10/20

LECTURE 13

The Bernoulli process

2nd half of class
—most classes end here
but next 6 lectures
probablistic models that
evolve over time

- **Readings:** Section 6.1

Lecture outline

- Definition of Bernoulli process —low sequence of inp. coin tosses
  discrete today
- Random processes
- Basic properties of Bernoulli process
- Distribution of interarrival times
- The time of the kth success
- Merging and splitting

↳family of 17th century mathematicians

The Bernoulli process

- A sequence of independent Bernoulli trials small experiments

- At each trial, i:

  0<p<1

  2 outcomes "binary"

  - $\mathbf{P}(\text{success}) = \mathbf{P}(X_i = 1) = p$

    ↳Heads

  - $\mathbf{P}(\text{failure}) = \mathbf{P}(X_i = 0) = 1 - p$

    ↳Tails

    ↑ each trial same as other independent

- Examples:
  - Sequence of lottery wins/losses
  - Sequence of ups and downs of the Dow Jones

    Some people make a little [illegible] $ to prove Dow is not Bernulli – find a better model/pattern

  - Arrivals (each second) to a bank

    or different time interval

    property not quite right – load depends on time

  - Arrivals (at each time slot) to server need $P_{X_i}(x_i)$ ← PMF of ith toss

Random processes

- First view:
  sequence of random variables $X_1, X_2, \ldots$

- $\mathbf{E}[X_t] =$ p

- $\text{Var}(X_t) =$ $p(1-p)$

not enough to describle property of process

– how are they related together? need joint PMF $P_{X_2 X_3 X_5 \dots X_{12}}$

– but told independent and told marginal PMF

- Second view:
  what is the right sample space?

Ω 11111111

- $\mathbf{P}(X_t = 1 \text{ for all } t) =$

$\approx$ subset of $P(X_1 = \dots X_k = 1)$ $\le$ $= p_k$ $\forall k$

0 ↑ must be less than p^k $\le$

↳ and as $k \to \infty$ this goes to 0

- Random processes we will study:
  - Bernoulli process
    (memoryless, discrete time)
  - Poisson process
    (memoryless, continuous time)
  - Markov chains
    (with memory/dependence across time)

as you go to ∞ each outcome becomes harder + harder

like uniform

↑ each point has $p=0$ but section has some prob.

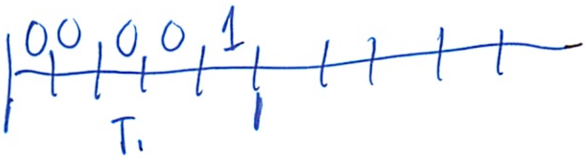
Number of successes S in $\underline{n}$ time slots

binomial

- $\mathbf{P}(S = k) = \binom{n}{k} p^k (1-p)^{n-k}$

- $\mathbf{E}[S] = np$

- $\mathsf{Var}(S) = np(1-p)$



Interarrival times

- T_1: number of trials until first success

geometric

then until 2nd sucess

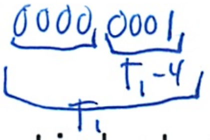
  - $\mathbf{P}(T_1 = t) = (1-p)^{t-1} p$ $\quad t = 1, 2, \ldots$
  - Memoryless property

$P(T_1 - 4 \mid T_1 > 4)$ $= t$ So $= P(T_1 = t)$

first 4 trials failures don't know about after $T-4$ remaing time till sucess

- its as if ~~pro~~ trials Starting just now

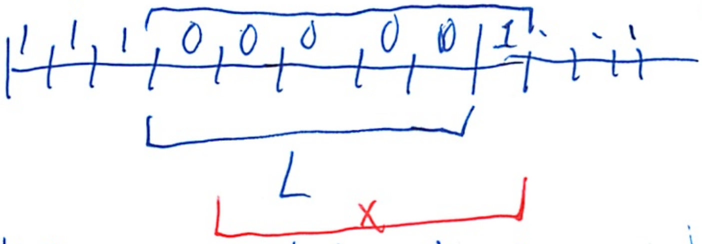
  - $\mathbf{E}[T_1] = 1/p$
  - $\mathrm{Var}(T_1) = \frac{1-p}{p^2}$



- If you buy a lottery ticket every day, what is the distribution of the length of the first string of losing days?

$= L$

memoryless can get a bit more confusing



Had ones - past does not matter

So just time until first sucess - 1

What is distribution of top bar

- must have at least a 0 ~~[illegible]~~
- must end w/ 1
- takes values 2, 3 ... - so not geometric
- Special info that 1 slot is 0
  - not ind. Bernouli trias

"call brother" after 1st 0

red line is geometric(p)

X is L shifted 1 unit in time

- gets harder in contineas time

Something new

Time of the kth arrival

independence + memoryless in play

$0\,0\,0\,0\,1\,\underbrace{0\,0\,0\,0\,0\,0\,1}$

- Given that first arrival was at time t i.e., $T_1 = t$: additional time, T_2, until next arrival

  T_2 - will also be geometric w/ p

  t_1 and t_2 are independent

  - has the same (geometric) distribution
  - independent of T_1

- Y_k: number of trials to kth success

  $= T_1 + T_2 + \ldots + T_k$

  each T_k geometric independent

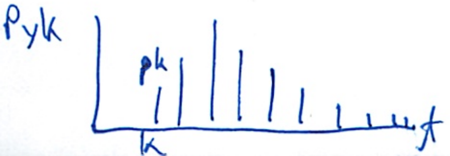
  - $\mathbf{E}[Y_k] =$ $k \cdot \frac{1}{p}$
  - $\text{Var}(Y_k) =$ $k \cdot \frac{1-p}{p^2}$
  - $\mathbf{P}(Y_k = t) =$

distribution something new harder

Convolutions several times too complicate look for shortcut

$$\underbrace{10101\,010\,10\,111}_{T_1}\,\underbrace{00\,000\,001}_{T_2}$$

Fix k - like if want 3rd arrival



$P(Y_k = t)$

$= P(X_1 + \ldots + X_{t-1} = k-1$

and $X_t = 1)$

$= P(X_1 + \ldots + X_{t-1} = k-1) \cdot P(X_t = 1$

$= p\binom{t-1}{k-1} p^{k-1} (1-p)^{(t-1)-(k-1)}$

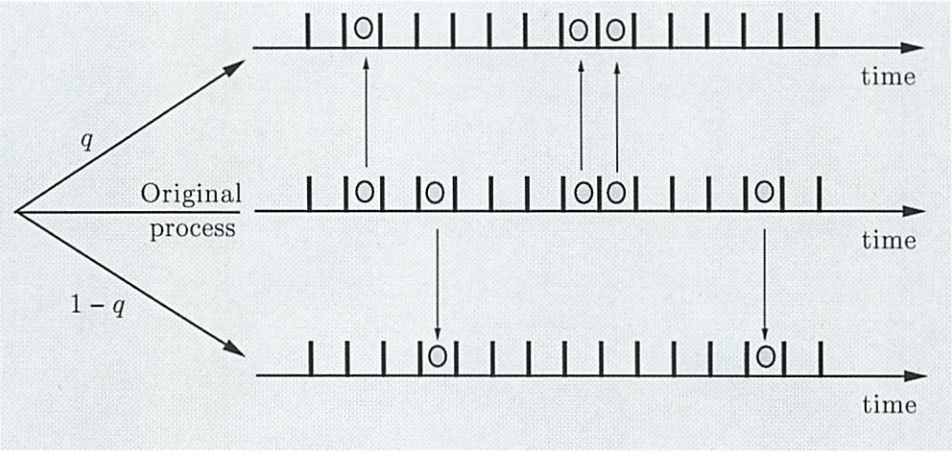
pascal pMF

$= \binom{t-1}{k-1} p^k (1-p)^{t-k}$

Splitting of a Bernoulli Process

(using independent coin flips)

jobs arrive in random process then you flip a coin to decide which server to send to



) pq

) $p(1-q)$

yields Bernoulli processes

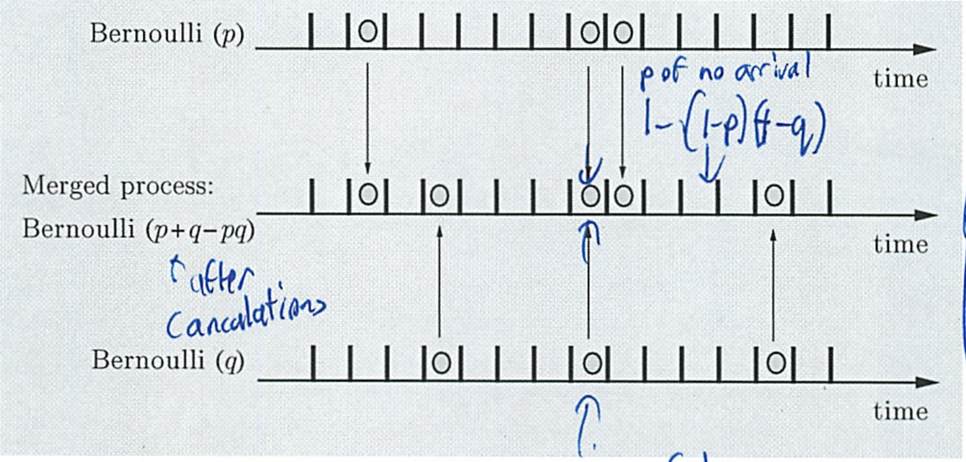
slots must be inp of each other
- depends on bit in orginal process and decided to send it their

Merging of Indep. Bernoulli Processes

each pair is independent
- pairwise ind.

is the top ind. of bottom
- no, knowing arrival on top means no arrival on bottom

dependent



p of no arrival $1-(1-p)(1-q)$

after cancalations

imperfect

yields a Bernoulli process
(collisions are counted as one arrival)

Then argue [illegible] top + bottom are ind.

does lose some resolution of info
- on continuous not possible

Poisson approximation to binomial

- Number of arrivals in n slots is binomial

$$p_S(k) = \frac{n!}{(n-k)!k!} \cdot p^k(1-p)^{n-k}, \qquad \text{for } k \geq 0$$

- Interesting to think of:
  $n \rightarrow \infty$ with $\lambda = np$ constant

$$\begin{aligned} p_S(k) &= \frac{n!}{(n-k)!k!} \cdot p^k(1-p)^{n-k} \\ &= \frac{n(n-1)\cdots(n-k+1)}{k!} \cdot \frac{\lambda^k}{n^k} \cdot \left(1 - \frac{\lambda}{n}\right)^{n-k} \\ &= \frac{n}{n} \cdot \frac{n-1}{n} \cdot \frac{n-k+1}{n} \cdot \frac{\lambda^k}{k!} \cdot \left(1 - \frac{\lambda}{n}\right)^{n-k} \end{aligned}$$

- For any fixed $k \geq 0$,
  $\lim_{n\rightarrow\infty} (1 - \lambda/n)^{n-k} = e^{-\lambda}$, so:

$$\lim_{n\rightarrow\infty} p_S(k) = \frac{\lambda^k}{k!} e^{-\lambda}, \qquad k = 1, 2, \ldots$$

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Fall 2010)

Recitation 14
October 26, 2010

1. You are visiting the rainforest, but unfortunately your insect repellent has run out. As a result, at each second, a mosquito lands on your neck with probability 0.5. If one lands, with probability 0.2 it bites you, and with probability 0.8 it never bothers you, independently of other mosquitoes.
   (a) What is the expected time between successive mosquito bites? What is the variance of the time between successive mosquito bites?
   (b) In addition, a tick lands on your neck with probability 0.1. If one lands, with probability 0.7 it bites you, and with probability 0.3, it never bothers you, independently of other ticks and mosquitoes. Now, what is expected time between successive bug bites? What is the variance of the time between successive bug bites?

2. Al performs an experiment comprising a series of independent trials. On each trial, he simultaneously flips a set of three fair coins.
   (a) Given that Al has just had a trial with 3 *tails,* what is the probability that both of the next two trials will also have this result?
   (b) Whenever all three coins land on the same side in any given trial, Al calls the trial a success.
      i. Find the PMF for K, the number of trials up to, but *not* including, the second success.
      ii. Find the expectation and variance of M, the number of tails that occur *before* the first success.
   (c) Bob conducts an experiment like Al's, except that he uses 4 coins for the first trial, and then he obeys the following rule: Whenever all of the coins land on the same side in a trial, Bob permanently removes one coin from the experiment and continues with the trials. He follows this rule until the *third* time he removes a coin, at which point the experiment ceases. Find $E[N]$, where N is the number of trials in Bob's experiment.

3. Suppose there are n papers in a drawer. You draw a paper and sign it, and then, instead of filing it away, you place the paper back into the drawer. If any paper is equally likely to be drawn each time, independent of all other draws, what is the expected number of papers that you will draw before signing all n papers? You may leave your answer in the form of a summation.

Recitation 14

10/26

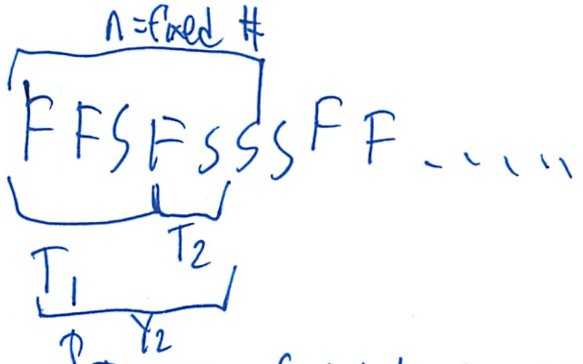
Bernoulli process

- Series of RV
- look at simplest one today
- Sequence of coin flipping
- each flip is independent

$\{X_1, X_2, X_3, \ldots\}$

X_i = Bernoulli, IID

$$P(X_i) = \begin{cases} p & \text{if } X_i = 1 \text{ (Sucess)} \\ 1-p & \text{if } X_i = 0 \text{ (failure)} \end{cases}$$



T_1 = # of trials to 1st sucess

- geometric(p)

$E[T] = \frac{1}{p}$ $var(T) = \frac{1-p}{p^2}$

S = # sucesses in n trials   Binomial(n, p)
↑fixed

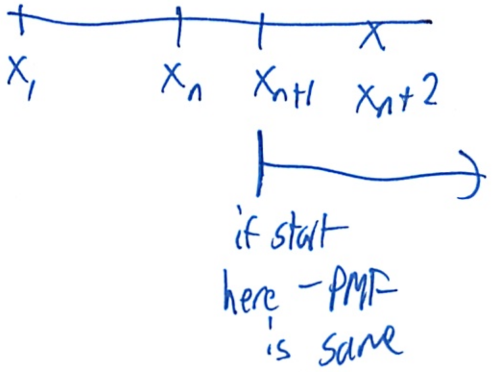
$p_S(k) = \binom{n}{k} p^k (1-p)^{n-k}$

$Y_k = T_1 + \dots + T_k$ # trials to k th sccess

Pascal of order k

$p_{Y_k}(t) = \binom{t-1}{k-1} p^k (1-p)^{t-k}$, for $t \geq k$

Fresh start



Values are independent of the past

Applies to all Bernouli processes

- But not all processes
- Applies also to Poisson process (contineous)

Example

every second a mosqeto falls from tree

lands on you $P = \frac{1}{2}$

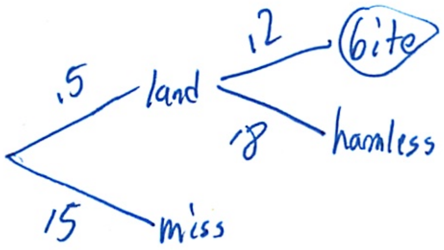
lands away $P = \frac{1}{2}$

③

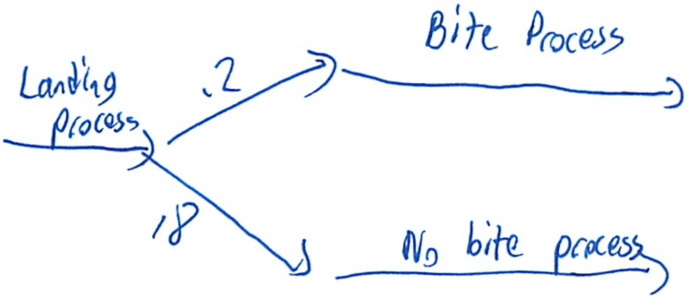
Ones that land bite w/ $p = .2$

Q: What is expected time b/w bites

Variance

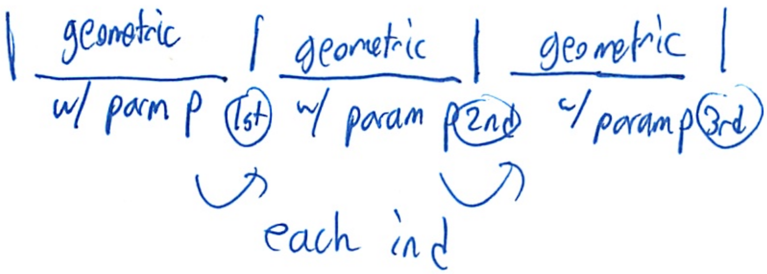


But is a process



each time instant independent

Split of bernoulli process = still bernoulli process



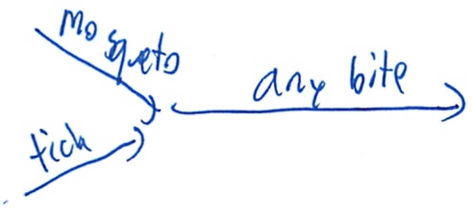
$$E[\text{bite}] = \frac{1}{p} = \frac{1}{.1} = 10$$

$$Var(\text{bite}) = \frac{1-p}{p^2} = \frac{.9}{(.1)^2} = 90$$

④

More complex

-add ticks

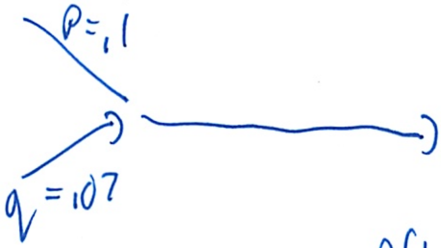


merged bernoulli process
still bernoulli proces

<u>tick</u>

$q = .1 \cdot .7 = .07$

So combined



Bernoulli w/ prob $p+q-pq = .163$

~~Disregard both bites~~
& disregard ~~b~~ both biting

$P(\text{both bite}) = 1-(1-p)(1-q)$ ↑algebra

$\boxed{P(\text{no bite}) = (1-p)\cdot(1-q)}$

$E[\text{time b/w bites}] = \dfrac{1}{p+q-pq} = 6.135$

⑤

Equivilant description

- don't get outcomes of every outcome
- but want amt of time b/w sucesses

start w/ $\{T_1, T_2, T_3 \dots\}$

T_i = geometric IID w/ prob p

= time b/w sucesses

Get corresponding x process

* $\{X_1, X_2 \dots\}$

- is Bernouli process w/ prob p

---

ex: time b/w ~~da~~ rainy days is geometric

T_i = # days b/w 2 rainy days

all identically distributed geometric

$P($Day 15 is raining, Day 321 is rain, Day 3,000,000 is rain$) =$???

$= p^3$ ← identically distributed and independent

equivilant bernouli process rain/no rain on each day

$= P(X_{15}=1, X_{321}=1, X_{3,000,000}=1)$

⑥

All independent so $p = p^3$

Can pass from one model to another

---

example Drawer w/ n documents

Pull doc at random, sign, replace

Repeat if signed just put it back

~~then~~ Stop when all signed

How long till done?

T_1 = time to sign 1st $\quad T_1 = 1$ will always get blank

T_2 - time to sign 2nd $\quad T_2$ = geometric w/ param $\frac{n-1}{n}$

T_i = time to sign ith $\quad T_i$ = geometric $\frac{n-(i-1)}{n}$

T_n = time to sign last $\quad T_n = \frac{1}{n}$

$$E[\text{time to sign all}] = E[T_1] + E[T_2] + E[T_{\dots}] \dots + E[T_n]$$

$$= \sum_{i=1}^{n} \frac{n}{n-(i-1)} = n \sum_{k=1}^{n} \frac{1}{k}$$

harmonic series up to time n

$\approx \log(n)$

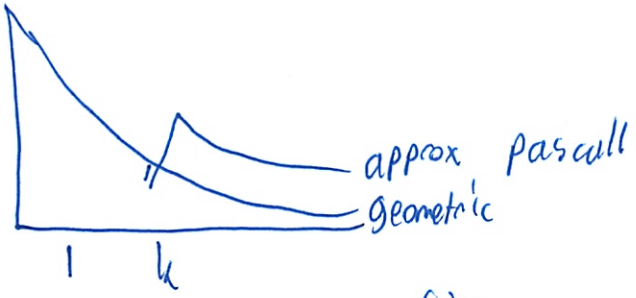
as time gets large

$\int_1^n \frac{1}{x} dx = \log n$ for n large

asymtotic calculation if some values get really big

Pascall

#~~Heads~~ trials to kth sucess

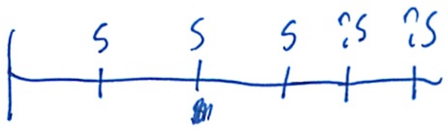


Suppose flip 3 fair coins at once

Sucess if all 3 are tails

$P(\text{sucess}) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8} = \left(\frac{1}{2}\right)^3 = p$

Q: Following a sucess, what is p next 2 trials are sucesses



Fresh start property - just like starting from scratch
(w/ condition 1st trial is sucess & but unconditional since fresh start)

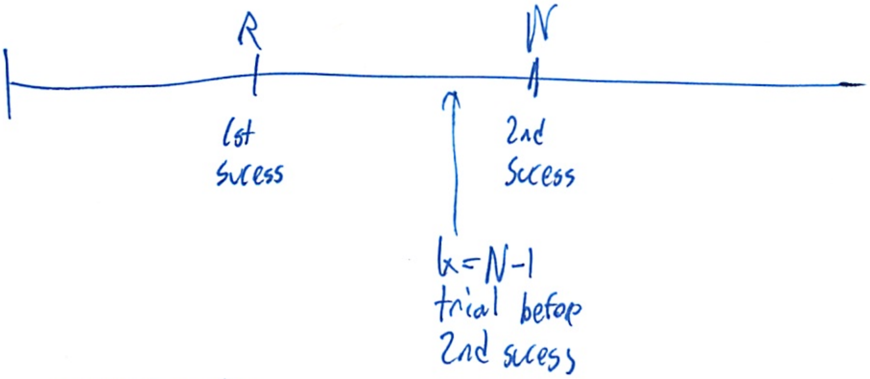
$= \frac{1}{8} \cdot \frac{1}{8} = \left(\frac{1}{8}\right)^2$

^ each trial independant (memoryless)

2. Now sucess if 3H or 3T

$P(\text{sucess}) = \frac{1}{8} + \frac{1}{8} = \frac{2}{8} = \frac{1}{4}$

^ disjoined



$PMF_k(k)$:

N = pascal of order 2

^ # of trials to 2nd sucess

Same as pascal - except shifted back by 1

$P_N(n) = \binom{\cancel{n}\, t-1}{1} \left(\frac{1}{4}\right)^2 \left(\frac{3}{4}\right)^{t-2}$ for $t \geq 2$

$P_k(k) = \binom{t}{1} \left(\frac{1}{4}\right)^2 \left(\frac{3}{4}\right)^{t-2}$ for $t \geq 1$

shift by 1

⑨

$U = R - 1$

time ~~be~~ just before 1 sucess

all failures up to that 2H, 1T

↳ $p(\text{failure}) = 3/4$ remember

$X_1 =$ # of Tails in that failure either 1 or 2

$p = 1/2$ for each

$$X_1 = \begin{cases} 1 & \text{w/ } p = 1/2 \\ 2 & \text{w/ } p = 1/2 \end{cases}$$

$M =$ Find sum of tails up to 1st sucess (but not including)

$= X_1 + X_2 + \dots + X_U$

$E[M] = ?$

$Var(M) = ?$

of X_s are fixed

- Sum of random # of RV
- last week!

$$E[M] = E[X] \cdot E[U]$$

$$\underset{\overset{\uparrow}{1 \cdot .5 + 2 \cdot .5}}{\frac{3}{2}} \cdot \underset{\overset{\uparrow}{\text{geometric}}}{E[R] - 1}$$

$\frac{1}{\frac{1}{4}} - 1$

$4 - 1$

3

$= \frac{3}{2} \cdot 3 = 4.5$

⑩

$$Var(M) = \underbrace{E[U]}_{3} \cdot \underbrace{var(x)}_{\text{Same as p Bernouli shifts don't cout}} + \underbrace{(E[X])}_{(\frac{3}{2})^2}{}^2 \, var(U)$$

$\frac{1}{2}(1-\frac{1}{2})$

$\frac{1}{2}$

U is $R-1$

$var(R) \leftarrow$ shifts don't matter

var of geometric w/ $\frac{1}{4}$

$$\frac{1-\frac{1}{4}}{(\frac{1}{4})^2}$$

$$= 3 \cdot \frac{1}{2} + \left(\frac{3}{2}\right)^2 \cdot \left(\frac{1-\frac{1}{4}}{(\frac{1}{4})^2}\right)$$

$$= \frac{111}{4}$$

Suppose 4 coins

when 4 heads, throw away a coin

repeat till all coins are gone

E[geometric w/ 4 coins] + E[geometric w/ 3 coins] + E[g w/ 2] + E[g w/ 1]

IID = independent identically distributed

Chap 6 Bernulli + Poisson Reading

10/26

Stochastic proces

- math model of probablistic experiment that evolves in time

ex

1. daily price of a stock
2. Seq. of scores in football
3. seq. of failure ~~A~~ times of a machine
4. the seq of hourly traffic loads at a node
5. seq of radar measurements

Can be finite or infinite

still RVs from a probability law

a) focus on dependencies of seq of values generated by process

b) interested in long term averages

c) want to know likelihood of boundry events

- ie when will all circuits on phone system be busy

We are interested in 2 categories ~~AA~~ of stochastic proces

arrival type processes

- message reciept
- look at interarrival time

indi: RV Bernuli discrete

~~markov~~ Poisson contineous

markov pocesses

- evolve over time
- next depends on past through current value
- chap 7

② 6.1 Bernoulli process

- seq of ind. coin tosses
  ↑ well think of like
- p sucess/1 = p

$$0 < p < 1$$

failure $= 1-p$

- could model arrival of customers
  - at each second $\begin{cases} 1 & \text{1 arrives} \\ 0 & \text{no arrival} \end{cases}$
- each ~~second~~ time interval is independent

$$P(X_i = 1) = P(\text{sucess ith trial}) = p$$

$$P(X_i = 2) = P(\text{failure ith trial}) = 1-p$$

- Remember

binomial

$$p_S(k) = \binom{n}{k} p^k (1-p)^{n-k} \qquad k = 0, 1, \ldots, n$$

$$E[S] = np \qquad Var(S) = np(1-p)$$

geometric

$$p_T(t) = (1-p)^{t-1} p \qquad t = 1, 2, \ldots$$

$$E[T] = \frac{1}{p} \qquad Var(T) = \frac{1-p}{p^2}$$

- Memoryless
  - one outcome does not effect other

③ run for n time steps

$X_1, X_2, \ldots X_n$

Fresh start

- Since past does not matter

$$P(T-n=t \mid T>n) = (1-p)^{t-1}p = P(T=t) \quad t=1,2,\ldots$$

- aka memoryless
- if started watching at time n, intinguisable from process that had just started

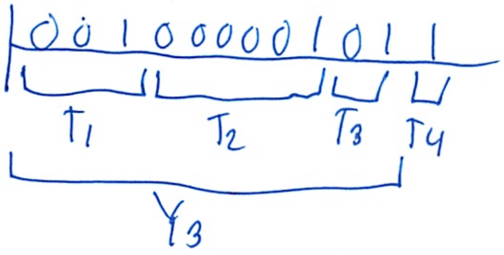
---

Intercarrival times

Y_k = time of the kth sucess

$$T_1 = Y_1 \quad T_k = Y_k - Y_{k-1} \quad k=2,3,\ldots$$

$$Y_k = T_1 + T_2 + \ldots + T_k$$



T_i = geometric RV w/ parameter p

remember each trial ind.

T_1, T_2, T_i ind. and have same geometric RV

I spent a lot of time, feel like understand

④ kth arrival time

time Y_k of kth sucess equal to sum of

(so same as last time)

$Y_k = T_1 + T_2 + \ldots + T_{\cancel{m}k}$ independent identically distributed (iid) geometric RV

$$E[Y_k] = E[T_1] + \ldots + E[T_k] = \frac{k}{p}$$

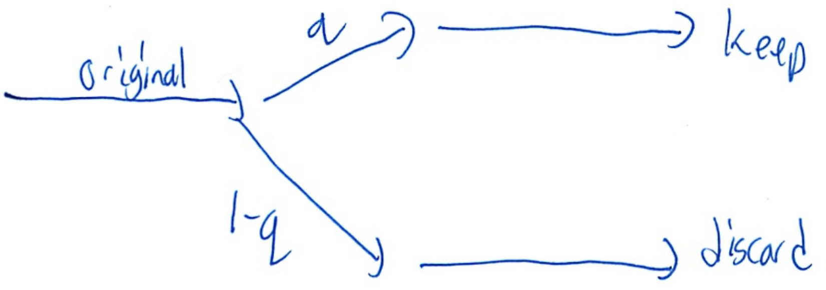
$$Var(Y_k) = Var(T_1) + \ldots + var(T_k) = \frac{k(1-p)}{p^2}$$

$$p_{Y_k}(t) = \binom{t-1}{k-1} p^k (1-p)^{t-k} \qquad t = k, k+1, \ldots$$

↑ know as Pascal PMF of order k

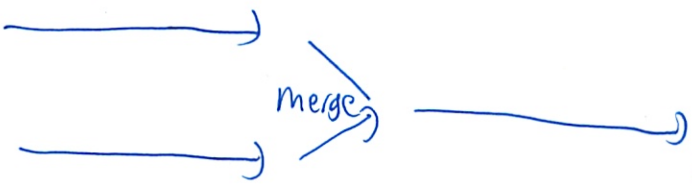
skipping verification

Splitting + Merging of Bernoulli Processes



$p = pq$

$p = p(1-q)$



no arrival = $(1-p)(1-q)$

arrival $= 1-(1-p)(1-q)$

$= p+q-pq$

⑤ Poisson Approx to Binomial

\# sucesses in n ind. Bernouli trials = binomial (n, p)

$E[\nearrow] = np$

when n is very large, but p is small, mean has moderate value

like when in contineous time

like # airplane crashes n high

p small

let n grow while decreasing p so $np = \lambda$

↑constant

this simplifies to poisson PMF

$$p_Z(k) = e^{-\lambda} \frac{\lambda^k}{k!} \qquad k = 0, 1, 2 \ldots$$

$$E[Z] = \lambda \qquad var(Z) = \lambda$$

for nonnegitive k

$$p_S(k) = \frac{n!}{(n-k)!\,k!} p^k (1-p)^{n-k}$$

converges to $p_Z(k)$ when take lim as $n \to \infty$

and $p = \frac{\lambda}{n}$ while keeping λ constant

(skipping verify)

10/27

LECTURE 14

Bernolli but contineous

The Poisson process

- **Readings:** Start Section 6.2.

Lecture outline

- Review of Bernoulli process
- Definition of Poisson process
- Distribution of number of arrivals
- Distribution of interarrival times
- Other properties of the Poisson process

Bernoulli review

- Discrete time; success probability p Sucess or failure
  arrival or no arrival

Subsets of

- Number of arrivals in n time slots: binomial pmf distribution
- Interarrival times: geometric pmf
- Time to k arrivals: Pascal pmf
- Memorylessness
  no psycic powers

Sub-sequent arrivals

$Y_3 = T_1 + T_2 + T_3$

each geometric ind.

convolve geometrics

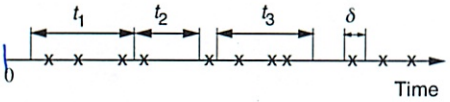
Shortcut: pascal dist

$P(Y_k = t) = \binom{t-1}{k-1} p^k (1-p)^{t-k} \quad k = t, t+1, \ldots$

time is contineous
-no arbitrary choice of lenght of slots

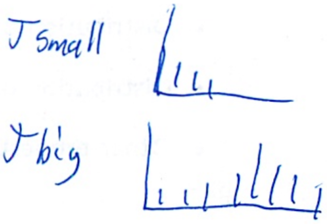
can't really talk about sucess or failure
just arrivals

Definition of the Poisson process



- **Time homogeneity:**
  $P(k,\tau)$ = Prob. of k arrivals in interval of duration τ - if intervals have same lenght - stats of the intervals are the same

τ = parameter of distribution
-Fix it and get a PMF
-for each a valid PMF
$$\sum_k P(k,\tau)=1$$



- Numbers of arrivals in disjoint time intervals are **independent** what happens in one does not affect other

- **Small interval probabilities:**
  For VERY small δ:

$$P(k,\delta)\approx\begin{cases}1-\lambda\delta, & \text{if } k=0;\\ \lambda\delta, & \text{if } k=1; \\ 0, & \text{if } k>1.\end{cases} + O(\delta^2)$$

enough to know # of arrivals in big interval

ignore 2nd order

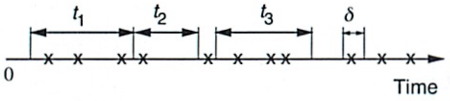
– λ: "arrival rate"

Expected # of arrivals per unit time

very small intervals -
Prob that no arrivals
if interval half size, then half expected # of arrivals

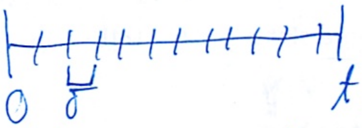
but this does not tell us arrival possibilites for big interval

PMF of Number of Arrivals N



$\tau = t$ (same thing)

Fix λ, t Want $P(k,t)$



Contineous analog of Bernoulli process

- Finely discretize $[0,t]$: approximately Bernoulli

$n = \frac{t}{\delta}$ slots

- N_t (of discrete approximation): binomial

$p = \lambda\delta + O(\delta^{-2}) = \frac{\lambda t}{n}$
↑prob of arrival in each slot

- Taking $\delta \to 0$ (or $n \to \infty$) gives:

$$P(k,\tau)=\frac{(\lambda\tau)^k e^{-\lambda\tau}}{k!}, \qquad k=0,1,\dots$$

$$P(k,t) = \binom{n}{k} p^k (1-p)^{n-k}$$
$$= \binom{n}{k}\left(\frac{\lambda t}{n}\right)^k\left(1-\frac{\lambda t}{n}\right)^{n-k}$$
$$= \frac{n(n-1)\dots(n-k+1)}{k!} \left(1-\frac{\lambda t}{n}\right)^{n-k}$$
$$= \frac{(\lambda t)^k e^{-\lambda t}}{k!}$$

$\delta \to 0$
$n \to \infty$
$np = \lambda t$ ← Expected value
↑# slots increase
prob of arrival in each slot ↓

- $E[N_t] = \lambda t$, $\text{var}(N_t) = \lambda t$

↑parameters, constants here

↑ $= np(1-p)$
but as $\delta \to 0$
then $p \to 0$
$np \to \lambda t$
so $= \lambda t$

note
$(1-\frac{1}{n})^n \to e^{-1}$

Example

- You get email according to a Poisson process at a rate of $\lambda = 5$ messages per hour. You check your email every thirty minutes.

$$\lambda t = 5 \cdot \overset{\text{hour}}{\tfrac{1}{2}} = 2.5$$

- Prob(no new messages) =

$$\frac{(\lambda t)^0}{0!} e^{-\lambda t} = e^{-2.5} = .08$$

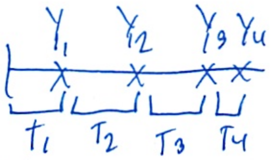
- Prob(one new message) =

$$\frac{(\lambda t)^1}{1!} e^{-\lambda t} = 2.5 e^{-2.5} = .20$$

* # arrivals over a given time distribution

Interarrival Times

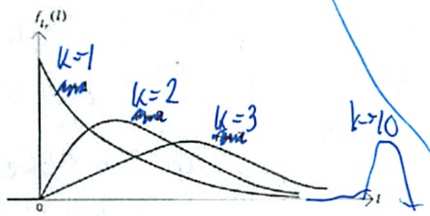
- Y_k time of kth arrival



$$Y_k = T_1 + \cdots + T_k$$

- **Erlang** distribution:

$$f_{Y_k}(y) = \frac{\lambda^k y^{k-1} e^{-\lambda y}}{(k-1)!}, \qquad y \geq 0$$

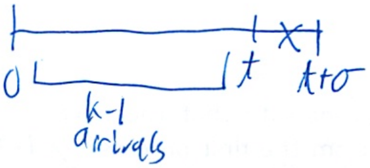


density is prob. of a tiny interval

$$f_{Y_k}(t)\delta \approx P(t \leq Y_k \leq t+\delta)$$

prob kth arrival in our little interval

very small intervals see 0 or 1 arrivals



disjoint so independent

$$= P(k-1, t)\lambda\delta$$

change dummy variable to y

- Time of first arrival ($k = 1$):
exponential: $f_{Y_1}(y) = \lambda e^{-\lambda y}, \quad y \geq 0$
  - Memoryless property: The time to the next arrival is independent of the past

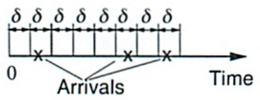
same

when k=1 its just exponential

$$f_{Y_1}(y) = f_{T_1}(y) = \lambda e^{-\lambda y} \quad y \geq 0$$

(discrete would be geometric)

Bernoulli/Poisson Relation



$n = t/\delta$

$p = \lambda\delta$

$np = \lambda t$

(nice)

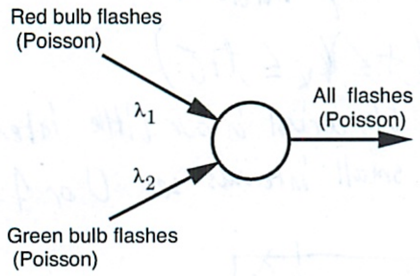
	POISSON	BERNOULLI
Times of Arrival	Continuous	Discrete
Arrival Rate	λ/unit time	p/per trial
PMF of # of Arrivals	Poisson	Binomial
Interarrival Time Distr.	Exponential	Geometric
Time to k-th arrival	Erlang	Pascal

at intuitive level they work the same way

Poission - very many mini slots each w/ a very small prob

Adding Poisson Processes

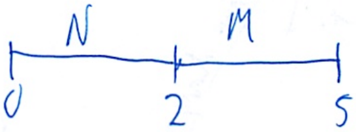
- Sum of independent Poisson **random variables** is Poisson
- Merging of independent Poisson **processes** is Poisson



$N =$ Poisson $E[N] = 2$

$M =$ " $E[M] = 3$

$\lambda = 1$



$N+M =$ Poisson $E[N+M] = 5$

b/c independent, 2 intervals are disjoint

- nicer that what happens in Bernoulli

* since "no" possibility of overlap *

$\lambda_1 \cdot \lambda_2 \cdot \delta^2$ ← too small to matter

- What is the probability that the next arrival comes from the first process?

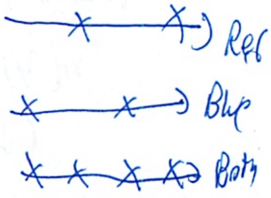
equally likely, if same

if diff - see tomorrow

$$\frac{\lambda_1\delta + \lambda_2\delta}{(\lambda_1+\lambda_2)\delta}$$

adding 2 processes also gets you a Poisson process

- ie Red + Blue bulbs flash randomly, ind. next to each other
- assume color blind
- See Poisson dist. of flashes



MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Fall 2010)

Recitation 15
October 28, 2010

1. Problem 6.14 (a)-(c),(h)-(j), page 330 in text.

   Beginning at time $t = 0$, we begin using bulbs, one at a time, to illuminate a room. Bulbs are replaced immediately upon failure. Each new bulb is selected independently by an equally likely choice between a type-A bulb and a type-B bulb. The lifetime, X, of any particular bulb of a particular type is a random variable, independent of everything else, with the following PDF:

$$\text{for type-A Bulbs:} \quad f_X(x) = \begin{cases} e^{-x}, & x \geq 0, \\ 0, & \text{otherwise;} \end{cases}$$

$$\text{for type-B Bulbs:} \quad f_X(x) = \begin{cases} 3e^{-3x}, & x \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

   (a) Find the expected time until the first failure.

   (b) Find the probability that there are no bulb failures before time t.

   (c) Given that there are no failures until time t, determine the conditional probability that the first bulb used is a type-A bulb.

   (d) Determine the probability that the total period of illumination provided by the first two type-B bulbs is longer than that provided by the first type-A bulb.

   (e) Suppose the process terminates as soon as a total of exactly 12 bulb failures have occurred. Determine the expected value and variance of the total period of illumination provided by type-B bulbs while the process is in operation.

   (f) Given that there are no failures until time t, find the expected value of the time until the first failure.

2. Problem 6.15 (a)-(c), p. 331 in text.

   A service station handles jobs of two types, A and B. (Multiple jobs can be processed simultaneously.) Arrivals of the two job types are independent Poisson processes with parameters $\lambda_A = 3$ and $\lambda_B = 4$ per minute, respectively. Type A jobs stay in the service station for exactly one minute. Each type B job stays in the service station for a random but integer amount of time which is geometrically distributed, with mean equal to 2, and independent of everything else. The service station started operating at some time in the remote past.

   (a) What is the mean, variance, and PMF of the total number of jobs that arrive within a given three-minute interval?

   (b) We are told that during a 10-minute interval, exactly 10 new jobs arrived. What is the probability that exactly 3 of them are of type A?

   (c) At time 0, no job is present in the service station. What is the PMF of the number of type B jobs that arrive in the future, but before the first type A arrival?

3. Let X, Y, and Z be independent exponential random variables with parameters λ, μ, and ν, respectively. Find $\mathbf{P}(X < Y < Z)$.

	Bernalli p	Poisson λ
PMF of # arrivals in interval of fixed lenght	Binomial w/ p $E\left[\begin{matrix}\text{\# arrivals in} \\ n \text{ slots}\end{matrix}\right] = np$	Lenght T $P_{N_t}(k) = \frac{e^{-\lambda T}(\lambda T)^k}{k!}$ $E[N_T] = var(N_T) = \lambda T$
Rate	p arrivals/unit time	λ arrivals/unit time
Time b/w arrivals	Exponental λ	Geometric p
Fresh start/memoryless	Yes	Yes
Alt. Description	Seq of Ind. Geometric w/ param p	Seq of Ind. Exp w/ param λ
kth arrival time	Pascal order k	Erlang of order k
Splitting/Multipling	Bernoulil	Poisson

Recitation 15

10/28

Poission process

- interesting theory + hw
- Connect intuition to Bernouli process
- also records arrivals
- contineaus
- but discretize it into small intervals δ

$\lim \delta \to 0$

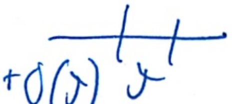
- prob of sucess $= \delta\lambda$

  ↑ constant of proportionalty defines process

- properties $P(k, t)$ $t = \tau$

1. Same and independent of start of interval t) time homogneity

2. Independentce of arrivals in disjoint intervals
3. Small interval probabilities $P(0,\tau) = 1 - \lambda\tau + o(\tau)$



$P(1,\tau) = \lambda\tau + o(\tau)$

$P(k,\tau) = o(\tau) \quad \forall k \geq 2$

λ = arrivals per unit time/ rate of arrivals of a process

②

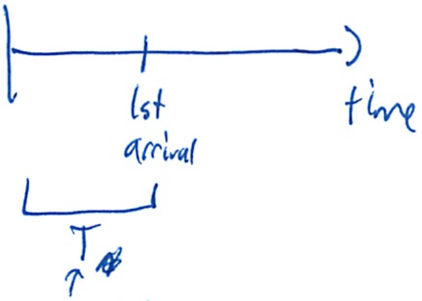
PMF # of arrivals in λ

- limit of Binomial as interval goes to 0
- so poisson
- $P_{N_\tau}(t) = \dfrac{e^{-\lambda\tau}(\lambda\tau)^k}{k!}$
- $E[N_\tau] = var(N_\tau) = \lambda\tau$

RVs of interest

N_τ: # of arrivals in interval τ

Poisson w/ param $\lambda\tau$



exponental
parameter λ
$E[T] = \frac{1}{\lambda}$
expected time
per unit
arrival

remember does not matter where you start

- time homonegiety

bernoulli

binomial w/p

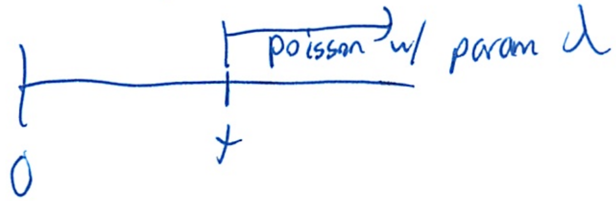
$E\left[\begin{matrix}\text{\# arrivals}\\ n \text{ sets}\end{matrix}\right] = n \cdot p$

geometric

ex

First start property / memoryless

-still applies



-time to 2nd arrival does not depend on how long 1st occured

Bernoulli

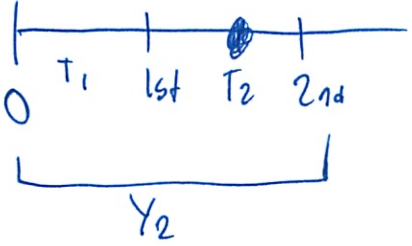
had X_1, X_2 ....

alternate process starts at 0, geometricly distributed time to heads

-geometric w/ param p

Poisson

Same but w/ exponential RV which represent inter-arrival time w/ param λ



$Y_2 = T_1 + T_2$

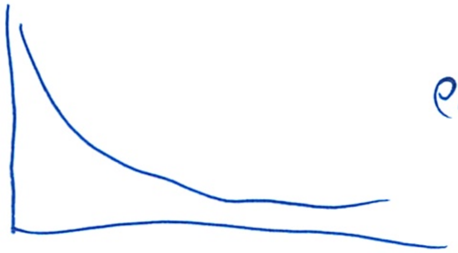
$Y_x = T_1 + T_2 + \dots + T_k$

called the erlang of order 2

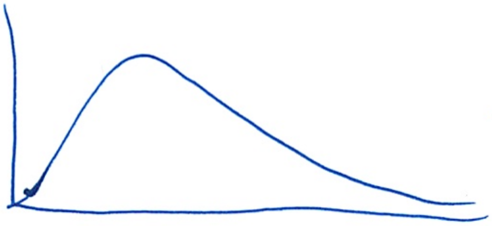
can get PDF w/ convolution

or look up erlang formula

④ Shape



erlang order 1 f_{Y_1}



erlang order 2 f_{Y_2} $E[\,] = \frac{2}{\lambda}$

$var = 2\,var(\,)$

f_{Y_3}

f_{Y_k} $E[\,] = \frac{k}{\lambda}$

$var(\,) = k\,var(\,)$

Bernoulli analog is pascal (of order k)

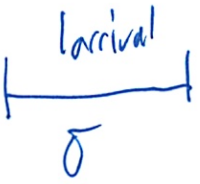
Splitting Poisson process

like Bernoulli

Merging

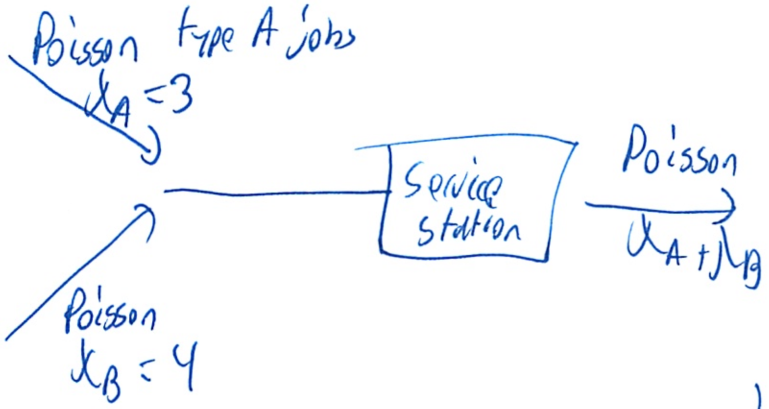
$\frac{\lambda_A}{\lambda_A+\lambda_B}$ Fraction/unit time

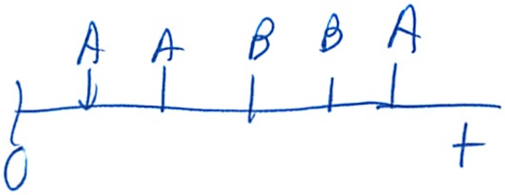
⑤



$$P(1 \text{ arrival from } A \mid \overset{\text{given}}{1 \text{ arrival}}) = \frac{\lambda_A \cdot \delta}{(\lambda_A+\lambda_B)\delta} = \frac{\lambda_A}{\lambda_A+\lambda_B}$$

Problem

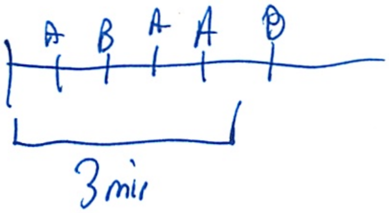




1. N = # of jobs w/ interval 3 min

$P_N(k) = ?$ $\quad var(N) = ?$

$E[N] = ?$



$\lambda\tau = (\lambda_A+\lambda_B)\tau$

$(3+4)\cdot 3 = 21$

$$\underbrace{p_N(k)}_{\text{poission}} = \frac{e^{-21}\cdot 21^k}{k!} \quad k \geq 0$$

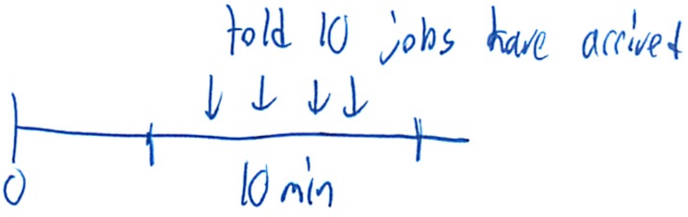
leave like that

$E[N] = \lambda\tau = 21$

$Var(N) = \lambda\tau = 21$

⑥

b)



P(3 out of 10 are A) =

view A as sucess

what is P(sucess)

$\frac{\lambda_A}{\lambda_A+\lambda_B}$ ~~#~~ $= \frac{3}{7}$ = P(sucess) = P(A)

$$1-\frac{3}{7}$$

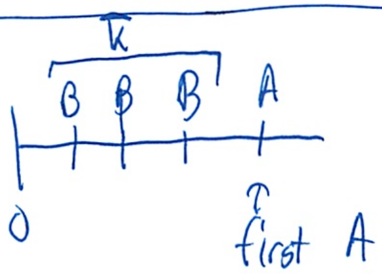
So

$$= \binom{10}{3}\left(\frac{3}{7}\right)^3\left(\frac{4}{7}\right)^4$$

the 10 min has nothing to do w/ it

---

c)



$\bar{k}$ = # B arrivals before A = k-1

k = # of arrivals to 1st A arrival

$P_k(k)$ ~~=?~~ view type A as sucess again

P(A) = $\frac{3}{7}$ again

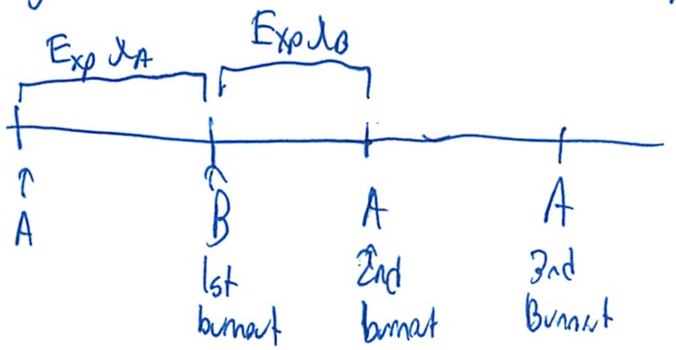
k = geometric w/ param $\frac{\lambda_A}{\lambda_A+\lambda_B} = \frac{3}{7}$

$P_{\bar{k}}(k) = \left(\frac{3}{7}\right)\left(\frac{4}{7}\right)^k$ k = 0, 1, 2, ... shifted geometrically

⑦

#1 (Problem 6.15)

lightbulb - life of described exponentially



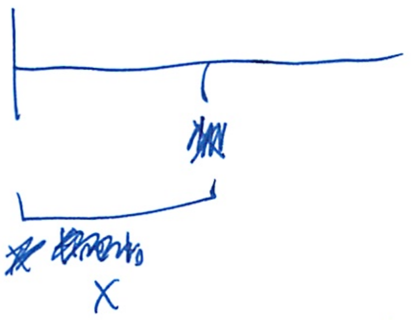
Type A

$\lambda_A = 1$

Type B

$\lambda_B = 3$ ← 3 times faster to burn out

a) X = time to 1st burnout



$E[X] = ?$

need to divide + conquor w/ total prob. theorm

$$E[X] = \underbrace{P(A)}_{1/2} \cdot \underbrace{E[X|A]}_{1/\lambda_A} + \underbrace{P(B)}_{1/2} \underbrace{E[X|B]}_{1/\lambda_B}$$

$$= \frac{1}{2} \cdot \frac{1}{1} + \frac{1}{2} \cdot \frac{1}{3}$$

$$= \frac{4}{6} = \frac{2}{3}$$

⑧

b)

D = event that 1st burnout after t

$P(D) = ?$

Use CDF

But don't know if have A or B

Divide + conquer w/ total prob theorm again

$$P(D) = \underbrace{P(A)}_{1/2} \underbrace{P(D|A)}_{\substack{1-CDF_A(t) \\ e^{-t}}} + \underbrace{P(B)}_{1/2} \underbrace{P(D|B)}_{e^{-3t}}$$

$$= \frac{1}{2}\left(e^{-t} + e^{-3t}\right)$$

c) $P(D|A) = ?$

now reverse

$P(A|D) = ?$ prob had an A given that it burned out in D time

Bayes rule

$$\frac{P(A \cap D)}{P(D)} = \frac{P(A)\,P(D|A)}{P(D)} = \frac{\frac{1}{2} \circ e^{-\lambda}}{\frac{1}{2}\left(e^{-\lambda} + e^{-3t}\right)}$$

⑨

d) tricky

-read solution

11/3

LECTURE 15

Poisson process — II

- **Readings:** Finish Section 6.2.

- Review of Poisson process
- Merging and splitting
- Examples
- Random incidence

Customers arriving over time

Review

- Defining characteristics
  - **Time homogeneity**: $P(k, \tau)$
  - **Independence**
  - **Small interval probabilities** (small δ):

$$P(k, \delta) \approx \begin{cases} 1 - \lambda\delta, & \text{if } k = 0, \\ \lambda\delta, & \text{if } k = 1, \\ 0, & \text{if } k > 1. \end{cases}$$

↳ more than 1 arrival

- N_τ is a Poisson r.v., with parameter $\lambda\tau$:

$$P(k, \tau) = \frac{(\lambda\tau)^k e^{-\lambda\tau}}{k!}, \qquad k = 0, 1, \ldots$$

$$\boxed{\mathrm{E}[N_\tau] = \mathrm{var}(N_\tau) = \lambda\tau}$$

- Interarrival times ($k = 1$): exponential:

$$f_{T_1}(t) = \lambda e^{-\lambda t}, \quad t \geq 0, \qquad \mathrm{E}[T_1] = 1/\lambda$$

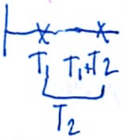
arrivals happening 1 customer at a time

- Time Y_k to <u>kth arrival</u>: <u>Erlang(k)</u>:

$$f_{Y_k}(y) = \frac{\lambda^k y^{k-1} e^{-\lambda y}}{(k-1)!}, \qquad y \geq 0$$

time to kth arrival

Memoryless



T_2 independent from t_1

Both exponentially distributed

$T_i \sim \exp(\lambda)$

$Y_k = T_1 + T_2 + \cdots + T_k$

Similar to Bernoulli process except very small slots

δ = tiny interval

τ = longer intervals

distribution of # of arrivals

λ = arrivals per unit time, arrival rate

τ = interval lenght

$\lambda\tau$ = unit customers

(why am I so bad at this)

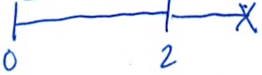
- just logic

- not really math

Poisson fishing

Simple example

- Assume: Poisson, $\lambda = 0.6$/hour.
  - Fish for two hours.
  - if no catch, continue until first catch.



a) P(fish for more than two hours)= nothing caught in first 2 hrs

$P(0,2) = e^{-\lambda t} = e^{-2 \cdot .6}$ ← or $P(T_1 > 2) = \int_2^{\infty} f_{T_1}(t)\,dt$

b) P(fish for more than two and less than five hours)= nothing caught in first 2 hrs + at least one fish 2-5 hrs

independent! $= P(0,2) \cdot (1 - P(0,3))$ ← or $P(2 < T_1 < 5) = \int_2^5 f_{T_1}(t)\,dt$

c) P(catch at least two fish)= must be in 1st 2 hrs

$\sum_{k=2}^{\infty} P(k,2) = 1 - P(0,2) - P(1,2)$ or $P(Y_2 \le 2) = \int_0^2 f_{Y_2}(y)\,dy$

d) E[number of fish]=

$2 \cdot 0.6 + P(0,2) \cdot 1$ → 1st part first 2 hrs + no catch alternative total probability therom

e) E[future fishing time | fished for four hours]= history does not matter since its a given!

$\exp(\lambda = .6) \rightarrow \frac{1}{.6}$

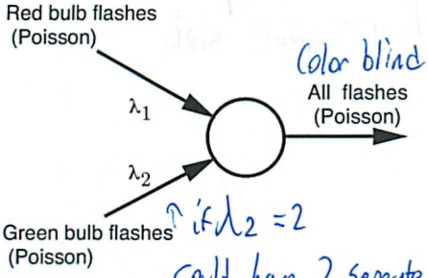
f) E[total fishing time]=

$2 + P(0,2) \cdot \frac{1}{.6}$

prob not done by 2

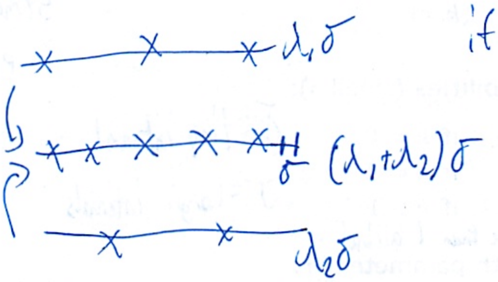
Merging Poisson Processes (again)

- Merging of independent Poisson **processes** is Poisson



↑ if $\lambda_2 = 2$

could have 2 seperate processes $\lambda = 1$ to merge together to form 3 processes with λ_1



if both were independent, merged process is independent

$\frac{\lambda_1}{\lambda_1 + \lambda_2} = P$ came from λ_1

  - What is the probability that the next arrival comes from the first process?

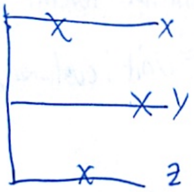
Origin of arrivals are independent RVs

Light bulb example

pretty much each problem has a simple solution

↳ short cut, get ans quickly

- Each light bulb has independent, exponential(λ) lifetime
- Install three light bulbs. Find expected time until last 3rd light bulb dies out.



all exp

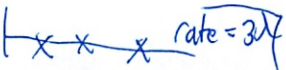
$E[\max(x, y, z)] =$

each is 1st arrival in a poisson process

time to some bulb burns out $\rightarrow \frac{1}{3\lambda}$ ← can be any of the light bulb

then take that away, 2 left $\rightarrow \frac{1}{2\lambda}$

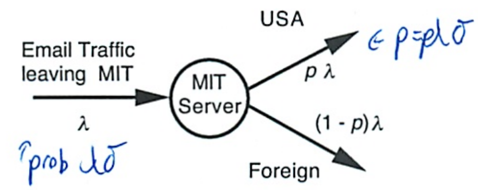
now time for last one $\rightarrow \frac{1}{\lambda}$



$= \frac{1}{3\lambda} + \frac{1}{2\lambda} + \frac{1}{\lambda}$

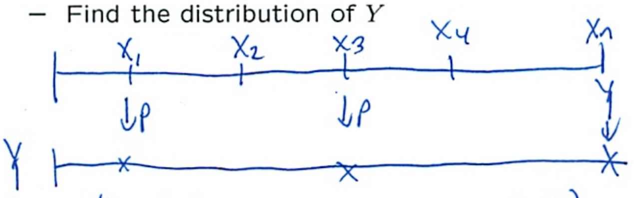
Splitting of Poisson processes

- Suppose email traffic through server is a Poisson process and destinations are independent.



- Each output stream is Poisson. (Why?)

- **Example:** $Y = X_1 + \cdots + X_N$
  $N, X_1, X_2, \ldots$ independent ~~exponential~~
  N: geometric(p); X_i: exponential(λ)
  - Find the distribution of Y



then take the sum $\sim \exp(p\lambda)$

(got confused on that)

Random incidence for Poisson

Paradox

[Subtle, counterintuitive

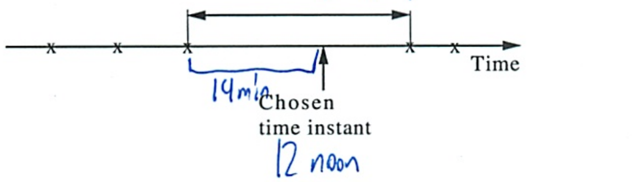
- Poisson process that has been running forever

Some bus 4/hr $\Rightarrow$ 15 min headway $= E[T_1]$
$\lambda = \frac{4}{Q_2}$

watching coin flip movie backwards
- still same!

- Show up at some "random time" (really means "arbitrary time")

$T_1 \sim \exp$ $E[T_1] = 15$



12 noon

$\overrightarrow{\text{how much}}$ time b/w bus arrivals

- What is the distribution of the length of the chosen interarrival interval?

ask when last bus came $\rightarrow$ 14 min
when will next bus come? 1 min ⓧ
its memory less!
You see times larger than 15

What is avg interarrival time?
↳ avg is 15 min - equal weight to all inter arrival times
- you are not giving = weight
- you are more likely to show up during large time intervals
- another example separate sheet

$\underbrace{E[T_1']}_{}$ $\underbrace{E[T_1]}_{}$
$= 15$ $= 15$

T_1, T_1' independent

so see avg interarrival times of 30 - but bus company is truthful

Random incidence in "renewal processes"

- Series of successive arrivals
  - i.i.d. interarrival times (but not necessarily exponential)

- **Example:**
  Bus interarrival times are equally likely to be 5 or 10 minutes

- If you arrive at a "random time":
  - what is the probability that you selected a 5 minute interarrival interval?
  - what is the expected time to next arrival?

Lecture 15 Add.

Its like if MIT has 10 classes → 100 people
100 classes → 1 person

So avg class size ≈ 2-3

But if you ask someone what is your average class size they will say ~90 since they are in a lot of 100 person classes and not many 1 person classes

6.041/6.431 Fall 2010 Quiz 2
Tuesday, November 2, 7:30 - 9:30 PM.

DO NOT TURN THIS PAGE OVER UNTIL YOU ARE TOLD TO DO SO

Name: Michael Plasmeier

Recitation Instructor: Dimitri

TA: Aliaa

Question	Score	Out of
1.1	6	10
1.2	0	10
1.3	6 PH	10
1.4	5	10
1.5	10	10
1.6	10	10
1.7	10	10
1.8	3	10
2.1	4	10
2.2	10	10
2.3	2	5
2.4	2	5
Your Grade	68	110

- **For full credit, answers should be algebraic expressions (no integrals), in simplified form. These expressions may involve constants such as π or e, and need not be evaluated numerically.** (so different than 1st
- This quiz has 2 problems, worth a total of 110 points.
- You may tear apart page 3, as per your convenience, **but you must turn them in together with the rest of the booklet.**
- Write your solutions in this quiz booklet, only solutions in this quiz booklet will be graded. Be neat! You will not get credit if we can't read it.
- You are allowed two two-sided, handwritten, 8.5 by 11 formula sheets. Calculators are not allowed.
- You have 120 minutes to complete the quiz.
- Graded quizzes will be returned in recitation on Thursday 11/4.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Fall 2010)

Problem 0: (0 points) Write your name, your assigned recitation instructor's name, and assigned TA's name on the cover of the quiz booklet. The Instructor/TA pairing is listed below.

Recitation Instructor	TA	Recitation Time
Vivek Goyal	Uzoma Orji	10 & 11 AM
Peter Hagelstein	Ahmad Zamanian	12 & 1 PM
Ali Shoeb	Shashank Dwivedi	2 PM
Dimitri Bertsekas (6.431)	Aliaa Atwi	[✓ 2 & 3 PM]

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Fall 2010)

Problem 1. (80 points) In this problem:
(i) X is a (continuous) uniform random variable on $[0, 4]$.
(ii) Y is an exponential random variable, independent from X, with parameter $\lambda = 2$.

1. **(10 points)** Find the mean and variance of $X - 3Y$.
2. **(10 points)** Find the probability that $Y \geq X$.
   (Let c be the answer to this question.)
3. **(10 points)** Find the conditional joint PDF of X and Y, given that the event $Y \geq X$ has occurred.
   (You may express your answer in terms of the constant c from the previous part.)
4. **(10 points)** Find the PDF of $Z = X + Y$.
5. **(10 points)** Provide a fully labeled sketch of the conditional PDF of Z given that $Y = 3$.
6. **(10 points)** Find $\mathbf{E}[Z \mid Y = y]$ and $\mathbf{E}[Z \mid Y]$.
7. **(10 points)** Find the joint PDF $f_{Z,Y}$ of Z and Y.
8. **(10 points)** A random variable W is defined as follows. We toss a fair coin (independent of Y). If the result is "heads", we let $W = Y$; if it is tails, we let $W = 2 + Y$. Find the probability of "heads" given that $W = 3$.

Problem 2. (30 points) Let $X, X_1, X_2, \ldots$ be independent normal random variables with mean 0 and variance 9. Let N be a positive integer random variable with $\mathbf{E}[N] = 2$ and $\mathbf{E}[N^2] = 5$. We assume that the random variables $N, X, X_1, X_2, \ldots$ are independent. Let $S = \sum_{i=1}^{N} X_i$.

1. **(10 points)** If δ is a small positive number, we have $\mathbf{P}(1 \leq |X| \leq 1 + \delta) \approx \alpha\delta$, for some constant α. Find the value of α.
2. **(10 points)** Find the variance of S.
3. **(5 points)** Are N and S uncorrelated? Justify your answer.
4. **(5 points)** Are N and S independent? Justify your answer.

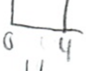
Each question is repeated in the following pages. Please write your answer on the appropriate page.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Fall 2010)

Problem 1. (80 points) In this problem:
(i) X is a (continuous) uniform random variable on $[0, 4]$.
(ii) Y is an exponential random variable, independent from X, with parameter $\lambda = 2$.

1. **(10 points)** Find the mean and variance of $X - 3Y$. $= p$

$X \sim$ uniform $[0,4]$



$Y \sim \exp(2)$

X, Y ind

$f_X(x) = \frac{1}{b-a} = \frac{1}{4} \quad 0 \le x \le 4$ $\qquad f_Y(y) = 2e^{-2y} \quad y \ge 0$

derived distribution?

$F_X(x) = \frac{x-a}{b-a} = \frac{x}{4} \quad 0 \le x \le 4$ $\qquad F_Y(y) = 1 - e^{-2y} \quad y > 0$

$E[X] = \frac{b-a}{2} = \frac{4}{2} = 2$ $\qquad E[Y] = \frac{1}{\lambda} = \boxed{\frac{1}{2}}$

$Var(X) = \frac{(b-a)^2}{12} = \frac{4^2}{12} = \frac{16}{12} = \frac{4}{3}$ $\qquad Var(Y) = \frac{1}{\lambda^2} = \frac{1}{4}$

$Var(X+Y) = var(X) + var(Y) + 2\,cov(X,Y)$

$var(X-3Y) = \boxed{var(X) - 3\,var(Y)}$? ind so 0

var(x)+9 Var(

$+1$

$= \frac{4}{3} - 3\frac{1}{4}$

$= \frac{4}{3} - \frac{3}{4}$

$= \frac{16}{12} - \frac{9}{12}$

$= \frac{7}{12}$

Or Linearity of Expectation

$E[X+Y] = E[X] + E[Y]$

$E[X-3Y] = E[X] - 3E[Y]$

$= \frac{b-a}{2} - 3\frac{1}{2}$

$= 2 - \frac{3}{2}$

$= \boxed{\frac{1}{2}}$ $\boxed{+5}$

First qu should be #1

2. **(10 points)** Find the probability that $Y \geq X$.
(Let c be the answer to this question.)

$P(Y \geq X)$

$1 - P(Y \leq X)$

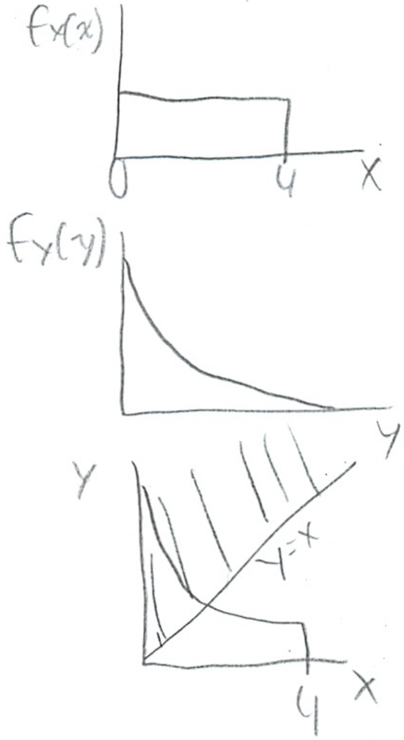
$1 - F_Y(y)$

$1 - 1 - e^{-2y}$

$2 - e^{-2y}$

???

how to do?
CDF?



??? can't visualize

and otherwise how done algebraically?

if $y > 4$

$$c = P(Y \geq X) = \begin{cases} 1 & Y \geq 4 \\ 2 - e^{-2y} & Y < 4 \end{cases}$$

+0

Should be small since $E[Y] = \frac{1}{2}$ and $E[X] = 2$

3. **(10 points)** Find the conditional joint PDF of X and Y, given that the event $Y \geq X$ has occurred.
$\uparrow A$
(You may express your answer in terms of the constant c from the previous part.)

X and Y independent

$$f_{X,Y|A} = \frac{f_{X,Y}(x,y)}{P(A)} = \frac{f_X(x)\,f_Y(y)}{c}$$

X 2/2

$$= \frac{\frac{1}{b-a} \cdot \lambda e^{-\lambda y}}{c}$$

6/10

$$= \frac{\frac{1}{4} \cdot 2e^{-2y}}{c}$$

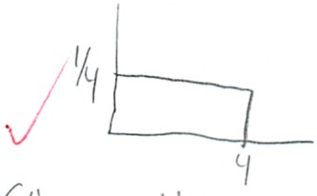
$$= \frac{e^{-2y}}{2c}$$

+4/4

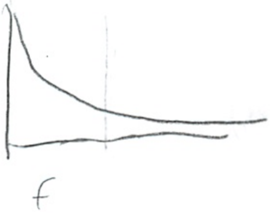
0/4 limits

4. **(10 points)** Find the PDF of $Z = X + Y$.

Convolution



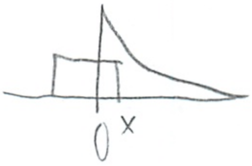
flip + slide



$f_z(z) =$

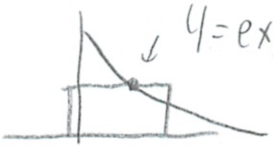
$0 \qquad z<0$

$\int_0^x \frac{1}{4} \cdot 2e^{-2(z-x)}\,dx \qquad z>0,\ z<e^2$



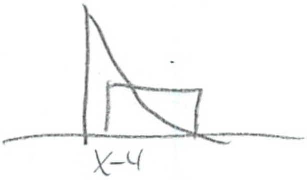
↓ same

$\int_0^x \frac{1}{4} \cdot 2e^{-2(z-x)} \qquad z>e^2,\ z-4<0,\ z<4$

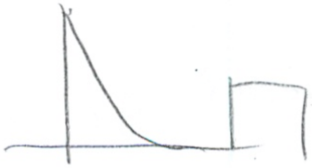


$4 = \exp(2) = e^2$

$\int_{x-4}^x \frac{1}{4} \cdot 2e^{-2(z-x)} \qquad z>e^2,\ z-4>0,\ z>4$



$0 \qquad$ Never happen



→ Over

$$\begin{cases} \int_0^x \frac{e^{-2(z-x)}}{2}\,dx & 0<z<4 \\ \int_{x-4}^x \frac{e^{-2(z-x)}}{2}\,dx. & z>4 \end{cases}$$

; but how suppose to take $\int$ of e^x
Oh its a common one, right? $\int e^x = e^x$

wrong limits & calculations

$$\int_0^x e^{-2z+2x}\,dx = \int_0^x e^{-2z} + \int_0^x e^{2x}\,dx = e^{-2z}\Big|_0^x + e^{+2x}\Big|_0^x$$

$$= e^{-2z+2x}$$

$$= e^{-2(z-x)}$$

$$\int_{x-4}^x e^{-2z+2x}\,dx = \int_0^x e^{-2z} + \int_0^x e^{2x}$$

$$e^{-2z}\Big|_{x-4}^x + e^{2x}\Big|_{x-4}^x \qquad z>4$$

$$e^{-2z} + e^{2x} - e^{2(x-4)}$$

$$e^{-2z} + e^{2x} - e^{2x-8}$$

$$e^{-2z+2x-2x+8}$$

$$e^{-2z+8}$$

$$0<z<4$$

$$f_z(z) = \begin{cases} e^{-2z+8} & 0<z<4 \\ e^{-2(z-x)} & z>4 \end{cases}$$

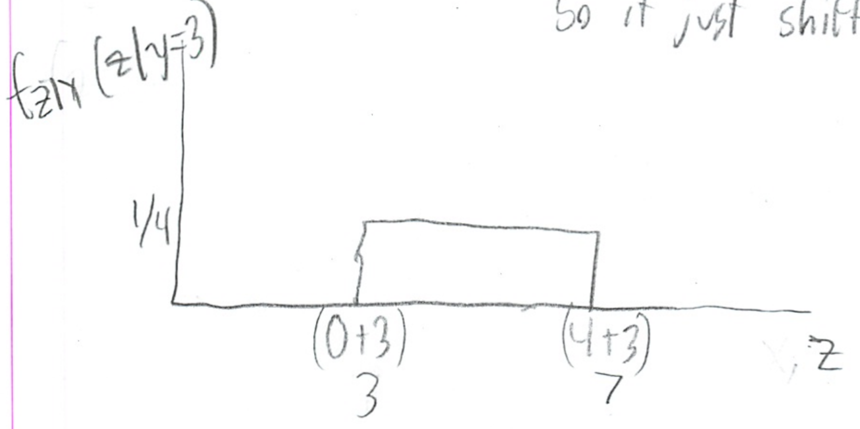
So unsure

5. **(10 points)** Provide a fully labeled sketch of the conditional PDF of Z given that $Y = 3$.

When $Y = y = 3$

Z is $X+3$

So it just shifts X to the right



d by graphin

$$f_{Z|Y}(z|y=3) = \begin{cases} 1/4 & 3 < z < 7 \\ 0 & \text{else} \end{cases}$$

$$= \frac{f_{Z,Y}(z,y)}{f_Y(y)}$$

← should build, this is later however

6. **(10 points)** Find $\mathbf{E}[Z \mid Y = y]$ and $\mathbf{E}[Z \mid Y]$. ✓function of Y

$E[Z|Y=y] = a$ #

$= \int_{-\infty}^{\infty} z \, f_{Z|Y}(z|Y=y) \, dz$

intuitivly from last problem

$= \int_{0+y}^{4+y} z \, 1/4 \; dz \qquad 0+y < z < 4+y$

$= \frac{z^2}{8}\Big|_{0+y}^{4+y}$

$= \frac{(4+y)^2}{8} - \frac{(y)^2}{8} = 2+y$ ✓

$E[Z|Y] = a$ RV, function of Y

$= \int_{0+Y}^{4+Y} 1/4 \, z \, dz \qquad 0+Y < Z < 4+Y$

$\frac{z^2}{4}\Big|_{0+Y}^{4+Y}$

$= \frac{(4+Y)^2}{8} - \frac{(Y)^2}{8} = 2+Y$

↑ function of Y

Urg on all

test using $y=3$, should $=5$ try $y=0$, should $=2$

$$\frac{(4+3)^2}{8} - \frac{4^2}{8}$$

$$\frac{49}{8} - 2$$

$$6\frac{1}{8} - 2$$

$$4\frac{1}{8} \otimes$$

close

- why the 1/8

pratice test
like this

$$\frac{(4+0)^2}{8} - 0$$

$$\frac{16}{8} = 2 \text{ ✓}$$

7. **(10 points)** Find the joint PDF $f_{Z,Y}$ of Z and Y.

Z and Y are not independent so can't $f_Z(z) f_Y(y)$ so how do otherwise

$f_Y(y) \, f_{Z|Y}(z|y)$ ✓

$2e^{-2Y} \cdot \frac{1}{4}$ $\quad 0+y < z < 4+y$

$y \geq 0$

$$f_{Z,Y}(z,y) = \frac{e^{-2y}}{2}$$

$Y \geq 0$

$0+y \leq z < 4+y$

$y < z < 4+y$ ✓

8. **(10 points)** A random variable W is defined as follows. We toss a fair coin (independent of Y). If the result is "heads", we let $W = Y$; if it is tails, we let $W = 2 + Y$. Find the probability of "heads" given that $W = 3$.

Bayes' Rule

$P(\text{heads}) = q$

$P(\text{tails}) = 1 - q$

know $P_{W|} = \begin{cases} Y & \text{if heads}, q \\ Y+2 & \text{if tails}, 1-q \end{cases}$

want $P_{\text{Heads}|W}(\text{heads} \mid W=3) =$

$P_{W|\text{heads}} = Y$

$$P_{\text{Heads}|W} = \frac{P(\text{heads})\, f_{W|\text{Heads}}(w)}{f_W(w)}$$

derived dist?

$$\frac{q \cdot Y}{2e^{-2Y}q}$$

3

$P_Y(y) = 2e^{-2x}$

$P_W(w) = 2e^{-2y}$ if q / heads

$2+2e^{-2y}$ if $1-q$ / tails

$= \begin{cases} 2e^{-2x} \cdot q \\ (2+2e^{-2y})(1-q) \end{cases}$

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Fall 2010)

I like when we get the #s

Problem 2. (30 points) Let $X, X_1, X_2, \ldots$ be independent normal random variables with mean 0 and variance 9. Let N be a positive integer random variable with $\mathbf{E}[N] = 2$ and $\mathbf{E}[N^2] = 5$. We assume that the random variables $N, X, X_1, X_2, \ldots$ are independent. Let $S = \sum_{i=1}^{N} X_i$.

Sum or random # of RVs

✓ Δ 1. **(10 points)** If δ is a small positive number, we have $\mathbf{P}(1 \leq |X| \leq 1+\delta) \approx \alpha\delta$, for some constant α. Find the value of α.

interval? into proofs?

Or is this just the constant thing? what is it asking?

aka $f_X(1)$

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

$$= \frac{1}{3\sqrt{2\pi}} e^{-\frac{(1-0)^2}{2\cdot 3^2}}$$

$$= \frac{1}{3\sqrt{2\pi}} e^{\frac{-1}{18}} \quad 2\cdot 3^2 = 18$$

6

-ah its $f_X(x)$ ← thank you formula sheet where $x = 1$

- But that is if the question is $P(1 \leq X \leq 1+\delta)$.
- to get $P(1 \leq |X| \leq 1+\delta)$ it would be $\frac{2}{178}$

$2 \cdot P(1 \leq X \leq 1+\delta) \approx 2 \cdot f_X(1)$

So

2. **(10 points)** Find the variance of S.

Sum of random # of RVs

$$\text{var}(S) = E[N]\,\text{var}(X) + (E[X])^2\,\text{var}(N)$$
$$2 \cdot 9 + 0^2 \;?$$
$$= 18 \checkmark$$

$E[X] = \mu = 0$

$\text{var}(X) = \sigma^2 = 9$

$$E[S] = E[N]\,E[X]$$
$$= 2 \cdot 0$$
$$= 0$$

3. **(5 points)** Are N and S uncorrelated? Justify your answer.

$$= E[(N-E[N])(S-E[S])]$$
$$= E[NS] - E[N]E[\boxed{S}]$$
$$0 \quad - \quad 2 \cdot 0$$

? just did today, but where ind.

$$0 - 0 = 0 = \text{Uncorrelated}$$

S will always center around 0, no matter how many tosses one has (N)

Not true!

+2

4. **(5 points)** Are N and S independent? Justify your answer.

No! S is $\sum_{i=1}^{N} X_i$. N is in the definition of S, thus it is ~~in~~dependent.

?

Oh wow duh

0/5

Uncorrelated does not imply independence.

→ Athough stuff I studied I don't think helped that much
Diff qu

→ only 1b → no clue

Utter disaster
- all abstract qu
- recitation one for easier

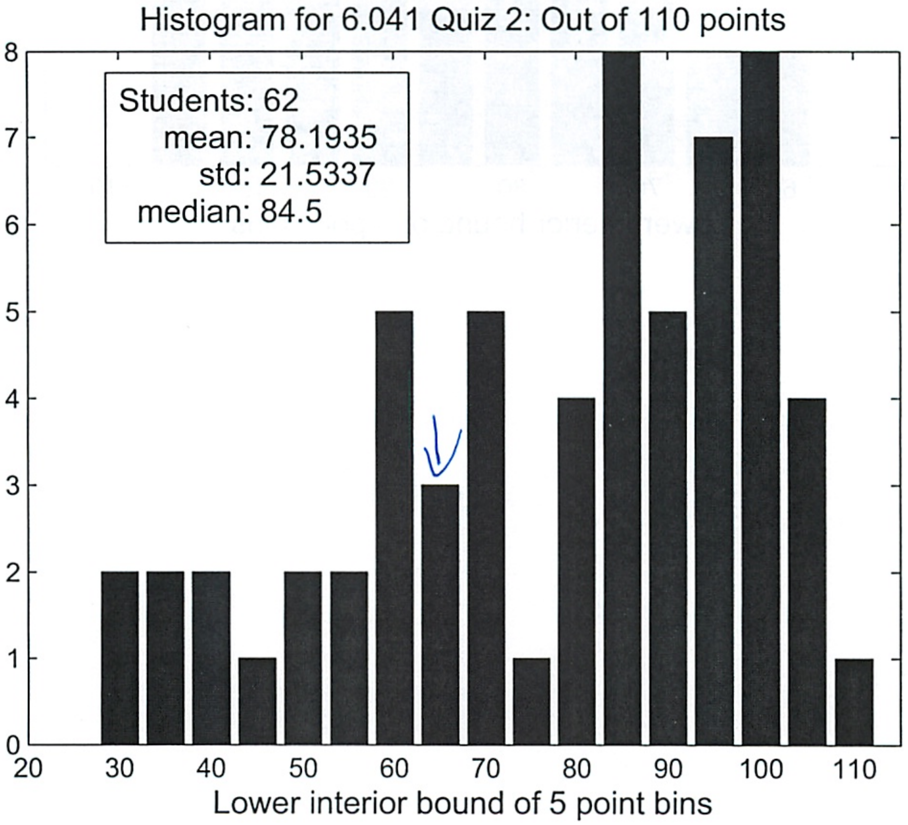
all for partial credit
not sure on a single one
and I thought I kinda knew

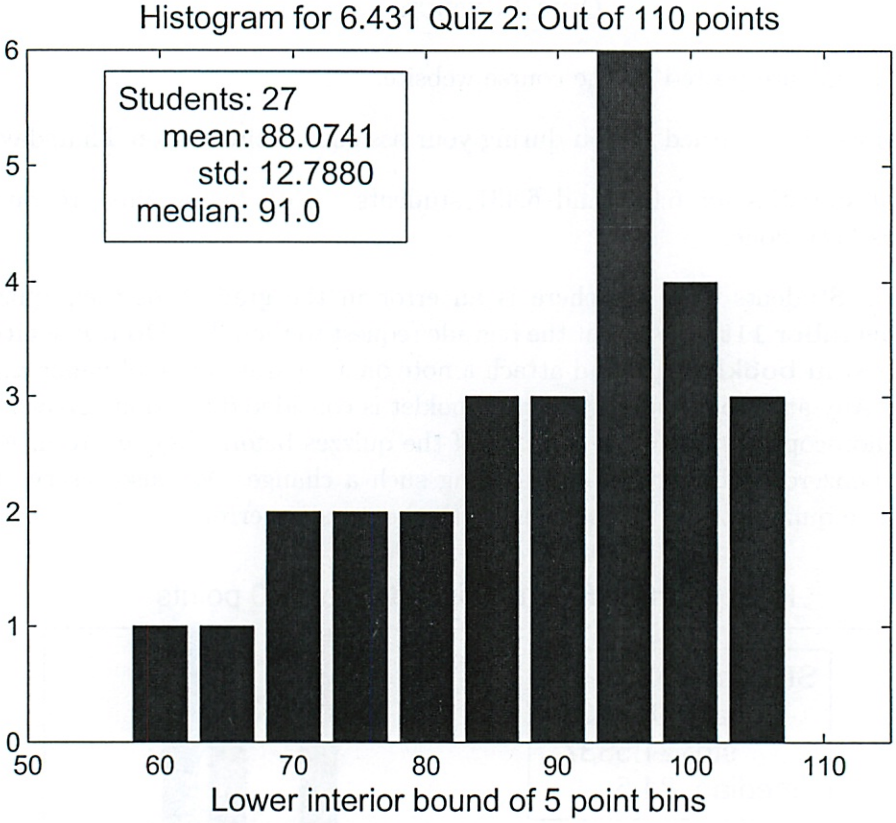
except #2b
really worked it for 2 hrs
though

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Fall 2010)

QUIZ 2 RESULTS

- Solutions to the quiz are posted on the course website.
- Graded quizzes will be returned to you during your assigned recitation on Thursday 11/4.
- Below are final statistics for 6.041 and 6.431 students. Both histograms are raw scores, no normalizing has been done.
- *Regrade Policy*: Students who feel there is an error in the grading of their quiz have until **Thursday November 11th** to submit the regrade request to their TA. **Do not write anything at all on the exam booklet!** Instead attach a note on a separate piece of paper explaining the putative error. Any attempt to modify a quiz booklet is considered a serious breach of academic honesty. We photocopy a substantial fraction of the quizzes before they are returned, implying there exists a nonzero probability of us catching such a change. We also reserve the right to regrade the entire quiz, not just the problem with the putative error.




Histogram for 6.431 Quiz 2: Out of 110 points
Students: 27
mean: 88.0741
std: 12.7880
median: 91.0
0
1
2
3
4
5
6
50
60
70
80
90
100
110
Lower interior bound of 5 point bins

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Fall 2010)

6.041/6.431 Fall 2010 Quiz 2 Solutions

Problem 1. (80 points) In this problem:
(i) X is a (continuous) uniform random variable on $[0, 4]$.
(ii) Y is an exponential random variable, independent from X, with parameter $\lambda = 2$.

1. **(10 points)** Find the mean and variance of $X - 3Y$.

$$\begin{aligned}\mathbf{E}[X-3Y] &= \mathbf{E}[X] - 3\mathbf{E}[Y] \\ &= 2 - 3\cdot\frac{1}{2} \\ &= \frac{1}{2}.\end{aligned}$$

$$\begin{aligned}\text{var}(X-3Y) &= \text{var}(X) + 9\text{var}(Y) \\ &= \frac{(4-0)^2}{12} + 9\cdot\frac{1}{2^2} \\ &= \frac{43}{12}.\end{aligned}$$

2. **(10 points)** Find the probability that $Y \geq X$.
(Let c be the answer to this question.)

The PDFs for X and Y are:

$$f_X(x) = \begin{cases} 1/4, & \text{if } 0 \leq x \leq 4, \\ 0, & \text{otherwise.}\end{cases}$$

$$f_Y(y) = \begin{cases} 2e^{-2y}, & \text{if } y \geq 0, \\ 0, & \text{otherwise.}\end{cases}$$

Using the total probability theorem,

$$\begin{aligned}\mathbf{P}(Y \geq X) &= \int_x f_X(x)\mathbf{P}(Y \geq X \mid X = x)\,dx \\ &= \int_0^4 \frac{1}{4}(1 - F_Y(x))\,dx \\ &= \int_0^4 \frac{1}{4}e^{-2x}\,dx \\ &= \frac{1}{8}\int_0^4 2e^{-2x}\,dx \\ &= \frac{1}{8}(1 - e^{-8}).\end{aligned}$$

3. **(10 points)** Find the conditional joint PDF of X and Y, given that the event $Y \geq X$ has occurred.

   (You may express your answer in terms of the constant c from the previous part.)

   Let A be the event that $Y \geq X$. Since X and Y are independent,

$$
\begin{aligned}
f_{X,Y|A}(x,y) &= \frac{f_{X,Y}(x,y)}{\mathbf{P}(A)} = \frac{f_X(x)f_Y(y)}{\mathbf{P}(A)} \text{ for } (x,y) \in A \\
&= \begin{cases} \frac{4e^{-2y}}{1-e^{-8}}, & \text{if } 0 \leq x \leq 4,\ y \geq x \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}
$$

4. **(10 points)** Find the PDF of $Z = X + Y$.

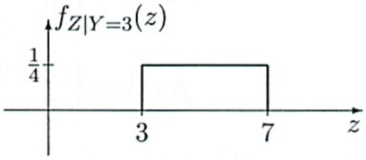
   Since X and Y are independent, the convolution integral can be used to find $f_Z(z)$.

$$
\begin{aligned}
f_Z(z) &= \int_{\max(0,z-4)}^{z} \frac{1}{4} 2e^{-2t}\, dt \\
&= \begin{cases} 1/4 \cdot \left(1 - e^{-2z}\right), & \text{if } 0 \leq z \leq 4, \\ 1/4 \cdot \left(e^{8} - 1\right) e^{-2z}, & \text{if } z > 4, \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}
$$

5. **(10 points)** Provide a fully labeled sketch of the conditional PDF of Z given that $Y = 3$.

   Given that $Y = 3$, $Z = X + 3$ and the conditional PDF of Z is a shifted version of the PDF of X. The conditional PDF of Z and its sketch are:

$$
f_{Z|\{Y=3\}}(z) = \begin{cases} 1/4, & \text{if } 3 \leq z \leq 7, \\ 0, & \text{otherwise.} \end{cases}
$$



6. **(10 points)** Find $\mathbf{E}[Z \mid Y = y]$ and $\mathbf{E}[Z \mid Y]$.

   The conditional PDF $f_{Z|Y=y}(z)$ is a uniform distribution between y and $y + 4$. Therefore,

$$
\mathbf{E}[Z \mid Y = y] = y + 2.
$$

   The above expression holds true for all possible values of y, so

$$
\mathbf{E}[Z \mid Y] = Y + 2.
$$

7. **(10 points)** Find the joint PDF $f_{Z,Y}$ of Z and Y.

   The joint PDF of Z and Y can be expressed as:

$$
\begin{aligned}
f_{Z,Y}(z,y) &= f_Y(y) f_{Z|Y}(z \mid y) \\
&= \begin{cases} 1/2 \cdot e^{-2y}, & \text{if } y \geq 0,\ y \leq z \leq y + 4, \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}
$$

8. **(10 points)** A random variable W is defined as follows. We toss a fair coin (independent of Y). If the result is "heads", we let $W = Y$; if it is tails, we let $W = 2 + Y$. Find the probability of "heads" given that $W = 3$.

   Let X be a Bernoulli random variable for the result of the fair coin where $X = 1$ if the coin lands "heads". Because the coin is fair, $\mathbf{P}(X = 1) = \mathbf{P}(X = 0) = 1/2$. Furthermore, the conditional PDFs of W given the value of X are:

$$\begin{aligned} f_{W|X=1}(w) &= f_Y(w) \\ f_{W|X=0}(w) &= f_Y(w-2). \end{aligned}$$

   Using the appropriate variation of Bayes' Rule:

$$\begin{aligned} \mathbf{P}(X = 1 \mid W = 3) &= \frac{\mathbf{P}(X=1)f_{W|X=1}(3)}{\mathbf{P}(X=1)f_{W|X=1}(3) + \mathbf{P}(X=0)f_{W|X=0}(3)} \\ &= \frac{\mathbf{P}(X=1)f_Y(3)}{\mathbf{P}(X=1)f_Y(3) + \mathbf{P}(X=0)f_Y(1)} \\ &= \frac{\mathbf{P}(X=1)f_Y(3)}{\mathbf{P}(X=1)f_Y(3) + \mathbf{P}(X=0)f_Y(1)} \\ &= \frac{e^{-6}}{e^{-6}+e^{-2}}. \end{aligned}$$

Problem 2. (30 points) Let $X, X_1, X_2, \ldots$ be independent normal random variables with mean 0 and variance 9. Let N be a positive integer random variable with $\mathbf{E}[N] = 2$ and $\mathbf{E}[N^2] = 5$. We assume that the random variables $N, X, X_1, X_2, \ldots$ are independent. Let $S = \sum_{i=1}^{N} X_i$.

1. **(10 points)** If δ is a small positive number, we have $\mathbf{P}(1 \le |X| \le 1+\delta) \approx \alpha\delta$, for some constant α. Find the value of α.

$$\begin{aligned} \mathbf{P}(1 \le |X| \le 1+\delta) &= 2\mathbf{P}(1 \le X \le 1+\delta) \\ &\approx 2f_X(1)\delta. \end{aligned}$$

   Therefore,

$$\begin{aligned} \alpha &= 2f_X(1) \\ &= 2 \cdot \frac{1}{\sqrt{9 \cdot 2\pi}} e^{-\frac{1}{2}\cdot\frac{(1-0)^2}{9}} \\ &= \frac{2}{3\sqrt{2\pi}} e^{-\frac{1}{18}}. \end{aligned}$$

2. **(10 points)** Find the variance of S.

   Using the Law of Total Variance,

$$\begin{aligned} \text{var}(S) &= \mathbf{E}[\text{var}(S \mid N)] + \text{var}(\mathbf{E}[S \mid N]) \\ &= \mathbf{E}[9 \cdot N] + \text{var}(0 \cdot N) \\ &= 9\mathbf{E}[N] = 18. \end{aligned}$$

3. **(5 points)** Are N and S uncorrelated? Justify your answer.

The covariance of S and N is

$$\begin{aligned}
\text{cov}(S,N) &= \mathbf{E}[SN] - \mathbf{E}[S]\mathbf{E}[N] \\
&= \mathbf{E}[\mathbf{E}[SN \mid N]] - \mathbf{E}[\mathbf{E}[S \mid N]]\mathbf{E}[N] \\
&= \mathbf{E}[\mathbf{E}[\sum_{i=1}^{N} X_i N \mid N]] - \mathbf{E}[\mathbf{E}[\sum_{i=1}^{N} X_i \mid N]]\mathbf{E}[N] \\
&= \mathbf{E}[X_1]\mathbf{E}[N^2] - \mathbf{E}[X_1]\mathbf{E}[N] \\
&= 0
\end{aligned}$$

since the $\mathbf{E}[X_1]$ is 0. Therefore, S and N are uncorrelated.

4. **(5 points)** Are N and S independent? Justify your answer.

S and N are not independent.

Proof: We have $\text{var}(S \mid N) = 9N$ and $\text{var}(S) = 18$, or, more generally, $f_{S|N}(s \mid n) = N(0, 9n)$ and $f_S(s) = N(0, 18)$ since a sum of an independent normal random variables is also a normal random variable. Furthermore, since $\mathbf{E}[N^2] = 5 \neq (\mathbf{E}[N])^2 = 4$, N must take more than one value and is not simply a degenerate random variable equal to the number 2. In this case, N can take at least one value (with non-zero probability) that satisfies $\text{var}(S \mid N) = 9N \neq \text{var}(S) = 18$ and hence $f_{S|N}(s \mid n) \neq f_S(s)$. Therefore, S and N are not independent.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Fall 2010)

Recitation 17
November 4, 2010

1. Iwana Passe is taking a multiple-choice exam. You may assume that the number of questions is infinite. *Simultaneously, but independently,* her conscious and subconscious faculties are generating answers for her, each in a Poisson manner. (Her conscious and subconscious are always working on different questions.) Conscious responses are generated at the rate λ_c responses per minute. Subconscious responses are generated at the rate λ_s responses per minute. Assume $\lambda_c \neq \lambda_s$. Each conscious response is an independent Bernoulli trial with probability p_c of being correct. Similarly, each subconscious response is an independent Bernoulli trial with probability p_s of being correct. Iwana responds only once to each question, and you can assume that her time for recording these conscious and subconscious responses is negligible.

   (a) Determine $p_K(k)$, the probability mass function for the number of *conscious responses* Iwana makes in an interval of T minutes.

   (b) If we pick any question to which Iwana has responded, what is the probability that her answer to that question:

      i. Represents a conscious response

      ii. Represents a conscious correct response

   (c) If we pick an interval of T minutes, what is the probability that in that interval Iwana will make exactly r conscious responses *and* s subconscious responses?

   (d) Determine the probability density function for random variable X, where X is the time from the start of the exam until Iwana makes her first conscious response which is preceded by at least one subconscious response.

2. Shem, a local policeman, drives from intersection to intersection in times that are independent and all exponentially distributed with parameter λ. At each intersection he observes (and reports) a car accident with probability p. (This activity does not slow his driving at all.) Independently of all else, Shem receives extremely brief radio calls in a Poisson manner with an average rate of μ calls per hour.

   (a) Determine the PMF for N, the number of intersections Shem visits up to and including the one where he reports his first accident.

   (b) Determine the PDF for Q, the length of time Shem drives between reporting accidents.

   (c) What is the PMF for M, the number of accidents which Shem reports in two hours?

   (d) What is the PMF for K, the number of accidents Shem reports between his receipt of two successive radio calls?

   (e) We observe Shem at a random instant long after his shift has begun. Let W be the total time from Shem's last radio call until his next radio call. What is the PDF of W?

3. Problem 6.27, page 337 in the textbook. **Random incidence in an Erlang arrival process.** Consider an arrival process in which the interarrival times are independent Erlang random variables or order 2, with mean $2/\lambda$. Assume that the arrival process has been ongoing for a very long time. An external observer arrives at a given time t. Find the PDF of the length of the interarrival interval that contains t.

Recitation 17

11/4

"Scores were pretty high"

soft recitation -> mostly review

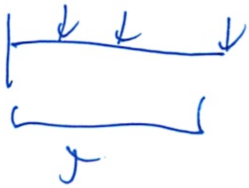
Poisson process

Rate λ

RV of interest

1. N_τ # of arrivals in interval τ

regardless of start time



$$P_{N_\tau}(k) = \text{Poisson} = \frac{e^{-\lambda\tau}(\lambda\tau)^k}{k!} \qquad k \geq 0$$

$E[N_\tau] = \lambda\tau \qquad \text{var}(N_\tau) = \lambda\tau$

2. $T =$ interarrival time

independent

exponential w/ λ

$E[T] = \frac{1}{\lambda} \qquad \text{var}(T) = \frac{1}{\lambda^2}$

3. See next sheet

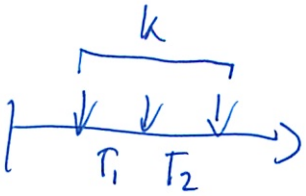
Memoryless / Fresh start

Alt description

sequence of indep. exponential

②

3. Time between k arrivals



$Y = T_1 + T_2$

Y_k = sum of ind. exponential so erlang

$$f_{Y_k}(y) = \frac{\lambda^k y^{k-1} e^{-\lambda y}}{(k-1)!}$$

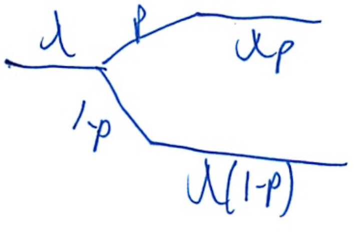
$$E[Y_k] = \frac{k}{\lambda} \qquad var(Y_k) = \frac{k}{\lambda^2}$$

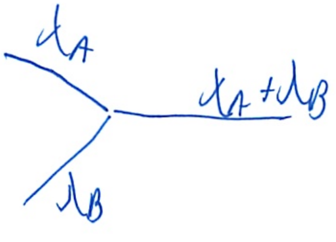
Lenght of ~~Interarrival~~ Interval of Random Inc

erlang of order 2

(Bernoulli Pascal order 2) (See next pg)

Splitting + Merging



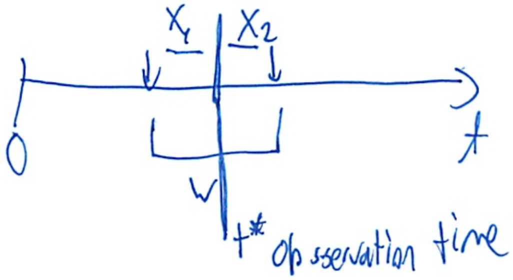


$$= \frac{\lambda_A}{\lambda_A + \lambda_B}$$

③

Random Incidence

- Only new thing from lecture

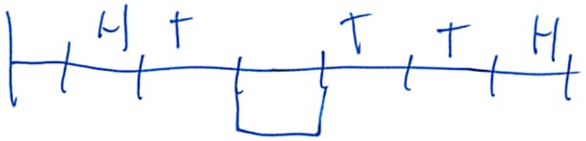


$W = X_1 + X_2$

$f_W(w)$ = erlang of order 2

Remember this is where you have a much larger ~~ob~~ chance of watching at large observation time

- can justify w/ algebra
- can argue heristically
  - add them up → see erlang order 2
- or look at Bernoulli proces - eaiser to understand



pick an observation slot at random

← look back

look forward →

to 1st head count slots (not time)

both are geometric, independent of each other

④

time bracket = $N_1 + N_2$ = geometric + geometric

= pascal of order 2

Now back to Poisson (flip coins ∞ fast, take limit)

exponential + exponental = erlang of order 2

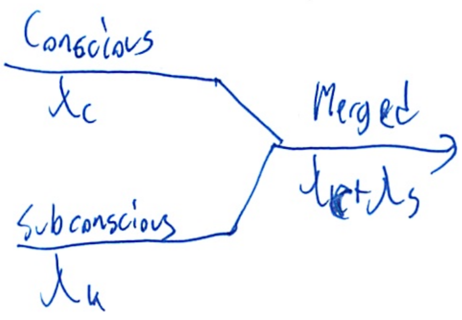
(semi rigerous argument)

Can talk about reversable processes

- he does not find it very intuitive

We made assumption that there was at least one head

#1, Multiple choice exam



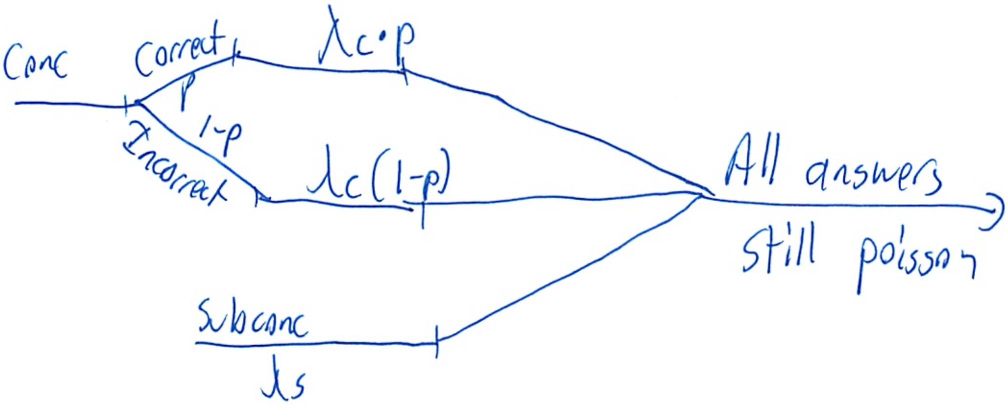
a) k = # of conscious ans within τ

$p_K(k) = ?$ poisson w/ param $\lambda_c \cdot \tau$

⑤

b) i) $P(\text{Ans being conc}) = \dfrac{\lambda_c}{\lambda_c + \lambda_s}$

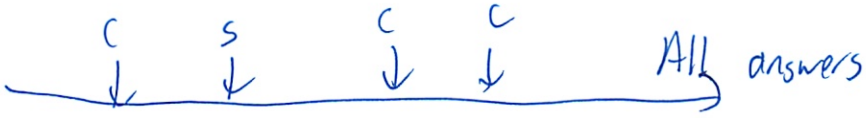
ii) $P(\text{Anc being conc + correct}) = ?$



$= \text{ratio of } \dfrac{\text{what we want}}{\text{all}}$

$$= \frac{\lambda_c \cdot p}{\lambda_c \cdot p + \lambda_c(1-p) + \lambda_s} = \frac{\lambda_c \cdot p}{\lambda_c + \lambda_s}$$

c)



$P(\overset{\text{exactly}}{r} \text{ conc ans within } \tau \text{ and } \overset{\text{exactly}}{s} \text{ subconc}_{\text{ans}} \text{ in } \tau) =$

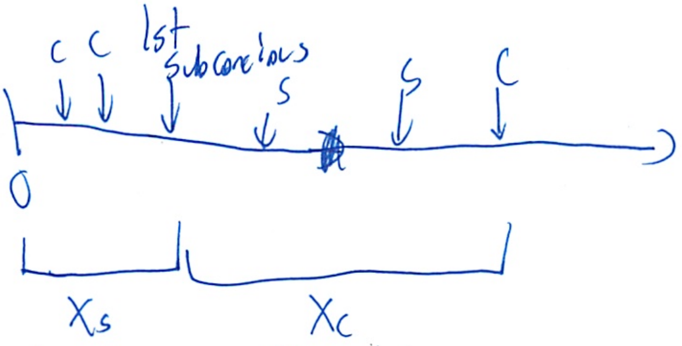
independent so

$$= \underbrace{P(r \text{ conc ans within } \tau)}_{\text{poisson}} \cdot \underbrace{P(s \text{ subconc ans in } \tau)}_{\text{poisson}}$$

$$= \frac{(\lambda_c \cdot \tau)^r e^{-\lambda_c \tau}}{r!} \cdot \frac{(\lambda_s \cdot \tau)^s e^{-\lambda_s \tau}}{s!}$$

⑥

d)



$X = X_s + X_c$ $= \underbrace{\exp \lambda_s + \exp \lambda_c}_{\text{Independent}}$

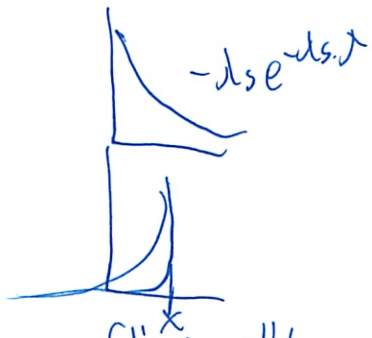
$f_X(x) = ?$

but not erlang since $\lambda_s \neq \lambda_c$ (order? above erlang)

if were same parameter

Could do - Convolution
- derived distribution
- have function of 2 dist
- ~~can~~ want dist of

Convolution



flip + slide by amt x

cross multiply over non 0 area

prof "I hate Convolution"

$$\int_0^x \lambda_s \lambda_c \, e^{-(x-t)} e^{-t} \, dt$$

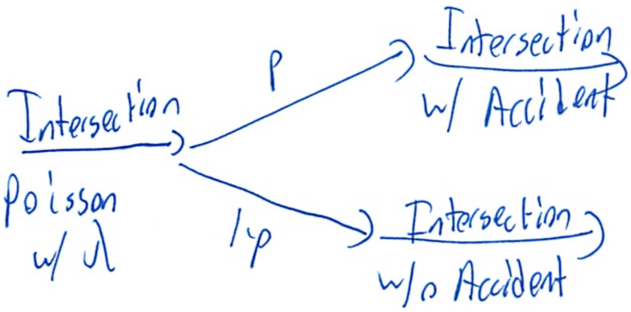
$$= \frac{\lambda_s \lambda_c}{\lambda_s - \lambda_c} \left(e^{-\lambda_c x} - e^{-\lambda_s x} \right)$$

⑦ If they were the same it would give you erlang
$\lambda_s = \lambda_c$

---

try problems
- inc tutorial + books

\# 2 Policeman traveling intersection to intersection
at each intersection could find accident or not



a) N = # intersections until 1st accident
$p_N(n) = ?$
Geometric w/ param p

b) Q = Time b/w sucessive accidents
$f_Q(q) = ?$
exponential, parameter λp

(8)

c) M= # of accidents he sees in 2hrs

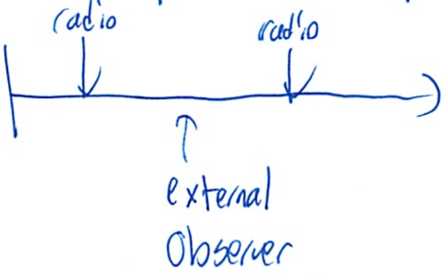
$p_M(m) = ?$

- poisson w/ param $(p \cdot \lambda) \cdot 2$

e) ↓

- gets radio calls, independent from accidents

↳ poisson w/ rate μ



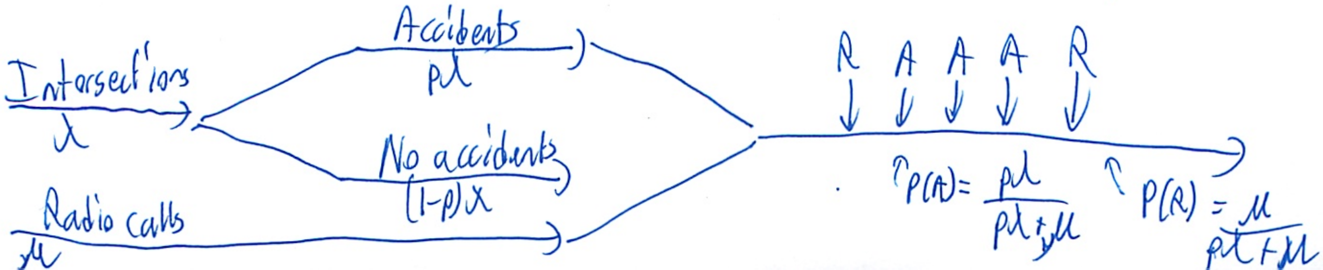
- calls and asks how long since last radio
- and please call me back when next call

So $f_W(w)$ = Erlang of order 2 w/ param μ

{back to d

d) Now radio calls and accidents

poisson $p\lambda + \mu$



⑨

k = # of accidents b/w 2 sucessive radio calls ← ie # of trials to first R (aka heads)

~~$p_K(k) =$~~

$$P(A) = \text{\# of trials to first R/ } \underbrace{\text{heads} - 1}_{\text{shifted geometric}}$$

since 1st trial is R ~~heads~~ — # of accidents not # of trials

$$p_K(k) = \left(\frac{p\lambda}{p\lambda + \mu}\right)^k \cdot \frac{\mu}{p\lambda + \mu}$$

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Fall 2010)

Tutorial 8
November 4/5, 2010

1. Type A, B, and C items are placed in a common buffer, each type arriving as part of an independent Poisson process with average arrival rates, respectively, of a, b, and c items per minute. For the first four parts of this problem, assume the buffer is discharged immediately whenever it contains a total of ten items.

   (a) What is the probability that, of the first ten items to arrive at the buffer, only the first and one other are type A?

   (b) What is the probability that any particular discharge of the buffer contains five times as many type A items as type B items?

   (c) Determine the PDF, expectation, and variance for the total time between consecutive discharges of the buffer.

   (d) Determine the probability that exactly two of each of the three item types arrive at the buffer input during any particular five minute interval.

2. A store opens at $t = 0$ and *potential* customers arrive in a Poisson manner at an average arrival rate of λ potential customers per hour. As long as the store is open, and independently of all other events, each particular potential customer becomes an *actual* customer with probability p. The store closes as soon as ten actual customers have arrived.

   (a) What is the probability that exactly three of the first five potential customers become actual customers?

   (b) What is the probability that the fifth potential customer to arrive becomes the third actual customer?

   (c) What is the PDF and expected value for L, the duration of the interval from store opening to store closing?

   (d) Given only that exactly three of the first five potential customers became actual customers, what is the conditional expected value of the *total* time the store is open?

   (e) Considering only customers arriving between $t = 0$ and the closing of the store, what is the probability that no two *actual* customers arrive within τ time units of each other?

3. Problem 6.24, page 335 in text.

   Consider a Poisson process with parameter λ, and an independent random variable T, which is exponential with parameter ν. Find the PMF of the number of Poisson arrivals during the time interval $[0, T]$.

Tutorial 8

11/5

Poisson Process (λ)

λ arrival rate

N_τ = # of arrivals in τ units of time

$$P_{N_\tau}(k) = \frac{\overset{(\lambda\tau}{(\lambda \cancel{t})^k} e^{-\lambda\tau}}{k!} \qquad k = 0, 1, \ldots$$

X = Interarrival times

exponential(λ)

$$f_X(x) = \lambda e^{-\lambda t} \qquad t \geq 0 \qquad F_X(t) = P(X \leq t) = 1 - e^{-\lambda t}$$

$$E[X] = \frac{1}{\lambda} \qquad var(X) = \frac{1}{\lambda^2} \qquad P(X > t) = e^{-\lambda t}$$

Y_k = Time until kth arrival

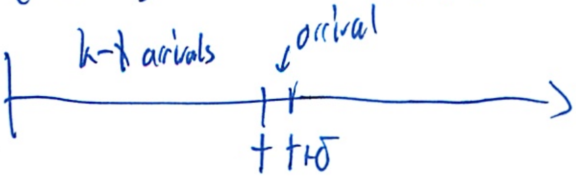
$Y_k = X_1 + X_2 + \ldots + X_k$

could convolve

erlang (order k)

approximate

$$P(Y_k = t) \approx f_{Y_k}(t)\delta = \lambda\delta \frac{(\lambda t)^{k-1} e^{-\lambda t}}{(k-1)!}$$



(δ so small it drops out)

②

$E[Y_k] = \frac{k}{\lambda}$ sum of expectations
linearity of expectation

$Var(Y_k) = \frac{k}{\lambda^2}$

1, a) Time of arrivals continuous
but arrivals still discrete

Given that there is an arrival

$P(A) = \frac{a}{a+b+c} = p$ (weighted average)
^
given that
arrival
$P(A \mid \text{arrival})$

$p \cdot \underbrace{\binom{9}{1} p (1-p)^8}_{\text{1 of next 9 is a Bernoulli} \Rightarrow \text{binomial}},$

b) P(discharge contains 5x as many type A or B)
Find all disjoint event that lead to this
5A, 1B, 4C / 0A, 0B, 10C
$\therefore = P(5A, 1B, 4C) + P(0A, 0B, 10C)$ add since disjoint

③

= binomial
p of common sequence $+ \left(\frac{c}{a+b+c}\right)^{10}$
-forget about order

$$= \frac{a^5 b^1 c^4}{(a+b+c)^{10}} + \frac{c^{10}}{(a+b+c)^{10}}$$

can rearrange
10! ways
but swapping as
does not matter

So $\frac{10!}{5!\,1!\,4!} = \binom{10}{5,1,4}$

$$= \frac{10!}{5!\,1!\,4!} \frac{a^5 b^1 c^4}{(a+b+c)^{10}} + \frac{c^{10}}{(a+b+c)^{10}}$$

---

c) PDF time b/w consecutive discharges
- waiting for 10 arrivals

$k=10$

10 arrivals $\lambda = a+b+c$

Z = time b/w consecutive discharges

Erlang of $k=10$ $\lambda = a+b+c$

$E[Z] = \frac{10}{a+b+c}$ $\quad var(Z) = \frac{10}{(a+b+c)^2}$

④

d) P[2 of each of 3 item types arrives during any 5 min interval)

↳ need 2A, 2B, 2C

processes are independent

P(2A, 2B, 2C in 5min) =

= P(2A in 5 min) · P(2B in 5 min) · P(2C in 5 min)

↑ poisson PMF
tells us p of k arrivals in τ units of time
(make sure units correct)

$$= \frac{(5a)^2 e^{-5a}}{2!} \cdot \frac{(5b)^2 e^{-5b}}{2!} \cdot \frac{(5c)^2 e^{-5c}}{2!}$$

---

2. a. just a binomial

$\binom{5}{3} p^3 (1-p)^2$ = p that they become an actual customer

b) P(5th potential customer becomes 3 actual buyer)

= P(2 out of the previous 4) · P(5th becomes cust)

$\binom{4}{2} p^2 (1-p)^2 \quad \cdot \quad p$

5

c) What is PDF + $E[\,]$ for L

L = duration store opening to closing when 10 th actual customer

← *key is to be able to convert these

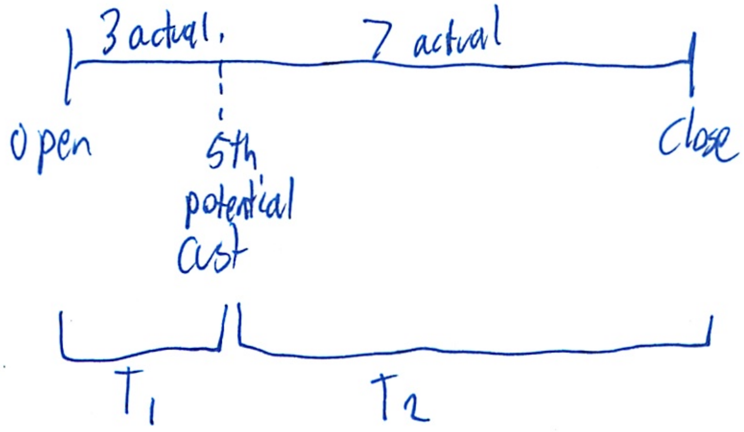
- split
- sum of 10 interarrival times
- erlang order $k=10$ $\lambda' = \lambda p$

$E[L] = \frac{10}{\lambda p}$ $Var(L) = \frac{10}{(\lambda p)^2}$

for $10 > 0$ need bounds

sidebar: think through exam
$E[\,]$ can't be −

d) Given 3 of 5 cus become actual cust what is conditional $E[\,]$ of total time store is open



$E[T] = E[T_1] + E[T_2]$

= erlang order 5

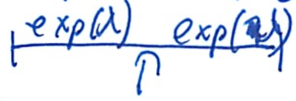
⑥

$$= \frac{5}{\lambda} + \frac{7}{\lambda p}$$

↑ here is split taken into account

e) Considering cust arriving b/w $t=0$ + Closing

What is p that no 2 actual cust arrive within τ units of time.



- time b/w cust $\sim \exp(\lambda p) = A_i$

$P(A_i > \tau) =$ look at exponential CDF

$$= e^{-\lambda p \tau}$$

$1 \to 2 > \tau$

$2 \to 3 > \tau$

$3 \to 4 > \tau$

⋮ etc

$P(A_1 > \tau, A_2 > \tau, A_3 > \tau, \ldots, A_9 > \tau)$

independent

multiply together

$(e^{-\lambda p \tau})^9$

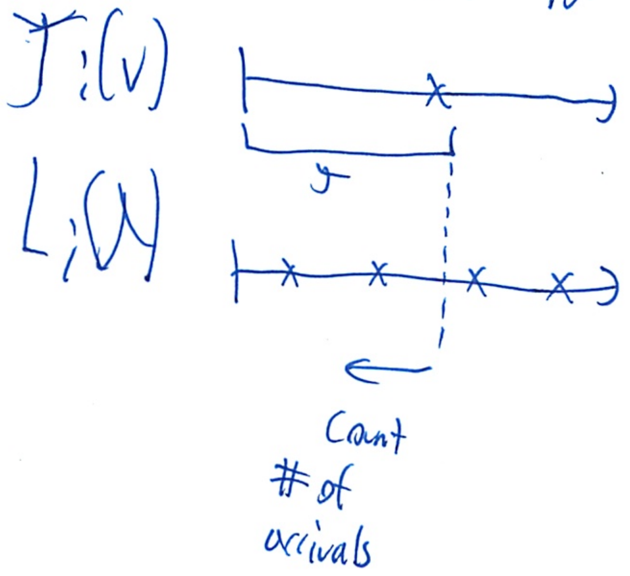
3. Not a ~~Random~~ <u>Sum</u> of Random # of RV

<u>interval</u> is a RV

split poisson?

don't think about $T \sim exp$

but as time until 1st arrival (which is $\sim exp$)



(tricky, clever)

need $P_N(n)$

Think of merged Poisson

- arrival from T or L

$N = \#$ of arrivals $[0, T]$

$P_N(n) = \{n$ "L" arrivals, 1 "T" arrival

$= \left(\frac{\lambda}{\lambda+v}\right)^n \left(\frac{v}{\lambda+v}\right), n = 0, 1, \ldots$

⑧

Consider $\overset{\uparrow}{p}$ $\overset{\uparrow}{1-p}$

Check

$$\sum_{n=0}^{\infty} p^n(1-p) = (1-p)\sum p^n = \frac{1-p}{1-p} = \underline{1}$$

so valid PDF

Random Incidence Paradox

11/7

Reading

Sum of two independent exponential RV (param λ)

$= $ erlang order 2

$$f_{Y_2}(y) = \frac{\lambda^2 y^{2-1} e^{-\lambda y}}{(2-1)!} \qquad y \geq 0$$

$$E[Y_2] = \frac{2}{\lambda}$$

$$\mathrm{Var}(Y_2) = \frac{2}{\lambda^2}$$

Oh

- inter arrival time → erlang k
  - ↳ poisson is erlang order 1
- interarrival time of random incidence → erlang order k+1

Michael Plasmeier 3.5+1.5+3 8/10 11/8

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Fall 2010)

Do you want it stapled or not!?!?
Too thick to staple

Binder clip is good.

Problem Set 7
Due November 8, 2010

#1, 3 and 5 most confused
#2,4 got about
#6 think got

1. Consider a sequence of mutually independent, identically distributed, probabilistic trials. Any particular trial results in either a success (with probability p) or a failure.

   (a) Obtain a simple expression for the probability that the ith success occurs before the jth failure. You may leave your answer in the form of a summation.

   (b) Determine the expected value and variance of the number of successes which occur before the jth failure.

   (c) Let L_{17} be described by a Pascal PMF of order 17. Find the numerical values of a and b in the following equation. Explain your work.

$$\sum_{l=42}^{\infty} p_{L_{17}}(l) = \sum_{x=0}^{a} \binom{b}{x} p^x (1-p)^{(b-x)}$$

2. Fred is giving out samples of dog food. He makes calls door to door, but he leaves a sample (one can) only on those calls for which the door is answered *and* a dog is in residence. On any call the probability of the door being answered is 3/4, and the probability that any household has a dog is 2/3. Assume that the events "Door answered" and "A dog lives here" are independent and also that the outcomes of all calls are independent.

   (a) Determine the probability that Fred gives away his first sample on his third call.

   (b) Given that he has given away exactly four samples on his first eight calls, determine the conditional probability that Fred will give away his fifth sample on his eleventh call.

   (c) Determine the probability that he gives away his second sample on his fifth call.

   (d) Given that he did not give away his second sample on his second call, determine the conditional probability that he will leave his second sample on his fifth call.

   (e) We will say that Fred "needs a new supply" immediately *after* the call on which he gives away his last can. If he starts out with two cans, determine the probability that he completes at least five calls before he needs a new supply.

   (f) If he starts out with exactly m cans, determine the expected value and variance of D_m, the number of homes with dogs which he passes up (because of no answer) before he needs a new supply.

3. Let T_1 and T_2 be exponential random variables with parameter λ, and let S be an exponential random variable with parameter μ. We assume that all three of these random variables are independent. Derive an expression for the expected value of $\min\{T_1+T_2, S\}$. *Hint:* See Problem 6.19 in the text.

4. A single dot is placed on a very long length of yarn at the textile mill. The yarn is then cut into lengths requested by different customers. The lengths are independent of each other, but all distributed according to the PDF $f_L(\ell)$. Let R be be the length of yarn purchased by that customer whose purchase included the dot. Determine the expected value of R in the following cases:

(a) $f_L(\ell) = \lambda e^{-\lambda\ell}, \quad \ell \geq 0$
(b) $f_L(\ell) = \frac{\lambda^3\ell^2e^{-\lambda\ell}}{2}, \quad \ell \geq 0$
(c) $f_L(\ell) = \ell e^{\ell}, \quad 0 \leq \ell \leq 1$

5. Consider a Poisson process of rate λ. Let random variable N be the number of arrivals in $(0, t]$ and M be the number of arrivals in $(0, t+s]$, where $t, s \geq 0$.

   (a) Find the conditional PMF of M given N, $p_{M|N}(m|n)$, for $m \geq n$.
   (b) Find the joint PMF of N and M, $p_{N,M}(n, m)$.
   (c) Find the conditional PMF of N given M, $p_{N|M}(n|m)$, for $n \leq m$, using your answer to part (b).
   (d) Rederive your answer to part (c) without using part (b). As a hint, consider what kind of distribution the k^{th} arrival time has if we are given the event $\{M = m\}$, where $k \leq m$.
   (e) Find $E[NM]$.

6. The interarrival times for cars passing a checkpoint are independent random variables with PDF

$$f_T(t) = \begin{cases} 2e^{-2t}, & \text{for } t > 0 \\ 0, & \text{otherwise.} \end{cases}$$

   where the interarrival times are measured in minutes. The successive experimental values of the durations of these interarrival times are recorded on small computer cards. The recording operation occupies a negligible time period following each arrival. Each card has space for three entries. As soon as a card is filled, it is replaced by the next card.

   (a) Determine the mean and the third moment of the interarrival times.
   (b) Given that no car has arrived in the last four minutes, determine the PMF for random variable K, the number of cars to arrive in the next six minutes.
   (c) Determine the PDF and the expected value for the total time required to use up the first dozen computer cards.
   (d) Consider the following two experiments:
      i. Pick a card at random from a group of completed cards and note the total time, Y, the card was in service. Find $\mathbf{E}[Y]$ and $\text{var}(Y)$.
      ii. Come to the corner at a random time. When the card in use at the time of your arrival is completed, note the total time it was in service (the time from the start of its service to its completion). Call this time W. Determine $\mathbf{E}[W]$ and $\text{var}(W)$.

G1†. Consider a Poisson process with rate λ, and let $N(G_i)$ denote the number of arrivals of the process during an interval $G_i = (t_i, t_i + c_i]$. Suppose we have n such intervals, $i = 1, 2, \cdots, n$, mutually disjoint. Denote the union of these intervals by G, and their total length by $c = c_1 + c_2 + \cdots + c_n$. Given $k_i \geq 0$ and with $k = k_1 + k_2 + \cdots + k_n$, determine

Commas = intersects

$$\mathbf{P}\Big(N(G_1) = k_1, N(G_2) = k_2, \ldots, N(G_n) = k_n \,|\, N(G) = k\Big).$$

multinomial probability $\overbrace{\binom{n}{k_1, k_2, k_3}()()(}$

partitioning

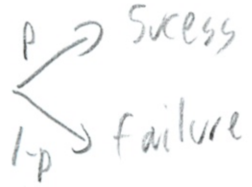
---

†Required for 6.431; optional for 6.041

6.041 P-Set 7

11/8

1. Sequence oin RV



? Bernalli trials ? - mentions summation

a) Obtain a simple expression that P(ith sucess occurs before jth failure)

- So the trick on these problems is to interpert the question into one of the 3 formats we know

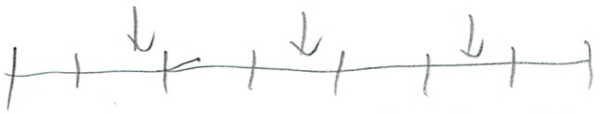
P(ith sucess) > P(jth failure)

? well that is underlying assumption

So want time until jth failure then find # successes inside ?

P that this is >< i ?

Not splitting!



X	0	1	1	2	2	3	3	= # sucesses
Y	1	1	2	2	3	3	4	= # failures

So at Expected value of jth failure, the X ?

remember discrete:

pascal: time to kth arrival

Y_k = amt of trials to kth sucess

② :

$$P(Y_{1k} = t) = P(X_1 + \dots + X_{t-1} = k-1 \text{ and } X_t = 1)$$

$$= P(X_1 + \dots + X_{t-1} = k-1)\,P(X_t = 1)$$

$$= P\binom{t-1}{k-1} p^{k-1} (1-p)^{(t-1)-(k-1)}$$

$$= \binom{t-1}{k-1} p^k (1-p)^{t-k}$$

the distribution of times to kth sucess

so if allso had distribution of jth Failure

somehow compare (CDF?) prob of one coming before the other

it must be a lot simpler than that

A First <u>event</u> would think of

For T_{i+j} - specific i, j

Then how event can come true

1. ith sucess before jth failure
2. ith " " j-1 failures ↓ continue

... One way ; if have i sucesses before jth failure

- binomial
- in first $\underbrace{i + j - 1}$ trials
  - have i sucesses
  - j-1 failures
- limiting way

②b:

Can we have more than i sucesses before jth failure

$i+1$ sucesses

j failure

Other condidition (extream) union of all of those events

Sum of finite terms

every event satisfies 1, 2, 3

(oh just confusing me more)

iterate over

Sum over

So we have 1, ith sucess before jth failure

2. ith " " $j-1$
3. i $j-2$

... ...

$i+1$. i $j-1$

Σ of those probabilities $P(i \leq j)$

or event $P(i \mid \text{event } 1)$

$$\sum_{n=0}^{i+1} P(i \mid \text{event } n)$$

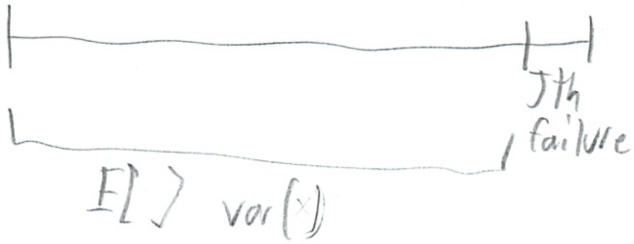
?? is that it?

③ :

b. Find E[ ] var( ) of # sucesses before jth failure

- So want time to jth failure
  - erlang order j wanting p-1

Remember discrete



# of sucesses in n time -> binomial

$P(S=k)=\binom{n}{k}p^k(1-p)^{n-k}$

$E[S] = np$

$var(S) = np(1-p)$

in this case n = what?

-expected time to jth failure -> pascal

$$E[Y_k] = \underset{\underset{j}{\uparrow}}{k} \cdot \underset{\underset{1-p}{\uparrow}}{\frac{1}{p}}$$

So together ?

$E[\#$ sucesses before jth failure$) = j \cdot \frac{1}{p-1} \cdot p$

$var($ " $) = j \cdot \frac{1}{p-1} p(1-p)$

(3b):

Say X sucesses before failure j

ith Failure

$E[X]$ var(X)

or more familar

Y = # trials until jth failure

↳not before

$X = Y - j$

now "easy" to find

Y_j is pascall of order j ← Sum of j iid geometric RVs $T_1\ T_2\ T_3 \dots$

prob failure = 1 − prob sucess

$$Y_j = \underbrace{T_1}_{\text{to 1st failure}} + \underbrace{T_2}_{\text{to 2nd failure}} + \dots + T_j \quad] \text{ inter arrival times}$$

normally can't multipy E[ ] by # of times it happens

④

C. L_{17} = Pascal $^{\text{PMF}}$ of order 17. Find a, b

$$\sum_{l=42}^{\infty} p L_{17}(l) = \sum_{x=0}^{a} \binom{b}{x} p^{x} (1-p)^{(b-x)}$$

So this qu is trying to get us to think about what a Pascal is.

It is the sum of interarrival times, which are geometric

Interarrival time is $(1-p)^{t-1}(p)$

they swap

$p\,(1-p)^{t-1}$

and have X I guess instead of 1

$p^{X}(1-p)^{b-X}$

So $b=t$; $\binom{t}{x} p^{x} (1-p)^{(t-x)}$

Makes sense.

Other/substiquent arrival times $T_k = Y_k - Y_{k-1}$

So we are summing those from 0 to a

↑ 17

But what does the 42 mean? Summing all of the pascals from 42 to ∞. Only thing I could think of is it must be t. But what is t anyway?

⑤ The time of the arrival - but 42 does not seem to fit in - although it is the parameter of the pascal

Oh - l is t, Textbook: $p_{Y_k}(t) = \binom{t-1}{k-1} p^k (1-p)^{t-k}$

$t = k, k+1, \ldots$

So $a = 17$

$b = t = 42 \to \infty$

~~Sum~~ So summing the pascalls

Still kinda confused

Oh

Pascal 17 - Prob time of 17th arrival ?

$P_{L_{17}}(l) = \text{Prob}(L_{17} = l)$

↑ time of 17th arrival

$\sum$ left

prob of 17th arrival

~~prob 42nd arrival~~

↳ occured time 42 or later

# of arrivals in b time intervals

↑ prob of x arrivals in b trials

(5b):

~~No arrivals in first 42~~

17th arrival does not occur before time 42 ①

(this OH confused me)

event 17th arrival occured after time 42

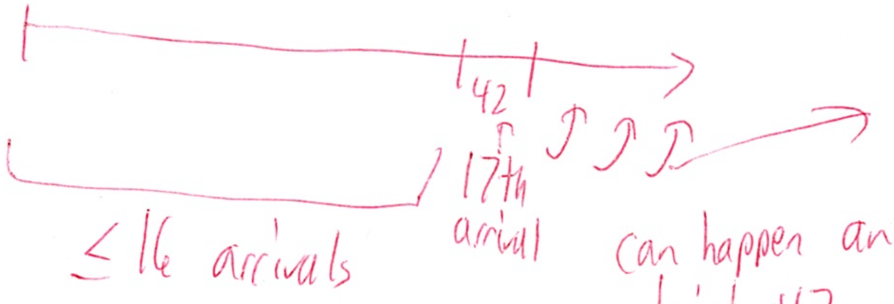
0 arrivals in 1st 100 trials satisfies ↑ certainly true

20 " " " " don't know if satisfies

0 " " 50 " " satisfies

how many arrivals can have

must have < 17 arrivals in time > 42) really satisfies



can happen anywhere trial 42 or later

think about pascal + binomial PMF

can extend to continuous erlang + Poisson PDF

I just don't get #1

⑥ :

2. Fred is giving at samples of dog food.

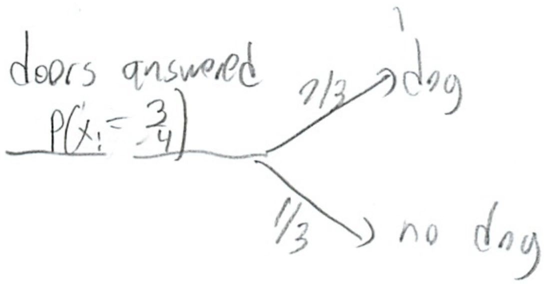
Calls door to door

leaves sample when door is answered and is doog

$P(\text{door answered}) = \frac{3}{4}$) ind.

$P(\text{dog}) = \frac{2}{3}$

So multisplitting Bernoulli



a) P(Fred gives away 1st sample on 3rd call) =

~~P(miss~~ does a call mean — knock on door ? ← will assume door answered

P(miss)P(miss) P(ans ∩ dog)

$$\frac{1}{4} \cdot \frac{1}{4} \cdot \left(\frac{3}{4} \cdot \frac{2}{3}\right) = \frac{1}{32}$$

↑ ↑
don't care doggie or not

no! See p8

$$\frac{1}{2} \cdot \frac{1}{2} \cdot \left(\frac{3}{4} \cdot \frac{2}{3}\right) = \frac{1}{8}$$

+0.5

finally qu I think I get

⑦ :

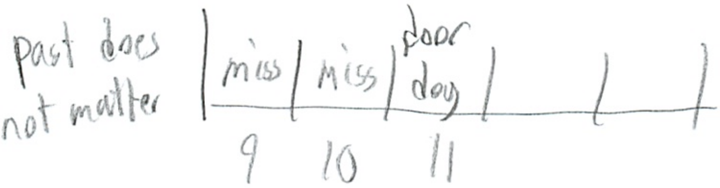
b) P(5th sample on 11th call | 4 samples on 8 calls) =

So can ~~tha~~ do this ways, right? $P(A|B) = \frac{P(A \cap B)}{P(B)}$

or just assume B happened and figure out.

? this should work here

So P



So the same as the previous question??

$$\frac{1}{4} \cdot \frac{1}{4} \cdot \left(\frac{3}{4} \cdot \frac{2}{3}\right) = \frac{1}{32}$$

↳ $\frac{1}{2} \cdot \frac{1}{2} \cdot \left(\frac{3}{4} \cdot \frac{2}{3}\right) = \frac{1}{8}$

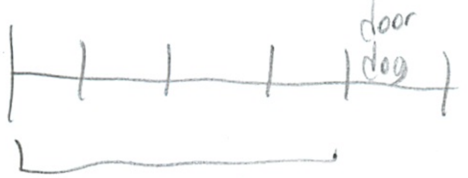
no
Since miss has broad def
see p8

+0.5

⑧ :

C. P(2nd sample 5th call)

(door means door opened)



3 misses

1 dog

$$\binom{4}{1}\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\cdot\frac{2}{3}\right)\cdot\left(\frac{3}{4}\cdot\frac{2}{3}\right)$$

But did I make a wrong assumption before?

failure $\Rightarrow \frac{1}{4}$ no door

↘ $\frac{3}{4}\cdot\frac{1}{3}$ door, no dog

Or am I "splitting" and only looking at doors w/ dogs

Now $p(\text{hit}) = pq = \frac{3}{4}\cdot\frac{2}{3}$

~~$p(\text{miss}) = p(1-q)$~~ that is door no dog miss

Our $p(\text{miss}) = 1 - \left(\frac{3}{4}\cdot\frac{2}{3}\right) = 1 - \frac{1}{2} = \frac{1}{2}$

So

$$= \underbrace{\binom{4}{1}\left(\frac{1}{2}\right)^3\left(\frac{3}{4}\cdot\frac{2}{3}\right)}_{\text{1 hit in 4 tries}}\cdot\underbrace{\left(\frac{3}{4}\cdot\frac{2}{3}\right)}_{\text{hit on 5th try}}$$

$$= 4\left(\frac{1}{2}\right)^5 = \frac{4}{64} = \frac{1}{8}$$ +0.5

(9)

d. P(2nd sample 5th call | did not give 2nd sample on 2nd call)

- So same as before
- except know 2nd call was a miss
  ↳ or! he gave away 1st sample on 2nd call

- So use $P(A|B) = \frac{P(A \cap B)}{P(B)}$

$P(B) = 1 - P(B)^c$

$P(B)^c = P($ gave 2nd sample 2nd call $)$

=

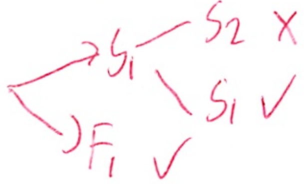

$= \frac{3}{4} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{2}{3}$

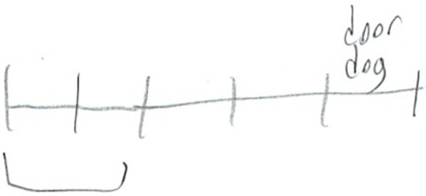
$= \frac{1}{2} \cdot \frac{1}{2}$

$= \frac{1}{4}$

$P(B) = 1 - \frac{1}{4} = \frac{3}{4}$

eiter 1st failure
1st sucess 2nd failure



$P(A \cap B) =$


or (miss miss $\frac{1}{4}$
miss hit $\frac{1}{4}$
hit mis $\frac{1}{4}$) $\frac{3}{4}$ right - established this already - but how related to other

I like these type of problems

⑩ :

Oh wait if he gave his 2nd sample on 2nd call then he could not give it on 5th call!

So condition is worthless, right?

Ans sames as before

3 b/c can't give on 2nd call

$$\binom{\overset{3}{\cancel{4}}}{1}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)$$

But what would it have been if $P(\text{3rd sample } \underset{\text{try}}{\text{5th}} \mid \underset{\text{sample}}{\text{2nd}} \text{ not } \underset{\text{try}}{\text{2nd}})$

What were $P(A \cap B^c)$ rules again?

$P(A \cap B)$ <u>not</u> independent

If A happend → B always happens

$P(A \cap B) = P(A)$

So $$\frac{P(A)}{P(B)} = \frac{\text{previous answer}}{3/4} = \frac{\binom{4}{1}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)}{3/4} = \frac{\frac{1}{8}}{3/4}$$

$$= \frac{1}{8} \cdot \frac{4}{3} = \frac{1}{6}$$

~~4~~ +0.5

(11) :

e.) Fred needs new supply after he gives away last can.

Starts w/ 2 cans, P(complete $\overset{\text{at least}}{5}$ calls)

So P(less than 2 hits in 5 calls) =

P(1 hit 5 calls) + P(2 hits 5 calls)

$$\binom{5}{1}\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right) + \binom{5}{2}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^2$$

<u>or</u> $P(\underset{\substack{\uparrow\\ T_1+T_2}}{Y_2} \geq 5)$

but how do you expand this again

- like the 1st problem I think

-summation?

$$\sum_{t=1}^{4} P(Y_2 = t)$$

$$= \sum_{t=1}^{4} \binom{t-1}{2-1} \underset{\substack{\uparrow\\ \frac{1}{2}}}{p}^{2} \underset{\substack{\uparrow\\ \frac{1}{2}}}{(1-p)}^{t-2}$$

$$= \sum_{t=1}^{4} \binom{t-1}{2-1} \frac{1}{2}^{2}\left(1-\frac{1}{2}\right)^{t-2}$$

does that match 1c ?

?? confused

⑪b

Same as previous problem

P(2nd sucess on or after 5th trial)

$$P(Y_2 \geq 5)$$

$$= 1 - P(Y_2 = 2) - P(Y_2 = 3) - P(Y_2 = 4)$$

$$= 1 - p_{Y2}(2) - p_{Y3}(3) - p_{Y2}(4)$$

$$= 1 - \binom{1}{1}\left(\tfrac{1}{2}\right)^2 - \binom{2}{1}\left(\tfrac{1}{2}\right)^3 - \binom{3}{1}\left(\tfrac{1}{2}\right)^4$$

$$= 1 - \frac{1}{4} - \frac{2}{8} - \frac{3}{16}$$

$$= \boxed{\frac{5}{16}}$$

Oh that matches what I had almost

-except without the second 1/2

+1

Well this is subtraction

¿ same ¿¿

⑫:

f. If he starts w/ M cars - find D_m → # of homes w/ dogs he passes up before needs new supply.

So for now assume M=2 (what is notation again)?

We want $E[D_2]$ D_2 = # of no door dog

$$\frac{1}{4} \cdot \frac{2}{3} = \frac{1}{6}$$

So what happens

$\frac{1}{4} \cdot \frac{1}{3} = \frac{1}{12}$ no door no dog

$\frac{1}{4} \cdot \frac{2}{3} = \frac{1}{6}$ no door dog ← care about

$\frac{3}{4} \cdot \frac{1}{3} = \frac{1}{4}$ door no dog

$\frac{3}{4} \cdot \frac{2}{3} = \frac{1}{2}$ door dog ← drops a sample

; so he leaves → a sample every other knock

So $E[\text{knocks}] = 2m$

and $\frac{1}{6}$ of knocks are no door dog

So $\frac{1}{6} \cdot 2m = \frac{1}{3} m$

$E[D_m] = \frac{m}{3}$ ← that is expects to add 1 to D every 3 doors we want to know how many doors he poasses up ... no think that is right

+0.5

but how in all world do you do that formally?

⑬.

And now need var ?

?? Need the to do formal way

time to the kth sucess -> pascal

sucess = ?, what - running out

- ~~par~~ ~~A~~no door, dog

but up to max of m tries

$$E[Y_k] = k \cdot \frac{1}{p}$$

$\uparrow$ runs out?

$\uparrow$ p no door dog

$E[k(m)]$ = what ?

Previous answer

well informally 2m

$2m \cdot \frac{1}{1/6}$ ← Close to what I had

$= 12m$

Oh that must control for me asking wrong qu

- want how many doors have to knock on

- ~~an~~ had the right instict

⑭

So now var

$$\text{var}() = k \cdot \frac{1-p}{p^2}$$

$$2m \cdot \frac{1-\frac{1}{6}}{\left(\frac{1}{6}\right)^2}$$

$$2m \cdot \frac{5}{216}$$

$$\frac{5m}{108}$$

?, seems weird

(−0.5)
See solution.

3.5/4

(1.5):

3. Let T_1, T_2 be ~ $\exp(\lambda)$ RV
$S \sim \exp(\mu)$) all ind

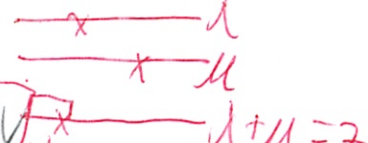
Want $E[\min\{T_1+T_2, S\}]$

Hint: like problem 6.19

- exp means continuous
- Seems like Recitation 15.1

T_1+T_2 = erlang order 2
↳ So what does that mean
- just merging?

6.19
- So same except max and no S
- direct but tedious is to find PDF of RV and integrate to find expectation
- easier to simply interpert RV in terms of underlying Poisson process
- merged poisson process $E[T] = \frac{1}{\lambda_1+\lambda_2}$

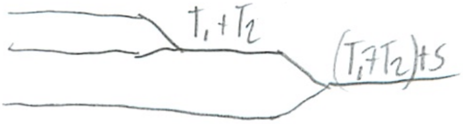


min (T_1, S)

$E[T] = \frac{1}{\lambda+\mu}$ time ind of which process it came fr

+0.5

So our problem $\frac{1}{\lambda+\lambda+\mu}$



So basically as soon as we get something in any one of the processes. Merge them and look for 1st one

Seemed too easy - want min T_1+T_2

15b:

$$A = \left\{ \begin{matrix} \text{It came from} \\ \lambda \text{ process} \end{matrix} \right\}$$

$$= \{ T_1 < S \}$$

$$E[Z] = P(A)\,E[Z|A] + P(A^c)\,E[Z|A^c]$$

$$= \frac{\lambda}{\lambda+\mu} \left(\quad \right) + \dots$$

$\uparrow$

not $\frac{1}{\lambda}$

$$E[\min t_1, S \mid T_1 < S]$$

$$= E[T_1 \mid T_1 < S]$$

$$\neq E[T_1]$$

$$= \frac{\lambda}{\lambda+\mu}\left(\frac{1}{\lambda+\mu}\right) + \frac{\mu}{\lambda+\mu}\left(\frac{1}{\lambda+\mu}\right)$$

$$= \frac{1}{\lambda+\mu}$$

Not quite.

+1

(−1.5)

1.5/3

See solutions

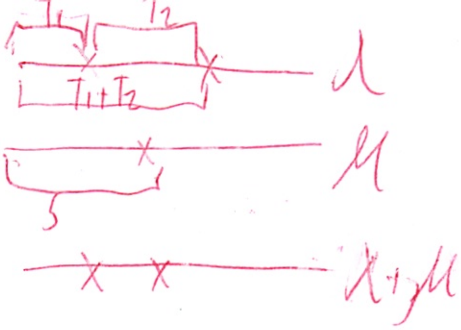
not what will use to solve problem

know that given from T_1, will not affect

will need to look at merged process

- look if came from λ or μ - independent

(15 c)



Remaining time

$\min(T_1+T_2, S)$

^what it is, what you contine adding up will change

= sum of some things

- expectation
  - S → just S
  - T_1 → time until next T_1 or S
- memoryless, start fresh
- waiting for S or T_2
- $\frac{1}{\lambda+\mu}$ each time you start no matter T_1, S

(verbal OH - confusing!)

⑯ :

4. Single dot placed on long length of yarn

Yarn then cut into different lenghts

↳ ind

L PDF $f_L(l)$

R = lenght of yarn purchased by cus w/ dot

~~and~~ Find E[R] when

a) $f_L(l) = \lambda e^{-\lambda t}$ $\quad l \geq 0$

So is this random incidence for Poisson?

-Seems like it since if R is big we are more likly to stumble on it - well that is not what asking

-but R is more likly to be on long string than short string - so will say Random Incidence for Poisson

-each piece l is cut w/ $\lambda e^{-\lambda t}$ (aka ~ exp())

So $E[\underset{\substack{\uparrow \\ \text{all } l \\ \text{distributed} \\ \text{same}}}{l_x}] = \frac{1}{\lambda}$

↑ is that it

Is correct, wrong reasoning see next pg

~~Or was it 2 times this $\frac{1}{\lambda} + \frac{1}{\lambda} =$~~ $\boxed{\frac{2}{\lambda}}$

Think was it $\frac{1}{\lambda}$ before and $\frac{1}{\lambda}$ after dot ↑ but confused why its 2

(16b.) No! erlang of order 2 – went over on diff problem

Actually what I had is right, but wrong reason!

We learned random incidence is erlang k+1

this was poisson – which is erlang order 1

So this is erlang order 2

$$E[l_x] = \frac{2}{\lambda}$$ ← same answer

⑰

b) $\frac{\lambda^3 l^2 e^{-\lambda l}}{2}$ $l \geq 0$

?, what is this →

how find E[ ] of ?

$$E[X] = \int_0^\infty l \cdot \frac{\lambda^3 \lambda^2 e^{-\lambda l}}{2} \, dl$$

$$= \frac{e^{-\lambda l}(\lambda^3 l^3 + 3\lambda^2 l^2 + 6\lambda l + 6)}{2b} + c$$

$$= \frac{3}{b} \text{ if } b > 0$$

and $\frac{3}{b} + \frac{3}{b} = \boxed{\frac{6}{b}}$

Oh is erlang $E[Y_l] = \frac{l}{\lambda}$

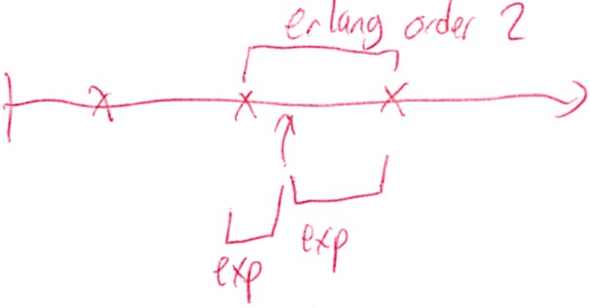
Order 3

$\cdot 2 = \boxed{\frac{2l}{\lambda}}$

See next pg

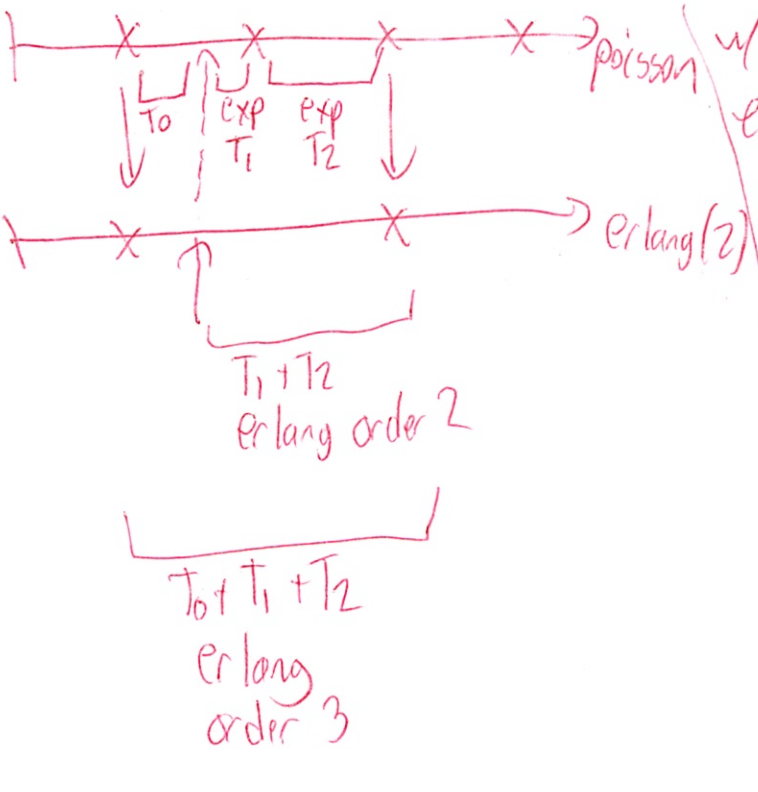
17b

Look at as poisson process



- like recitation problem #3

Interarrival times
~ Erlang(2)



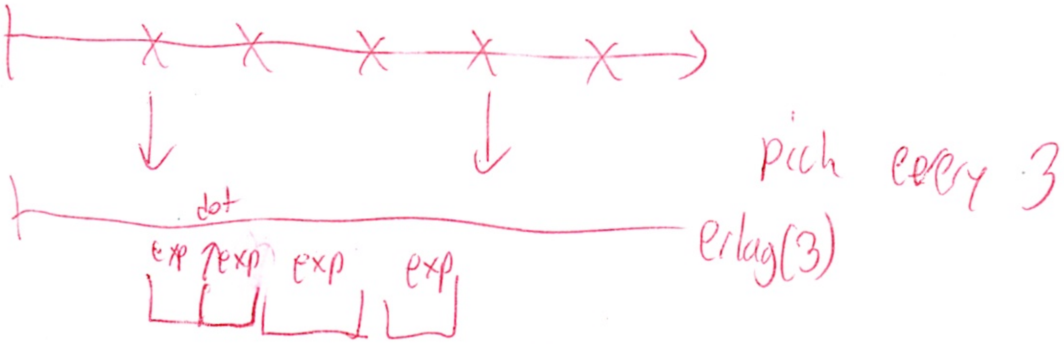
arrials
can always be associated
w/ other poisson process
even + odd
picking
deterministically
not random

← heristic
did not
go over

except in this erlang order 3 so pick every 3rd one

(17c) So just $\frac{3}{\lambda}$? No

But want E[ ] interarrival time w/ dot



So erlang order 4

$\boxed{\frac{4}{\lambda}}$

---

remember exp memoryless

So life left in lightbulb = exponential

When come back = light still expoential

#4c don't need to do

↳ would take like 40 min to explain

(1.8):

c. $f_L(l) = le^l \qquad 0 \le l \le 1$

don't need to do

See email

$E[L] = \int_0^\infty l \cdot le^l \, dl$

$= e^l (l^2 - 2l + 2) + C \Big|_0^\infty$

= does not converge

on right bounds

$= \int_0^1 l \cdot le^l \, dl$

$= e^l (l^2 - 2l + 2) \Big|_0^1$

$= e - 2$

$\approx .7182$

$.7182 + .7182 = 1.436$

⑲

I have not really been using lessons from class

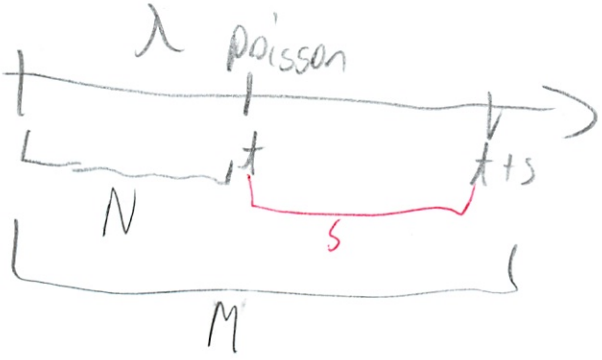
5, Consider poisson process w/ rate λ

$N = \#$ of arrivals in $[0, t]$

$M = \#$ of arrivals in $(0, t+s]$ $t, s \geq 0$

a). Find conditional PMF M given N $p_{M|N}(m|n)$ $m \geq n$

↑ discrete



So we know N → so case about distribution of arrivals from $(0, s)$ since fresh start

PMF of arrivals → binomial

but contineous $\delta \to 0$

$$P(k, \tau) = \frac{(\lambda\tau)^k e^{-\lambda\tau}}{k!} \qquad k = 0, 1, \ldots$$

$M - N = 0$

$k = M - N$ # arrivals

$\tau = s$

here is one from class

⑳

$$P(M-N, s) = \frac{(\lambda s)^{M-N} e^{-\lambda s}}{(M-N)!}$$

↑ can just leave like that

$k = ?$ what
^ Kinda confused

well k is all possible values M-N
but those are RVs!

Now do I need to add constant M or value of N to offset?
^ is a RV

can't just add it at end - since this is prob

$$\underset{??}{P(M-N+N, s)} = P(M, s)$$

No N is given to us!
So n is lower

$$P(M-n, s) = \frac{(\lambda s)^{M-n} e^{-\lambda s}}{(M-n)!}$$

What I did

$= P_{M|N}(m|n)$

$= P(M=m \mid N=n)$

$= P(M$ arrivals in $(0, t+s] \mid N$ arrivals in $(0,t])$

Conditions drops out??

$= P(m-n$ arrivals in $(t, t+s]) \mid n$ arrivals in $(0,t])$

$= P(m-n$ arrivals in $(t, t+s])$

(21):

b) Find joint PMF of N, M $p_{N,M}(n,m)$

~~$\iint_A$~~ continuous

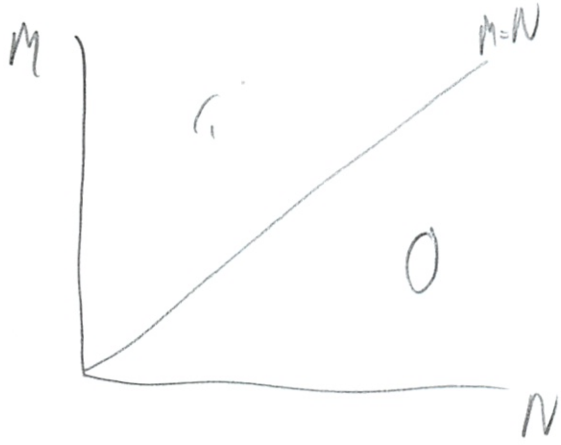
$p_{N,M}(n,m) = P(N=n, M=m)$

poisson PMF
Divide certain things

not ind.

M is ind from N and M-N

don't need table →



Some sort of constant?

is it even evenly distributed?

joint of one · conditional of other

$p_{M,N}(m,n) = p_{M|N}(m|n)\, p_N(n)$ no matter if ind

↑ should know both

↑ don't forget about this trick!

- product only non 0 if both are non 0 at each expression

= ~~$P(M=m \mid N=n)$~~

= ~~P(M arrivals in (0, t+s] | N arrivals in (0, t])~~

(2.1b)

know both parts, just plugin

$$= \frac{(\lambda s)^{M-N} e^{-\lambda s}}{(M-N)!} \frac{(\lambda t)^{N} e^{-\lambda t}}{N!}$$

c) Find the conditional PMF of n given M using ans from b $P_{N|M}(n|m) = ?$

Also Bayes Rule

$$= \frac{P_{M|N}(m|n)\, P_N(N)}{P_M(m)}$$

-but that does not use b

$$\frac{P_N(n)\, P_{M|N}(n|m)}{P_M(m)}$$

$$\uparrow \sum_n P_N(n)\, P_{M|N}(m|n)$$

OH

?? answer looks like some we saw

$P_{N|M}(n|m) =$ look at poisson property $n \le m$

Understand some property it has

like 61

Given any 1 particular arrivals

if $M=N \to S=0$

if arrival inside $0, t+s$

P that inside $0,t$ $= \frac{t}{t+s}$ time ind. property

?1 arrival

(oh I see kinda obvious)

22b

Prob than N arrivals occured here

like chap 2, will be like something you saw before

— like binomial

$\binom{m}{n} (p)^n (1-p)^{m-n}$

└ but how does that help?

Actually now have all the parts for Bayes

$$= \frac{\frac{(\lambda s)^{M-N} e^{-\lambda s}}{(M-N)!} \frac{(\lambda t)^N e^{-\lambda t}}{N!}}{\frac{(\lambda(t+s))^M e^{-\lambda(t+s)}}{M!}}$$

but is that supposed to simplify?

㉓

d) Redaived your answer to c w/o using b

As a hint consider what kind of dist. the kth arrival time has if we are given event {M=m)

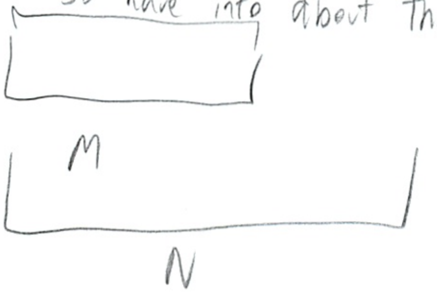
note k≤m

; how does that help?

Ok this is where we are supposed do the simplifed version?

So

So have info about this much of it



- but how docs that help?

㉔

e) Find $E[NM]$

Iterated expectations

$$E[X] = E[E[X|Y]]$$

so how does that help?

㉕

6. The interarrival time for cars passing a checkpoint

$$f_T(t) = \begin{cases} 2e^{-2t} & t>0 \\ 0 & \text{otherwise} \end{cases}$$ measured in min

Sucessive experimental value of durations are recorded on computer cards w/ slots for 3 entries.

a) Determine mean and third moment of inter arrival times

↑ $E[x^3]$

exponential so $E[T] = \frac{1}{\lambda} = \frac{1}{2}$ minute

$$E[T^3] = \int_0^\infty t^3 \cdot 2e^{-2t}\, dt$$

need to solve integral

shortcut

← kinda looks like erlang PDF $\frac{\lambda^k t^{k-1} e^{-\lambda t}}{(k-1)!}$ $t \geq 0$

$$= -\frac{1}{4} e^{-2t}(4t^3 + 6t^2 + 6t + 3)$$

+0.5

$$= \frac{3}{4}$$

↑ So erlang of order 4 with proper constants

$\frac{\lambda^4}{3!}$

note

$e^t = e^t$

$e^{2t} = \frac{e^{2t}}{2}$

$e^{2t+1} = \frac{e^{2t+1}}{2}$

PDF should integrate to 1 then constants

$\frac{2^4}{3!} = \frac{16}{6} = \frac{8}{3}$

$E[Y_k] = \frac{k}{\lambda} \rightarrow \frac{8}{3} \cdot \frac{4}{2} = \frac{16}{3}$ Still not what got

Should $\int$ to PDF or $E[\,]$

↑ could also like $E[\,]$ erlang order 3

(26)

b) Given that no car in last 4 min
Find PMF of k = # of cars in 6 min

- we don't care about the past (memoryless)
- so find $P(k, 6) = \frac{(\lambda \tau)^k e^{-\lambda \tau}}{k!}$

do we know λ - is it $\frac{1}{2}$?
- no we know it is $\underline{2}$

$$= \frac{(2 \cdot 6)^k e^{-2 \cdot 6}}{k!}$$

$$= \frac{12^k e^{-12}}{k!}$$

+0.5

?So can leave like that

Yeah function of k

$0 \le k \le 6$

well k is supposed to be the result

㉗

c) Determine the PDF, E[ ] for total time to use 12 cards

So arrival of $3 \cdot 12 = 36$ cars

~~[illegible]~~

So time of 36th arrival $= Y_{36}$ = Erlang order 36

$$f_{Y_k}(y) = \frac{\lambda^k y^{k-1} e^{-\lambda y}}{(k-1)!} \qquad y \geq 0$$

$\lambda = 2$
$k = 36$
$y = t$

$$= \frac{2^{36} y^{36-1} e^{-2t}}{(36-1)!}$$

$$f_{Y_{36}}(t) = \frac{68719476736\, t^{35} e^{-2t}}{35!}$$

$$E[Y_{36}] = \frac{k}{\lambda} = \frac{36}{2} = 18$$

+0.5

㉘

d) Consider the following experiments.

i) Pick card at random note Y. Find E[Y] var(Y)

So find $Y = T_1 + T_2 + T3 = Y_3$

So erlang order 3

$E[Y_3] = \frac{k}{\lambda} = \frac{3}{2}$

$Var(Y_3) = \frac{k}{\lambda^2} = \frac{3}{2^2} = \frac{3}{4}$

✓

+0.5

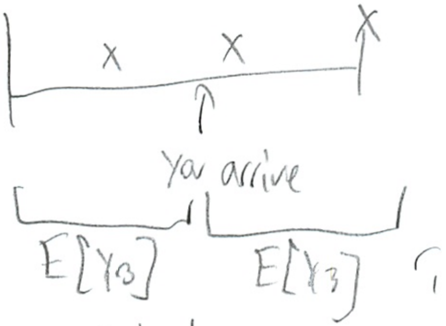
(29)

ii) Come to corner at random time, ← random incidence When the card in use is completed note the total time in service $= W$

$E[W]$ $\mathrm{var}(W)$

So this is a random incidence problem

~~$2E[Y_3]$~~ remember not



but that is not Y_3

poison PMF
divide [illegible]
things?

not [illegible]

- Would it not just be once Y_3 - but no - thats the paradox its ~~$2E[Y_3]$, right~~

$E[Y_3] = \frac{3}{2}$ ← erlang order 3, $\mathrm{Var}(Y_3) = \frac{3}{4}$

+1

~~$2E[Y_3] = 2 \cdot \frac{3}{2} = 3$~~ So

have not learned how Var is affected

3/3

we want erlang order 4 $= \frac{4}{\lambda} = \frac{4}{2} = 2$

Or is random incidence always sum of 2 exponential RV

~~So always sum of erlang order 2~~

So $\mathrm{var} = \frac{\cancel{2}^{4}}{\lambda^2}$ erlang order 4 $= \frac{\cancel{2}^{4}}{4} = \cancel{\tfrac{1}{2}}\ 1$ - ~~So I did rest of P-set wrong - opps clarify in OH~~ that was not it either

(290)

Now poisson — like previous qv

— erlang of order 4

~~Poisson~~

interarrival times → erlang k

" " of random incidence → erlang order k+1

poisson is ~~the~~ erlang 1

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Fall 2010)

Problem Set 7: Solutions

1. (a) The event of the ith success occuring before the jth failure is equivalent to the ith success occurring within the first $(i + j - 1)$ trials (since the ith success must occur no later than the trial right before the jth failure). This is equivalent to event that i or more successes occur in the first $(i + j - 1)$ trials (where we can have, at most, $(i + j - 1)$ successes). Let S_i be the time of the ith success, F_j be the time of the jth failure, and N_k be the number of successes in the first k trials (so N_k is a binomial random variable over k trials). So we have:

$$\mathbf{P}(S_i < F_j) = \mathbf{P}(N_{i+j-1} \geq i) = \sum_{k=i}^{i+j-1} \binom{i+j-1}{k} p^k (1-p)^{i+j-1-k}$$

   (b) Let K be the number of successes which occur before the jth failure, and L be the number of trials to get to the jth failure. L is simply a jth order Pascal, with probability of $1 - p$ (since we are now interested in the failures, not the successes.) Plugging into the formula for jth order Pascal random variable,

$$\mathbf{E}[L] = \frac{j}{1-p}, \sigma_K^2 = \frac{p}{(1-p)^2} j$$

   Since $K = L - j$,

$$\mathbf{E}[K] = \frac{p}{1-p} j, \sigma_K^2 = \frac{p}{(1-p)^2} j$$

   (c) This expression is the same as saying we need at least 42 trials to get the 17th success. Therefore, it can be rephrased as having a maximum of 16 successes in the first 41 trials. Hence $b = 41$, $a = 16$.

2. A successful call occurs with probability $p = \frac{3}{4} \cdot \frac{2}{3} = \frac{1}{2}$.

   (a) Fred will give away his first sample on the third call if the first two calls are failures and the third is a success. Since the trials are independent, the probability of this sequence of events is simply

$$(1-p)(1-p)p = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$$

   (b) The event of interest requires failures on the ninth and tenth trials and a success on the eleventh trial. For a Bernoulli process, the outcomes of these three trials are independent of the results of any other trials and again our answer is

$$(1-p)(1-p)p = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$$

   (c) We desire the probability that L_2, the time to the second arrival is equal to five trials. We know that $p_{L_2}(\ell)$ is a Pascal PMF of order 2, and we have

$$p_{L_2}(5) = \binom{5-1}{2-1} p^2 (1-p)^{5-2} = 4 \cdot \left(\frac{1}{2}\right)^5 = \frac{1}{8}$$

(d) Here we require the conditional probability that the experimental value of L_2 is equal to 5, given that it is greater than 2.

$$\begin{aligned}\mathbf{P}(L_2 = 5 | L_2 > 2) &= \frac{p_{L_2}(5)}{P(L_2 > 2)} = \frac{p_{L_2}(5)}{1 - p_{L_2}(2)} \\ &= \frac{\binom{5-1}{2-1}p^2(1-p)^{5-2}}{1 - \binom{2-1}{2-1}p^2(1-p)^0} = \frac{4 \cdot \left(\frac{1}{2}\right)^5}{1 - \left(\frac{1}{2}\right)^2} = \frac{1}{6}\end{aligned}$$

(e) The probability that Fred will complete at least five calls before he needs a new supply is equal to the probability that the experimental value of L_2 is greater than or equal to 5.

$$\begin{aligned}\mathbf{P}(L_2 \geq 5) = 1 - P(L_2 \leq 4) &= 1 - \sum_{\ell=2}^{4} \binom{\ell - 1}{2 - 1} p^2 (1-p)^{\ell-2} \\ &= 1 - (\tfrac{1}{2})^2 - \binom{2}{1}(\tfrac{1}{2})^3 - \binom{3}{1}(\tfrac{1}{2})^4 = \frac{5}{16}\end{aligned}$$

(f) Let discrete random variable F represent the number of failures before Fred runs out of samples on his mth successful call. Since L_m is the number of trials up to and including the mth success, we have $F = L_m - m$. Given that Fred makes L_m calls before he needs a new supply, we can regard each of the F unsuccessful calls as trials in another Bernoulli process with parameter r, where r is the probability of a success (a disappointed dog) obtained by

$$\begin{aligned} r &= \mathbf{P}(\text{dog lives there} \mid \text{Fred did not leave a sample}) \\ &= \frac{\mathbf{P}(\text{dog lives there AND door not answered})}{1 - \mathbf{P}(\text{giving away a sample})} = \frac{\frac{1}{4} \cdot \frac{2}{3}}{1 - \frac{1}{2}} = \frac{1}{3}\end{aligned}$$

We define X to be a Bernoulli random variable with parameter r. Then, the number of dogs passed up before Fred runs out, D_m, is equal to the sum of F Bernoulli random variables each with parameter $r = \frac{1}{3}$, where F is a random variable. In other words,

$$D_m = X_1 + X_2 + X_3 + \cdots + X_F.$$

Note that D_m is a sum of a random number of independent random variables. Further, F is independent of the X_i's since the X_i's are defined in the conditional universe where the door is not answered, in which case, whether there is a dog or not does not affect the probability of that trial being a failed trial or not. From our results in class, we can calculate its expectation and variance by

$$\begin{aligned}\mathbf{E}[D_m] &= \mathbf{E}[F]\mathbf{E}[X] \\ \text{var}(D_m) &= \mathbf{E}[F]\text{var}(X) + (\mathbf{E}[X])^2\text{var}(F),\end{aligned}$$

where we make the following substitutions.

$$\begin{aligned}\mathbf{E}[F] &= \mathbf{E}[L_m - m] = \frac{m}{p} - m = m. \\ \text{var}(F) &= \text{var}(L_m - m) = \text{var}(L_m) = \frac{m(1-p)}{p^2} = 2m. \\ \mathbf{E}[X] &= r = \frac{1}{3}. \\ \text{var}(X) &= r(1-r) = \frac{2}{9}.\end{aligned}$$

Finally, substituting these values, we have

$$\begin{aligned}\mathbf{E}[D_m] &= m \cdot \frac{1}{3} = \frac{m}{3} \\ \text{var}(D_m) &= m \cdot \frac{2}{9} + \left(\frac{1}{3}\right)^2 \cdot 2m = \frac{4m}{9}\end{aligned}$$

3. We view the random variables T_1 and T_2 as interarrival times in two independent Poisson processes both with rate λ S as the interarrival time in a third Poisson process (independent from the first two) with rate μ. We are interested in the expected value of the time Z until either the first process has had two arrivals or the second process has had an arrival.

   Given that the first arrival was from the second process, the expected wait time for that arrival would be $\frac{1}{\mu+\lambda}$. The probability of an arrival from the second process is $\frac{\mu}{\mu+\lambda}$. Given that the first arrival time was from the first process, the expected wait time would be that for first arrival, $\frac{1}{\mu+\lambda}$, plus the expected wait time for another arrival from the merged process. Similarily, the probability of an arrival from the first process is $\frac{\lambda}{\mu+\lambda}$. Thus,

$$\begin{aligned}\mathbf{E}[Z] = \mathbf{P}(\text{Arrival from second process})\mathbf{E}[\text{wait time}|\text{Arrival from second process}] +\\ \mathbf{P}(\text{Arrival from first process})\mathbf{E}[\text{wait time}|\text{Arrival from first process}]\\ = \frac{\mu}{\mu+\lambda} \cdot \frac{1}{\mu+\lambda} + \frac{\lambda}{\mu+\lambda} \cdot \left(\frac{1}{\mu+\lambda} + \frac{1}{\mu+\lambda}\right).\end{aligned}$$

   After some simplifications, we see that

$$\mathbf{E}[Z] = \frac{1}{\mu+\lambda} + \frac{\lambda}{\mu+\lambda} \cdot \frac{1}{\mu+\lambda}$$

4. The dot location of the yarn, as related to the size of the pieces of the yarn cut for any particular customer, can be viewed in light of the random incident paradox.

   (a) Here, the length of each piece of yarn is exponentially distributed. As explained on page 298 of the text, due to the memorylessness of the exponential, the distribution of the length of the piece of yarn containing the red dot is a second order Erlang. Thus, the $\mathbf{E}[R] = 2\mathbf{E}[L] = \frac{2}{\lambda}$.

   (b) Think of exponentially-spaced marks being made on the yarn, so the length requested by the customers each involve *three* such sections of exponentially distributed lengths (since the PDF of L is third-order Erlang). The piece of yarn with the dot will have the dot in any one of these three sections, and the expected length of that section, by (a), will be $2/\lambda$, while the expected lengths of the other two sections will be $1/\lambda$. Thus, the total expected length containing the dot is $4/\lambda$.

   In general, for processes, in which the interarrival intervals with distribution $F_X(x)$ are IID, the expected length of an arbitrarily chosen interval is $\frac{\mathbf{E}[X^2]}{\mathbf{E}[X]}$. We see that for the above parts, this formula is certainly valid.

   (c) Using the formula stated above, $\mathbf{E}[L] = \int_0^1 \ell^2 e^\ell \, d\ell = e^\ell(\ell^2 - 2\ell + 2)]_0^1 = e - 2$
   $E[L^2] = \int_0^1 \ell^3 e^\ell \, d\ell = e^\ell(\ell^3 - 3\ell^2 + 6\ell - 6)]_0^1 = 6 - 2e$
   Hence,

$$\mathbf{E}[R] = \boxed{\frac{6-2e}{e-2}}.$$

5. (a) We know there are n arrivals in t amount of time, so we are looking for how many extra arrivals there are in s amount of time.

$$p_{M|N}(m|n) = \frac{(\lambda s)^{m-n}e^{-\lambda s}}{(m-n)!} \quad \text{for } m \geq n \geq 0$$

(b) By definition:

$$\begin{aligned} p_{N,M}(n,m) &= p_{M|N}(m|n)p_N(n) \\ &= \frac{\lambda^m s^{m-n} t^n e^{-\lambda(s+t)}}{(m-n)!n!} \quad \text{for } m \geq n \geq 0 \end{aligned}$$

(c) By definition:

$$\begin{aligned} p_{N|M}(n|m) &= \frac{p_{M,N}(m,n)}{p_M(m)} \\ &= \binom{m}{n}\frac{s^{m-n}t^n}{(s+t)^m} \quad \text{for } m \geq n \geq 0 \end{aligned}$$

(d) We want to find: $\mathbf{P}(N=n|M=m)$. Given M=m, we know that the m arrivals are uniformly distributed between 0 and t+s. Consider each arrival a success if it occurs before time t, and a failure otherwise. Therefore given M=m, N is a binomial random variable with m trials and probability of success $\frac{t}{t+s}$. We have the desired probability:

$$\mathbf{P}(N=n|M=m) = \binom{m}{n}\left(\frac{t}{t+s}\right)^n\left(\frac{s}{t+s}\right)^{m-n} \quad \text{for } m \geq n \geq 0$$

(e) We can rewrite the expectatation as:

$$\begin{aligned} \mathbf{E}[NM] &= \mathbf{E}[N(M-N)+N^2] \\ &= \mathbf{E}[N]\mathbf{E}[M-N]+\mathbf{E}[N^2] \\ &= (\lambda t)(\lambda s) + \left(\text{var}(N) + (\mathbf{E}[N])^2\right) \\ &= (\lambda t)(\lambda s) + \lambda t + (\lambda t)^2 \end{aligned}$$

where the second equality is obtained via the independent increment property of the poisson process.

6. The described process for cars passing the checkpoint is a Poisson process with an arrival rate of $\lambda = 2$ cars per minute.

(a) The first and third moments are, respectively,

$$\mathbf{E}[T] = \frac{1}{\lambda} = \frac{1}{2} \qquad \mathbf{E}[T^3] = \int_0^\infty t^3 2e^{-2t}dt = \frac{3!}{2^3}\underbrace{\int_0^\infty \frac{2^4t^3e^{-2t}}{3!}dt}_{=1} = \frac{3}{4}$$

where we recognized the integrand to be a 4th-order Erlang PDF and therefore integrating it over the entire range of the random variable must sum to unity.

(b) The Poisson process is memoryless, and thus the history of events in the previous 4 minutes does not affect the future. So, the conditional PMF for K is equivalent to the unconditional PMF that describes the number of Poisson arrivals in an interval of time, which in this case is $\tau = 6$ minutes and thus $(\lambda\tau) = 12$:

$$p_K(k) = \frac{12^k e^{12}}{k!}, \quad k = 0, 1, 2, \ldots \quad .$$

(c) The first dozen computer cards are used up upon the 36th car arrival. Letting D denote this total time, $D = T_1 + T_2 + \ldots + T_{36}$, where each independent T_i is exponentially distributed with parameter $\lambda = 2$, the distribution for D is therefore a 36th-order Erlang distribution with PDF and expected value of, respectively,

$$f_D(d) = \frac{2^{36} d^{35} e^{-2d}}{35!}, \quad d \geq 0 \qquad \mathbf{E}[D] = 36\mathbf{E}[T] = 18$$

(d) In both experiments, because a card completes after registering three cars, we are considering the amount of time it takes for three cars to pass the checkpoint. In the second experiment, however, note that the manner with which the particular card is selected is biased towards cards that are in service longer. That is, the time instant at which we come to the corner is more likely to fall within a longer interarrival period – one of the three interarrival times that adds up to the total time the card is in service is selected by *random incidence* (see the end of Section 6.2 in text).

i. The service time of any particular completed card is given by $Y = T_1 + T_2 + T_3$, and thus Y is described by a 3rd-order Erlang distribution with paramater $\lambda = 2$:

$$\mathbf{E}[Y] = \frac{3}{\lambda} = \frac{3}{2} \qquad \text{var}(Y) = \frac{3}{\lambda^2} = \frac{3}{4} \quad .$$

ii. The service time of a particular completed card with one of the three interarrival times selected by random incidence is $W = T_1 + T_2 + L$, where L is the interarrival period that contains the time instant we arrived at the corner. Following the arguments in the text, L is Erlang of order two and thus W is described by a 4th-order Erlang distribution with parameter $\lambda = 2$:

$$\mathbf{E}[W] = \frac{4}{\lambda} = 2 \qquad \text{var}(W) = \frac{4}{\lambda^2} = 1 \quad .$$

G1†. For simplicity, introduce the notation $N_i = N(G_i)$ for $i = 1, \ldots, n$ and $N_G = N(G)$. Then

$$\begin{aligned}
\mathbf{P}(N_1 = k_1, \ldots, N_n = k_n | N_G = k) &= \frac{\mathbf{P}(N_1 = k_1, \ldots, N_n = k_n, N_G = k)}{\mathbf{P}(N_G = k)} \\
&= \frac{\mathbf{P}(N_1 = k_1) \cdots \mathbf{P}(N_n = k_n)}{\mathbf{P}(N_G = k)} \\
&= \frac{\left(\frac{(c_1\lambda)^{k_1} e^{-c_1\lambda}}{k_1!}\right) \cdots \left(\frac{(c_n\lambda)^{k_n} e^{-c_n\lambda}}{k_n!}\right)}{\left(\frac{(c\lambda)^{k} e^{-c\lambda}}{k!}\right)} \\
&= \frac{k!}{k_1! \ldots k_n!} \left(\frac{c_1}{c}\right)^{k_1} \cdots \left(\frac{c_n}{c}\right)^{k_n} \\
&= \left({k \atop k_1 \;\cdots\; k_n} \right) \left(\frac{c_1}{c}\right)^{k_1} \cdots \left(\frac{c_n}{c}\right)^{k_n}
\end{aligned}$$

The result can be interpreted as a *multinomial distribution.* Imagine we throw an n-sided die k times, where Side i comes up with probability $p_i = c_i/c$. The probability that side i comes up k_i times is given by the expression above. Now relating it back to the Poisson process that we have, each side corresponds to an interval that we sample, and the probability that we sample it depends directly on its relative length. This is consistent with the intuition that, given a number of Poisson arrivals in a specified interval, the arrivals are uniformly distributed.

---

†Required for 6.431; optional for 6.041

<u>OH</u> After PSet Due 11/14

1. ith sucess before jth Failure

Exactly i sucesess in 1st $i+j-1$ trials

$\underbrace{\_\_\ \_\_\ \_\_\ \_\_}_{\substack{i+j-1\\ \text{sucesses}}}\ \underset{\substack{(i+j)^{th}\\ \text{trial}}}{\_\_}\ \_\_\ \_\_$ <u>best way</u>

If $\underbrace{\qquad\qquad\qquad}_{\substack{i\text{th sucess}\\ j-1 \text{ failures}}}$ j will occur w/ $p=1$ <u>worst way</u>

$\underbrace{\qquad\qquad\qquad}_{\substack{i+1 \text{ sucesses}\\ j-2 \text{ failures}}}$ ith sucess not before jth Failure

... ↓ all disjoint

P(i sucesses in 1st $i+j-1$ trials)

$$=\binom{i+j-1}{i} p^{i} (1-p)^{j-1}$$

What if k sucesses

P(k sucesses in 1st $i+j-1$ trials)

$$\sum_{k=1}^{i+j-1} \binom{i+j-1}{k} p^{k} (1-p)^{(i+j-1)-k}$$ <u>worst</u>

$\underline{k}$ $k = i, \ldots, i+j-1$

?# of sucesses

②

$< i$ sucesses
$> j + j - 1$ " would not work

1b) # of sucesses before j th failure $= S_J$
trials ← eaiser

↳ looking at Bernalli process time as jth arrival

Arrival ≡ Failure

now this is time of jth arrival $= Y_J$
trial #

$S_J = Y_J - j$

trials − # failures

$E[S_J] = E[Y_j] - j$ ← Pascal of order j w/ parameter $1-p$

↑ failure

Sucess

$Var(S_J) = Var(Y_J)$

$Y_J = T_1 + T_2 + \dots + T_J$

↑ $E[T_j] = \frac{1}{p}$

$E[Y_J] = \frac{j}{1-p}$

③

$$\mathrm{var}(Y_j) = \text{add } \mathrm{var}(T_j)$$

$$= \frac{1-(1-p)}{(1-p)^2} \; j$$

$$= \frac{p}{(1-p)^2} \; j$$

$$E[S_j] = \frac{j}{1-p} - j$$

$$\mathrm{var}(Y_j) = \frac{jp}{(1-p)^2}$$