12

Mini-Quiz Feb. 16 #1

Your name: Michael Plasmeier

Circle the name of your TA:

Ali Nick Oscar (Oshani)

- This quiz is **closed book**. Total time is 25 minutes.
- Write your solutions in the space provided. If you need more space, write on the back of the sheet containing the problem. Please keep your entire answer to a problem on that problem's page.
- GOOD LUCK!

20 min · 1st

DO NOT WRITE BELOW THIS LINE

Problem	Points	Grade	Grader
1	5	2	OS
2	5	5	OM
3	5	2	OM
4	5	4	OS
Total	20	13	OS

Problem 1 (5 points).
Prove that $\log_9 12$ is irrational. *Hint:* Proof by contradiction.

Seems like you have not proved this by WOP!

Proof by WOP. Assume that you can write $\log_9 12$ as the quotent of two integers ($\frac{m}{n}$) ✓
This quotient shold be reduced to its lowest form so it can not be redced again.

However

$$\log_9 12 = \frac{m}{n}$$ ✓ 9 to the power of x

$$9^{\log_9 12} = 9^{m/n}$$ ✓

$$12 = 9^{m/n}$$ ✓

$\sqrt[n]{\ }$ $\sqrt[n]{\ }$

$12^n = 9^m$
LHS is even, but 9^m is odd.
This is a contradiction, ∴ it proves that $\log_9 12$ is irrational.

$$9 = \sqrt[n/m]{12}$$

$$9n = 12m$$ — ? really? is this possible?

$$n = 4m$$

There can never be $\frac{m}{n}$ written in lowest form because m is a factor of n. Thus $\log_9 12$ is irrational

should reviewed P-Set 1

Problem 2 (5 points).
Show that there are exactly two truth assignments for the variables P,Q,R,S that satisfy the following formula:

$$(\overline{P} \text{ OR } Q) \text{ AND } (\overline{Q} \text{ OR } R) \text{ AND } (\overline{R} \text{ OR } S) \text{ AND } (\overline{S} \text{ OR } P)$$

Hint: A truth table will do the job, but it will have a bunch of rows. A proof by cases can be quicker; if you do use cases, be sure each one is clearly specified.

P	Q	R	S	$\overline{P}$ or Q	$\overline{Q}$ or R	$\overline{R}$ or S	$\overline{S}$ or P	
T	T	T	T	T	T	T	T	(T)
T	T	T	F	T	T	F	T	F
T	T	F	T	T	F	T	T	F
T	T	F	F	T	F	T	T	F
T	F	T	T	F	T	T	T	F
T	F	T	F	F	T	F	T	F
T	F	F	T	F	T	T	T	F
T	F	F	F	F	T	T	T	F
F	T	T	T	T	T	T	F	F
F	T	T	F	T	T	F	T	F
F	T	F	T	T	F	T	F	F
F	T	F	F	T	F	T	T	F
F	F	T	T	T	T	T	F	F
F	F	T	F	T	T	F	T	F
F	F	F	T	T	T	T	F	F
F	F	F	F	T	T	T	T	(T)

Only 2 rows are true

Problem 3 (5 points).
The (flawed) proof below uses the Well Ordering Principle to prove that every amount of postage that can be paid exactly, using only 10 cent and 15 cent stamps, is divisible by 5. Let $S(n)$ mean that exactly n cents postage can be paid using only 10 and 15 cent stamps. Then the proof shows that

other way around

$$S(n) \text{ IMPLIES } 5 \mid n, \quad \text{for all nonnegative integers } n. \qquad (*)$$

Fill in the missing portions (indicated by "...") of the following proof of (*), and at the final line point out where the error in the proof is.

Let C be the set of *counterexamples* to (*), namely

$$C ::= \{n \mid S(n) \text{ and NOT}(5 \mid n)\}$$

Assume for the purpose of obtaining a contradiction that C is nonempty. Then by the WOP, there is a smallest number, $m \in C$. Then $S(m-10)$ or $S(m-15)$ must hold, because the m cents postage is made from 10 and 15 cent stamps, so we remove one.

So suppose $S(m-10)$ holds. Then $5 \mid (m-10)$, because...

You can remove 10 cents and it would not change if it's divisable by 5 since 10/5 is T - even when removing 10 cents since 10/5 is T

But if $5 \mid (m-10)$, then $5 \mid m$, because...

Again, you can always divide by 5 - never having a smallest counterexample

contradicting the fact that m is a counterexample.

Next suppose $S(m-15)$ holds. Then the proof for $m-10$ carries over directly for $m-15$ to yield a contradiction in this case as well. Since we get a contradiction in both cases, we conclude that C must be empty. That is, there are no counterexamples to (*), which proves that (*) holds.

What was wrong/missing in the argument? Your answer should fit in the line below.

-1 m must be larger than a certain value (10)

larger than 0.

Never really got mechanics of this

Problem 4 (5 points).

The following predicate logic formula is invalid:

$$\forall x, \exists y. P(x, y) \longrightarrow \exists y, \forall x. P(x, y)$$

Which of the following are counter models for the implication above?

✗ 1. X The predicate $P(x, y) =$ '$yx = 1$' where the domain of discourse is $\mathbb{Q}$.

not always an X

✓ 2. X The predicate $P(x, y) =$ '$y < x$' where the domain of discourse is $\mathbb{R}$.

not all x for that y

✓ 3. X The predicate $P(x, y) =$ '$yx = 2$' where the domain of discourse is $\mathbb{R}$ without 0.

✓ 4. ______ The predicate $P(x, y) =$ '$yxy = x$' where the domain of discourse is the set of all binary strings, including the empty string.

Works y is empty

Fair quiz

Solutions to Mini-Quiz Feb. 16

Problem 1 (5 points).
Prove that $\log_9 12$ is irrational. *Hint:* Proof by contradiction.

Solution. *Proof.* Suppose to the contrary that $\log_9 12 = m/n$ for some integers m and n. Since $\log_9 12$ is positive, we may assume that m and n are also positive. So we have

$$\begin{aligned} \log_9 12 &= m/n \\ 9^{\log_9 12} &= 9^{m/n} \\ 12 &= \left(9^m\right)^{1/n} \\ 12^n &= 9^m \end{aligned} \tag{1}$$

But this is impossible, since left hand side of (1) is even, but, because m is positive, the right hand side is odd.

This contradiction implies that $\log_9 12$ must be irrational. ■

Problem 2 (5 points).
Show that there are exactly two truth assignments for the variables P,Q,R,S that satisfy the following formula:

$$(\overline{P} \text{ OR } Q) \text{ AND } (\overline{Q} \text{ OR } R) \text{ AND } (\overline{R} \text{ OR } S) \text{ AND } (\overline{S} \text{ OR } P)$$

Hint: A truth table will do the job, but it will have a bunch of rows. A proof by cases can be quicker; if you do use cases, be sure each one is clearly specified.

Solution. You can deduce the only two possibilities by cases:

If P is false, then in order to have any chance of satisfying clause 4, S must be false. Similarly, if S is false, then in order to satisfy clause 3, R must be false. And similarly, Q must be false. On the other hand, if P is true, then Q must be true to make clause 1 true and have any chances of making the overall expression true. Similarly, If Q is true, then R must be true and if R is true then S is true.

Those arguments prove there are at most 2 cases, but you need to show the assignments we are left with actually satisfy the formula. This can be easily done, by plugging the values into the formula:

If all variables are set to true, then since clause 1 has Q clause 2 has R, clause 3 has S, and clause 4 has P, then every clause is satisfied, and the full AND is satisfied. If all are false, then since clause 1 has $\overline{P}$, clause 2 has $\overline{Q}$, clause 3 has $\overline{R}$ and clause 4 has $\overline{S}$, then again every clause is satisfied and the overall proposition is satisfied. So both of those satisfy the proposition.

■

Problem 3 (5 points).
The (flawed) proof below uses the Well Ordering Principle to prove that every amount of postage that can be paid exactly, using only 10 cent and 15 cent stamps, is divisible by 5. Let $S(n)$ mean that exactly n cents postage can be paid using only 10 and 15 cent stamps. Then the proof shows that

$$S(n) \text{ IMPLIES } 5 \mid n, \quad \text{for all nonnegative integers } n. \tag{*}$$

Fill in the missing portions (indicated by "...") of the following proof of (*), and at the final line point out where the error in the proof is.

Let C be the set of *counterexamples* to (*), namely

$$C ::= \{n \mid S(n) \text{ and NOT}(5 \mid n)\}$$

Assume for the purpose of obtaining a contradiction that C is nonempty. Then by the WOP, there is a smallest number, $m \in C$. Then $S(m-10)$ or $S(m-15)$ must hold, because the m cents postage is made from 10 and 15 cent stamps, so we remove one.

So suppose $S(m-10)$ holds. Then $5 \mid (m-10)$, because...

Solution. ...if $\text{NOT}(5 \mid (m-10))$, then $m-10$ would be a counterexample smaller than m, contradicting the minimality of m. ■

But if $5 \mid (m-10)$, then $5 \mid m$, because...

Solution. ...$5 \mid (m-10)$ and $5 \mid 10$, so $5 \mid (m-10+10)$. ■

contradicting the fact that m is a counterexample.

Next suppose $S(m-15)$ holds. Then the proof for $m-10$ carries over directly for $m-15$ to yield a contradiction in this case as well. Since we get a contradiction in both cases, we conclude that C must be empty. That is, there are no counterexamples to (*), which proves that (*) holds.

What was wrong/missing in the argument? Your answer should fit in the line below.

Solution. We didn't check $m > 0$, if $m = 0$ neither $S(m-10)$ nor $S(m-15)$ hold. ■

Problem 4 (5 points).

The following predicate logic formula is invalid:

$$\forall x, \exists y. P(x,y) \longrightarrow \exists y, \forall x. P(x,y)$$

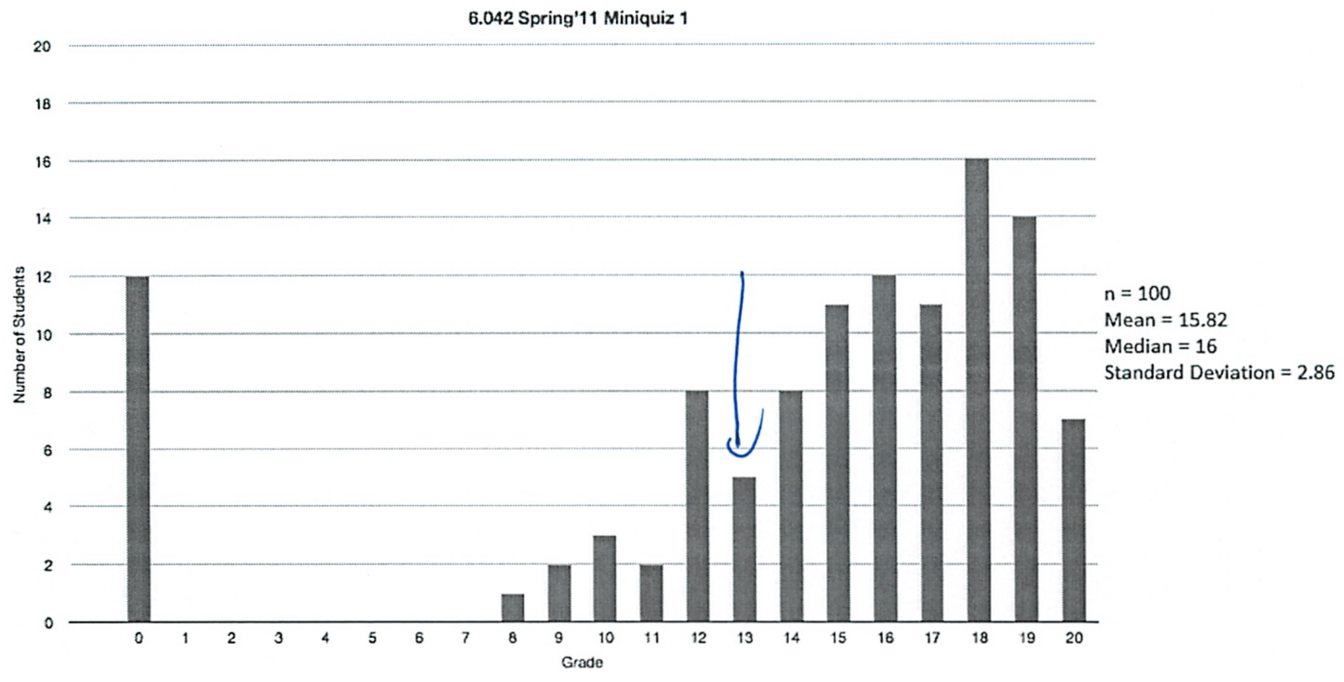
Which of the following are counter models for the implication above?

1. The predicate $P(x,y) =$ '$yx = 1$' where the domain of discourse is $\mathbb{Q}$.
2. The predicate $P(x,y) =$ '$y < x$' where the domain of discourse is $\mathbb{R}$.
3. The predicate $P(x,y) =$ '$yx = 2$' where the domain of discourse is $\mathbb{R}$ without 0.
4. The predicate $P(x,y) =$ '$yxy = x$' where the domain of discourse is the set of all binary strings, including the empty string.

Solution.

1. In the rationals, 0 has no inverse. Hence the hypothesis is false, since not all rationals have inverses. An implication with a false hypothesis is automatically true, so this is not a countermodel.
2. COUNTERMODEL. For every real number x, there exists a real number y which is strictly less than x. So while the antecedent of the implication is true, the consequence is not since there is no minimum element for the partial order, the strictly less than relation, $<$, on $\mathbb{R}$.
3. COUNTERMODEL. in this case the hypothesis is true, but the conclusion is not: its not possible to find a single number that will do this.
4. In the set of binary strings, both sides of the implication are true if we let $y = \lambda$, the empty string.

■



Bit lower than my usual position

Mini-Quiz Mar. 2 #2

Your name: Michael Plasmeier

Circle the name of your TA:

Ali Nick Oscar (Oshani)

- This quiz is **closed book**. Total time is 25 minutes.
- Write your solutions in the space provided. If you need more space, write on the back of the sheet containing the problem. Please keep your entire answer to a problem on that problem's page.
- GOOD LUCK!

T/R

DO NOT WRITE BELOW THIS LINE

Problem	Points	Grade	Grader
1	5	5	AK
2	5	0	AK
3	5	2	AK
4	5	0	NS
Total	20	7	NS

Problem 1 (5 points).
Set equalities such as the one below can be proved with a chain of *iff*s starting with "$x \in$ left-hand-set" and ending with "$x \in$ right-hand-set," as done in class and the text. A key step in such a proof involves invoking a propositional equivalence. State a propositional equivalence that would do the job for this set equality:

$$\overline{A - B} = (\overline{A} - \overline{C}) \cup (B \cap C) \cup ((\overline{A} \cup B) \cap \overline{C})$$

Do not simplify or prove the propositional equivalence you obtain.

For example, to prove $A \cup (B \cap A) = A$, we would have the following "iff chain":

$x \in A \cup (B \cap A)$ iff $x \in A$ OR $x \in (B \cap A)$
iff $x \in A$ OR ($x \in B$ AND $x \in A$)
iff $x \in A$ (Since P OR (Q AND P) is equivalent to P.)

5

$x \in \overline{A - B}$ IFF $x \notin A - B$
IFF NOT($x \in A$ AND $x \notin B$)

$x \in (\overline{A} - \overline{C})$ IFF $x \notin A$ AND NOT($x \in C$)
$x \in (B \cap C)$ IFF $x \in B$ And $x \in C$
$x \in ((\overline{A} \cup B) \cap \overline{C})$ IFF ($x \notin A$ or $x \in B$) AND $x \notin C$)

right side = ($x \notin A$ AND $x \in C$) or [$x \in B$ AND $x \in C$)
Or (($x \notin A$ or $x \in B$) And ($x \notin C$)

See online solution.

Problem 2 (5 points).
Let A and B denote two countably infinite sets:

$$A = \{a_0, a_1, a_2, a_3, \ldots\}$$
$$B = \{b_0, b_1, b_2, b_3, \ldots\}$$

✓ programatically ?

Show that their product, $A \times B$, is also a countable set by showing how to list the elements of $A \times B$. You need only show enough of the initial terms in your sequence to make the pattern clear — a half dozen or so terms usually suffice.

$A \times B = f(a(), b())$

(a_0, b_0)	(a_1, b_0)	(a_2, b_0)	$(a_{\ldots}, b_0)$
(a_0, b_1)	(a_1, b_1)	(a_2, b_1)	$(a_{\ldots}, b_1)$
(a_0, b_2)	(a_1, b_2)	(a_2, b_2)	$(a_{\ldots}, b_2)$
$(a_0, b_{\ldots})$	$(a_0, b_{\ldots})$	$(a_2, b_{\ldots})$	$(a_{\ldots}, b_{\ldots})$

Build matrix like this
Matrix of ∞ size

We want a list, not a matrix!

These are countable, because each a_0, b_0 is countable

ϕ

well then just read across ... silly
- went over in review session
- but should be partial at least

Problem 3 (5 points).
The nth Fibonacci number, F_n, is defined recursively as follows:

$$F_n = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ F_{n-1} + F_{n-2} & \text{if } n \geq 2 \end{cases}$$

These numbers satisfy many unexpected identities, such as

$$F_0^2 + F_1^2 + \cdots + F_n^2 = F_n F_{n+1} \tag{1}$$

Equation (1) can be proved to hold for all $n \in \mathbb{N}$ by induction, using the equation itself as the induction hypothesis, $P(n)$.

(a) Prove the **base case** ($n = 0$).

Hyp: $P(n) = F_0^2 + F_1^2 + \ldots + F_n^2 = F_n F_{n+1}$

$F_n = 0$

$F_0^2 = F_0 F_1$

$0^2 = 0 \cdot 1$

$0 = 0$

(2)

(b) Now prove the **inductive step**.

?

$F_0^2 + F_1^2 + \ldots + F_n^2 + (F_{n+1})^2 = F_n F_{n+1} (F_{n+1})$

$F_0^2 + F_1^2 + (F_{n-1} + F_{n-2})^2 + (F_n + F_{n-1})^2 = F_n (F_{n+1})^2$

$\ldots + F_{n-1}^2 + F_{n-1}F_{n-2} + F_{n-2}^2 + \frac{F_n^2 + F_n F_{n-1}}{F_n} + (F_{n-1})^2 = \frac{F_n (F_{n+1})}{F_n}$

(0)

$\ldots + F_n + F_{n-1} + \frac{(F_{n-1})^2}{F_n} = (F_{n+1})^2$

Shift time

$\frac{F_{n-1} + F_{n-2}}{F_n} + \frac{(F_{n-2})^2}{F_{n-1}} = F_n^2$

$F_n + \frac{(F_{n-2})^2}{F_{n-1}} = F_n^2$

Problem 4 (5 points).
The set, M, of strings of brackets is recursively defined as follows:
Base case: $\lambda \in M$.
Constructor cases: If $s, t \in M$, then

- $[\, s \,] \in M$, and
- $s \cdot t \in M$.

The set, RecMatch, of strings of matched brackets was defined recursively in class. Recall the definition:
Base case: $\lambda \in$ RecMatch.
Constructor case: If $s, t \in$ RecMatch, then $[\, s \,]\, t \in$ RecMatch.

Fill in the following parts of a proof by structural induction that

$$\text{RecMatch} \subseteq M. \tag{2}$$

(a) State an **induction hypothesis** suitable for proving (2) by structural induction.

P(n) ::= RecMatch ⊆ M ∀s,t ∈ M X see sols
If P(b) is true for each base case element b ∈ R for all 2 argument constructors c [P(r) and P(s) → P[c(r,s)]] for all r, s ∈ R then P(c) is true for all r ∈ R

(b) State and prove the **base case(s)**.

X Base case s = λ
There are no [] in base case, so base case of RecMatch definition that λ ∈ RecMatch
λ ∈ M ?

(c) Prove the **inductive step**.

X If s,t ∈ RecMatch, then [s]t ∈ RecMatch
Proof by cases
[s]t ∈ M
then remove the brackets on the outside, recursively
[s']t ∈ M
then remove brackets on the inside, recursively
want to show [s]t ∈ M,

did not look at class fraught

As a matter of fact, M = RecMatch, though we won't prove this. An advantage of the RecMatch definition is that it is *unambiguous*, while the definition of M is ambiguous.

(d) Give an example demonstrating that M is ambiguously defined.

X We don't know what is in M. It could be an empty set or it could be a set where [ ] do not match. no. see sols.

(e) Briefly explain what advantage unambiguous recursive definitions have over ambiguous ones. (Remember that "ambiguous definition" has a technical mathematical meaning which does not imply that the ambiguous definition is unclear.)

We know that we have proved for all cases. There are no cases that can be considered X that might lead to a different outcome.

did not study end well

Solutions to Mini-Quiz Mar. 2

Problem 1 (5 points).
Set equalities such as the one below can be proved with a chain of *iff*s starting with "$x \in$ left-hand-set" and ending with "$x \in$ right-hand-set," as done in class and the text. A key step in such a proof involves invoking a propositional equivalence. State a propositional equivalence that would do the job for this set equality:

$$\overline{A - B} = (\overline{A} - \overline{C}) \cup (B \cap C) \cup ((\overline{A} \cup B) \cap \overline{C})$$

Do not simplify or prove the propositional equivalence you obtain.

For example, to prove $A \cup (B \cap A) = A$, we would have the following "iff chain":

$$\begin{aligned} x \in A \cup (B \cap A) \quad &\text{iff} \quad x \in A \text{ OR } x \in (B \cap A) \\ &\text{iff} \quad x \in A \text{ OR } (x \in B \text{ AND } x \in A) \\ &\text{iff} \quad x \in A \qquad \text{(since } P \text{ OR } (Q \text{ AND } P) \text{ is equivalent to } P\text{).} \end{aligned}$$

Solution. The stated set equality holds iff membership in $\overline{A - B}$ implies and is implied by membership in $(\overline{A} - \overline{C}) \cup (B \cap C) \cup ((\overline{A} \cup B) \cap \overline{C})$. That is, the set equality holds iff, for all x,

$$x \in \overline{A - B} \quad \text{iff} \quad x \in (\overline{A} - \overline{C}) \cup (B \cap C) \cup ((\overline{A} \cup B) \cap \overline{C}).$$

Define three propositions describing the membership of x in each of the sets A, B, and C:

$$\begin{aligned} P &::= x \in A \\ Q &::= x \in B \\ R &::= x \in C \end{aligned}$$

Now, express membership in $\overline{A - B}$ in terms of P, Q, and R:

$$\begin{aligned} & x \in \overline{A - B} \\ &\quad \text{iff} \quad \text{NOT}\,(x \in (A \cap \overline{B})) \\ &\quad \text{iff} \quad \text{NOT}\,(x \in A \text{ AND } x \in \overline{B}) \\ &\quad \text{iff} \quad \text{NOT}\,(x \in A \text{ AND NOT}\,(x \in B)) \\ &\quad \text{iff} \quad \text{NOT}\,(P \text{ AND NOT}\,(Q)) \end{aligned}$$

Then express membership in

$$(\overline{A} - \overline{C}) \cup (B \cap C) \cup ((\overline{A} \cup B) \cap \overline{C})$$

in terms of P, Q, and R:

$$x \in (\overline{A} - \overline{C}) \cup (B \cap C) \cup ((\overline{A} \cup B) \cap \overline{C})$$

$$\text{iff} \quad x \in (\overline{A} - \overline{C}) \text{ OR } x \in (B \cap C) \text{ OR } x \in ((\overline{A} \cup B) \cap \overline{C})$$

$$\text{iff} \quad x \in \left(\overline{A} \cap \overline{\overline{C}}\right) \text{ OR } x \in (B \cap C) \text{ OR } (x \in (\overline{A} \cup B) \text{ AND } x \in \overline{C})$$

$$\text{iff} \quad x \in (\overline{A} \cap C) \text{ OR } x \in (B \cap C) \text{ OR } (x \in (\overline{A} \cup B) \text{ AND } x \in \overline{C})$$

$$\text{iff} \quad (x \in \overline{A} \text{ AND } x \in C) \text{ OR } (x \in B \text{ AND } x \in C) \text{ OR } ((x \in \overline{A} \text{ OR } x \in B) \text{ AND } x \in \overline{C})$$

$$\text{iff} \quad (\text{NOT}(x \in A) \text{ AND } x \in C) \text{ OR } (x \in B \text{ AND } x \in C) \text{ OR } ((\text{NOT}(x \in A) \text{ OR } x \in B) \text{ AND NOT}(x \in C))$$

$$\text{iff} \quad (\overline{P} \text{ AND } R) \text{ OR } (Q \text{ AND } R) \text{ OR } ((\overline{P} \text{ OR } Q) \text{ AND } \overline{R})$$

So the stated set equality holds if and only if the following two propositional formulas are equivalent

$$\text{NOT}(P \text{ AND } \overline{Q})$$

and

$$((\overline{P} \text{ AND } R) \text{ OR } (Q \text{ AND } R) \text{ OR } ((\overline{P} \text{ OR } Q) \text{ AND } \overline{R})).$$

Notice that you were **not** expected to write out a proof like this. We've written this out to remind you how the propositional equivalence would be used in such a proof.

The point is that there is a clear correspondence between the set equality and the needed propositional equivalence in such proofs, and once you've recognized this, you can read off the propositional equivalence from the set equality without having to go through any long derivation.

■

Problem 2 (5 points).
Let A and B denote two countably infinite sets:

$$A = \{a_0, a_1, a_2, a_3, \ldots\}$$
$$B = \{b_0, b_1, b_2, b_3, \ldots\}$$

Show that their product, $A \times B$, is also a countable set by showing how to list the elements of $A \times B$. You need only show enough of the initial terms in your sequence to make the pattern clear — a half dozen or so terms usually suffice.

Solution. The elements of $A \times B$ can be arranged as follows:

$$\begin{array}{ccccc}
(a_0, b_0) & (a_0, b_1) & (a_0, b_2) & (a_0, b_3) & \cdots \\
(a_1, b_0) & (a_1, b_1) & (a_1, b_2) & (a_1, b_3) & \cdots \\
(a_2, b_0) & (a_2, b_1) & (a_2, b_2) & (a_2, b_3) & \cdots \\
(a_3, b_0) & (a_3, b_1) & (a_3, b_2) & (a_3, b_3) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

Traversing this grid along successive lower-left to upper-right diagonals yields the required list:

$$(a_0, b_0), (a_1, b_0), (a_0, b_1), (a_2, b_0), (a_1, b_1), (a_0, b_2), (a_3, b_0), (a_2, b_1), (a_1, b_2), (a_0, b_3), \ldots$$

Obviously, travelling in the opposite direction along each diagonal yields an equally acceptable list:

$$(a_0, b_0), (a_0, b_1), (a_1, b_0), (a_0, b_2), (a_1, b_1), (a_2, b_0), (a_0, b_3), (a_1, b_2), (a_2, b_1), (a_3, b_0), \ldots$$

■

Problem 3 (5 points).
The nth Fibonacci number, F_n, is defined recursively as follows:

$$F_n = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ F_{n-1} + F_{n-2} & \text{if } n \geq 2 \end{cases}$$

These numbers satisfy many unexpected identities, such as

$$F_0^2 + F_1^2 + \cdots + F_n^2 = F_n F_{n+1} \tag{1}$$

Equation (1) can be proved to hold for all $n \in \mathbb{N}$ by induction, using the equation itself as the induction hypothesis, $P(n)$.

(a) Prove the **base case** ($n = 0$).

Solution.

$$\sum_{i=0}^{0} F_i^2 = (F_0)^2 = 0 = (0)\,(1) = F_0 F_1$$

Therefore, $P(0)$ is true. ■

(b) Now prove the **inductive step**.

Solution. We need to prove that $P(n)$:

$$\sum_{i=0}^{n} F_i^2 = F_n F_{n+1}$$

implies $P(n+1)$:

$$\sum_{i=0}^{n+1} F_i^2 = F_{n+1} F_{n+2}$$

Proof.

$$\begin{aligned} \sum_{i=0}^{n+1} F_i^2 &= \sum_{i=0}^{n} F_i^2 + F_{n+1}^2 \\ &= F_n F_{n+1} + F_{n+1}^2 && \text{By } P(n). \\ &= F_{n+1}\,(F_n + F_{n+1}) \\ &= F_{n+1} F_{n+2} && \text{By the definition of the Fibonacci sequence.} \end{aligned}$$

■

Problem 4 (5 points).
The set, M, of strings of brackets is recursively defined as follows:

Base case: $\lambda \in M$.

Constructor cases: If $s, t \in M$, then

- $[\, s\,] \in M$, and
- $s \cdot t \in M$.

The set, RecMatch, of strings of matched brackets was defined recursively in class. Recall the definition:
Base case: $\lambda \in$ RecMatch.
Constructor case: If $s, t \in$ RecMatch, then $[\, s\,]\, t \in$ RecMatch.

Fill in the following parts of a proof by structural induction that

$$\text{RecMatch} \subseteq M. \tag{2}$$

(a) State an **induction hypothesis** suitable for proving (2) by structural induction.

Solution.

$$P(x) ::= x \in M$$

■

(b) State and prove the **base case(s)**.

Solution. Base case ($x = \lambda$): By definition of M, the empty string is in M.

■

(c) Prove the **inductive step**.

Solution. *Proof.* **Constructor case** ($x = [\, s\,]\, t$ for $s, t \in$ RecMatch): By structural induction hypothesis, we may assume that $s, t \in M$. By the first constructor case of M, it follows that $[s] \in M$. Then, by the second constructor case of M, it follows that $[\, s\,]\, t \in M$.

■

As a matter of fact, $M =$ RecMatch, though we won't prove this. An advantage of the RecMatch definition is that it is *unambiguous*, while the definition of M is ambiguous.

(d) Give an example demonstrating that M is ambiguously defined.

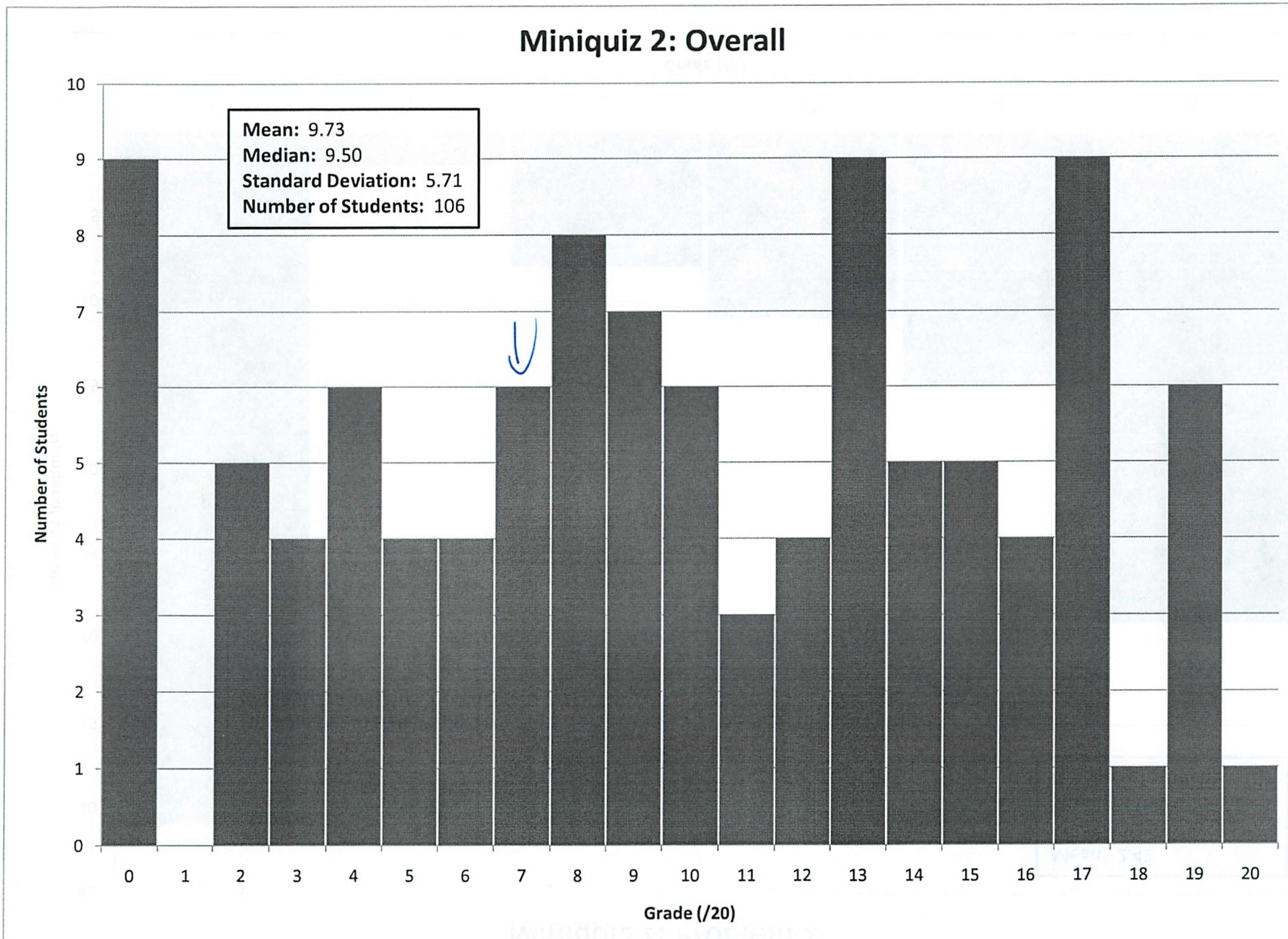
Solution. Consider derivations of the empty string. This could be derived directly from the base case λ, or by starting with λ and then constructing $\lambda\lambda$ through the second constructor case.

■

(e) Briefly explain what advantage unambiguous recursive definitions have over ambiguous ones. (Remember that "ambiguous definition" has a technical mathematical meaning which does not imply that the ambiguous definition is unclear.)

Solution. If a definition is ambiguous, functions defined recursively on it may not be well-defined.

■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Mean: 9.73
Median: 9.50
Standard Deviation: 5.71
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Number of Students
0
1
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3
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7
8
9
10
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1
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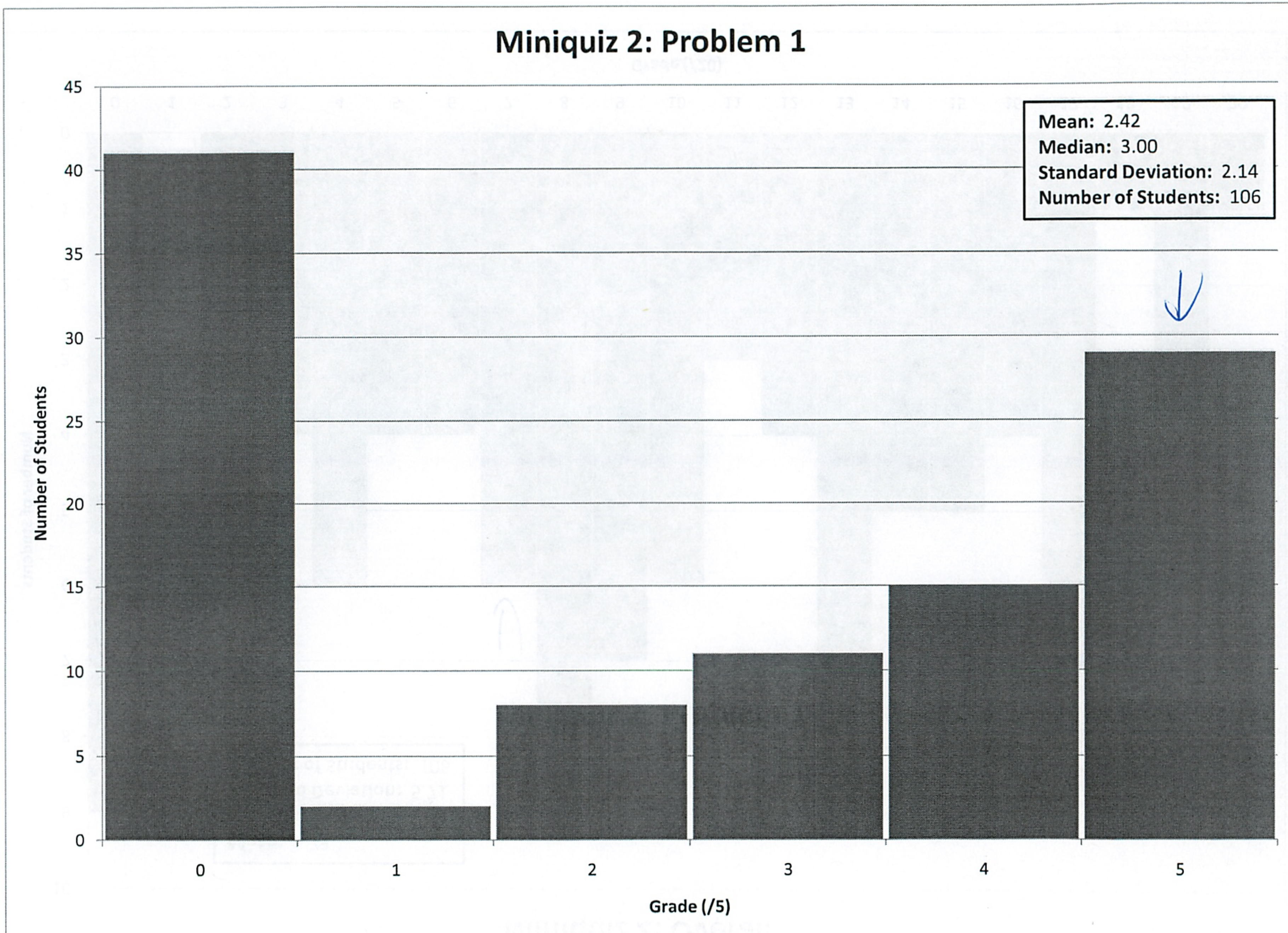
18
19
20
Grade (/20)


Miniquiz 2: Problem 1
Mean: 2.42
Median: 3.00
Standard Deviation: 2.14
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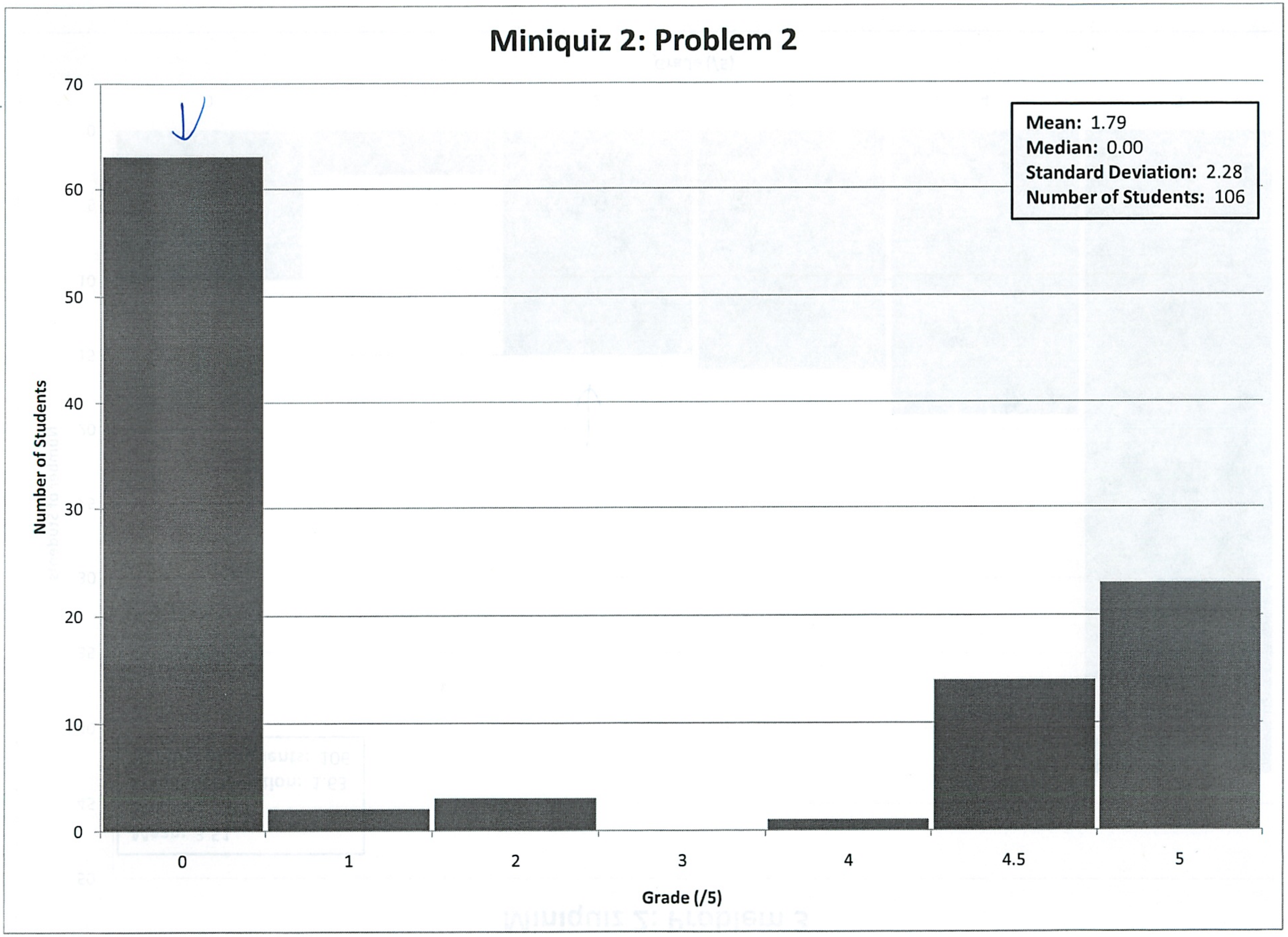
Grade (/5)


Miniquiz 2: Problem 2
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Median: 0.00
Standard Deviation: 2.28
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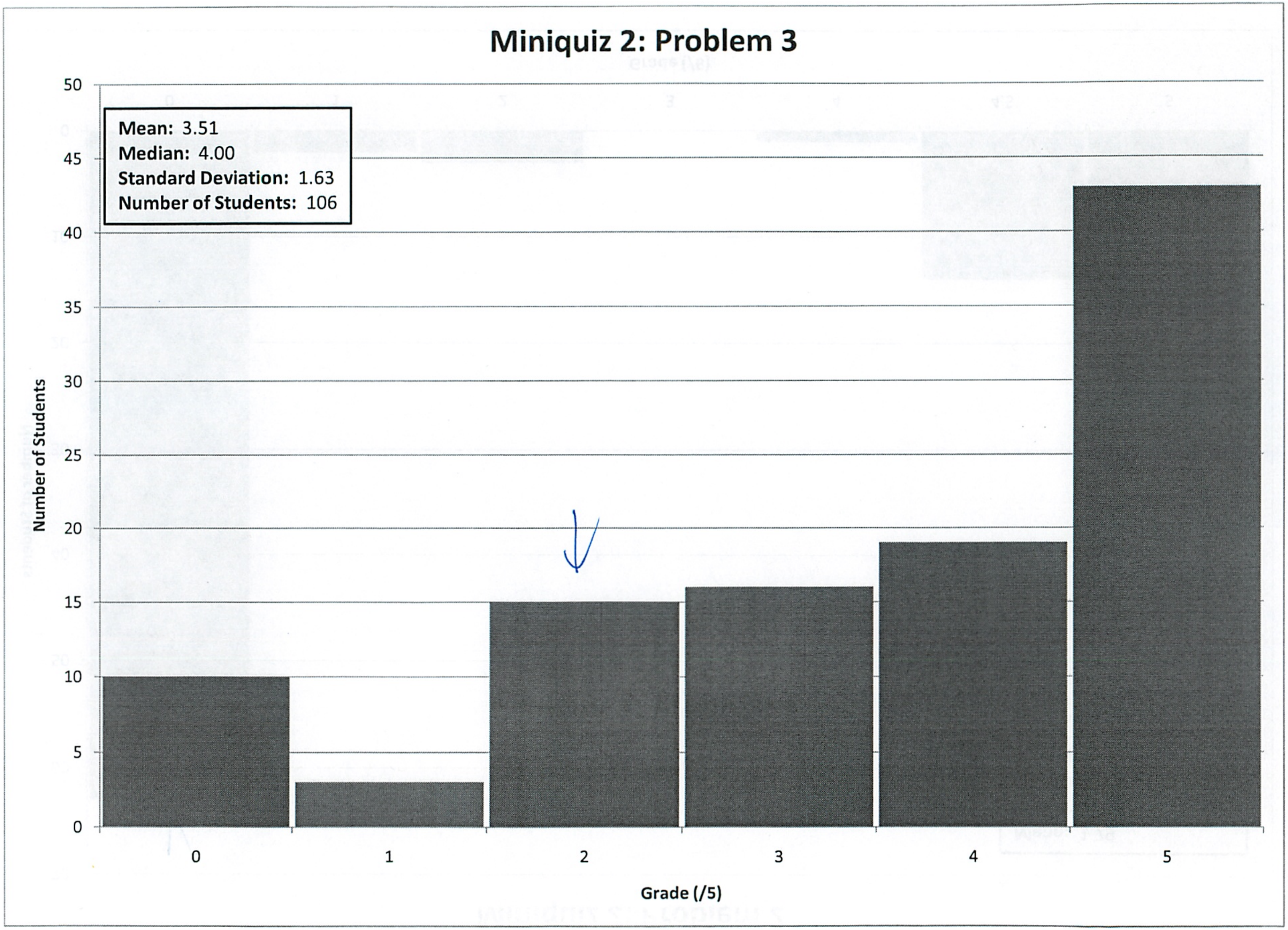
5
Grade (/5)



Miniquiz 2: Problem 3
Mean: 3.51
Median: 4.00
Standard Deviation: 1.63
Number of Students: 106
Number of Students
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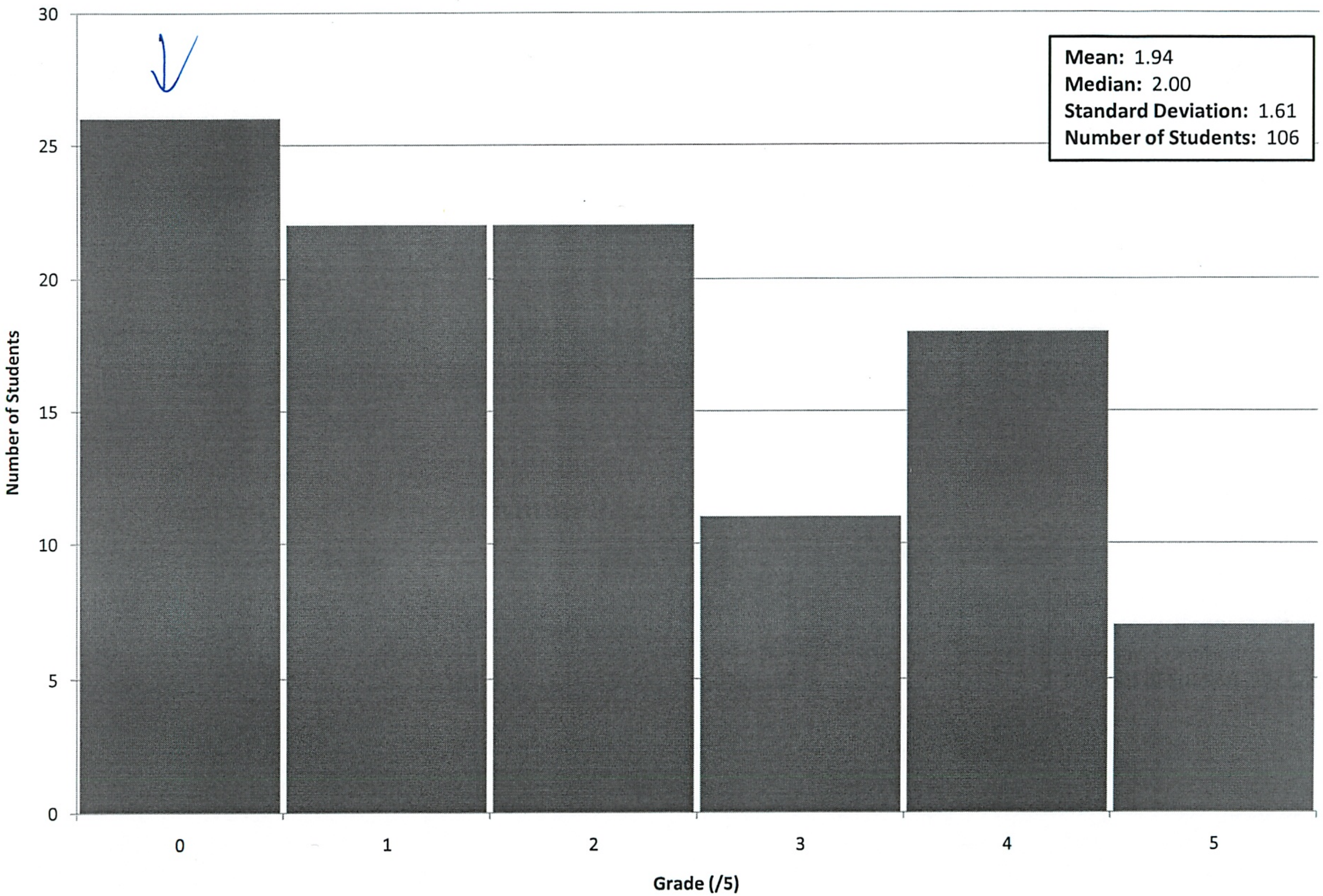
15
20
25
30
35
40
45
50
0
1
2
3
4
5
Grade (/5)

Miniquiz 2: Problem 4



Mini-Quiz Mar. 16 #3

Your name: Plasmeier

Circle the name of your TA and write your table number:

Ali Nick Oscar (Oshani) **Table number** 12

- This quiz is **closed book**. Total time is 30 minutes.
- Write your solutions in the space provided. If you need more space, write on the back of the sheet containing the problem. Please keep your entire answer to a problem on that problem's page.
- GOOD LUCK!

DO NOT WRITE BELOW THIS LINE

Problem	Points	Grade	Grader
1	5	3	NJ
2	5	1	AK
3	5	4.5	OM
4	5	5	OS
Total	20	13.5	OS

3/5

Problem 1 (5 points). (a) Calculate the value of $\phi(100)$.

$\phi(100) = \phi(2^2 \cdot 5^2)$
$= \phi(2^2) \cdot \phi(5^2)$
$= (2^2 - 2^1)(5^2 - 5^1)$
$= (4-2)(25-5)$
$= 2 \cdot 20$
$= 40$ ✓

24
3
72

✓

(b) Assume an integer $k > 9$ is relatively prime to 100. Explain why the last two digits of k and k^{121} are the same.

13

Hint: Use your solution to part (a).

Since $k^{\phi(100)} \equiv 1 \bmod 100$

So since $k^{\phi(100)}$ is congruent to 1 mod 100, the last two digits (which is the remainder mod 100) will be the same

no, want last 2 digits of k^{121} and k^1 same.

$k^{121} \equiv k \bmod 100$

$k^{\phi(p)-1}$ is the mul. inverse of k mod n

k is rel. prime because gcd(k, 100) = 1

−2

$a \equiv b \pmod{n}$ iff $n \mid (a-b)$ iff rem(a, n) = rem(b, n)

$k^{40} \equiv 1 \bmod 100$ — the remainder, subtract out

121 − 1 is a multiple of 40

So when take the power of − it will be the same mod 100

Problem 2 (5 points).
Prove that if $a \equiv b \pmod{14}$ and $a \equiv b \pmod{5}$, then $a \equiv b \pmod{70}$.

~~Since $a \equiv b \pmod{n}$ iff $n \mid (a-b)$~~

~~$p \mid ab$ iff $p \mid a$ or $p \mid b$~~ (prime) (prime)

$\frac{14 \cdot 5}{70}$

Assign $x = 14$
$y = 5$
Notice $xy = 70$

You've got this backward.

Now note that since 70 is a factor of both 14 and 5, values that are congruent mod 14 and mod 5 will also be congruent mod 70. Why?

1

How to word?

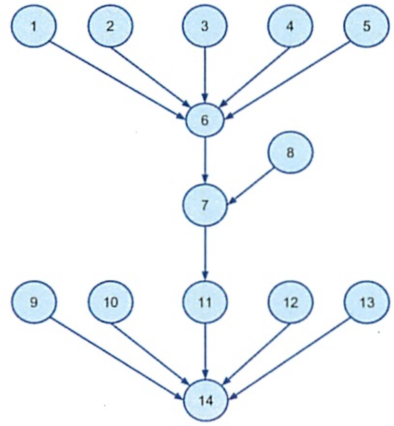


Figure 1 Task DAG

Problem 3 (5 points).
Answer the following questions about the dependency DAG shown in figure 1. Assume each node is a task that takes 1 second.

1. What is the largest chain in this DAG, if there is more than one, only show one.

   3 → 6 → 7 → 11 → 14.

2. What is the largest antichain? (again, pick one if you find there is more than one).

   9 10 11 12 13 —1 all one big chain

3. How much time would be required to complete all the tasks with a single processor.

   14

4. How much time would be required to complete all the tasks if there are unlimited processors available.

   5

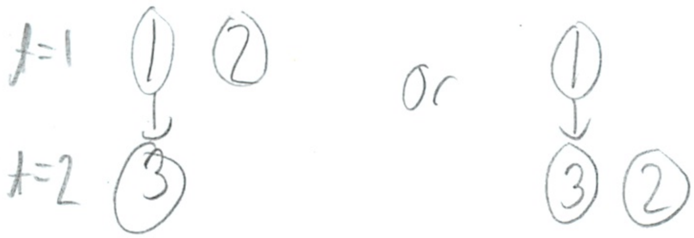
5. What is the smallest number of processors that would still allow to complete all the tasks in optimal time. Show a schedule proving it.

   t=1 1 2 3 4 5
   t=2 6 8
   3 7
   4 9 10 11 12 13
   5 14

   5 processors.
   We need to complete 1,2,3,4,5 before moving on to 6, so in order to get all 5 done in one time step, like we did in optimal solution, we need 5 processors

Problem 4 (5 points).
What is the smallest number of partially ordered tasks for which there can be more than one minimum time schedule, if there are unlimited number of processors? Explain your answer.

3 tasks



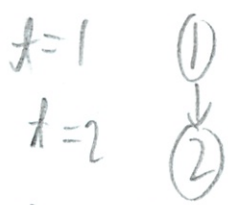
It can not work for 0 tasks
- only 1 schedule: do nothing

It can not work for 1 task
- only one schedule: do task!

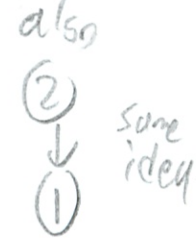
t=1 ①

It can not work for 2 tasks

t=1 ① ② since just do both time 1



Since just do time 1 and 2



And it can work for 3 tasks, an example is above

Solutions to Mini-Quiz Mar. 16

Problem 1 (5 points). (a) Calculate the value of $\phi(100)$.

Solution.

$$\phi(100) = \phi(25)\phi(4) = \phi(5^2)\phi(2^2) = (5^2 - 5)(2^2 - 2) = 40.$$

■

(b) Assume an integer $k > 9$ is relatively prime to 100. Explain why the last two digits of k and k^{121} are the same.

Hint: Use your solution to part (a).

Solution. Notice that all we have to prove is that k and k^{121} are congruent mod 100, implying they have the same last two digits.

$$k^{121} \equiv k^{40 \cdot 3+1} \equiv k(k^{40})^3 \pmod{100}.$$

By Euler's Theorem, since k and 100 are relatively prime, $k^{\phi(100)} \equiv 1 \pmod{100}$. By part (a), we have that $\phi(100) = 40$, implying $k^{40} \equiv 1 \pmod{100}$. Hence, $k(k^{40})^3 \equiv k(1^3) \equiv k \pmod{100}$. ■

Problem 2 (5 points).
Prove that if $a \equiv b \pmod{14}$ and $a \equiv b \pmod{5}$, then $a \equiv b \pmod{70}$.

Solution. We know $a \equiv b \pmod{14}$ means $14 | a - b$. Likewise, $a \equiv b \pmod{5}$ means $5 | a - b$. Also 14 and 5 are relatively prime.

For any p, q and x, if $p | x$ and $q | x$ and p and q are relatively prime, we know from class that $pq | x$. So, applying that reasoning with $x = a - b$, $p = 14$ and $q = 5$ yields $70 | a - b$, which is what we were trying to prove. ■

Problem 3 (5 points).
Answer the following questions about the dependency DAG shown in figure 1. Assume each node is a task that takes 1 second.

1. What is the largest chain in this DAG, if there is more than one, only show one.
2. What is the largest antichain? (again, pick one if you find there is more than one).
3. How much time would be required to complete all the tasks with a single processor.
4. How much time would be required to complete all the tasks if there are unlimited processors available.
5. What is the smallest number of processors that would still allow to complete all the tasks in optimal time. Show a schedule proving it.

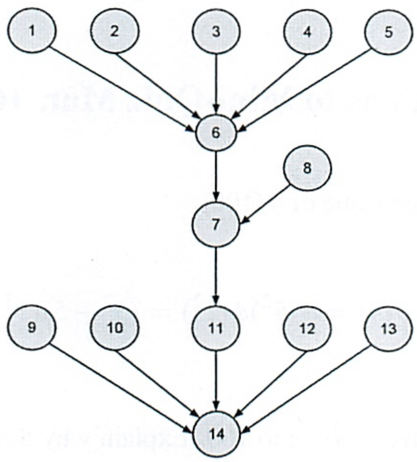


Figure 1 Task DAG

Solution. 1. One largest chain is $\{1, 6, 7, 11, 14\}$

2. One largest antichain is $\{1, 2, 3, 4, 5, 8, 9, 10, 12, 13\}$

3. There are 14 nodes, so a single processor would take 14 seconds.

4. With unlimited processors, we can take 5 seconds. This is the length of the longest chain.

5. With 5 processors, we can still finish everyting in 5 seconds. A schedule showing this is $\{1, 2, 3, 4, 5\}$, $\{6, 8\}$, $\{7\}$, $\{9, 10, 11, 12, 13\}$, $\{14\}$. We cannot do this with less than 5 processors because in order to make progress on the longest chain at every time step, we need to process $\{1, 2, 3, 4, 5\}$ in step 1.

■

Problem 4 (5 points).
What is the smallest number of partially ordered tasks for which there can be more than one minimum time schedule, if there are unlimited number of processors? Explain your answer.

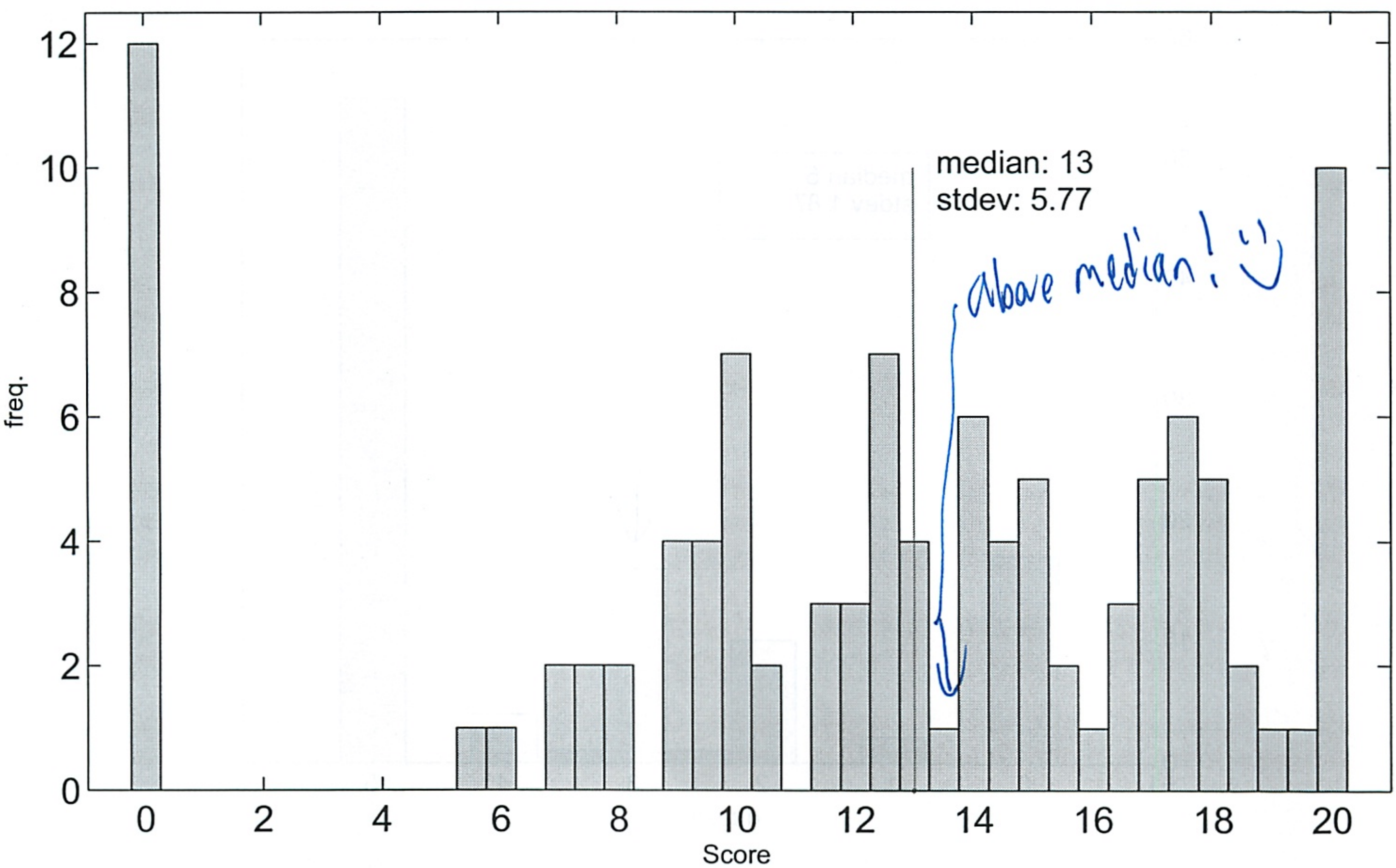
Solution. Three tasks.

With one task, there is only one possible schedule. Two tasks that are incomparable can both be completed in one step, and this is the unique minimum step schedule. For two tasks that are comparable, there is only one possible schedule, which therefore is the unique minimum time schedule.

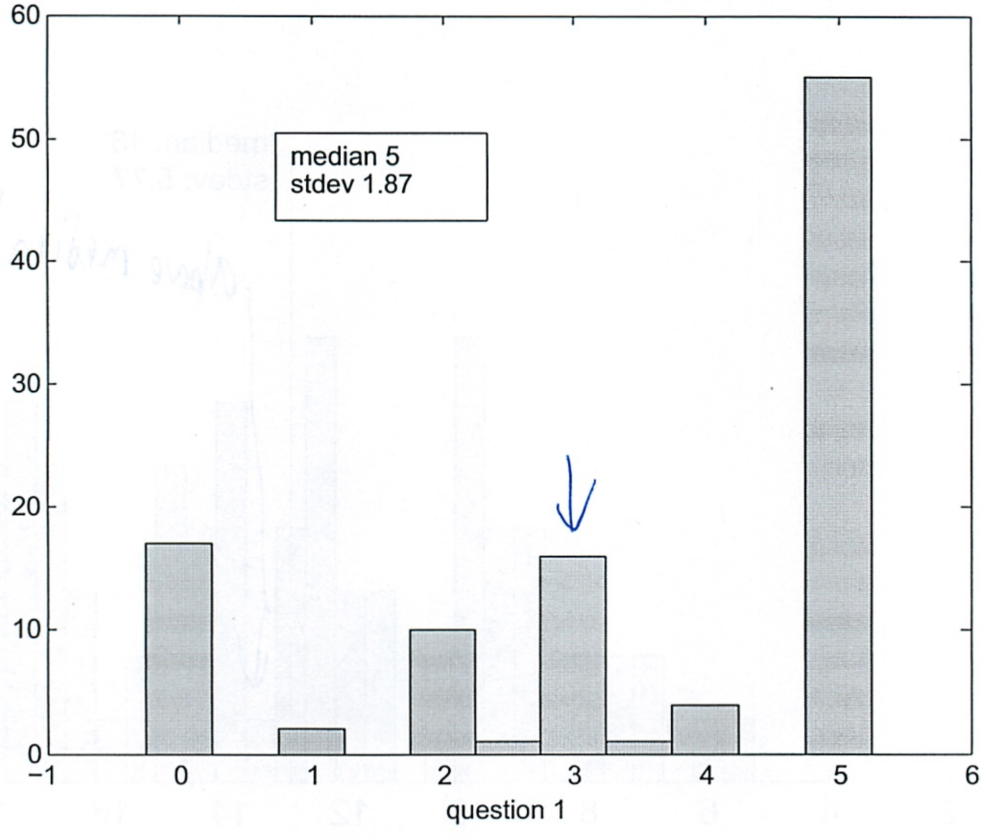
For an example with three tasks with two minimum time schedules, let two of the tasks be comparable and the third task incomparable to the other two. The two comparable tasks have a unique minimum time schedule that takes two steps. So any schedule for the three tasks that also takes only two steps will certainly be minimum time for the three. But the third task can be scheduled at the same time as either the first or the second of the comparable tasks, giving two minimum schedules for the three tasks.

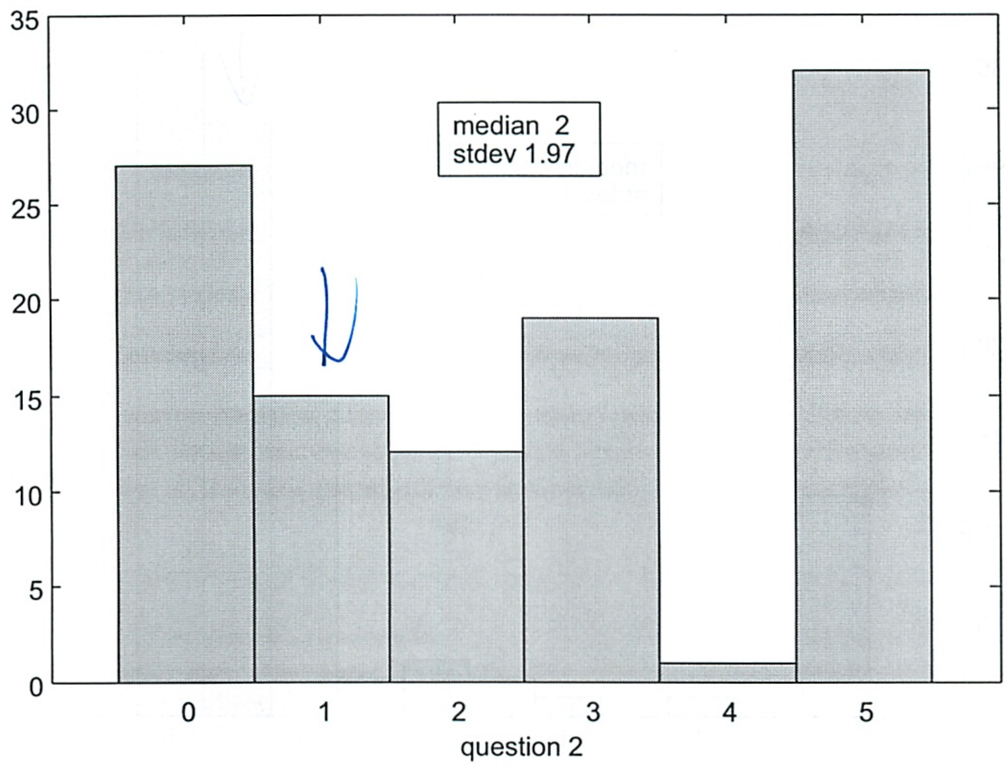
■

MQ3 grades

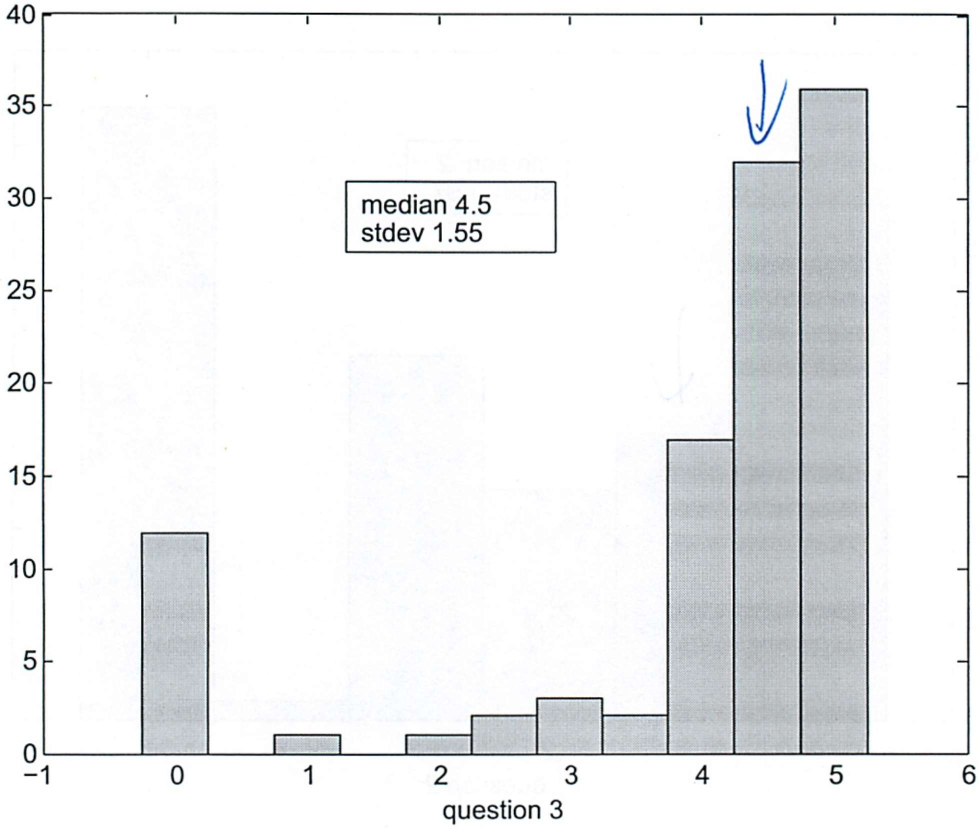


12
10
8
6
4
2
0
freq.
median: 13
stdev: 5.77
Above median! :)
0
2
4
6
8
10
12
14
16
18
20
Score

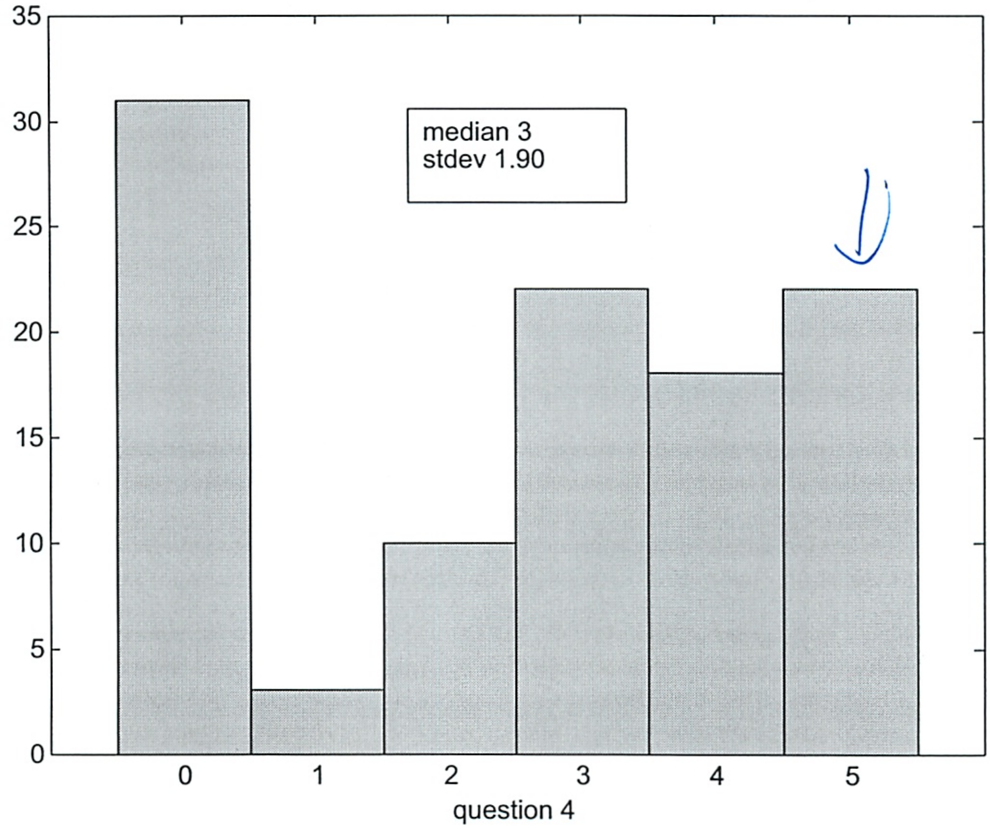

median 5
stdev 1.87
question 1



35
30
25
20
15
10
5
0
median 2
stdev 1.97
0
1
2
3
4
5
question 2



median 4.5
stdev 1.55
question 3



median 3
stdev 1.90
35
30
25
20
15
10
5
0
0
1
2
3
4
5
question 4

Mini-Quiz Apr. 6 #4

Your name: Michael Plasmeier

Circle the name of your TA and write your table number:

Ali Nick Oscar (Oshani) **Table number** 12

- This quiz is **closed book**. Total time is 30 minutes.
- Write your solutions in the space provided. If you need more space, write on the back of the sheet containing the problem. Please keep your entire answer to a problem on that problem's page.
- GOOD LUCK!

DO NOT WRITE BELOW THIS LINE

Problem	Points	Grade	Grader
1	6	3	OS
2	3	0	NJ
3	3	2	OM
4	5	2	AIC
5	3	2	OS
Total	20	9	OS

Problem 1 (6 points). (a) A simple graph has 8 vertices and 24 edges. What is the average degree per vertex?

✓ Handshake: $\sum \deg = 2|E|$

$8 \cdot avg = 2 \cdot 24$

$avg = \frac{48}{8} = 6$ ✓

(b) A connected planar simple graph has 5 more edges than it has vertices. How many faces does it have?

✓ Euler's $v - e + f = 2$

$1 - 6 + f = 2$

$-5 + f = 2$

$f = 7$ ✓

(c) A connected simple graph has one more vertex than it has edges. Is it necessarily planar?

~~$2 - 1 + f = 2$~~

Because of the condition that $v > 2$ this does not always hold – only when $v > 2$ proved w/ base + constructor

Base $e \leq 3v - 6$

say $v = 3$, $e = 2$

$2 \leq 3(3) - 6$

$2 \leq 12$ ✓ can work

Iterative

$v = +1$

$e = +1$

will work – right hand side ↑ at factor of 3 compared to left

?

try $v = 2$ $e = 1$

$1 \leq 3(2) - 6$

$1 \leq 0$ ⊗ won't work.

Must $v \geq 3$

(d) If your answer to the previous part was *yes*, then how many faces can such a graph have? If your answer was *no*, then give an example of a nonplanar connected simple graph whose vertices outnumber its edges by one.

$v - e + f = 2$

So $f = 2 - v + e$

How it holds when $v > 2$

when $v = 3$ $e = 2$

$f = 2 - 3 + 2$

$f = 1$

$V = 4$ $e = 3$

$f = 2 - 4 + 3$

$= 1$ ✓

$V = 5$ $e = 4$

$f = 2 - 5 + 4$

$= 1$

(e) Consider the graph shown in Figure 1. How many distinct isomorphisms exist between this graph and itself? (Include the identity isomorphism.)

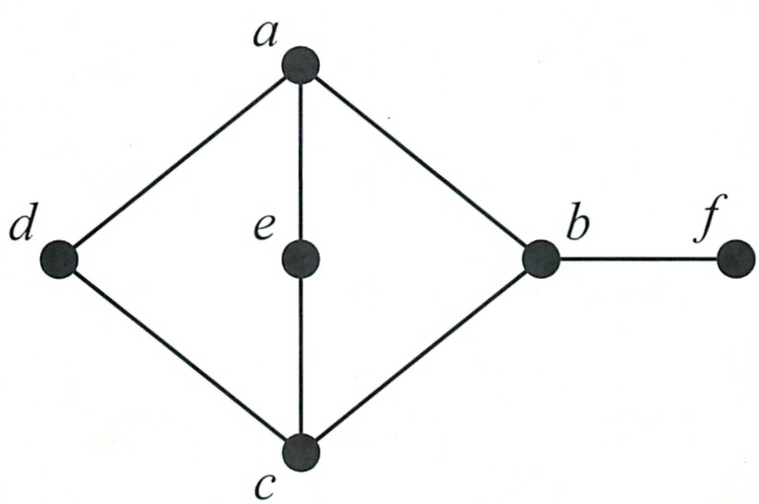


Figure 1

Just 1 by definition ✗

- Since can't move or relable anything

a → a
b → b
c → c
d → d
e → e
f → f

Problem 2 (3 points).
The n-dimensional hypercube, H_n, is a simple graph whose vertices are the binary strings of length n. Two vertices are adjacent if and only if they differ in exactly one bit. Consider for example H_3, shown in Figure 2. (Here, vertices `111` and `011` are adjacent because they differ only in the first bit, while vertices `101` and `011` are not adjacent because they differ in both the first and second bits.) Hemming distance

Explain why it is impossible to find two spanning trees of H_3 that have no edges in common.

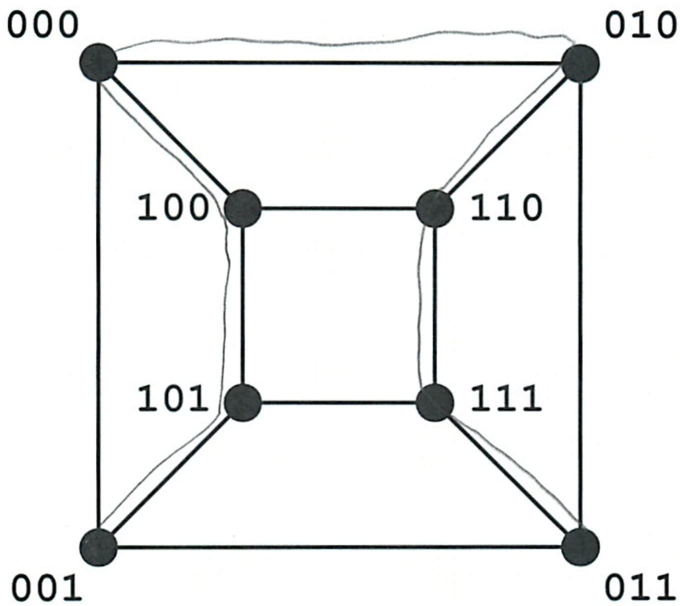


Figure 2 H_3.

0/3

Once you start in a certain way, there are very limited choices as to what you can do next in the spanning tree.

or ... or etc rotated

Its just the same pattern rotated
Each point can is, degree 3, so there are a limited # of cut edges possible to find different spanning trees. that's not a general argument

Problem 3 (3 points).

✓ Consider the graph shown in Figure 3. Determine a valid coloring of the graph, using as few colors as possible. (Simply write your proposed color for each vertex next to that vertex. You may use *R* for red, *G* for green, etc.)

Will use #s

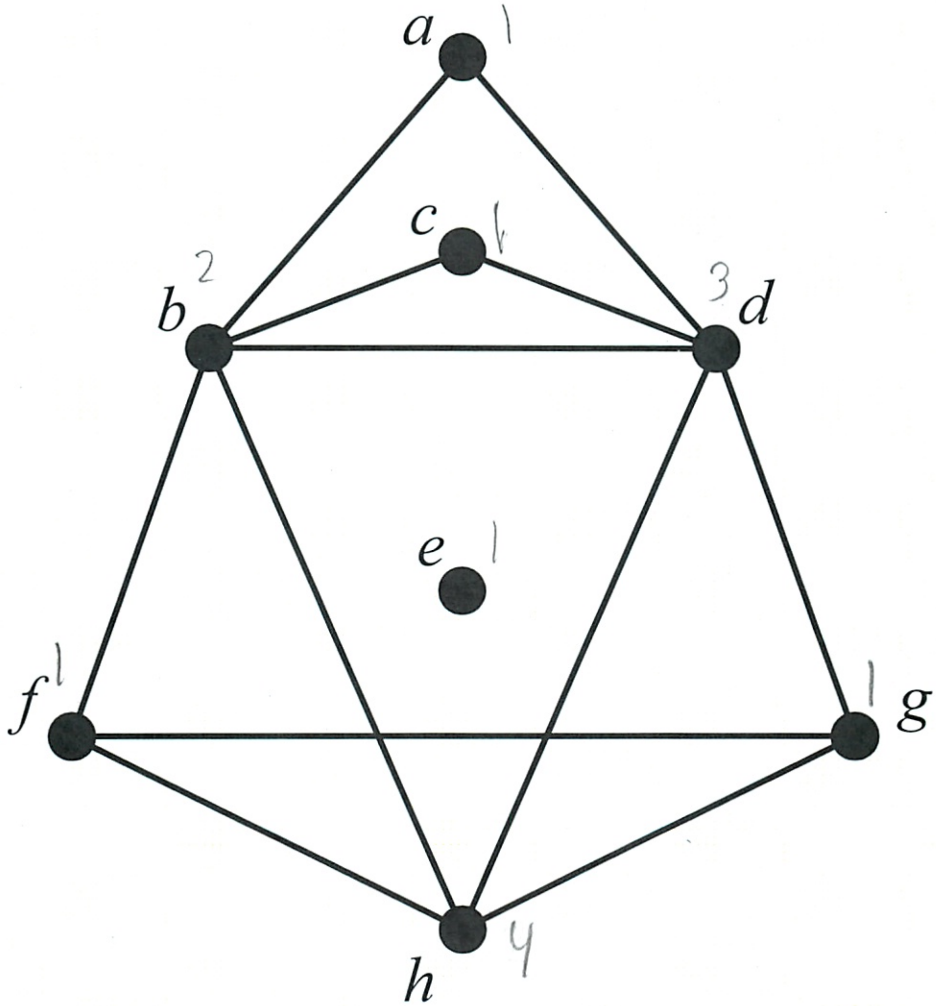


Figure 3

Say
1=Red
2=Green
3=Yellow
4=Orange

Since max degree =4

−1 3 is enough

Problem 4 (5 points). (a) Consider the bipartite graph G in Figure 4. Is it possible to find a matching that covers $L(G)$? If yes, explain what property of the graph guarantees the existence of such a matching. (Show that the graph exhibits this property and what this implies. Full credit will not be given for merely identifying a matching.) If no, identify a bottleneck that prevents a matching.

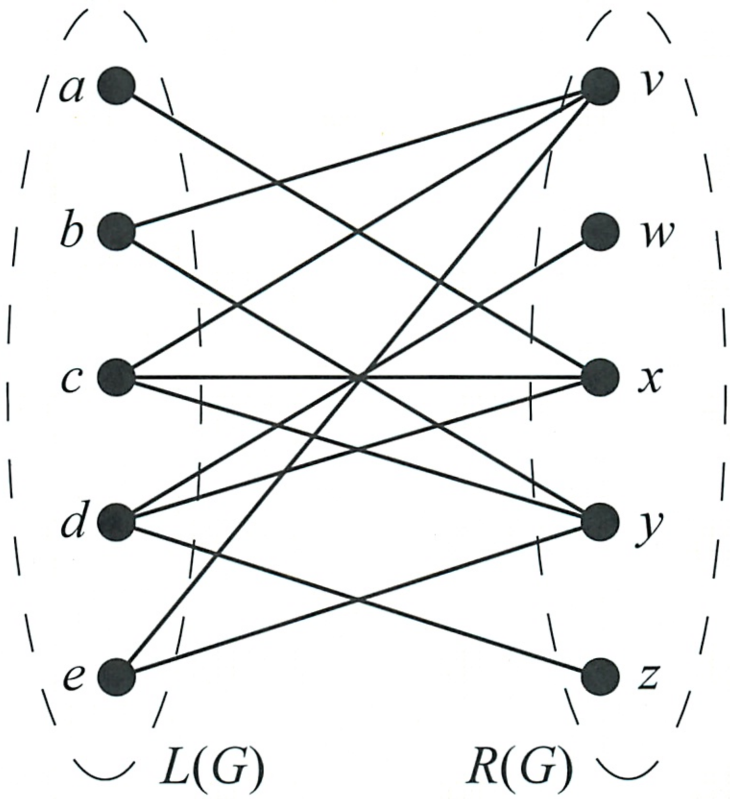


Figure 4 G.

Matching - set of M edges G s.t. no vertex is incident to ≥ 1 edges in M

Matching Condition - every subset of L(G) is connected to at least as large a subset of R(G)

bottle neck |S| > |N(S)| neighbors too

Covers - all vertices included (perfect)

Hall's theorem - Matching in G (bipartite) that covers L(G) if no subset of L(G) is a bottleneck.

There is no bottleneck. For all subsets of L(G) there exists a subset of equal or larger size in R(G)

√

(b) Consider the bipartite graph H in Figure 5. Is it possible to find a matching that covers $L(H)$? If yes, explain what property of the graph guarantees the existence of such a matching. (Show that the graph exhibits this property and what this implies. Full credit will not be given for merely identifying a matching.) If no, identify a bottleneck that prevents a matching.

Covers - all verticies included

2

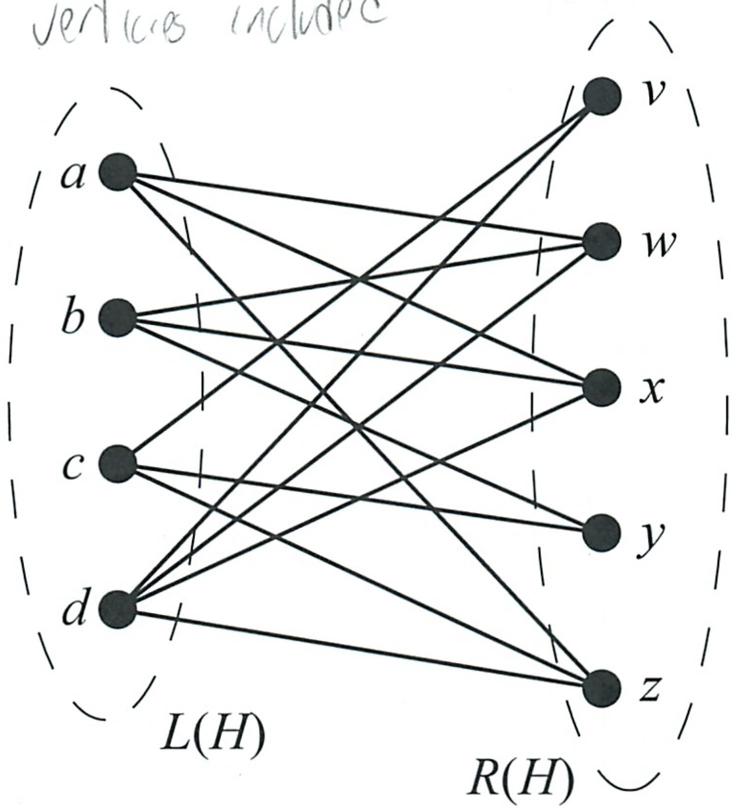


Figure 5 H.

Yes. Since the deg(l) ≥ deg(r) for all l ∈ L and r ∈ R so that it is degree constrained

This means there is a matching that covers ~~the entire graph,~~ No! L(H).

See def in previous page.

Problem 5 (3 points).
In the Mating Ritual, suppose Tiger is one of the boys and Elin is one of the girls. Which of the following are preserved invariants **in general**?

2/3

1. Tiger is Elin's only suitor.
2. On Tiger's current list, the girl whom he prefers to all the others is his optimal wife[1].
3. Elin's name has been crossed off by Tiger and by everyone whom she prefers to him.

1. We don't know that w/ info we have been given.

2. Yes, true. Of the names remaining on the list (the current names) the name at the top would be the girl he perfers to all others. This is defined as his optimal wife.
He stays with this girl till they get married or she kicks him out and then he is off the list (no longer on current list)

3. False. Everyone who Elin perfers to Tiger has no relation to who Tiger crosses off his list. Elin's name is crossed off by Tiger when she rejects him. There is no relation between Elin's name on Tiger's list and Elin's personal preferences.

[1] His *optimal wife* in the usual sense: Given some particular instance of the Stable Marriage Problem, consider all possible stable perfect matchings, including that which is generated by the Mating Ritual. In each of these, Tiger has a wife. Of these "possible wives," he prefers one to all others. This girl, to whom he is married in one of the matchings but not necessarily all of them, is his *optimal wife*.

Solutions to Mini-Quiz Apr. 6

Problem 1 (6 points). (a) A simple graph has 8 vertices and 24 edges. What is the average degree per vertex?

Solution. By the Handshaking Lemma, the sum of the degrees of the vertices in any graph is equal to twice the number of edges. So in this case, the sum of the degrees of the vertices is $2 \times 24 = 48$. With 8 vertices, the average degree per vertex is $\frac{48}{8} = 6$. ■

(b) A connected planar simple graph has 5 more edges than it has vertices. How many faces does it have?

Solution. Denoting the number of vertices by v, the number of edges by e, and the number of faces by f, Euler's Formula states that $v - e + f = 2$. But here, $e = v + 5$. Substituting gives $v - (v + 5) + f = 2$ and hence $f = 7$. ■

(c) A connected simple graph has one more vertex than it has edges. Is it necessarily planar?

Solution. Let G denote any such graph. Now, any graph with v vertices but fewer than $v - 1$ edges cannot possibly be connected. So every edge in G is a cut edge, and therefore G is acyclic. So G is a tree and must be planar. ■

(d) If your answer to the previous part was *yes*, then how many faces can such a graph have? If your answer was *no*, then give an example of a nonplanar connected simple graph whose vertices outnumber its edges by one.

Solution. Since the graph is connected and acyclic, it only has one face. ■

(e) Consider the graph shown in Figure 1. How many distinct isomorphisms exist between this graph and itself? (Include the identity isomorphism.)

Solution. Only vertex f has degree 1, so in any self-isomorphism, f must map to itself. b is the only vertex to be adjacent to a degree-1 vertex, so b must also map to itself. a and c are both degree-3 vertices, and d and e are both degree-2 vertices. It is clear from examining the graph that a can be mapped to c and c to a, or each of a and c can be mapped to itself. Independently, and similarly, d can be mapped to e and e to d, or each of d and e can be mapped to itself. The only possible isomorphisms, then, are obtained by choosing one of the two possible mappings for a and c and, independently, one of the two possible mappings for d and e. The result is $2 \times 2 = 4$ possible isomorphisms. ■

Problem 2 (3 points).
The n-dimensional hypercube, H_n, is a simple graph whose vertices are the binary strings of length n. Two vertices are adjacent if and only if they differ in exactly one bit. Consider for example H_3, shown in Figure 2. (Here, vertices `111` and `011` are adjacent because they differ only in the first bit, while vertices `101` and `011` are not adjacent because they differ in both the first and second bits.)

Explain why it is impossible to find two spanning trees of H_3 that have no edges in common.

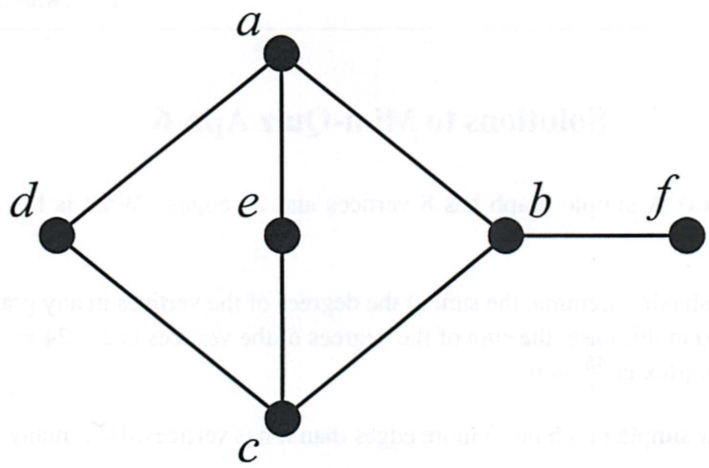


Figure 1

Solution. H_3 has 8 vertices, so any spanning tree must have $8 - 1 = 7$ edges. But H_3 has only 12 edges, so any two sets of 7 edges must overlap. ■

Problem 3 (3 points).
Consider the graph shown in Figure 3. Determine a valid coloring of the graph, using as few colors as possible. (Simply write your proposed color for each vertex next to that vertex. You may use R for red, G for green, etc.)

Solution. There are odd-length cycles in the graph, so at least three colors will be needed. So assume that three colors are sufficient. (If we encounter a contradiction under this assumption, we will need to use more colors.) Start with the length-3 cycle $abda$. All of its vertices must be colored differently, so assign red to a, blue to b, and green to d. The length-3 cycle $bdhb$ now forces h to be colored red. f must now be colored green and g must be colored blue. The coloring is valid so far. c is adjacent to a blue vertex and a green vertex, and no others, it must be colored red. Finally, e is not adjacent to any other vertices, so it can be assigned any of the three colors. Choosing red for e, the result is shown in Figure 4. There is no pair of like-colored adjacent vertices, so this coloring is valid. ■

Problem 4 (5 points). (a) Consider the bipartite graph G in Figure 5. Is it possible to find a matching that covers $L(G)$? If yes, explain what property of the graph guarantees the existence of such a matching. (Show that the graph exhibits this property and what this implies. Full credit will not be given for merely identifying a matching.) If no, identify a bottleneck that prevents a matching.

Solution. It is not possible. One bottleneck is $S = \{a, b, c, e\}$, since $N(S) = \{v, x, y\}$ and hence $|S| = 4 > 3 = |N(S)|$. (It is easy to see that there are no bottlenecks of size 1, 2, 3, or 5.) ■

(b) Consider the bipartite graph H in Figure 6. Is it possible to find a matching that covers $L(H)$? If yes, explain what property of the graph guarantees the existence of such a matching. (Show that the graph exhibits this property and what this implies. Full credit will not be given for merely identifying a matching.) If no, identify a bottleneck that prevents a matching.

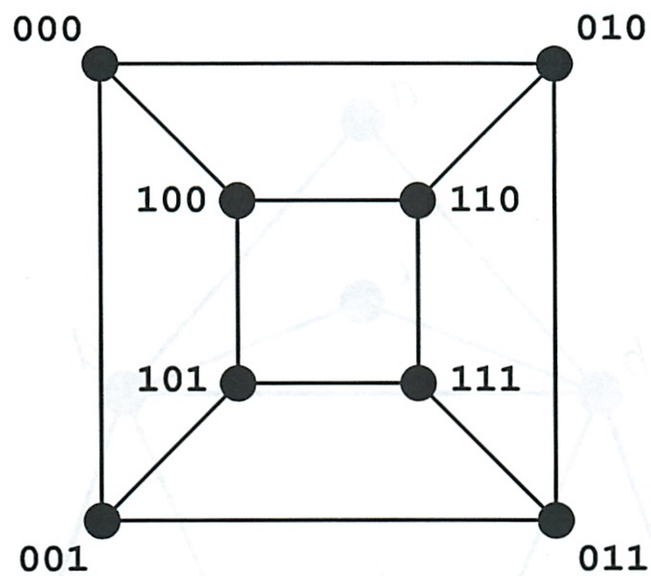


Figure 2 H_3.

Solution. A matching is guaranteed to exist. Each vertex in $L(H)$ has degree at least 3, while each vertex in $R(H)$ has degree at most 3. Consequently, the graph is degree-constrained. There are therefore no bottlenecks and a matching must exist by Hall's Theorem. ■

Problem 5 (3 points).
In the Mating Ritual, suppose Tiger is one of the boys and Elin is one of the girls. Which of the following are preserved invariants **in general**?

1. Tiger is Elin's only suitor.
2. On Tiger's current list, the girl whom he prefers to all the others is his optimal wife[1].
3. Elin's name has been crossed off by Tiger and by everyone whom she prefers to him.

Solution. The statements that are preserved invariants in general appear in boldface below:

1. Tiger is Elin's only suitor. (This would certainly make Tiger Elin's favorite that day, but one or more of the boys who got rejected by another girl that day may visit Elin the following day.)
2. **On Tiger's current list, the girl whom he prefers to all the others is his optimal wife.** (The Mating Ritual gives each boy his optimal wife. Tiger must therefore ultimately marry his optimal wife, so once she becomes the most preferred girl on his list – and thus the girl he is serenading – she must remain the top girl on his list.)
3. **Elin's name has been crossed off by Tiger and by everyone whom she prefers to him.** (Note that this is a preserved invariant because it cannot ever be true. Were it true on some day, Tiger would have crossed Elin's name off his list, so he would end up marrying a woman he finds less desirable.

[1]His *optimal wife* in the usual sense: Given some particular instance of the Stable Marriage Problem, consider all possible stable perfect matchings, including that which is generated by the Mating Ritual. In each of these, Tiger has a wife. Of these "possible wives," he prefers one to all the others. This girl, to whom he is married in one of the matchings but not necessarily all of them, is his *optimal wife*.

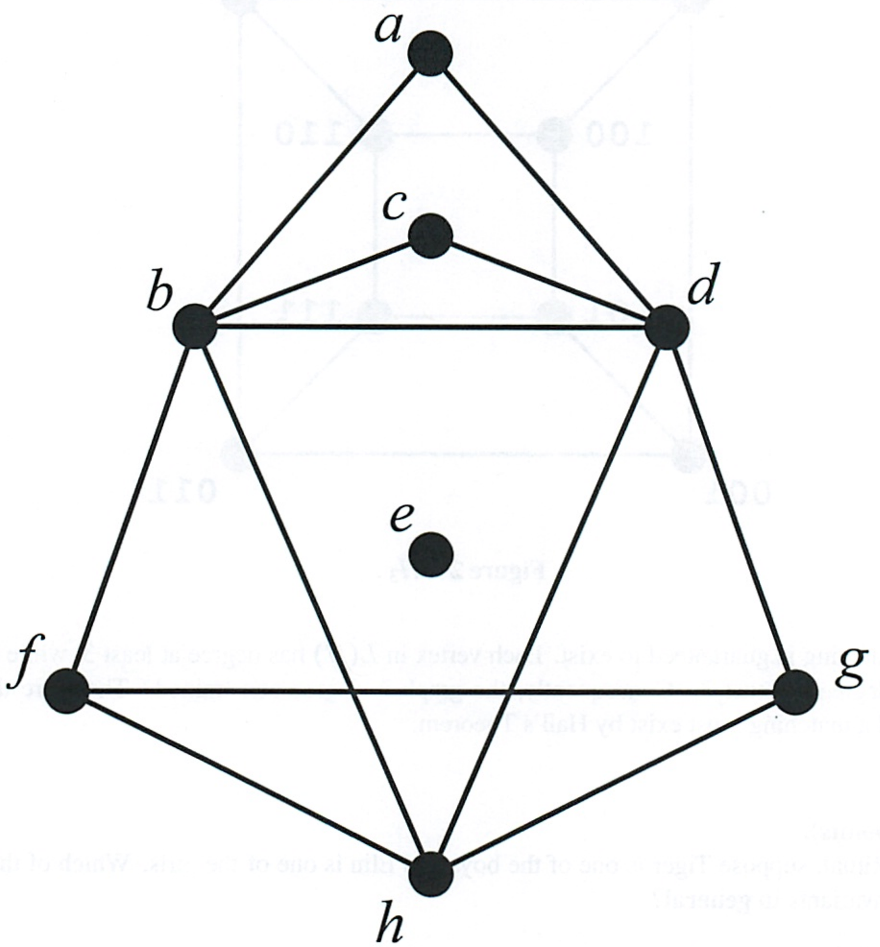


Figure 3

She would also have removed from contention everyone she finds more desirable than Tiger. So she would end up marrying someone she finds less desirable than Tiger. Consequently, Tiger and Elin would constitute would a rogue couple. Another way to think about it is this: If Elin's name was crossed off by Tiger and all the boys Elin prefers to him, then she must have a current favorite whom she prefers to all of them. But Tiger and his betters in Elin's eyes are the top boys on her list: there is no one she prefers to them.)

■

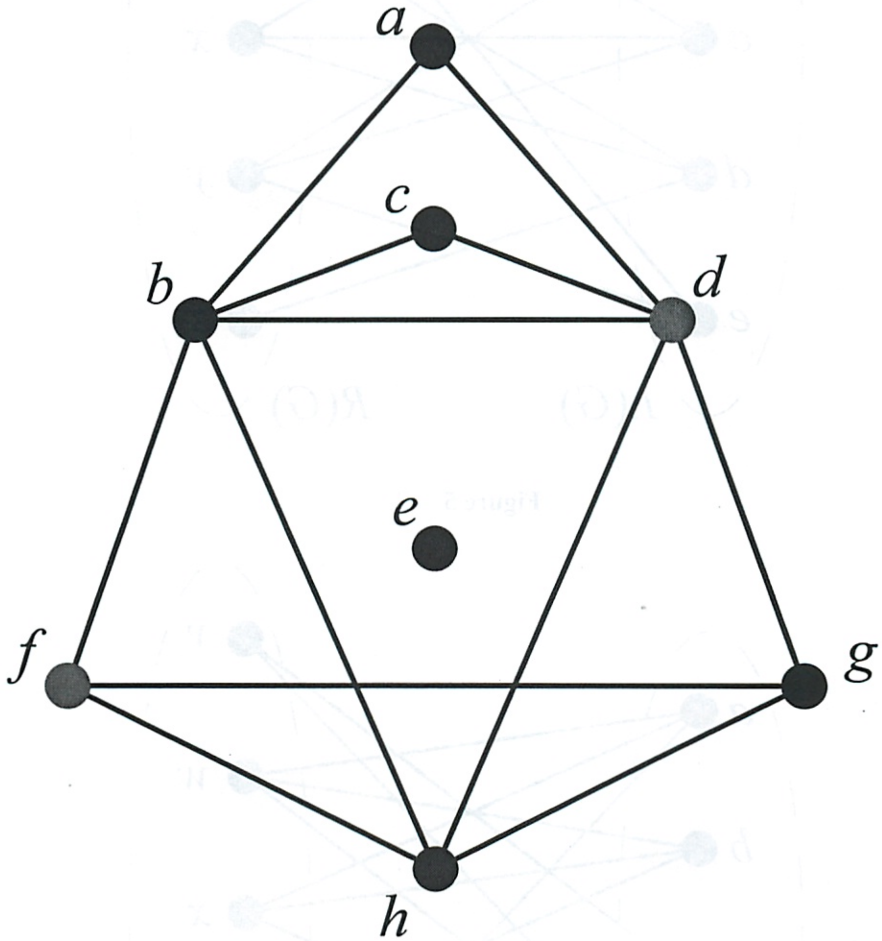


Figure 4 A valid coloring.

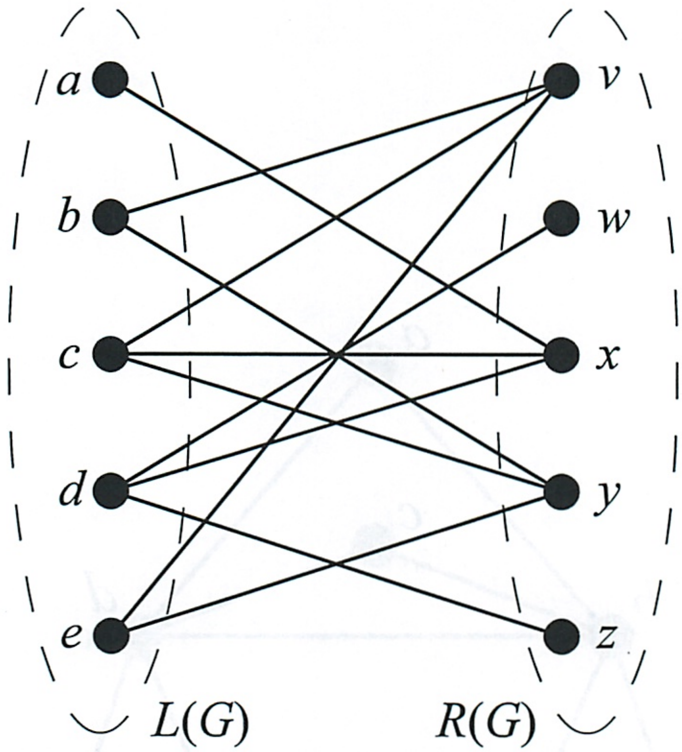


Figure 5 G.

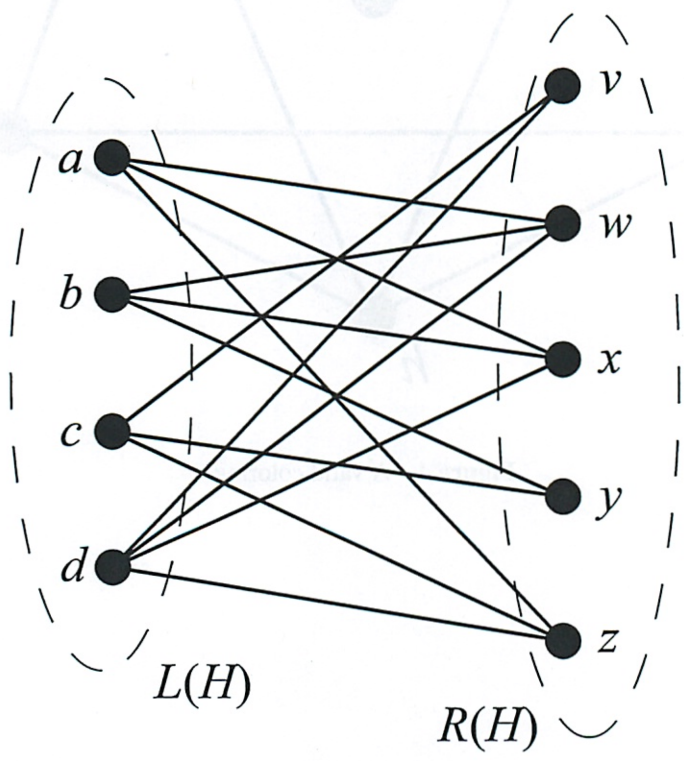
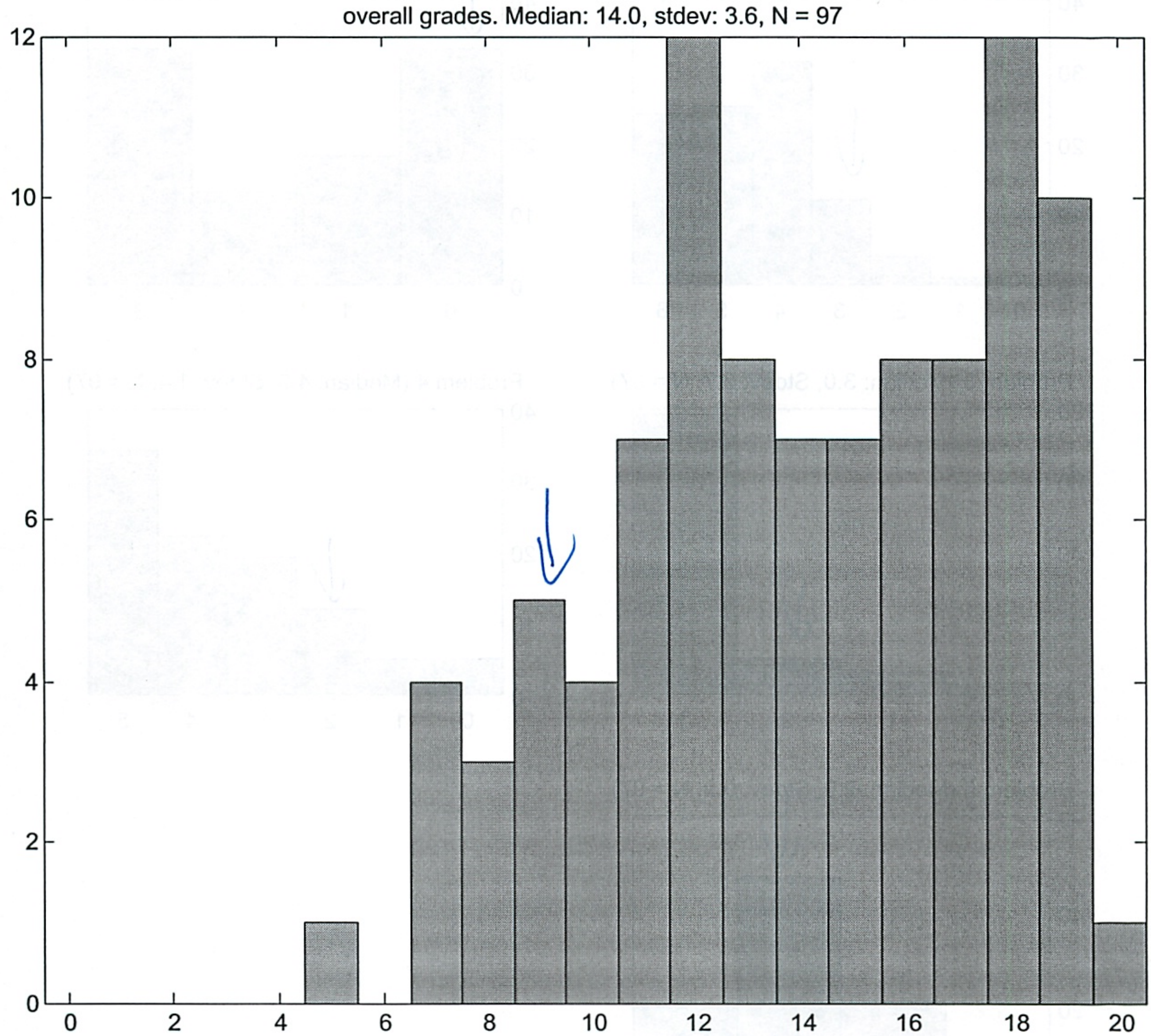


Figure 6 H.


overall grades. Median: 14.0, stdev: 3.6, N = 97
12
10
8
6
4
2
0
0
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8
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12
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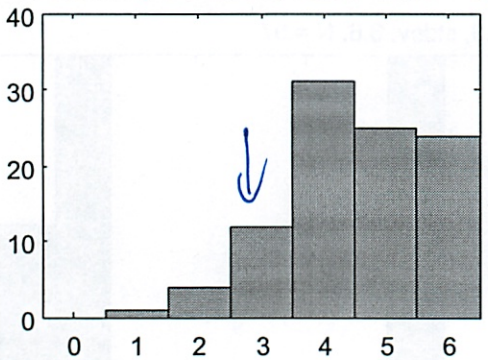
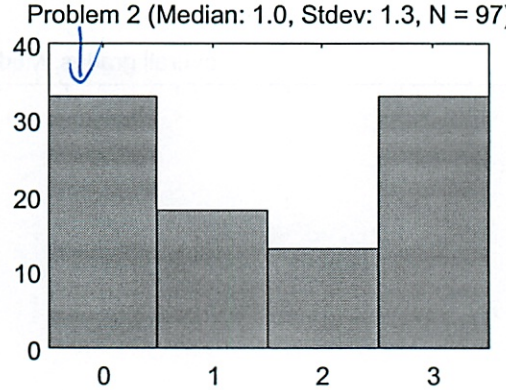
16
18
20

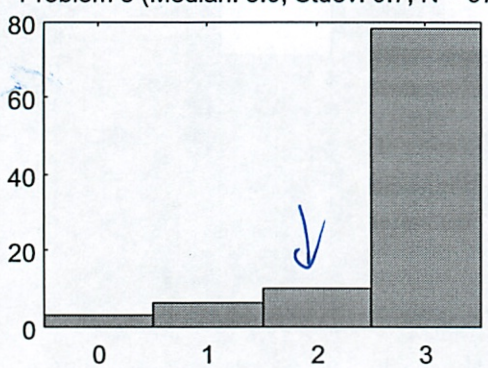
Problem 1 (Median: 5.0, Stdev: 1.1, N = 97)



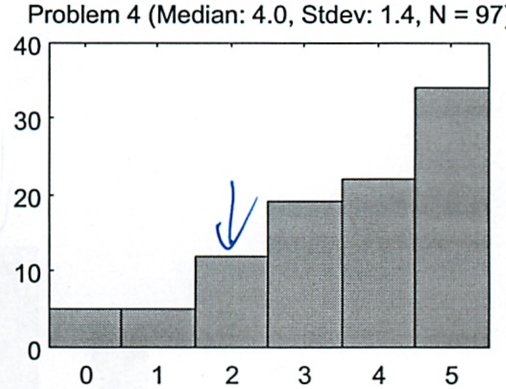
Problem 2 (Median: 1.0, Stdev: 1.3, N = 97)



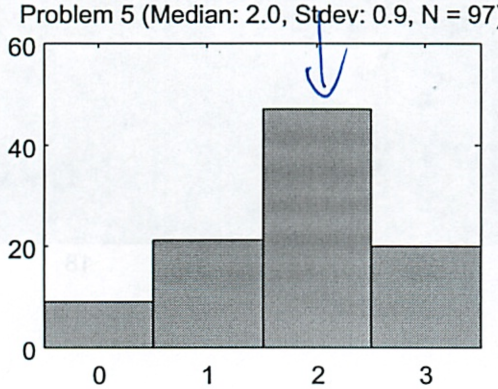
Problem 3 (Median: 3.0, Stdev: 0.7, N = 97)



Problem 4 (Median: 4.0, Stdev: 1.4, N = 97)



Problem 5 (Median: 2.0, Stdev: 0.9, N = 97)



Mini-Quiz Apr. 20 #5

Your name: Michael Plasmeier

Circle the name of your TA and write your table number:

Ali Nick Oscar (Oshani) **Table number** 12

- This quiz is **closed book**. Total time is 30 minutes.
- Write your solutions in the space provided. If you need more space, write on the back of the sheet containing the problem. Please keep your entire answer to a problem on that problem's page.
- GOOD LUCK!

DO NOT WRITE BELOW THIS LINE

Problem	Points	Grade	Grader
1	5	2	ORI
2	7	4	OS
3	3	1	NJ
4	5	∅	AK
Total	20	7	

Every card is different except 2 pair

Problem 1 (5 points). (a) Suppose two identical 52-card decks[1] are mixed together. Write a simple expression for the number of different arrangements of the 104 cards that could possibly result from such a mixing.

$104!$ permutations

$104 \cdot 103 \cdot 102 \cdots$

— Oh but 2 cards will be same 2-to-1

$\frac{104!}{2}$ −1

(b) Using only integers from the interval $[1, n]$, how many different **strictly increasing** length-m sequences can be formed?

Correction $m \le n$ on cheat sheet

$\binom{m+n}{n}$ −1

bij to seq $x_1, x_1+x_2, x_1+x_2+x_3, \ldots$

ans follows from $x_1+x_2+\cdots+x_m \le n \to \binom{m+n}{n}$

[1] Standard decks of playing cards, without jokers.

Problem 2 (7 points).
For each pair of functions, $f : \mathbb{N}^+ \to \mathbb{N}$ and $g : \mathbb{N}^+ \to \mathbb{N}$, in the table below, indicate which of the listed asymptotic relations hold **and** which do not.

Fill **every** cell in the table. You may use checkmarks and crosses, "T" and "F", "TRUE" and "FALSE", "Y" and "N", or "YES" and "NO". Look at largest term

$f(n)$	$g(n)$	$f = O(g)$	$f = o(g)$	$g = O(f)$	$g = o(f)$
$\log_4 n$	$\sqrt[3]{n}$	✗ ✗	✓	✓ ✗	✗
$n^2 + 3^n$	$n^3 + 2^n$	✓	✗	✓	✓
$n \ln n!$	$n^2 \log_{10} n^2$	✗ ✗	✓ ✗	✓	✗
$n^{2\cos(\pi n/2)+3}$	$5n^5 + 3n^3 + n$	✓	✓ ✗	✓ ✗	✗

10/16

didn't write which smaller
exp > log
poly > log
on cheat sheet

$f = O(g)$ iff $\lim \frac{f(x)}{g(x)} < \infty$ finite

$n^2 + 3^n = f$
$n^3 + 2^n = g$

f bigger

g bigger means $f = o(g)$
f " " $g = o(f)$

(how tell if finite?

guessing

Problem 3 (3 points).
Give an example of a pair of strictly increasing total functions, $f : \mathbb{N}^+ \to \mathbb{N}^+$ and $g : \mathbb{N}^+ \to \mathbb{N}^+$, that satisfy $f \sim g$ but **not** $3^f = O(3^g)$.

kinda remember

asy = to

upper bound on growth

$\lim_{x \to \infty} \frac{f(x)}{g(x)} = 1$

if leading terms =

correct.

how to do?

$f = 2^x$

$g = 3^x$

$2^\infty = \infty$

$3^\infty = \infty$

$\frac{\infty}{\infty} = 1$

you can't take limits like that; $\lim_{x\to\infty} \frac{f}{g} = \lim_{x\to\infty} \left(\frac{2}{3}\right)^x = 0.$

$\frac{3^{2^x}}{3^{3^x}}$ Not $< \infty$

So not finite

actually $\lim_{x\to\infty} \frac{3^{2^x}}{3^{3^x}} < \infty.$

common base rule.

$\frac{1}{3}$

guessing

not as much (anting as thought

Problem 4 (5 points).
A spacecraft is traveling through otherwise-empty three-dimensional space. It can move along only one dimension at a time, stepping precisely one unit in the positive direction along that dimension with each movement. For any two points, P and Q, in space, let $p_{P,Q}$ denote the number of distinct paths the spacecraft can follow to go from P to Q.

(a) Let P and Q have coordinates (x_P, y_P, z_P) and (x_Q, y_Q, z_Q), respectively. Assuming that $p_{P,Q}$ is positive, express $p_{P,Q}$ as a **single multinomial coefficient**.

- did not write - was on pset; was not multinomial before?

Can be any order

$X_q - X_p = k_x \quad Y_q - Y_p = k_y \quad Z_q - Z_p = k_z$

∅

$$P_{p,q} = \sum_{k_x, k_y, k_z \in N} \binom{k_x \cdot k_y \cdot k_z}{k_x, k_y, k_z} X_p^{k_x} \cdot Y_p^{k_y} \cdot Z_p^{k_z}$$

example

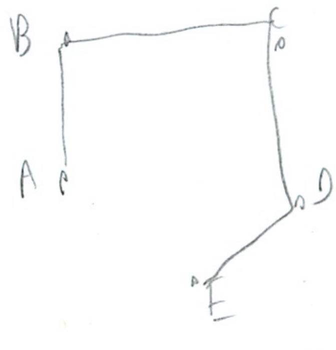
(b) Suppose there exist five points in space, A, B, C, D, and E, such that it is possible for the spacecraft to travel from A to B, from B to C, from C to D, and from D to E. Write an expression for the number of distinct paths the spacecraft can follow to go from A to E while **avoiding** B, C, and D. Your expression **must** be written entirely in terms of symbols of the form $p_{P,Q}$, where $P, Q \in \{A, B, C, D, E\}$.

Hint: Inclusion-Exclusion. how is it a non disjoint set? TA: don't go there at all might be able to go A → E

Can go A → E
A → D → E
A → C → D → E
but want to avoid

∅

Can go



But how do we know if it can go A → E directly?

$|S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5| = |S_1| + |S_2| + |S_3| + |S_4| + |S_5| - |$ the sets of 2 $S_1 \cap S_2|$ etc
$+ |$ sets of 3 $S_1 \cap S_2 \cap S_3| - |$ sets of 4 $S_1 \cap S_2 \cap S_3 \cap S_4| + |S_1 \cap S_2 \cap S_3 \cap S_4 \cap S_5|$

Don't get what qu asking!

Oh related to previous problem

So can go direct

$P_{A,E}$ or through stops $+2$ stop $+ 3$ stop $+ 4$ stop $+ 5$ stop

↑ but not allowed

So just direct $P_{A,E}$ which is multinomial from part a

Solutions to Mini-Quiz Apr. 20

Problem 1 (5 points). (a) Suppose two identical 52-card decks[1] are mixed together. Write a simple expression for the number of different arrangements of the 104 cards that could possibly result from such a mixing.

Solution. In the mixed deck, there are precisely two copies of each of 52 distinct cards. By the Bookkeeper Rule and the definition of multinomial coefficients, the number of possible arrangements of cards in the mixed deck is therefore just

$$\frac{104!}{(2!)^{52}}.$$

■

(b) Using only integers from the interval $[1, n]$, how many different **strictly increasing** length-m sequences can be formed?

Solution.

$$\binom{n}{m}$$

Justification: Given any m-element subset of $\{1, 2, ..., n\}$, listing its elements in increasing order yields a sequence that is strictly increasing and has length m. By collecting in a set the terms of any strictly increasing length-m sequence whose terms have been drawn from $\{1, 2, ..., n\}$, an m-element subset of $\{1, 2, ..., n\}$ is formed. Thus there is a bijection between the set of all strictly increasing length-m sequences with terms drawn from $\{1, 2, ..., n\}$ and the set of all size-m subsets of $\{1, 2, ..., n\}$.

■

Problem 2 (7 points).
For each pair of functions, $f : \mathbb{N}^+ \to \mathbb{N}$ and $g : \mathbb{N}^+ \to \mathbb{N}$, in the table below, indicate which of the listed asymptotic relations hold **and** which do not.

Fill **every** cell in the table. You may use checkmarks and crosses, "T" and "F", "TRUE" and "FALSE", "Y" and "N", or "YES" and "NO".

$f(n)$	$g(n)$	$f = O(g)$	$f = o(g)$	$g = O(f)$	$g = o(f)$
$\log_4 n$	$\sqrt[3]{n}$				
$n^2 + 3^n$	$n^3 + 2^n$				
$n \ln n!$	$n^2 \log_{10} n^2$				
$n^{2\cos(\pi n/2)+3}$	$5n^5 + 3n^3 + n$				

[1]Standard decks of playing cards, without jokers.

Solution.

$f(n)$	$g(n)$	$f = O(g)$	$f = o(g)$	$g = O(f)$	$g = o(f)$
$\log_4 n$	$\sqrt[3]{n}$	*YES*	*YES*	*NO*	*NO*
$n^2 + 3^n$	$n^3 + 2^n$	*NO*	*NO*	*YES*	*YES*
$n \ln n!$	$n^2 \log_{10} n^2$	*YES*	*NO*	*YES*	*NO*
$n^{2\cos(\pi n/2)+3}$	$5n^5 + 3n^3 + n$	*YES*	*NO*	*NO*	*NO*

Justification:

$f(n)$	$g(n)$	$f = O(g)$	$f = o(g)$	$g = O(f)$	$g = o(f)$
$\log_4 n$	$\sqrt[3]{n}$	*YES*	*YES*	*NO*	*NO*

Using either (1) l'Hôpital's Rule or (2) the fact that $\log n = o(n^\epsilon)$ for all $\epsilon > 0$ (see the Notes), conclude that $f = o(g)$. This implies that $f = O(g)$, $g \neq o(f)$, and $g \neq O(f)$.

$f(n)$	$g(n)$	$f = O(g)$	$f = o(g)$	$g = O(f)$	$g = o(f)$
$n^2 + 3^n$	$n^3 + 2^n$	*NO*	*NO*	*YES*	*YES*

Intuitively, 3^n grows far faster than n^2 and 2^n grows far faster than n^3, as n grows large. (Any power of n is asymptotically smaller than any increasing exponential in n.) Also, 3^n grows far faster than 2^n. (Given two increasing exponentials, the one with the smaller base will be asymptotically smaller.) A bit more rigorously,

$$\begin{aligned}\lim_{n\to\infty} \frac{g(n)}{f(n)} &= \lim_{n\to\infty} \frac{n^3 + 2^n}{n^2 + 3^n} \\ &= \lim_{n\to\infty} \frac{\frac{n^3}{3^n} + \left(\frac{2}{3}\right)^n}{\frac{n^2}{3^n} + 1} \\ &= \frac{\lim_{n\to\infty} \frac{n^3}{3^n} + \lim_{n\to\infty} \left(\frac{2}{3}\right)^n}{\lim_{n\to\infty} \frac{n^2}{3^n} + \lim_{n\to\infty} 1} \\ &= \frac{0+0}{0+1} \\ &= 0\end{aligned}$$

Where $\lim_{n\to\infty} \frac{n^3}{3^n}$ and $\lim_{n\to\infty} \frac{n^2}{3^n}$ can be found to be zero by l'Hôpital's Rule, and $\lim_{n\to\infty} \left(\frac{2}{3}\right)^n$ is zero because $\left|\frac{2}{3}\right| < 1$. Thus $g = o(f)$, which implies $g = O(f)$, $f \neq o(g)$, and $f \neq O(g)$.

$f(n)$	$g(n)$	$f = O(g)$	$f = o(g)$	$g = O(f)$	$g = o(f)$
$n \ln n!$	$n^2 \log_{10} n^2$	*YES*	*NO*	*YES*	*NO*

Using Stirling's formula, $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$, it is easy to show that $\ln n! \sim n \ln n$ and hence that $f(n) \sim n^2 \ln n$. Now,

$$\begin{aligned}n^2 \log_{10} n^2 &= 2n^2 \log_{10} n \\ &= 2n^2 \frac{\ln n}{\ln 10}\end{aligned}$$

It should be evident now that $g(n) \sim \frac{2}{\ln 10} f(n)$. Hence $f \neq o(g)$ and $g \neq o(f)$, but $f = O(g)$ and $g = O(f)$.

$f(n)$	$g(n)$	$f = O(g)$	$f = o(g)$	$g = O(f)$	$g = o(f)$
$n^{2\cos(\pi n/2)+3}$	$5n^5 + 3n^3 + n$	*YES*	*NO*	*NO*	*NO*

Notice that

$$f(n) = \begin{cases} n^5 & \text{if } n \equiv 0 \bmod 4 \\ n^3 & \text{if } n \equiv 1 \bmod 4 \text{ or } n \equiv 3 \bmod 4 \\ n & \text{if } n \equiv 2 \bmod 4. \end{cases}$$

Because $f(n)$ is thus clearly bounded above by n^5 and $g(n)$ is a polynomial of degree 5, have $f = O(g)$. The behavior of $f(n)$ when n is not a multiple of 4 leads to $g \neq O(f)$. It is obvious that $\lim_{n\to\infty} \frac{f(n)}{g(n)}$ and $\lim_{n\to\infty} \frac{g(n)}{f(n)}$ are both nonzero, so $f \neq o(g)$ and $g \neq o(f)$. ■

Problem 3 (3 points).
Give an example of a pair of strictly increasing total functions, $f : \mathbb{N}^+ \to \mathbb{N}^+$ and $g : \mathbb{N}^+ \to \mathbb{N}^+$, that satisfy $f \sim g$ but **not** $3^f = O(3^g)$.

Solution. The pair

$$f(n) = n^2 + n$$
$$g(n) = n^2$$

satisfies these criteria. Since n^2 is the term that dominates the behavior of $n^2 + n$ as n grows large, it is obvious that $n^2 + n \sim n^2$. (Applying the limit definition of asymptotic equality readily establishes this result.) Clearly, $3^{f(n)} = 3^{n^2+n} = 3^n 3^{n^2}$, while $3^{g(n)} = 3^{n^2}$. Thus $3^{f(n)} = 3^n 3^{g(n)}$. From this, it is obvious that $3^f \neq O(3^g)$. (It is very easy to check that, in fact, $3^g = o(3^f)$.) ■

Problem 4 (5 points).
A spacecraft is traveling through otherwise-empty three-dimensional space. It can move along only one dimension at a time, stepping precisely one unit in the positive direction along that dimension with each movement. For any two points, P and Q, in space, let $p_{P,Q}$ denote the number of distinct paths the spacecraft can follow to go from P to Q.

(a) Let P and Q have coordinates (x_P, y_P, z_P) and (x_Q, y_Q, z_Q), respectively. Assuming that $p_{P,Q}$ is positive, express $p_{P,Q}$ **as a single multinomial coefficient.**

Solution. Because each of the spacecraft's permissible atomic movements involves incrementing precisely one of its three position coordinates, $p_{P,Q} > 0$ implies that $x_Q - x_P$, $y_Q - y_P$, and $z_Q - z_P$ are all nonnegative integers. (The converse is also true.) To go from P to Q, the spacecraft must increment its first position coordinate $x_Q - x_P$ times, its second $y_Q - y_P$ times, and its third $z_Q - z_P$ times. So it must undergo precisely $(x_Q - x_P) + (y_Q - y_P) + (z_Q - z_P)$ atomic movements, $x_Q - x_P$ of them along the first dimension, $y_Q - y_P$ of them along the second, and $z_Q - z_P$ of them along the third.

So, number the spacecraft's atomic movements: $1, 2, \ldots, (x_Q - x_P) + (y_Q - y_P) + (z_Q - z_P)$. Partition the set $T = \{1, 2, \ldots, (x_Q - x_P) + (y_Q - y_P) + (z_Q - z_P)\}$ into three sets, T_x, T_y, and T_z, such that $|T_x| = x_Q - x_P$, $|T_y| = y_Q - y_P$, and $|T_z| = z_Q - z_P$. T_x then specifies which atomic movements are along the first dimension, T_y does the same for the second dimension, and T_z for the third. Each distinct partition corresponds to a single permissible path from P to Q, and each permissible path from P to Q corresponds to a single partition. So the number of permissible paths from P to Q is just the number of distinct partitions – that is, the number of $(x_Q - x_P, y_Q - y_P, z_Q - z_P)$-splits of the $\left((x_Q - x_P) + (y_Q - y_P) + (z_Q - z_P)\right)$-element set T. And of course this number is just:

$$p_{P,Q} = \binom{(x_Q - x_P) + (y_Q - y_P) + (z_Q - z_P)}{x_Q - x_P, y_Q - y_P, z_Q - z_P}$$

Alternatively, consider a bijection between the set of possible paths from P to Q and the set of sequences of length $(x_Q - x_P) + (y_Q - y_P) + (z_Q - z_P)$ that contain $(x_Q - x_P)$ 1s, $(y_Q - y_P)$ 2s, and $(z_Q - z_P)$ 3s. The kth term of each sequence specifies the dimension associated with the kth atomic movement in the corresponding path. The Bookkeeper Rule then leads directly to the expression for $p_{P,Q}$. ■

(b) Suppose there exist five points in space, A, B, C, D, and E, such that it is possible for the spacecraft to travel from A to B, from B to C, from C to D, and from D to E. Write an expression for the number of distinct paths the spacecraft can follow to go from A to E while **avoiding** B, C, and D. Your expression **must** be written entirely in terms of symbols of the form $p_{P,Q}$, where $P, Q \in \{A, B, C, D, E\}$.

Hint: Inclusion-Exclusion.

Solution. First, note that since it is possible for the spacecraft to travel from A to B, from B to C, from C to D, and from D to E, therefore paths exist from A to each of A, B, C, D, and E, from B to each of C, D, and E, ..., and from E to E. Thus, because of the way in which the spacecraft must move, positive-length paths cannot exist from E to A, B, C, D, or E, from D to A, B, C, or D, ..., or from A to A. (This is why, in what follows, terms like $p_{B,D}$ appear, but terms like $p_{D,B}$ do not. If B and D are distinct, $p_{B,D}$ is positive and $p_{D,B}$ is zero, so including $p_{D,B}$ would affect nothing. If B and D are coincident, both $p_{B,D}$ and $p_{D,B}$ are equal to one, but considering both would amount to counting every path through B twice.) In a very loose sense, and if cases involving coincident points are ignored, this essentially means that the spacecraft only moves "forward" and that B is "ahead" of A, C is "ahead" of B, D is "ahead" of C, and E is "ahead" of D.

Let S denote the set of all paths from A to E. Clearly, $|S| = p_{A,E}$.

Let S_X denote the set of all paths that go from A to E, through X, where $X \in \{B, C, D\}$. Evidently, $|S_X| = p_{A,X}p_{X,E}$.

Now, $S_X \cap S_Y$ is the set of paths that go from A to E, through both X and Y, where $X, Y \in \{B, C, D\}$. Obviously, $|S_B \cap S_C| = p_{A,B}p_{B,C}p_{C,E}$, $|S_B \cap S_D| = p_{A,B}p_{B,D}p_{D,E}$, and $|S_C \cap S_D| = p_{A,C}p_{C,D}p_{D,E}$.

Also, $S_B \cap S_C \cap S_D$ is the set of all paths that go from A to E, through all three of B, C, and D. Obviously, $|S_B \cap S_C \cap S_D| = p_{A,B}p_{B,C}p_{C,D}p_{D,E}$.

Now, the set of paths that go from A to E and pass through at least one of B, C, and D, is just $S_B \cup S_C \cup S_D$. By inclusion-exclusion,

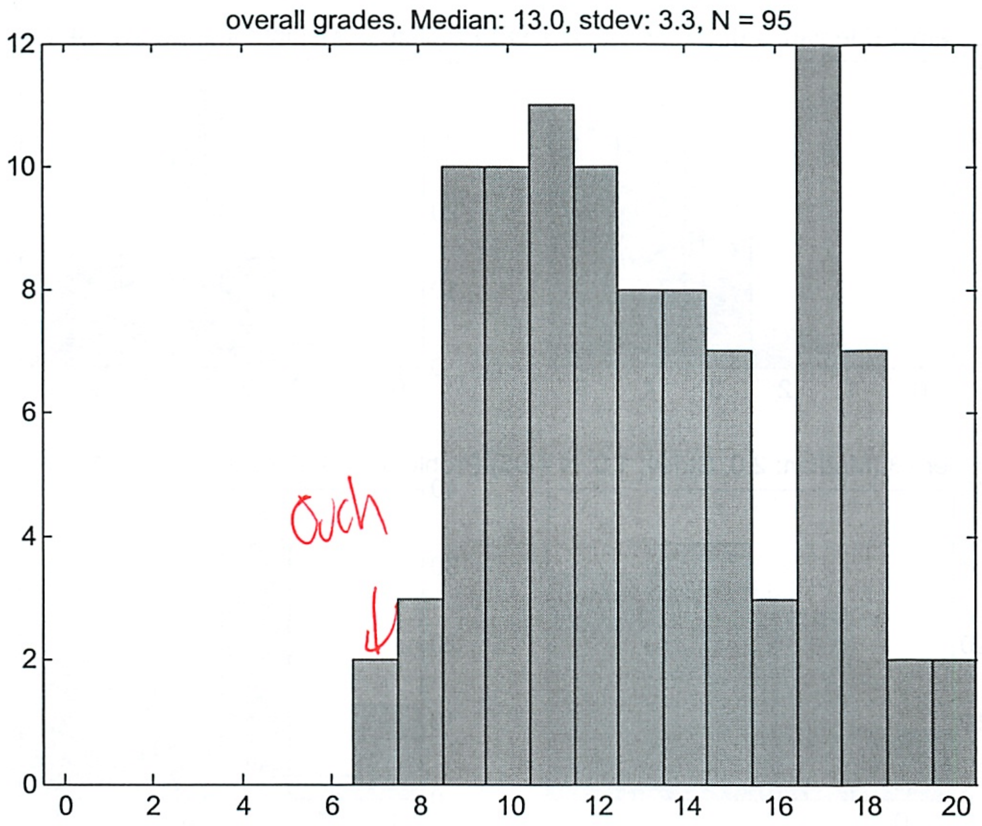
$$\begin{aligned}|S_B \cup S_C \cup S_D| =& |S_B| + |S_C| + |S_D| - |S_B \cap S_C| - |S_B \cap S_D| - |S_C \cap S_D| + |S_B \cap S_C \cap S_D| \\ =& p_{A,B}p_{B,E} + p_{A,C}p_{C,E} + p_{A,D}p_{D,E} \\ & - p_{A,B}p_{B,C}p_{C,E} - p_{A,B}p_{B,D}p_{D,E} - p_{A,C}p_{C,D}p_{D,E} + p_{A,B}p_{B,C}p_{C,D}p_{D,E}\end{aligned}$$

Let R denote the set of all paths from A to E that go through neither B, nor C, nor D. Evidently, $S = R \cup (S_B \cup S_C \cup S_D)$ and $R \cap (S_B \cup S_C \cup S_D) = \emptyset$. Therefore $|S| = |R| + |S_B \cup S_C \cup S_D|$, so the number of distinct paths the spacecraft can follow to go from A to E while avoiding B, C, and D is

$$\begin{aligned}|R| =& |S| - |S_B \cup S_C \cup S_D| \\ =& p_{A,E} - p_{A,B}p_{B,E} - p_{A,C}p_{C,E} - p_{A,D}p_{D,E} \\ & + p_{A,B}p_{B,C}p_{C,E} + p_{A,B}p_{B,D}p_{D,E} + p_{A,C}p_{C,D}p_{D,E} - p_{A,B}p_{B,C}p_{C,D}p_{D,E}\end{aligned}$$

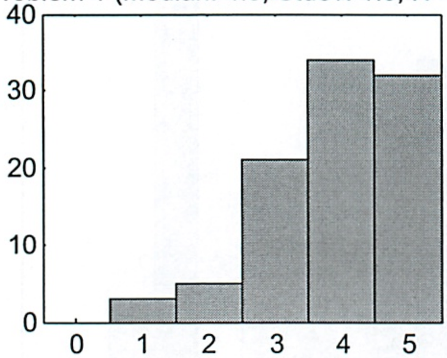
■

MQ 5

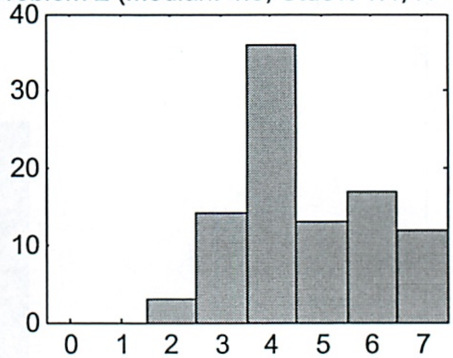

overall grades. Median: 13.0, stdev: 3.3, N = 95
12
10
8
6
4
2
0
0
2
4
6
8
10
12
14
16
18
20
Ouch

MQ5

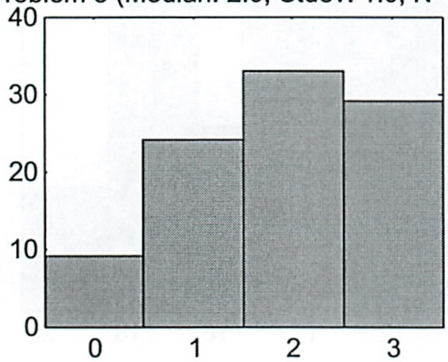
Problem 1 (Median: 4.0, Stdev: 1.0, N = 95)



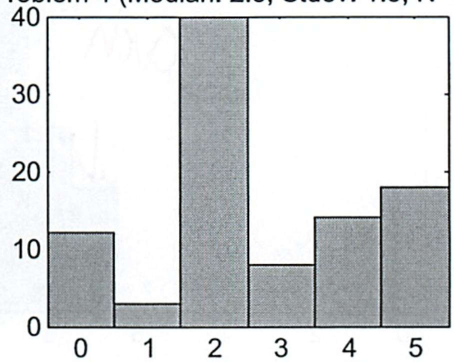
Problem 2 (Median: 4.0, Stdev: 1.4, N = 95)



Problem 3 (Median: 2.0, Stdev: 1.0, N = 95)



Problem 4 (Median: 2.5, Stdev: 1.5, N = 95)



6.042 Grade Report for *Plasmeier, Michael*

Problem Sets

id ▲	adjusted score	raw score	max	statistics
PS.01	**35.15**	28.00	50.00	link
PS.02	**35.98**	33.00	50.00	link
PS.03	**22.00**	18.50	40.00	link
PS.04	**26.02**	24.00	30.00	link
PS.05	**34.83**	32.20	40.00	link
PS.06	**36.82**	33.00	50.00	link
PS.07	**33.72**	29.00	50.00	link

Note: The psets' adjusted scores reflect the psets scores after being adjusted by its corresponding MQ's score. The adjusted scores will be further increased according to final exam's performance.

Mini Quizzes

id ▲	pts	max	statistics
MQ.01	**13.00**	20.00	link
MQ.02	**7.00**	20.00	link
MQ.03	**13.50**	20.00	link
MQ.04	**9.00**	20.00	link
MQ.05	**7.00**	20.00	link

Reading Assignments

No grades available yet.

Tutor Problems

id ▲	pts	max
T.01	**1.00**	1.00
T.02	**1.00**	1.00
T.03	**1.00**	1.00
T.04	**1.00**	1.00
T.05	**1.00**	1.00
T.06	**1.00**	1.00
T.07	**1.00**	1.00
T.08	**1.00**	1.00
T.09	**1.00**	1.00

Class Participation

id ▲	pts	max	pending makeup
CP.01	**2.00**	2.00	
CP.02	**2.00**	2.00	
CP.03	**2.00**	2.00	
CP.04	**2.00**	2.00	
CP.05	**2.00**	2.00	
CP.06	**2.00**	2.00	
CP.07	**2.00**	2.00	
CP.08	**2.00**	2.00	
CP.09	**2.00**	2.00	
CP.10	**1.00**	2.00	
CP.11	**2.00**	2.00	
CP.12	**2.00**	2.00	
CP.13	**1.00**	2.00	
CP.14	**1.00**	2.00	
CP.15	**2.00**	2.00	
CP.16	**1.00**	2.00	
CP.17	**2.00**	2.00	
CP.18	**2.00**	2.00	
CP.19	**2.00**	2.00	
CP.20	**2.00**	2.00	
CP.21	**2.00**	2.00	
CP.22	**1.00**	2.00	
CP.23	**1.00**	2.00	
CP.24	**2.00**	2.00	
CP.25	**2.00**	2.00	
CP.26	**2.00**	2.00	
CP.27	**0.00**	2.00	
CP.28	**1.00**	2.00	

4/21

Final Exam

No grades available yet.

Totals

id ▲	pts	max	weight	mean	median	stddev
Problem Set	228.59	300.00	0.25	255.65	269.58	41.00
Final Exam	0.00	0.00	0.30	0.00	0.00	0.00
Class participation	36.00	38.00	0.20	36.82	38.00	3.47
Miniquiz	42.50	80.00	0.17	57.65	57.50	12.37
Reading Comments	0.00	0.00	0.03	0.00	0.00	0.00
Tutorial	9.00	9.00	0.05	8.23	9.00	1.55
Grand Total	**52.03**	**67.00**	**1.00**	**57.51**	**59.04**	**6.94**

Note: The totals only reflect grades that have been completely entered for the class. A grade with gray background signifies that the grade has not been completely entered yet.

Note: A grade with red font signifies that the grade has been dropped.

Grade Quartile

Your current rank is: ***4th quartile (79th - 101th)*** out of 101 students.

Grades compiled at: 4/21/11 8:28 AM

Please contact your TA if there is any problem with the grade report.

Mini-Quiz May 5 #6

Your name: Michael Plasmeier

Circle the name of your TA and write your table number:

Ali Nick Oscar (Oshani) **Table number** 12

- This quiz is **closed book**. Total time is 30 minutes.
- Write your solutions in the space provided. If you need more space, write on the back of the sheet containing the problem. Please keep your entire answer to a problem on that problem's page.
- GOOD LUCK!

DO NOT WRITE BELOW THIS LINE

Problem	Points	Grade	Grader
1	6	1	OS
2	6	4	ON
3	4	1	NJ
4	4	4	AK
Total	20	10	

Avg = 12.5

1	2
3	4

Problem 1 (6 points).
Suppose there are 4 desks in a classroom, laid out in the corners of a square with corners 1 2 3 and 4.

Each desk is occupied by a male with probability $p > 0$ or a female with probability $q ::= 1 - p > 0$. A male and a female *flirt* when they occupy desks in adjacent corners of the square. Let $I_{12}, I_{23}, I_{34}, I_{41}$ be the indicator variables that there is a flirting couple at the indicated adjacent desks.

(a) Show that if $p = q$ then the events $I_{12} = 1$ and $I_{23} = 1$ are independent.

Each desk has a M, F determined independly w/ P() = p

0/2

So $P(I_{12}=1) = \frac{2pq}{p^2+2pq+q^2}$ $P(I_{12}=1 \text{ AND } I_{23}=1) = P(I_{12}=1) \cdot P(I_{23}=1)$

$\frac{2p_1q_1}{p^2+2pq+q^2} \cdot \frac{2p_2q_2}{p^2+2pq+q^2} = \frac{2pq}{p^2+2pq+q^2} + \frac{2pq}{p^2+2pq+q^2}$

Same for other one I_{23}

Over

(b) Show rigorously that if the events $I_{12} = 1$ and $I_{23} = 1$ are independent then $p = q$. Hint: work from the definition of independence, set up an equation and solve.

$\frac{2pq}{p^2+2pq+q^2}$ Otherwise, would be bias - and would not be ind

If you have B G Then the third one can be B or G w/o bias of before, But if one gender is more biased, then the two I_{12}, I_{23} are not as likely to be evenly split - will be biased

0/2

Two events ind iff indicator variables ind

(c) What is the expected number of flirting couples in terms of <u>p and q</u>?

$E = \frac{1}{n} = 2$

1/2

more rigorous

$$\frac{p_1 q_2 + p_2 q_1}{p_1 p_2 + p_1 q_2 + p_2 q_1 + q_1 q_2}$$

$p_1 = p_2$ as defined in problem

$q_1 = q_2$

So can simplify

One does not effect the other

Each child determined ind, means each couple picked independetly

Problem 2 (6 points).

Consider the following 2 player game. A coin is tossed repeatedly. Turns alternate between the two players. The game stops after the first Heads come up. If the first time the coin came up Heads is during one of player 1's turns, player 1 wins. On the other hand, if the first time the coin came up Heads is during one of player 2's turns then player 2 wins.

Assume p = prob heads $= \frac{1}{2}$

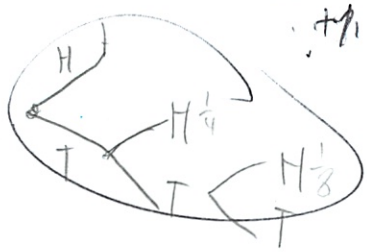
(a) What is the expected number of turns N until the game ends?

$S = p \cdot 1 + q(1+s)$

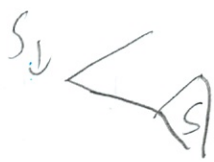
Mean time to failure $= \frac{1}{p} = \frac{1}{1/2} = 2$

(b) What is the probability p_1 that player 1 wins? (Hint: draw an event tree)

1 2 1



or



1 $p^1 + p^3 + p^5 + \ldots$

2 $p^2 + p^4 + p^6$

1 $\sum_{n = \text{odd}\#} p^n$

2 $\sum_{n \text{ even}\#} p^n$

→ over

(c) What is Ex[N|1], the expected number N of rounds in the game given player 1 wins? You can assume that the game ends with probability 1 and that Ex[N|2] = Ex[N|1] + 1. Hint: Law of total Expectation.

$E[N] = E[N|1] \cdot P(1) + E(N|2) P(2)$

$\frac{1}{p} = E[N|1] \sum_{n \in \text{odd}} p^n + (E[N|1]+1) \sum_{n \in \text{even}\#} p^n$

+1

$E[N|1] = \dfrac{(E[N|1]+1) \sum_{n = \text{even}\#} p^n}{\sum_{n = \text{odd}\#} p^n}$

$E(N|1) = \dfrac{(E(N|1)+1)\, p_1}{p_2}$

Over

1) $\frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \dots$

2) $\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \dots$

forgot sums formula - not this unit

$$\frac{1}{p} = p_2 E[N|1] + p_1 E[N|1] + p_1$$

$$1/p = E[N|1](p_1 + p_2) + p_1$$

$$E[N|1] = \frac{\frac{1}{p} - p_1}{p_1 + p_2}$$

Problem 3 (4 points). (a) Write the term of $(x+y)^{40}$ which includes x^3.

$(x+y)^{40} = x^0 y^{40} + x^1 y^{39} + \dots + x^3 y^{37}$

X

you need the binom coefficients too though. study more

$\binom{40}{3} x^3 y^{37}$

(b) Write the term of $(x+y+z)^{40}$ which includes x^3y^5.

$x^3 y^5 z^{32}$

X

(c) Give a combinatorial proof that

$$\sum_{i=0}^{n} \binom{n}{i} = 2^n.$$

$\binom{3}{0} + \binom{3}{1} + \binom{3}{2} + \binom{3}{3}$

Hint: Begin by finding a set whose cardinality is equal to the right hand side of the equation.

RHS 2^n is the number of bit strings (either 1 or 0) of length n

good

LHS Is all of the possible combinations of 1s, i is the number of 1s. The binomial term is the # of possible combinations of bit strings with i 1s. Add up all possible 1s,

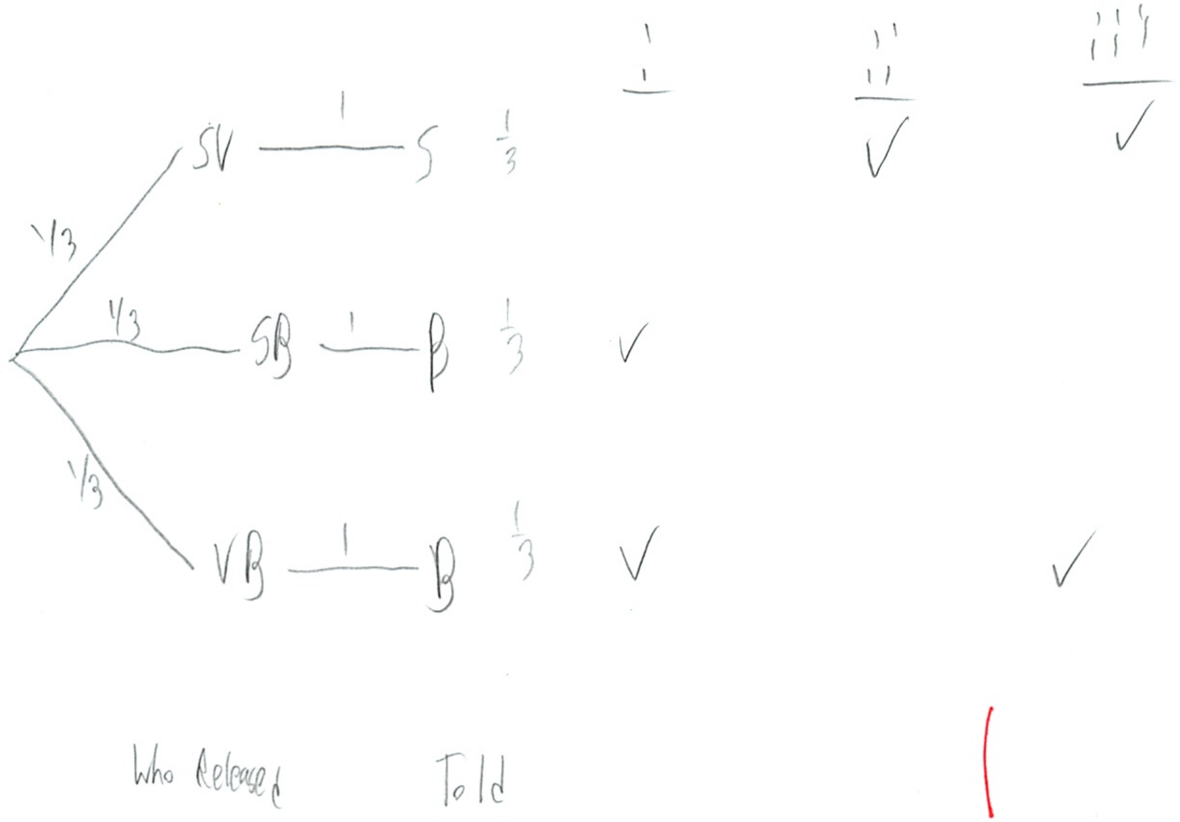
$\frac{1}{2}$.

Conc The LHS = RHS, so both count the same thing.

seems like you have right idea but you say things which don't have any clear meaning

Problem 4 (4 points).
We revisit Sauron, Voldemort, and Bunny Foo Foo as in the class problem. As before, the guard is going to release exactly two of the three prisoners; he's equally likely to release any set of two prisoners. The guard offers to tell Voldemort the name of one of the prisoners to be released. The guards rule for which name he chooses:
1. The guard will never say that Voldemort will be released.
2. If both Foo Foo and Sauron are getting released, the guard will always give Foo Foos name.
Were interested in which characters are released, and in which character the guard says will be released.

(a) Draw a tree to represent the sample space. Indicate, in your drawing, which outcomes correspond to the following events:
i. The guard tells Voldemort that Foo Foo will be released
ii. The guard tells Voldemort that Sauron will be released
iii. Voldemort is released



(b) What is the probability that Voldemort is released, given that the guard says Foo-foo will be released?

$$P(iii \mid i) = \frac{P(iii \cap i)}{P(i)} = \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{3} \cdot \frac{3}{2} = \frac{1}{2}$$

(c) What is the probability Voldemort is released, given that the guard says Sauron will be released?

$$P(\text{iii} \mid \text{ii}) = \frac{P(\text{iii} \cap \text{ii})}{P(\text{ii})} = \frac{\frac{1}{3}}{\frac{1}{3}} = 1$$

(d) Use the above calculations, and the Law of Total Probability, to find the total probability that Voldemort will be released.

$$P(\text{iii}) = P(\text{iii} \mid \text{i})\, P(\text{i}) + P(\text{iii} \mid \text{ii})\, P(\text{ii})$$

$$\frac{1}{2} \cdot \frac{2}{3} + 1 \cdot \frac{1}{3}$$

$$= \frac{2}{6} + \frac{1}{3}$$

$$= \frac{2}{3}$$

Solutions to Mini-Quiz May 4

Problem 1 (6 points).
Suppose there are 4 desks in a classroom, laid out in the corners of a square with corners 1 2 3 and 4.

Each desk is occupied by a male with probability $p > 0$ or a female with probability $q ::= 1 - p > 0$. A male and a female *flirt* when they occupy desks in adjacent corners of the square. Let $I_{12}, I_{23}, I_{34}, I_{41}$ be the indicator variables that there is a flirting couple at the indicated adjacent desks.

(a) Show that if $p = q$ then the events $I_{12} = 1$ and $I_{23} = 1$ are independent.

Solution. If $p = q = 1/2$ then $\Pr[I_{12} = 1] = \Pr[I_{23} = 1] = 1/2$ and $\Pr[I_{12} = 1 \& \; I_{23} = 1]$ can be calculated from the fact that only F-M-F and M-F-M are possible when both couples are flirting. In that case, we have $\Pr[I_{12} = 1 \& \; I_{23} = 1] = 2/8 = 1/4 = \Pr[I_{12} = 1] \cdot \Pr[I_{12} = 1]$. ■

(b) Show rigorously that if the events $I_{12} = 1$ and $I_{23} = 1$ are independent then $p = q$. Hint: work from the definition of independence, set up an equation and solve.

Solution. We can again compare $\Pr[I_{12} = 1 \& \; I_{23} = 1]$ and $\Pr[I_{12} = 1] \cdot \Pr[I_{23} = 1]$.

As in the previous part, $I_{12} = 1 \& \; I_{23} = 1$ only happen when we have a pattern of F-M-F or M-F-M for students 1 2 and 3 respectively. These occur with total probablity $p^2q + pq^2$. On the other hand, I_{12} happens with probability $2pq$ total, accouting for the two patterns possible, M-F and F-M. Hence, I_{12} and I_{23} are independent iff $p^2q + pq^2 = pq(p + q) = 4p^2q^2$. By manipulating the expression we get $p + q = 4pq$. Recall $p + q = 1$. Hence, we are dealing with $1 = 4p - 4p^2$. The equation can be factored into $(2p - 1)^2 = 0$, yielding $p = 1/2$. ■

(c) What is the expected number of flirting couples in terms of p and q?

Solution. The expected number of couples is $8pq$ by linearity of expectation. ■

Problem 2 (6 points).
Consider the following 2 player game. A coin is tossed repeatedly. Turns alternate between the two players. The game stops after the first Heads come up. If the first time the coin came up Heads is during one of player 1's turns, player 1 wins. On the other hand, if the first time the coin came up Heads is during one of player 2's turns then player 2 wins.

(a) What is the expected number of turns N until the game ends?

Solution. This is just mean time to failure (a Head), so by Lemma 17.4.8, the expected number of steps is $\text{Ex}[N] = 1/(1/2) = 2$. ■

(b) What is the probability p_1 that player 1 wins? *Hint:* draw an event tree.

Solution. The tree can be described by $A = H_1 + T_1(H_2 + T_2 A)$. The probability of winning can be found via the law of total probability.

$p_1 = (1/2) \cdot 1 + (1/2)(1/2 \cdot 0 + 1/2 \cdot p_1)$

Hence $(3/4) \cdot p_1 = 1/2$, so $p_1 = 2/3$

■

(c) What is $\text{Ex}[N \mid 1]$, the expected number N of rounds in the game given player 1 wins? You can assume that the game ends with probability 1 and that $\text{Ex}[N \mid 2] = \text{Ex}[N \mid 1] + 1$. *Hint:* Law of total Expectation.

Solution. From the law of total expectation, we know $\text{Ex}[N] = \text{Ex}[N \mid 1]p_1 + \text{Ex}[N \mid 2]p_2$. Now we know $p_1 = 2/3$, $p_2 = 1/3$ and $\text{Ex}[N] = 2$ and the hint.

We get $(2/3 + 1/3)\,\text{Ex}[N \mid 1] = 2 - 1/3$ so $\text{Ex}[N \mid 1] = 5/3$. ■

Problem 3 (4 points). (a) Write the term of $(x + y)^{40}$ which includes x^3.

Solution.

$$\binom{40}{3} x^3 y^{37}.$$

■

(b) Write the term of $(x + y + z)^{40}$ which includes $x^3 y^5$.

Solution.

$$\binom{40}{3,5,32} x^3 y^5 z^{32}$$

■

(c) Give a combinatorial proof that

$$\sum_{i=0}^{n} \binom{n}{i} = 2^n.$$

Hint: Begin by finding a set whose cardinality is equal to the right hand side of the equation.

Solution. Count the number of n-length bit strings. For the LHS, we consider the ith term of the sum to represent the bit strings which have i zeros. ■

Problem 4 (4 points).
We revisit Sauron, Voldemort, and Bunny Foo Foo as in the class problem. As before, the guard is going to release exactly two of the three prisoners, and he's equally likely to release any set of two prisoners.

The guard offers to tell Voldemort the name of one of the prisoners to be released. The guard's rule for which name he chooses:

1. The guard will never say that Voldemort will be released.
2. If both Foo Foo and Sauron are getting released, the guard will always give Foo Foo's name.

We're interested in which characters are released, and in which character the guard says will be released.

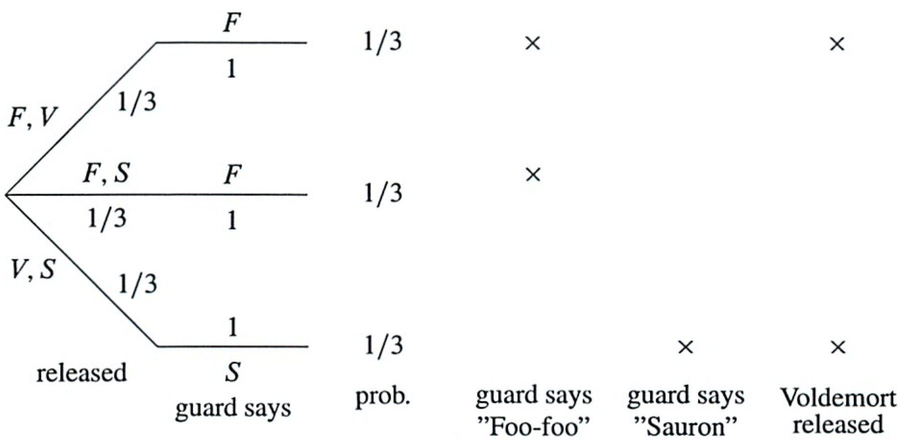
(a) Draw a tree to represent the sample space. Indicate, in your drawing, which outcomes correspond to the following events:

i. The guard tells Voldemort that Foo Foo will be released

ii. The guard tells Voldemort that Sauron will be released

iii. Voldemort is released

Solution.



■

(b) What is the probability that Voldemort is released, given that the guard says Foo-foo will be released?

Solution. $\frac{1}{2}$ ■

(c) What is the probability Voldemort is released, given that the guard says Sauron will be released?

Solution. 1 ■

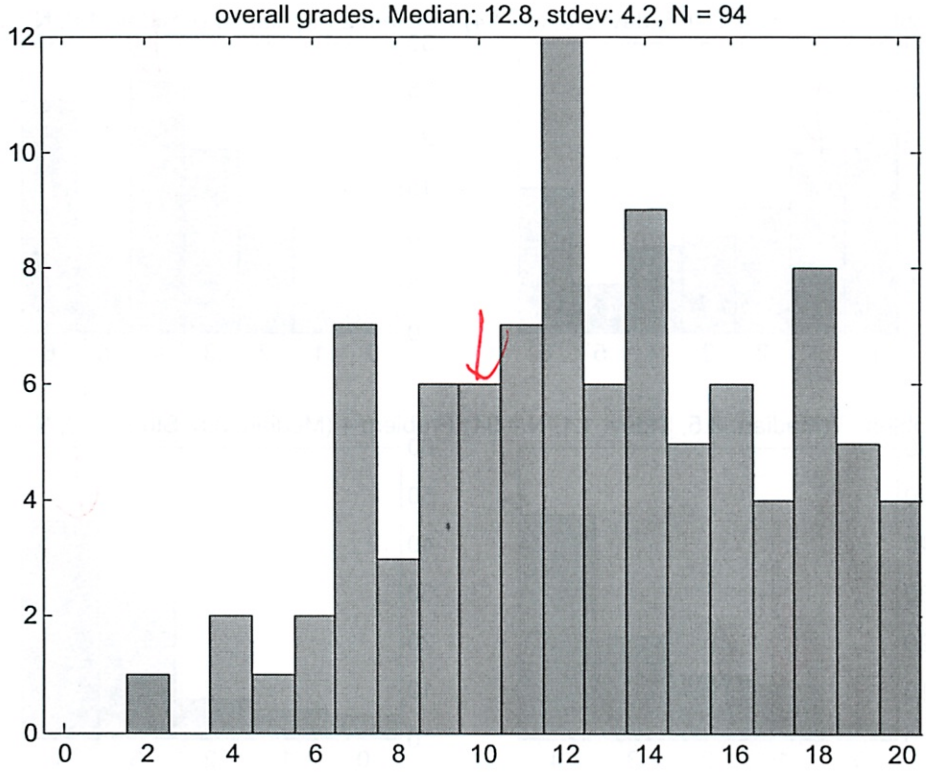
(d) Use the above calculations, and the Law of Total Probability, to find the total probability that Voldemort will be released.

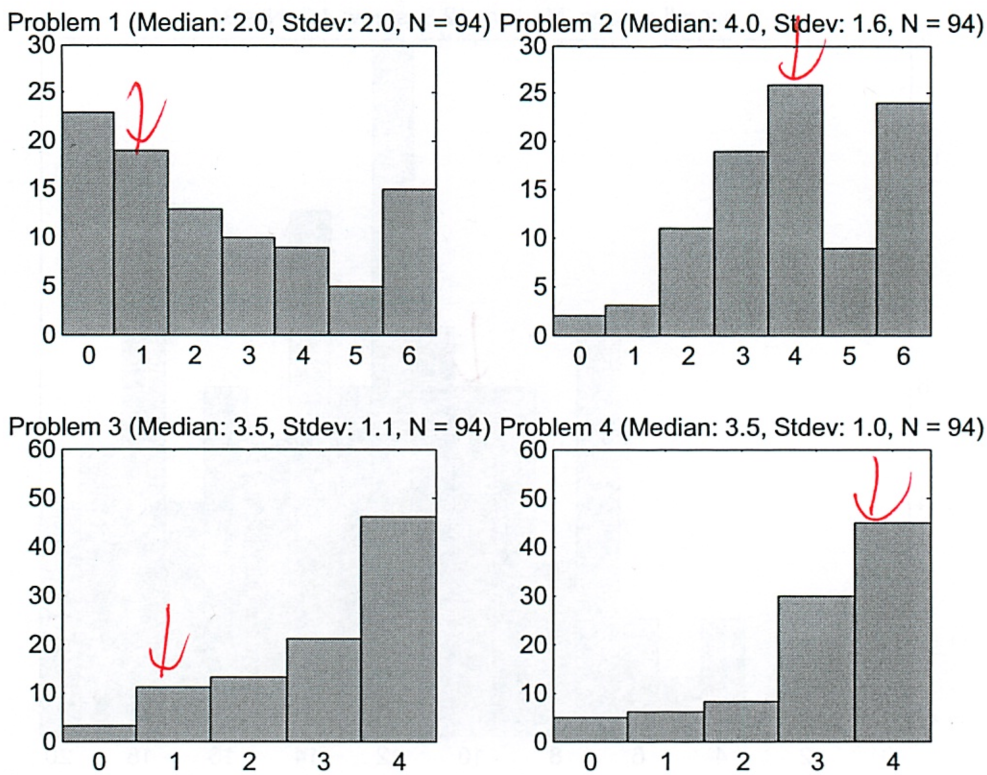
Solution. Still 2/3, by law of total probability.

$$\begin{aligned}\Pr[\texttt{V released}] &= \Pr\left[\texttt{V released} \mid \texttt{says foofoo}\right] \cdot \Pr[\texttt{says foofoo}] \\ &\quad + \Pr\left[\texttt{V released} \mid \texttt{says sauron}\right] \cdot \Pr[\texttt{says sauron}] \\ &= \end{aligned}$$

■

MQ 6




Problem 1 (Median: 2.0, Stdev: 2.0, N = 94)
Problem 2 (Median: 4.0, Stdev: 1.6, N = 94)
Problem 3 (Median: 3.5, Stdev: 1.1, N = 94)
Problem 4 (Median: 3.5, Stdev: 1.0, N = 94)
0 5 10 15 20 25 30
0 10 20 30 40 50 60
0 1 2 3 4 5 6
0 1 2 3 4

Old Mini Quizes

5/16

Quiz 1

#1 Prove $\log_9 12$ is irrational

- what is irrational?

~~when divided~~ # that can not be represented by # a fixed length

ie $\frac{1}{3}$

(can't be ~~pre~~ represented as fraction) think

So thats $\frac{\log 12}{\log 9}$

I am not good at logs

where to start...

Proof by contridiction. ← they said to do that

↙ this is all part of the template

Suppose $\log_9 12 = \frac{m}{n}$ for some int m and n

Can assume m, n (+)

$\log_9 12 = m/n$

$9^{\log_9 12} = 9^{m/n}$ ← see would not have known to do that

$12 = (9^m)^{1/n}$

$12^n = 9^m$ ↙ would not have the algebra knowledge to do this

But this impossible - one side even + odd

(2)

2. Show exactly 2 truth assignments
- did table

- don't think I could do proof

Proof by cases (I didn't even look at the problem!)

If P is false, then in order to have any chance clause 4

S must be false

if S is false then Q must be F for clause 3

~~[illegible]~~ And Q

or the other cases

Show what you have satisfies formula

3. Flawed WOP proof
↑ I hate the WOP!

5 is divisable by n

n | (5n) and Not 5|N

- but that does not make sense

Assume C not empty so 5(m - 10) or 5(m-10) ↑ remove one

So must be 0 at end"

(3)

Suppose S(m-10) holds

$5 \mid (m-10)$ must hold since $5 \mid m$ held ?

? but this is going wrong dict

And need to show that set of examples is empty

If not$(5 \mid (m-10))$ then m-10 would be

a counter example smaller than m - Contradiction

that m is smallest.

? oh missed that part

Oh and $S(n) \rightarrow 5 \mid n$

n is divisable by 5

$n \mid S(n)$ S(n) is divisable by n

n is divisable by 5 is divisable by n ???

I just don't get this

TA focus on some stuff

But if $5 \mid (m-10)$ then $5 \mid m$ ...

... $5 \mid (m-10)$ and $5 \mid 10$ so $5 \mid (m-10+10)$

Contridicting that m is counterexample

Didn't check m > 0 if m=0 - does not hold!

the weird exception would not find in real life

(4)

4. Following predicate invalid

$$\forall x \exists y\ P(x,y) \to \exists y\ \forall x\ P(x,y)$$

~~W~~ What are counter models?

? where it works?

1. $P(x,y)$ $yx=1$ on $\mathbb{Q}$

for all x, there is a y so ~~x~~ $yx=1$

- true w/ $\mathbb{Q}$

does that imply ← real qu

~~f~~or a y, for all x this is true

- remember true if if false or then true

if part true then part false X

No 0 has no inverse so hyp is false - so true

? again how suppose to think that

~~R~~ Try all inputs -1 $\frac{1}{2}$ 1 0 $\sqrt{2}$ etc

(5)

b) ↓ real # test

$y < x$ for $\mathbb{R}$ -1 0 1, $\sqrt{2}$, π

for all x there is a y that is bigger

✓ true for -1 0 1

since Reals ∞

does there exist a y so that all x are bigger

no

– no "rock bottom #"

False ✓ well true that is a counter model

c) $yx = 2$ for $\mathbb{R}$ w/o 0

for all x is there a y

– yeah since not 0

π: $\frac{2}{\pi}$

$\sqrt{2}$ would be: $\frac{2}{\sqrt{2}}$ will say T

for a y there ~~[illegible]~~ all x work

– no?

Counterexample ✓

(I'm starting to get the point of this)

(6)

d) $yxy = x$ all binary strings including empty
↑ concat

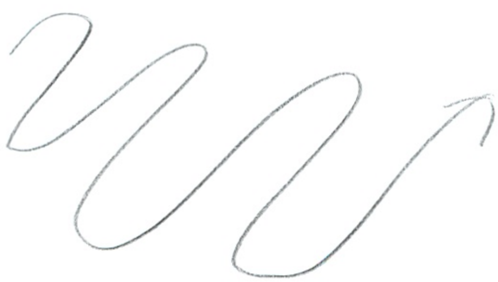
$\forall x$ there is a y where this is true
Yeah the X empty string

$\exists y \; \forall x$ yeah
True, true
Not a counterexample ✓

(Oh I got these right before - darn - thought I learned something)

Quiz #2 1. Chain of iffs
(I used to like tese problems - not anymore)
Skipping

2. A, B are countably ∞ sets
Show $A \times B$ also countable



This is that thing

(7)

3. nth Fib # F_n

$$F_n \begin{cases} 0 & n=0 \\ 1 & n=1 \\ F_{n-1} + F_{n-2} & n \geq 2 \end{cases}$$

$$F_0^2 + F_1^2 + \cdots + F_n^2 = F_n F_{n+1}$$

Prove by ind

<u>Base case n=0</u>

be clear about what induction is over
here its # of terms

$$0^2 = 0 \cdot 1$$

<u>n=1</u> (I like to do a few more - not required)
to test

$$0^2 + 1^2 = 1 \cdot 1$$

<u>n=2</u>

$$0^2 + 1^2 + 1^2 = 1 \cdot 2$$

<u>Ok Ind</u> Assume $P(n)$, Show $P(n+1)$

$$F_0^2 + F_1^2 + \cdots + F_n^2 + F_{n+1}^2 = F_{n+1} F_{n+2}$$

Subtract $F_n F_{n+1}$

$$F_{n+1}^2 = F_{n+1} F_{n+2} - F_{n+1} F_n$$

This does not seem right

Fib
0
1
1
2
3
5
8
12
20
32
52
84

(8)

$F_0^2 + F_1^2 + \ldots + F_n^2 + F_{n+1}^2 = F_n F_{n+1} + F_{n+1}$

subtract $F_n F_{n+1}$ — ? just add to both sides

$F_{n+1}^2 = F_{n+1}$ (No)

Well added ~~we~~ diff things to both sides

~~$F_{n+1}^2 +$~~ or what suppose to do

$F_0^2 + \ldots F_n^2 + F_{n+1}^2 = F_{n+1} F_{n+2}$

subtract $F_n F_{n+1}$

$F_{n+1}^2 = F_{n+1} F_{n+2} - F_n F_{n+1}$

Thats what I did first?

Oh divide both sides by F_{n+1}

$F_{n+1} = F_{n+2} - F_n$

$F_n = F_{n+2} - F_{n+1}$

ohhh - close

Oh + not - inside

equiv of

$F_{n+2} = F_{n+1} + F_n$

can I do that

$F_{n+2} = F_n + F_{n+1}$

(9) I think I just figured out my first actual induction proof

- well very messy
- straighten out

Hopefully they will give people much more time to do the exam.

Their proof was more elegant - Hopefully mine would be acceptable

---

4. Recursive brackets
   - I'm bad at this too

Base $\lambda \in M$

If $s, t \in M$

$[s] \in M$

$s \cdot t \in M$

Rect match $[s]t$

a) Show RectMatch can be built from M
   I don't think I got this before

Well if $s, t \in M$
Then we can do $[s]$
And since that is in M, define it as s and do $s \cdot t$
What else?

$P(x) ::= x \in M$ oh they just want the ind hyp

(10)

b) Base cases

$\lambda \in$ Rectmatch is $\subseteq$ M since M base $\lambda \in$

$\{\lambda\} \subseteq \{\lambda\}$ ✓

I think that works

c) inductive

Did that above

~~Since s~~ Assume s,t $\in$ Rectmatch

Then take s and [s] which $\in$ M

and define this as s and $s \cdot t \in M$

so rectmatch $\subseteq$ M ✓

d) M = Rect Match - but RecMatch unamb - M is amb

?? what is 'ambigeous'

When a recursive def allows a el to be constructed in more than one way

Could do

[s] + [s]:

by [s]

[s] +

[s] + [s]

or + [s] [s]

[s] + [s]

(1!)

Not sure about that

Their ans: Consider derivation of the empty string
- from base case λ or $\lambda\lambda$

I would not have thought of

e) Why is unamb clearer

if amb fns can be defined ∞ recursively
or not be well-defined ← threw away ans!

Quiz #3 1. Calc $\phi(100)$

(I used to love these - actual do - but now I am in a proof mode)

$\phi(100)$ ← product of rel primes
↑ for a,b if $\gcd(a,b)=1$
so rel primes of 100

2, 3, 4, 7, 11, 13, 17,

$2 \cdot 50$
↑
$2 \cdot 25$

$\phi(2^2 \cdot 5^2)$

was my intuition on rel prime wrong? ← also coprime ← I like better

GCD largest possible # that divides both # w/o remainder

← must be for primes?

(12)

Rel prime - integers w/ no prime factors in common

- having no ~~ex~~ common ~~den~~ divisor > 1

$gcd(a,b)=1$

8, 15 rel prime since $gcd(8,15)=1$

for prime - every int > 1 < p is rel prime

- So things are invertable + cancelable

WA Coprime(100)

1 3 7 9 11 13 17 19 21 2~~3~~3 27

29 etc

↑ But 2, 5 not on list!

Or are 2, 5 rel prime?

Oh for 100 used my example

Oh ϕ is # rel prime #

- use example

* Only pair must be rel prime

(13)

b) Assume $k > 9$ is rel prime to 100
Explain why last 2 digits of k k^{121} are same

No clue - What is the rule here
(This chap is all about rules)
- That I thought I knew
- But obviously don't

Hint is to use above

So # rel prime to 100 ~~to~~ 9
are ~~1, 3 7 9~~ oh > 9
So not 1, 3, 7, 9 - 36 choices

Its some rule

~~Since~~ Just need to prove congruent mod 100
for last 2 digits Ohhh

So $a \equiv b \pmod n$ iff $n \mid (a-b)$
$100 \mid (k^{121} - k)$ ← k^{120}? $x^3 - x$
if $x=5$ $125-5$ ⊗ No

$k^{121} \equiv k^{40 \cdot 3 + 1} = k(k^{40})^3$ mod 100
By Euler's Theorem since k, 100 rel prime $k^{\phi(100)} \equiv 1 \pmod{100}$
by a $\phi(100) = 40$ so $k^{40} \equiv 1$ mod 100 so $k(k^{40})^3$ is $k(1^3) \equiv k$

Very clear
I should reread rules

(14)

~~Ch~~ (this chap ~~well~~ has nice story - good to learn but bad ref)

2. Prove that if $a \equiv b \pmod{14}$ and $a \equiv b \pmod{5}$
then $a \equiv b \pmod{70}$

does $5 \cdot 14 = 70$?

Isn't this just a rule?

We know $a \equiv b \pmod{14}$ means $14 \mid a-b$

" $5 \mid a-b$

14, 5 rel prime

So if $p \mid x$ ^ $q \mid x \Rightarrow pq \mid x$

(p, q rel prime)

Was that on my sheet?

Is now

(15)

3. a) Largest chain

3 6 7 11 14 ✓

b) Largest antichain

9 10 11 12 13 ✓

c) How much time w/ single processor

14 ✓

(giveaway q.)

d) How much time if ∞ processors

5 ✓

e) Smallest # processors to do in 5

5 ✓

4. Oh I remember this HW 3

Quiz #4 1a. 8 vertices 24 edges Avg deg per vertex

So $\frac{48}{8} = 6$ ✓

"Handshake" lemma

(16)

b) 5 more e than v f?

$v - e + f = 2$

$e = v + 5$

$v - (v+5) + f = 2$

~~$5 + f = 2$~~

~~$f = -3$~~

Oh distribute

$v - v - 5 + f = 2$

$-5 + f = 2$

$f = 7$ ✓

c) Has 1 more e than v. Is it planer

$V = 1 + e$

$v - e + f = 2$

$(1+e) - e + f = 2$

$1 + f = 2$

$f = 3$

? yes 3 faces

(17)

V vertices v-1 edges can't be connected ← I like that logic

So every G is cut edge, acyclic so G is tree - must be planar

? but then f = 1

- not what I got

They had $v - (v-1) + f = 2$

$1 + f = 2$

$f = 3$ ← still

? I don't get how can have both

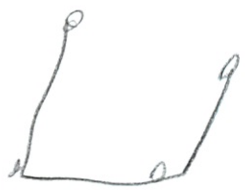
formula only applies if connected graph has a planar embedding

↑ non empty set of closed walks = faces

tree must be planar

(18)

d) If no give example of non planar connected simple graph whose vertices > edges by 1



Since graph is connected + acyclic, only 1 face

answer to previous was yes
question was is it nessarily planar?

e) How many isomorphism b/w graph + itself
— are there formulas?   ?; what does this mean;

Only vertex f has deg 1 so any self isomorphisms must map to itself. b is only vertex adj, so must map to itself

a, c both deg flip so a → a, c → c ; a → c, c → a (+2)

Ind

d → d, e → e →; d → e, e → d → (+2)

oh I see

So 2 × 2 = 4

(19)

2. n dimension Hypercube H_n

vertices if binary string 1 diff
↑ Hamming dist

Why is it impossible to find 2 spanning trees w/ no edges in common

Spanning tree = min # lines so all vertices are connected

Since its graph chap - prob has something to do w/

Always deg 3

- So you come in 1 way

Can leave one of 2 ways

Pick one at random 1st time

Sub times ... oh - does not help

Its some property I forget....

H_3 has 8 vertices so spanning tree $8-1=7$ edges

But H_3 has 12 edges - so any 2 sets of 7, must overlap

Oh I was thinking was possible - pay attention! -1

(20)

~~But they don't prove~~ that

Oh I guess since spanning tree

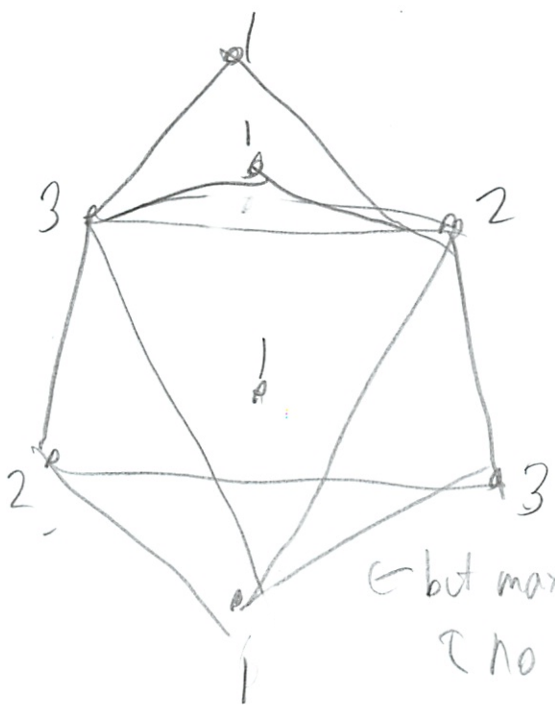
Put on cheat sheet
- triggers my memory!

---

3. Valid coloring
 - max k + 1

- I just do

Can you do it in 3



← but max degree is 4
? no rule about degree?
k is max degree
? so at most 5

no two adj vertices same color

Oh there we go

Know the rule
I ~~was~~ keep thinking rule is broader

(21)

4. Can there be a matching

Set M of edges so no vertex is incident to ≥1 edge in M

Sa bottleneck $|S| > |N(S)|$

↑ need to ture up my defn

First is there one

if degree constrained $\deg(l) \geq \deg(r)$ for all subsets

ie can you pick l, r where degrees wrong

b   v

? deg(2)   deg(3)

So no?

Not possible $S = \{a, b, c, e\}$ $N(S) = \{v, x, y\}$

so $|S| \geq$ ~~[illegible]~~ $|N(S)|$ ↑ neighbors of S

So I think my b, v would work $\deg(l) \geq \deg(r)$

$|S| = 1$ but $N(S) = 2$

—

Plus I thing degree constrained ~~Imp~~ Matching

but not

↑ Oh that makes things clearer

(22)

b) Can you find matching here?

Deg(l) ≥ deg(r)

~~4~~ 3 ← minimum   ← max 3

3 ≥ 3 ✓ So matching

But not "covering" whole graph
← can't say

Covers - every one on left has one out to right
did not prove here

---

5. Preserved invariants - if true at start true for all

1. ~~Don't know~~ ↗
where is he
~~No Tiger is for suitor~~

She can never ↓ suitor below 1 so true

No

a boy can come over

Oh the boy is being a suitor
← Think through the problem!

(23)

b) Yes - def of optimal wife

c) (I should review ritual - ? they should provide rules)

~~If true at begining - stays true~~

Can not ever be true

Were it true Tiger would have crossed elin's name off - so he would end up w/ less desirable woman

---

Quiz #5 1. Counting

1. 2 identical 52 cards - how many combo
↑ so cant tell apart

$\frac{104!}{2}$ ← 2 to 1 matching

↑ did same thing on quiz

$\frac{104!}{(2!)^{52}}$ ← oh the 52 makes sense
- not just one or two
- but why factorial - not times

↑ book keeper

2! 2! 2! 2! ... 52 times ? So can have in any pos

(why the am I so bad at this?)

(24)

b) Using int [1, n] how many strictly ↑ lenght m seq

So if n = 3

1 2 3 = 1 ← (n-2) 1

1 3 +

2 3 ← (n-1) 2

3 ← n +

2 3

1

n!

$\binom{n}{m} = \frac{n!}{k!(n-k)!}$

Given any m-el subset of {1, 2, ..., n} listing els in order is len m

By collecting in a set the terms of any strictly ↑ lenght m seq whose terms are drawn from {1, 2, ... n} an m el subset of {1, 2, ... n} is formed

Bij b/w the set of length m seq w/ terms drawn from {1, 2, .. n} and size of m{ subsets}

Oh only lenght m! — But even if I new that screwed up above? No — not w/ what I drew

should have bigger example

(25)

2. for each pair which hold or not

- I remember this one - guessed a lot!

Need pairs

Well the O( ) notes helped me

- look for dom term

Oh so $\frac{n^3}{n^2} = n \not\to 0$ so Not o

$\frac{n^2}{n^3} = \frac{1}{n} = 0$ so o( )

And finite $n < \infty$ as $n \infty$

but $\frac{1}{n} < \infty$

But I don't know on these weird $\frac{\log_4 n}{\sqrt[3]{n}}$

(26)

3. Strictly ↑ total fn

total always 1 arrow ∀ x ≠ y ∈ A

So arrows everywhere?

def on every arrow of domain

But how to come up w/ something?

$f(n) = n^2 + n$

$g(n) = n^2$

↑ dominates as terms grow large

$n^2 + n \sim n^2$

But $3^{f(n)} = 3^{n^2+n} = 3^n 3^{n^2} \neq 3^{n^2}$

So $3^{f(n)} = 3^n 3^{g(n)}$

$3^f \neq O(3^g)$

↑? which term dominates?

$3^n 3^{n^2}$

WA $\lim_{n \to \infty} \frac{3^n 3^{n^2}}{3^{n^2}} \neq 1$ × so Finite

But what does Than mean in our formula?

Forget it

(27)

4. space craft points

- mutinomial coeff

- Oh crazy ans

Massachusetts Institute of Technology
6.042J/18.062J, Spring '10: Mathematics for Computer Science — May 18
Prof. Albert R. Meyer — revised May 18, 2010, 1101 minutes

Final Examination

Your name:____

- This exam is **closed book** except for a three page, 2-sided crib sheet. Total time is 3 hours.
- Write your solutions in the space provided with your name on every page. If you need more space, write on the back of the sheet containing the problem. Please keep your entire answer to a problem on that problem's page.
- GOOD LUCK!

DO NOT WRITE BELOW THIS LINE

Problem	Points	Grade	Grader
1	10		
2	10		
3	10		
4	10		
5	10		
6	10		
7	10		
8	10		
9	10		
10	10		
Total	100		

Problem 1 (Probable Satisfiability) (10 points).

A *literal* is a propositional variable or its negation. A *k-clause* is an OR of k literals, with no variable occurring more than once in the clause. For example,

$$P \text{ OR } \overline{Q} \text{ OR } \overline{R} \text{ OR } V,$$

is a 4-clause, but

$$\overline{V} \text{ OR } \overline{Q} \text{ OR } \overline{X} \text{ OR } V,$$

is not, since V appears twice.

Let S be a set of n distinct k-clauses involving v variables. The variables in different k-clauses may overlap or be completely different, so $k \leq v \leq nk$.

A random assignment of true/false values will be made independently to each of the v variables, with true and false assignments equally likely. Write formulas in n, k, and v in answer to the first two parts below.

(a) (2 points) What is the probability that the last k-clause in S is true under the random assignment?

(b) (3 points) What is the expected number of true k-clauses in S?

(c) (5 points) A set of propositions is *satisfiable* iff there is an assignment to the variables that makes all of the propositions true. Use your answer to part (b) to prove that if $n < 2^k$, then $\mathcal{S}$ is satisfiable.

Problem 2 (Asymptotic Bounds and Partial Orders) (10 points).

For each of the relations below, indicate whether it is *transitive* but not a partial order (**Tr**), a *total order* (**Tot**), a *strict partial order* that is not total (**S**), a *weak partial order* that is not total (**W**), or *none* of the above (**N**).

- the "is a subgraph of" relation on graphs. ______
  (Note that every graph is considered a subgraph of itself.)

Let f, g be nonnegative functions on the real numbers.

- the "Big Oh" relation, $f = O(g)$, ______
- the "Little Oh" relation, $f = o(g)$, ______
- the "asymptotically equal" relation, $f \sim g$. ______

Problem 3 (Graph Coloring & Induction) (10 points).
Recall that a *coloring* of a graph is an assignment of a color to each vertex such that no two adjacent vertices have the same color. A k*-coloring* is a coloring that uses at most k colors.

False Claim. *Let G be a graph whose vertex degrees are all $\leq k$. If G has a vertex of degree strictly less than k, then G is k-colorable.*

(a) (2 points) Give a counterexample to the False Claim when $k = 2$.

(b) (4 points) Underline the exact sentence or part of a sentence that is the first unjustified step in the following "proof" of the False Claim.

False proof. Proof by induction on the number n of vertices:

Induction hypothesis:

$P(n)$::= "Let G be an n-vertex graph whose vertex degrees are all $\leq k$. If G also has a vertex of degree strictly less than k, then G is k-colorable."

Base case: ($n = 1$) G has one vertex, the degree of which is 0. Since G is 1-colorable, $P(1)$ holds.

Inductive step:

We may assume $P(n)$. To prove $P(n+1)$, let G_{n+1} be a graph with $n+1$ vertices whose vertex degrees are all k or less. Also, suppose G_{n+1} has a vertex, v, of degree strictly less than k. Now we only need to prove that G_{n+1} is k-colorable.

To do this, first remove the vertex v to produce a graph, G_n, with n vertices. Let u be a vertex that is adjacent to v in G_{n+1}. Removing v reduces the degree of u by 1. So in G_n, vertex u has degree strictly less than k. Since no edges were added, the vertex degrees of G_n remain $\leq k$. So G_n satisfies the conditions of the induction hypothesis, $P(n)$, and so we conclude that G_n is k-colorable.

Now a k-coloring of G_n gives a coloring of all the vertices of G_{n+1}, except for v. Since v has degree less than k, there will be fewer than k colors assigned to the nodes adjacent to v. So among the k possible colors, there will be a color not used to color these adjacent nodes, and this color can be assigned to v to form a k-coloring of G_{n+1}. ■

(c) (4 points) With a slightly strengthened condition, the preceding proof of the False Claim could be revised into a sound proof of the following Claim:

Claim. *Let G be a graph whose vertex degrees are all $\leq k$. If* ⟨statement inserted from below⟩ *has a vertex of degree strictly less than k, then G is k-colorable.*

Circle each of the statements below that could be inserted to make the Claim true.

- G is connected and
- G has no vertex of degree zero and
- G does not contain a complete graph on k vertices and
- every connected component of G
- some connected component of G

Problem 4 (Planar Embeddings) (10 points).
The planar graph embeddings in class (repeated in the Appendix) were only defined for connected planar graphs. The definition can be extended to planar graphs that are not necessarily connected by adding the following additional constructor case to the definition:

- **Constructor Case:** (collect disjoint graphs) Suppose $\mathcal{E}$ and $\mathcal{F}$ are planar embeddings with no vertices in common. Then $\mathcal{E} \cup \mathcal{F}$ is a planar embedding.

Euler's Planar Graph Theorem now generalizes to unconnected graphs as follows: if a planar embedding, $\mathcal{E}$, has v vertices, e edges, f faces, and c connected components, then

$$v - e + f - 2c = 0. \tag{1}$$

This can be proved by structural induction on the definition of planar embedding.

(a) (4 points) State and prove the base case of the structural induction.

(b) (2 points) Carefully state what must be proved in the new constructor case (collect disjoint graphs) of the structural induction.

(c) (4 points) Prove the (collect disjoint graphs) case of the structural induction.

Problem 5 (Euler's Function) (10 points).

(a) (2 points) What is the value of $\phi(175)$, where ϕ is Euler's function?

(b) (3 points) Call a number from 0 to 174 *powerful* iff some positive power of the number is congruent to 1 modulo 175. What is the probability that a random number from 0 to 174 is powerful?

(c) (5 points) What is the remainder of $(-12)^{482}$ divided by 175?

Problem 6 (Magic Trick Redux) (10 points).
In this problem we consider the famous 6.042 magic trick. Unlike the one performed in class by the TAs, this time the Assistant will be choosing 4 cards and revealing 3 of them to the Magician (in some particular order) instead of choosing 5 and revealing 4.

(a) Show that the Magician could not pull off this trick with a deck larger than 27 cards.

(b) Show that, in principle, the Magician could pull off the Card Trick with a deck of exactly 27 cards. (You do not need to describe the actual method.)

Problem 7 (Combinatorial Proof) (10 points).

(a) (2 points) Let S be a set with i elements. How many ways are there to divide S into a pair of subsets?

(b) (4 points)

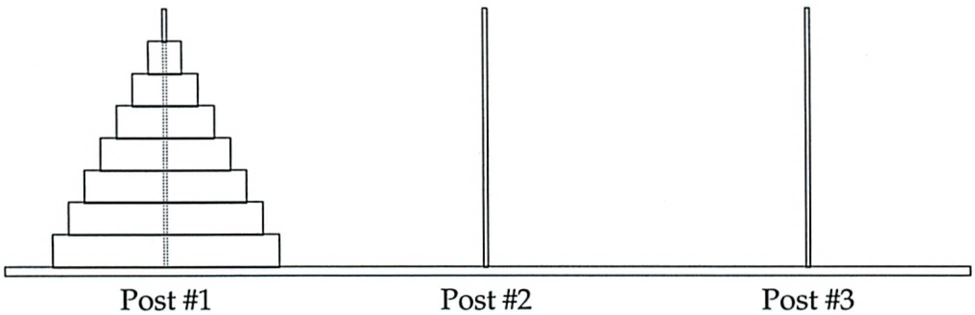
Here is a combinatorial proof of an equation giving a closed form for a certain summation $\sum_{i=0}^{n}$:

> There are n marbles, each of which is to be painted red, green, blue, or yellow. One way to assign colors is to choose red, green, blue, or yellow successively for each marble. An alternative way to assign colors to the marbles is to
>
> - choose a number, i, between 0 and n,
> - choose a set, S, of i marbles,
> - divide S into two subsets; paint the first subset red and the other subset green.
> - divide the set of all the marbles not in S into two subsets; paint the first subset blue and the other subset yellow.

What is the equation?

(c) (4 points) Now use the binomial theorem to prove the same equation.

Problem 8 (Linear Recurrence) (10 points).
Less well-known than the Towers of Hanoi —but no less fascinating —are the Towers of Sheboygan. As in Hanoi, the puzzle in Sheboygan involves 3 posts and n disks of different sizes. Initially, all the disks are on post #1:



The objective is to transfer all n disks to post #2 via a sequence of moves. A move consists of removing the top disk from one post and dropping it onto another post with the restriction that a larger disk can never lie above a smaller disk. Furthermore, a local ordinance requires that *a disk can be moved only from a post to the next post on its right —or from post #3 to post #1*. Thus, for example, moving a disk directly from post #1 to post #3 is not permitted.

(a) (2 points) One procedure that solves the Sheboygan puzzle is defined recursively: to move an initial stack of n disks to the next post, move the top stack of $n - 1$ disks to the furthest post by moving it to the next post two times, then move the big, nth disk to the next post, and finally move the top stack another two times to land on top of the big disk. Let s_n be the number of moves that this procedure uses. Write a simple linear recurrence for s_n.

(b) (4 points) Let $S(x)$ be the generating function for the sequence $\langle s_0, s_1, s_2, \ldots \rangle$. Carefully show that

$$S(x) = \frac{x}{(1-x)(1-4x)}.$$

(c) (4 points) Give a simple formula for s_n.

Problem 9 (Variance & Deviation) (10 points).
The hat-check staff has had a long day serving at a party, and at the end of the party they simply return people's hats at random. Assume that n people checked hats at the party.

Let X_i be the indicator variable for the ith person getting their own hat back. Let S_n be the total number of people who get their own hat back.

(a) (1 point) What is the expected number of people who get their own hat back?

(b) (2 points) Write a simple formula for $\mathrm{E}[X_i X_j]$ for $i \neq j$. *Hint:* What is $\Pr\{X_j = 1 \mid X_i = 1\}$?

(c) (3 points) Show that $\mathrm{E}[S_n^2] = 2$. *Hint:* $X_i^2 = X_i$.

(d) (1 point) What is the variance of S_n?

(e) (3 points) Use the Chebyshev bound to show that the probability that 11 or more people get their own hat back is at most 0.01.

Problem 10 (Sampling & Confidence) (10 points).
Yesterday, the bakers at a local cake factory baked a huge number of cakes. To estimate the fraction, b, of cakes in this program that are improperly prefared, the cake-testers will take a small sample of cakes chosen randomly and independently (so it is possible, though unlikely, that the same cake might be chosen more than once). For each cake chosen, they perform a variety of non-destructive tests to determine if the cake is improperly prepared, after which they will use the fraction of bad cakes in their sample as their estimate of the fraction b.

The factory statistician can use estimates of a binomial distribution to calculate a value, s, for a number of cakes to sample which ensures that with 97% confidence, the fraction of bad cakes in the sample will be within 0.006 of the actual fraction, b, of bad cakes in the back.

Mathematically, the *batch* is an actual outcome that already happened. The *sample* is a random variable defined by the process for randomly choosing s cakes from the batch. The justification for the statistician's confidence depends on some properties of the batch and how the sample of s cakes from the batch are chosen. These properties are described in some of the statements below. Mark each of these statements as **T** (true) or **F** (false), and then briefly explain your answer.

1. The probability that the ninth cake in the *batch* is bad is b. ______

2. All cakes in the batch are equally likely to be the third cake chosen in the *sample*. ______

3. The probability that the ninth cake chosen for the *sample* is bad, is b. ______

4. Given that the first cake chosen for the *sample* is bad, the probability that the second cake chosen will also be bad is greater than b.

   ______

5. Given that the last cake in the *batch* is bad, the probability that the next-to-last cake in the batch will also be bad is greater than b.

   ______

6. Given that the first two cakes selected in the *sample* are the same kind of cake —they might both be chocoloate, or both be angel food cakes,...—the probability that the first cake is bad may be greater than b.

   ____

7. The expectation of the indicator variable for the last cake in the *sample* being bad is b._______

8. There is zero probability that all the cakes in the *sample* will be different. _______

6.042 Cheat Sheet 1

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::= defined to be
$\cap$ = AND
$\cup$ = OR
$\rightarrow$ = implies
$\neg$ = not
$\leftrightarrow$ = iff, equivilant
$\oplus$ - XOR
$\exists$ = Exists
$\forall$ = for all
$\in$ is a member of
$\subseteq$ subset
$\subset$ subset proper
P(A) power set 2^n items
$\mathbb{N}$ = non neg
$\mathbb{Z}$ = int
$\mathbb{Z}^+$ = pos int
$\mathbb{Z}^-$ = neg int
$\mathbb{R}$ = real - not complex int, rat, irrat, trans.
$\mathbb{Q}$ = rational - can $\frac{a}{b}$ b not 0
$\mathbb{C}$ = complex - add imag # a+bj
λ = empty string

A AND B $\leftrightarrow$ B AND A commutativity
(A AND B) AND C $\leftrightarrow$ A AND (B AND C) associtivity
T AND A $\leftrightarrow$ A identity
F AND A $\leftrightarrow$ F zero
A AND A $\leftrightarrow$ A idempotence
A AND $\bar{A}$ $\leftrightarrow$ F contridictions
Not ($\bar{A}$) $\leftrightarrow$ A double negation
A or $\bar{A}$ $\leftrightarrow$ T validity
A AND (B or C) $\leftrightarrow$ (A AND B) or (A AND C) distributive
NOT (A AND B) $\leftrightarrow$ $\bar{A}$ or $\bar{B}$ DeMorgan AND
NOT (A or B) $\leftrightarrow$ $\bar{A}$ AND $\bar{B}$ Demorgan or

function A $\geq$ 1 arrow out
total A $\leq$ 1 arrow out
function + total A = 1 out
surjective B $\leq$ 1 in
injective B $\geq$ 1 in
bijective A = 1 and B = 1 in/out

axiom - basic assumption
theorm - important proposition
lemma - prelim proposition
corollary - prop after steps after theorm
predicate - depends on value of variable

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valid - always true (Universal)
satisfiable - true sometimes (existential)

Proof
1. Assume P
2. Show that Q logically follows
3. So if P, then Q

iff Prove P→Q and Q→P

WOP Every nonempty set of nonneg int has a smallest element
To prove P(n) holds for every non neg int n
1. Define a set of counterexamples to P being true
   $C ::= \{n \in \mathbb{N} \mid P(n) \text{ is false}\}$
2. Assume for proof that this is nonempty
3. By WOP must be smallest el n in C
4. Reach a contridiction (somehow)
   - often by showing how to use n to find another member of C that is smaller than n
5. Conclude C must be empty
   - watch for special cases 1, 0

Implies true if part false or then true
Contrapositive NOT(Q) → NOT(P) iff P→Q
NOT($\forall$x P(x)) iff $\exists$x. NOT(P(x))

Mapping Rules

1. $|A| \geq |B|$ A surj B
2. $|A| \leq |B|$ A inj B
3. $|A| = |B|$ A bij B
4. $|A| > |B|$ A strict B

Bij ∞

$e(b) ::= a_0$

$e(a_n) ::= a_{n+1}$ for $n \in N$

$e(a) ::= a$ for $a \in A - \{b, a_0, \ldots\}$

Russell's Paradox

$W ::= \{S \mid S \notin S\}$

So $S \in W$ iff $S \notin S$

for every S

Cont: $W \in W$ iff $W \notin W$

So W not a set - can't be a member of itself

ZFC

Extensionality 2 sets = if same members

$(\forall z. (z \in x \text{ IFF } z \in y)) \to x = y$

Pairing For 2 sets x, y there is a set $\{x, y\}$ w/ x, y as only eles

$\forall x, y \exists u \forall z [z \in u \text{ iff } (z = x \text{ or } z = y)]$

Union Union of z is also a set

$\forall z. \exists u \forall x (\exists y. x \in y \text{ AND } y \in z) \text{ iff } x \in u.$

Infinity There is an ∞ set. A nonempty set x, such that for any set $y \in x$ the set $\{y\}$ is also a member of x.

Power Set All subsets form another set.

$\forall x. \exists p \forall u. u \subseteq x \text{ iff } u \in p$

6.042 Cheat Sheet 2

Replacement A formula ϕ of set theory defines the graph of a fn:

$\forall x, y, z. [\phi(x, y) \text{ and } \phi(x, z)] \to y = z$

The image of any set s under that fn is also a set t.

$\forall s \exists t \forall y [\exists x. \phi(x, y) \text{ iff } y \in t]$

Foundation There can not be an ∞ seq $\ldots \in x_n \in \cdots \in x_1 \in x_0$ of sets where each one is member of previous.

$\text{member-minimal}(m, x) = [m \in x \text{ and } \forall y \in x, y \notin m]$

So $\forall x. x \neq \emptyset \to \exists m \text{ member-minimal}(m, x)$

Choice Given a set s, whose members are non empty sets, no 2 of which have any elm in common, then is set c, consisting of one el from each set in s.

Induction

P() = predicate

If P(0)

$P(n) \to P(n+1)$ for $\forall n \in N$

then P(m) for $\forall m \in N$

Invariant Principle

If the preserved invariant of a SM is true for the start state, then it is true for all reachable states

Strong Induction

P() = predicate

If P(0)

for all $n \in N$, $P(0), P(1), \ldots, P(n)$ together $\to P(n+1)$

then P(m) is true for all $m \in N$

Recursive

- construct new data els from previous ones

Structural induction - w/ constructor

⟨1, ⟨0, ⟨1, ⟨1, λ⟩⟩⟩⟩

Concatination

Expression parsing

Structural Induction

P() predicate, R = data type

If P(b) is true for each base case el $b \in R$

for all 2 argument constructors c

$[P(r) \text{ and } P(s) \to P(c(r, s))]$ for all $r, s \in R$

then P(r) is true for all $r \in R$

3/1

Factorial $f(0) = 1$

$f(n+1) = (n+1) \cdot fac(n)$ for $n \geq 0$

Fib $Fib(0) = 0$

$Fib(1) = 1$

$Fib(n) = Fib(n-1) + F(n-2)$

Countable ∞ if N bij C

5/15

Halting Problem can't perfectly check itself for all inputs if it will halt

Cheat Sheet 3

3/15

$a|b = [ak = b$ for some $k]$
  └ or b/a = integer

$n = q \cdot d + r$ and $0 \le r < d$

$qcnt(n,d) \Rightarrow \frac{n}{d}$

$rem(n/d) \rightarrow \frac{n}{d}$

Euclid Algorithm

$gcd(a,b) = gcd(b, rem(a,b))$

Pulverizer

$gcd(a,b) = sa + tb \quad \exists s,t$

$gcd(259, 70)$

		rem(x,y)	x − q·y
259	70	49	= 259 − 3·70
70	49	21	= 70 − 1·49
			= 70 − 1(259 − 3·70)
			= −1·259 + 4·70
49	21	7	= 49 − 2·21
			= (259 − 3·70) − 2(−1·259 + 4·70)

$7 = 3 \cdot 259 - 11 \cdot 70$
(↑ s, ↑ t)

Fund. Theorem of Algebra Every pos integer is a product of uniquely weakly decreasing seq. of primes

If $p|ab$ then $p|a$ or $p|b$
p = prime

$a \equiv b \pmod{n}$ iff $n|(a-b)$
iff $rem(a,n) = rem(b,n)$

Multiplicative Inverse of 7 mod 5

$7 \cdot \_ \equiv 1 \pmod{5}$

- guess + check
- Pulverizer

$sp = 1 - tk$

$p|(1-tk) \rightarrow$ so $tk \equiv 1 \pmod{p}$

$r = x - qy$

any thing multiple of gcd $\rightarrow r = ax + by$

aka $gcd(5,3)$
↑k ↑p
take b result

Fermat's Little Theorem

$k^{p-1} \equiv 1 \pmod{p}$

$k^{p-2} \cdot k \equiv 1 \pmod{p}$

So find $rem(3^3, 5)$
↑ k=3, ↑ p=5

Euler's Theorem ← generalization

$\phi(p) = p - 1$

$k^{\phi(n)} \equiv 1 \pmod{n}$ if k rel prime to n

$k^{\phi(n)-1} \equiv$ mul. inverse k mod n

So $3^{\phi(5)-1}$

Eulers Theorem

$\phi(pq) = (p-1)(q-1)$ ↓prime↓

$\phi(p^k) = p^k - p^{k-1}$ ↑primes

$\phi(ab) = \phi(a)\phi(b)$
↑rel prime

$\phi(300) = \phi(2^2 \cdot 3 \cdot 5^2)$
$= \phi(2^2) \cdot \phi(3) \cdot \phi(5^2)$
$= (2^2 - 2^1)(3^1 - 3^0)(5^2 - 5^1)$
$= 80$

= # of rel prime #

Rel Prime for a,b if $gcd(a,b) = 1$

For primes: all except 1

RSA

Before

1. Generate 2 primes p q
2. $n = pq$
3. Select e from $gcd(e, (p-1)(q-1)) = 1$
   - solve for e, guess + check
   - Smallest prime that does divide

   public (e,n)
4. Compute d = inverse e mod $(p-1)(q-1)$

   Secret (d,n)

Encoding Check $gcd(m,n) = 1$

$m^* = rem(m^e, n)$ using other parties key

Decoding Use your private key

$m = rem((m^*)^d, n)$

If p,q rel prime and $p|x$ and $q|x$ then $pq|x$

Combine w/ $a \equiv b \pmod{c} \rightarrow c|(a-b)$

gcd - largest common seq of two factorization, take all primes that appear in both factorization raised to the <u>min</u> power of each respective <u>prime</u>

lcm - max instead

Chinese Remainder Theorem

for all m, n $\exists X$ such that

$X \equiv m \bmod a$

$X \equiv n \bmod b$

$X' \equiv X \bmod ab$

Proof $e_a = b^{-1}b$ $e_b = a^{-1}a$ $x = me_a + ne_b$

TBA

Another way to exponentiate:

split up

$13^{21} = 13^{16} \cdot 13^{2 \cdot 4} \cdot 13$

↳ $2^2 \bmod 23$ ...

Verifying <u>Preserved Invariant</u> - once it enters it stays - show value does not change

Find rem $26^{1818181}$ $\quad 1818181 = (180 \cdot 10101) + 1$

$\phi(297) = \phi(3^3 \cdot 11) = \phi(3^3) \cdot \phi(11) = (3^3 - 3^2)(11-1)$

$= 180$

$26^{1818181} = 26^1 \cdot 26^{180 \cdot 10101}$ $\quad \leftarrow 26^{180} \equiv 1 \bmod 297$

$= 26 \cdot 1^{10101}$

$= 26$

6.042 Cheat Sheet 4

graph = network

directed = digraph = 1 way = arrows

DAG = directed, acyclic [no cycles (+ lenght)]

dots = nodes = verticies

two sets vertices + edges

$e = \langle u \to v \rangle$

$|v|$ = lenght

$\text{indeg}(v) ::= |\{e \in E(G) \mid \text{head}(e) = v\}|$

$\text{outdeg}(v) ::= |\{e \in E(G) \mid \text{tail}(e) = v\}|$

$$\sum_{v \in V(G)} \text{indeg}(v) = \sum_{v \in V(G)} \text{outdeg}(v)$$

V(G) = verticies E(G) = edges

walk - can repeat points

path - all pts must be unique

merge = $f \hat{v} r$ combine 2 walks

dist(u,v) = lenght of shortest path

Adj Matrix $(A_G)_{ij}$ if $\langle v_i \to v_j \rangle \in V(G)$

$(A_G)^k$ count of lenght k walks b/w = C_{uv} for a certain point

$u G^* v$ is a path G^+ = pos lenght G_n path

Reflexity every node in G has self loop
$\forall x \in A, \; xRx$

Irreflexity no self loops in G
NOT $\exists x \in A \; xRx$

Symmetry $\forall x, y \in A \; xRy \to yRx$
if edge $x \to y$ also $y \to x$

Asymmetry at most 1 edge everywhere
No self loops
$\forall x, y \in A \; xRy \to \text{NOT}(yRx)$

Antisymmetry at most 1 edge, can be self loops
$\forall x \neq y \in A \; xRy \to \text{Not}(yRx)$

Transitive if pos path $u \to v$ then $u \to v$
$\forall x, y, z \in A (xRy \text{ and } yRz) \to xRz$

Total Given any 2 vertices in G, there is an edge in 1 dir or the other b/w them
$\forall x \neq y \in A (xRy \text{ or } yRx)$

Critical path = lenght longest chain

depth = size of critical path

Closed walk - starts + ends at same vertex

Cycle - closed walk w/ distinct verticies

$\subseteq$ = subset

isomorphic if relation presering bij

total - always 1 arrow $\forall x \neq y \in A (xRy \text{ or } yRx)$
└ defined on every el of domain (6.042)

product order $R_1 \times R_2$ - where every item comparable - not 8.02 (?)

domain $(R_1 \times R_2)$ = domain $(R_1) \times$ domain (R_2)
codomain " = co " $\times$ co "

$(a_1, a_2)(R_1 \times R_2)(b_1, b_2)$ iff $[a_1 R_1 b_1$ and $a_2 R_2 b_2]$

- where both are true

topological sort $a \preceq b \to a \sqsubseteq b$
makes it a total ordering partial total

antichain - all items incomparable

equivilance = reflexive, symmetric, transitive

$\subset$ = proper subset $A \subset B$ means B has everything + more
- asymmetric
- transitive

SPO - transitive + asymmetric
$\prec$ [less then, ranked higher
if (+) path relation of a DAG

WPO - same as SPO but $a R a$ also holds
$\subseteq$ on sets $\leq$ on $\mathbb{R}$ for all
- reflexive
- transitive
- antisymmetric
$\preceq$ or $\sqsubseteq$

total - like a path/chain

Symmetric $\forall x, y \in A \; xRy \to yRx$
- arrow in both dirs

Simple Graphs - undireted (no arrows)
$v - w$ = undirected edge
no self loops (from u to u)

two pts adjacent if edge

edge is incident to end pts

deg(v) = # edges incident to vertice

$$\sum_{x \in M} \deg(x) = \sum_{y \in F} \deg(y)$$

Handshake Sum of deg of verties = 2 x # edges

$2|E| = \sum_{v \in V} \deg(v)$

K_n = complete graph - every arrow

L_n = line graph
- if add 1 edge, would have cycle

isomorphism is a edge-persevering bij $f: V(G) \to V(H)$ s.t.
$u - v \in E(G)$ iff $f(u) - f(v) \in E(H)$
for all $u, v \in V(G)$

biparte - can split into 2 graps

Matching cond - every subset men likes at least as large as subset of men

Matching - set M of edges G s.t. no vertex is incident to ≥ 1 edge in M.

covers - if all vertices included = perfect

bottleneck $|S| > |N(S)|$
↑ neighbors

Hall's Theorem: Matching in G (biparte) that covers L(G) iff no subset of L(G) is a bottleneck

if degree constrained - is a matching - but not other way around!

degree constrained $\deg(l) \geq \deg(r)$ for all l, r

regular - each node has same degree

Every reg biparte graph has perfect matching

Stable - no rage couples - pair that likes each other more

if w is off m's list w has suitor she prefers over m

men = optimal
girls = pessimal

termination # remaining names strictly ↓

Coloring - adj vertices diff color

$\chi()$ = chromatic # = min # colors

$\chi(K_n) = n$ $\chi(\text{biparte}) = 2$

$\chi(C_{even}) = 2$ $\chi(C_{odd}) = 3$

$\chi(\text{max degree} = k) = k+1$

Subgraphs

Connected - every pair verticies connected
- path exists somewhere

Connected components

k-edge connected = # edges can remove till splits
- called cut edge if

Tree - connected acyclic graph

Connected component of trees = forest

leaf = node w/ deg(1)

1. Each connected subgraph = tree
2. Unique simple path b/w every pair of vertices
3. Adding edge b/w nonadj nodes creates a cycle
4. Removing any edge - disconnects
   └ All edges = cut edges
5. If ≥ 2 vertices ≥ 2 leaves
6. # vertices = #edges +1

Spanning tree - min # of lines so all vertices still connected
$V-1 = \#edges$

if branches weighted → Min-weight tree (MST)

Planar - no lines crossing
drawing - one particular set of curves
face - continuous
 - but divide up into discrete
 - don't forget outside
bridge
dangle

discrete face = planar embeddings
 - either split a face or add a bridge

Euler's Formula $V - e + f = 2$ | if connected ≥1 face
 - proof w/ the 2 constructors

$e \leq 3v - 6$ limit of planar if $v \geq 3$

minor - delete vertices, edges, merge vertices

every planar graph has degree ≤ 5
 - so 5-colorable

At most 5 regular polyhedra

Power set -- set of all subsets

So $P\{1,2,3\} = \{1\}, \{2\}, \{3\}$
$\{1,2\} \{2,3\} \{3,1\}$
$\{1,2,3\}$

If longest chain is of size t, can partition into t antichains

For all t > 0 every partially ordered set must have chain of size > t or antichain of size n/t

6.042 Cheet-Sheat 5

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Sums

closed form ↓

$1+2+3+\dots+n = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

$1+x+x^2+x^3+\dots+x^n = \frac{1-x^{n+1}}{1-x}$

for product → take log to convert to sum

to find → Pertubation Method

$V = \sum_{i=1}^{n} \frac{m}{(1+p)^{i-1}} = m \sum_{j=0}^{n-1} \left(\frac{1}{1+p}\right)^j$ sub $j = i-1$

$= m \sum_{i=0}^{n-1} x^i$ sub $x = \frac{1}{1+p}$

$S = 1+x+x^2+\dots+x^n$

$xS = x+x^2+\dots+x^{n+1}$

Subtract $S - xS \to = 1-x^n$

Solve for $S \to S = \frac{1-x^{n+1}}{1-x}$

$V = m\left(\frac{1-x^n}{1-x}\right)$

$= m\left(\frac{1+p-(1/(1+p))^{n-1}}{p}\right)$

If $|x|<1$: $\sum_{i=0}^{\infty} x^i = \frac{1}{1-x}$ ∞ Geometric Seq

find by taking lim $n \to \infty$

$V = m \sum_{i=0}^{\infty} x^i$

$= m \cdot \frac{1}{1-x}$

$= m \cdot \frac{1+p}{p}$

$1+\frac{1}{2}+\frac{1}{4}+\dots = \sum_{i=0}^{\infty}\left(\frac{1}{2}\right)^i = \frac{1}{1-\frac{1}{2}} = 2$

$.999\dots = .9\sum_{i=0}^{\infty}\left(\frac{1}{10}\right)^i = .9\left(\frac{1}{1-1/10}\right) = .9\frac{10}{9} = 1$

$1-\frac{1}{2}+\frac{1}{4}-\dots = \sum_{i=0}^{\infty}\left(-\frac{1}{2}\right)^i = \frac{1}{1-(-1/2)} = \frac{2}{3}$

$1+2+4+\dots+2^{n-1} = \sum_{i=0}^{n-1} 2^i = \frac{1-2^n}{1-2} = 2^n-1$

$1+3+9+\dots+3^{n-1} = \sum_{i=0}^{n-1} 3^i = \frac{1-3^n}{1-3} = \frac{3^n-1}{2}$

(can also differentiate/integrate)

$\sum_{i=1}^{n-1} i x^i = \frac{x-nx^n+(n-1)x^{n+1}}{(1-x)^2}$

$\sum_{i=1}^{\infty} i x^i = \frac{x}{(1-x)^2}$

Sum of Powers $\sum_{i=1}^{n} i^2 = \frac{(2n+1)(n+1)n}{6}$

Approximating - find closed-form upper+lower bounds

Weakly ↑ $S = \sum_{i=1}^{n} f(i) \to I = \int_1^n f(x)\,dx$

$I + f(1) \le S \le I + f(n)$

Weakly ↓ $I + f(n) \le S \le I + f(1)$

nth Harmonic # $H_n = \sum_{i=1}^{n} \frac{1}{i}$

So $S_n = \frac{H_n}{2}$ - no closed form - so can get first few terms

- or upper/lower bounds

$\int_1^n \frac{1}{x}\,dx = \ln(x)\Big|_1^n = \ln(n)$

$\ln(n) + \frac{1}{n} \le H_n \le \ln(n) + 1$

So $\gamma = .577215664$

Asymptotic Inequality

~ leading term = iff $\lim \frac{f(x)}{g(x)} = 1$

Products

$P = \prod_{i=1}^{n} f(i)$ take log $\ln(P) = \sum_{i=1}^{n} \ln(f(i))$

to approximate $= \sum_{i=1}^{n} \ln(i)$

$n\ln(n) - n + 1 \le \Sigma \le n\ln(n) - n + 1 + \ln(n)$

exponentiate

$n^n/e^{n-1} \le n! \le n^{n+1}/e^{n-1}$

Stirling's Formula for $n \ge 1$

$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\epsilon(n)}$

where

$\frac{1}{12n+1} \le \epsilon(n) \le \frac{1}{12n}$ but $\epsilon \to 0$ so

$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$

Little Oh asy. smaller

$f = o(g(x))$ iff $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$

Big Oh upper bound on growth

$f = O(g(x))$ iff $\limsup_{x\to\infty} \frac{f(x)}{g(x)} < \infty$ ← finite

There exists constant c, such that for $\forall x \ge x_0$

$|f(x)| \le c\,g(x)$

- ignoring some leading term

Theta - precise up to constant terms

- upper + lower bounds

$f = \Theta(g)$ iff $f = O(g)$ and $g = O(f)$

every constant is 1

base of exponent matters

big O always upper bound

never = to, bad notation

Omega lower bound of running time

$f = \Omega(g)$ is $g = O(f)$

Little Omega - one grows strictly faster than other

$f = \omega(g)$ is $g = o(f)$

Cardinality/Counting Rules

- Count one thing by counting another

- that is related as a bij

- like encode w/ 1s and 0s

Product Rule size of product of sets

if finite just multiply sizes

2^n = # bit string subsets

Sum Rule - if disjoint - just add

# of possible arrangements of 3 prizes $= n^3$

but if prizes must go unique people $= n(n-1)(n-2)$

Or $\frac{n!}{(n-3)!}$

Permutations (order matters) each item once $= n!$

$\sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$

Divison Rule k-to-1 function

- like fingers to person = 10 to 1 relation

$|A| = k \cdot |B|$

So $|B| = \frac{|A|}{k}$

Knights of Round Table - only who next to who matters

$|B| = \frac{|A|}{n} = \frac{n!}{n} = (n-1)!$, n cyclic shifts ok

Counting Subsets

How many k-el subsets from n-el set.

$\binom{n}{k} = \frac{n!}{k!(n-k)!}$

Also by division rule $n! = k!(n-k)!\binom{n}{k}$

example # n bit seq w/ k 1s $= \binom{n}{k}$

its like putting k in 1 subset and n-k in another

n from permutations - then $k!(n-k)!$ to 1 fn

Can have m subsets

$\binom{n}{k_1, k_2, \ldots k_m} = \frac{n!}{k_1! \cdot k_2! \cdots k_m!}$ multinomial coefficient

So # of splits of n el subset

Subset split rule

example Bookkeeper rule

$\frac{10!}{1!2!2!3!1!1!} \rightarrow \binom{k_1+k_2+\ldots+k_m}{k_1, k_2, \ldots k_m}$

Binomial Theorem = sum of 2 terms

$(a+b)^4 = 2^4$ terms

terms w/ k copies of b is

$\frac{n!}{k!(n-k)!} = \binom{n}{k}$

So $a^{n-k}b^k = \binom{n}{k}$

$(a+b)^4 = \binom{4}{0}a^4b^0 + \ldots \binom{4}{4}a^0b^4$

So $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k}b^k$

Multinomial - Extension above

- want coefficients for $bo^2k^2e^3pr$

$(z_1+z_2+\ldots+z_m)^n = \sum_{\substack{k_1, \ldots k_m \in \mathbb{N} \\ k_1+\ldots+k_m=n}} \binom{n}{k_1, k_2, \ldots k_m} z_1^{k_1} z_2^{k_2} \ldots z_m^{k_m}$

Poker

Think of clear rules to represent what is specified or not

Inclusion - Exclusion

adding non-disjoint sets

$|S_1 \cup S_2| = |S_1| + |S_2| - |S_1 \cap S_2|$

$|S_1 \cup S_2 \cup S_3| = |S_1| + |S_2| + |S_3| - |S_1 \cap S_2| - |S_1 \cap S_3| - |S_2 \cap S_3| + |S_1 \cap S_2 \cap S_3|$

- must remove duplicates

Proving

1. Define S
2. Show $|S| = n$ by counting 1 way
3. Show $|S| = m$ " " other "
4. Conclude $n = m$

Pascal's Identity

Boxer story

$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

Pigeonhole Principle

If more pigons than holes, then at least 2 pigons in some hole

Identify A = Pigeons $\rightarrow n$

B = Pigeonholes $\rightarrow h$

$f \; A \rightarrow B$

Prove size of both

$\lceil \frac{n}{h} \rceil$ pigons in some hole

Magic Trick

- info on which card kept hidden
- what is degree constrained?

log	$O(\log n)$	
poly	n^x	n is what ?
exp	x^n	

non neg integer solutions

$x_1 + x_2 + \ldots + x_m = k \rightarrow \binom{m+k-1}{k}$

$x_1 + x_2 + \ldots + x_m \leq k \rightarrow \binom{m+k}{k}$

length $k+m-1$ bit strings w/ k 0s

add m+1

length-m weakly ↑ seq non neg int $\leq k$

$\binom{m+k}{k}$

↳ bij to seq $x_1, x_1+x_2, x_1+x_2+x_3, \ldots$

So ans follows from

6.042 Cheat Sheet 6

Binomial Theorm

binomial = sum of 2 terms $a+b$

One term for each seq of a, b

terms is $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ (bookkeeper)

Can expand terms to

$$\sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^{k}$$

Can have multinomial for all $n \in N$

$$\sum_{k_1, \ldots k_m \in N} \binom{n}{k_1, k_2, \ldots k_m} z_1^{k_1} z_2^{k_2} \ldots$$

$k_1 + k_2 + \ldots k_m = n$

Pascal's Identity

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

Combinatorial Proof tell a story

1. Define a set S
2. Show $|S| = n$ by 1 way
3. Show $|S| = m$ by other way
4. Conclude $n = m$

Probability

Draw a tree!

outcome

events = set of outcomes

4 step method

1. Find the sample space
   - draw the tree
2. Find the events of interest
   - set of outcomes we are looking at
3. Determine outcome probabilites
   - Multiply along tree
4. Compute event probabilites
   - add up the outcomes

For any $n \geq 2$ there is a set of n dice, for any n-node diagraph w/ exactly 1 directed edge b/w every 2 distinct nodes, there is a # of rolls k s.t. the sum of k rolls of the ith die is 7 sum for jth die w/ $P \geq \frac{1}{2}$ iff edge $i \to j$ in graph

$P[w] \geq 0$ for all $w \in S$

$$\sum_{w \in S} P(w) = 1$$

For event $E \subseteq S$ $P(E) = \sum_{w \in E} P(w)$ Uniform $P(E) = \frac{|E|}{|S|}$

Sum Rule

$$P\left(\bigcup_{n \in N} E_n\right) = \sum_{n \in N} P(E_n) \text{ disjoint}$$

Complement Rule $P(\overline{A}) = 1 - P(A)$

Diff Rule $P(B - A) = P(B) - P(A \cap B)$

In-Ex $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Boole's Ineq $P(A \cup B) \leq P(A) + P(B)$

Monotonicity If $A \subseteq B$ then $P(A) \leq P(B)$

Union Bound $P(E_1 \cup E_2 \cup \ldots \cup E_n) \leq P(E_1) + \ldots + P(E_n)$

∞ Prob Space Same ∞ sums as before

$S(T^n H \mid n \in N)$ $P(T^n H) = \frac{1}{2^{n+1}}$

Conditional $P(A \mid B) = \frac{P(A \cap B)}{P(B)}$

Product Rule $P(A \cap B) = P(B \mid A) P(A)$

Always draw tree!

Law of Total Prob $P(A) = P(A \mid E) \cdot P(E) + P(A \mid \overline{E}) P(\overline{E})$

$$= \sum_{i=1}^{n} P(A \mid E_i) P(E_i)$$

Independence $P(A \mid B) = P(A)$

If disjoint $\Rightarrow$ know not ind trick!

$P(A \cap B) = P(A) \cdot P(B)$

$$P\left(\bigcap_{j \in S} E_j\right) = \prod_{j \in S} P(E_j)$$

k-wise ind if every set k ind

Pairwise ind = 2-wise ind

mutually ind = all subsets ind

$2^n - (n+1)$ to check

5/3

Random Variables map outcomes to #

Bernoulli = if 1 or 0

Independence $P(C = x_1 \text{ AND } M = x_2) = P(C = x_1) \cdot P(M = x_2)$

Two events ind if indicator variables ind

PDF

$$PDF_R(x) = \begin{cases} P(R = x) & \text{if } x \in range(R) \\ 0 & \text{if } x \notin range(R) \end{cases}$$

$$\sum_{x \in Range} PDF_R(x) = 1$$

$$CDF_R(x) = P(R \leq x) = \sum_{y \leq x} P(R = y) = \sum_{y \leq x} PDF_R(y)$$

Binomial $f_p(0) = p$ $f_p(1) = 1 - p$

$$F_p(x) = \begin{cases} 0 & \text{if } x < 0 \\ p & \text{if } 0 \leq x < 1 \\ 1 & \text{if } 1 \leq x \end{cases}$$

Uniform $f: V \to [0,1]$ $f(v) = \frac{1}{n}$ for all $v \in V$

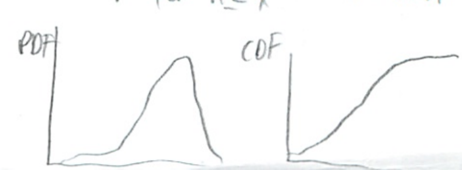
$$F(x) = \begin{cases} 0 & \text{if } x < 1 \\ k/n & \text{if } k \leq x < k+1 \text{ for } 1 \leq k < n \\ 1 & \text{if } n \leq x \end{cases}$$

Binomial k # heads in n flips

$\binom{n}{k} 2^{-n}$

seq, prob of each seq

$$F(n) = \begin{cases} 0 & \text{if } x < 1 \\ \sum_{i=0}^{k} & \text{if } k \leq x \leq k+1 \text{ for } 1 \leq k < n \\ 1 & \text{for } n < x \end{cases}$$



General Binomial Dist Fn

if coins biased $P(\text{heads}) = p$

$$f_{n,p} = \binom{n}{k} p^k (1-p)^{n-k}$$

$\binom{n}{k}$: # seqs; $p^k(1-p)^{n-k}$: prob of each seq

$$F_{n,p}(x) = \begin{cases} 0 & \text{if } x<1 \\ \sum_{i=0}^{k} \binom{n}{i} p^i (1-p)^{n-i} & \text{for } k\le x< k+1 \text{ for } 1\le k<n \\ 1 & \text{if } n\le x \end{cases}$$

Expectations aka mean or avg

$$E[R] = \sum_{w\in S} R(w)\, P(w)$$

^weighted avg of value

$$E[\text{uniform}] = \sum_{i=1}^{n} i \cdot \frac{1}{n} = \frac{n(n+1)}{2n} = \frac{n+1}{2} = \frac{b+a}{2}$$

$$E[I_A] = p(A)$$

^indicator: 1 or 0

Median $P(R\le x)\le \frac{1}{2}$ and $P(R\ge x) < \frac{1}{2}$

Cond'l Expectation Same as before

Law Total Ex $E[R] = \sum_i E[R|A_i]P(A_i)$

Mean Time to Failure

$$E[C] = 1\cdot p + (1+E[C])(1-p)$$
$$= p + 1 - p + (1-p)E[C]$$
$$= 1 + (1-p)E[C]$$
$$1 = E[C] - (1-p)E[C] = pE[x]$$
$$E[C] = \frac{1}{p}$$

in gambling - weight to payoffs

Linearity of Expectations

$$E[R_1 + R_2] = E[R_1] + E[R_2]$$
$$E[a_1R_1 + a_2R_2] = a_1E[R_1] + a_2E[R_2]$$

ind or not!

Sum of indicator RV to show 1 person gets hat back

Sum the prob of each event occurring

$E[\text{Binomial}] = np$

Coupon Collector

$P(\text{we have already}) = \frac{k}{n}$

So $P(\text{new}) = 1 - \frac{k}{n} = \frac{n-k}{n}$

So $E[\text{\# meals till new}] = \frac{n}{n-k}$

Sum these up

$$E[T] = E[X_0] + E[X_1] + \dots E[X_{n-1}]$$
$$= \frac{n}{n-0} + \frac{n}{n-1} + \dots \frac{n}{3} + \frac{n}{2} + \frac{n}{1}$$
$$= n\left(\frac{1}{n} + \frac{1}{n-1} + \dots \frac{1}{3} + \frac{1}{2} + \frac{1}{1}\right)$$
$$= n\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n-1} + \frac{1}{n}\right)$$
$$= nH_n$$
$$\sim n\ln(n)$$

If ∞ sum converges

$$E\left[\sum_{i=0}^{\infty} R_i\right] = \sum_{i=0}^{\infty} E[R_i]$$

$E[R^2] = (E[R])^2$ only if ind

If A, B ind, $A, \bar{B}$ ind

$$P(A\cap \bar{B}) = P(A) - P(A\cap B)$$
$$= P(A) - P(A)\cdot P(B)$$
$$= P(A)(1-P(B))$$
$$= P(A)\,P(\bar{B})$$

Variation Cheat-Sheet 7

Markov $P(R\ge x) \le \frac{E[R]}{x}$ $R\ge 0$ RV, $x>0$

very basic bound

$$P(R \ge c\cdot E[R]) \le \frac{1}{c} \quad c\ge 1$$

Chebyshev

$$P(|R|\ge x) \le \frac{E[|R|^\alpha]}{x^\alpha}$$
$$P(|R-E[R]|\ge x) \le \frac{E[|R-E[R]|^\alpha]}{x^\alpha}$$

When $\alpha = 2$ called var

$$P(|R-E[R]|\ge x) \le \frac{Var(R)}{x^2}$$
$$P(|R-E[R]|\ge c\sigma_R) \le \frac{1}{c^2}$$

$$Var = E[(R-E[R])^2]$$
$$= E[R^2] - E^2[R]$$
$$\sigma = \sqrt{Var}$$

mean time to failure $E[C^2] = \frac{2-p}{p^2}$

$$var(aR) = a^2 var(R)$$
$$var(R+b) = var(R)$$
$$var(R_1+R_2) = var(R_1) + var(R_2) \quad \text{ind}$$

Sampling - estimate how close we are to true mean

$$P\left(\left|\frac{S_n}{n} - p\right| \le .04\right) \ge .95$$

within 4% of true prob $= p$ 95% of time

Note no prob in what p is - its known, prob of estimation procedure

Chernoff Sum of T_i $0\le T_i\le 1$ $c\ge 1$

$$P(T\ge c\,E[T]) \le e^{-k\cdot E[T]}$$
$$k = c\ln(c) - c + 1$$

Murphy's that there are no errors

T_i = indicator RV 1 = error

$$P(T=0)\le e^{-E[T]}$$

∞ Expectations Can some time add to ∞

Random Processes/Random Walks - up + down

If start n, aim for $T\ge n$, then $p(\text{reach}) = \frac{n}{T}$ ^fair game or not fair $\frac{r^n - 1}{r^T - 1}$ $r = \frac{q = 1-p}{p}$

$P(\text{we win game}) \le \left(\frac{p}{1-p}\right)^{T-n}$

Like Google Random walks on graph

$P(\text{follow link out at pg } x) = \frac{1}{\text{outdeg}(x)}$

$P(\text{go to } y) = \sum \text{links to } y \text{ at } x$

$= \sum \frac{P(\text{at } x)}{\text{out}(x)}$ for all x

So solve system for y - w/ supernode if none at

Given out at final

Notes for 6.042 Final

- The exam is long, so a good strategy would be to skip hard parts and come back to them later after moving quickly through easy parts.
- On **true/false** questions, you will get part credit for a correct answer of **false** even without the called-for counter-example. So if you don't quickly see a counter-example, go on to the next question.
- Problem 1(k) the b should be **1**.
- Problem 2(b), these are statements about **finite trees**. The last item should read
  - For every finite graph (not necessarily a tree), there is one (a finite tree) that spans it.
- Problem 2(e), "stable" should be "stationary."
- Problem 12(d), Write a formula solely in terms of the expressions given in part (a): $\Pr[B]$, $\Pr\left[Y \mid B\right]$ and $\Pr\left[\overline{Y} \mid \overline{B}\right]$.
- In your proof for Problem 13, intelligible abuse of $\Theta()$ notation will be accepted. For example, writing "$\Theta(x) \cdot \Theta(y) = \Theta(x \cdot y)$" to abbreviate some proof steps would be OK.