D)



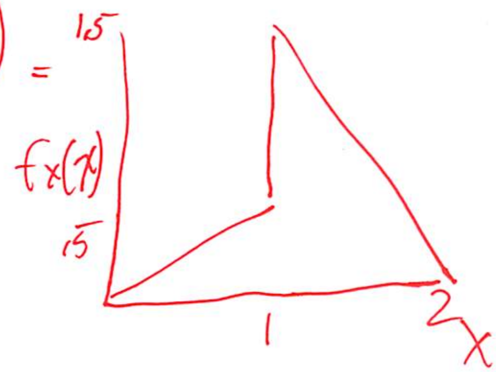
Well find $f_x(x)$ first

- marginal PMF $f_x(x)$

$\int_{-\infty}^{\infty}$ joint dy

oh joint is given!

need to remember graphical way



$x/2$ $0 \le x < 1$

$-3x/2 + 3$ $1 < x \le 2$

0 otherwise

$\int_0^1 \frac{1}{2}\, dy$

↑ x?

no needs to be y

$\frac{y}{2}\Big|_0^1$

same in this case

$\frac{y}{2}$ ← height of $\frac{y}{2}$

go to each x and add up

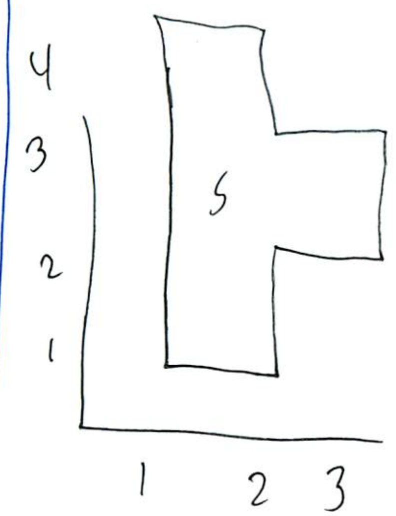
1/2 now I realize

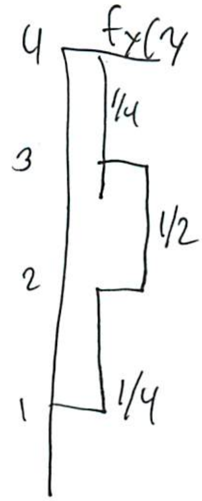
$\int_0^x \frac{1}{2}\, dy$

$\frac{x}{2}$

right?

(4)



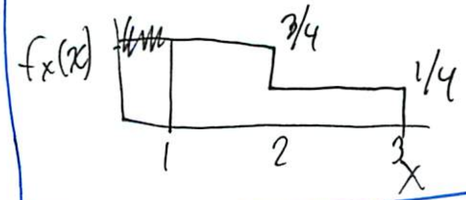


~~So at each point~~

So for $f_Y(y)$

go to each y → look across

and add up



(from the book)

Now try second part

- graphically

well line is

$\cdot 2 - y$

$\frac{3}{2}(2-y)$

$3 - \frac{3}{2}y$ ✓ got it graphically

↑ and x not y

$\int_0^1 \frac{3}{2}\, dy$

$\frac{3}{2}y$

$\frac{3}{2}$ ⊗ why does algebra not work out

5

try with x

$$\int_2^3 \frac{3}{2}\, dy$$

$$\frac{3}{2}y \Big|_2^3$$

$$\frac{9}{2} - 3$$

oh I get it

$$\int_0^y \frac{3}{2}\, dy$$

no

$$\int_1^x \frac{3}{2}\, dy$$

$$\frac{3}{2}y \Big|_1^x$$

$$\frac{3x}{2} - \frac{3}{2}$$ X

why not this?

$$\int_x^2 \frac{3}{2}\, dy$$ ← the value that y is valid but this seems wrong

y?

$$\frac{3}{2}y \Big|_x^2$$

$$3 - \frac{3x}{2}$$ ✓ here we are

(6) what would have book example have been

for x $\int_1^4 1 \, dy$ $\quad 1 \leq x \leq 2$

watch =

$4-1$

3

want $3/4$

Or back to problem

$$\int_0^{2-y} \frac{3}{2} \, dy$$

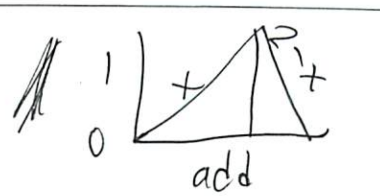
$$\frac{3}{2} y \Big|_0^{2-y}$$

$$\frac{3}{2}(2-y)$$

there we are ✓

So mark every line on the graph

Now try for y



$f(y)$ $\displaystyle\int_x^1 \frac{1}{2} \, dx + \int_1^{2-x} \frac{3}{2} \, dx$

(7)

$\frac{1}{2}x\Big|_x^1 + \frac{3x}{2}\Big|_1^{2-x}$ $\quad 0 \le x \le 1$

$\frac{1}{2} - \frac{x}{2} + 3 - \frac{3x}{2} - \frac{3}{2}$

$4 - 2x \quad 0 \le y \le 1$

but supposed to have y

Call the $x = y$ here

but in general replace variables $\int \quad dx$

to be same as $f_Y(y)$

opposet here

important part

- never same

and when 2 $\Rightarrow$

have the complex one inside

(remembering 18.02)

$f_{Y|X}(y|.5)$

so just do at $x = .5$

or take ans to f_Y and put it in graphically

$f_Y(y) = \begin{cases} \cancel{\frac{1}{2}}\ 2 & 0 \le y \le .5 \\ 0 & \text{else} \end{cases}$ up and down at that x

rescale

(8)

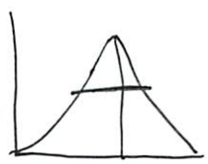
reamber conditioning $\dfrac{f_{X,Y}(x,y)}{f_X(x)} = f_{Y|X}(y|x)$

remember to rescale $\rightarrow$

So plug values in

$$\frac{\frac{1}{2}}{x/2} = \frac{1}{2}\cdot\frac{2}{x} = \frac{1}{x} = \frac{1}{.5} = 2$$

$f_{X|Y}(x|.5)$



left right

$$\frac{1/2}{4-2y} + \frac{3/2}{4-2y} = \frac{1/2}{3} + \frac{3/2}{3} = \frac{1}{6} + \frac{1}{2} = \frac{4}{6} = \frac{2}{3}$$

no! two different answers

$$\begin{cases} 1/2 & 1/2 < x < 1 \\ 3/2 & 1 < x < 3/2 \\ 0 & \text{else} \end{cases}$$

why not need to rescale?

~~Oh still complete~~

or can see still 1

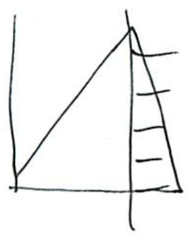
$$\frac{1}{2}\cdot\frac{1}{2} + \frac{3}{2}\cdot\frac{1}{2}$$

$$\frac{1}{4} + \frac{3}{4} = 1$$

my $f_Y(y)$ is likely wrong

(9) I love how I do <u>worse</u> on these now

( ) $R = XY$ $\quad A =$ event $X < .5$ $\quad$ Find $E[R \mid A]$



but how to find $E[XY]$

; law of iterated expectations ;

no just find it

Conditional PDF $= \frac{1}{\text{area}} = 8$

; since constant but need to rescale

$\to$ oh c!

$E[XY \mid A]$

$$= \int_0^{.5} \int_y^{.5} 8xy \; dx \; dy$$

$\cancel{\frac{1}{2} \cdot 1 \cdot 1 \cdot \frac{3}{2}}$

no evenly distributed in area!

$xy \cdot f(x,y)$

How did they get 8!

; something w/ it multiplied together ;

Oh! $x < .5$ not 1!

So area

$.5 \cdot .5 + \frac{1}{2} .5 \cdot .5 + \frac{1}{2} \cdot 1 \cdot 1$

$\frac{1}{4} + \frac{1}{8} + 1$

and take PDF into account, right ;

$.5 \cdot .5 \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} + \frac{1}{2} \cdot 1 \cdot 1 \cdot \frac{3}{2}$

$\frac{1}{8} \quad + \frac{1}{16} \quad + \frac{3}{4}$

$\frac{2}{16} \quad + \frac{1}{16} \quad + \frac{12}{16}$

$= \frac{15}{16}$

(10)

$f_{X|A}(x) = \dfrac{f_X(x)}{P(X \in A)} \rightarrow$

P want $= 15/16$

P total $= \frac{1}{2}\cdot\frac{1}{2}\cdot 1 \cdot 1 + \frac{3}{2}\cdot\frac{1}{2}\cdot 1 \cdot 1$

$= \frac{1}{4} + \frac{3}{4}$

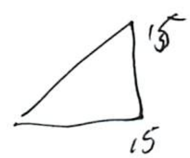
$$\frac{\frac{1}{2}}{\frac{15}{16}} = \frac{1}{2}\cdot\frac{16}{15} = \frac{16}{30} = \frac{8}{15}$$

But it says ~~X???~~ $X < .5$ is a right triangle w/ constant height

Oh I had wrong direction! Why do I screw up so much!

$\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \cancel{\frac{1}{2}}$

triangle height width forget that

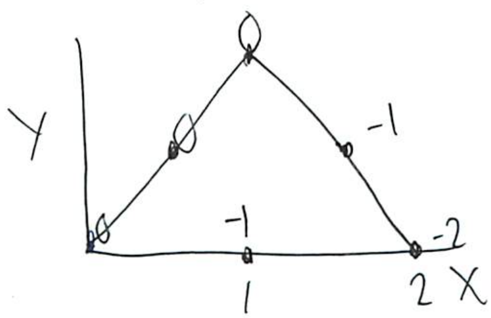


prob ~~...~~ does not mater

So I see now   We are assuming it happened!

Ok 8 pages later the main question

$W = Y - X$ find CDF of W



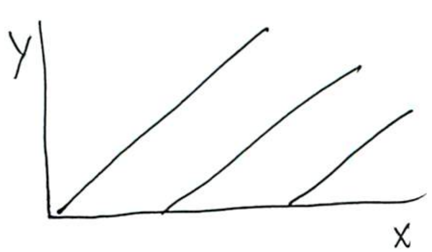
So something /// like that

draw pts

wish had that p-set to review this

this is derived dist, right?

Something about CDF



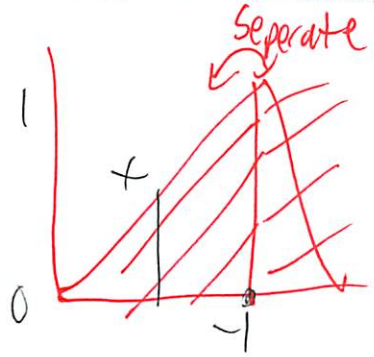
(11)

$F_W(w) = P(W \le w) = P(Y - \cancel{X} \le w) = P(Y \le X + w)$

Integrate area below $Y = X + w$

↑ those values I made

Remember different densities



4 regions of interest

$w < -2$

$-2 \le w \le -1$

$-1 < w \le 0$

$w > 0$

↑ remember increasing!!!

$w > 0 \rightarrow \cancel{0}\ 1$

So what do CDF on definition $P(X \le x)$

$\int_0^x x + w \quad dy$

$w = Y - X \qquad -1 < w \le 0$

$\int_0^x y \, dy$

$\frac{y^2}{2}\Big|_0^x$

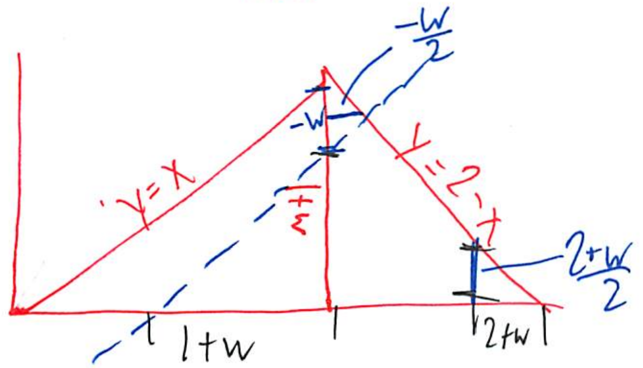
$\frac{x^2}{2} \quad -1 < w \le 0$

$$\boxed{\frac{1}{2} \cdot \frac{1}{2}(1+w)^2 + \frac{3}{2}\left(\frac{1}{2} \cdot 1 \cdot 1 - \frac{1}{2}\left(-\frac{w}{2} \cdot -w\right)\right.}$$

↳ forgot 2nd half

— add here, I was going to do later

(12) draw out lines



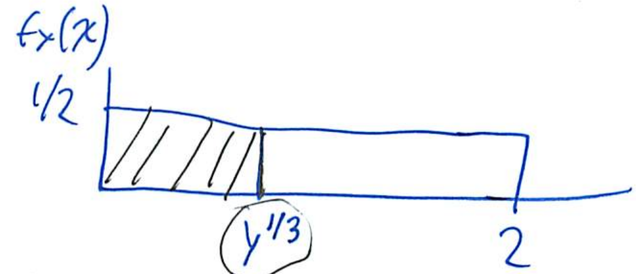
Where in all world did we get all this?

So for each w add up area?
Wish they showed more work

Look at a recitation

Lecture 10

X = Uniform $[0, 2]$



~~M~~ $Y = X^3$

$$F_Y(y) = P(Y \le y) = P(X^3 \le y) = P(X \le y^{1/3})$$

I always miss this step
Why do it?

(1/2) now $F_X(y^{1/3})$ right?

I guess to get it in terms of what you want in the problem
In our problem it was Y

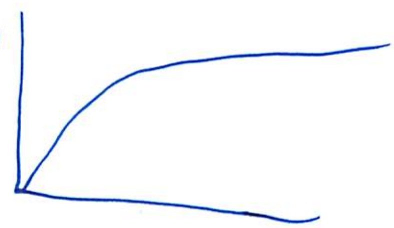
~~Well~~

Think about it Prob that X is $\le y^{1/3}$
aka in the shaded version
and add up over all possible Y

13

Or wait → do you?
lower case y means it has already been "picked"
is that in our problem too?
No! we want for each $W = w$

$= \frac{1}{2} y^{1/3}$ ← so integer answer



No this is still $F_Y(y)$

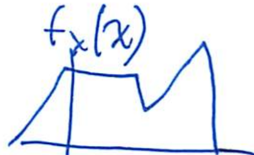
Differeiate

Then do bounds, where 1, 0

$$\frac{F_Y(y)}{dy} = \frac{1}{2} \cdot \frac{y^{4/3}}{\frac{4}{3}} = \frac{1}{2} \cdot \frac{3}{4} \cdot y^{4/3} = \frac{3}{8} y^{4/3}$$ ✓

Another

$Y = 2x + 5$



variables here

$$F_Y(y) = P(Y \le y) = P(aX + b \le y) = P\left(X \le \frac{y-b}{a}\right)$$

how choose?

$$= F_X\left(\frac{y-b}{a}\right)$$

11/2

based on what have $F_X(x)$ at

? so what is that

$$\int_{-\infty}^{\infty} \frac{\cancel{x}-b}{a}\, dt$$

??

↑ I think setting bounds was hard part on ours

14

So back to ours

$\int$

Well lot ~~this~~ in recitation 11 shortcut

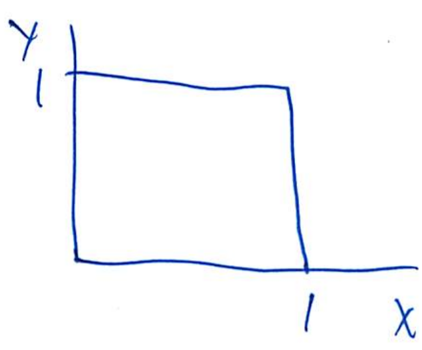
$$F_X(y^2) - F_X(-y)$$

$$\frac{d}{dx} = 2y f_X(y^2) + f_X(-y)$$

; how in all world do you that ?

(the problem on this is they spent too much time on Bayes' rule)

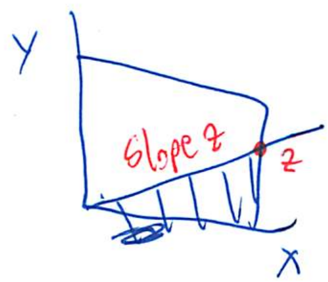
Book example 4.8



$$Z = \frac{Y}{X}$$

2 cases

$0 \leq z \leq 1$



$z > 1$ ~~$0 \leq z \leq$~~

(15)

$$F_Z(z) = P\left(\frac{Y}{X} \le z\right) = \begin{cases} z/2 & 0 \le z \le 1 \\ 1 - 1/2z & z > 1 \\ 0 & \text{other} \end{cases}$$ ← this I remember — do 1 − CDF

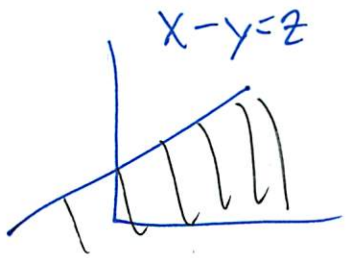
↓ differentiate

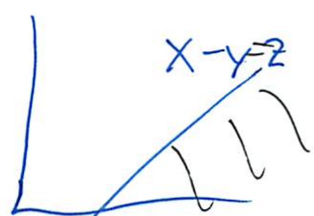
$$f_Z(z) = \begin{cases} 1/2 & 0 \le z \le 1 \\ 1/(2z^2) & z > 1 \\ 0 & \text{other} \end{cases}$$

Example 4.9

X = Romeo late

Y = Juliet late





this one

↓ how decide what goes here

$$F_Z(z) = P(X - Y \le z)$$

find the P of this area!

$$= 1 - P(X - Y > z)$$

$$= 1 - \int_0^\infty \left(\int_{x+y}^\infty f_{X,Y}\, dx \right) dy$$ Ok I think I get it

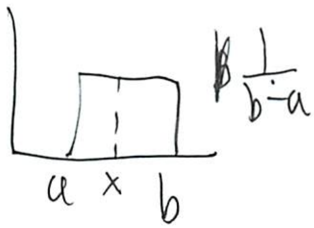
↑ sum pdf over area

for each z?

that is what confuses me

16

Well what is proof $F_X(x)$ uniform = $\frac{x-a}{b-a}$



$$\int_{-\infty}^{x} \frac{1}{b-a}\,dt$$

$$\left(\frac{1}{b-a}\right)t \Big|_{\cancel{\infty}\, a}^{x}$$ a is start

$$\frac{x}{b-a} - \frac{a}{b-a}$$

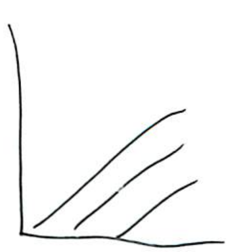
$$\frac{x-a}{b-a}$$

Ok get it cdf more now

but the derived part still throws me off

It might be that each shape −/÷ throws me off

No I had that



need to integrate over the lines

$$P(Y \le X + w)$$

$$\int \int_{1+w}^{1} \frac{1}{2}\,dx\,dy$$

I don't get this - why have we not gotten P-set back

(17)

$$\underbrace{\frac{1}{2}}_{\text{PDF}} \cdot \underbrace{\frac{1}{2}(1+w)(1+w)}_{\text{area}} + \underbrace{\frac{3}{2}}_{\text{PDF}}\Big(\underbrace{\frac{1}{2}\,1\cdot 1}_{\text{area}} - \underbrace{\frac{1}{2}\left(-\frac{w}{2}\cdot -w\right)}_{\cdot 1 \text{ area}}\Big)$$

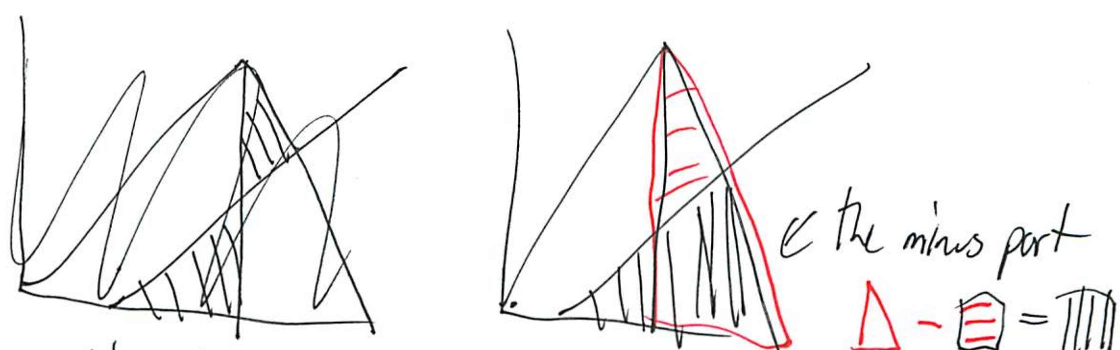
So as w increases ↑ goes
So area gets larger

why is this one taking away

Oh whatever is the triangle

remember integrate area not each line
(Part of why this is hard is prob double interals Unsurity)

So this is area

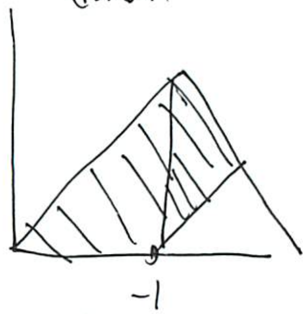


Oh they ~~[illegible]~~ are going the other way ?
in doing CDF
-actually makes sense lowest value (-2) to highest (+0)

$-1 \leq z \leq 0$

(18) Still not sure could find all the lines - but that is geometry
So let me try next part

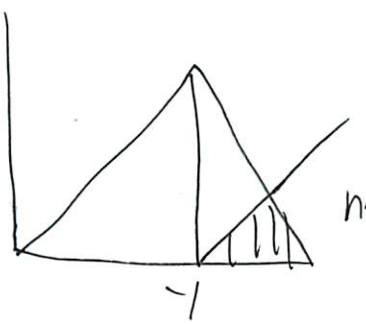
domain



? so wait what was other part

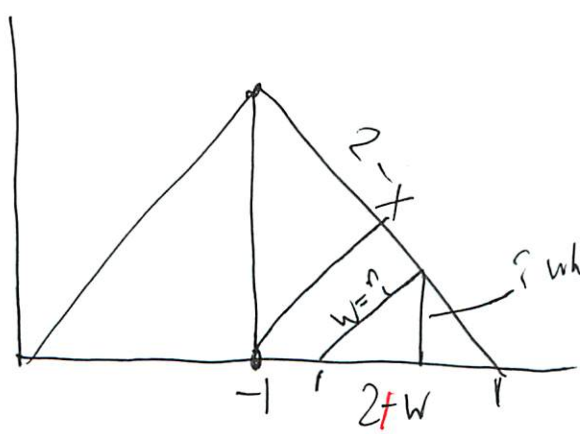
Oh wait I did that

I am all out of order!



now this ~~one~~

need geometry - try w/o looking



? what is this $\frac{2+w}{2}$ ~~Where~~ Where in all world

Oh duh always half of triangle

know your geometry well all are +

(19)

$\underbrace{\frac{3}{2}}_{\text{PDF}} \cdot \frac{1}{2} \cdot (2+w) \cdot \left(\frac{2+w}{2}\right)$ that it?

$-1 \le w \le -2$

and then $w > 0$

CDF $= 1$

17 pages later I think I kinda got that!

Would like to do another practice of that

If only the test was on that - but other things, back to exam

b) i) independent

ii) never saw

iii) no distributed - or this convolution? well only if ind

iv) ind

So no for all?

Say 2 since can expectations I think

linearity of expectations

---

c) X = uniform $[0,4]$ Y = uniform $[0,4]$

X, Y independent $Z = X + Y$

$f_Z(4.5) = ?$

(20) Convolution!

this does not matter, right?

$\int f_X(x)\, f_Y(\cancel{z}-x)$

$\frac{1}{4-0} \cdot \frac{1}{1-0} = \cancel{1/4}$ $\frac{1}{8}$ ← they did not show this step

$\int_{-\infty}^{\infty} f_X(\alpha)\, f_Y(4.5-\alpha)\, d\alpha$

$\int_{3.5}^{4} \frac{1}{4}\, d\alpha = \frac{1}{8}$

Oh I just did this part and got

this is the overlapping part

but rest of problem!

Must ~~cross~~ integrate this over the overlap

Well is where 3.5 from? ← flip + slide??

and I am sure of graphical

IDK - can't find ?? in book notes

---

d) For RV from C ~~(hate these dependecies)~~ just problems

$P(\max(X,Y) > 3)$

So Y is always > 3

So P $X > 3$ $\frac{1}{4}$ ✓

(2c)

They have a much more complex solution

$$P(\max(X,Y) > 3) = 1 - P(\max(X,Y) \leq 3)$$

$$= 1 - P(X \leq 3 \cap Y \leq 3)$$

independent

much more formal

$$= 1 - P(X \leq 3) \cap P(Y \leq 3)$$

$$1 - \frac{3}{4} \cap 1$$

$$\frac{1}{4}$$

---

e) Consider variant of hat problem from lecture & don't remember
very bad they ref lecture - should be able to do class w/o lecture
and poisson - not on quiz so skip

---

f) poisson aswell

---

g) transform skip

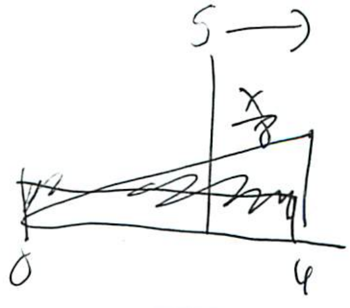
---

h) X_i $i = 1, 2, \ldots$

$$S = \frac{1}{100} \sum_{i=1}^{100} X_i \quad P(S > 3)$$

$$f_X(x) = \frac{x}{8} \quad 0 \leq x \leq 4$$

so avg of 100 X_is
each has prob $\frac{x}{8}$



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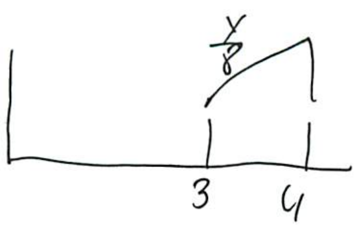


but 100 of these

and they say normal?

? is the normal way

guess ~~iv~~

iii

$$P(S>3) = 1-P(S\leq 3)$$

? isn't that flipped?
Opps wrong on cheat sheet!!!
fixed!

$$= 1-\Phi\left(\frac{3-E(S)}{\sqrt{Var(S)}}\right)$$

just do!

$$E[X_i] = \int_0^4 \underbrace{x}_{X} \underbrace{\frac{x}{8}}_{f_X(x)} = \frac{x^3}{24}\Big|_0^4 = \frac{8}{3}$$

↑ the area

$$Var(X_i) = E[X_i^2] - (E[X_i])^2$$

$$= \int_0^4 \underbrace{x^2}_{\text{just } x^2} \frac{x}{8}\,dx - \left(\frac{8}{3}\right)^2$$

just x^2 sometimes I forget how easy this is

$$= \frac{x^4}{32}\Big|_0^4 - \left(\frac{8}{3}\right)^2$$

$$= \frac{8}{9}$$

(23)

$E[S] = \frac{1}{100} E[X_i] + \dots + \frac{1}{10} E[X_{100}] = \frac{8}{3}$

$Var(S) = \frac{1}{100^2} Var(X_1) + \dots + \frac{1}{100^2} Var(X_i) = \frac{8}{9} \cdot \frac{1}{100}$

$$P(S > 3) = 1 - \Phi\left(\frac{3 - \frac{8}{3}}{\sqrt{\frac{8}{9} \cdot \frac{1}{100}}}\right)$$

$$= 1 - \Phi\left(\frac{5}{\sqrt{2}}\right)$$

how in all world know that?

Well I guess right ~~[illegible]~~ (if my cheat sheet was correct)

But I was not really thinking through $E[X_i]$

and then $E[S]$ and then normal of that

Did we ever do one of those?

---

1) $X_i = 1, 2, \dots, i$ RV

distributed $f_X(x) = 1, 0 \leq x \leq 1$

$Y = X_1 X_2 X_3 \dots X_n$? so inger

$Var(Y_n)$ is = to?

? is this Sum or Random # of RV

but its product

? did not cover

(24)

Is this like an ~~ill~~ intuition/guess project?

will involve N

and <u>not</u> independently distributed

never did, skip

$\frac{1}{3^n} - \frac{1}{4^n}$

$$E[Y_n] = E[X_1] \times \cdots \times E[X_n]$$

I think I need to know expectation + var rules

Only one I remember is $var(x+2) = var(x)$

yeah never did this

or is this squared?

$$E[Y_n^{2}] = E[X_1^2 \times \cdots X_n^2]$$

$$E[X_i] = \frac{1}{2}$$

$$E[X_i^2] = \frac{1}{3} \quad \text{for } i = 1, 2, 3, \ldots n$$

expected value - is the same! for all

$$Var(S_n) = E[Y_n^2] - (E[Y_n])^2 = \frac{1}{3^n} - \frac{1}{4^n}$$

$$\underbrace{\frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3}}_{n} \qquad \underbrace{\frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{4}}_{n}$$

$$\left(\frac{1}{2}\right)^2 \cdot \left(\frac{1}{2}\right)^2 \cdot \left(\frac{1}{2}\right)^2$$

Need to get better at seeing these type of problems

(25) Finally past the multiple choice

2. ~~###~~

MacBooks have exponentially distributed life param λ

life of MacBooks ind

T_1 = 1st failure T_2 = 2nd

why is laptop crossed out?

$f_{T_1}(t_1) = \lambda e^{-\lambda x}$

$= \lambda e^{-\lambda t_1}$

no bounds needed

(X) See next pg

b) ~~d~~ $X = T_2 - T_1$ Compute $f_{X|T_1}(x|t_1)$

life independently distributed, so same?

$= \lambda e^{-\lambda t}$

its not total time, or is it – what is X?

no ~~#~~ should = t_1

c) Yes memoryless

Now need "math justification" – oh boy!

(2) Given that $x > t$ (3) $X - t \sim \exp(\lambda)$

(1) $X \sim \exp(\lambda)$ $X - t \sim \exp(t)$

(26)

d) Compute $f_{T_2}(t_2)$

here it is $\neq$ than t_1 since

or is it t_2 ?

need to review

$E[T_2] = \frac{1}{\lambda}$

~~[illegible]~~

No! Read the whole thing carefully!

Only have 2 macbooks

M_1 = life time macbook 1

M_2 = " " 2

$F_M(m) = 1 - e^{-\lambda m}$

$f_M(m) = \lambda e^{-\lambda m}$ for each

T_1 time of first failure

find CDF 1st then differentiate

$F_{T_1}(t) = P(\min\{M_1, M_2\} < t)$

$= 1 - P(\min\{M_1, M_2\} \geq t)$

$= 1 - P(M_1 \geq t) P(M_2 \geq t)$ independent

Don't confuse w/ process qu!

What section is this anyway?

← ? so derived dist?

← 2nd problem like this make sure know how to do this

27

$= 1-(1-F_M(t))^2$

$= 1-e^{-2\lambda t} \quad t \geq 0$

since 2

differentiate

$f_{T_1}(t) = 2\lambda e^{-2\lambda t} \quad t \geq 0$

all I was missing

need this bounds

and the deeper insight ~~that~~ of how they got that don't really know/would not have thought of that deep way they did it - would have gotten wrong

---

So of course that will change the other results
Re think for a min

intuitivly; does knowing 1 broke help?

- not really
- lets you know you are some amout into time period →
- right
- need to study these type

Memoryless → so always exponential

$\lambda e^{-\lambda x}$ Actually I got this exactly correct

So maybe I do understand these

28

c) ✓ got right

and my math was kinda right

d) Never did - lets try again

$f_{T_2}(t_2)$ $\cancel{t - t_2}$

Since independent $E[T_2]$ $\frac{1}{\cancel{\lambda}} = \cancel{\frac{1}{t_2}}$ no I don't think you can say that

$\lambda e^{\lambda x} \rightarrow \lambda e^{-\lambda t_2}$

X I always make the problem too simple

$$f_{T_2}(t) = \int_0^\infty f_{T_1}(\tau) f_X(t-\tau)\, d\tau$$

since $t_2 = T_1 + X$

independent so convolution

oh here is where I forgot about the time ~~the~~ that passed already

but cool how $T_2 = T_1 + (T_2 - T_1)$

but how get $f_X(t-\tau)$ as X

oh convolution

$f_X(\ )$ normal

Convolution adds t

try to solve from here

(29)

$$f_{T_2}(t) = \int_0^\infty \lambda e^{-\lambda\tau} \cdot \lambda e^{-\lambda(t-\tau)}\, d\tau$$

now exponents add, right

$$-\lambda\tau - \lambda(t-\tau)$$
$$-\lambda(\tau - t - \tau)$$
$$-\lambda(\cancel{0} - t)$$

$$\int_0^\infty \lambda e^{+\lambda t}\, d\tau$$

$$\lambda\tau e^{\lambda t}\Big|_0^\infty$$

$$\lambda\infty e^{\lambda t} - 0$$

$$\infty \quad \otimes$$

need to fill in of course

$$= \int_0^t 2(\lambda)^2 e^{-2\lambda\tau} e^{-\lambda(t-\tau)}\, d\tau$$

that was a disaster

also remember $f_{T_1}(t)$ was 1st problem

and any laptop can break so $2\lambda e^{-2\lambda t}$

So can't add exponents

$$= \underbrace{2\lambda e^{-\lambda t}}_{\text{constant take out}} \int_0^t \lambda e^{-\lambda\tau}\, d\tau$$

$$= 2\lambda e^{-\lambda t}\left(1 - e^{-\lambda t}\right)$$

(30)

Also linearity of expectations

$$E[T_2] = E[T_1] + E[X]$$
$$= \frac{1}{2\lambda} + \frac{1}{\lambda} = \frac{3}{2\lambda}$$

~~First~~ qu
2nd MC qu

Another way to solve

T_2 is $\max(M_1, M_2)$

derive distribution of T_2

$$F_{T_2}(t) = P(\max(M_1, M_2) < t)$$
$$= P(M_2 \le t)\, P(M_2 \le t)$$
$$= M_m(t)^2$$
$$= 1 - 2e^{-\lambda t} + e^{-2\lambda t}$$

differeiate w/ respect to t

$$f_{T_2}(t) = 2\lambda e^{-\lambda t} - 2\lambda e^{-2\lambda t}$$

31

e) Now have 100 Macbooks

Y is time of 1st failure

- ~~X~~ was like this one

One

- this is CDF

- no its some varient

$P(Y < .01)$

(I'm not focusing)

Oh duh like 1st problem

$f_Y(y) = 100\lambda e^{-100\lambda y} \quad t \geq 0$

integrate over (this is right dir)

$\int_0^{.1} 100\lambda e^{-100\lambda y}$

$= 1 - e^{-\lambda}$

So simple, why did I not recognize

Since not staying on topic

Charlie buys ~~n~~ S MBs every day

$P_S(S) = \begin{cases} 4 & 1/2 \\ 8 & 1/2 \end{cases}$

S_{100} = # baught over 100 days

? will this be Bayes's Rule?

since mix discrete + contineous

(32) Find best approx for $P(S_{100} \le 608)$ answer in $\Phi(\ )$

$$\Phi\left(\frac{X - E[\cancel{x}S]}{\sqrt{vars}}\right)$$

- this variables are wrong I think

$S = \#$ new MB

$S_{100} = \#$ of ~~M~~MB bought 100th day

$E[\cancel{x}S] =$

↑ what E do we want ? The sum, ~~but should be same ?~~ ?

no cause its sum

$E[X] = \frac{1}{2} \cdot 4 + \frac{1}{2} \cdot 8$

$\frac{4}{2} + \frac{8}{2}$

$2 + 4 = 6$

↑ obvious

$E[S] = 100 \cdot 6$

this is some expectation rule or something

this is sum of R # RV

but we know 100 days

? any way call iterated expectations

$Var(X) = E[X^2] - E[X]^2$

$\frac{1}{2} \cdot 16 + \frac{1}{2} \cdot 64 - 6^2$

$8 + 32 - 36$

$40 - 36$

4

(33)

$$\mathrm{var}(S) = \cancel{400}\, E[N] \cdot \mathrm{var}(x) + (E[x])^2\, \mathrm{var}(N)$$

$$100 \cdot 4 \quad + \quad - \quad 0$$

$$400$$

ϕ right way

$$\phi\left(\frac{608-600}{\sqrt{400}}\right)$$

$$\cancel{\phi(\ )} \quad \phi\left(\frac{8}{20}\right) = \phi\left(\frac{4}{10}\right) = \phi\left(\frac{2}{5}\right)$$ nice, is it right

$\otimes$

They just use "De Moivre Laplace Approx"

Step size = 4

$$\phi\left(\frac{608+2-100\cdot 6}{\sqrt{100\cdot 4}}\right) = \phi\left(\frac{10}{20}\right) = \phi(.5)$$

I had except the + 2!

why that? ½ step size

this is also in chap 5!

grr

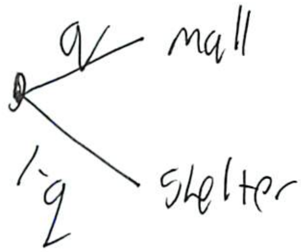
- well close
- but why +2?

(34)

also this test seemed easier than last one

3. After doing this for 5 hrs I'm so burnt out!

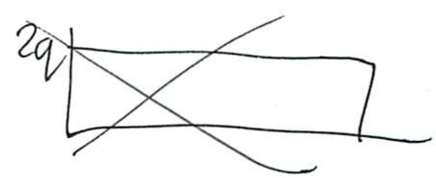
a) P goes to shelter

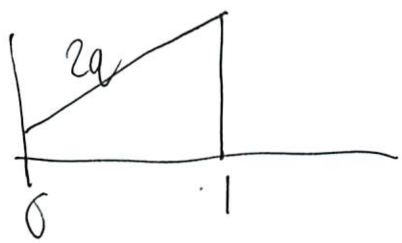


$1-q$

modeled by $2q$

So what does this mean





$P = 1 - f_Q(q)$

~~head~~ wait what is what

$f_Q(a) = P(\text{heads})$

So $1-$

Solved on next pg

just do 1st

(35)

b. he gets \$4 when tails

so ? this is law of expected ?

or fixed # RV

$$E[q] = \int_{-\infty}^{\infty} 2q \cdot q \, dq$$

$$2q^2 \, dq$$

$$\frac{2q^3}{3}\Big|_0^1$$

$$\frac{2}{3}$$ ← so P ~~head~~ tail $= \frac{1}{3}$ ✓

$Y =$ X_i = amt win each toss · 30 days

$$\begin{cases} 4 & 1/3 \\ 0 & 2/3 \end{cases}$$

$E[X_i] = 4 \cdot \frac{1}{3}$

$\frac{4}{3}$

$$Y = X_1 + X_2 + \dots + X_{30}$$

$$E[Y] = 30 \cdot \frac{4}{3}$$

$$\frac{120}{3} = 40$$

(X) See next pgs

$Var(X) = E[N] \, var(X_i)$ keep variables straight

(36)

$$Var(x_i) = E[x^2] - (E[x])^2$$

$$\frac{1}{3} \cdot 4^2 - \frac{4}{3}^2$$

$$\frac{16}{3} - \frac{16}{9}$$

$$\frac{32}{9}$$

Watch variables, I used different

c) B = goes 1 in k days

- need to review these patterns

$f_{Q|B}(q)$ - this is like the machine is broken problem

i forget
do rest 1st - since independent

d) At mall

Call from Mais

↑ Z min late uniform [0, 10]

pay R = exp(Z+2) derived dist

(37)

Urg is this the CDF + differentiate one!

- no convolution

$F_R(r) = P(\exp(z+2) \leq r)$

Idk skip this

answer to last 1st

note this is what expmeans

$E[R] = \int_0^{10} e^{z+2} f(z)\, dz$

$= \frac{e^2}{10} \int_0^{10} e^z\, dz$

$= \frac{e^{12} - e^2}{10}$

here is derived dist → $F_r(r) = P(e^{z+2} \leq r)$ had!

$= P(z+2 \leq \ln(r))$ need to know what to do next

$= \int_0^{\boxed{\ln(r)-2}} \frac{1}{10}\, dz$

then $f_X(x)$

Oh is it other part there, thats why move all to one side

?? isolate what have $f_X(x)$ for right?

(38)

$= \frac{\ln(r)-2}{10}$ $\quad e^2 \le r \le e^{12}$ exactly

differentiate

$f_R(r) = \frac{1}{10r}$ $\quad e^2 \le r \le e^{12}$

How was able to find E w/out doing whole part

? could have done after finding density

- no needed $f_Z(z)$ not $f_R(r)$
- but why ??

total expectation theorem

or before was

---

a. ✓ got it, but only via next problem

b) Var is wrong $\frac{1}{2}$

$Y_i = 1$ shelter
$Y_i = 0$ mall $\quad Q = q \quad P(Y_i = 1) = q \quad i = 1, 2, \ldots, 30$

$P(Y_i = 0) = 1 - q$ did not have this formal

$X = 4(Y_1 + Y_2 + \ldots + Y_{30})$

$\text{var}(X) = 16\,\text{var}(Y_1 + \ldots + Y_{30})$

$= 16\,\text{var}(E[Y_1 + \ldots + Y_{30}] \mid Q) + E[\text{var}(Y_1 + Y_2 + \ldots Y_{30} \mid Q)]$

39

Conditioned on Q, Y are ind.

$= var(Y_1|Q) + var(Y_2|Q) + \dots + var(Y_{30}|Q)$

? ~~what does~~ why conditioned on Q ?

$var(X) = 16\, var(30Q) + 16\, E[var(Y_1|Q) + \dots + var(Y_{30}|Q)]$

$= 16 \cdot 30^2 var(Q) + 16 \cdot 30\, E[Q(1-Q)]$

$= 16 \cdot 30^2 (E[Q^2] - E[Q]^2) + 16 \cdot 30 (E[Q] - E[Q^2])$

$= 16 \cdot 30^2 (\frac{1}{2} - \frac{4}{9}) + 16 \cdot 30 (\frac{2}{3} - \frac{1}{2})$

$= 880$

$E[Q] = \int_0^1 2q^2\, dq = \frac{2}{3}$

$E[Q^2] = \int_0^1 2q^3\, dq = 1/2$

? so oh ans is really 880

- I did not follow that

40

c) This was the one I skipped

$$f_{Q|B}(q) = \frac{P(B|Q=q)\, f_Q(q)}{\int P(B|Q=q)\, f_Q(q)\, dq}$$

Oh duh just from Bayes rule

fill in

$$= \frac{(1-q^k)\, 2q}{\int_0^1 (1-q^k)\, 2q\, dq}$$

$$= \frac{2q(1-q^k)}{1 - \frac{2}{k+2}} \qquad 0 \le q \le 1$$

don't forget about using these
Understand how

key thing $\quad P(B \mid \underbrace{Q=q}_{\text{says we know } q})$

$= (1-q^k)$

Ok so think about

q = don't go

q^k = go to mall k days in a row, right?

$1-q^k$ = 1 day don't go, right?

(4i) Look at machine example from tutorial
Or was that Bernoulli process?

~~ds~~ did not do a real Bayes Rule practice

No Tutorial 5 10/15 #1

Ok the machine was Bayes' rule

Observing days gives you an idea of q
So if see machine not working for 10 days
On 11th day likely to be not working
(even if each day independent)
Can infer prob of machine working or not
then once you know q - the past does not matter

a) A = machine working

$P(A) = ?$

$P(A \mid Q = q) = q$ ← is this a given??? no
or something we can define?

? Total prob theorm
(? how - only 1 item?)

$$P(A) = \int_0^1 P(A \mid Q = q) \cdot f_Q(q)\, dq$$

$$= \int_0^1 q \cdot 1\, dq = \left[\frac{q^2}{2}\right]_0^1 = \frac{1}{2}$$

(42)

What does that tell you?

How does that work?

And what is $f_Q(a)$?

$$= \frac{1}{b-a} = \frac{1}{1-0} = 1$$

---

b) B = machine is working m out of n days

What does this tell us about the PDF?

$$P_{Q|B}(q) = ?$$

- have B
- want to find q

$$= \frac{P(B \mid Q=q)\, f_Q(q)}{\int_{-\infty}^{\infty} P(B \mid Q=q)\, f_Q(q)\, dq}$$

✓ got off formula sheet

$$= \frac{\binom{n}{m} q^m (1-q)^{n-m} \cdot 1}{\int \binom{n}{m} q^m (1-q)^{n-m} \cdot 1 \, dq}$$

← oh so have to know that of course

$$= \frac{q^m (1-q)^{n-m}}{\frac{m!\,(n-m)!}{(n+1)!}}$$

← what is this? algebra

Can leave??

All for night - worked 8 hrs on this - breaks

6.041/6.431 Spring 2008 Quiz 2
Wednesday, April 16, 7:30 - 9:30 PM.

SOLUTIONS

Name: ____________________

Recitation Instructor: ____________________

TA: ____________________

6.041/6.431: ____________________

Question	Part	Score	Out of
0			3
1	all		36
2	a		4
	b		5
	c		5
	d		8
	e		5
	f		6
3	a		4
	b		6
	c		6
	d		6
	e		6
Total			100

- Write your solutions in this quiz packet, only solutions in the quiz packet will be graded.
- Question one, multiple choice questions, will receive no partial credit. Partial credit for question two and three will be awarded.
- You are allowed 2 two-sided 8.5 by 11 formula sheet plus a calculator.
- You have 120 minutes to complete the quiz.
- Be neat! You will not get credit if we can't read it.
- We will send out an email with more information on how to obtain your quiz before drop date.
- **Good Luck!**

Massachusetts Institute of Technology
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Spring 2008)

Problem 0: (3 pts) Write your name, your assigned recitation instructor's name, and assigned TA's name on the cover of the quiz booklet. The Instructor/TA pairing is listed below. Also write the class you are registered for: 6.041 or 6.431.

Recitation Instructor	TA	Recitation Time
Vivek Goyal	Natasa Blitvic	10 & 11 AM
Michael Collins	Danielle Hinton	10 & 11 AM
Shivani Agarwal	Stavros Valavanis	12 & 1 PM
Dimitri Bertsekas (6.431)	Aman Chawla (6.431)	1 & 2 PM

Question 1

Multiple choice questions. **CLEARLY** circle the best answer for each question below. Each question is worth 4 points each, with no partial credit given.

a. Let X_1, X_2, and X_3 be independent random variables with the continuous uniform distribution over $[0, 1]$. Then $\mathbf{P}(X_1 < X_2 < X_3) =$

   (i) $\boxed{1/6}$
   (ii) 1/3
   (iii) 1/2
   (iv) 1/4

   Solution: To understand the principle, first consider a simpler problem with X_1 and X_2 as given above. Note that $\mathbf{P}(X_1 < X_2) + \mathbf{P}(X_2 < X_1) + \mathbf{P}(X_1 = X_2) = 1$ since the corresponding events are disjoint and exaust all the possibilities. But $\mathbf{P}(X_1 < X_2) = \mathbf{P}(X_2 < X_1)$ by symmetry. Furthermore, $\mathbf{P}(X_1 = X_2) = 0$ since the random variables are continuous. Therefore, $\mathbf{P}(X_1 < X_2) = 1/2$.

   Analogously, omitting the events with zero probability but making sure to exhaust all other possibilities, we have that $\mathbf{P}(X_1 < X_2 < X_3) + \mathbf{P}(X_1 < X_3 < X_2) + \mathbf{P}(X_2 < X_1 < X_3) + \mathbf{P}(X_2 < X_3 < X_1) + \mathbf{P}(X_3 < X_1 < X_2) + \mathbf{P}(X_3 < X_2 < X_1) = 1$. And, by symmetry, $\mathbf{P}(X_1 < X_2 < X_3) = \mathbf{P}(X_1 < X_3 < X_2) = \mathbf{P}(X_2 < X_1 < X_3) = \mathbf{P}(X_2 < X_3 < X_1) = \mathbf{P}(X_3 < X_1 < X_2) = \mathbf{P}(X_3 < X_2 < X_1)$. Thus, $\mathbf{P}(X_1 < X_2 < X_3) = 1/6$.

b. Let X and Y be two continuous random variables. Then

   (i) $\mathbf{E}[XY] = \mathbf{E}[X]\mathbf{E}[Y]$
   (ii) $\boxed{\mathbf{E}[X^2 + Y^2] = \mathbf{E}[X^2] + \mathbf{E}[Y^2]}$
   (iii) $f_{X+Y}(x + y) = f_X(x)f_Y(y)$
   (iv) $\text{var}(X + Y) = \text{var}(X) + \text{var}(Y)$

   Solution: Since X^2 and Y^2 are random variables, the result follows by the linearity of expectation.

c. Suppose X is uniformly distributed over $[0, 4]$ and Y is uniformly distributed over $[0, 1]$. Assume X and Y are independent. Let $Z = X + Y$. Then

   (i) $f_Z(4.5) = 0$
   (ii) $\boxed{f_Z(4.5) = 1/8}$
   (iii) $f_Z(4.5) = 1/4$
   (iv) $f_Z(4.5) = 1/2$

   Solution: Since X and Y are independent, the result follows by convolution:

$$f_Z(4.5) = \int_{-\infty}^{\infty} f_X(\alpha) f_Y(4.5 - \alpha)\, d\alpha = \int_{3.5}^{4} \frac{1}{4}\, d\alpha = \frac{1}{8}\,.$$

d. For the random variables defined in part (c), $\mathbf{P}(\max(X, Y) > 3)$ is equal to

(i) 0

(ii) 9/4

(iii) 3/4

(iv) $\boxed{1/4}$

Solution: Note that $\mathbf{P}(\max(X, Y) > 3) = 1 - \mathbf{P}(\max(X, Y) \leq 3) = 1 - \mathbf{P}(\{X \leq 3\} \cap \{Y \leq 3\})$. But, X and Y are independent, so $\mathbf{P}(\{X \leq 3\} \cap \{Y \leq 3\}) = \mathbf{P}(X \leq 3)\mathbf{P}(Y \leq 3)$. Finally, computing the probabilities, we have $\mathbf{P}(X \leq 3) = 3/4$ and $\mathbf{P}(X \leq 3) = 1$. Thus, $\mathbf{P}(\max(X, Y) > 3) = 1 - 3/4 = 1/4$.

e. Recall the hat problem from lecture: N people put their hats in a closet at the start of a party, where each hat is uniquely identified. At the end of the party each person randomly selects a hat from the closet. Suppose N is a Poisson random variable with parameter λ. If X is the number of people who pick their own hats, then $\mathbf{E}[\mathrm{X}]$ is equal to

(i) λ

(ii) $1/\lambda^2$

(iii) $1/\lambda$

(iv) $\boxed{1}$

Solution: Let $X = X_1 + \ldots + X_N$ where each X_i is the indicator function such that $X_i = 1$ if the i^{th} person picks their own hat and $X_i = 0$ otherwise. By the linearity of the expectation, $\mathbf{E}[X \mid N = n] = \mathbf{E}[X_1 \mid N = n] + \ldots + \mathbf{E}[X_N \mid N = n]$. But $\mathbf{E}[X_i \mid N = n] = 1/n$ for all $i = 1, \ldots, n$. Thus, $\mathbf{E}[X \mid N = n] = n\mathbf{E}[X_i \mid N = n] = 1$. Finally, $\mathbf{E}[X] = \mathbf{E}[\mathbf{E}[X \mid N = n]] = 1$.

f. Suppose X and Y are Poisson random variables with parameters λ_1 and λ_2 respectively, where X and Y are independent. Define $W = X + Y$, then

(i) W is Poisson with parameter $\min(\lambda_1, \lambda_2)$

(ii) $\boxed{W \text{ is Poisson with parameter } \lambda_1 + \lambda_2}$

(iii) W may not be Poisson but has mean equal to $\min(\lambda_1, \lambda_2)$

(iv) W may not be Poisson but has mean equal to $\lambda_1 + \lambda_2$

Solution: The quickest way to obtain the answer is through transforms: $M_X(s) = e^{\lambda_1(e^s - 1)}$ and $M_Y(s) = e^{\lambda_2(e^s-1)}$. Since X and Y are independent, we have that $M_W(s) = e^{\lambda_1(e^s-1)}e^{\lambda_2(e^s-1)} = e^{(\lambda_1+\lambda_2)(e^s-1)}$, which equals the transform of a Poisson random variable with mean $\lambda_1 + \lambda_2$.

g. Let X be a random variable whose transform is given by $M_X(s) = (0.4 + 0.6e^s)^{50}$. Then

(i) $\mathbf{P}(X = 0) = \mathbf{P}(X = 50)$

(ii) $\mathbf{P}(X = 51) > 0$

(iii) $\boxed{\mathbf{P}(X = 0) = (0.4)^{50}}$

(iv) $\mathbf{P}(X = 50) = 0.6$

Solution: Note that $M_X(s)$ is the transform of a binomial random variable X with $n = 50$ trials and the probability of success $p = 0.6$. Thus, $\mathbf{P}(X = 0) = 0.4^{50}$.

h. Let X_i, $i = 1, 2, \ldots$ be independent random variables all distributed according to the PDF $f_X(x) = x/8$ for $0 \le x \le 4$. Let $S = \frac{1}{100}\sum_{i=1}^{100} X_i$. Then $\mathbf{P}(S > 3)$ is approximately equal to

(i) $1 - \Phi(5)$

(ii) $\Phi(5)$

(iii) $\boxed{1 - \Phi\left(\frac{5}{\sqrt{2}}\right)}$

(iv) $\Phi\left(\frac{5}{\sqrt{2}}\right)$

Solution: Let $S = \frac{1}{100}\sum_{i=1}^{100} Y_i$ where Y_i is the random variable given by $Y_i = X_1/100$. Since Y_i are *iid*, the distribution of S is approximately normal with mean $\mathbf{E}[S]$ and variance $\text{var}(S)$. Thus, $\mathbf{P}(S > 3) = 1 - \mathbf{P}(S \le 3) \approx 1 - \Phi\left(\frac{3 - \mathbf{E}(S)}{\sqrt{\text{var}(S)}}\right)$. Now,

$$\begin{aligned}
\mathbf{E}[X_i] &= \int_0^4 x\frac{x}{8}\,dx = \left.\frac{x^3}{24}\right|_0^4 = \frac{8}{3} \\
\text{var}(X_i) &= \mathbf{E}[X_i^2] - (\mathbf{E}[X_i])^2 = \int_0^4 x^2\frac{x}{8}\,dx - \left(\frac{8}{3}\right)^2 = \left.\frac{x^4}{32}\right|_0^4 - \left(\frac{8}{3}\right)^2 = \frac{8}{9}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathbf{E}[S] &= \frac{1}{100}\mathbf{E}[X_i] + \ldots + \frac{1}{100}\mathbf{E}[X_{100}] = 8/3. \\
\text{var}(S) &= \frac{1}{100^2}\text{var}(X_i) + \ldots + \frac{1}{100^2}\text{var}(X_i) = \frac{8}{9} \times \frac{1}{100}.
\end{aligned}$$

and

$$\mathbf{P}(S > 3) \approx 1 - \Phi\left(\frac{3 - 8/3}{\sqrt{\frac{8}{9} \times \frac{1}{100}}}\right) = 1 - \Phi\left(\frac{5}{\sqrt{2}}\right).$$

i. Let X_i, $i = 1, 2, \ldots$ be independent random variables all distributed according to the PDF $f_X(x) = 1$, $0 \le x \le 1$. Define $Y_n = X_1 X_2 X_3 \ldots X_n$, for some integer n. Then $\text{var}(Y_n)$ is equal to

(i) $\frac{n}{12}$

(ii) $\boxed{\frac{1}{3^n} - \frac{1}{4^n}}$

(iii) $\frac{1}{12^n}$

(iv) $\frac{1}{12}$

Solution: Since $X_1, \ldots, X_n$ are independent, we have that $\mathbf{E}[Y_n] = \mathbf{E}[X_1] \times \ldots \times \mathbf{E}[X_n]$. Similarly, $\mathbf{E}[Y_n^2] = \mathbf{E}[X_1^2] \times \ldots \times \mathbf{E}[X_n^2]$. Since $\mathbf{E}[X_i] = 1/2$ and $\mathbf{E}[X_i^2] = 1/3$ for $i = 1, \ldots, n$, it follows that $\text{var}(S_n) = \mathbf{E}[Y_n^2] - (\mathbf{E}[Y_n])^2 = \frac{1}{3^n} - \frac{1}{4^n}$.

Question 2

Each Mac book has a lifetime that is exponentially distributed with parameter λ. The lifetime of Mac books are independent of each other. Suppose you have two Mac books, which you begin using at the same time. Define T_1 as the time of the first ~~laptop~~ failure and T_2 as the time of the second ~~laptop~~ failure.

a. Compute $f_{T_1}(t_1)$.

Solution

Let M_1 be the life time of mac book 1 and M_2 the lifetime of mac book 2, where M_1 and M_2 are iid exponential random variables with CDF $F_M(m) = 1 - e^{-\lambda m}$. T_1, the time of the first mac book failure, is the minimum of M_1 and M_2. To derive the distribution of T_1, we first find the CDF $F_{T_1}(t)$, and then differentiate to find the PDF $f_{T_1}(t)$.

$$
\begin{aligned}
F_{T_1}(t) &= P(\min(M_1, M_2) < t) \\
&= 1 - P(\min(M_1, M_2) \ge t) \\
&= 1 - P(M_1 \ge t)P(M_2 \ge t) \\
&= 1 - (1 - F_M(t))^2 \\
&= 1 - e^{-2\lambda t} \qquad t \ge 0
\end{aligned}
$$

Differentiating $F_{T_1}(t)$ with respect to t yields:

$$f_{T_1}(t) = 2\lambda e^{-2\lambda t} \qquad t \ge 0$$

b. Let $X = T_2 - T_1$. Compute $f_{X|T_1}(x|t_1)$.

Solution

Conditioned on the time of the first mac book failure, the time until the other mac book fails is an exponential random variable by the memoryless property. The memoryless property tells us that regardless of the elapsed life time of the mac book, the time until failure has the same exponential CDF. Consequently,

$$f_{X|T_1}(x) = \lambda e^{-\lambda x} \qquad x \geq 0.$$

c. Is X independent of T_1? Give a mathematical justification for your answer.

Solution

Since we have shown in 2(c) that $f_{X|T_1}(x \mid t)$ does not depend on t, X and T_1 are independent.

d. Compute $f_{T_2}(t_2)$ and $\mathbf{E}[T_2]$.

Solution

The time of the second laptop failure T_2 is equal to $T_1 + X$. Since X and T_1 were shown to be independent in 2(b), we convolve the densities found in 2(a) and 2(b) to determine $f_{T_2}(t)$.

$$\begin{aligned} f_{T_2}(t) &= \int_0^\infty f_{T_1}(\tau) f_X(t-\tau) d\tau \\ &= \int_0^t 2(\lambda)^2 e^{-2\lambda\tau} e^{-\lambda(t-\tau)} d\tau \\ &= 2\lambda e^{-\lambda t} \int_0^t \lambda e^{-\lambda\tau} d\tau \\ &= 2\lambda e^{-\lambda t}(1 - e^{-\lambda t}) \quad t \geq 0 \end{aligned}$$

Also, by the linearity of expectation, we have that $\mathbf{E}[T_2] = \mathbf{E}[T_1] + \mathbf{E}[X] = \frac{1}{2\lambda} + \frac{1}{\lambda} = \frac{3}{2\lambda}$.

An equivalent method for solving this problem is to note that T_2 is the maximum of M_1 and M_2, and deriving the distribution of T_2 in our standard CDF to PDF method:

$$\begin{aligned} F_{T_2}(t) &= \mathbf{P}(\max(M_1, M_2) < t) \\ &= \mathbf{P}(M_2 \leq t) P(M_2 \leq t) \\ &= F_M(t)^2 \\ &= 1 - 2e^{-\lambda t} + e^{-2\lambda t} \quad t \geq 0 \end{aligned}$$

Differentiating $F_{T_2}(t)$ with respect to t yields:

$$f_{T_2}(t) = 2\lambda e^{-\lambda t} - 2\lambda e^{-2\lambda t} \quad t \geq 0$$

which is equivalent to our solution by convolution above.

Finally, from the above density we obtain that $\mathbf{E}[T_2] = \frac{2}{\lambda} - \frac{1}{2\lambda} = \frac{3}{2\lambda}$, which matches our earlier solution.

e. Now suppose you have 100 Mac books, and let Y be the time of the first laptop failure. Find the best answer for $\mathbf{P}(Y < 0.01)$

Solution

Y is equal to the minimum of 100 independent exponential random variables. Following the derivation in (a), we determine by analogy:

$$f_Y(y) = 100\lambda e^{-100\lambda y} \qquad t \geq 0$$

Integrating over y from 0 to .01, we find $\mathbf{P}(Y < .01) = 1 - e^{-\lambda}$.

Your friend, Charlie, loves Mac books so much he buys S new Mac books every day! On any given day S is equally likely to be 4 or 8, and all days are independent from each other. Let S_{100} be the number of Mac books Charlie buys over the next 100 days.

f. (6 pts) Find the best approximation for $\mathbf{P}(S_{100} \leq 608)$. Express your final answer in terms of $\Phi(\cdot)$, the CDF of the standard normal.

Solution

Using the De Moivre - Laplace Approximation to the Binomial, and noting that the step size between values that S can take on is 4,

$$\begin{aligned} P(S_{100} \leq 608) &= \Phi\left(\frac{608 + 2 - 100 \times 6}{\sqrt{100 \times 4}}\right) \\ &= \Phi\left(\frac{10}{20}\right) \\ &= \Phi\left(.5\right) \end{aligned}$$

Question 3

Saif is a well intentioned though slightly indecisive fellow. Every morning he flips a coin to decide where to go. If the coin is heads he drives to the mall, if it comes up tails he volunteers at the local shelter. Saif's coin is not necessarily fair, rather it possesses a probability of heads equal to q. We do not know q, but we do know it is well-modeled by a random variable Q where the density of Q is

$$f_Q(q) = \begin{cases} 2q & \text{for } 0 \leq q \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Assume conditioned on Q each coin flip is independent. Note parts a, b, c, and $\{d, e\}$ may be answered independent of each other.

a. (4 pts) What's the probability that Saif goes to the local shelter if he flips the coin once?

Solution

Let X_i be the outcome of a coin toss on the i^{th} trial, where $X_i = 1$ if the coin lands 'heads', and $X_i = 0$ if the coin lands 'tails.' By the total probability theorem:

$$\begin{aligned}
\mathbf{P}(X_1 = 0) &= \int_0^1 P(X_1 = 0 \mid Q = q) f(q) dq \\
&= \int_0^1 (1-q) 2q \, dq \\
&= \frac{1}{3}
\end{aligned}$$

In an attempt to promote virtuous behavior, Saif's father offers to pay him \$4 every day he volunteers at the local shelter. Define X as Saif's payout if he flips the coin every morning for the next 30 days.

b. Find $\text{var}(X)$

Solution Let Y_i be a Bernoulli random variable describing the outcome of a coin tossed on morning i. Then, $Y_i = 1$ corresponds to the event that on morning i, Saif goes to the local shelter; $Y_i = 0$ corresponds to the event that on morning i, Saif goes to the mall. Assuming that the coin lands heads with probability q, i.e. that $Q = q$, we have that $P(Y_i = 1) = q$, and $P(Y_i = 0) = 1 - q$ for $i = 1, \ldots, 30$.

Saif's payout for next 30 days is described by random variable $X = 4(Y_1 + Y_2 + \cdots + Y_{30})$.

$$\begin{aligned}
\text{var}(X) &= 16 \, \text{var}(Y_1 + Y_2 + \cdots + Y_{30}) \\
&= 16 \, \text{var}(\mathbf{E}[Y_1 + Y_2 + \cdots + Y_{30} \mid Q]) + \mathbf{E}[\text{var}(Y_1 + Y_2 + \cdots + Y_{30} \mid Q)]
\end{aligned}$$

Now note that, conditioned on $Q = q$, $Y_1, \ldots, Y_{30}$ are independent. Thus, $\text{var}(Y_1 + Y_2 + \cdots + Y_{30} \mid Q) = \text{var}(Y_1 \mid Q) + \ldots + \text{var}(Y_{30} \mid Q)$. So,

$$\begin{aligned}
\text{var}(X) &= 16 \, \text{var}(30Q) + 16 \, \mathbf{E}[\text{var}(Y_1 \mid Q) + \ldots + \text{var}(Y_{30} \mid Q)] \\
&= 16 \times 30^2 \text{var}(Q) + 16 \times 30 \, \mathbf{E}[Q(1-Q)] \\
&= 16 \times 30^2 (\mathbf{E}[Q^2] - (\mathbf{E}[Q])^2) + 16 \times 30 (\mathbf{E}[Q] - \mathbf{E}[Q^2]) \\
&= 16 \times 30^2 (1/2 - 4/9) + 16 \times 30 (2/3 - 1/2) = 880
\end{aligned}$$

since $\mathbf{E}[Q] = \int_0^1 2q^2 \, dq = 2/3$ and $\mathbf{E}[Q^2] = \int_0^1 2q^3 \, dq = 1/2$.

Let event B be that Saif goes to the local shelter at least once in k days.

c. Find the conditional density of Q given B, $f_{Q|B}(q)$

Solution
By Bayes Rule:

$$\begin{aligned} f_{Q|B}(q) &= \frac{P(B \mid Q = q) f_Q(q)}{\int P(B \mid Q = q) f_Q(q) dq} \\ &= \frac{(1-q^k)2q}{\int_0^1 (1-q^k)2q\, dq} \\ &= \frac{2q(1-q^k)}{1-2/(k+2)} \qquad 0 \le q \le 1 \end{aligned}$$

While shopping at the mall, Saif gets a call from his sister Mais. They agree to meet at the Coco Cabana Court yard at exactly 1:30PM. Unfortunately Mais arrives Z minutes late, where Z is a continuous uniform random variable from zero to 10 minutes. Saif is furious that Mais has kept him waiting, and demands Mais pay him R dollars, where $R = \exp(Z + 2)$.

e. Find Saif's expected payout, $\mathbf{E}[R]$.

Solution

$$\begin{aligned} \mathbf{E}[R] &= \int_0^{10} e^{z+2} f(z) dz \\ &= \frac{e^2}{10} \int_0^{10} e^z dz \\ &= \frac{e^{12} - e^2}{10} \end{aligned}$$

f. Find the density of Saif's payout, $f_R(r)$.

Solution

$$\begin{aligned} F_R(r) &= \mathbf{P}(e^{Z+2} \le r) \\ &= \mathbf{P}(Z + 2 \le \ln(r)) \\ &= \int_0^{\ln(r)-2} \frac{1}{10} dz \\ &= \frac{\ln(r) - 2}{10} \qquad e^2 \le r \le e^{12} \end{aligned}$$

$$f_R(r) = \frac{1}{10r} \qquad e^2 \le r \le e^{12}$$

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Fall 2010)

Recitation 16
(6.041/6.431 Spring 2007 Quiz 2)
November 2, 2010

Problem 1: Xavier and Wasima are participating in the 6.041 MIT marathon, where race times are defined by random variables[1]. Let X and W denote the race time of Xavier and Wasima respectively. All race times are in hours. Assume the race times for Xavier and Wasima are independent (i.e. X and W are independent). Xavier's race time, X, is defined by the following density

$$f_X(x) = \begin{cases} 2c, & \text{if } 2 \le x < 3, \\ c, & \text{if } 3 \le x \le 4, \\ 0, & \text{otherwise,} \end{cases}$$

where c is an unknown constant. Wasima's race time, W, is uniformly distributed between 2 and 4 hours. The density of W is then

$$f_W(w) = \begin{cases} \frac{1}{2}, & \text{if } 2 \le w \le 4, \\ 0, & \text{otherwise.} \end{cases}$$

(a) (i) Find the constant c

(ii) Compute $\mathbf{E}[X]$

(iii) Compute $\mathbf{E}[X^2]$

(iv) Provide a fully labeled sketch of the PDF of $2X + 1$

(b) Compute $\mathbf{P}(X \le W)$.

(c) Wasima is using a stopwatch to time herself. However, the stopwatch is faulty; it over-estimates her race time by an amount that is uniformly distributed between 0 and $\frac{1}{10}$ hours, which is independent of the actual race time. Thus, if T is the time measured by the stopwatch, then we have

$$f_{T|W}(t|w) = \begin{cases} 10, & \text{if } w \le t \le w + \frac{1}{10} \text{ and } 2 \le w \le 4, \\ 0, & \text{otherwise.} \end{cases}$$

Find $f_{W|T}(w|t)$, when $t = 3$.

(d) Wasima realizes her stopwatch is faulty and buys a new stopwatch. Unfortunately, the new stopwatch is also faulty; this time, the watch adds random noise N that is normally distributed with mean $\mu = \frac{1}{60}$ hours and variance $\sigma^2 = \frac{4}{3600}$. Find the probability that the watch over-estimates the actual race time by more than 5 minutes, $\mathbf{P}(N > \frac{5}{60})$. For full credit express your final answer as a number.

(e) Wasima has a sponsor for the marathon! If Wasima finishes the marathon in w hours, the sponsor pays her $\frac{24}{w}$ thousand dollars. Define

$$S = \frac{24}{W}$$

Find the PDF of S.

[1] A runner's race time is defined as the time required for a given runner to complete the marathon.

Problem 2. Consider the following family of **independent** random variables $N, A_1, B_1, A_2, B_2, \ldots$, where N is a nonnegative discrete random variable and each A_i or B_i is normal with mean 1 and variance 1. Let $A = \sum_{i=1}^{N} A_i$ and $B = \sum_{i=1}^{N} B_i$. Recall that the sum of a fixed number of independent normal random variables is normal.

(a) Assume N is geometrically distributed with a mean of $1/p$.

(i) Find the mean, μ_a, and the variance, σ_a^2, of A.

(ii) Find c_{ab}, defined by $c_{ab} = \mathbf{E}[AB]$.

(b) Now assume that N can take only the values 1 (with probability 1/3) and 2 (with probability 2/3).

(i) Give a formula for the PDF of A.

(ii) Find the conditional probability $\mathbf{P}(N = 1 \mid A = a)$.

(c) Is it true that $\mathbf{E}[A \mid N] = \mathbf{E}[A \mid B, N]$? Either provide a proof, or an explanation why the equality does not hold.

Recitation 16

Quiz 2 Review

"not quiz review" → "review of a quiz" → Spring 07

Contineous RV PDF, Mean, Var
Uniform, Normal, exp
CDF
Joint PDFs, Conditioning
Bayes's Rule

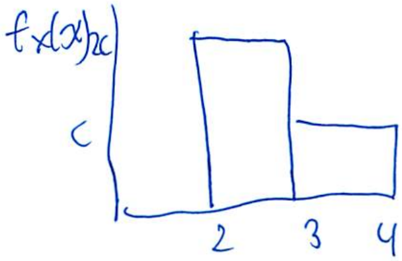
Derived Distribution
Covariance/Correlation
Iterated Expectations/Random Sums

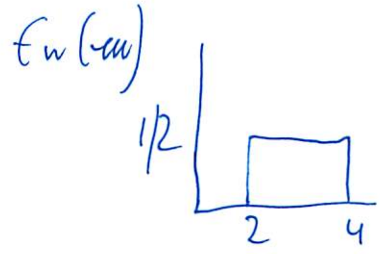
---

1. X, W run race

X = Xavier's race time
W = Wasima's race time

$$f_X(x) = \begin{cases} 2c & 2 \le x < 3 \\ c & 3 \le x \le 4 \\ 0 & \text{else} \end{cases} \qquad f_W(w) = \begin{cases} 1/2 & 2 \le w \le 4 \\ 0 & \text{otherwise} \end{cases}$$





(2)

a) Find c

area should integrate to 1 $\rightarrow 2c + c = 1$

$c = \frac{1}{3}$

b) $E[X] = \int_2^4 x f_X(x)\,dx$

contineous! not $2\cdot\frac{2}{3} + 3\cdot\frac{1}{3}$!!!

$= \int_2^3 \frac{2}{3} x\,dx + \int_3^4 \frac{1}{3} x\,dx = \frac{17}{6}$

prof: lots of qv
Must work fast

c) $E[X^2] = \int_2^3 \frac{2}{3} x^2\,dx + \int_3^4 \frac{1}{3} x^2\,dx = \frac{25}{3}$

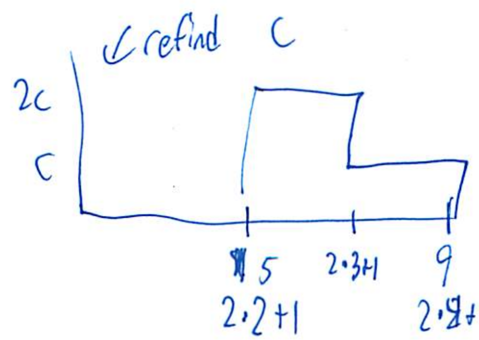
d) $Y = 2X + 1$

$f_Y(y) = ?$

- deried dist + plot
- formula

$= \frac{1}{2} f_X\left(\frac{y-1}{2}\right)$

could also do it graphically!



3

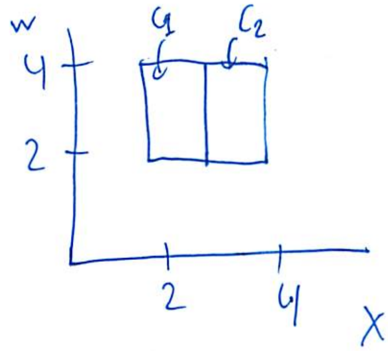
Now both

e) Compute $P(X \le W)$

bring in joint PDF

key the 2 are ind.

So $f_{X,W}(x,w) = f_X(x) \cdot f_W(w)$



$C_1 = 2C_2$

area under joint CDF should integrate to 1

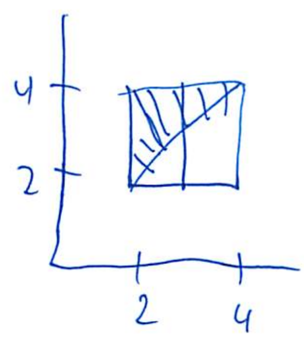
$2c_1 + 2 \cdot c_2 = 1$

area (4-2)(3-2)

solve

So $c_1 = \frac{1}{3}$ $c_2 = \frac{1}{6}$

Now need the event $X \le W$



Find area $\cdot$ PDF

$$= \int_2^3 \int_x^4 \underbrace{\frac{1}{3}}_{\text{PDF}} \, dw\, dx + \int_3^4 \int_x^4 \frac{1}{6} \, dw\, dx$$

$$= \frac{7}{12}$$

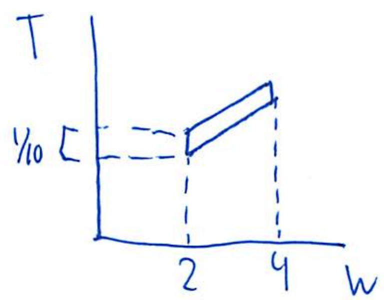
(4)

f) Stopwatch time — over estimate

T = stopwatch time

$f_{T|W}(t|w) \sim$ uniform b/w $[w, w+\frac{1}{10}]$, ind of w

can write joint given the marginal



so T, W not ind

slice + normalize

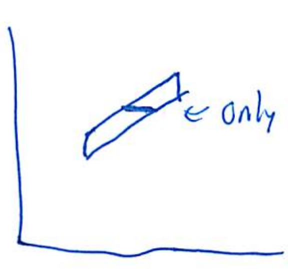
prob uniformly distributed over the strip

What was inference of running time?

Bayes's rule problom

$f_{W|T}(w|3) =$

$T=3$



← only that interval

slice and normalize

$=$ Uniform within $[3-\frac{1}{10}, 3]$

= same answer

Could also do w/ Bayes's rule $\frac{f_{W,T}(w,3)}{f_T(3)} = \frac{f_{T|W}(3|w) \cdot f_W(w)}{f_T(3)}$

(5) # is off

9a) She gets new stopwatch
Guassian distributed

$$\text{Error} \sim N\left(\text{mean} = \frac{1}{60}, \text{var} = \frac{4}{3600}\right)$$

$$P\left(N > \frac{5}{60}\right) = ?$$

this qu just has to do w/ N

Standrize + use table

$$= 1 - P\left(N \leq \frac{5}{60}\right)$$

$$= 1 - P\left(\frac{N - \frac{1}{60}}{2/60} \leq \underbrace{\frac{5/60 - 1/60}{2/60}}_{2}\right)$$

$$= 1 - \Phi(2)$$

table

$$= 1 - .9772$$

$$= .0028$$

(6) ✓ Derived dist CDF

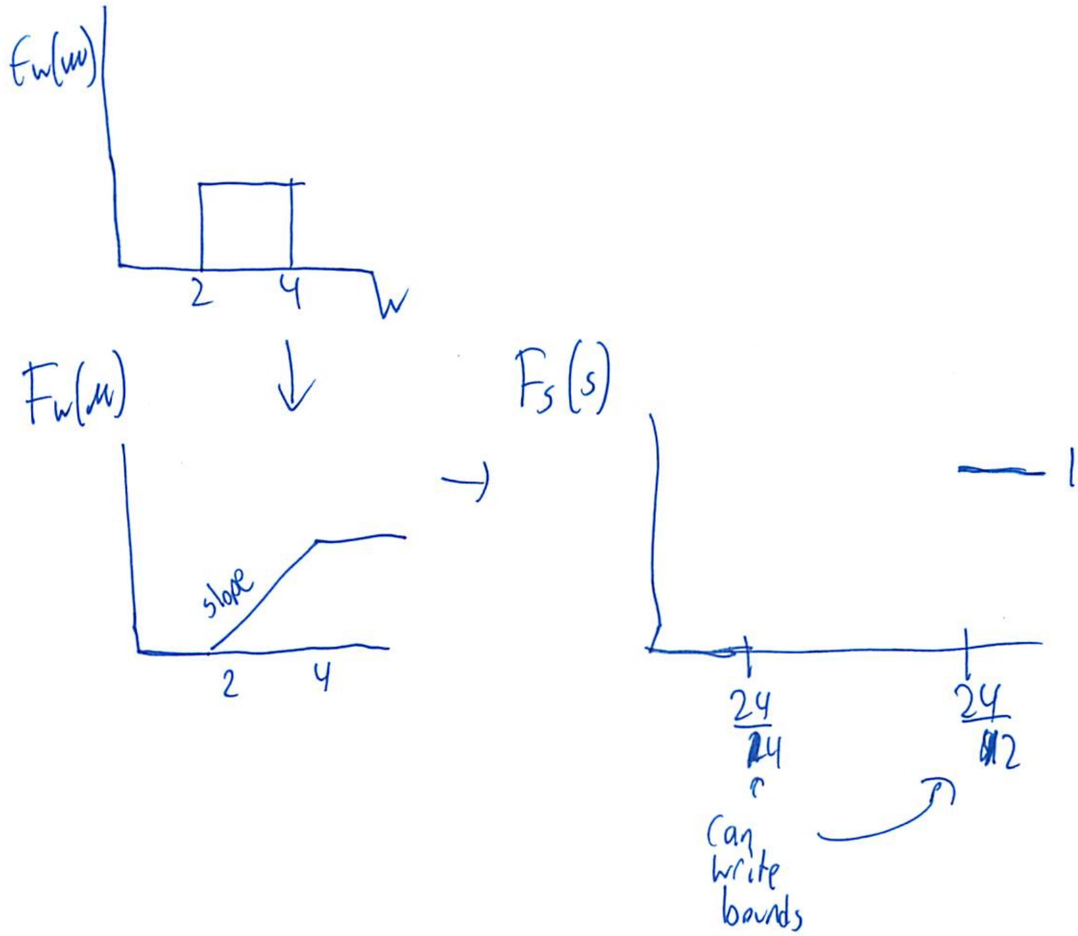
h) Sponsor - pays $S = \frac{24}{W}$

Find PDF of S

Could use monotonic ~~...~~ trick

But full out

(this exam is fairly easy I think - I just need to memorize + execute now)



$$F_S(s) = P(S \leq s) = P\left(\frac{24}{W} \leq s\right)$$

$$= 1 - P\left(W \leq \frac{24}{s}\right)$$

have CDF of W - oh what we have already!

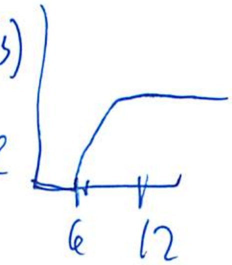
$$= 1 - F_W\left(\frac{24}{s}\right)$$

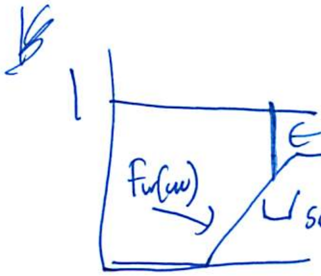
7

$$= 1 - \underbrace{\left(\frac{24}{5} - 2\right)}_{\cancel{\text{Equation of line}}} \cdot \underbrace{\frac{1}{2}}_{\cancel{\text{Slope}}}$$

$$= 2 - \frac{12}{5}$$ within [6, 12)

need to calculate



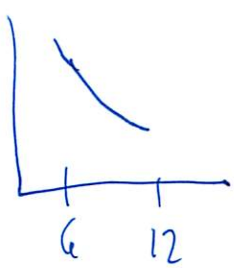


1 - Fw(w)

so want the height of this line!

dh so this is whatever the problem calls for

differentiate



$$f_s(s) = \frac{d}{ds} F_s(s) = \frac{12}{s^2}$$ within [6,12]

(8)

Iterated expectation can make your life a lot eaiser

2, RVs $N, A_1, B_1, A_2, B_2, \ldots\ldots$

all independent

$A_i \sim \text{normal}(1,1)$

$A = A_1 + A_2 + \ldots + A_N$

a) Assume N is geometric

So $E[N] = \frac{1}{p}$ $\quad var(N) = \frac{1-p}{p^2}$

$E[A]$ = Sum or random # of RVs
iterated expectations ....

$= E[A_i]\, E[N] =$

$= 1 \cdot \frac{1}{p}$

$= \frac{1}{p}$

$var(A) = E[N] \cdot var(A_i) + (E[A_i])^2\, var(N)$

$= \frac{1}{p} \cdot 1 + 1 \quad \frac{1-p}{p^2}$

$= \cancel{} = \frac{1}{p^2}$

(9)

ii) Find $E[AB]$ ← still don't know

$\stackrel{?}{=} E[A]\,E[B]$ <u>No</u> → not independent
Coupled through N

But for fixed <u>N</u>, can use iterated expectations

$= E[E[AB|N]]$ ← I mentioned indicator RVs - totally wrong

$= E[E[A|N] \cdot E[B|N]]$

$= E[N \cdot \underbrace{E[A_i]}_{1} \cdot N \cdot \underbrace{E[B_i]}_{1}]$

$= E[N \cdot 1 \cdot N \cdot 1]$

$= E[N^2]$

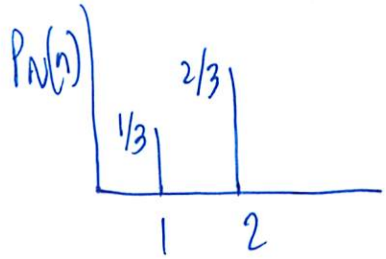
$= (E[N])^2 + var(N)$ ← 2nd moment

$= \frac{1}{p^2} + \frac{1-p}{p^2}$

$= \frac{2-p}{p^2}$

(10)

b) Now N no longer geometric



Find $f_A(a) =$

?A is sum

find PDF

write guassian on sheet

total prob theorm

$$= \frac{1}{3} f_{A|N}(a|1) + \frac{2}{3} f_{A|N}(a|2)$$

$$= \frac{1}{3} \cdot N(1,1) + \frac{2}{3} N(2,2)$$

$$= \frac{1}{3} \cdot \frac{1}{\sqrt{2\pi}} e^{-(a-1)^2/2} + \frac{2}{3} \frac{1}{\sqrt{}} e^{-(a-2)^2/4}$$

?4π

ii) Inference question Find $P(N=1 | A=a) =$

n can be 1 or 2

what will be more likly

Bayer's rule

$$= \frac{p_N(1) \cdot f_{A|N}(a|1)}{f_A(a)}$$

$$= \frac{\frac{1}{3} \cdot N(1,1)}{\text{previous ans}} = \frac{1/3 \, \frac{1}{2\pi} e^{-(a-1)^2/2}}{\text{previous ans}}$$

11 Confused prof at 1st

c) $E[A|N] = E[A|B,N]$?

- is true
- but why
- what is av even?

- are they independent
  jumping to ans

function of N
RV

expression
function of B,N

could be
e^N $\frac{N+5}{n^2}$

could be
e^{BN} $\frac{NB}{1+B}$

B does not come into picture
<u>if</u> they are =

So they are independent of B
does not tell you anything

is what independence is

$E[A] \neq E[A|B]$

#

RV

so can't be =

More Review

$f_Y(y) = ?$

$$Y = 2X + 1$$

$$Y - 1 = 2x$$

$$\frac{Y-1}{2} = x$$

$$cf\left(\frac{Y-1}{2}\right)$$

need to refind

2·2+1 2·3+1 2·4+1 just shifts

5 7 9

$$\int_5^7 2c$$

but how get that ?

$$\int_5^7 c f_x\left(\frac{Y-1}{2}\right) dx + \int_7^9 c f_x\left(\frac{Y-1}{2}\right) dx$$

what is it ?

Oh he did not find c here

$- \cancel{c} = \frac{1}{2}$

- but how would have found that ???
- he used linear formula $\frac{1}{|a|} f_X\left(\frac{x-b}{a}\right)$

② So no finding c there

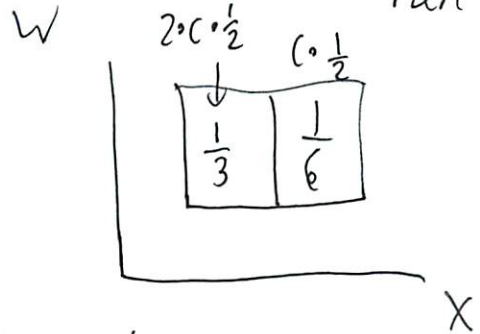
just $\frac{1}{|a|} = \frac{1}{2}$

---

e Now when do joint

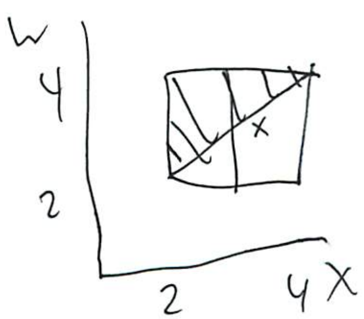
First Need $f_{X,Y}(x,y) = f_X(x) f_Y(y)$

~~X~~c he always drew it
(which I was good at 1st
then focused on algebra)



oh! I never thought of it so simple!

Now just want P $X \le W$



integrate over area (w/ density)
lable lies 1st

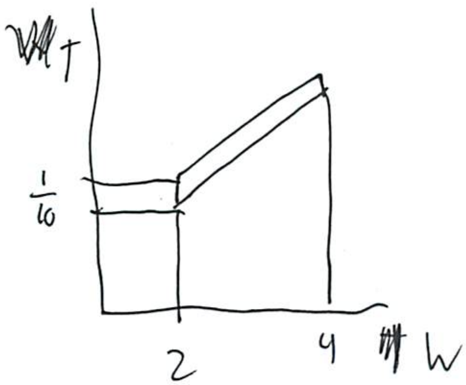
+ w/ variable inside

$$\int_2^3 \int_x^4 \frac{1}{3} \, d\cancel{y}w \, dx + \int_3^4 \int_x^4 \frac{1}{6} \, dw \, dx$$

w is our variable

③ Then make sure to actually do!

c) I think I need to think clearer and more explicitly intuitively about the variables - which I used to be good at.



← just remember this

But always think how to graph something like this!

What does $f_T(t) = \begin{cases} 10 \\ 0 \end{cases}$ each dih interval is $1 \to \frac{1}{10}$ so this normalizes to 1 evenly distributed

Find $f_{W|T}(w|t)$ - Bayes's Rule
$t=3$

Bayes rule
- both RV

$$f_{W|T}(w|t=3) = \frac{f_{T|W}(t|w)\, f_W(w)}{f_T(t)}$$

(4)

$$\frac{10 \cdot \frac{1}{2}}{?}$$

how get just $f_T(t)$?

- what is given is marginal ?
- ? is it

$$f_{T|W} = \frac{f(t \cap w)}{f(w)}$$ does that apply to RV?

- or say: $$\frac{1}{b-a} = \frac{1}{\frac{1}{10}} = 10 \ ?$$

hmm they (in recitation) did not really give an answer

said "a uniform b/w $[w, w+\frac{1}{10}]$, ind of w

can write joint given marginal"

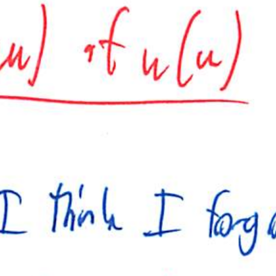
Oh never mind see below

$$= \frac{f_{T|W}(3|w) \cdot f_W(w)}{f_T(3)}$$

^ I think I forgot to fill in

= Uniform within $[3-\frac{1}{10}, 3]$

don't forget graph



slice + normalize

(5)

So if uniform

$$f_{W|X}(w \mid x=3) = \frac{1}{b-a} = \frac{1}{3-2.9} = \frac{1}{.1} = 10$$

so bottom ? must have been a 2

-? how get that ?

I wish review would have been more rigorous

---

9) Normally distributed stopwatch

their # ? my # ?

$$\mu = \frac{1}{60} \qquad \sigma^2 = \frac{4}{3600}$$

Ok - so need to remember this

Oh right off formula sheet

↓ but see recognized it in a minute

$$1 - \Phi\left(\frac{x - 1/60}{4/3600}\right)$$

$$1 - \Phi\left(\frac{5/60 - 1/60}{4/3600}\right) \quad ✓$$

reduce

1 - ~ .9767 - its just off the table

that would have freaked me out on exam

- thought I was wrong

6

e) Pays $\frac{24}{W} = S$

Find PDF of S

Ok this is the derived dist. qv

So we have

↓

could look up on chart $\left(\frac{x-a}{b-a}\right)$

See if can do

$P(X \leq x)$

$\int_2^{x} f_x(x)\,dx$ leave as x!

replace good ✓

$\int_2^{x} \frac{1}{4-2}\,dx$

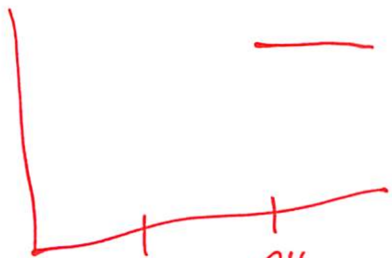
$\frac{x}{2}\Big|_2^{x}$

$\frac{x}{2} - \frac{2}{2}$

$\frac{x-2}{2}$ ✓ matches

7

Now need $F_S(s)$



$\frac{24}{4}$ $\frac{24}{2}$

→ know bounds

(notice 4 on left, its actually smaller)

$$F_S(s) = P(S \leq s) = P\left(\frac{24}{W} \leq s\right)$$

how did we get to 1 − ?

$$= 1 - P\left(W \leq \frac{24}{s}\right)$$

→ have CDF of W

how to covert

$$= 1 - F_W\left(\frac{24}{s}\right)$$

$\frac{24}{w} \leq s$

$24 \leq sw$

$\frac{24}{s} \leq w$

Oh I see

now flip

$1 - W \leq \frac{24}{s}$

→ w needs to be less than

(fairly sure this is right, I'm working backwards)

→ So have

$$= 1 - \frac{x-2}{2}$$

$$= 1 - \frac{24/s - 2}{2}$$

Oh same as in class

I like his graphical explination better

Then differenitate

8 Now back

N = RV

A_i, B_i normal

A, B the sums

a) N = geometric

A mean of a bunch of Normal

So $E[N] = \frac{1}{p}$

Oh iterated expectations!

So find $E[N]$ and $E[A_i]$ seperatly

Then multiply together since ind.

- like sum of Random # RV

get this

but iterated expectations not really - review

looked over

- usually given $E[X|Y] = nY$

$E[Y] = \frac{1}{2}$

Use it to find $E[X]$

by $E[E[X|Y]] = E[nY] = n E[Y] = n\frac{1}{2} = \frac{n}{2}$

9

then fill the rest in

- should be fairly easy
- just each component

---

ii. Find $E[AB]$

↳ if ind $= E[A]E[B]$

here is iterated expectation

$= E[E[AB|N]]$

why must fix N

: thats how problem is possible

$= E[E[A|N]\, E[B|N]]$

oh now they are ind can split!

~~[illegible]~~

$= E[N \cdot E[A_i] \cdot N E[B_i]]$

: why can you bring N out?

etc
to problem

skipping estimation errors

What is this "2nd moment" thing?

— transform stuff

⑩ Next ~~tota~~

i) total prob theorm ✓

ii) and then Bayes rule + plug + chug
should have # unlike last time

~~M~~.. 30 min left review yesterday's notes
viewing this a lot bettor than last time