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LECTURE 16

Markov Processes – I

Still random processes

- **Readings:** Sections 7.1–7.2

new chap

past matters / memory

Lecture outline

- Checkout counter example
- Markov process definition
- n-step transition probabilities
- Classification of states

Many real world processes described by
1. Simple example
2. Formal def
3. Examples

Checkout counter model

at supermarket

also jobs – arrive randomly
processing time random

- Discrete time $n = 0, 1, \ldots$
- Customer arrivals: Bernoulli(p)
  - geometric interarrival times by definition
- Customer service times: geometric(q)

assume independent

since random # of items

- "State" X_n: number of customers at time n

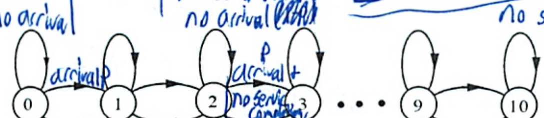
$pq + (1-p)(1-q)$

no completion + no arrival

or completion and arrival

no arrival

no service completion $(1-q)$



no service

1 being served

amt of time served geometric(q)

no arrival + service completion $q(1-p)$

service completion q

only 10 people can be in line

– like cashier flipping coin after every item to continue service if heads

Finite state Markov chains

States don't have to be #

good day bad date

State decides to stay or jump

- X_n: state after n transitions
  - belongs to a finite set, e.g., $\{1, \ldots, m\}$
  - X_0 is either given or random
- **Markov property/assumption:** (key property)
  (given current state, the past does not matter)

$$\begin{aligned} p_{ij} &= \mathbf{P}(X_{n+1} = j \mid X_n = i) \\ &= \mathbf{P}(X_{n+1} = j \mid X_n = i, \underbrace{X_{n-1}, \ldots, X_0}_{}) \end{aligned}$$

past history does not matter

$P_{ij} \geq 0 \quad \sum_j P_{ij} = 1$ | given transition property P_{23} — but does not depend on history

$② \rightarrow ③$

current state matters

- Model specification:
  - identify the possible states
  - identify the possible transitions
  - identify the transition probabilities

$X_{n+2} \quad X_{n+1} \quad \underbrace{X_n} \quad X_{n-1} \quad X_{n-2} \quad \ldots \quad X_0$

given X_n
past and future are ind.

ie a ball in air
- only need balls current pos X
- and velocity dX

together → are the state (X, dX)

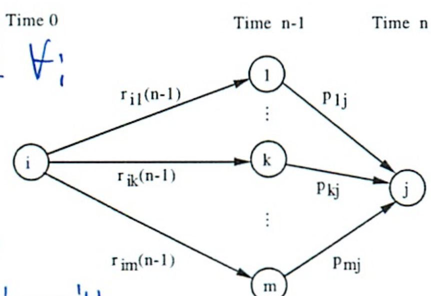
n-step transition probabilities

Models useful for given predictions probabilities of various options of where it will be n time steps later

- State occupancy probabilities, given initial state i:

$r_{ij}(n) \geq 0 \qquad r_{ij}(n) = \mathbf{P}(X_n = j \mid X_0 = i)$

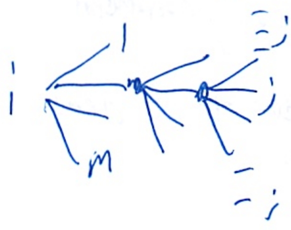
$\sum_j r_{ij}(n) = 1 \ \forall i$



$r_{ii}(0) = 1$ certain

$r_{ij}(0) = 0$ impossible

$r_{ij}(1) = P_{ij}$



lots of leaves come and to state j
- must add all of the possibilities of j
- but too much calculation!

Markoff allows us to lump leaves together
add up scenarios
... (missed some)

- Key recursion:

$$r_{ij}(n) = \sum_{k=1}^{m} r_{ik}(n-1) p_{kj}$$

know start at i

- With random initial state:

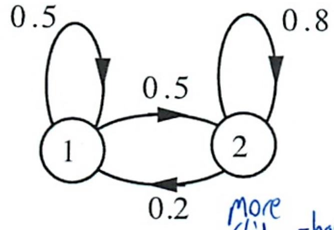
$$\mathbf{P}(X_n = j) = \sum_{i=1}^{m} \mathbf{P}(X_0 = i) r_{ij}(n)$$

Unknown start
consider all possibilities

→ extra sheet

Example Simplist Markoff chain

$r_{11}(n) = r_{11}(n-1) \cdot .5 + r_{12}(n-1) \cdot .2$



more sticky - harder to get out

$r_{12}(n) = 1 - r_{11}(n)$

everything except 1

not moving

	$n=0$	$n=1$	$n=2$	$n=100$	$n=101$
$r_{11}(n)$	1	.5	$\frac{.5}{.2}$ .35 $\frac{.5}{.2}$	$\approx 2/7$	$\approx 2/7$
$r_{12}(n)$	0	.5	.65	$\approx 5/7$	$\approx 5/7$
$r_{21}(n)$	0	.2		$\approx 2/7$	$\approx 2/7$
$r_{22}(n)$	1	.8		$\approx 5/7$	$\approx 5/7$

in the limit start state does not matter!

start at state 2

initial state makes a difference w/ small n

so can't move outcome certain

just the branches

here can happen in multiple ways each event can happen in multiple ways

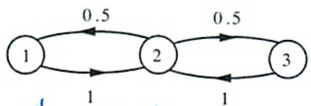
repeat a bunch of times

almost the same since they settle "markoff ~~change~~ chain reached steady state" - still changing - but we know prob of finding state in each place

Generic convergence questions:

- Does $r_{ij}(n)$ converge to something?

Will they always converge? no not always



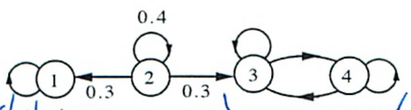
always come back

n odd: $r_{22}(n) = 0$ n even: $r_{22}(n) = 1$

no way to be at 2 - at 1 or 3

- Does the limit depend on initial state?

Can have a huge impact



stuck here stuck here

$r_{11}(n) = 1$ (always)

$r_{31}(n) = 0$

$r_{21}(n) = \frac{1}{2}$

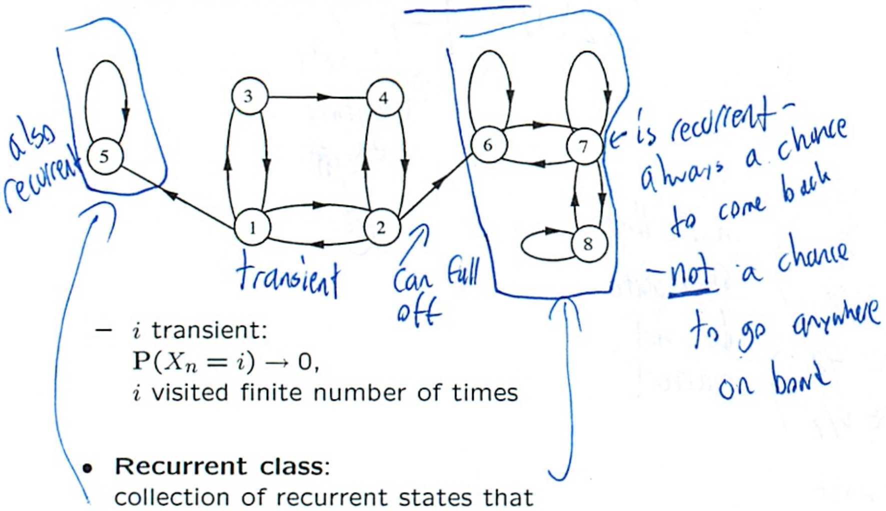
does depend where we start

→ if n very large impossible to be on 2 will leave both directions = likly

above - at any state can get to any other state → recurrent

Recurrent and transient states

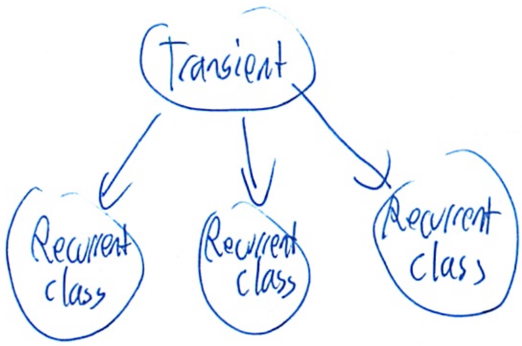
- State i is **recurrent** if:
  starting from i,
  and from wherever you can go,
  there is a way of returning to i
- If not recurrent, called **transient**

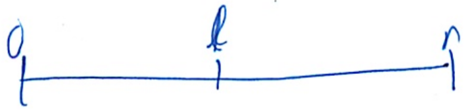


- i transient:
  $P(X_n = i) \to 0$,
  i visited finite number of times

- **Recurrent class**:
  collection of recurrent states that "communicate" with each other and with no other state

two different recurrent sets





Could split here

Can ~~the~~ condition on step l (intermediate step)

$$r_{ij}(n) = \sum_k r_{ik}(l)\, r_{kj}(\underbrace{n-l}_{\text{\# of steps}})$$

all possible ways to get to state j

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Department of Electrical Engineering & Computer Science

6.041/6.431: Probabilistic Systems Analysis

(Fall 2010)

Recitation 18

November 9, 2010

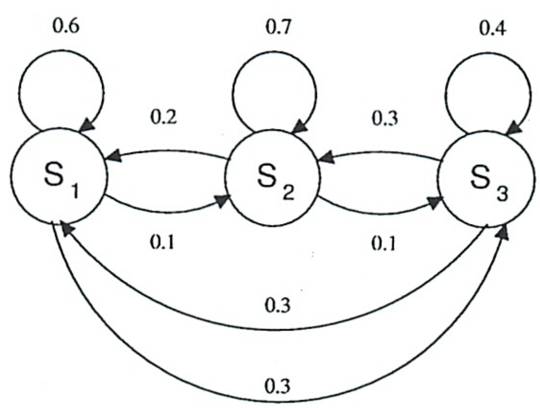
1. There are n fish in a lake, some of which are green and the rest blue. Each day, Helen catches 1 fish. She is equally likely to catch any one of the n fish in the lake. She throws back all the fish, but paints each green fish blue before throwing it back in. Let G_i denote the event that there are i green fish left in the lake.

   (a) Show how to model this fishing exercise as a Markov chain, where $\{G_i\}$ are the states. Explain why your model satisfies the Markov property.

   (b) Find the transition probabilities $\{p_{ij}\}$.

   (c) List the transient and the recurrent states.

2. Problem 5.02, from *Fundamentals of Applied Probability* (Drake).
   Consider the following three-state discrete-transition Markov chain:



Determine the three-step transition probabilities $r_{11}(3)$, $r_{12}(3)$, and $r_{13}(3)$ both from a sequential sample space and by using the equation $r_{ij}(n+1) = \sum_k r_{ik}(n)p_{kj}$ in an effective manner.

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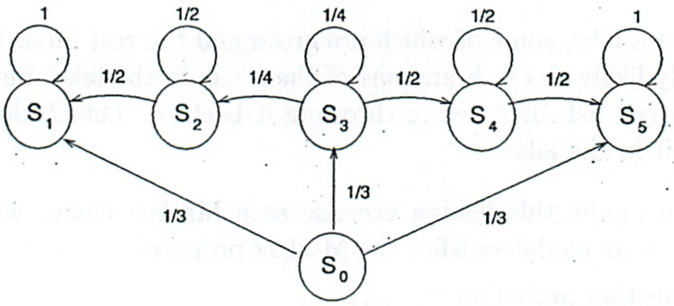
3. Consider the following Markov chain, with states labelled from $s_0, s_1, \ldots, s_5$:



Given that the above process is in state s_0 just before the first trial, determine by inspection the probability that:

(a) The process enters s_2 for the first time as the result of the kth trial.

(b) The process never enters s_4.

(c) The process enters s_2 and then leaves s_2 on the next trial.

(d) The process enters s_1 for the first time on the third trial.

(e) The process is in state s_3 immediately after the nth trial.

Recitation 18

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- in some way generallizing processes
- now ~~[illegible]~~ some very structured memory
- looking at discrete time, discrete valued RVs

"chain"

$$P(X_{k+1} = j \mid X_k = i, X_{k-1} = i_{k-1}, X_{k-2} = i_{k-2} \ldots)$$

more distant past is irrelevant.

Markoff property

$$= P(X_{k+1} = j \mid X_k = i)$$

$= P_{ij}$ depends ~~only~~ not on k → time invariance

$X_1, X_2, \ldots$ is a Bernoulli process

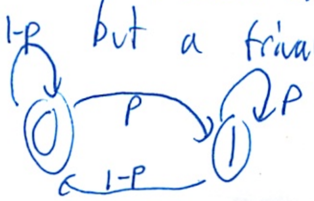
w/ $P(X_i = 1) = p$

is a Markoff chain - a very simple markoff chain

$$P(X_{k+1} = j \mid X_k = i, X_{k-1} = i_{k-1})$$

$$= \begin{cases} p & \text{for } j = 1 \\ 1-p & \text{for } j = 0 \end{cases}$$

is a Markoff chain - but a trival markoff chain



no memory in process

3) Don't have to have memory for it to be Markoff chain

1. $G_i = i$ green fish

$Y_i = \begin{cases} 1 & \text{when Helen catches a green fish on day } i \\ 0 & \text{otherwise} \end{cases}$

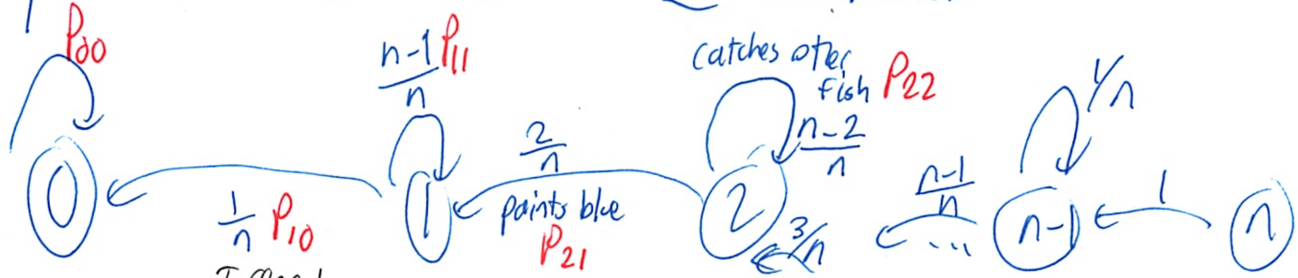
$Y_1, Y_2, \ldots$ = Bernoulli RVs

Not a bernoulli process since state changes

also $P(Y_i = 1)$ is a complicated function of i

$\{Y_1 = 1\}$ and $\{Y_2 = 1\}$ are not independent

G_i:



↑ remember how intuitive these are!

~ = lables

first figure out transition properties

getting endpoints right are critical!

markoff b/c $G_i = 2$ is a state

- does not depend what happened before (so Markoff)
- but transition possibilities are based on state (not ind (Bernoulli))

(3)

transient: ~~ma~~ can always return → all but 0

recurrent: will always be~~a~~ at least one → 0

---

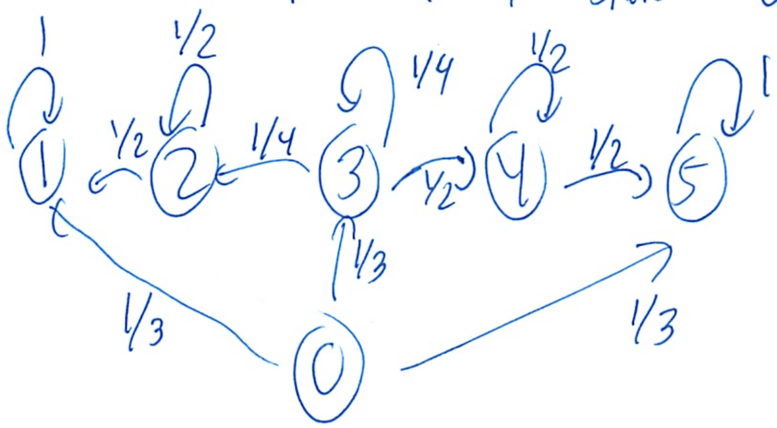
preview In steady state will always be in a recurrent state

Prob of being in transient state = 0

↳ $\lim_{n \to \infty} P(X_n = s) = 0$

#3 (in order different than sheet)

- in #1 were told nothing about start state
- here we are told initial state = 0



d) P(enters S_1 on 3rd trial)

can only happen if goes $3 \to 2 \to 1$

$= P_{03} P_{32} P_{21}$

$= \frac{1}{3} \cdot \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{24}$

(4) (I asked if any shortcuts - he did not really answer)

e) P(in ~~3~~ 3 after n trials)

$= P(X_n = 3)$

$= P_{03}(P_{33})^{n-1}$ ← only seq. that results in this

$= \frac{1}{3}\left(\frac{1}{4}\right)^{n-1}$ & leave n as a variable

a) P(enters S_2 for 1st time as result of kth trial)

$= P_{03}(P_{33})^{k-2}P_{32}$

and then no other way

$= \frac{1}{3}\left(\frac{1}{4}\right)^{k-2}\left(\frac{1}{4}\right)$

$= \frac{1}{3}\left(\frac{1}{4}\right)^{k-1}$

- Prof: should write given $|X_0=0$ condition on all to be clear

b) P(prob never enters 4) =

$= 1 - P(\text{ever enters } 4)$

$= 1 - \sum_{k=2}^{\infty} P(\text{enters } 4 \text{ at kth time for 1st time})$

↑ first opp to enter

$= 1 - \sum_{k=2}^{\infty} \underbrace{\frac{1}{3}\left(\frac{1}{4}\right)^{k-2}\frac{1}{2}}_{\text{like previous ans}}$

$= 1 - \frac{1}{6}\sum_{k=2}^{\infty}\left(\frac{1}{4}\right)^{k-2}$

5

$= 1 - \frac{1}{6} \cdot \frac{1}{1-\frac{1}{4}}$

$= 1 - \frac{1}{6} \cdot \frac{4}{3}$ ~~ans~~

$= \frac{7}{9}$

---

or

$P(\text{ever } s_4 \mid X_1 = 3)$

$= P(X_{k+1} = 4 \mid X_{k+1} \neq X_k, X_1 = 3)$

$= \frac{P(X_{k+1} = 4, X_k = 3)}{P(X_{k+1} \neq X_k, X_k = 3)}$ use def. conditional prob

$= \frac{P(X_{k+1} = 4 \mid X_k = 3) \cancel{P(X_k = 3)}}{P(X_{k+1} \neq X_3 \mid X_k = 3) \cancel{P(X_k = 3)}}$

$= \frac{P_{34}}{P_{34} + P_{32}} = \frac{\frac{1}{2}}{\frac{1}{2} + \frac{1}{4}} = \frac{2}{3}$

$= 1 - \frac{1}{3} \cdot \frac{2}{3} = \frac{7}{9}$

(6)

c) P(process enters S_2 and leaves S_2 on next trial)

" $1/2$ - too basic ans!

$= P(\text{ever entering } S_2) \cdot P(\text{leaving } S_2 \mid \text{are in } S_2)$

$\frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{2}$ $= \frac{1}{18}$

must go $0 \to 3$

when it leaves ~~then~~ prob of leaving is $\frac{1}{3}$

or more formally

$$\sum_{k=2}^{\infty} \left(\frac{1}{4}\right)^{k-2}$$

So together

$$\frac{1}{2} \sum_{k=2}^{\infty} \frac{1}{3} \left(\frac{1}{4}\right)^{k-2} \frac{1}{4} = \frac{1}{18}$$

#2. $r_{ij}(n) = P(X_{k+n} = j \mid X_k = i)$

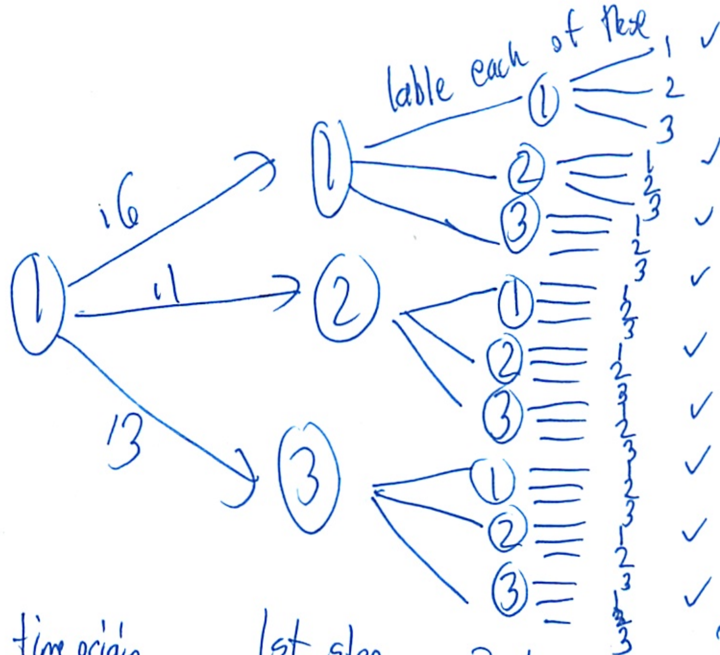
How to get 3 step transition possibilities

$r_{11}(3)$

$r_{12}(3)$

$r_{13}(3)$

(7)



time origin (kinda) $X_k = 1$

1st step

2nd step

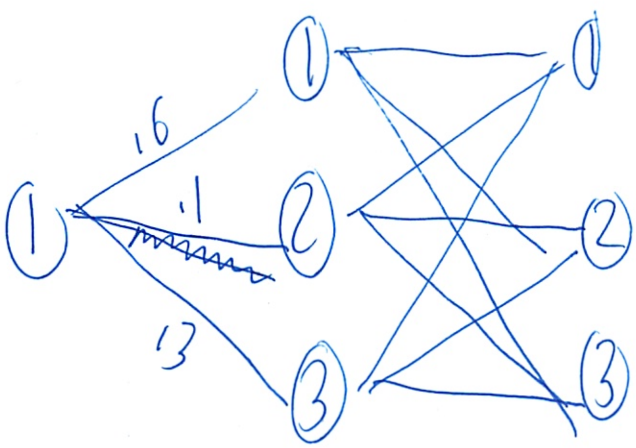
3rd step

← Chap 1 thinking

↑ these end in state 1
could find all these
add up to get r_{11}

but Markoff property!

- how got to state 2 does not matter!



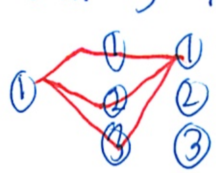
time origin

1st step

2nd step

$P(X_{k+2} = 1 \mid X_k = 1)$

sum 3 terms



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$= P(X_{k+1} = 1 | X_k = 1) p_{11} + P(X_{k+2} = 2 | X_k = 1) p_{21} + P(X_{k+3} = 3 | X_k = 1) p_{31}$

so this diagram # of possibilities = # of states

not n^n

$= \underbrace{r_{11}(1)}_{\text{other name for above}} p_{11} + r_{12}(1) p_{12} + r_{13}(1) p_{13}$

$= r_{11}(2)$

- ~~If~~ we solved for 2, problem is for 3
  - tedious
  - but use key recursion

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LECTURE 17

Markov Processes – II

- **Readings:** Section 7.3

Lecture outline

- Review
- Steady-State behavior
  - Steady-state convergence theorem
  - Balance equations
- Birth-death processes

Review

- Discrete state, discrete time, time-homogeneous
  - Transition probabilities p_{ij}
  - Markov property
- $r_{ij}(n) = \mathbf{P}(X_n = j \mid X_0 = i)$ ↓ starting at i
  of being in a state j after n (any) trial
- Key recursion:
  $r_{ij}(n) = \sum_k r_{ik}(n-1)p_{kj}$

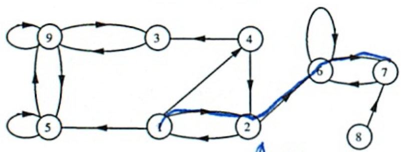
↑ law of total probability
consider all possibilities of where
I can be previous time

X_n = State at time n
(when a #) like a RV, can find E[ ]

P_{23}
↳ transition property
- does not depend on time (time homogenity)
- or how you got to current state
  (past does not matter)

$r_{ij}(n) = (X_{100+n} = j \mid X_{100} = i)$
still an n-step transition possibility

Warmup



↑of this blue path

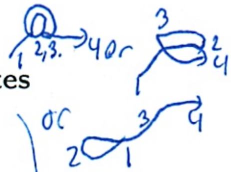
just multiplication rule
- multiply transition possibilities along a path

$P(X_1 = 2, X_2 = 6, X_3 = 7 \mid X_0 = 1) =$

$= P_{12} \cdot P_{26} \cdot P_{67}$

$P(X_4 = 7 \mid X_0 = 2) =$

three ~~two~~ ways of doing this



$= P_{26} P_{66} P_{66} P_{67}$
$+ P_{26} P_{67} P_{76} P_{67}$
$+ P_{21} P_{12} P_{26} P_{67}$) add probabilities

but when have a lot
- do it using markoff rule

Recurrent and transient states

- State i is **recurrent** if:
  starting from i,
  and from wherever you can go,
  there is a way of returning to i
- If not recurrent, called **transient**
- **Recurrent class**:
  collection of recurrent states that "communicate" to each other and to no other state

given enough time you will come back

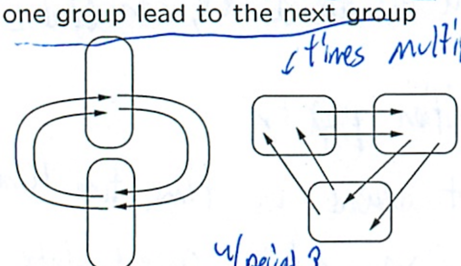
5, 9, 3 set is recurrent
6, 7 set

Periodic states

- The states in a recurrent class are **periodic** if they can be grouped into $d > 1$ groups so that all transitions from one group lead to the next group

$r_{ii}(n) \neq 0$ if $n = 3k$
$= 0$ if otherwise

↙times multiple of 3, be here

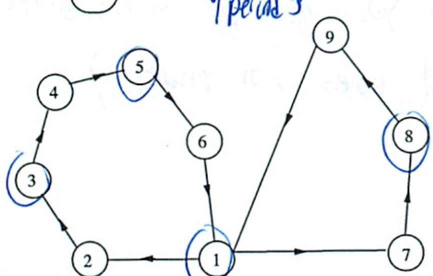


← multiple of 3 + 1 be here

w/ period 3

1 circle that it goes around ?!

where you start makes diff. in transition possibilities



- does the $P_{ii}(n)$ converge to something
  - does not depend on i ?!
  ↑ does depend on i

? periodic d = 2 (w/ circled)
- can be grouped -> hot is group
- hard to do visually - trick: but if has self arc → never periodic

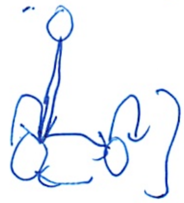
Steadу-State Probabilities

"is nice if it has this property" at any time

- Do the $r_{ij}(n)$ converge to some π_j? (independent of the initial state i)

- Yes, if:
  - recurrent states are all in a single class, and
  - single recurrent class is not periodic

  ← those in which bad ~~things~~ things do not happen



this is a single recurrent class, so yes
start pt does not matter
prob of being in any 1 steady state

- Assuming "yes," start from key recursion

$$r_{ij}(n) = \sum_k r_{ik}(n-1)p_{kj}$$

  - take the limit as $n \to \infty$

$$\pi_j = \sum_k \pi_k p_{kj}, \qquad \text{for all } j$$

  - Additional equation:

$$\sum_j \pi_j = 1$$

as many eq as there are states, unknowns
trivial solution $\pi = 0$
so ∞ many solutions
but
now has a unique solution

Visit frequency interpretation

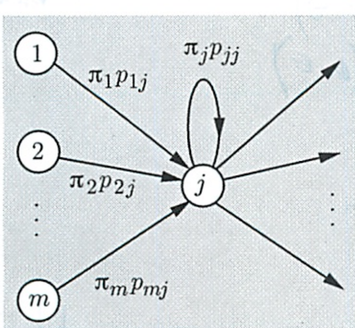
Balance equation

$$\pi_j = \sum_k \pi_k p_{kj}$$

- (Long run) frequency of being in j: π_j ← steady state probability
  prob that I find myself in j in a long time
  also freq - how often you will be in j

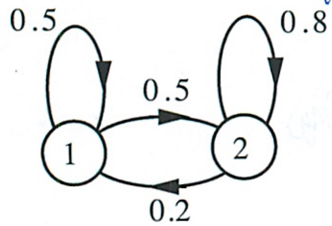
- Frequency of transitions $k \to j$: $\pi_k p_{kj}$

- Frequency of transitions into j: $\sum_k \pi_k p_{kj}$



flows - trajectories flow from one to another

Example Same as last time



Balance eq

$\pi_1 = \pi_1 \cdot .5 + \pi_2 \cdot .2$

$\pi_2 = \pi_1 \cdot .5 + \pi_2 \cdot .8$

$\pi_1 \cdot .5 = \pi_2 \cdot .2$

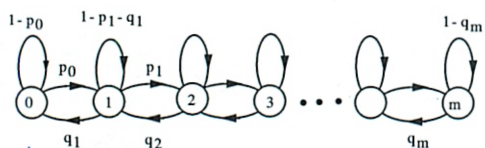
- not enough info

but $\pi_1 + \pi_2 = 1$

so $\pi_1 = \frac{2}{7}$ $\pi_2 = \frac{5}{7}$ are probabilities but can also think of them as freq.

* State has not setted / stop - prob of each state settles to some value

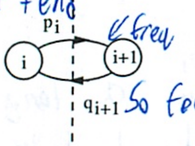
Birth-death processes



Can do ^ for million of rows on computer

← but if want to do more analysis by hand

1. Find start + end
2. Fill in middle



← freq with which transitions of this type occur

$\pi_i p_i = \pi_{i+1} q_{i+1}$

So focus on just pair - think in terms of freq

recursive structure - local balance eq

- Special case: $p_i = p$ and $q_i = q$ for all i

$\rho = p/q$ = load factor transitions

* what goes up (or →) must come down (or ←)

if = 1 up and down =ly likly

$$\pi_{i+1} = \pi_i \frac{p}{q} = \pi_i \rho \quad \text{recursion}$$

$$\pi_i = \pi_0 \rho^i, \qquad i = 0, 1, \ldots, m$$

- Assume $p < q$ and $m \approx \infty$ end does not matter

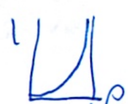
tends to stay where it is

$\pi_0 = 1 - \rho = \frac{1}{1/1-\rho}$ stable queues

$\mathrm{E}[X_n] = \frac{\rho}{1-\rho}$ (in steady-state)

$$\sum_{i=0}^{m} \pi_0 \rho^i = 1 \to \pi_0 = \frac{1}{\sum_{i=0}^{m} \rho^i}$$

- queing theory



$\rho > 1 \to$ unstable $\to$ keeps growing

$\rho < 1 \to$ stable $\to$ moves towards 0

still have randomness

$$E[X_n] = \lim_{n \to \infty} \sum_i P(X_n = i)\, i = \sum_{i=0}^{\infty} i (1-\rho) \rho^i \quad \text{≈ geometric}$$

Chap 6.2 Poisson Process Reading

11/11

Continuous time analog to Bernoulli process except the interval $\to \infty$

(doing after ~~the~~ hw due on this)

$P(k, \tau) = P(\text{there are } k \text{ arrivals during interval } \tau)$

$\lambda =$ arrival rate / intensity

Definition

a) Time homogeneity - same for all intervals of same lenght

b) Independence - history does not mater

c) Small interval probabilities

$$P(0,\tau) = 1 - \lambda\tau + o(\tau)$$
$$P(1,\tau) = \lambda\tau + o_1(\tau)$$
$$P(k,\tau) = o_k(\tau) \quad \text{for } k = 2,3,4,\ldots$$

$$\lim_{\tau \to 0} \frac{o(\tau)}{\tau} = 0 \qquad \lim_{\tau \to 0} \frac{o_k(\tau)}{\tau} = 0$$

~~[illegible]~~ negligible

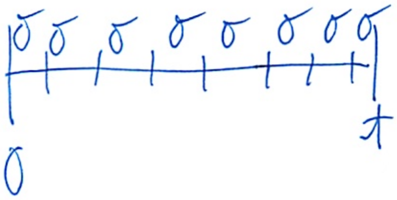
↑ basically only 1 or 0 arrivals at a time

me

$$P(1) = \lambda\tau$$
$$P(0) = 1 - \lambda\tau$$

for really small intervals

② # of Arrivals in an Interval



δ = interval lenght ($\lim \delta \to 0$)
$n = \frac{\tau}{\delta}$ ← # of intervals

P(arrival in each period) $= \lambda\delta$
E[total arrivals] $= \lambda\tau =$ var(total arrivals)

$P(k,\tau)$ = binomial prob of k sucesses in $n = \frac{\tau}{\delta}$

$\delta \to 0$ so $n \to \infty$ but np stays constant

$$P(k,\tau) = e^{-\lambda\tau}\frac{(\lambda\tau)^k}{k!} \qquad k = 0, 1, \ldots$$

~~doing~~ doing the Taylor series expansion gets result on previous page
what is this again

$E[N_\tau] = \lambda\tau \qquad \text{var}(N_\tau) = \lambda\tau$
# of arrivals during time interval of length τ

CDF $F_T(t) = P(T \le t) = 1 - P(T > t) = 1 - P(0,t) = 1 - e^{-\lambda t} \quad t \ge 0$
assuming no arrivals $[0,t]$
$f_T(t) = \lambda e^{-\lambda t}$

λ = time T till first arrival
$f_T(t) = \lambda e^{-\lambda t} \quad t \ge 0 \quad E[T] = \frac{1}{\lambda} \quad \text{var}(t) = \frac{1}{\lambda^2}$

(3)

Independence + Memoryless

- (pretty much went over this)

t = given time $\bar{T}$ = time of first arrival after t

$\bar{T} - t$ has exponential distribution w/ param λ

$$P(\bar{T} - t > s) = P(0 \text{ arrivals during } [t, t+s]) = P(0, s) = e^{-\lambda s}$$

Interarrival times

time of kth arrival $= Y_k$

kth interarrival time $= T_k$

$T_1 = Y_1$ $T_k = Y_k - Y_{k-1}$ $k = 2, 3, \ldots$

$$Y_k = T_1 + T_2 + \ldots + T_k$$

T_2 is independent of T_1 and all have exponential distribution

kth Arrival Time

erlang of order k   ?: circular definition??

$$Y_k = T_1 + T_2 + \ldots + T_k$$

$$E[Y_k] = E[T_1] + E[T_2] + \ldots + E[T_k] = \frac{k}{\lambda}$$

$$\text{var}(Y_k) = \text{var}(T_1) + \text{var}(T_2) + \ldots + \text{var}(T_k) = \frac{k}{\lambda^2}$$

$$f_{Y_k}(y) = \frac{\lambda^k y^{k-1} e^{-\lambda y}}{(k-1)!} \qquad y \geq 0$$

(4) skipping proof

example call IRS 56th person in line

2 calls answered / minute

How long on average ($E[\,]$) till your service starts?

Prob have to wait > 30 min

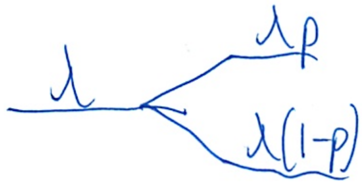
So waiting time in minutes (Y) = erlang of order 56

$$E[Y] = \frac{56}{\lambda} = \frac{56}{2} = 28$$

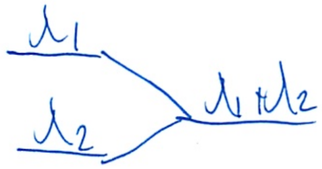
$$P(Y \geq 30) = \int_{30}^{\infty} \frac{\lambda^{56} y^{55} e^{-\lambda y}}{55!}\, dy = \text{do on computer}$$

(why can't I think of that!)

Splitting + Merging of Poisson Process



← poisson process w/ rate λp



$P(\text{came from } 1) = \dfrac{\lambda_1}{\lambda_1+\lambda_2}$ $\quad P(\text{came from } 2) = \dfrac{\lambda_2}{\lambda_1+\lambda_2}$

$P(0 \text{ arrivals in merged process}) \approx (1-\lambda_1\delta)(1-\lambda_2\delta) \approx 1-(\lambda_1+\lambda_2)\delta$

$P(1 \text{ arrival in merged process}) \approx \lambda_1\delta(1-\lambda_2\delta) + (1-\lambda_1\delta)\lambda_2\delta \approx (\lambda_1+\lambda_2)\delta$

5

Sums of RVs

N, X_1, X_2 independent RVs

N only takes integers

$Y = X_1 + X_2 + \ldots + X_n$ for $N > 0$

$Y = 0$ $N = 0$

1. if $X_i \sim$ Bernolli (p) and $N \sim$ Binomial (m, q)

   w/ parameter

   $Y \sim$ Binomial (m, pq)

2. if $X_i \sim$ Bernoulli (p) and $N \sim$ ~~Binomi~~ Poisson (λ)

   $Y \sim$ Poisson (λp)

3. if $X_i \sim$ Geometric (p) and $N \sim$ geometric (q)

   $Y \sim$ geometric (pq)

4. if $X_i \sim$ Exponential (λ) and $N \sim$ Geometric (q)

   $Y \sim$ exponential (λq)

Some things may not be Poisson

But when you take a very large # of independent arrivals

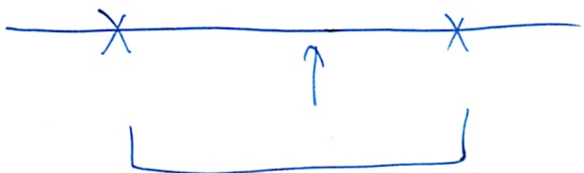
Poisson matches it closely

⑥

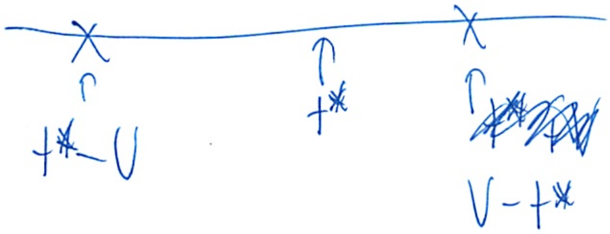
Random Incidence Paradox

(I understand fairly well)

— explained Mon's office hrs



erlang of order 2



$L = V - U$

arrivals

$= (t^* - U) + (V - t^*)$

elapsed + time to next

independent

exponential order 2   exponential order 2

erlang order 2

(7)

~~Subtle~~ bias towards larger interarrival times

- a different probabilistic mechinism
  - ~~this that~~ a fixed nonrandom k and RV of kth interarrivaltime ≠ fixing a tim t and cosidering random k such that kth interarrvial time contains t

(so which is the paradox?)

picking a bus at random + counting riders ≠ picking a bus rider and seeing how many people on his bus

(I liked the class size example from lecture better)

Chap 7 Reading Markoff Chains

11/11

in the preceding chaps Bernoulli + Poisson processes were memoryless
future did not depend on past
- occurences of new "arrivals" does not depend on past history of process

but here the current state (but not anything before that ?!) matters
- we analize the probabilistic sequence of state values

lots of dynamic systems can be modeled this way

7.1 Discrete Time Markoff-Chains

time $= n$

current state $= X_n$

states belong to finite set S (state space)
weird letter

$S = \{1, 2, \ldots, m)$

transition probabilities p_{ij}
- whenever state is at i
- that prob next state will be j

$$p_{ij} = P(X_{n+1} = j \mid X_n = i) \quad i, j \in S$$

- * key assumption: this applies whenever i is visited no matter how state i was reached

(2)

<u>Markoff property</u> $P(X_{n+1} = j \mid X_n = i, \underbrace{X_{n-1} = i_{n-1}, \ldots, X_0 = i_0}_{\text{basically this does not matter}})$

$= P(X_{n+1} = j \mid X_n = i)$

$= P_{ij}$

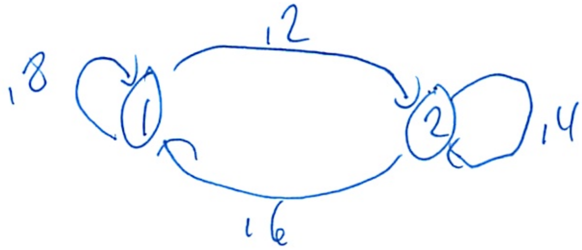
$\sum_{j=1}^{m} P_{ij} = 1$ for all i

add up all of the probs of going to all of the states

Can all be encoded into <u>transition probability matrix</u>

$$\begin{array}{cc} & i \\ j & \begin{bmatrix} P_{11} & P_{12} & \cdots & P_{1m} \\ P_{21} & P_{22} & \cdots & P_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ P_{m1} & P_{m2} & \cdots & P_{mm} \end{bmatrix} \end{array}$$

or <u>transition probability graph</u>



$$\begin{bmatrix} .8 & .2 \\ .6 & .4 \end{bmatrix}$$

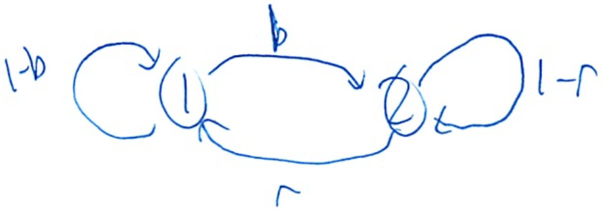
Very basic example

$[1.0]$

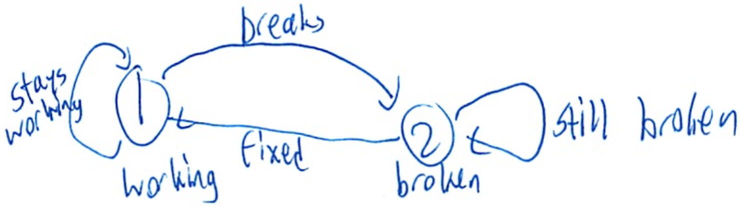
(I made - may not be right)

(3)

State 1 = machine is working
" 2 = " " not "



$$\begin{bmatrix} 1-b & b \\ r & 1-r \end{bmatrix}$$



Probability of a Path

- by following a path
- multiplication rule like leaves on a tree

$$P(X_0 = i_0, X_1 = i_1, \ldots, X_n = i_n) = P(X_0 = i_0)\, P_{i_0 i_1} P_{i_1 i_2} \cdots P_{i_{n-1} i_n}$$

- ~~Here is~~ so probability of a path

$P(X_0 = 1, X_1 = 2, X_2 = 3) =$
$P(X_0 = 1) P_{12} P_{23}$

- ~~Note~~

?note need to give prob of initially here unless question conditioned on it being here

4 n-Step Transition Probabilities

- many problems want prob law of <u>state</u> at a certain time conditioned on current step

$$r_{ij}(n) = P(X_n = j \mid X_0 = i)$$

$\uparrow$

*note letters "r"

- aka probability that the state after n time periods will be j given that current state is i

(realize the subtile differenses – just think about it)

–(be interested in it!)

– Calculate w/ Chapman – Kolmogorov equation

$$r_{ij} = \sum_{k=1}^{m} r_{ik}(n-1)\, p_{kj} \quad \text{for } n > 1 \text{ for all } i, j$$

starting w/ $r_{ij}(1) = p_{ij}$

(so basically add up all the possible states)

(also think seperatly about the math notation and what it actually means)

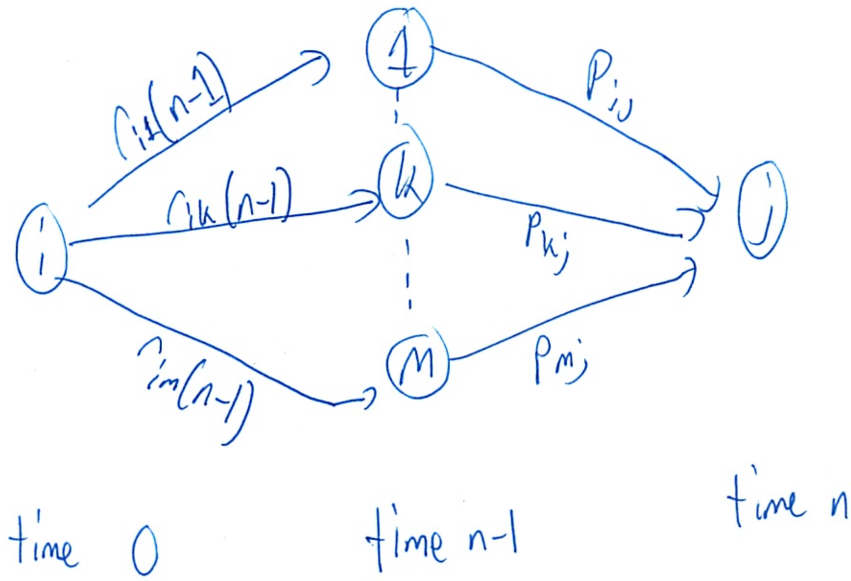
(visualize!!!)

– to verify

$$P(X_n = j \mid X_0 = i) = \sum_{k=1}^{m} P(X_{n-1} = k \mid X_0 = i)\, P(X_n = j \mid X_{n-1} = k, X_0 = i)$$

$$= \sum_{k=1}^{m} r_{ik}(n-1)\, p_{kj}$$

(5)



(wish there were animations for this)

[So sum of all the different ways of reaching j from starting at i

n-step transition probability matrix

$r_{ij}(1)$

	U	B
U	.8	.2
B	.6	.4

$r_{ij}(2)$

.76	.24
.72	.28

$r_{ij}(3)$

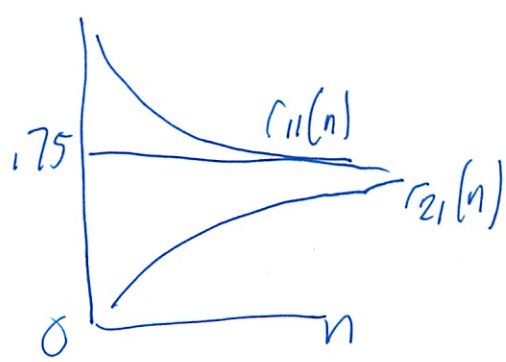
.752	.248
.744	.256

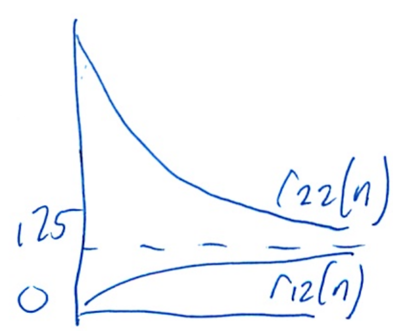
$r_{ij}(4)$

.7504	.2496
.7488	.2512

$r_{ij}(5)$

.7501	.2499
.7498	.2502





[? so is this what we did in 6.01 on Tue?]
(I think was slightly different but closest to anything we did)

⑥

we see $r_{ij}(n)$ converge to a limit as $n \to \infty$

this is the <u>steady state</u> probability

(still moves around - but that is probability of being ~~what?~~ ~~posit~~ at that position)

State j is <u>accessible</u> from a state i if for n the probability $r_{ij}(n)$ is positive

state i <u>recurrent</u> if for every j that is accessible from i, i is ~~all~~ also accessible from j

When we start at recurrent state i we can only visit states $j \in A(i)$ from which i is accessible

↑ a set/class ~~yeah~~

if a recurrent state is visited once it is certain to be revisited an ∞ # of times

<u>transient</u> → not recurrent

(I dislike this math language - so hard to decipher)

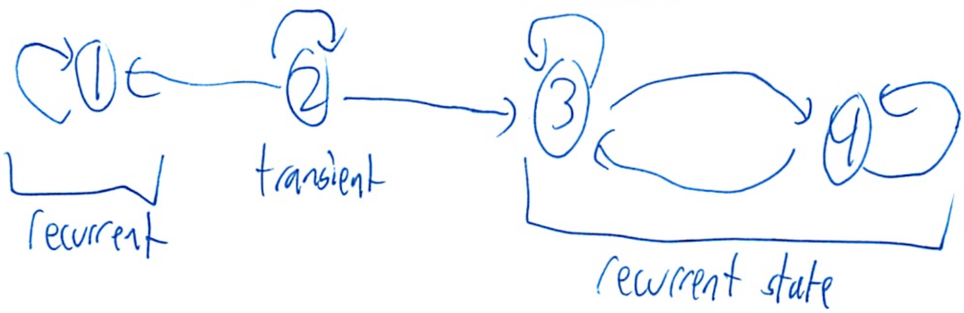
If i is a recurrent state the set of states $A(i)$ that are accessible from i form a <u>recurrent class</u> (class)

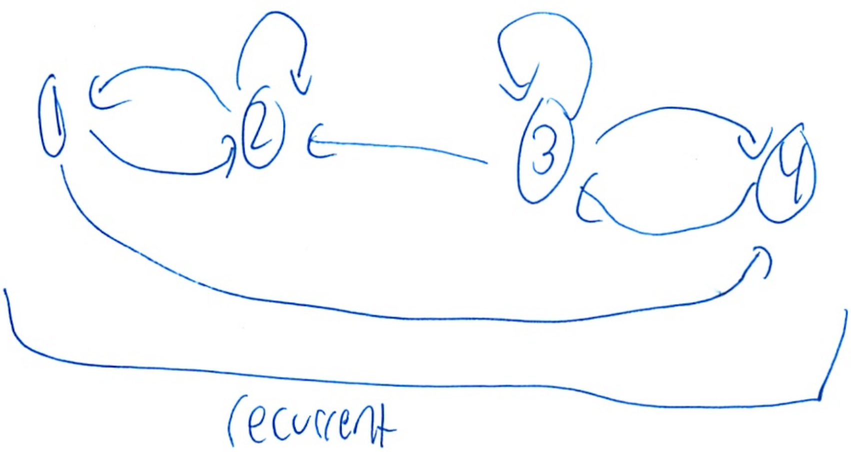
- all states inside $A(i)$ are accessible from eachother
- no state outside $A(i)$ is accessible from them
- $A(i) = A(i)$ ← same set

7

Markov Chain Decomposition

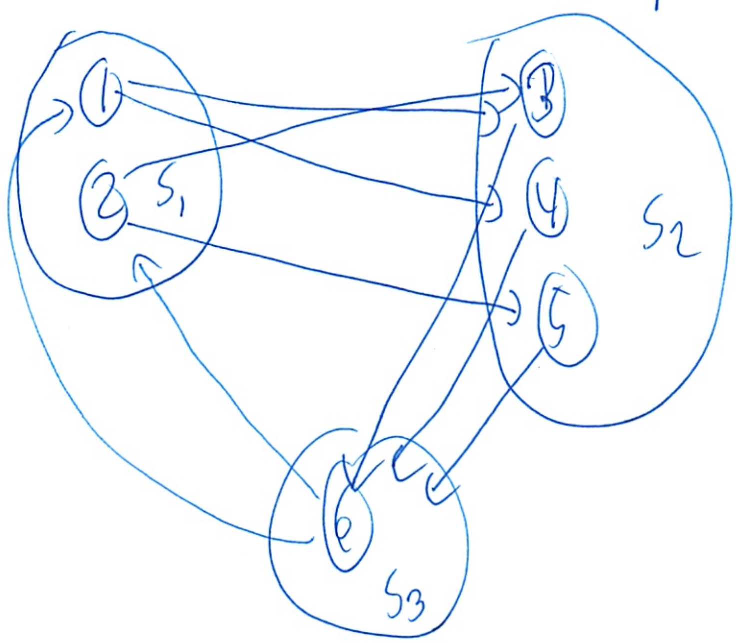
- can be decomposed into 1 or more recurrent classes plus possibly some transient states
- a recurrent state is accessible from all states in its class, but not accessible from recurrent states in other classes
- a transient state is not accessible from any recurrent state
- at least one, possibly more, recurrent states are accessible from a given transient state





8 Periodicity

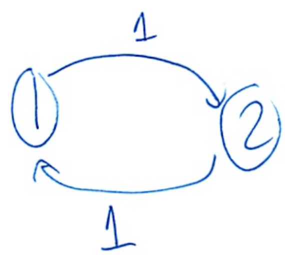
- if states can be graped in $d > 1$ disjoint Subsets $S_1, \ldots, S_d$ so that all transitions from one subset lead to the next subset
- if $i \in S_k$ and $p_{ij} > 0$ then $\begin{cases} j \in S_{k+1} & \text{if } k = 1, \ldots, d-1 \\ j \in S & \text{if } k = d \end{cases}$
- a recurrent class that is not periodic = aperiodic



- so it always gets passed from 1 set to another

(9) 7.3 Steady State Behavior

- the long-term state occupancy behavior
- independent of the inital state
- if two or more classes of recurrent states then it must depend on the inital state
- so restrict to 1 recurrent class (but also consider some transient states)
- even w/ a single recurrent class it may still fail to converge



$P_{12} = P_{21} = 1$

But if start at 1 will always be at
2 when n = odd
1 " " = even

$r_{ii}(n) = \begin{cases} 1 & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$

- so if it is periodic $r_{ij}(n)$ will generically oscillate

- so the probability $r_{ij}(n)$ of being at state j approaches a limiting value independent of inital state ; (and no multiple recurrent classes or aperiodic)
$\pi_j \approx P(X_n = j)$ when n is large

(10)

That is the called the <u>steady-state probability of j</u>

- proof complicated

<u>Properties</u>

a) for every j we have

$$\lim_{n \to \infty} r_{ij}(n) = \pi_j \quad \text{for all } i$$

b) The π_j ~~are~~ are the unique solutions of the equations below

$$\pi_j = \sum_{k=1}^{m} \pi_k p_{kj} \quad j = 1, 2, \ldots, m$$

$$1 = \sum_{k=1}^{m} \pi_k$$

c) We have

$\pi_j = 0$ for all transient states j

$\pi_j > 0$ for all recurrent states j

(note how to represent things in math - I just don't think like this - need to change!)

- these steady state probabilites π_j sum to 1 and form a probability distribution on the state space called the <u>stationary distribution</u> of the chain

- so as long as the initial state is chosen according to the <u>stationary</u> distribution, the state at any future time will have the same distribution

(11)

The equations: are the balance equations

$$\pi_j = \sum_{k=1}^{m} \pi_k p_{kj} \qquad j = 1, \ldots, m$$

(know what is theorem and what is proof)

So as convergence $r_{ij}(n)$ taken for granted

$$r_{ij}(n) = \sum_{k=1}^{m} r_{ik}(n-1)\, p_{kj}$$

along w/ normilization equation

$$\sum_{k=1}^{m} \pi_k = 1$$

Solve the eq like described in lecture
w/ linear algebra
↑ (also know what these terms mean)

Long Term Frequency Interpretations

Probabilities are often interpreted as relative freq in an ∞ly long chain of indep. trials

$$\pi_j = \lim_{n \to \infty} \frac{V_{ij}(n)}{n}$$

← expected value of # of visits to state j within the first n transitions, starting from state i

expected value of repair cost on a day randomly chosen far in the future $= \dfrac{\text{total expected repair cost in } n \text{ days}}{n \text{ days}}$

both are the same

(12)

π_j is the long term expected fraction of time that the state is = to j

each time that the state j is visited there is probability P_{jk} that the next transition takes us to state k

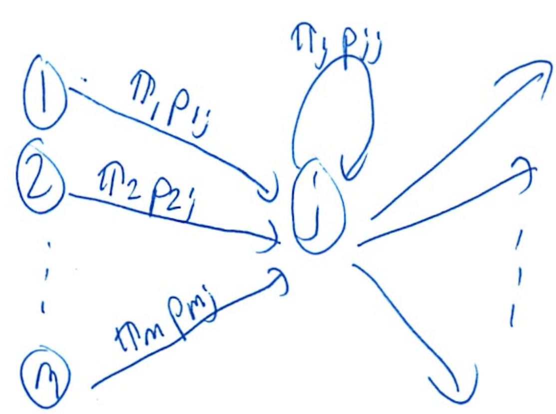
$\pi_j P_{jk}$ is the long-term expected fraction of transitions that move the state from j to k

Expected Freq of Particular Transition

Consider n transitions of a Markov chain w/ a single class which is ~~ap~~ aperiodic starting from a given inital state. Let $q_{jk}(n)$ be the expected # of transitions that take the state from j to k, So regardless of inital state

$$\lim_{n\to\infty} \frac{q_{jk}(n)}{n} = \pi_j P_{jk}$$

So



$$\pi_j = \sum_{k=1}^{m} \pi_k P_{kj}$$

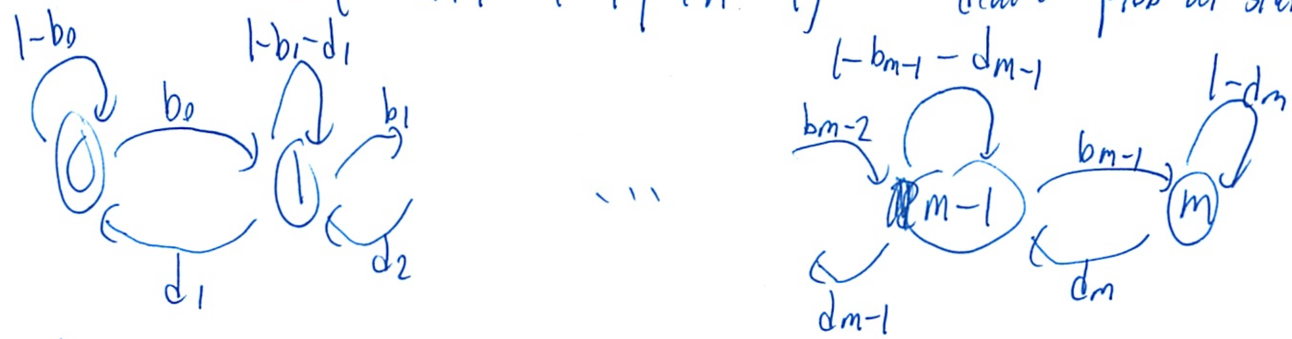
(13)

Birth-Death Process

- Markov chain in which states are linearly arranged and transitions can only occur to a neighboring state or else leave the state unchanged
- popular in queing theory
- some generic notion for transition possibilities

$$b_i = P(X_{n+1} = i+1 \mid X_n = i) \quad \text{"birth" prob at state } i$$

$$d_i = P(X_{n+1} = i-1 \mid X_n = i) \quad \text{"death" prob at state } i$$



- balance eq can be simplified
  - lets look at i and $i+1$
    - any transition from i to $i+1$ has to be followed by $i+1$ to i before i to $i+1$ again
    - (lecture makes more sense now!)
    - So expected freq of transitions from $i \to i+1$ which is $\pi_i b_i$ must be $=$ to expected freq of transition from $i+1 \to i$ which is $\pi_{i+1} d_{i+1}$
    - So <u>local balance</u>

$$\pi_i b_i = \pi_{i+1} d_{i+1} \qquad i = 0, 1, \dots, m-1$$

(14)

$$\pi_i = \pi_0 \frac{b_0 b_1 \cdots b_{i-1}}{d_1 d_2 \cdots d_i} \qquad i = 1, \ldots, m$$

So along w/ normalization eq can find the steady state probs π_i

Queveing Ex 7.9

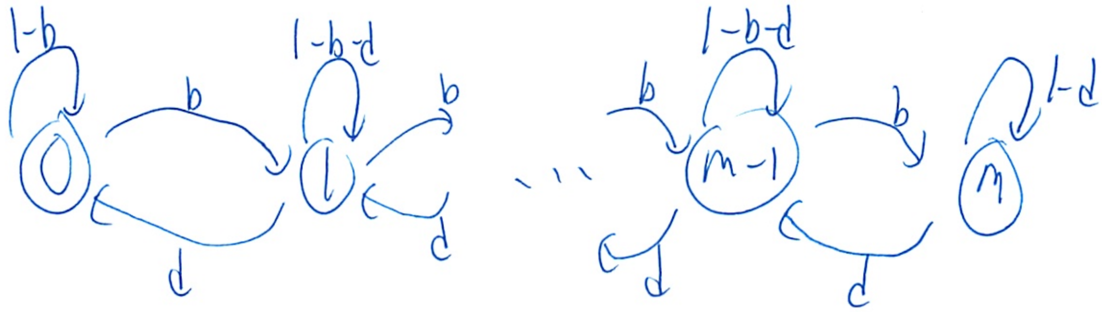
m = storage capacity

b = prob packet arrives

d = prob packet departs

$1 - b - d$ = prob nothing happens if ≥ 1 packets in node

$1 - b$ " " " if 0 packets " "



$$\pi_i b = \pi_{i+1} d \qquad i = 0, 1, \ldots, m-1$$

$$\rho = \frac{b}{d} \leftarrow \text{define}$$

$$\pi_{i+1} = \rho \pi_i$$

$$\pi_i = \rho^i \pi_0 \qquad i = 0, 1, \ldots, m$$

Normalization eq $1 = \pi_0 + \pi_1 + \cdots + \pi_m$

$1 = \pi_0 (1 + \rho + \cdots + \rho^m)$

(15)

$$\pi_0 = \begin{cases} \dfrac{1-\rho}{1-\rho^{m+1}} & \text{if } \rho \neq 1 \\ \dfrac{1}{m+1} & \text{if } \rho = 1 \end{cases}$$

Using $\pi_i = \rho^i \pi_0$ steady state probabilities

$$\pi_i = \begin{cases} \dfrac{1-\rho}{1-\rho^{m+1}} \rho^i & \text{if } \rho \neq 1 \\ \dfrac{1}{m+1} & \text{if } \rho = 1 \end{cases} \quad i = 0, 1, \ldots, m$$

When m buffer size ∞ly large

a) If $b < d$ or $\rho < 1 \rightarrow$ ~~arrival~~ departures more likly so π_i will decrease w/ i as in a truncated geometric PMF

$$\pi_i \rightarrow \rho^i (1-\rho) \text{ for all } i$$

b) If $b > d$ or $\rho > 1 \rightarrow$ arrivals more likly so # of packets grows ∞. Steady state prob π_i increases w/ i

As m grows larger $\pi_i \rightarrow 0$ for all i

Every state is transient

(math interested at the limits)

(it's my confusion like physics where derivitives velocity + acceleration - keeping them apart)

16

7.4 Asorbtion Probabilities and Expected Time to Asorbtion

- Short term behavior of a Markoff chain
- Starting at a transient state
- interested in the 1st recurrent state to be entered and time until this happens

- every recurrent state is asorbing

$P_{kk} = 1$ $\qquad P_{kj} = 0$ for all $j \neq k$

steady state prob is 1

- all other states are transient
- will be reached w/ prob = 1 (certenty)
- from <u>any</u> inital state
- or prob of <u>every</u> asorbing state is added up

$a_i = P(X_n$ eventualy becomes = to asorbing state $s \mid X_0 = i)$

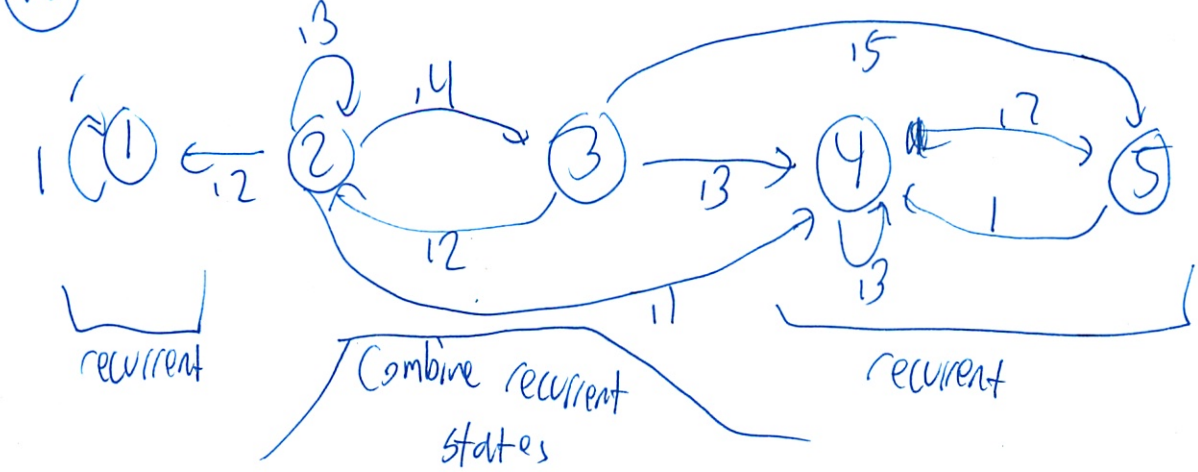
$a_s = 1$

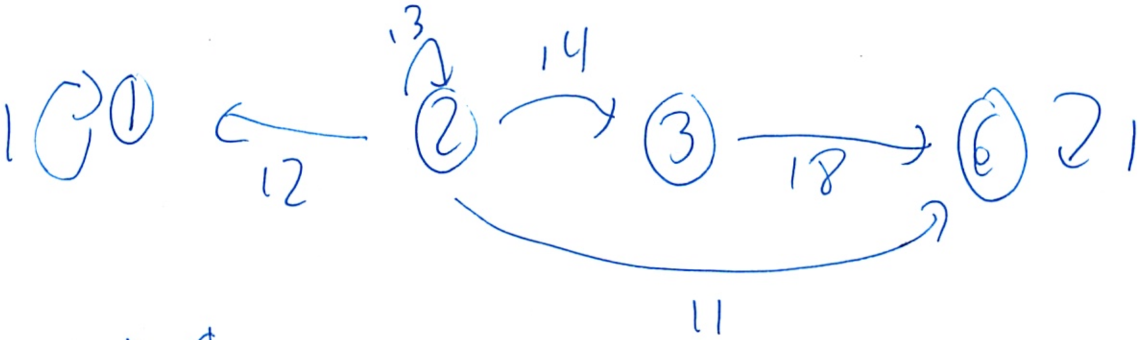
$a_i = 0$ for all absorbing $i \neq s$

$a_i = \sum_{j=1}^{m} P_{ij} a_j$ for all transient i

(skipping proof)

17





each of the transient states

$a_2 = .2a_1 + .3a_2 + .4a_3 + .1a_6$

$a_3 = .2a_2 + .8a_6$

$a_1 = 0$ ← ?? why 0, or qu asked prob of entering $\{4,5\}$

$a_6 = 1$

$a_2 = .3a_2 + .4a_3 + .1$

$a_3 = .2a_2 + .8$

Solve for a_2, a_3

$a_2 = \frac{21}{31}$

$a_3 = \frac{29}{31}$

(17b) Confirm

11/25

$a_2 = .3a_2 + .4a_3 + .1$

$a_3 = .2a_2 + .8$

$$\frac{a_3 - .8}{.2} = a_2$$

$$\frac{a_3 - .8}{.2} = .3\left(\frac{a_3 - .8}{.2}\right) + .4a_3 + .1$$

$\cdot .2 \quad \cdot .2 \quad \cdot .2 \quad \cdot .2$

$a_3 - .8 = .3(a_3 - .8) + .08a_3 + .02$

$a_3 = .3a_3 - .24 + .08a_3 + .02 + .8$

$a_3 = .38a_3 - .58$

$.62a_3 = -.58$

$$a_3 = -\frac{.58}{.62} = \frac{29}{31}$$

$$a_2 = \frac{\frac{29}{31} - .8}{.2} = \frac{21}{31}$$

Ok can do this

(18) ## Expected Time to Absorption

for any state i

$$\mu_i = E[\text{\# of transitions until asorbtion, starting from } i]$$
$$= E[\min\{n \geq 0 \mid X_n \text{ is recurrent}\} \mid X_0 = i)$$

So if i is recurrent $\mu_i = 0$

derived from total expectation theorm

time to asorbtion starting from transient state i is = to 1 plus the expected time to asorbtion starting from the next state which is j w/ prob p_{ij}

then get system of linear eq

$$\mu_i = 0 \quad \text{for all recurrent states } i$$
$$\mu_i = 1 + \sum_{j=1}^{m} p_{ij}\mu_j \quad \text{for all transient states } i$$

Mean First Passage + Recurrence times

- this ~~case~~ expected time to asorbtion can also be used to reach a particular recurrent state starting from any other state
- for Markov chain w/ a single recurrent state
- state s
- t_i = mean first passage time from state i to state s

(19)

t_i = E[# of transitons to reach s for the first time, starting from i]

$= E[\min\{n \geq 0 \mid X_n = s\} \mid X_0 = i]$

transitions out of state s are irrelevant to calculation of the mean first passage chain

So new Markov chain ~~+~~ identical to original - except special state s is converted into an absorbing state

$p_{ss} = 1 \quad p_{sj} = 0$ for all $j \neq s$

ie all other states become transient

$$t_i = 1 + \sum_{j=1}^{m} p_{ij} t_j \quad \text{for all } i \neq s$$

$$t_s = 0$$

this system of linear eq can be solved

Mean recurrence time of special state s

---

t_s^* = E[# of transitions up to 1st return to s, starting from s]

$= E[\min\{n \geq 1 \mid X_n = s\} \mid X_0 = s]$

Can obtain once have 1st passage time t_i

$$t_s^* = 1 + \sum_{j=1}^{m} p_{sj} t_j$$

- time to return to s, starting from s, equals 1 + expected time to reach s from next state, which is j w/ prob p_{sj}

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering & Computer Science

6.041/6.431: Probabilistic Systems Analysis
(Fall 2010)

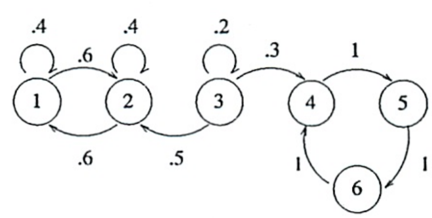
Tutorial/Recitation 9
November 12, 2010

1. Problem 7.13, page 385 in textbook.
The times between succesive customer arrivals at a facility are independent and identically distributed ranom variables with the following PMF:

$$p(k) = \begin{cases} 0.2, & k = 1 \\ 0.3, & k = 3 \\ 0.5, & k = 4 \\ 0, & \textit{otherwise} \end{cases}$$

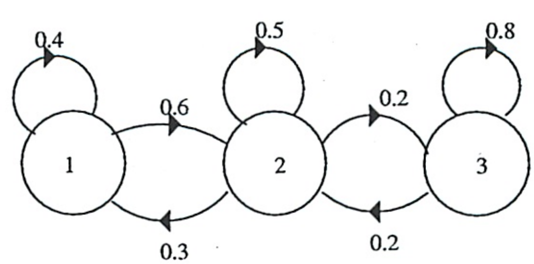
Construct a four-state Markov chain model that describes the arrival process. In this model, one of the states should correspond to the times when an arrival occurs.

2. The Markov chain shown below is in state 3 immediately before the first trial.



(a) Indicate which states, if any, are recurrent, transient, and periodic.
(b) Find the probability that the process is in state 3 after n trials.
(c) Find the expected number of trials up to and including the trial on which the process leaves state 3.
(d) Find the probability that the process never enters state 1.
(e) Find the probability that the process is in state 4 after 10 trials.
(f) Given that the process is in state 4 after 10 trials, find the probability that the process was in state 4 after the first trial.

3. Problem 7.13, page 385 in textbook.
Consider the Markov chain below. Let us refer to a transition that results in a state with a higher (respectively, lower) index as a birth (respectively, death). Calculate the following quantities, assuming that when we start observing the chain, it is already in steady-state.



(a) For each state i, the probability that the current state is i.

(b) The probability that the first transition we observe is a birth.

(c) The probability that the first change of state we observe is a birth.

(d) The conditional probability that the process was in state 2 before the first transition that we observe, given that this transition was a birth.

(e) The conditional probability that the process was in state 2 before the first change of state that we observe, given that this change of state was a birth.

(f) The conditional probability that the first observed transition is a birth given that it resulted in a change of state.

(g) The conditional probability that the first observed transition leads to state 2, given that it resulted in a change of state.

Tutorial 9

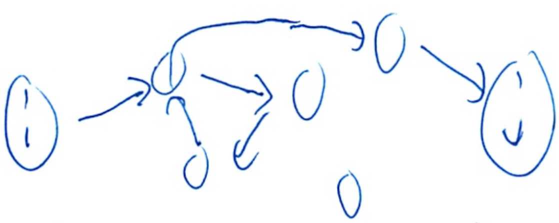
11/12

(5 min late)

Markov only depends on current state

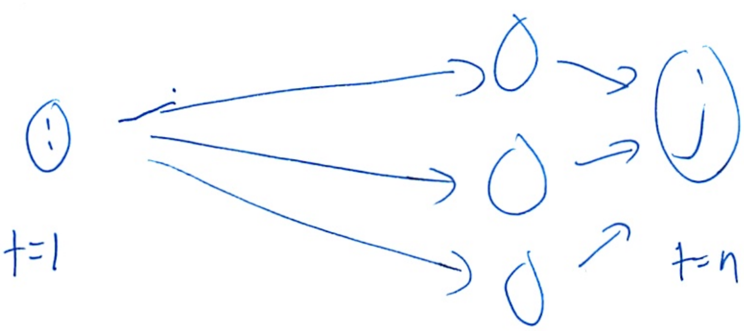
← 1 step

But what about prob of being there in n steps



$t=1$ $t=n$

So represent as



$t=1$ $t=n$

$t=n-1$

So iterate it back if you actually have n

$$r_{ij}(n) = \sum_{k=1}^{m} r_{ik}(n-1) \cdot p_{kj}$$

$$= \sum_{k=1}^{m} p_{ik} \cdot r_{kj}(n-1) \quad \leftarrow \text{other way}$$

m = # states
i = start } state
j = end } state
k = transition
m = # of steps

2

<u>recurrent</u> - can always get back to the state

See #2

1, 2 recurrent class

4, 5, 6 recurrent class

3 → transient

4, 5, 6 are periodic
- can divide into distinct groups
- always move from 1 group to another
- in this case each state is its own group

Steady state

- after Markov chain runs $n \to \infty$ How does it behave
- <u>conditions</u> - single ~~...~~ recurrent class
  - aperiodic

- $P(X_n = j) \approx \pi_j$ large n

- $\lim_{n \to \infty} r_{ij}(n) = \pi_j \quad \forall_i$

  for all

- How to find?

  local balance ⊗ only for birth-death

  total balance ✓

(3)

$$\begin{cases} \pi_j = \sum_{k=1}^{m} \pi_k \cdot p_{kj} & j = 1, 2, \dots, m \\ 1 = \sum_{k=1}^{m} \pi_k \end{cases}$$

$$\lim_{n \to \infty} r_{ij}(n) = \sum_{k=1}^{m} r_{ik}(n-1)\, p_{kj}$$

$$\pi_j = \sum_{k=1}^{m} \pi_k \, p_{kj}$$

$\pi_2 = \pi_1 \cdot .6 + \pi_2 \cdot .4$ ← one eq

steady state prob of transient state $= 0$

recurrent > 0

---

freq + E[ ] n always long term

- freq of being in state i

  $= \pi_i$ so prob and freq are same thing!

- freq of transitions $i \to j$

  $= \pi_i \cdot p_{ij}$

- freq of transitions into j

  $= \sum_i \pi_i \, p_{ij} = \pi_j$ j ← all of the ways of entering j

6

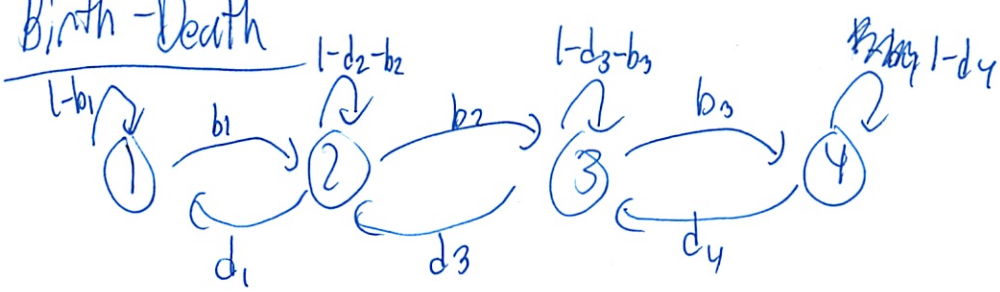
$E[\#$ of visits to j $\underbrace{\text{starting from } i,}_{\text{does not make a difference}}$ in n steps$]$

$= n \cdot \pi_j$

$E[\#$ of transitions from $i \to j$ in n steps$]$

$= n \cdot \pi_i \cdot p_{ij}$

Birth-Death



How to find balance eq?

- can not transition b_i again until d_i happened

$\pi_i \cdot b_i = \pi_{i+1} d_{i+1}$

$\sum_i \pi_i = 1$

and more formulas in book

(5) Start in 3 for all

#2 b) Prob(in state 3 after n trials) =

$= r_{33}(n)$ ← but don't think about this here

$= (.2)^n$

c) E[trials to leave state 3] =

~geometric

- sucess is leaving state 3 $p = .5 + .3$

$= \dfrac{1}{p(\text{sucess})} = \dfrac{1}{.8}$

d) Prob (process never enters state 1) =

= P(That it enters 4,5,6 recurrent class)

$= \dfrac{.3}{.5 + .3} = \dfrac{3}{8}$

$\lim_{n \to \infty} r_{34}(n)$

π only valid in 1 recurrent class

this is called asorbtion - doing later

e) P(in State 4 at Trial 10)

$= \left[(.2)^9 \cdot .3\right] + \left[.3 \cdot \underset{1^9}{\text{cycle 3 times}}\right] + \left[(.2)^3 \cdot .3 \cdot \underset{1^6}{\text{cycle 2 times}}\right] + \left[(.2)^6 \cdot .3 \cdot \underset{1^3}{\text{cycle once}}\right.$

(6)

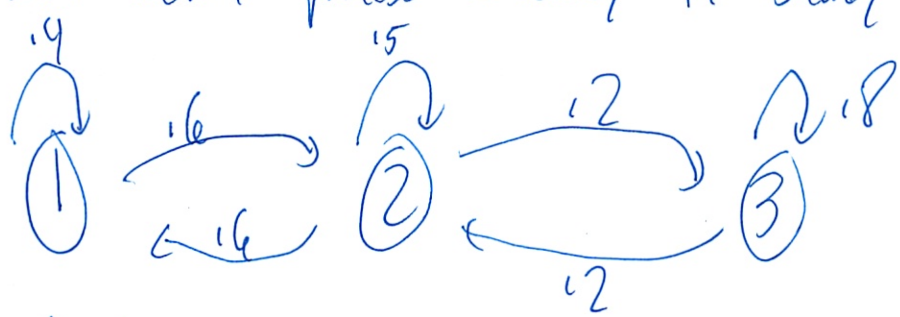
f) $P($process in state 4 after 1st trial $\overbrace{}^{B}$ | process in state 4 after first 10 trials$\overbrace{}^{A})$

$$\frac{P(B \cap A)}{P(A)} = \frac{.3}{.3024}$$

- This whole problem was ~~steady~~ short term - no steady state

---

3. Birth-death process already in steady state



a) So local balance

$\pi_1 \cdot .6 = \pi_2 \cdot .3$ $\qquad \sum \pi_i = 1$

$\pi_2 \cdot .2 = \pi_3 \cdot .2$ $\qquad \pi_1 + \pi_2 + \pi_3 = 1$

So 3 eq 3 unknowns solve

$\pi_1 = 1/5$

$\pi_2 = \pi_3 = 2/5$

b) $P(\overbrace{\text{1st transition observed is a birth}}^{A})$

= lots of ways to have a birth

- Condition on where you start S_i = starting state before transition is i

7

So total prob theorm of all states

starting in 3

$= P[A|S_1] \cdot P[S_1] + P[A|S_2] P(S_2) + P(A|S_3) \cdot P(S_3)$

$= .6 \cdot \pi_1 + .2 \pi_2 + 0 \cdot \pi_3$

$= 1/5$

c) $P(\overbrace{\text{1st change of state is a birth}}^{B})$

no self transitions

$= P(B|S_1) P(S_1) + P(B|S_2) \cdot P(S_2) + P(B|S_3) P(S_3)$

$= 1 \quad \pi_1 + \frac{.2}{.2+.3} \quad \pi_2 + 0 \quad \pi_3$

self transition not allowed

self transition not allowed

$= \frac{9}{25}$

1st transition is a birth

d) $P(S_2 | A) = \frac{P(S_2 \cap A)}{P(A)} = \frac{\pi_2 (.2)}{.2}$

$= \frac{2}{5}$

take that piece from eq

e) $P(S_2 | B) = \frac{P(S_2 \cap B)}{P(B)} = \frac{\frac{.2}{.2+.3} \pi_2}{9/25} = 4/9$

(8)

last parts can be solved w/ math or w/ intuitive answer

f) $P[A \mid \text{1st transition is change of state}] = \frac{1}{2}$

since freq from $\curvearrowright$ = freq $\curvearrowleft$

equally likely birth or death

what goes up must come down

or classical way

$$= \cancel{\frac{P[A \wedge \text{1st transition change of state}]}{P[\text{1st transition change of state}]}}$$

$$= \frac{P[\text{birth}]}{P[\text{change of state}]} = \frac{1/5}{\frac{1}{2}\cdot.6 + \frac{2}{5}\cdot.5 + \frac{2}{5}\cdot.2} = \frac{1}{2}$$

← answer to part b

g) $P(\text{first observed transition is to state } 2 \mid \text{change of state}] =$

$$= \frac{\pi_1 \cdot .6 + \pi_3 \cdot .2}{2/5} = 1/2$$

transitioning into 2 has to have same freq as leaving it

what goes up ~~what~~ ~~come~~ must come down

Very unlikely to go ∞ grow

all assumes finite # of steps

Michael Plasmeier

3/10

11/15

Massachusetts Institute of Technology
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Fall 2010)

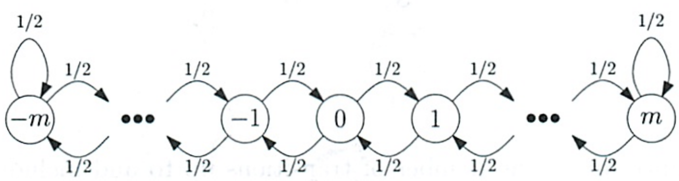
Problem Set 8
Due November 15, 2010

1. Oscar goes for a run each morning. When he leaves his house for his run, he is equally likely to go out either the front or back door; and similarly, when he returns, he is equally likely to go to either the front or back door. Oscar owns only five pairs of running shoes which he takes off immediately after the run at whichever door he happens to be. If there are no shoes at the door from which he leaves to go running, he runs barefooted. We are interested in determining the long-term proportion of time that he runs barefooted.

   (a) Set the scenario up as a Markov chain, specifying the states and transition probabilities.

   (b) Determine the long-run proportion of time Oscar runs barefooted.

2. Consider a Markov chain $X_1, X_2, \ldots$ modeling a *symmetric simple random walk with barriers*, as shown below:



   (a) Explain why $|X_1|, |X_2|, |X_3|, \ldots$ also satisfies the Markov property and draw the associated chain.

   (b) Suppose that we also wish to keep track of the largest deviation from the origin, i.e., define the largest deviation at time t as $Y_t = \max\{|X_1|, |X_2|, \ldots, |X_t|\}$. Draw a Markov chain that keeps track of the largest deviation and explain why it satisfies the Markov property.

3. As flu season is upon us, we wish to have a Markov chain that models the spread of a flu virus. Assume a population of n individuals. At the beginning of each day, each individual is either infected or susceptible (capable of contracting the flu). Suppose that each pair (i, j), $i \neq j$, independently comes into contact with one another during the daytime with probability p. Whenever an infected individual comes into contact with a susceptible individual, he/she infects him/her. In addition, assume that overnight, any individual who has been infected for at least 24 hours will recover with probability $0 < q < 1$ and return to being susceptible, independently of everything else (i.e., assume that a newly infected individual will spend at least one restless night battling the flu).

   (a) Suppose that there are m infected individuals at daybreak. What is the distribution of the number of new infections by day end?

   (b) Draw a Markov chain with as few states as possible to model the spread of the flu for $n = 2$. In epidemiology, this is called an SIS (Susceptible-Infected-Susceptible) model.

   (c) Identify all recurrent states.

Due to the nature of the flu virus, individuals almost always develop immunity after contracting the virus. Consequently, we improve our model and assume that individuals become infected at most one time. Thus, we consider individuals as either infected, susceptible, or recovered.

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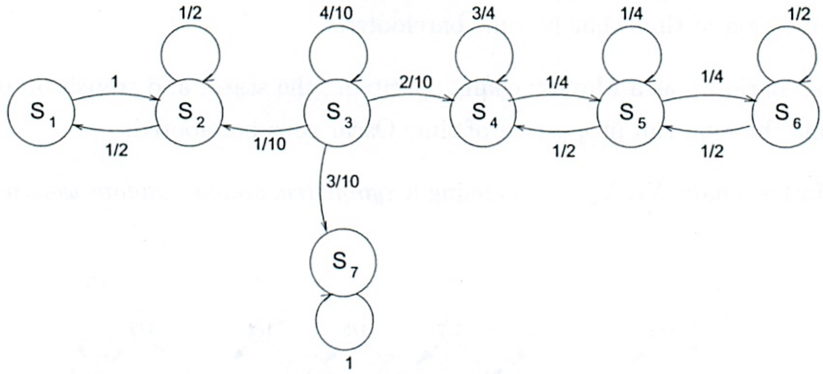
(Fall 2010)

(d) Draw a Markov chain to model the spread of the flu for $n = 2$. In epidemiology, this is called an SIR (Susceptible-Infected-Recovered) model.

(e) Identify all recurrent states.

4. Consider the Markov chain below. For all parts of this problem, the process is in state 3 immediately before the first transition. Be sure to comment on any unusual results.



(a) Find the variance for J, the number of transitions up to and including the transition on which the process leaves state 3 for the last time.

(b) Find the expectation for K, the number of transitions up to and including the transition on which the process enters state 4 for the first time.

(c) Find π_i for $i = 1, 2, \ldots, 7$, the probability that the process is in state i after 10^{10} transitions.

(d) Given that the process never enters state 4, find the π_i's as defined in part (c).

G1†. Consider a Markov chain $\{X_k\}$ on the state space $\{1, \ldots, n\}$, and suppose that whenever the state is i, a reward $g(i)$ is obtained. Let R_k be the total reward obtained over the time interval $\{0, 1, \ldots, k\}$, that is, $R_k = g(X_0) + g(X_1) + \cdots + g(X_k)$. For every state i, let

$$m_k(i) = E[R_k \mid X_0 = i],$$

and

$$v_k(i) = \text{var}(R_k \mid X_0 = i)$$

respectively be the conditional mean and conditional variance of R_k, conditioned on the initial state being i.

(a) Find a recursion that, given the values of $m_k(1), \ldots, m_k(n)$, allows the computation of $m_{k+1}(1), \ldots, m_{k+1}(n)$.

(b) Find a recursion that, given the values of $m_k(1), \ldots, m_k(n)$ and $v_k(1), \ldots, v_k(n)$, allows the computation of $v_{k+1}(1), \ldots, v_{k+1}(n)$. *Hint*: Use the law of total variance.

†Required for 6.431; optional for 6.041

Michael Plasmeier **P-Set 8** 11/15

1. Oscar goes for a run each morning.

=ly likly to go back or front door

=ly licly to return to either door.

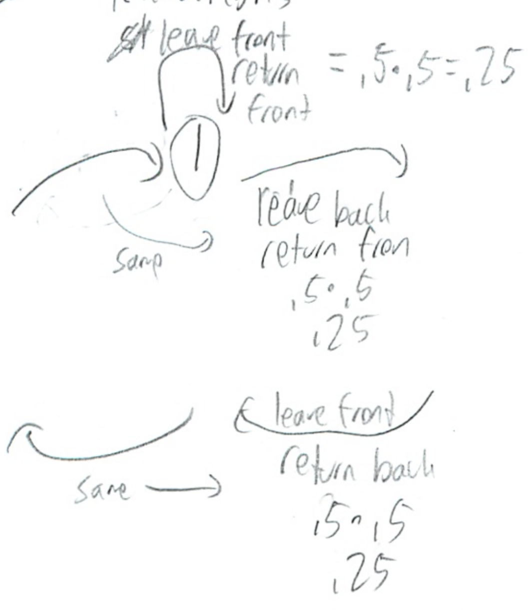
Owns 5 shoes. Leaves at door

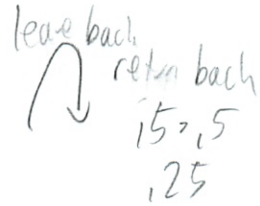
Will go barefoot

Want long term prob barefoot $\pi_{barefoot}$

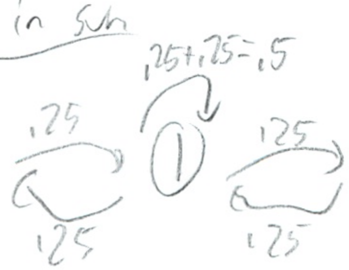
S = So # of shoes at door front

So transitions



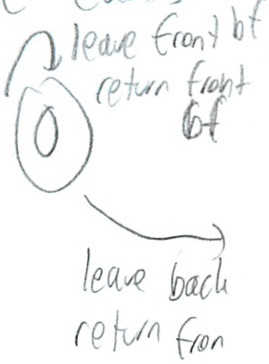


So in sum



②

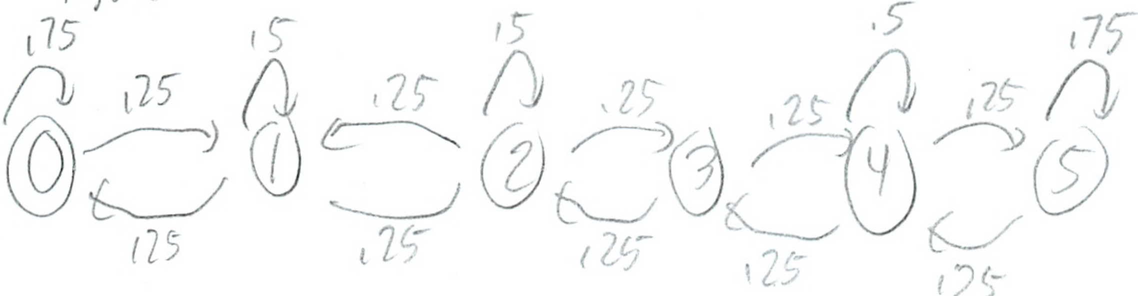
End cases



+ leave back return back .25 + leave front bf return back .25

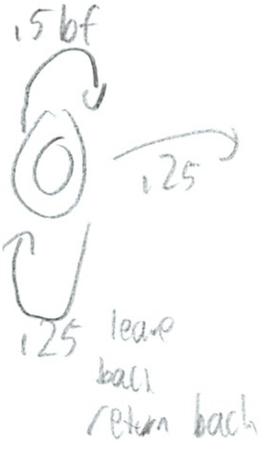
- other by symmetry

Put together



b) How long run proportion he runs barefoot

Actually have to split



We want $r_{00b} + r_{55b}$ ← but only when barefoot

? long run transition posibilities

$\lim_{n\to\infty} r_{ij}(n) = \pi_j$ $\underbrace{\text{for all } i}_{\text{not true here}}$ ← transition posibilities

We are not interested in long run of 0 but long run that those trasitions are made

(3) Oh wait that is

$$\lim_{n\to\infty} \frac{q_{jk}(n)}{n} = \pi_j P_{jk}$$ ← expected # of transitions

$= \pi_0 P_{00bf} + \pi_5 P_{55bf}$

↑ .5 ↑ .5

now I need these

so need balance eq

$$\pi_j = \sum_{k=1}^{m} \pi_k P_{kj} \qquad j = 1, \dots, m$$

↑ so that is all of the probabilities in, right

did we ever do a practical example?

↑ is it birth-death

- example 7.6 textbooks

$\pi_0 = .5\pi_0 + .25\pi_0 + .25\pi_1$

↑ all arrival possibilities

$\pi_1 = .25\pi_0 + .5\pi_1 + .25\pi_2$

$\pi_2 = .25\pi_1 + .5\pi_2 + .25\pi_3$

$\pi_3 = .25\pi_2 + .5\pi_3 + .25\pi_4$

$\pi_4 = .25\pi_3 + .5\pi_4 + .25\pi_5$

$\pi_5 = .5\pi_5 + .25\pi_5 + .25\pi_4$

④ Now need solve somehow

- system of linear eq
- Wolfram Alpha

$\pi_0 = \pi_1 = \pi_2 = \pi_3 = \pi_4 = \pi_5$ ← ? correct ?

so $\frac{1}{5}$ for each

So

$= \pi_0 \, P_{00} \, bf + \pi_5 \, P_{55} \, by$

$\frac{1}{5} \cdot \frac{1}{2} + \frac{1}{5} + \frac{1}{2}$

$= \frac{2}{10} = \frac{1}{5}$

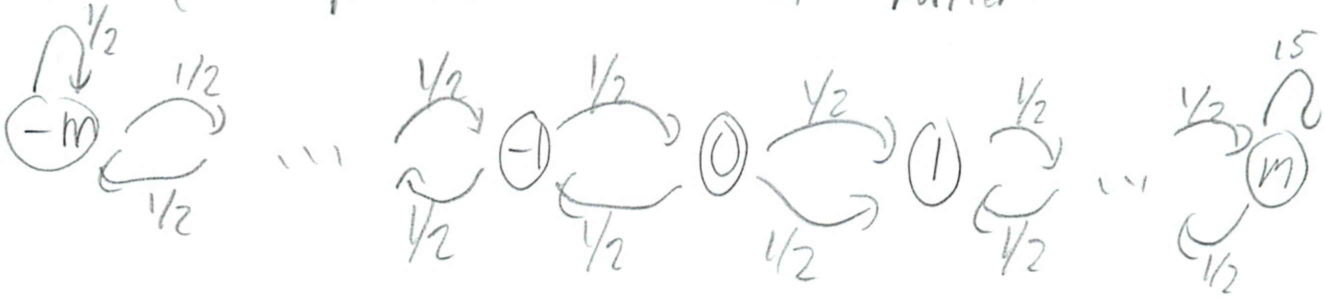
? could we know that all π were same since all recurrent ?

No

5

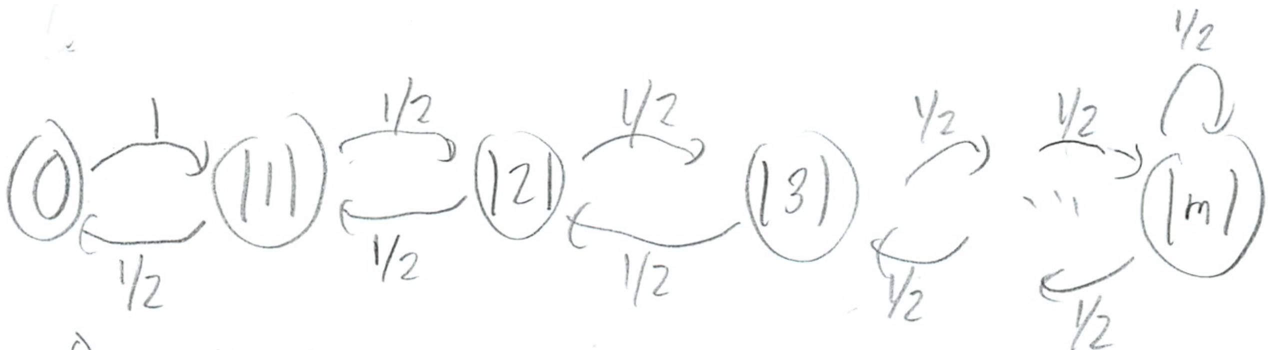
2. Following Markoff chain

Symmetric simple random walk w/ barriers



a) Explain why $|X_1|, |X_2|, |X_3|$ aslo satisfies the Markov property. + draw chain

?SD



From 0 can go

$\begin{cases} \frac{1}{2} & -1 \\ \frac{1}{2} & 1 \end{cases}$

From +1 can go

$\begin{cases} \frac{1}{2} & 0 \\ \frac{1}{2} & +2 \end{cases}$

From +2 can go

$\begin{cases} \frac{1}{2} & +3 \\ \frac{1}{2} & +1 \end{cases}$

Repeat

From -1 can go

$\begin{cases} \frac{1}{2} & 0 \\ \frac{1}{2} & -2 \end{cases}$

from -2 can go

$\begin{cases} \frac{1}{2} & -1 \\ \frac{1}{2} & -3 \end{cases}$

(6)

Markov property

$$P(X_{n+1} = j \mid X_n = i, \underbrace{X_{n-1} = i_{n-1}, \cdots, X_0 = i_0}_{\text{basically } \to \text{ this does not matter}})$$

$$= P(X_{n+1} = j \mid X_n = i)$$

$$= P_{ij}$$

ii. IDK

The same past does not matter because it just forms a simple Markov Chain where the past continues not to matter

iii.

(7)

b) Suppose we wish to keep track of the largest deviation from the origin

$$Y_t = \max\{|X_1|, |X_2|, \ldots, |X_t|\}$$

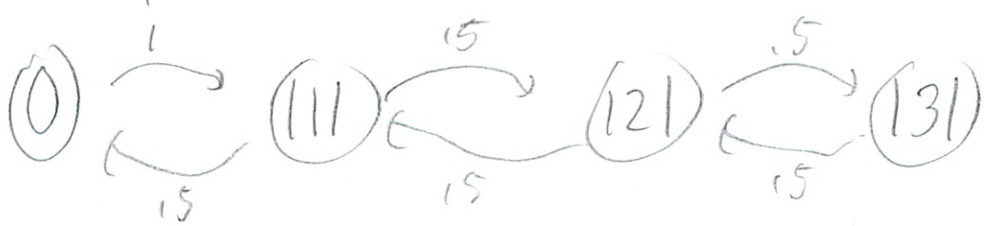
; So essentially it never goes down
So lets assume we have 2 part chain

$Y_0 = |0|$ always
$Y_1 = |1|$ always / w/ certanity

Now 3 parts

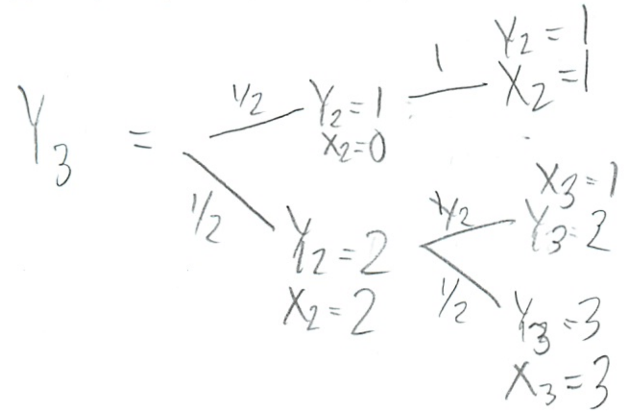
$Y_2 = |1|$ w/ p ½ (if goes to 0) $\downarrow X_2 = 0$
$\quad = |2|$ w/ p ½ (if goes to 2)
$X_2 = 2$

Now 4

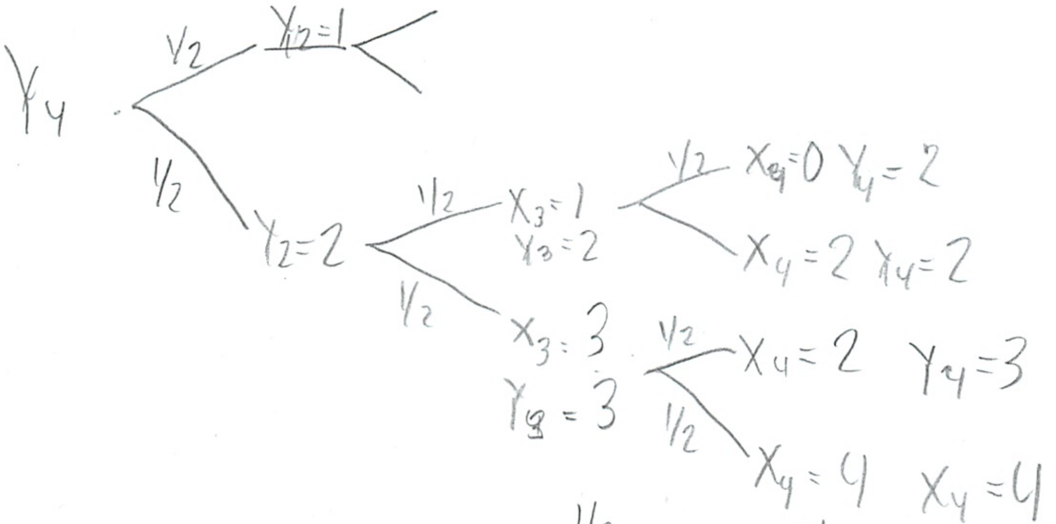


8

Starts to break off



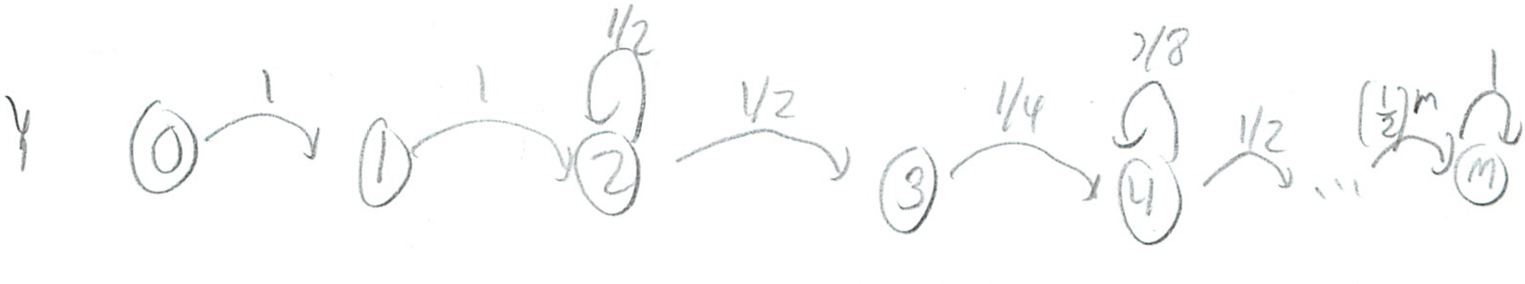
all abs value



So $p=\left(\frac{1}{2}\right)^3$

To m which is

← can't get any bigger



(9)

It follows the Markov property because I was able to draw a Markov chain to represent it. At a particular state the transition probabilites are fixed (for that given state). History does not matter

¿What else to say?

in addition to math on last pg

(10)

3. As flu season is upon us have a Markov chain

Population $= n$

Each individual is infected or susceptiable each day

Each day pair (i,j) $i \neq j$ come into contact w/ each other w/ prob p + if they are infected, will infect other

Sepertly any individual will recover w/ p $0 < q < 1$ and return to being susceptible

Individuals stay infected at least 24 hrs (basically to next day - don't need to worry)

? Birth death ??

What is chain - # people infected

start ?

$$\begin{tikzcd} \textcircled{0} & \textcircled{1} & \textcircled{2} \arrow[l, bend left, "2q"] \end{tikzcd}$$

↑ each person ind

So $q = 1$ cured

So kq

↑ current state

11 So what is prob of infection

Choose i indp.

j indp

~~Both suceptiple~~

Oh wait <u>multiple pairs</u>

	sick i	
sick j	—	+1
	+1	—

So each pair has prob p of happening

So if 1 infected

n people

each person infected w/ prob p $(n-1)p$

If 2 infected

will infect each of the n-2 people

if they meet w/ prob

$$(n-2)p + (n-2)p$$

person A infecting each other person

$$2((n-2)p)$$

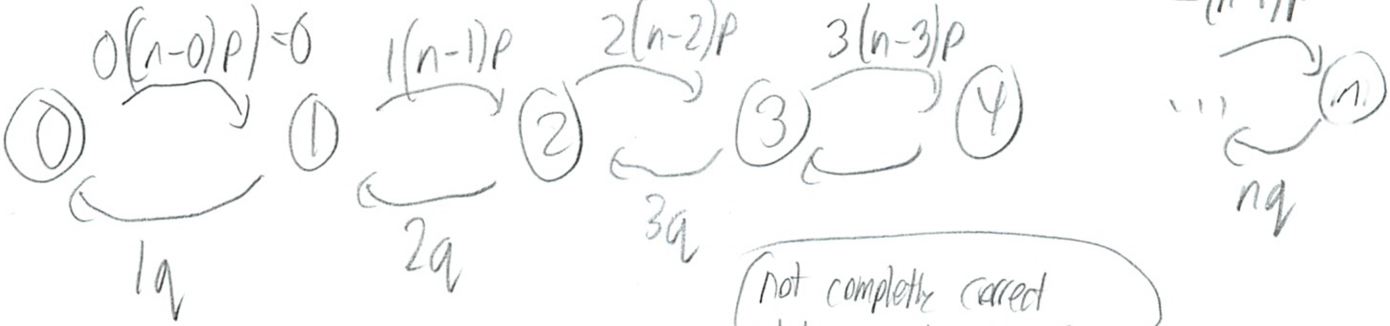
(12)

(I really like taking the time to figure this stuff out)
(I am much better - I hope)

So if k people infected

$k((n-k)p)$

$(n-1)(n-[n-1])p$
$(n-1)(-1)p$
$-(n-1)p$



not completly correct
- but never have to draw

a) Suppose m individuals at start of day (starting condition)
What is distribution of # of new infections at days end
? ie what is next step

First off edge case if $m=0$
than will be 0 cases next day

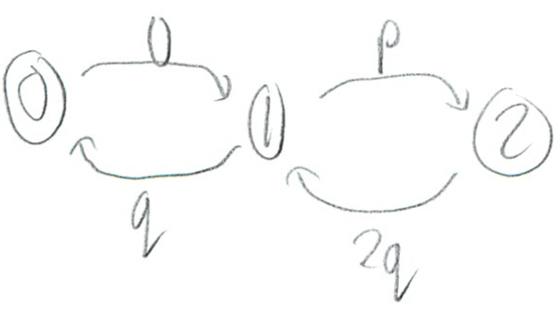
(-1)

Otherwise there are $m(n-m)p$ new cases
(see work on last page)

b) Draw a Markov chain w/ as few states as possible
for $n=2$. This is a SIS Susceptible - Infected - Susceptible

b/c you go back to Susptible

(-1)



See p14 for correction

(13)

c) Identify all recurrent states

0 is the only recurrent state because you can not transfer out of it

And you can transfer into 0 from all states no matter the starting state +1

Due to nature of flu viruses people become immune after being infected once. (SIR) Suceptible - infected - recovered

So now 1 person actively infected

$n-c-1$ people suceptable

c people cured

but this is not Markov anymore

$(n-c)p$

2 people infected

$(n-c-2)p + (n-c-2)p$ to infected

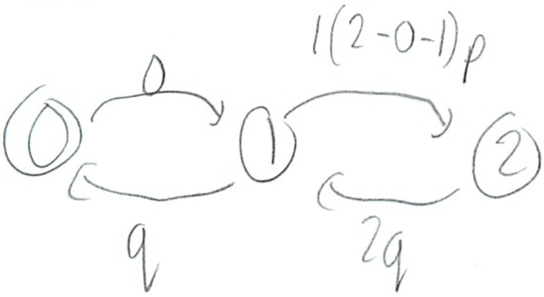
$2(n-c-2)p$

k people infected

$k(n-c-k)p$

(14)

So how to represent in a Markov chain?



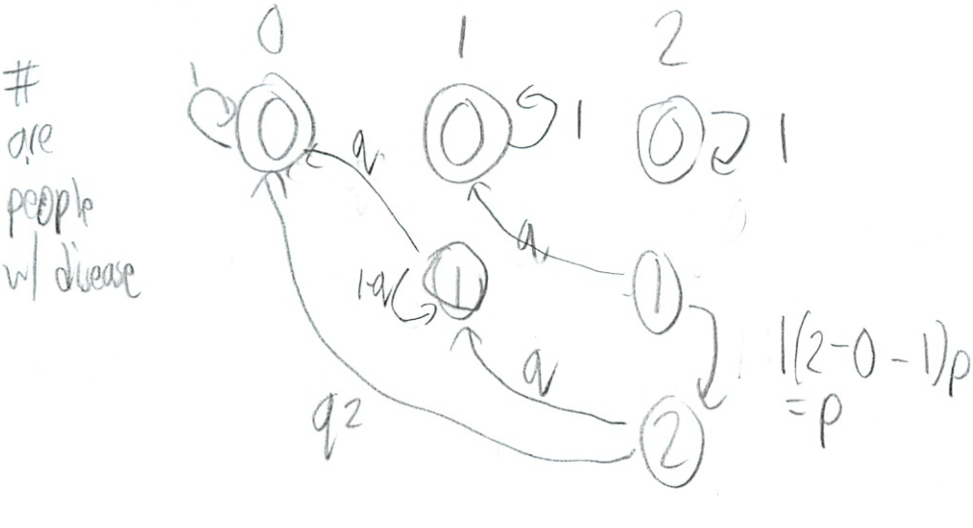
But how to represent people not sickened anymore

And my model above was wrong - I only had 1/0 people being sickened/cured at once

b revisit)

← also forgot the self loops

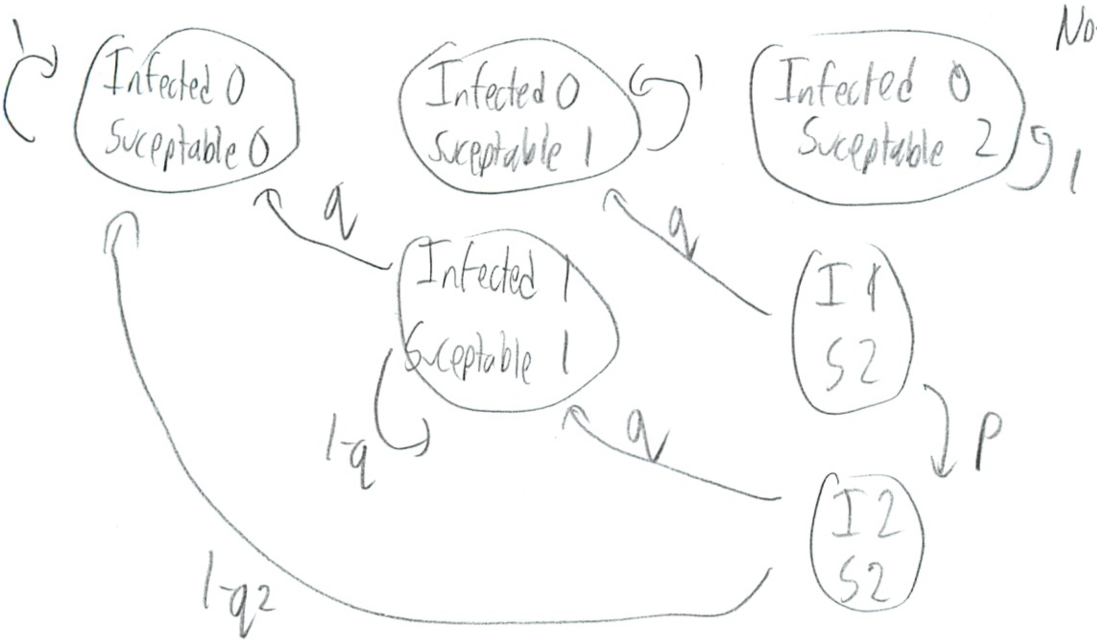
back to d) Need some sort of matrix?



(-1)

(15)

well the suceptable should be inside the bubble



Note suceptable = suceptable + infected

e) Identify all recurrent states

All of the 0 people infected states were recurrent, All other states led down to them

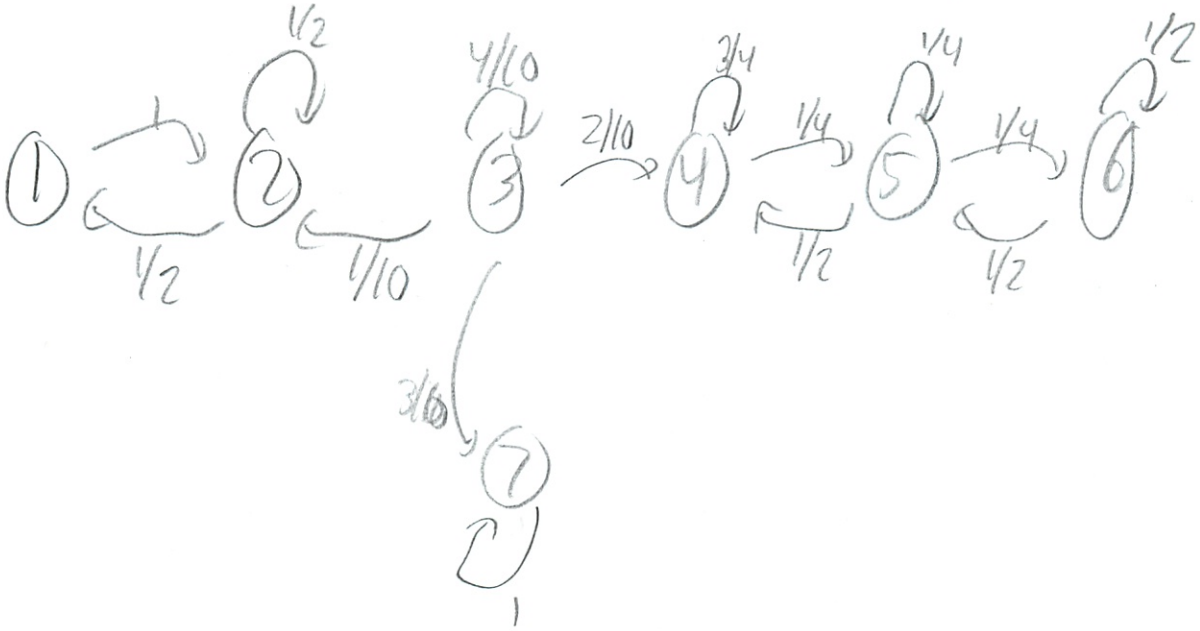
+1

2/5

(16)

(This is either much easier than before -or I really get it)
(spent 2 hrs up to now)

4. Consider this Markov chain
Process starts in state 3 $X_0 = 3$



a) Find the variance for J = # of transitions up to and including the transition on which process leaves J for last time

- Once it leaves J it does not come back
- Starts at 3

- So $P(\text{at 3 after 1 time}) = \frac{4}{10}$

$P(\quad " \quad 2 \text{ times}) = \left(\frac{4}{10}\right)^2$

$n \qquad \left(\frac{4}{10}\right)^n$

But want Var

(17)

Where is var in here

Don't see it anywhere in book ??

Unless some general variance format

$$E[J] = \frac{4}{10} \cdot 1 + \left(\frac{4}{10}\right)^2 \cdot 2 + \left(\frac{4}{10}\right)^3 \cdot 3 + \dots + \left(\frac{4}{10}\right)^n \cdot n$$

$$= \sum_{n=1}^{\infty} \left(\frac{4}{10}\right)^n \cdot n$$

wolfram alpha

$$= \frac{10}{9}$$

$$E[J^2] = \left(\frac{4}{10}\right) \cdot 1^2 + \left(\frac{4}{10}\right)^2 \cdot 2^2 + \dots + \left(\frac{4}{10}\right)^n \cdot n^2$$

$$= \sum_{n=1}^{\infty} \left(\frac{4}{10}\right)^n \cdot n^2$$

wolfram alpha

$$= \frac{70}{27} \approx 2.59259$$

$$\mathrm{Var}(J) = E[J^2] - E[J]^2$$

$$= \frac{70}{27} - \left(\frac{10}{9}\right)^2$$

$$= \frac{110}{81}$$

$$\approx 1.358$$

(−1) See soln.

(hope was able to figure out)

(18)

b) ~~#~~ Find $E[k]$

# of transitions up to + including when enters 4 for 1st time

So $\frac{2}{10} + \frac{4}{10} \cdot \frac{2}{10} + \left(\frac{4}{10}\right)^2 \cdot \frac{2}{10} + \dots + \left(\frac{4}{10}\right)^{n-1} \cdot \frac{2}{10}$

1 2 3 n

$$E[k] = \frac{2}{10} \cdot 1 + \frac{4}{10} \cdot \frac{2}{10} \cdot 2 + \left(\frac{4}{10}\right)^2 \cdot \frac{2}{10} \cdot 3 + \dots + \left(\frac{4}{10}\right)^{n-1} \cdot \frac{2}{10} \cdot n$$

$$= \sum_{n=1}^{\infty} \frac{2}{10} \cdot \left(\frac{4}{10}\right)^{n-1} \cdot n$$

wolfram alpha

$$= \frac{5}{9} \approx .5556$$

That's all right - no fancy math

(−1)

See soln.

(19)

c) Find π_i for 1, 2, ... 7 the probability the process is in state i after 10^{10}

- so just find long term state
- balance eq
  - remember what arrives there

a $\pi_1 = .5\pi_2$

b $\pi_2 = \pi_1 + .5\pi_2 + .1\pi_3$

c $\pi_3 = 0$ transient

d $\pi_4 = .2\pi_3 + .75\pi_4 + .5\pi_5$

e $\pi_5 = .25\pi_4 + .25\pi_5 + .5\pi_6$

f $\pi_6 = .25\pi_5 + .5\pi_6$

g $\pi_7 = .3\pi_3 + \pi_7$

Wolfram alpha system of linear eq

$\pi_1 = \frac{\pi_2}{2}$

$\pi_2 = 2\pi_1$

$\pi_3 = 0$

$\pi_4 = 2\pi_5$

$\pi_5 = \frac{\pi_4}{2}$

$\pi_6 = \frac{\pi_4}{4}$

$\pi_7 = .3$ ← had to do manually

oh see p ~~8~~ 25

(20)

d) Given that process never enters State 4, find π_is

Well just chop off states 4,5,6, so $\pi_{4,5,6} = 0$

Readjust probabilities for leaving 3

$$\frac{1/10}{8/10} = \frac{1}{8} \qquad \frac{3/10}{8/10} = \frac{3}{8}$$

$$\frac{4/10}{8/10} = \frac{1}{2}$$

$\pi_1 = .5\,\pi_2$

$\pi_2 = \pi_1 + .5\,\pi_2 + \frac{1}{8}\pi_3$

$\pi_3 = 0$

$\pi_{4,5,6} = 0$

$\pi_7 = \frac{3}{8}\pi_3 + \pi_7$

Solved p 26

1/5

~~Wolfram alpha: no solutions exist~~ opps typed in wrong

$\pi_1 = ?$

$\pi_2 = 2\pi_1$

$\pi_3 = 0$

$\pi_{4,5,6} = 0$

$\pi_7 = ??$ no response

- ? am I doing this wrong?

(21)

Well forgot to add the constraint eq

$\pi_1 = .$

$\pi_2 = 2\pi_1$

$\pi_3 = 0$

$\pi_{4,5,6} = 0$

$\pi_7 = 1 - 3\pi_1$

---

for C revisit)

$\pi_1 = ?$ not given

Solved p 25

$\pi_2 = 2\pi_1$

$\pi_3 = 0$

$\pi_4 = ?$

$\pi_5 = \frac{\pi_4}{2}$

$\pi_6 = \frac{\pi_4}{4}$

$\pi_7 = -\pi_1 - 1.75\pi_4 + 1$

~~oh $\pi_1 = 0$ since~~

$\pi_1 \neq 0$ will spend some time there

Still confused, ask in OH

So balance eq only thing struggled w/ on P-set

ask in OH

Otherwise happy this was done in 3 hrs

Or is Wolfram alpha screwing up

Ran example problem through + it errored

(22) OH 11/14

4c)

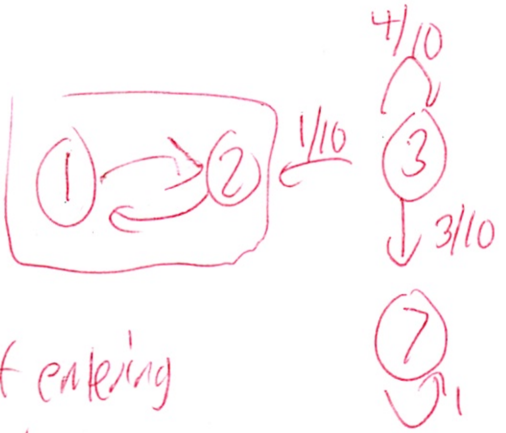
No single recurrent class

3 " "

if started in 1,2 would be easy ($\pi_{3-7}=0$)

---

if in recurrent class what balance eq for that class
 x prob of entering it



P. of entering

$\frac{1/10}{6/10} = \frac{1}{6}$ then do normal bance for each

all ans should be # and add up to 1

Then p of 7 = $\frac{3/10}{6/10} \cdot 1$

$= \frac{1}{2}$

for others just solve the simple system

(23)

So I already did

$\pi_3 = 0$ ← was right

$\pi_7 = \frac{1}{2}$

Now

$\pi_1 = \frac{1}{2}\pi_2$

$\pi_2 = \pi_1 + \frac{1}{2}\pi_2 + \frac{1/10}{6/10}$

Solve

$2\pi_1 = \pi_2$

$2\pi_1 = \pi_1 + \frac{1}{2}(2\pi_1) + \frac{1}{6}$

$2\pi_1 = \pi_1 + \pi_1 + \frac{1}{6}$

$2\pi_1 = 2\pi_1 + \frac{1}{6}$

I don't think so

→ i Leave the $\frac{1}{6}$ at at first + multiply

$2\pi_1 = \pi_2$

$2\pi_1 = \pi_1 + \pi_1$

Yes... does not help

Confirmed this solving on a past problem

Must be the constaint eq

Lets do the rest

(24)

$\pi_4 = \frac{3}{4}\pi_4 + \frac{1}{2}\pi_5$ + skipping π_3 ← well $= 0$ so does not matter

$\pi_5 = \frac{1}{4}\pi_4 + \frac{1}{4}\pi_5 + \frac{1}{2}\pi_6$

$\pi_6 = \frac{1}{4}\pi_5 + \frac{1}{2}\pi_6$

$\frac{1}{2}\pi_6 = \frac{1}{4}\pi_5$

$\cdot 4 \quad \cdot 4$

$2\pi_6 = \pi_5$

$\frac{1}{4}\pi_4 = \frac{1}{2}\pi_5$

$\cdot 2 \quad \cdot 2$

$\frac{1}{2}\pi_4 = \pi_5$

$\frac{1}{2}\pi_4 = \pi_5 = 2\pi_6$ ← ok same thing wolfram alpha gave

- but now need to add up

well p of entering

$\frac{2/10}{6/10} = \frac{2}{6} = \frac{1}{3}$

So

$\frac{1}{6}(2\pi_1) + \frac{1}{6}\pi_2 + \frac{1}{2} + \frac{1}{3}\cdot\frac{1}{2}\pi_4 + \frac{1}{3}\pi_5 + \frac{1}{3}\cdot 2\pi_6 = 1$

$\frac{1}{3}\pi_1 + \frac{1}{6}\pi_2 + \frac{1}{2} + \frac{1}{6}\pi_4 + \frac{1}{3}\pi_5 + \frac{2}{3}\pi_6 = 1$

and $2\pi_1 = \pi_2$ $\qquad \frac{1}{2}\pi_4 = \pi_5 = 2\pi_6$

(28)

Opps simplification was wrong

$$\frac{1}{6}\pi_2 + \frac{1}{6}\pi_2 + \frac{1}{2} + \frac{1}{3}\pi_5 + \frac{1}{3}\pi_5 + \frac{1}{3}\pi_5 = 1$$

$$\frac{1}{3}\pi_2 + \pi_5 = \frac{1}{2}$$

? ok better

but how do π_2, π_5 relate ??

this is why wolfram alpha could not do better

Or solve inside each recurrent class

$2\pi_1 = \pi_2$

$\pi_1 + \pi_2 = 1$

inside $\begin{cases} \pi_1 = \frac{1}{3} \\ \pi_2 = \frac{2}{3} \end{cases}$ Outside $\cdot\frac{1}{6}$ $\pi_1 = \frac{1}{18}$ $\pi_2 = \frac{2}{18}$

??) ok this makes more sense

glad finally figured out

$\frac{1}{2}\pi_4 = \pi_5 = 2\pi_6$

$\pi_4 + \pi_5 + \pi_6 = 1$

(−2)

* balance eq for each recurrent class = 1

inside $\pi_4 = .5714$ $\pi_5 = .2857$ $\pi_6 = .1428$

outside $\cdot\frac{1}{3}$ $\pi_4 = .1904$ $\pi_5 = .0952$ $\pi_6 = .0476$

Ok verify $\frac{1}{18} + \frac{2}{18} + 0 + .1904 + .0952 + .0476 + \frac{1}{2} = 1$

(26)

d (revisit, revisit)

Now $P(\text{state } 1 \text{ and } 2) = \frac{1/10}{4/10} = \frac{1}{4}$

$P(\text{state } 7) = \frac{3/10}{4/10} = \frac{3}{4}$

local $\pi_1 = \frac{1}{3}$

$\pi_2 = \frac{2}{3}$

Outside $\cdot \frac{1}{4}$

$\pi_1 = \frac{1}{12}$

$\pi_2 = \frac{2}{12} = \frac{1}{6}$

$\pi_3 = 0$

$\pi_{4,5,6} = 0$

$\pi_7 = \frac{3}{4} \cdot 1 = \frac{3}{4}$

+1

Check

$\frac{1}{12} + \frac{1}{6} + 0 + \frac{3}{4} = 1$ ✓

So in the future can use wolfram alpha

but not $\pi_1 + \pi_2 + \pi_3 + \pi_4 + \pi_5 + \pi_6$

but each recurrent case

$\pi_1 + \pi_2 = 1$

$\pi_4 + \pi_5 + \pi_6 = 1$

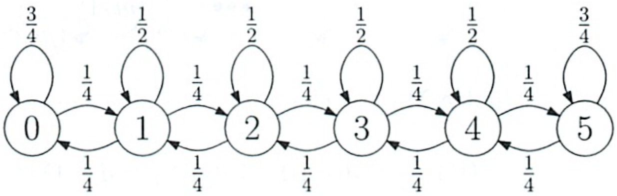
$\pi_7 = 1$

Massachusetts Institute of Technology
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Fall 2010)

Problem Set 8: Solutions

1. (a) We consider a Markov chain with states $0, 1, 2, 3, 4, 5$, where state i indicates that there are i shoes available at the front door in the morning before Oscar leaves on his run.

   Now we can determine the transition probabilities. Assuming i shoes are at the front door before Oscar sets out on his run, with probability $\frac{1}{2}$ Oscar will return to the same door from which he set out, and thus before his next run there will still be i shoes at the front door. Alternatively, with probability $\frac{1}{2}$ Oscar returns to a different door, and in this case, with equal probability there will be $\min\{i+1, 5\}$ or $\max\{i-1, 0\}$ shoes at the front door before his next run. These transition probabilities are illustrated in the following Markov chain:



   (b) When there are either 0 or 5 shoes at the front door, with probability $\frac{1}{2}$ Oscar will leave on his run from the door with 0 shoes and hence run barefooted. To find the long-term probability of Oscar running barefooted, we must find the steady-state probabilities of being in states 0 and 5, π_0 and π_5, respectively. Note that the steady-state probabilities exist because the chain is recurrent and aperiodic.

   Since this is a birth-death process, we can use the local balance equations. We have

$$\pi_0 p_{01} = \pi_1 p_{10} \ ,$$

   implying that

$$\pi_1 = \pi_0$$

   and similarly,

$$\pi_5 = \ldots = \pi_1 = \pi_0 \ .$$

   As

$$\sum_{i=0}^{5} \pi_i = 1 \ ,$$

   it follows that $\pi_i = \frac{1}{6}$ for $i = 0, 1, \ldots, 5$. Hence,

$$\mathbf{P}(\text{Oscar runs barefooted in the long-term}) = \frac{1}{2}(\pi_0 + \pi_5) = \frac{1}{6} \ .$$

2. (a) Consider any possible sequence of values $x_1, x_2, \ldots, x_{t-1}, i$ for $X_1, X_2, \ldots, X_t$, and note that

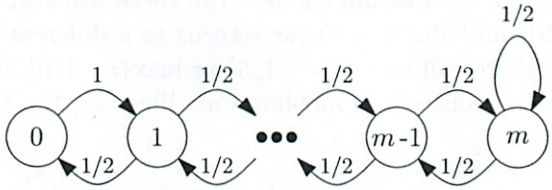
$$\mathbf{P}(|X_{t+1}| = |i| + 1 \,|\, X_t = i, X_{t-1} = x_{t-1}, \ldots X_1 = x_1) = \begin{cases} \frac{1}{2} & 0 < |i| < m \\ 1 & i = 0 \\ 0 & |i| = m \end{cases} \ ,$$

$$\mathbf{P}(|X_{t+1}| = |i| \,|\, X_t = i, X_{t-1} = x_{t-1}, \ldots X_1 = x_1) = \begin{cases} \frac{1}{2} & |i| = m \\ 0 & |i| \neq m \end{cases} \ ,$$

$$\mathbf{P}(|X_{t+1}| = |i| - 1\,|X_t = i, X_{t-1} = x_{t-1}, \ldots X_1 = x_1\,) = \begin{cases} \frac{1}{2} & 0 < |i| \le m \\ 0 & i = 0 \end{cases},$$

$$\mathbf{P}(|X_{t+1}| = j\,|X_t = i, X_{t-1} = x_{t-1}, \ldots X_1 = x_1\,) = 0\,, \quad ||i| - j| > 1\,.$$

As the conditional probabilities above only depend on $|i|$, where $|X_t| = |i|$, it follows that $|X_1|, |X_2|, \ldots$ satisfy the Markov property. The associated Markov chain is illustrated below.



(b) Note that $Y_1, Y_2, \ldots$ is not a Markov chain for $m > 1$, because

$$\mathbf{P}(Y_{t+1} = d + 1|Y_t = d, Y_{t-1} = d - 1) = \frac{1}{2}$$

does not equal

$$\mathbf{P}(Y_{t+1} = d + 1|Y_t = d, Y_{t-1} = d, Y_{t-2} = d - 1) = 0\,,$$

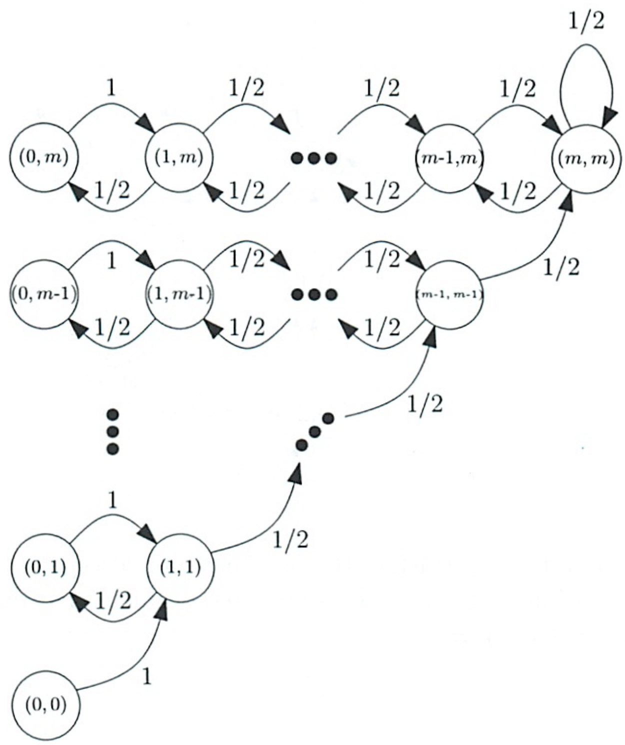
for $0 < d < m$ (the idea is that if $Y_{t-2} = d - 1, Y_{t-1} = d$, and $Y_t = d$, then $|X_t| = d - 1$, while if $Y_{t-1} = d - 1$, and $Y_t = d$, then $|X_t| = d$). If, however, we keep track of $|X_t|$ and Y_t, we do have a Markov chain, because for any possible sequence of pairs of values $(x_1, y_1), \ldots, (x_{t-1}, y_{t-1}), (i_1, i_2)$ for $(|X_1|, Y_1), \ldots, (|X_{t-1}|, Y_{t-1}), (|X_t|, Y_t)$,

$$\begin{aligned}\mathbf{P}((|X_{t+1}|, Y_{t+1}) = (i_1 + 1, i_2 + 1) \quad | \quad & (|X_t|, Y_t) = (i_1, i_2), \ldots (|X_1|, Y_1) = (x_1, y_1)) \\ = & \begin{cases} \frac{1}{2} & 0 < i_1 = i_2 < m \\ 1 & i_1 = i_2 = 0 \\ 0 & \text{otherwise} \end{cases},\end{aligned}$$

$$\begin{aligned}\mathbf{P}((|X_{t+1}|, Y_{t+1}) = (i_1 - 1, i_2) \quad | \quad & (|X_t|, Y_t) = (i_1, i_2), \ldots (|X_1|, Y_1) = (x_1, y_1)) \\ = & \begin{cases} \frac{1}{2} & 0 < i_1 \le i_2 \le m \\ 0 & \text{otherwise} \end{cases},\end{aligned}$$

$$\begin{aligned}\mathbf{P}((|X_{t+1}|, Y_{t+1}) = (i_1, i_2) \quad | \quad & (|X_t|, Y_t) = (i_1, i_2), \ldots (|X_1|, Y_1) = (x_1, y_1)) \\ = & \begin{cases} \frac{1}{2} & i_1 = i_2 = m \\ 0 & \text{otherwise} \end{cases},\end{aligned}$$

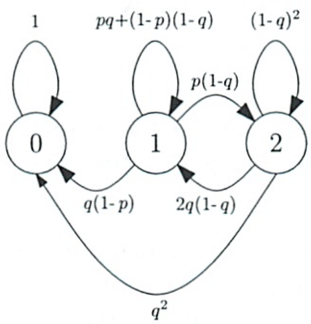
from which it is clear that the conditional probabilities only depend on (i_1, i_2), the values of $|X_t|$ and Y_t, respectively. The corresponding Markov chain is illustrated below.



3. (a) If m out of n individuals are infected, then there must be $n - m$ susceptible individuals. Each one of these individuals will be independently infected over the course of the day with probability $\rho = 1 - (1 - p)^m$. Thus the number of new infections, I, will be a binomial random variable with parameters $n - m$ and ρ. That is,

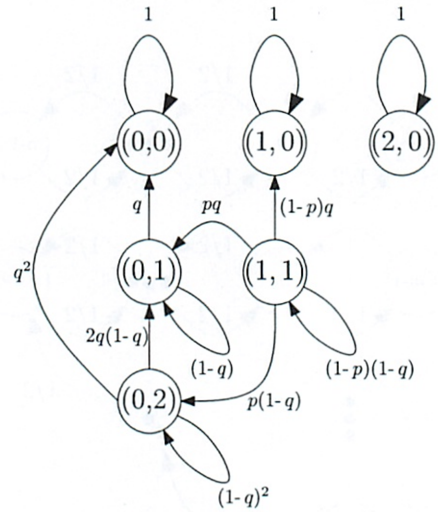
$$p_I(k) = \binom{n-m}{k} \rho^k (1-\rho)^{n-m-k} \qquad k = 0, 1, \ldots, n - m\,.$$

(b) Let the state of the SIS model be the number of infected individuals. For $n = 2$, the corresponding Markov chain is illustrated below.

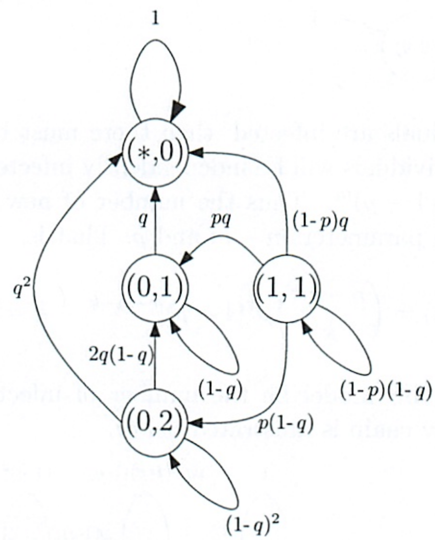


(c) The only recurrent state is the state with 0 infected individuals.

(d) Let the state of the SIR model be (S, I), where S is the number of susceptible individuals and I is the number of infected individuals. For $n = 2$, the corresponding Markov chain is illustrated below.



If one did not wish to keep track of the breakdown of susceptible and recovered individuals when no one was infected, the three states free of infections could be consolidated into a single state as illustrated below.



(e) Any state where the number of infected individuals equals 0 is a recurrent state. For $n = 2$, there are either one or three recurrent states, depending on the Markov chain drawn in part (d).

4. (a) The process is in state 3 immediately before the first transition. After leaving state 3 for the first time, the process cannot go back to state 3 again. Hence J, which represents the number of transitions up to and including the transition on which the process leaves state 3 for the last time is a geometric random variable with success probability equal to 0.6. The variance for J is given by:

$$\sigma_J^2 = \frac{1-p}{p^2} = \frac{10}{9}$$

(b) There is a positive probability that we never enter state 4; i.e., $P(K < \infty) < 1$. Hence the expected value of K is ∞.

(c) The Markov chain has 3 different recurrent classes. The first recurrent class consists of states $\{1, 2\}$, the second recurrent class consists of state $\{7\}$ and the third recurrent class consists of states $\{4, 5, 6\}$. The probability of getting absorbed into the first recurrent class starting from the transient state 3 is,

$$\frac{1/10}{1/10 + 2/10 + 3/10} = \frac{1}{6}$$

which is the probability of transition to the first recurrent class given there is a change of state. Similarly, probability of absorption into second and third recurrent classes are $\frac{3}{6}$ and $\frac{2}{6}$ respectively.

Now, we solve the balance equations within each recurrent class, which give us the probabilities conditioned on getting absorbed from state 3 to that recurrent class. The unconditional steady-state probabilities are found by weighing the conditional steady-state probabilities by the probability of absorption to the recurrent classes.

The first recurrent class is a birth-death process. We write the following equations and solve for the conditional probabilities, denoted by p_1 and p_2.

$$p_1 = \frac{p_2}{2}$$

$$p_1 + p_2 = 1$$

Solving these equations, we get $p_1 = \frac{1}{3}$, $p_2 = \frac{2}{3}$. For the second recurrent class, $p_7 = 1$. The third recurrent class is also a birth-death process, we can find the conditional steady-state probabilities as follows,

$$p_4 = 2p_5$$

$$p_5 = 2p_6$$

$$p_4 + p_5 + p_6 = 1$$

and thus, $p_4 = \frac{4}{7}$, $p_5 = \frac{2}{7}$, $p_6 = \frac{1}{7}$.

Using these data, the unconditional steady-state probabilities for all the states are found as follows:

$$\pi_1 = \frac{1}{3} \cdot \frac{1}{6} = \frac{1}{18}$$

$$\pi_2 = \frac{2}{3} \cdot \frac{1}{6} = \frac{1}{9}$$

$$\pi_3 = 0 \text{ (transient state)}$$

$$\pi_7 = 1 \cdot \frac{3}{6} = \frac{1}{2}$$

$$\pi_4 = \frac{4}{7} \cdot \frac{2}{6} = \frac{4}{21}$$

$$\pi_5 = \frac{2}{7} \cdot \frac{2}{6} = \frac{2}{21}$$

$$\pi_6 = \frac{1}{7} \cdot \frac{2}{6} = \frac{1}{21}$$

(d) The given conditional event, that the process never enters state 4, changes the absorption probabilities to the recurrent classes. The probability of getting absorbed to the first recurrent class is $\frac{1}{4}$, to the second recurrent class is $\frac{3}{4}$, and to the third recurrent class is 0. Hence, the steady state probabilities are given by,

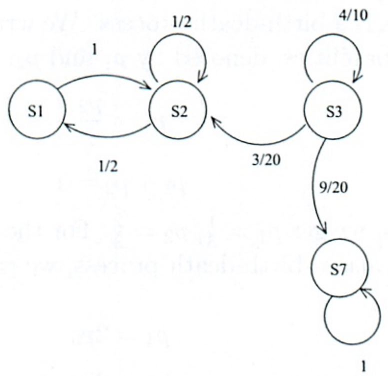
$$\pi_1 = \frac{1}{3} \cdot \frac{1}{4} = \frac{1}{12}$$

$$\pi_2 = \frac{2}{3} \cdot \frac{1}{4} = \frac{1}{6}$$

$$\pi_3 = \pi_4 = \pi_5 = \pi_6 = 0$$

$$\pi_7 = 1 \cdot \frac{3}{4} = \frac{3}{4}$$

For pedagogical purposes, let us actually draw what the new Markov chain would look like, given the event that the process never enters state 4. The resulting chain is shown below. Let us see how we came up with these transition probabilities.



We need to be careful when rescaling the new transition probabilities. First of all, it is clear that the probabilities within the recurrent classes $\{S1, S2\}$ and $\{S7\}$ don't get affected. We also note that the self loop transition probability of the transient state $S3$ doesn't get changed either.(this would be true for any other transient state)
To see that the self loop probability $p_{3,3}$ doesn't get changed, we condition on the event that we eventually enter S2 or S7. Let's call the new self loop probability, $q_{3,3}$.
Then,

$$q_{3,3} = P(X_1 = S3| \text{ absorbed into 2 or 7}, X_0 = S3) = \frac{p_{3,3} * P(\text{absorbed into 2 or 7} \mid X_1 = S3, X_0 = S3)}{P(\text{absorbed into 2 or 7} \mid X_0 = S3)}$$

$$= \frac{p_{3,3} * (a_{3,2} + a_{3,7})}{(a_{3,2} + a_{3,7})} = p_{3,3} = \frac{4}{10}$$

Now, we calculate $q_{3,7}$ and $q_{3,2}$.

$$q_{3,7} = P(X_1 = S7| \text{ absorbed into 2 or 7}, X_0 = S3) = \frac{p_{3,7} * P(\text{absorbed into 2 or 7} \mid X_1 = S7, X_0 = S3)}{P(\text{absorbed into 2 or 7} \mid X_0 = S3)}$$

$$= \frac{p_{3,7} * 1}{(a_{3,2} + a_{3,7})} = \frac{\frac{3}{10}}{\frac{1}{6} + \frac{1}{2}} = \frac{9}{20}$$

$q_{3,2} = P(X_1 = S2|$ absorbed into 2 or 7, $X_0 = S3) = \frac{p_{3,2} * P(\text{absorbed into 2 or 7} \mid X_1 = S2,\ X_0 = S3)}{P(\text{absorbed into 2 or 7} \mid X_0 = S3)}$

$= \frac{p_{3,2}*1}{(a_{3,2}+a_{3,7})} = \frac{\frac{1}{10}}{\frac{1}{6}+\frac{1}{2}} = \frac{3}{20}$

Now, we can calculate the absorption probabilities of this new Markov chain.

The probability of getting absorbed into the recurrent class $\{1,2\}$, starting from $S3$, is $\frac{\frac{3}{20}}{\frac{3}{20}+\frac{9}{20}} = \frac{1}{4}$. The probability of getting absorbed into the recurrent class $\{7\}$, starting from $S3$, is $\frac{\frac{9}{20}}{\frac{3}{20}+\frac{9}{20}} = \frac{3}{4}$. Thus, our calculated absorption probabilities match the probabilities we intuited earlier. The important thing to take away from this example is that, when doing problems of this sort, (i.e given we do/don't enter a particular set of recurrent classes), it is neccessary to rescale the transition probabilities of the new chain, coming out of ALL the transient states. In other words, to find each of the new transition probabilities, we condition on the given event, that we do or do not enter particular recurrent classes.

G1†. a) First let the p_{ij}'s be the transition probabilities of the Markov chain.

Then

$$\begin{aligned}
m_{k+1}(1) &= E[R_{k+1}|X_0 = 1] \\
&= E[g(X_0) + g(X_1) + \ldots + g(X_{k+1})|X_0 = 1] \\
&= \sum_{i=1}^{n} p_{1i} E[g(X_0) + g(X_1) + \ldots + g(X_{k+1})|X_0 = 1, X_1 = i] \\
&= \sum_{i=1}^{n} p_{1i} E[g(1) + g(X_1) + \ldots + g(X_{k+1})|X_1 = i] \\
&= g(1) + \sum_{i=1}^{n} p_{1i} E[g(X_1) + \ldots + g(X_{k+1})|X_1 = i] \\
&= g(1) + \sum_{i=1}^{n} p_{1i} m_k(i)
\end{aligned}$$

and thus in general $m_{k+1}(c) = g(c) + \sum_{i=1}^{n} p_{ci} m_k(i)$ when $c \in \{1, \ldots, n\}$.

Note that the third equality simply uses the total expectation theorem.

b)

$$\begin{aligned}
v_{k+1}(1) &= Var[R_{k+1}|X_0 = 1] \\
&= Var[g(X_0) + g(X_1) + \ldots + g(X_{k+1})|X_0 = 1] \\
&= Var[E[g(X_0) + g(X_1) + \ldots + g(X_{k+1})|X_0 = 1, X_1]] +
\end{aligned}$$

$$\begin{aligned}
& E[Var[g(X_0) + g(X_1) + ... + g(X_{k+1})|X_0 = 1, X_1]] \\
= \; & Var[g(1) + E[g(X_1) + ... + g(X_{k+1})|X_0 = 1, X_1]] + \\
& E[Var[g(1) + g(X_1) + ... + g(X_{k+1})|X_0 = 1, X_1]] \\
= \; & Var[E[g(X_1) + ... + g(X_{k+1})|X_0 = 1, X_1]] + E[Var[g(X_1) + ... + g(X_{k+1})|X_0 = 1, X_1]] \\
= \; & Var[E[g(X_1) + ... + g(X_{k+1})|X_1]] + E[Var[g(X_1) + ... + g(X_{k+1})|X_1]] \\
= \; & Var[m_k(X_1)] + E[v_k(X_1)] \\
= \; & E[(m_k(X_1))^2] - E[m_k(X_1)]^2 + \sum_{i=1}^{n} p_{1i} v_k(i) \\
= \; & \sum_{i=1}^{n} p_{1i} m_k^2(i) - (\sum_{i=1}^{n} p_{1i} m_k(i))^2 + \sum_{i=1}^{n} p_{1i} v_k(i)
\end{aligned}$$

so in general $v_{k+1}(c) = \sum_{i=1}^{n} p_{ci} m_k^2(i) - (\sum_{i=1}^{n} p_{ci} m_k(i))^2 + \sum_{i=1}^{n} p_{ci} v_k(i)$ when $c \in \{1, ..., n\}$.

11/15

LECTURE 18

Markov Processes – III

Last part

Readings: Section 7.4

Lecture outline

Model stuff that evolve w/ time

- Review of steady-state behavior
- Probability of blocked phone calls (real-world)
- Calculating absorption probabilities
- Calculating expected time to absorption

long-term behavior

Review

- Assume a single class of recurrent states, aperiodic; "nice"
  plus transient states. Then,

$$\lim_{n\to\infty} r_{ij}(n) = \pi_j$$

where π_j does not depend on the initial conditions:

$$\lim_{n\to\infty} \mathrm{P}(X_n = j \mid \underbrace{X_0 = i}_{\text{does not matter}}) = \pi_j$$

effect of inital state gets wiped out

- $\pi_1, \ldots, \pi_m$ can be found as the unique solution to the balance equations

$$\pi_j = \sum_k \pi_k p_{kj}, \qquad j = 1, \ldots, m,$$

together with

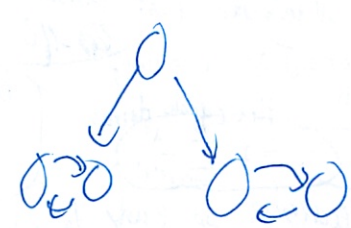
$$\sum_j \pi_j = 1$$

Summry of what happens in long term

solve system of linear equations

If multiple recurrent classes (like the hw)

depends on where you start

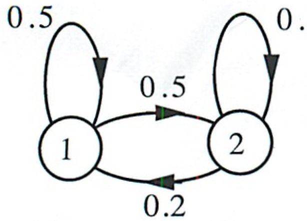


Each has steady state probabilities on its own

have a different set

- Concentrate on 1 at a time

Example



long run $\pi_1 = 2/7,\ \pi_2 = 5/7$ no transient state

- Assume process starts at state 1.

- $P(X_1 = 1,\ \text{and } X_{100} = 1) = P(X_1 = 1 \mid X_0 = 1) \cdot P(X_{100} = 1 \mid X_1 = 1, X_0 = 1)$

multiplication rule

does not matter

$= p_{11}\, r_{11}(99)$

$\approx p_{11}\pi_1 = \frac{1}{2}\cdot\frac{2}{7} = \frac{2}{14} = \frac{1}{7}$

- $P(X_{100} = 1 \text{ and } X_{101} = 2)$

$\approx \pi_1 p_{12}$

What if

.998 .002 .999

1 2

.001

- will take a 1000 steps to start moving
- needs several thousand steps to reach steady state
- no formula

$r_{ij}(n) \approx \pi_j$

when n is very large.

Is 100 very large?
- how quickly does it get into steady state / forgot where started
- depends on randomness in chain

The phone company problem

famous problem

- Calls originate as a Poisson process, rate λ
  - Each call duration is exponentially distributed (parameter μ)
  - B lines available

n people, B lines

Need assumption:

How are calls placed? Poisson λ - check how many calls are coming in - empirical

How long do they last?

each call lasts for diff time, ind.

exponential - analytically simple
- good approx of reality
- except when dial-up internet

Example $\lambda = 30$

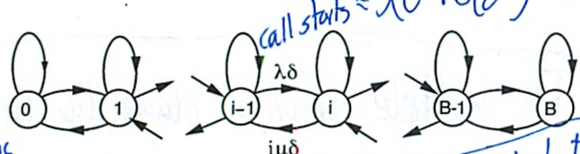
$\mu = \frac{1}{3}$ so $E[T] = 3$

so avg 90 lines
- look at table if error rate = 1% then $B = 106$

- Discrete time intervals of (small) length δ

call starts $= \lambda\delta + O(\delta^2)$

state: # of active conversations



Call terminated first arrival time

$i\mu\delta$ like many poisson processes at same time

sum the rates - so $i\mu\delta$

- Balance equations: $\lambda\pi_{i-1} = i\mu\pi_i$

so birth-death

then cancel deltas

like I did on p-set except this supposed to do continuous time

- $\pi_i = \pi_0 \dfrac{\lambda^i}{\mu^i i!}$ $\quad \pi_0 = 1 / \sum_{i=0}^{B} \dfrac{\lambda^i}{\mu^i i!}$

recursive so easy to solve

$\pi_1 = \frac{\lambda}{\mu}\pi_0$

$\pi_2 = \frac{\lambda}{2\mu}\pi_1 = \frac{\lambda^2}{2\mu^2}\pi_0$

then solve

$\sum \pi_i = 1$

but this is continuous
- won't do theory in class
- discretize time into small slots

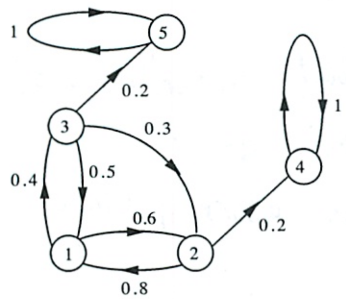
Call failed if at state B

so $P(\text{call fails}) = \pi_B$

each phone engineer >20 years ago had a book w/ this table -

Calculating absorption probabilities

- What is the probability a_i that: process eventually settles in state 4, given that the initial state is i?



Some states are asorbing
once get in, can't get out

4 recurrent
5 recurrent
1, 2, 3 transient

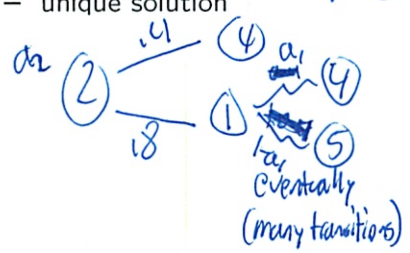
For $i = 4$, $a_i =$ 1 ← you start there + recurrent
For $i = 5$, $a_i =$ 0 ← you start in other recurrent class

$$a_i = \sum_j p_{ij} a_j, \quad \text{for all other } i$$

– unique solution ← won't prove

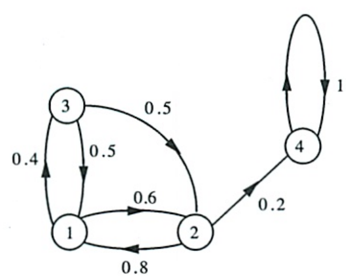
← starting state!

$a_2 = 0.2 + .8 a_1$



if it's a recurring class – just find p of entering the whole class – like it is a point

Expected time to absorption



no matter where we start, we end up here

- Find expected number of transitions μ_i, until reaching the absorbing state, given that the initial state is i?

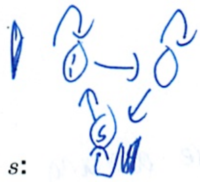
$\mu_1 = 1 + .6\mu_2 + .4\mu_3$

$\mu_i = 0$ for $i =$ 4

For all other i: $\mu_i = 1 + \sum_j p_{ij} \mu_j$

– unique solution

Mean first passage and recurrence times



Same problem as before

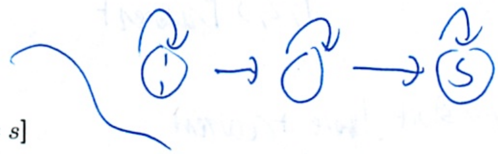
- Chain with one recurrent class; fix s recurrent
- **Mean first passage time from i to s:**

  $t_i = \mathrm{E}[\min\{n \geq 0 \text{ such that } X_n = s\} \mid X_0 = i]$

don't care about transition $s \to i$
Since world ends at s
So s becomes asorbing class

- $t_1, t_2, \ldots, t_m$ are the unique solution to

$$t_s = 0,$$
$$t_i = 1 + \sum_j p_{ij} t_j, \qquad \text{for all } i \neq s$$



- **Mean recurrence time of s:**

  $t_s^* = \mathrm{E}[\min\{n \geq 1 \text{ such that } X_n = s\} \mid X_0 = s]$

- $t_s^* = 1 + \sum_j p_{sj} t_j$

Reviewed in this week's tutorial

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Fall 2010)

Recitation 19: November 16, 2010

1. Josephina is currently a 6-1 student. On each day that she is a 6-1 student, she has a probability of 1/2 of being a course 6-1 student the next day. Otherwise, she has an equally likely chance of becoming a 6-2 student, a 6-3 student, a course 9 student or a course 15 student the next day. On any day she is a 6-3 student, she has a probability of 1/4 of switching to course 9, a probability of 3/8 of switching to 6-1 and a probability of 3/8 of switching to 6-2 the next day. On any day she is a 6-2 student, she has a probability of 1/2 of switching to course 15, a probability of 3/8 of switching to 6-1 and a probability of 1/8 of switching to 6-3 the next day.

   In answering the questions below, assume Josephina will be a student forever. Also assume, for parts (a)-(f) that if Josephina switches to course 9 or course 15, she will stay there and will not change her course again.

   (a) What is the probability that she eventually will leave course 6?

   (b) What is the probability that she will eventually be in course 15?

   (c) What is the expected number of days until she leaves course 6?

   (d) Every time she switches into 6-1 from 6-2 or 6-3, she buys herself an ice cream cone at Tosci's. She can only afford so much ice cream, so after she's eaten 2 ice cream cones, she stops buying herself ice cream. What is the expected number of ice cream cones she buys herself before she leaves course 6?

   (e) Her friend Oscar started out just like Josephina. He is now in course 15. You don't know how long it took him to switch. What is the expected number of days it took him to switch to course 15?

   (f) Josephina decides that course 15 is not in her future. Accordingly, when she is a course 6-1 student, she stays 6-1 for another day with probability 1/2, and otherwise she has an equally likely chance of becoming any of the other options. When she is 6-2, her probability of entering 6-1 or 6-3 are in the same proportion as before. What is the expected number of days until she is in course 9?

   (g) For this part only, assume that when Josephina is in course 9 she is equally likely to stay in course 9 or switch to course 15. Similarly, if she is in course 15, she is equally likely to stay in course 15 or switch to course 9. Calculate the probability of Josephina being in each course on any given day far into the future.

   (h) Suppose that if she is course 9 or course 15, she has probability 1/8 of returning to 6-1, and otherwise she remains in her current course. What is the expected number of days until she is 6-1 again? (Notice that we know today she is 6-1, so if tomorrow she is still 6-1, then the number of days until she is 6-1 again is 1).

Recitation 19

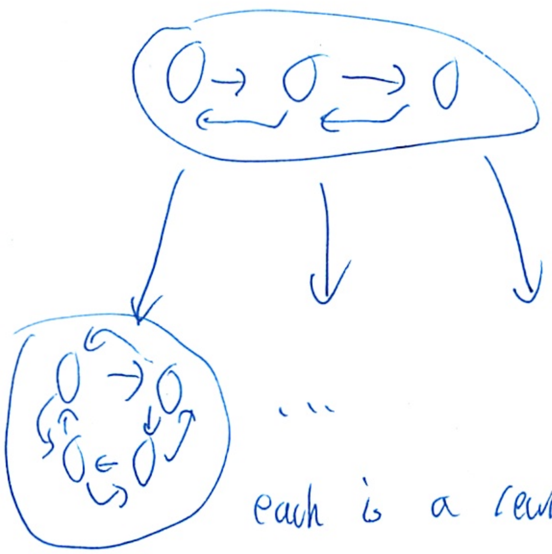
11/16

(5 min late) - but know this

Markov $X_0, X_1, \ldots, X_n$

X_i values $1, 2, \ldots, m$

$P_{ij} = P(X_{n+1} = j \mid X_n = i)$



each is a recurrent ~~state~~ class

- once you get there - can't get back

Can ask long term or short term questions?

? the steady state probabilities

Long-Term behavior (of chain w/ single recurrent class)

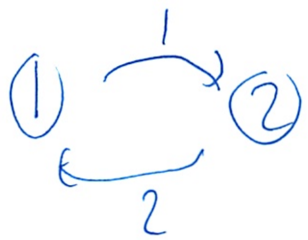
$\pi_j = \lim_{n \to \infty} r_{ij}(n)$

= n-step transition possibility $i \to j$

- no matter where you start

(2)

Except if periodic



$r_{12}(n) = 1$ if n is odd

$r_{12}(n) = 0$ if n is even

no limit to ∞ because go up + down

π_j = also = long term freq

- The expected # of visits to j given that you start at i
- normalized into future
  - but we are doing long term

this works also in periodic cases

$$\pi_j = \sum_{k=1}^{m} \pi_k p_{kj} \quad \text{for all } j = 1, 2, \ldots, n$$

and also $\pi_1 + \ldots + \pi_n = 1$

- so $\pi_j = 0$ if transient

③

Transient / Short-Term Behavior

- Absorption prob

a_i = Prob (absorption to special state s | starting at i)

$a_s = 1$ ← process gets trapped (ie a recurrent class)

$a_i = 0$ $\forall$ absorbing $i \neq s$

$$a_i = \sum_{j=1}^{m} p_{ij} a_j \quad \forall \text{ transient } i$$

So lump entire recurring class into a single state

Solve → get a_is from it

Only involves the transient states

Or how long before I get absorbed? = ~~#~~

↳ Expected Time to Asorbtion

(applies to case of single asorbing state)

μ_i = E[# of steps until asorbtion | start at i]

$\mu_i = 0$ for absorbing state

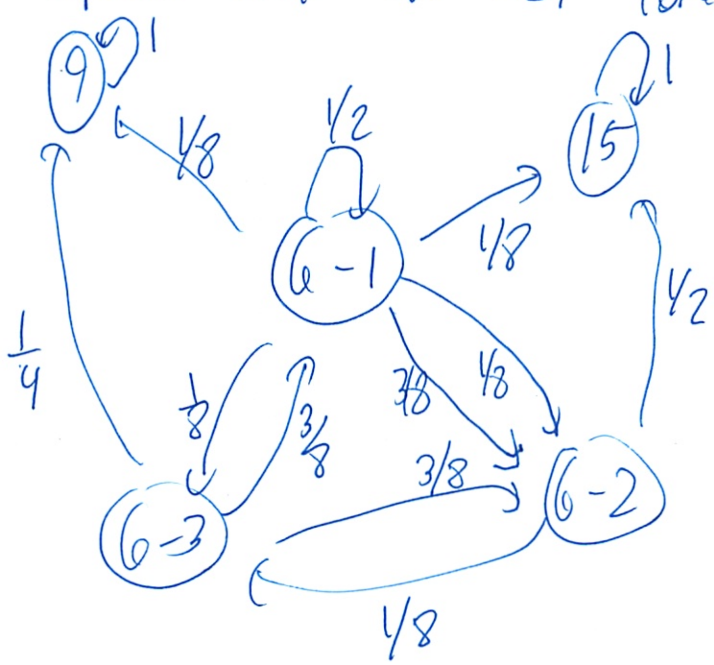
$$\mu_i = 1 + \sum_{j=1}^{m} p_{ij} \mu_j \quad \text{for nonasorbing } i$$

↳ Total expectation theorm

(4)

Basically lump all recurrent states into 1 and find expected time to asorbtion

1. Josephina stays at MIT forever



9, 15 are the recurring, asorbing states

a) prob (will leave Course 6)?

1 → all they are transient

b) P(will end up in 15) = asorbtion (is short term)

a_{6-1} ← absorbtion to 15 from 6-1

$a_{15} = 1$

$a_9 = 0$

(5)

$\swarrow$ all of the places can go

$$a_{6-1} = \frac{1}{2} a_{6-1} + \frac{1}{8}\left(\underset{\substack{\uparrow \\ 0}}{a_9} + \underset{\substack{\uparrow \\ 1}}{a_{15}} + a_{6-2} + a_{6-3}\right)$$

$$a_{6-2} = \frac{1}{2} \cdot \underset{\substack{\uparrow \\ 1}}{a_{15}} + \frac{3}{8} \cdot a_{6-1} + \frac{1}{8} a_{6-3}$$

$$a_{6-3} = \frac{1}{4} \underset{\substack{\uparrow \\ 0}}{a_9} + \frac{3}{8} a_{6-1} +$$

Solving system of eq
- 3 eq
- 3 unknowns

$a_{6-1} = .571$

$a_{6-2} \approx .77$

$a_{6-3} \approx .505$

So this is P(asorbed into 15) if you start at each of these

P(asorbed into 9) = 1 − those ans

(6) E[# of steps to leave 6] =
- lump the asorbing states into 1

$\mu_{15} = 0$

$\mu_9 = 0$

$\uparrow$ if start there, of course 0

$\swarrow$ all of the places can go

$$\mu_{6-1} = 1 + \frac{1}{2}\mu_{6-1} + \frac{1}{8}\left(\underset{\substack{\uparrow \\ 0}}{\mu_9} + \underset{\substack{\uparrow \\ 0}}{\mu_{15}} + \mu_{6-1} + \mu_{6-2}\right)$$

(e)

$$\left.\begin{aligned} M_{6-2} &= 1 + \tfrac{1}{2} M_{15} + \tfrac{3}{8} M_{6-1} + \tfrac{1}{8} M_{6-3} \\ M_{6-3} &= 1 + \text{etc} \end{aligned}\right\}$$

Solve $M_{6-1} = 3.522$ months

d) Must be some # b/w 0 and 2 → discrete so 0,1,2

Ice Cream (up to 2) each time transition $\left.\begin{matrix} 6-2 \\ 6-3 \end{matrix}\right\} \to 6-1$

N = # transitions

Want $E[N \mid \text{start at } 6-1]$

Calculate using total expectation theorm

$V_i(m) = P\left(\text{transitioning } m \text{ additional times } \underset{6-2 \to 6-1 \text{ or } 6-3 \to 6-1}{} \mid \text{start at } i\right)$

?(always strange remembering all of the letters)

$$E[N] = 0 \cdot V_{6-1}(0) + 1 \cdot V_{6-1}(1) + \underbrace{2\left(1 - V_{6-1}(0) - V_{6-2}(1)\right)}_{\text{prob 2 or more transitions (given started 6-1)}}$$

. by meaning of expectation

Calculate using total probability theorm

$V_{15}(0) = V_q(0) = 1$ ← won't be any transitions if start there

$V_{15}(1) = V_q(1) = 0$

(7)

$V_{6-1}(0) = \frac{1}{2} V_{6-1}(0) + \frac{1}{8}\left(\underset{\uparrow = 1}{V_9(0)} + \underset{\uparrow 1}{V_{15}(0)} + V_{6-2}(0) + V_{6-3}(0)\right)$

$V_{6-2}(0) = \frac{3}{8} \cdot \underbrace{0}_{} + \frac{1}{8} V_{6-3}(0) + \frac{1}{2} \underset{\uparrow 1}{V_{15}(0)}$

0 add. transitions
- won't make any more transitions ∴

$V_{6-3}(0) = \frac{3}{8} \cdot 0 + \frac{3}{8} V_{6-2}(0) + \frac{1}{4} \underset{\uparrow 1}{V_9}$

Solve system

$V_{6-1}(0) = .764$

$V_{6-2}(0) = \ldots$

$V_{6-3}(0) = \ldots$

again outgoing arcs

$V_{6-1}(1) = \frac{1}{2} V_{6-1}(1) + \frac{1}{8}\left(V_{6-2}(1) + V_{6-3}(1) + \underset{\uparrow 0}{V_{15}(1)} + \underset{\uparrow 0}{V_9(1)}\right)$

$V_{6-2}(1) = \frac{3}{8} \underset{\uparrow \text{note}}{V_{6-1}(0)} + \frac{1}{8} V_{6-3}(1) + \frac{1}{2} \underset{\uparrow 0}{V_{15}(1)}$

$V_{6-3}(1) \; \frac{3}{8} V_{6-1} + \frac{1}{8} V_{6-2}(1) + \frac{1}{4} \cdot \underset{\uparrow 0}{V_9(1)}$

Solve system

$V_{6-1}(1) = .185$

Now back to the original problem + plug in #

$E[N] = .306$

(8)

8. If 9 + 15 become recurrent class

$\pi_9 = \pi_{15} = 1/2$