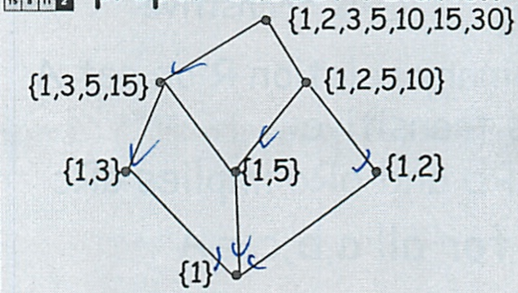
Mathematics for Computer Science
MIT 6.042J/18.062J

Partial Orders

proper subset relation



proper subset relation

$A \subset B$ means
B has everything
that A has
and more: $B \not\subset A$

properties of $\subset$

$A \subset B$ implies $B \not\subset A$
asymmetry

Since has more

$\subset$ is asymmetric

binary relation R on set A is asymmetric:

aRb implies NOT(bRa)
for all $a,b \in A$

properties of $\subset$

[$A \subset B$ and $B \subset C$]
implies $A \subset C$
transitivity

$\subset$ is transitive

binary relation R on set A is transitive:
aRb and bRc implies aRc
for all a,b,c $\in$ A

strict partial orders

transitive & asymmetric

strict partial orders

examples:

- "less than," <, on the real numbers
- "ranked higher than" on professional tennis players
- "indirect prereq for" on MIT subjects

weak partial orders

same as a strict partial order R, except that
aRa always holds

examples:

- $\subseteq$ is weak p.o. on sets
- $\leq$ is weak p.o. on $\mathbb{R}$

Reflexivity

relation R on set A is reflexive iff aRa for all a $\in$ A

antisymmetry

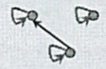
binary relation R on set A is antisymmetric iff it is asymmetric except for aRa case.

weak partial orders

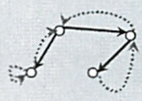
transitive
antisymmetric
& reflexive

Graphical Properties of Relations

Reflexive



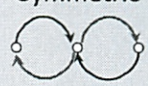
Asymmetric



NO

Transitive

Symmetric



Graphical Properties of Relations

Total
(looks like a path/chain)

Team Problems

Problems
1-4

(7 min late)

Weak Partial Order

- same as strict partial order R, except that aRa always holds
  - add a self loop on every

$\subseteq$ could be equal - weak PO on sets

$\leq$ is weak P.O. on $\mathbb{R}$

Reflexive if aRa holds for all A

Antisymmetry - asymmetric plus aRa

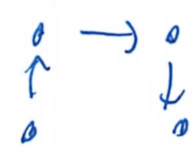
WPO - transitive, Reflexive, antisymmetric

Relexive

(oh the P-set problems!)

Transitive

Asymmetric



no self loops

Symmetric

Total/Full
- looks like a path

②

If strict partial order → must be DAG

But every DAG is not a strict partial order

But positive partial order is

(Class problem)

Be able to understand what they mean + tell one from other

Massachusetts Institute of Technology
6.042J/18.062J, Spring '11: Mathematics for Computer Science — March 14
Prof. Albert R Meyer — revised Monday 14th March, 2011, 07:06

In-Class Problems Week 7, Mon.

Problem 1.
For each of the binary relations below, state whether it is a strict partial order, a weak partial order, or neither. If it is not a partial order, indicate which of the axioms for partial order it violates.

(a) The superset relation, $\supseteq$ on the power set $\mathcal{P}\{1, 2, 3, 4, 5\}$.

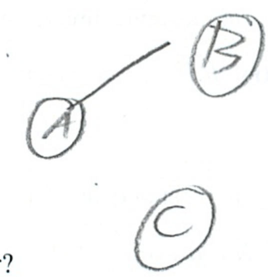
(b) The relation between any two nonegative integers, a, b that $a \equiv b \pmod{8}$.

(c) The relation between propositional formulas, G, H, that G IMPLIES H is valid.

(d) The relation 'beats' on Rock, Paper and Scissor (for those who don't know the game Rock, Paper, Scissors, Rock beats Scissors, Scissors beats Paper and Paper beats Rock).

(e) The empty relation on the set of real numbers.

(f) The identity relation on the set of integers.



Problem 2. **(a)** Why is every strict partial order a DAG?

(b) Give an example of a DAG that is not a strict partial order.

(c) Why is the positive path relation of a DAG a strict partial order?

Problem 3. **(a)** Verify that the divisibility relation on the set of nonnegative integers is a weak partial order.

(b) What about the divisibility relation on the set of integers?

Problem 4.
Consider the nonnegative numbers partially ordered by divisibility.

(a) Show that this partial order has a unique minimal element.

(b) Show that this partial order has a unique maximal element.

(c) Describe an infinite chain in this partial order.

(d) Describe an infinite antichain in this partial order.

(e) What are the minimal elements of divisibility on the integers greater than 1? What are the maximal elements?

Properties of a Relation $R : A \to A$ / Digraph G with $V(G) = A$

Reflexivity R is *reflexive* when

$$\forall x \in A.\ x\ R\ x.$$

Every node in G has a self-loop.

Irreflexivity R is *irreflexive* when

$$\text{NOT}\,\exists x \in A.\ x\ R\ x.$$

There are no self-loops in G.

Symmetry R is *symmetric* when

$$\forall x, y \in A.\ x\ R\ y \text{ IMPLIES } y\ R\ x.$$

If there is an edge from x to y in G, then there is an edge back from y to x in G as well.

Asymmetry R is *asymmetric* when

$$\forall x, y \in A.\ x\ R\ y \text{ IMPLIES NOT}(y\ R\ x).$$

There is at most one directed edge between any two nodes in G; there are no self-loops.

Antisymmetry R is *antisymmetric* when

$$\forall x \neq y \in A.\ x\ R\ y \text{ IMPLIES NOT}(y\ R\ x).$$

There is at most one directed edge between any two nodes; there may be self-loops.

Transitivity R is *transitive* if

$$\forall x, y, z \in A.\ (x\ R\ y \text{ AND } y\ R\ z) \text{ IMPLIES } x\ R\ z.$$

If there is a positive length path from u to v, then there is an edge from u to v.

Total R is *total* when

$$\forall x \neq y \in A.\ (x\ R\ y \text{ OR } y\ R\ x)$$

Given any two vertices in G, there is an edge in one direction or the other between them.

Strict Partial Order R is a *strict partial order* iff it is transitive and asymmetric iff it is transitive and irreflexive.

Weak Partial Order R is a *weak partial order* iff it is transitive and anti-symmetric and reflexive.

In Class Week 7 Mon

3/14

1. a) weak

b) none symetric

c) TA kinda weird – skip
   - neither – not antisymetric

Then H won't imply G
- implication goes both ways
  So thats why **not** antisymmetric

breaks: $(G \to H) \to \text{not}(H \to G)$

Prof: means $G \to H$

~~We~~ So that is ~~iff~~ $H \to G$ **not** =
(which I got wrong on previous quiz
– They are not ~~the~~ same)

For it to be anti sym $H = G$ – we know that is not true

See other sheet

d) violates transitivity
   - unless would not be fun game
   - one would always win
   - missing one axiom

e) empty – no arrows
   - irreflexive
   - is transative – pre condition never satisfied
   - asymetric " " "
   - strict

f) weak
   - same as above
   - but ^self loops

Only self loops

b) TA: Just cause symetric – does not mean weak – need to give counterexample

2. If you had cycle and is transitive could loop to self – violating transitivity and ~~asym~~ asymmetry

②

directed since binary relation
irreflexive + transitive - can't have cycle

1c cont A and (B or C) → (A and B) or (B and C)
(A and B) or (B and C) → A and (B or C)

But A and (B or C) ≠ (A and B) or (B and C)
So not antisymmetric

2a board) Transitive requires edge from one point
to every other vertex reachable from that pt.
If there is a cycle in strict partial order,
then there is a vertex that is reachable from itself
Transitivy would require an edge from the vertex
to itself which would violate irreflexivity.
There are no cycles in SPO.
So SPO must be DAG

2b

$A \to B \to C$

is aRb and bRc but <u>not</u> aRc

③

c) A DAG is by def. irreflexive + asym.
The pos path relation adds a ~~vertex~~ edge from each vertex to every downstream vertex, which guarentees transitivity.
∴ A pos path relation is transitive, asymetric + irreflexive so its SPO.

TA said something

~~TA use Relation instead of edge~~

Positive path relation = non zero lenght path

2c) The positive path relation is defined as uRv for verticies u, v which has a positive lenght path from u to v in the DAG
This is irreflexive + antisymmetric b/c the DAG has no cycles, so uRv → not vRu for all verticies u, v. It is transitive b/c if uRv and vRw then uRw b/c there would have to be a path from u to w in DAG.

(9)

3. transitive $a|b$ and $b|c \to a|c$

$c = bx$ $b = cy$

so $a = cyx \to a|c$

antisymmetric $a \neq b$ $a|b \to b \nmid a$

$a = bx \to b = \frac{1}{x} a$

$\frac{1}{x}$ is not an int b/c x must be int > 1 so $b \nmid a$

reflexive $a \mid a$

by def of divides

b) is not a WPO or SPO

violates antisymmetry

$1|-1$ and $-1|1$

$a|b$ means $\exists c$ $ac = b$

so $0|0$

5

4a) 1 $\forall n \in \mathbb{N}$ $1 | n$

Divisability is a WPO (see #3) so by antisymmetry $\forall n \neq 1$ not $(n | 1)$

by refexitivity $\forall n$ $(n | n)$

$\therefore$ every int has at least 1 and n as divisors

1 only has 1 as a divisor

b) 0, $\forall n \in \mathbb{N}$ $n | 0$

~~if there was a nonzero max element you could multiply it and get~~

c) $1 \to 2 \to 2^2 \to 2^3$ - - -

d) primes, no prime divides any other prime

e) primes are minimal for $n > 1$

no maximal el greater than 1

$\forall n \neq 0$ not $(n+1 | 1)$

Solutions to In-Class Problems Week 7, Mon.

Problem 1.
For each of the binary relations below, state whether it is a strict partial order, a weak partial order, or neither. If it is not a partial order, indicate which of the axioms for partial order it violates.

(a) The superset relation, $\supseteq$ on the power set $\mathcal{P}\{1, 2, 3, 4, 5\}$.

Solution. This is a weak partial order, but not a total one. For example, the sets of size 3 form an antichain. ■

(b) The relation between any two nonegative integers, a, b that $a \equiv b \pmod 8$.

Solution. Violates antisymmetry: $8\ R\ 16$ and $16\ R\ 8$ but $8 \neq 16$. It is transitive, though. ■

(c) The relation between propositional formulas, G, H, that G IMPLIES H is valid.

Solution. Violates antisymmetry: P and NOT(NOT(P)) imply each other but are different expressions. It is transitive, though. ■

(d) The relation 'beats' on Rock, Paper and Scissor (for those who don't know the game Rock, Paper, Scissors, Rock beats Scissors, Scissors beats Paper and Paper beats Rock).

Solution. Violates transitivity: obviously. Also violates antisymmetry. ■

(e) The empty relation on the set of real numbers.

Solution. It's vacuously asymmetric and transitive, so it's a strict partial order. It's irreflexive. It's not total. Every element is vacuously both minimal and maximal. ■

(f) The identity relation on the set of integers.

Solution. It's obviously reflexive, antisymmetric and transitive, so it's a weak partial order. It's not total. Every element is vacuously both minimal and maximal.

■

Problem 2. (a) Why is every strict partial order a DAG?

Solution. If a the strict partial was not a DAG, then it has a vertex v that is on positive length cycle. So there is a positive length path from v to v, which implies that v is related to itself n the partial order. This contradicts assymetry. ■

in

(b) Give an example of a DAG that is not a strict partial order.

Solution. $\langle 1 \to 2 \rangle$, $\langle 2 \to 3 \rangle$ but not $\langle 1 \to 3 \rangle$. ■

(c) Why is the positive path relation of a DAG a strict partial order?

Solution. In a DAG, there is no positive length path from a vertex to itself, so it's positive path relation is irreflexive. If there a a positive length path from u to v and another from v to w, thenn the merge of the paths goes from u to w, so the positive path relation is transitive. These two properties make it a strict partial order. ■

Problem 3. **(a)** Verify that the divisibility relation on the set of nonnegative integers is a weak partial order.

Solution. Divisibility is reflexive since $n \mid n$.

It is transitive by Lemma 8.1.3.1.

It is anti-symmetric since if $n \mid m$, then $n \le m$ for all positive integers m and nonnegative n. So if $n \mid m$ and $m \mid n$, then $m \le n$ and $n \le m$, that is, $n = m$. Also, if $n \mid 0$ then $n = 0$, which confirms anti-symmetry when $m = -0$.

■

(b) What about the divisibility relation on the set of integers?

Solution. Divisibility is not antisymmetric on the integers, since $n \mid -n$. ■

Problem 4.
Consider the nonnegative numbers partially ordered by divisibility.

(a) Show that this partial order has a unique minimal element.

Solution. 1 is minimal as there is no other natural number that divides 1. It is unique because all other numbers are divisible by 1 and therefore are not minimal. ■

(b) Show that this partial order has a unique maximal element.

Solution. 0 is maximal: all nonnegative integer divide zero. It is the only maximal element, because for every positive natural number, n, we have that n is strictly "smaller" than $2n$ under divisibility. ■

(c) Describe an infinite chain in this partial order.

Solution. 1 2 4 8 16 ... is a chain with infinite length. ■

(d) Describe an infinite antichain in this partial order.

Solution. The set of prime numbers is infinite. Since no prime divides another, any two primes are incomparable. So the set of prime numbers is an antichain. ■

(e) What are the minimal elements of divisibility on the integers greater than 1? What are the maximal elements?

Solution. The primes are the minimal elements. There are no maximal elements. ■

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Cheat Sheet 3

$a|b = [ak = b$ for some $k]$
└ or b/a = integer

$n = q\cdot d + r$ and $0 \le r < d$

$qcnt(n,d) \Rightarrow \frac{n}{d}$

$rem(n/d) \to \frac{n}{d}$

Euclid Algorithm

$gcd(a,b) = gcd(b, rem(a,b))$

Pulverizer

$gcd(a,b) = sa + tb \ \exists s,t$

$gcd(259, 70)$

		rem(x,y)	x − q·y
259	70	49	$= 259 - 3\cdot 70$
70	49	21	$= 70 - 1\cdot 49$
			$= 70 - 1(259 - 3\cdot 70)$
			$= -1\cdot 259 + 4\cdot 70$
49	21	7	$= 49 - 2\cdot 21$
			$= (259 - 3\cdot 70) - 2(-1\cdot 259 + 4\cdot 70)$
			$= \underset{s}{3}\cdot 259 - \underset{t}{11}\cdot 70$

Fund. Theorem of Algebra Every pos integer is a product of uniquly weakly decreasing seq of primes

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If $p|ab$ then $p|a$ or $p|b$
p = prime

$a \equiv b \pmod n$ iff $n | (a-b)$
iff $rem(a,n) = rem(b,n)$

Multiplicative Inverse of 7 mod 5

$7 \cdot \_ \equiv 1 \pmod 5$

- guess + check
- Pulverizer

$sp = 1 - tk$

$p | (1 - tk) \to$ so $tk \equiv 1 \pmod p$

$r = x - qy$ aka $gcd(\underset{k}{5}, \underset{p}{3})$

any thing multiple of gcd $\to r = ax + by$

take b result

Fermat's Little Theorem

$k^{p-1} \equiv 1 \pmod p$

$k^{p-2}\cdot k \equiv 1 \pmod p$

So find $rem(3^3, 5)$ $k=3$, $p=5$

Euler's Theorem ↑ generalization

$\phi(p) = p - 1$

$k^{\phi(n)} \equiv 1 \pmod n$

$k^{\phi(n)-1} \equiv$ mul. inverse k mod n

So $3^{\phi(5)-1}$

Eulers Theorem

$\phi(pq) = (p-1)(q-1)$ ↑ primes

$\phi(p^k) = p^k - p^{k-1}$

$\phi(ab) = \phi(a)\phi(b)$ ↑ rel prime

$\phi(300) = \phi(2^2\cdot 3\cdot 5^2)$
$= \phi(2^2)\cdot\phi(3)\cdot\phi(5^2)$
$= (2^2 - 2^1)(3^1 - 3^0)(5^2 - 5^1)$
$= 80$

= # of rel prime

Rel Prime for a,b if $gcd(a,b) = 1$
For Primes all except 1

RSA

Before 1. Generate 2 primes p q
2. $n = pq$
3. Select e from $gcd(e, (p-1)(q-1)) = 1$
   - solve for e, guess + check
   - Smallest prime that does divide

   public (e,n)
4. Compute d = inverse e mod$(p-1)(q-1)$

   secret (d,n)

3/15

Encoding Check $gcd(m,n) = 1$

$m^* = rem(m^e, n)$ using other parties key

Decoding Use your private key

$m = rem((m^*)^d, n)$

Diagraph = directed graph (have arrow)

dots = nodes/vertices

lines = directed edge/arrow

$e = \langle u \to v \rangle$

indegree = # arrows in
$|\{e \in E(G) \mid head(e) = v\}|$

out degree = # arrows out
$|\{e \in E(G) \mid tail(e) = v\}|$

two sets = vertices and edges

$|v|$ = lenght

$f\,\hat{v}\,r$ = merging walk at v

walk = repeat vertices

path = no repeats
- Shortest walk b/w a pair of vertices is path

$dist(u,v)$ = shortest path

Adj. Matrix

$(A_G)_{ij} = \begin{cases} 1 & \text{if } \langle v_i \to v_j \rangle \in V(G) \\ 0 & \text{otherwise} \end{cases}$

$u\,G^*\,v$ = path relation

$u\,G^+\,v$ = pos. path relation

Reflexity every node in G has self-loop
$\forall x \in A,\ xRx$

Irreflexity no self-loops in G
NOT $\exists x \in A,\ xRx$

Symmetry $\forall x,y \in A.\ xRy \to yRx$
if edge $x \to y$ also $y \to x$

Asymmetry at most one edge everywhere no self loops
$\forall x,y \in A.\ xRy \to Not(yRx)$

Antisymmetry at most one edge, can be self loops
$\forall x \ne y \in A\ xRy \to Not(yRx)$

Transitive if pos path $u \to v$ then $v \to u$
$\forall x,y,z \in A\ (xRy \text{ and } yRz) \to xRz$

Total Given any 2 vertices in G there is an edge in 1 dir or other b/w them
$\forall x \neq y \in A.(xRy \text{ or } yRx)$

SPO transitive and asymmetric
iff transitive and irreflexive
positive path relation of DAG

WPO transitive and anti-sym and reflexive
aRb iff ($a \underset{\text{strict}}{S} b$ or $a=b$)

Closed walk - digraph starts + ends same

cycle - closed walk, but other pts distinct

DAG - directed graph w/ no pos len cycles

$\preceq$ partial order

$\sqsubseteq$ total order

antichain - incomparable

critical path = length of longest chain

depth = size of critical path

If largest chain is size of t, can partition into t antichains

For all $t > 0$ every partially ordered set must have chain of size greater than t or antichain of size $\frac{n}{t}$

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gcd - largest common seq of two factorization, take all primes that appear in both factorization raised to the min power of each respective prime

lcm - max instead

Chinese Remainder Theorem
for all m, n $\exists x$ such that
$x \equiv m \bmod a$
$x \equiv n \bmod b$
$x' \equiv x \bmod ab$

proof $e_a = b^{-1}b$ $e_b = a^{-1}a$ $x = me_a + ne_b$
TBA

Another way to exponentiate:
split up
$13^{21} = 13^{16} \cdot 13^{4} \cdot 13$
$\hookrightarrow 2^2 \bmod 23$ etc

Verifying Preserved Invariant - once it enters it stays - show value does not change

---

Find rem $26^{1818181}$ $\quad 1818181 = (180 \cdot 10101) + 1$

$\phi(297) = \phi(3^3 \cdot 11) = \phi(3^3) \cdot \phi(11) = (3^3 - 3^2)(11-1)$
$= 180$

$26^{1818181} = 26^1 \cdot 26^{180 \cdot 10101}$ $\quad 26^{180} \equiv 1 \bmod 297$
$= 26 \cdot 1^{10101}$
$= 26$

Midterm 3

3/15

I know the material the most of any unit
- studied befor
- interested
- But can I do the proofs?

For path just copying def.
- ? Theroms too

Need to review problems
And spent a lot of time on last P-set

Mini-Quiz Mar. 16

Your name: Plasmeier

Circle the name of your TA and write your table number:

Ali   Nick   Oscar   Oshani   **Table number** 12

- This quiz is **closed book**. Total time is 30 minutes.
- Write your solutions in the space provided. If you need more space, write on the back of the sheet containing the problem. Please keep your entire answer to a problem on that problem's page.
- GOOD LUCK!

DO NOT WRITE BELOW THIS LINE

Problem	Points	Grade	Grader
1	5	3	NJ
2	5	1	AK
3	5	4.5	OTh
4	5	5	OS
Total	20	13.5	OS

3/5

Problem 1 (5 points). (a) Calculate the value of $\phi(100)$.

$$\phi(100) = \phi(2^2 \cdot 5^2)$$
$$= \phi(2^2)\cdot\phi(5^2)$$
$$= (2^2-2^1)(5^2-5^1)$$
$$= (4-2)(25-5)$$
$$= 2\cdot 20$$
$$= 40$$

24
3
72

(b) Assume an integer $k > 9$ is relatively prime to 100. Explain why the last two digits of k and k^{121} are the same.

13

Hint: Use your solution to part (a).

Since $k^{\phi(100)} \equiv 1 \bmod 100$

So since $k^{\phi(100)}$ is congruent to 1 mod 100, the last two digits (which is the remainder mod 100) will be the same

no, want last 2 digits of k^{121} and k^1 same.

$k^{121} \equiv k \bmod 100$

~~$k^{\phi(n)-1}$ is the mul. inverse of k mod n~~

-2

k is rel. prime because gcd $(k, 100) = 1$

$a \equiv b \pmod n$ iff $n \mid (a-b)$ iff $\text{rem}(a,n) = \text{rem}(b,n)$

$k^{40} \equiv 1 \bmod 100$ the remainder, subtract out

$121-1$ is a multiple of 40

So when take the power of – it will be the same mod 100

Problem 2 (5 points).
Prove that if $a \equiv b \pmod{14}$ and $a \equiv b \pmod 5$, then $a \equiv b \pmod{70}$.

~~Since $a \equiv b \pmod n$ iff $n | (a-b)$~~

~~$p|ab$ iff $p|a$ or $p|b$~~
prime prime

$\frac{14 \cdot 5}{70}$

Assign $x = 14$
$y = 5$
Notice $xy = 70$

You've got this backward.

Now note that since $\boxed{70 \text{ is a factor of both } 14 \text{ and } 5,}$ values that are congruent mod 14 and mod 5 will also be congruent mod 70. Why?

1

How to word?

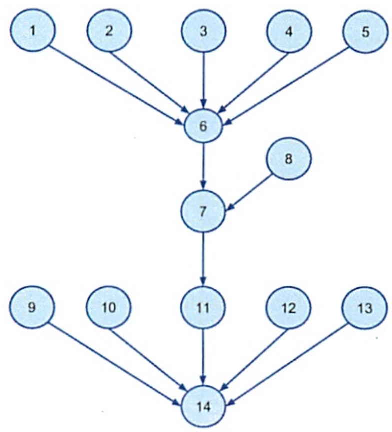


Figure 1 Task DAG

Problem 3 (5 points).
Answer the following questions about the dependency DAG shown in figure 1. Assume each node is a task that takes 1 second.

1. What is the largest chain in this DAG, if there is more than one, only show one.

   3 → 6 → 7 → 11 → 14.

2. What is the largest antichain? (again, pick one if you find there is more than one).

   9 10 11 12 13 — 1/2

   all one big chain

3. How much time would be required to complete all the tasks with a single processor.

   14

4. How much time would be required to complete all the tasks if there are unlimited processors available.

   5

5. What is the smallest number of processors that would still allow to complete all the tasks in optimal time. Show a schedule proving it.

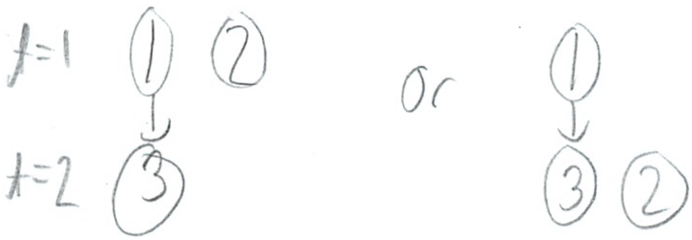
t=1 1 2 3 4 5
t=2 6 8
3 7
4 9 10 11 12 13
5 14

5 processors.

We need to complete 1,2,3,4,5 before moving on to 6, so in order to get all 5 done in one time step, like we did in optimal solution, we need 5 processors

Problem 4 (5 points).
What is the smallest number of partially ordered tasks for which there can be more than one minimum time schedule, if there are unlimited number of processors? Explain your answer.

3 tasks



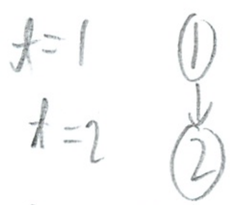
It can not work for 0 tasks
- only 1 schedule: do nothing

It can not work for 1 task
- only one schedule: do task!

t=1 (1)

It can not work for 2 tasks

t=1 (1) (2) since just do both time 1



since just do time 1 and 2

also

same idea

And it can work for 3 tasks, an example is above

Solutions to Mini-Quiz Mar. 16

Problem 1 (5 points). (a) Calculate the value of $\phi(100)$.

Solution.

$$\phi(100) = \phi(25)\phi(4) = \phi(5^2)\phi(2^2) = (5^2 - 5)(2^2 - 2) = 40.$$

■

(b) Assume an integer $k > 9$ is relatively prime to 100. Explain why the last two digits of k and k^{121} are the same.

Hint: Use your solution to part (a).

Solution. Notice that all we have to prove is that k and k^{121} are congruent mod 100, implying they have the same last two digits.

$$k^{121} \equiv k^{40\cdot3+1} \equiv k(k^{40})^3 \pmod{100}.$$

By Euler's Theorem, since k and 100 are relatively prime, $k^{\phi(100)} \equiv 1 \pmod{100}$. By part (a), we have that $\phi(100) = 40$, implying $k^{40} \equiv 1 \pmod{100}$. Hence, $k(k^{40})^3 \equiv k(1^3) \equiv k \pmod{100}$. ■

Problem 2 (5 points).
Prove that if $a \equiv b \pmod{14}$ and $a \equiv b \pmod{5}$, then $a \equiv b \pmod{70}$.

Solution. We know $a \equiv b \pmod{14}$ means $14 | a - b$. Likewise, $a \equiv b \pmod{5}$ means $5 | a - b$. Also 14 and 5 are relatively prime.

For any p, q and x, if $p | x$ and $q | x$ and p and q are relatively prime, we know from class that $pq | x$. So, applying that reasoning with $x = a - b$, $p = 14$ and $q = 5$ yields $70 | a - b$, which is what we were trying to prove. ■

Problem 3 (5 points).
Answer the following questions about the dependency DAG shown in figure 1. Assume each node is a task that takes 1 second.

1. What is the largest chain in this DAG, if there is more than one, only show one.
2. What is the largest antichain? (again, pick one if you find there is more than one).
3. How much time would be required to complete all the tasks with a single processor.
4. How much time would be required to complete all the tasks if there are unlimited processors available.
5. What is the smallest number of processors that would still allow to complete all the tasks in optimal time. Show a schedule proving it.

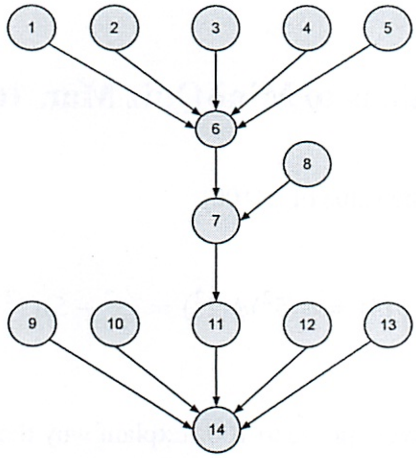


Figure 1 Task DAG

Solution.
1. One largest chain is $\{1, 6, 7, 11, 14\}$
2. One largest antichain is $\{1, 2, 3, 4, 5, 8, 9, 10, 12, 13\}$
3. There are 14 nodes, so a single processor would take 14 seconds.
4. With unlimited processors, we can take 5 seconds. This is the length of the longest chain.
5. With 5 processors, we can still finish everyting in 5 seconds. A schedule showing this is $\{1, 2, 3, 4, 5\}$, $\{6, 8\}$, $\{7\}$, $\{9, 10, 11, 12, 13\}$, $\{14\}$. We cannot do this with less than 5 processors because in order to make progress on the longest chain at every time step, we need to process $\{1, 2, 3, 4, 5\}$ in step 1.

■

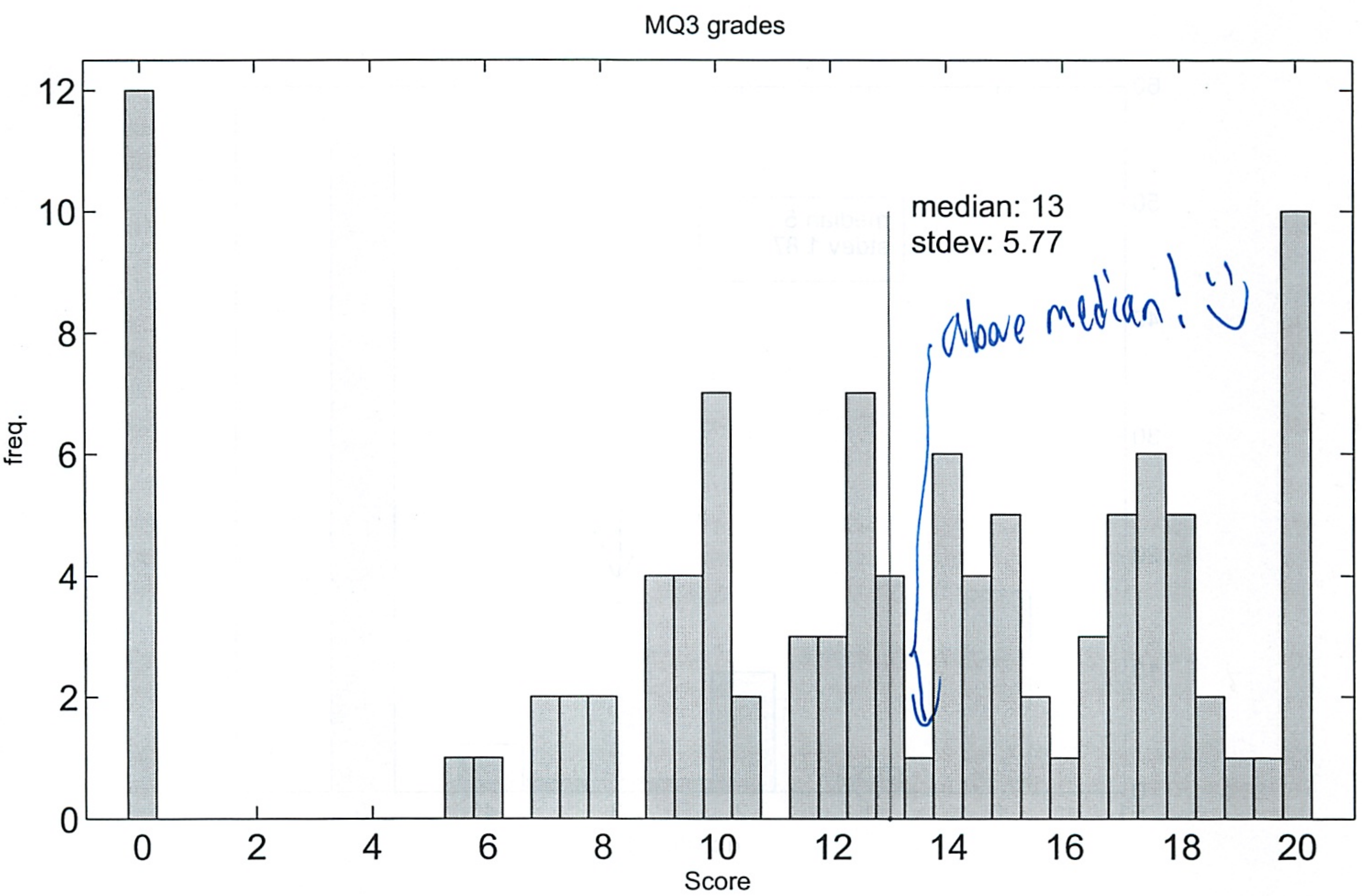
Problem 4 (5 points).
What is the smallest number of partially ordered tasks for which there can be more than one minimum time schedule, if there are unlimited number of processors? Explain your answer.

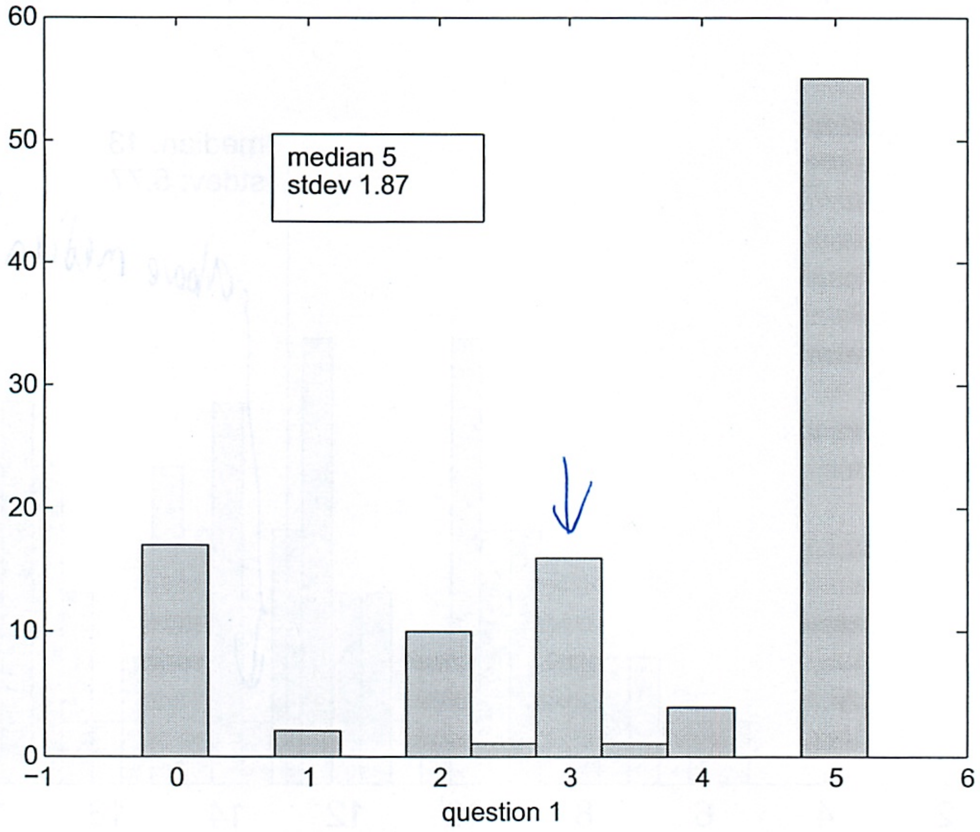
Solution. Three tasks.

With one task, there is only one possible schedule. Two tasks that are incomparable can both be completed in one step, and this is the unique minimum step schedule. For two tasks that are comparable, there is only one possible schedule, which therefore is the unique minimum time schedule.

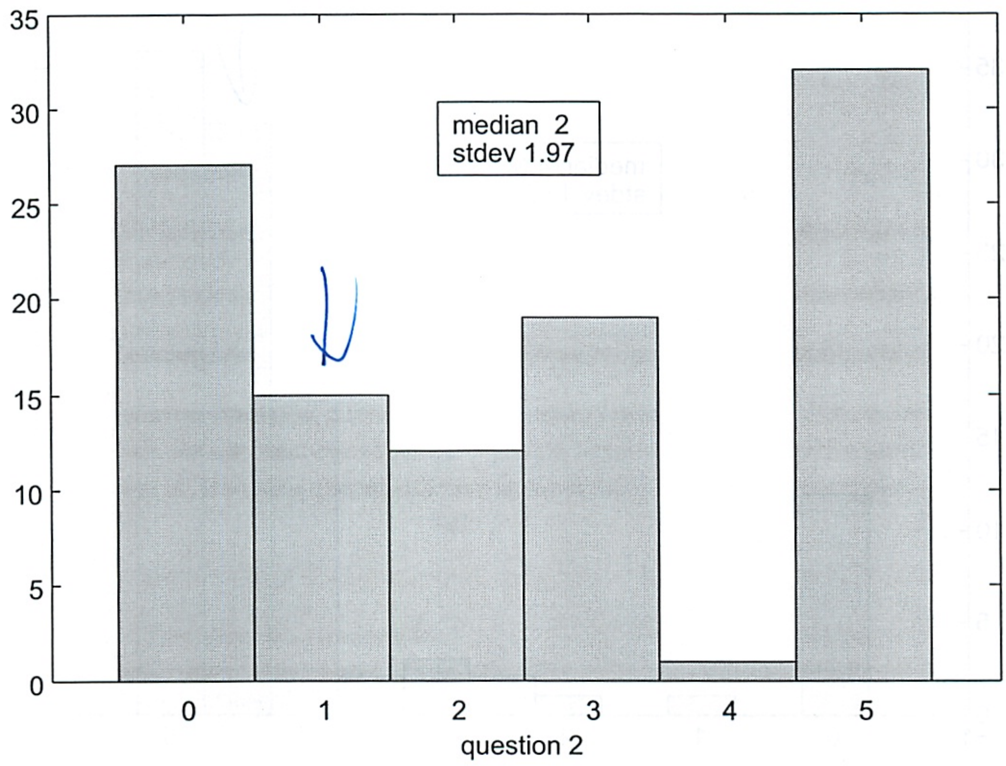
For an example with three tasks with two minimum time schedules, let two of the tasks be comparable and the third task incomparable to the other two. The two comparable tasks have a unique minimum time schedule that takes two steps. So any schedule for the three tasks that also takes only two steps will certainly be minimum time for the three. But the third task can be scheduled at the same time as either the first or the second of the comparable tasks, giving two minimum schedules for the three tasks.

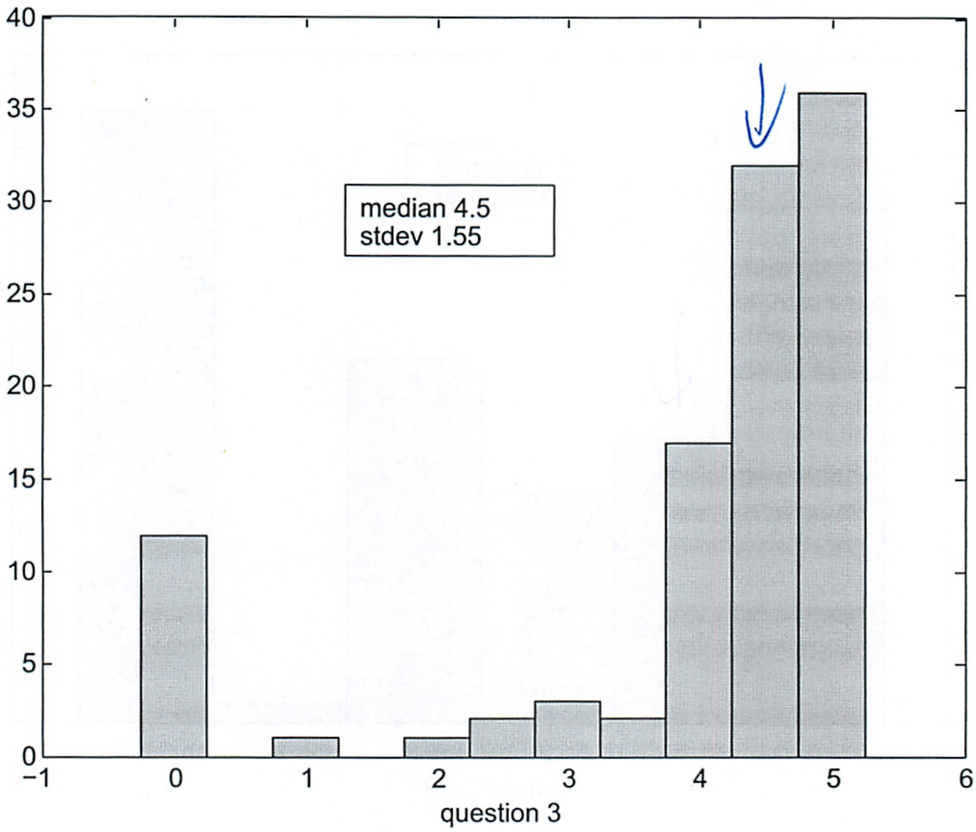
■


MQ3 grades
median: 13
stdev: 5.77
Above median! :)
freq.
Score
0
2
4
6
8
10
12
14
16
18
20

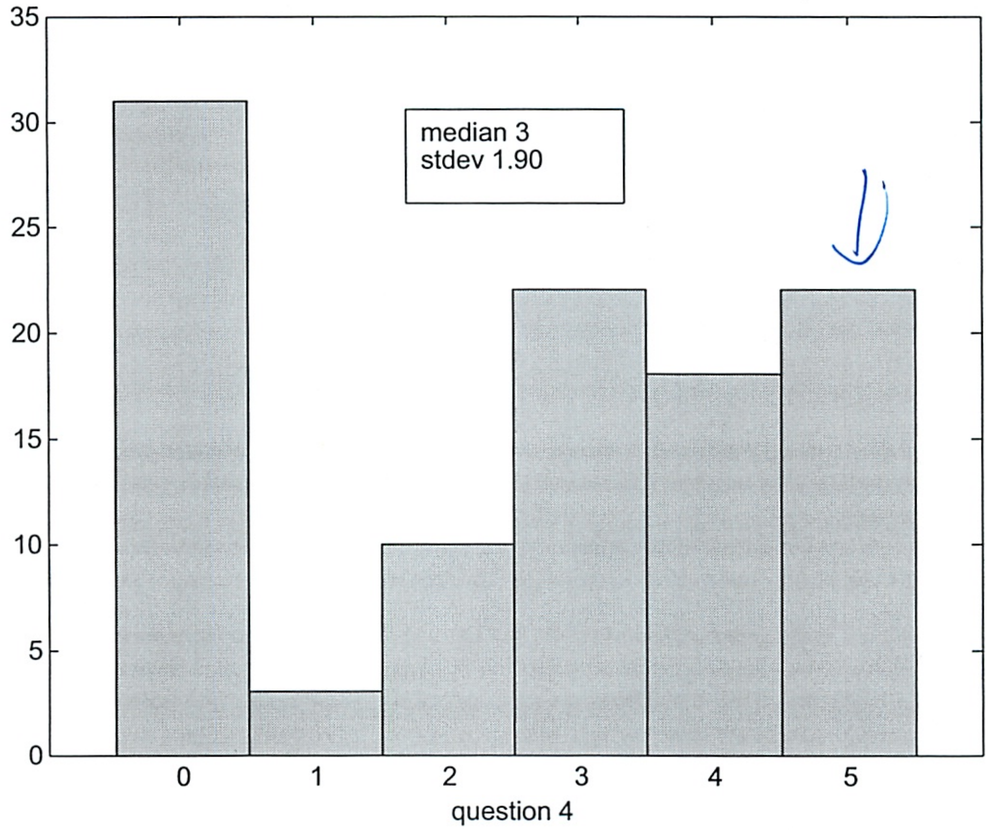


median 5
stdev 1.87
question 1


median 2
stdev 1.97
35
30
25
20
15
10
5
0
0
1
2
3
4
5
question 2



median 4.5
stdev 1.55
40
35
30
25
20
15
10
5
0
−1
0
1
2
3
4
5
6
question 3

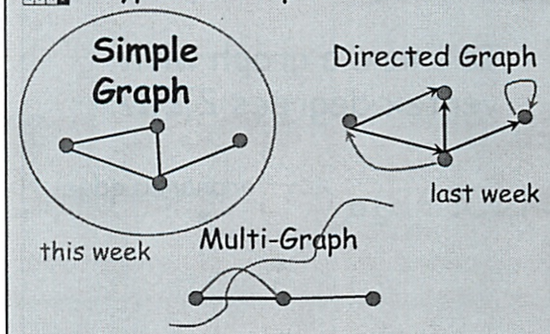

35
30
25
20
15
10
5
0
median 3
stdev 1.90
0
1
2
3
4
5
question 4

3/16

Mathematics for Computer Science
MIT 6.042J/18.062J

Simple Graphs Degrees

Types of Graphs

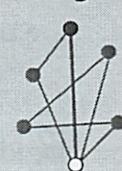


A simple graph:

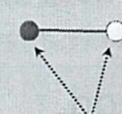
Definition:
A simple graph *G* consists of
- a nonempty set, V, of vertices, and
- a set, E, of edges such that each edge has two endpoints in V

A Simple Graph

edge



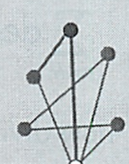
vertices, V
undirected edges, E



::= {●,○}

Vertex degree

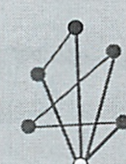
degree of a vertex is # of incident edges



deg(●) = 2

Vertex degree

degree of a vertex is # of incident edges

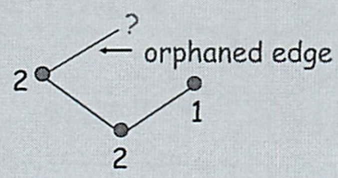


deg(○) = 4

Impossible Graph

Is there a graph with vertex degrees 2,2,1?

NO!



Handshaking Lemma

sum of degrees is twice # edges

$$\boxed{2|E| = \sum_{v \in V} \deg(v)}$$

Proof: Each edge contributes 2 to the sum on the right

Handshaking Lemma

sum of degrees is twice # edges

$$\boxed{2|E| = \sum_{v \in V} \deg(v)}$$

2+2+1 = odd, so impossible

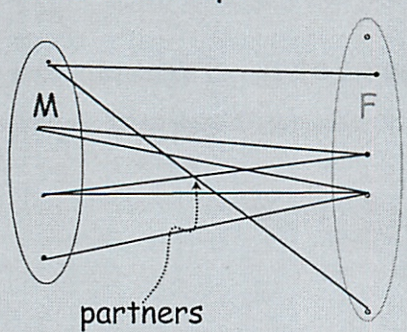
Sex in America: Men more Promiscuous?

Study claims:

Men average many more partners than women.

Graph theory shows this is nonsense

Sex Partner Graph



Counting pairs of partners

$$\sum_{m \in M} \deg(m) = |E| = \sum_{f \in F} \deg(f)$$

now divide by both sides by |M|

$$\underbrace{\frac{\sum_{m \in M} \deg(m)}{|M|}}_{\text{avg-deg}(M)} = \frac{|F|}{|M|} \cdot \underbrace{\frac{\sum_{f \in F} \deg(f)}{|F|}}_{\text{avg-deg}(F)}$$

Average number of partners

$$\text{avg-deg}(M) = 1.035 \cdot \text{avg-deg}(F)$$

Averages differ solely by ratio of females to males.
No big difference
Nothing to do with promiscuity

Team Problems

Problems 1 & 2

6.042 Simple Graphs 3/16

Degrees

(Miniquiz 3)

technical meaning: no arrows

Can also have multigraphs - we don't cover

non empty set of verticies + edges

each edge has 2 endpoints

"undirected"

no self loops by convention in simple graphs

 ∷ {•, o}

if 2 verticies have an edge = <u>adjecent</u>

edge is <u>incident</u> to its end points

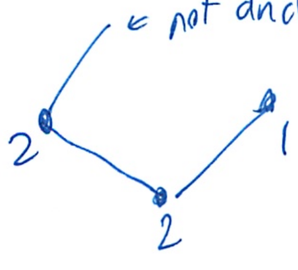
degree of vertex = <u># of incident edges</u>

- aka # arrows in/out

but no distinction b/w in/out

Is there a graph w/ vertex degrees 2, 2, 1?

No ← not anchored → illegal



②

<u>Handshaking Lemma</u> – sum of degrees is twice # edges

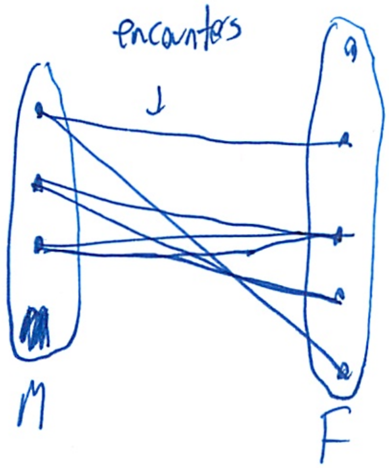
$$2|E| = \sum_{v \in E} \deg(v)$$

so sum of degrees must be even

– not every ~~even~~ even sum can be done

Sex Study: men have many more partners than womens

– hetrosexual, one on one only



Counting pairs of partners

$$\sum_{m \in M} \deg(m) = |E| = \sum \deg(f)$$

must be equal! (invarient)

divide both side by |M| to find # of partners

$$\underbrace{\sum_{m \in M} \deg(m)}_{\text{avg for men}} = \frac{|F|}{|M|} \cdot \underbrace{\frac{\sum \deg(f)}{|F|}}_{\text{avg partners for men}}$$

↑ so cancles

3

Only has to do w/ ratio of men vs females

* Averages only differ by ratio of females to males

In-Class Problems Week 7, Wed.

Problem 1.
A researcher analyzing data on heterosexual sexual behavior in a group of m males and f females found that within the group, the male average number of female partners was 10% larger that the female average number of male partners.

(a) Comment on the following claim. "Since we're assuming that each encounter involves one man and one woman, the average numbers should be the same, so the males must be exaggerating."

(b) For what constant c is $m = c \cdot f$?

(c) The data shows that approximately 20% of the females were virgins, while only 5% of the males were. The researcher wonders how excluding virgins from the population would change the averages. If he knew graph theory, the researcher would realize that the nonvirgin male average number of partners will be $x(f/m)$ times the nonvirgin female average number of partners. What is x?

(d) For purposes of further researach, it would be helpful to pair each female in the group with a unique male in the group. Explain why this is not possible.

Problem 2. **(a)** Prove that in every graph, there are an even number of vertices of odd degree.

Hint: The Handshaking Lemma 11.2.1.

(b) Conclude that at a party where some people shake hands, the number of people who shake hands an odd number of times is an even number.

(c) Call a sequence of two or more different people at the party a *handshake sequence* if, except for the last person, each person in the sequence has shaken hands with the next person in the sequence.

Suppose George was at the party and has shaken hands with an odd number of people. Explain why, starting with George, there must be a handshake sequence ending with a different person who has shaken an odd number of hands.

Hint: Just look at the people at the ends of handshake sequences that start with George.

In-Class 7 Wed

3/16

1. a. This argument was made in class

more females then males means males more promiscuous slightly

↑ a diff proportion not exactly more

b. $m = c \cdot f$

just $c = \frac{m}{f} = 1.1$ - the ratio of males to females

c. Exclude virgins

Would it change average

$X \left(\frac{f}{m}\right)$

or something about ratio of virgin-ness

or not related at all - if trick

- but how could that work?

$\frac{.8}{.95} \approx .8421$

want it to be close to 1

d. 1.1 m

So there are more women than men

So there would need to be some three-somes

②

Notation	
$\therefore$	therefore
$\because$	because

20. Explained in lecture as well.

Sum of the degrees of the vertices must be even. The sum of any two odd # is even So every odd-degree vertex must be paired with another for the total to be even

$\therefore$ ~~All~~ There must be an even # of odd degree vertices

b) Yeah b/c pairing?

Induction?

Write out a bit more

Handshakes can be considered eges in a graph where each person is a vertex. The degree of each Vertex is the # of ~~[illegible]~~ hands the corresponding person has shaken

So by part a, there must be an even # of people who have shaken hands w/ an odd #

Yeah so reflect it back to graph theory

c) A handshake seq beginning at an odd-degree vertex can loop back to the starting vertex, but each loop req two edges on the start vertex.

(3)

~~The last edge~~ can not return to the ~~shart~~ start vertex. False

Only for handshake seq of some kind

Any even-order vertex reached on this seq must have another edge leaving the vertex

- these edges can be followed untill an odd-degree vertex is reached (guarenteed by part a)

Correct - but not well written

Solutions to In-Class Problems Week 7, Wed.

Problem 1.
A researcher analyzing data on heterosexual sexual behavior in a group of m males and f females found that within the group, the male average number of female partners was 10% larger that the female average number of male partners.

(a) Comment on the following claim. "Since we're assuming that each encounter involves one man and one woman, the average numbers should be the same, so the males must be exaggerating."

Solution. The averages won't be the same. According to equation (11.1),

$$\text{Avg. \# male partners} = \frac{|F|}{|M|} \cdot \text{Avg. \# female partners} \tag{1}$$

So the averages simply reflect the relative sizes of the male and female populations. This means that the males could truthfully report a higher average if there where more females.

Of course if the males exaggerate, then their reported average could be as large as they choose to fantasize, whatever the size of the female population. ■

(b) For what constant c is $m = c \cdot f$?

Solution. By equation (1), the men's average number of partners is f/m times the female's average, so $f/m = 1.1$ which implies $m = (1/1.1)f$ and $c = 10/11$. ■

(c) The data shows that approximately 20% of the females were virgins, while only 5% of the males were. The researcher wonders how excluding virgins from the population would change the averages. If he knew graph theory, the researcher would realize that the nonvirgin male average number of partners will be $x(f/m)$ times the nonvirgin female average number of partners. What is x?

Solution. The male average number of partners is f/m times the female average number of partners. (According to part (b), $f/m = 1.1$, but this number isn't needed here.) When virgins are excluded, the ratio of the male's average to the females' average will be

$$\frac{f - .2f}{m - .05m} = \frac{.8f}{.95m} = \frac{4/5}{19/20} \cdot \frac{f}{m},$$

so $x = 80/95 = 16/19$. ■

(d) For purposes of further researach, it would be helpful to pair each female in the group with a unique male in the group. Explain why this is not possible.

Solution. There are more females than males, so there cannot be an injective function from the females to the males. ■

Problem 2. **(a)** Prove that in every graph, there are an even number of vertices of odd degree.

Hint: The Handshaking Lemma 11.2.1.

Solution. *Proof.* Partitioning the vertices into those of even degree and those of odd degree, we know

$$\sum_{v \in V} d(v) = \sum_{d(v) \text{ is even}} d(v) + \sum_{d(v) \text{ is odd}} d(v)$$

By the Handshaking Lemma, the value of the lefthand side of this equation equals twice the number of edges, and so is even. The first summand on the righthand side is even since it is a sum of even values. So the second summand on the righthand side must also be even. But since it is entirely a sum of odd values, it must must contain an even number of terms. That is, there must be an even number of vertices with odd degree. ■

(b) Conclude that at a party where some people shake hands, the number of people who shake hands an odd number of times is an even number.

Solution. We can represent the people at the party by the vertices of a graph. If two people shake hands, then there is an edge between the corresponding vertices. So the degree of a vertex is the number of handshakes the corresponding person performed. The result in the first part of this problem now implies that there are an even number of odd-degree vertices, which translates into an even number of people who shook an odd number of hands. ■

(c) Call a sequence of two or more different people at the party a *handshake sequence* if, except for the last person, each person in the sequence has shaken hands with the next person in the sequence.

Suppose George was at the party and has shaken hands with an odd number of people. Explain why, starting with George, there must be a handshake sequence ending with a different person who has shaken an odd number of hands.

Hint: Just look at the people at the ends of handshake sequences that start with George.

Solution. The handshake graph between just the people at the ends of handshake sequences that start with George is a graph, so by part (b), it must have an even number of people who shake an odd number of hands. In particular, there must be at least one other person besides George, call him Harry, who has also shaken an odd number of hands. So the handshake sequence from George that ends with Harry is what we were looking for. ■

6.042 Tutor 7 3/17

TP.7.1 Edges + Degrees

How many edges if degree of vertices are

4, 3, 3, 2, 2

Sum = 14

~~Ab~~ Take Half = 7

TP 7.2 Matching

A. A Perfect Matching

- if covers V(G)

not technical def, too bad

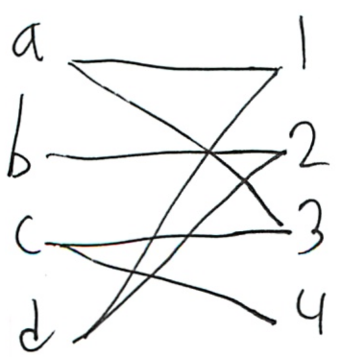
- regulare matching = vertex only has 1 edge
- Cover = every vertex has ~~only~~ at least 1 edge in the set
- Perfect = " covers for S of ~~ver~~ all verticles

~~All vertices conn~~

the minimum size edge cover

Still confused, emailed in

Find the perfect ~~edge~~ matching



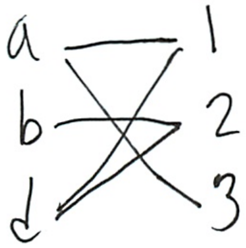
So find a Graph
- a selection of edges so true?

wp: lots of rules about finding

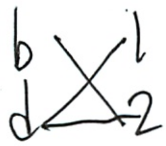
② Try to find

4 must go to c
since only line

then cross them out



next 3 - a



b - 2

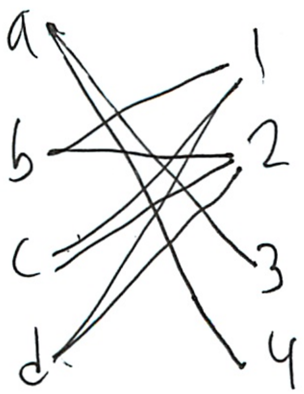
d - 1

I found the algorithm!

Part 2

→ No Perfect Matching

Which property makes perfect matching impossible?



(3)

Well let me try

oh a goes to 3,4

But 7 possible ans

#1 False

#2 ? Don't think

#3 True

#4 8 edges - must be deg(l) ≥ deg(r)

? Can't you flip - so must be = ?

8 8

well = - not what meant

but a smaller subset

not regular

will say no

#5. True

#6. Perhaps - when do w/ subsets say True

#7 True - wrong if flip

3 5 6 7 ✗ Can you even do this?

3 5 7 ✗

3 5 ✓

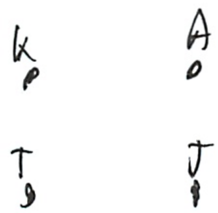
④

TP.7.3 Stable Marrage Invarients

What are invarient?

(Always false is invariant)

#1



Well first part may not be true — we don't have the graph

~~But if graph says~~

No I see. True

#2. T → J

Yes, it above

#3 Not in beginning

#4 Never

#5 No one can move down

1	2	3	4
A	B	A	A
B	A	B	B
C	C	C	C
D	D	D	D
A pref	B pref	C pref	D pref

#5

#6 True - he prefed her - but ~~ca~~not possible now

So is that true?

#7. True - if above

#8. No set not true Oh did not read whole qu

1 2 6 7 X

Tried a bunch ... Give up

1 ~~2~~ 4 6 7 8

✓ ? ? ✓ ✓ ?

not 2

~~M~~ Long reasoning ...

(6)

TP 7.4 Graph Isomorphism

How many bij are there

- I was always bad w/ these!

1?; (X)

No it means how many diff sets of arrows

Lables do not matter

$\cancel{(N^2 - \frac{N}{2})}$; $\frac{N^2}{2} - \frac{N}{2}$ $\quad \frac{N^2 - N}{2}$

Say for 3

A. B. .C

A - B
A - C
B - C

	A	B	C
A		X	X
B			X
C			

$\frac{9-3}{2} = 3$

4

A. B. C. D.

A - B
A - C
A - D
B - C
B - D
C - D

	A	B	C	D
A		X	X	X
B			X	X
C				X
D				

$\frac{16-4}{2} = 6$

So for 5

$\frac{5^2-5}{2} = \frac{20}{2} = 10$ ✓ (Pool)

7 Part 2 Non Iso Graphs

What sentances prove that not iso

- So must always ans comparing 2 graphs

#1. ~~Three~~ Matters + True

#2 Does not matter

#3 Matters + True

#4 Who cares labling

#5 " drawling

#6 Matters + ~~True~~

↑ did not look fully - got lazy

1 3 6 ✗

1 3 ✓

TP 7.5 BiPartite Graphs

Which are biparte

a ✓

b x ✓

c x

easy!

TP 7.6 Connected Components

G = graph w/ vertices ~~are~~ are the integers

w/ edges b/w i-j iff $|i-j| = 6$

↑ size or abs?

So 7 − 1
8 − 2
9 − 3
10 − 4

i connected components
− not edges

Oh continue, I see

11 − 5
12 − 6
13 − 7 ← repeat
14 − 8
15 − 9

Oh

13 — 7 — 1

14 — 8 — 2
15 — 9 — 3

Won't loop

6 ✓

P. 7.7 Graph Coloring

— future reading

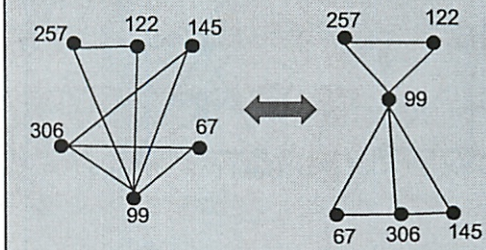
3/19

Mathematics for Computer Science
MIT 6.042J/18.062J

Simple Graphs
Isomorphism
Stable Marriage

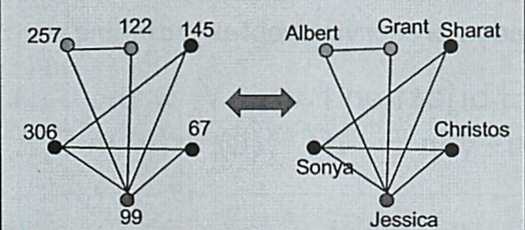
The Graph Abstraction

Same graph, different layouts



The Graph Abstraction

Same layout, different vertices



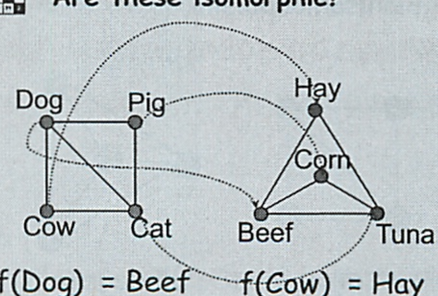
The Graph Abstraction

All that matters are the connections: graphs with the same connections are isomorphic

Isomorphism

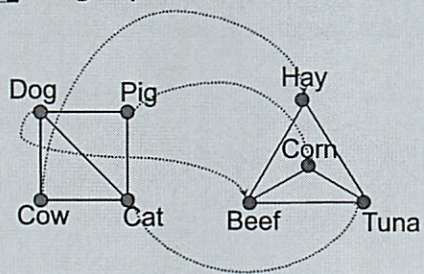
two graphs are isomorphic when there is an edge-preserving bijection between their vertices.

Are these isomorphic?



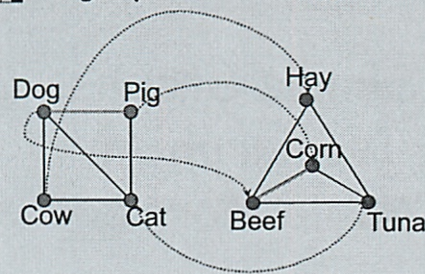
f(Dog) = Beef   f(Cow) = Hay
f(Cat) = Tuna   f(Pig) = Corn

Edges preserved?



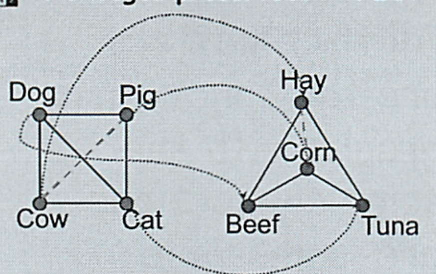
Albert R Meyer March 18, 2011 lec 7F.14

Edges preserved? YES!



Albert R Meyer March 18, 2011 lec 7F.15

Nonedges preserved? YES!



isomorphic!

Albert R Meyer March 18, 2011 lec 7F.17

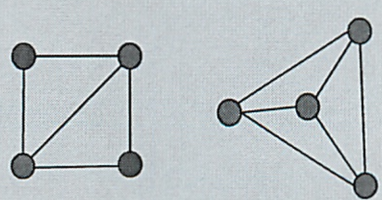
Formal Def of Graph Isomorphism

G_1 isomorphic to G_2 means edge-preserving vertex matching:

$\exists$ bijection $f: V_1 \to V_2$ with
$u—v$ in E_1 iff $f(u)—f(v)$ in E_2

Albert R Meyer March 18, 2011 lec 7F.18

Nonisomorphism



degree 2   all degree 3

Albert R Meyer March 18, 2011 lec 7F.19

Proving nonisomorphism

If some property preserved by isomorphism differs for two graphs, then they're not isomorphic:

- # of nodes,
- # of edges,
- degree distributions, ...

Albert R Meyer, March 18, 2011 lec 7F.20

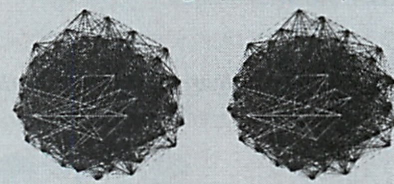
Finding an isomorphism?

many possible mappings: large search
can use properties *preserved* by isomorphisms as a guide, for example:

- a deg 4 vertex adjacent to a deg 3 can only match with
- a deg 4 vertex also adjacent to a deg 3

but even so...

Are these two graphs isomorphic?



...nothing known is *sure* to be much faster than searching thru all bijections for an isomorphism

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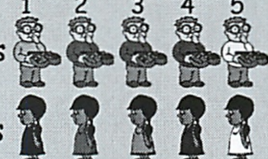
Stable Matching

Stable Marriage

A Marriage Problem

1 2 3 4 5

Boys



Girls

A B C D E

Stable Marriage

Preferences

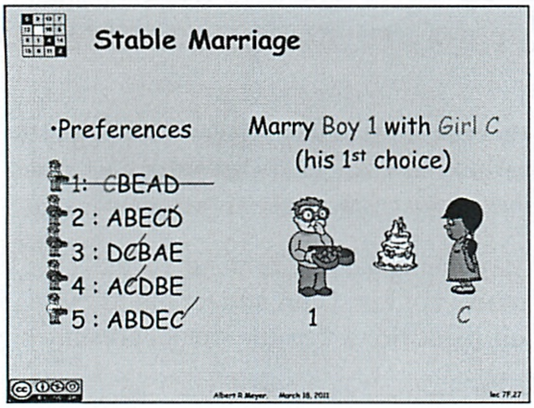
Boys	Girls
1 : CBEAD	A : 35214
2 : ABECD	B : 52143
3 : DCBAE	C : 43512
4 : ACDBE	D : 12345
5 : ABDEC	E : 23415

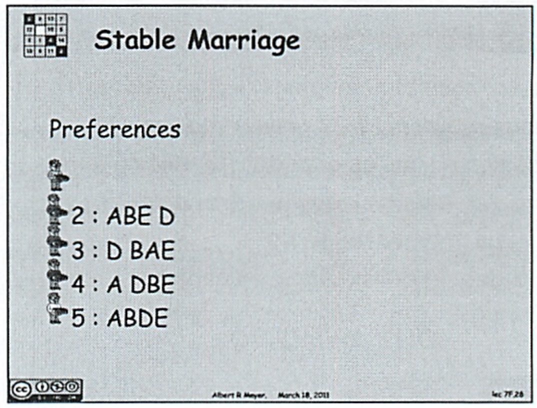
Stable Marriage

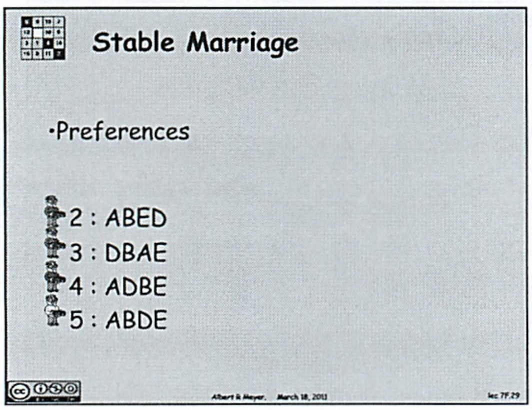
Preferences

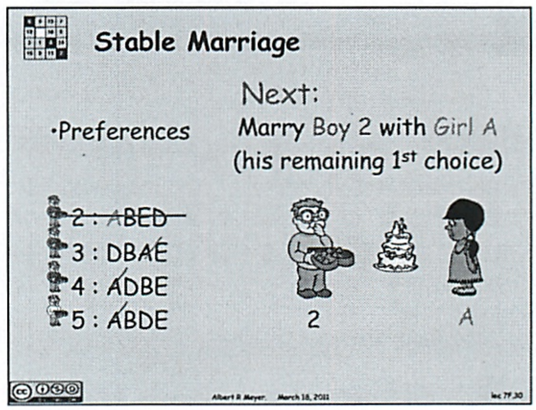
1: CBEAD
2 : ABECD
3 : DCBAE
4 : ACDBE
5 : ABDEC

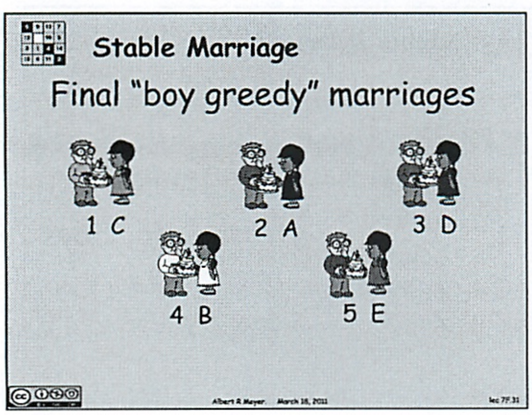
Try "greedy" strategy for boys

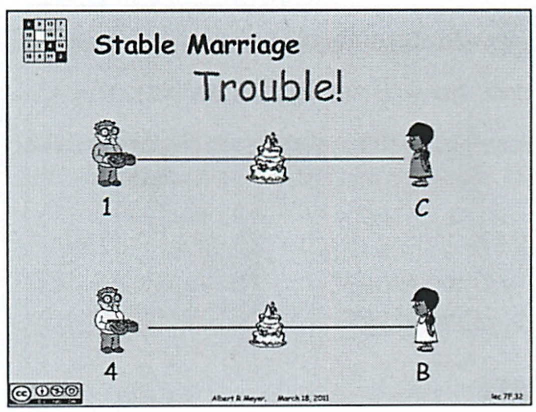

Stable Marriage
•Preferences
~~1: CBEAD~~
2 : ABECD
3 : DCBAE
4 : ACDBE
5 : ABDEC
Marry Boy 1 with Girl C
(his 1st choice)
1
C
Albert R Meyer, March 18, 2011
lec 7F.27

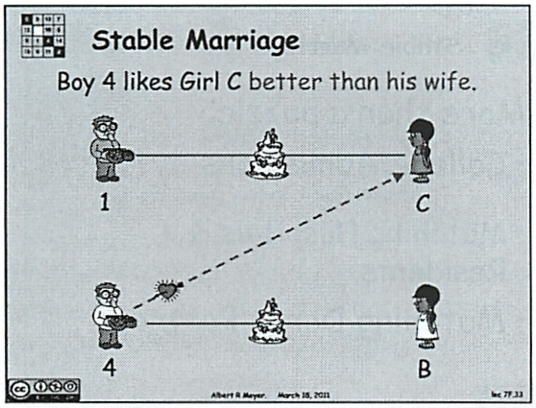

Stable Marriage
Preferences
2 : ABE D
3 : D BAE
4 : A DBE
5 : ABDE
Albert R Meyer, March 18, 2011
lec 7F.28

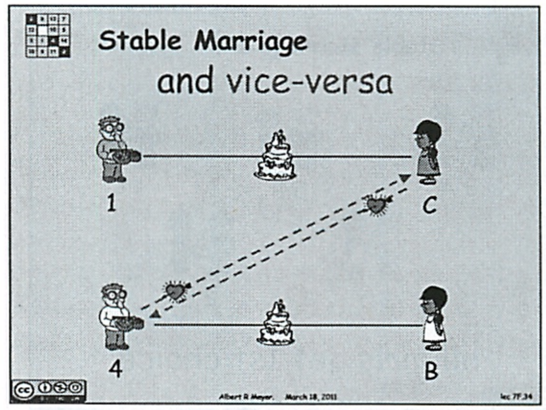

Stable Marriage
•Preferences
2 : ABED
3 : DBAE
4 : ADBE
5 : ABDE
Albert R Meyer, March 18, 2011
lec 7F.29

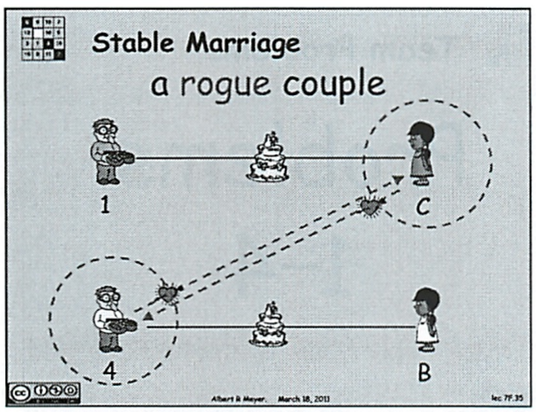

Stable Marriage
Next:
Marry Boy 2 with Girl A
(his remaining 1st choice)
•Preferences
~~2 : ABED~~
3 : DBAE
4 : ADBE
5 : ABDE
2
A
Albert R Meyer, March 18, 2011
lec 7F.30

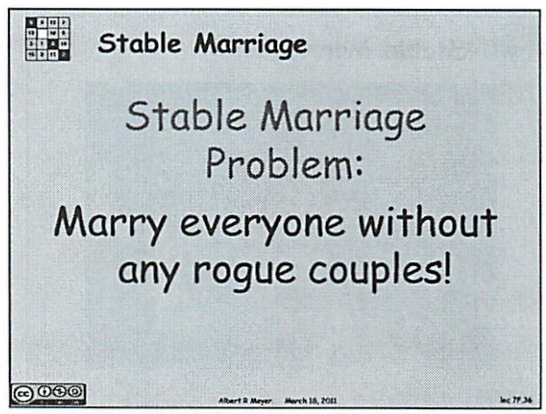

Stable Marriage
Final "boy greedy" marriages
1 C
2 A
3 D
4 B
5 E
Albert R Meyer, March 18, 2011
lec 7F.31

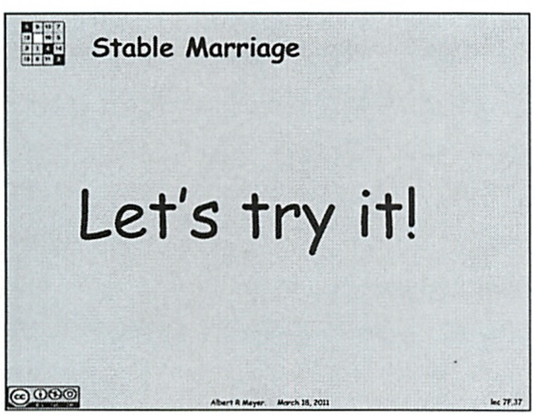

Stable Marriage
Trouble!
1
C
4
B
Albert R Meyer, March 18, 2011
lec 7F.32

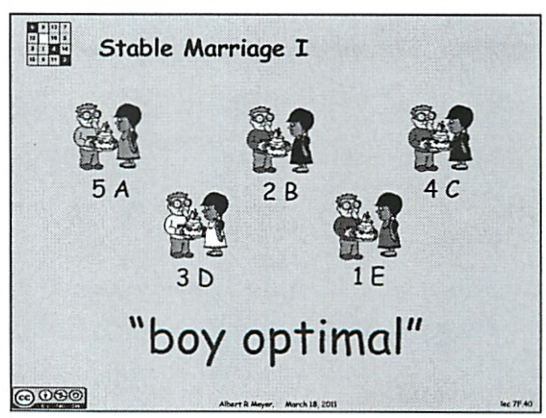

Stable Marriage
Boy 4 likes Girl C better than his wife.
1
C
4
B
Albert R Meyer. March 18, 2011
lec 7F.33


Stable Marriage
and vice-versa
1
C
4
B
Albert R Meyer. March 18, 2011
lec 7F.34


Stable Marriage
a rogue couple
1
C
4
B
Albert R Meyer. March 18, 2011
lec 7F.35


Stable Marriage
Stable Marriage Problem:
Marry everyone without any rogue couples!
Albert R Meyer. March 18, 2011
lec 7F.36


Stable Marriage
Let's try it!
Albert R Meyer. March 18, 2011
lec 7F.37


Stable Marriage I
5 A
2 B
4 C
3 D
1 E
"boy optimal"
Albert R Meyer. March 18, 2011
lec 7F.40

Stable Marriage II.

all girls get 1st choice

Stable Marriage

More than a puzzle:

- College Admissions
  (original Gale & Shapley paper, 1962)
- Matching Hospitals & Residents.
- Matching Dance Partners.

Stable Marriage

Team Problems

Problems

1–4

6.042

3/19

TA teaching Oshani:

Simple Graphs
Isomorphism
Stable Marriage

lots of uses

Isomorphism - same # of verticies

Same connection = isomorphic
  └ edge-presering bijection

diff lables and lines

non edges are presered as well

$$\boxed{\begin{array}{l} \exists \text{ bij } f: V_1 \to V_2 \text{ with} \\ u-v \text{ in } E_1 \text{ iff } f(u)-f(v) \text{ in } E_2 \end{array}}$$

\# of verticies w/ each degree must match

\# nodes " "

\# edges " "

(2)

But hard to tell

$2^{\sqrt{n}\log n}$

Stable Matching

The Marriage Problem

1	2	3	4	5	Boys
A	B	C	D	E	~~Girls~~ Girls

Preferences

1	CBEAD	A	35214
2	etc	B	52143
3		C	
4		D	
5		E	

"Greedy" strategy for boys

(Just what was in book)

Can do online paper

College Applications

Internship Assigns

Alkami

In-Class Problems Week 7, Fri.

Problem 1.
See if you can come up with a stable marriage assignment given the following preferences. You are not expected to know/remember the Mating Ritual that solves this problem in general. (And if you do remember the protocol, don't spoil your teammates' fun by telling them.)

boys	*girls*
$1 : CBEAD$	$A : 35214$
$2 : ABECD$	$B : 52143$
$3 : DCBAE$	$C : 43512$
$4 : ACDBE$	$D : 12345$
$5 : ABDEC$	$E : 23415$

Problem 2.
For each of the following pairs of graphs, either define an isomorphism between them, or prove that there is none. (We write ab as shorthand for a—b.)

(a)

$$G_1 \text{ with } V_1 = \{1,2,3,4,5,6\},\ E_1 = \{12,23,34,14,15,35,45\}$$
$$G_2 \text{ with } V_2 = \{1,2,3,4,5,6\},\ E_2 = \{12,23,34,45,51,24,25\}$$

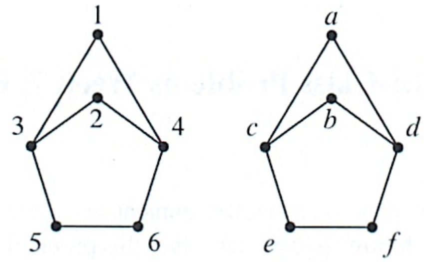
(b)

$$G_3 \text{ with } V_3 = \{1,2,3,4,5,6\},\ E_3 = \{12,23,34,14,45,56,26\}$$
$$G_4 \text{ with } V_4 = \{a,b,c,d,e,f\},\ E_4 = \{ab,bc,cd,de,ae,ef,cf\}$$

(c)

$$G_5 \text{ with } V_5 = \{a,b,c,d,e,f,g,h\},\ E_5 = \{ab,bc,cd,ad,ef,fg,gh,he,dh,bf\}$$
$$G_6 \text{ with } V_6 = \{s,t,u,v,w,x,y,z\},\ E_6 = \{st,tu,uv,sv,wx,xy,yz,wz,sw,vz\}$$

Problem 3.
There are four isomorphisms between these two graphs. List them.



Problem 4.
The most famous application of stable matching was in assigning graduating medical students to hospital residencies. Each hospital has a preference ranking of students and each student has a preference order of hospitals, but unlike the setup in the notes where there are an equal number of boys and girls and monogamous marriages, hospitals generally have differing numbers of available residencies, and the total number of residencies may not equal the number of graduating students.

What would be a *rogue couple* when matching medical students and hospitals?

Modify the definition of stable matching so it applies in this situation.

(B)

In-Class

3/19

1. Just do the protocal - it was in the req reading

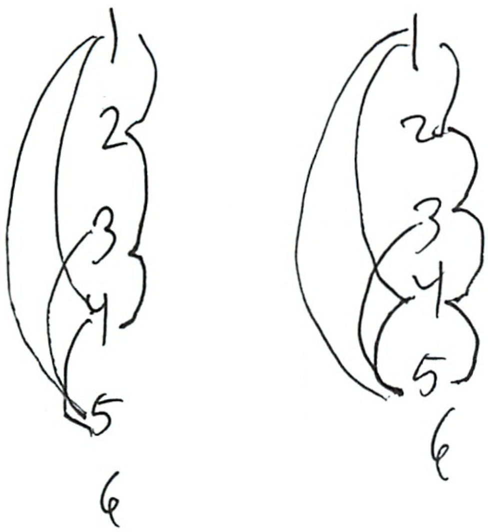
A	B	C	D	E
35214	52143	43512	12345	23415
~~2 ABECD~~		1 CBEAD	3 BDCBAE	
~~4 ACDBE~~				
5 ABDEC				

5 ABDEC	2 BECD	~~1 CBEAD~~	3 DCBAE	
		4 CDBE		
5	2	4	3	
	~~1 BEAD~~			
5	2	4	3	1 ✓

Worst stable pairings for ~~boys~~ girls
- Best guy they can get

(2)

2. Means writing bij for what = what? Yes

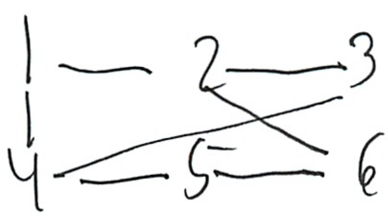


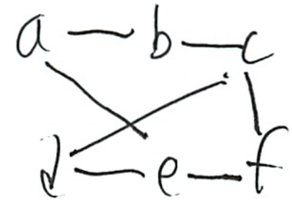
← has a 4-deg

exists no 4 degre

No isomorphism

b.





So ignore 1-2-3-4-5-6 - i can do

no - perhaps have # of deg

1 → c

4 → f

a → 2

e → 6

b → 5

d → 3

(3)

Just draw and see

$1 \to f$
$2 \to e$
$3 \to d$
$4 \to c$
$5 \to b$
$6 \to a$

– if draw in certain way easy to flip

Hard to see quickly

b.c No bij

G_6 degree 3 – adj – to degree 3

but G_5 does not

3. 4 isomorphisms possible

So 4 dif possibilites for bij

first obv		deg	mirror		or just mirror one		or other	
1	a	2	1	a	1	a	1	a
2	b	2	2	b	2	b	2	b
3	c	3	3	d	3	d	3	c
4	d	3	4	c	4	c	4	d
5	e	2	5	f	5	e	5	f
6	f	2	6	e	6	f	6	e

no flip ce and df together

(4)

4. Grad School

Rage caple - ~~a~~ a student would perfer a hospital more

and the hospital would perfer the student more

= unhappines

def centers around happiness

; schools may drop out of program if

they don't see good matches

Break up ~~into~~ hospitals into ind. spots

for each student in Hospitals

there ∄ a hospital that it prefers that would take him/her

Perfers him/her over a student it already has

Solutions to In-Class Problems Week 7, Fri.

Problem 1.
See if you can come up with a stable marriage assignment given the following preferences. You are not expected to know/remember the Mating Ritual that solves this problem in general. (And if you do remember the protocol, don't spoil your teammates' fun by telling them.)

boys	*girls*
$1 : CBEAD$	$A : 35214$
$2 : ABECD$	$B : 52143$
$3 : DCBAE$	$C : 43512$
$4 : ACDBE$	$D : 12345$
$5 : ABDEC$	$E : 23415$

Solution.

$5A\ 2B\ 4C\ 3D\ 1E$	a boy optimal matching
$3A\ 5B\ 4C\ D2\ 2E$	girls get their 1st choice

■

Problem 2.
For each of the following pairs of graphs, either define an isomorphism between them, or prove that there is none. (We write ab as shorthand for a—b.)

(a)

$$G_1 \text{ with } V_1 = \{1,2,3,4,5,6\},\ E_1 = \{12,23,34,14,15,35,45\}$$
$$G_2 \text{ with } V_2 = \{1,2,3,4,5,6\},\ E_2 = \{12,23,34,45,51,24,25\}$$

Solution. Not isomorphic: G_2 has a node, 2, of degree 4, but the maximum degree in G_1 is 3. ■

(b)

$$G_3 \text{ with } V_3 = \{1,2,3,4,5,6\},\ E_3 = \{12,23,34,14,45,56,26\}$$
$$G_4 \text{ with } V_4 = \{a,b,c,d,e,f\},\ E_4 = \{ab,bc,cd,de,ae,ef,cf\}$$

Solution. Isomorphic (two isomorphisms) with the vertex correspondences:
$1f, 2c, 3d, 4e, 5a, 6b$
or $1f, 2e, 3d, 4c, 5b, 6a$

■

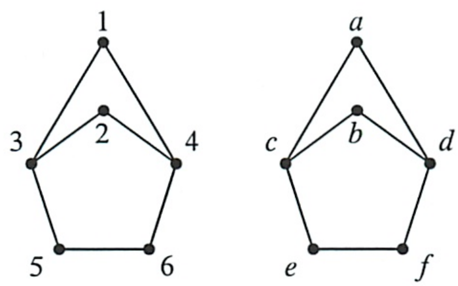
(c)

$$G_5 \text{ with } V_5 = \{a,b,c,d,e,f,g,h\},\ E_5 = \{ab,bc,cd,ad,ef,fg,gh,he,dh,bf\}$$
$$G_6 \text{ with } V_6 = \{s,t,u,v,w,x,y,z\},\ E_6 = \{st,tu,uv,sv,wx,xy,yz,wz,sw,vz\}$$

Solution. Not isomorphic: they have the same number of vertices, edges, and set of vertex degrees. But the degree 2 vertices of G_1 are all adjacent to two degree 3 vertices, while the degree 2 vertices of G_2 are all adjacent to one degree 2 vertex and one degree 3 vertex.

■

Problem 3.
There are four isomorphisms between these two graphs. List them.



Solution. These are the vertex correspondences for the four isomorphisms:
1A, 2B, 3C, 4D, 5E, 6F
1A, 2B, 3D, 4C, 5F, 6E
1B, 2A, 3C, 4D, 5E, 6F
1B, 2A, 3D, 4C, 5F, 6E ■

Problem 4.
The most famous application of stable matching was in assigning graduating medical students to hospital residencies. Each hospital has a preference ranking of students and each student has a preference order of hospitals, but unlike the setup in the notes where there are an equal number of boys and girls and monogamous marriages, hospitals generally have differing numbers of available residencies, and the total number of residencies may not equal the number of graduating students.

What would be a *rogue couple* when matching medical students and hospitals?

Modify the definition of stable matching so it applies in this situation.

Solution. A matching is an assignment of medical students to residencies in each of the hospitals (an injection, A : students $\rightarrow$ residencies) such that every student has a residency (A is total), or every residency has an assigned student (A is a surjection). A stable assignment is one with no *rogue couples*, where a rogue couple is a hospital student pair (H, S) such that S is not assigned to one of the residencies at H, which she prefers over her current assignment, and

- H has some students assigned to some of its residencies and prefers S to at least one of its assigned students, or
- H has none of its residencies assigned.

■

3/28

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Stable Matching: Mating Ritual

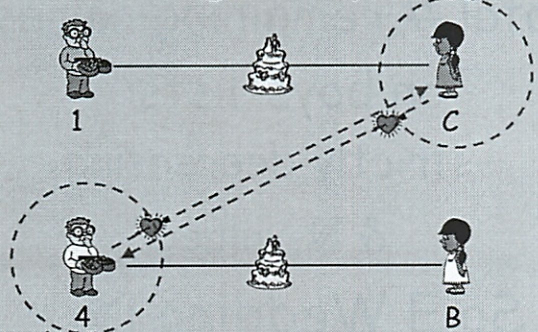
Stable Marriage

Stable Marriage Problem:

Marry everyone without any rogue couples!

Stable Marriage

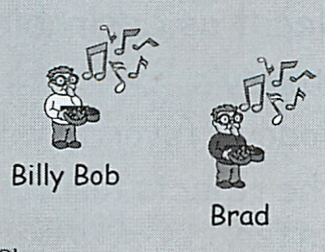
a rogue couple



The Mating Ritual (day by day)

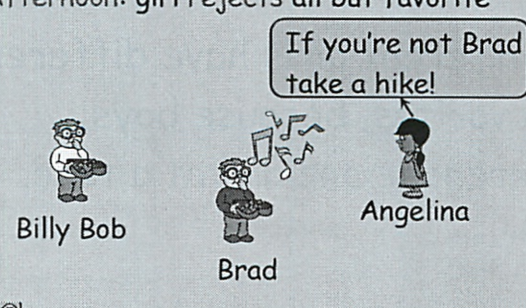
Mating Ritual

Morning: boy serenades favorite girl



Mating Ritual

Morning: boy serenades favorite girl
Afternoon: girl rejects all but favorite



Mating Ritual

- Morning: boy serenades his favorite girl
- Afternoon: girl rejects all but her favorite boy
- Evening: rejected boy writes off girl

Billy Bob

Mating Ritual

Stop when no girl rejects.
Each girl marries her favorite suitor (if any).

Mating Ritual

Termination:

There exists a Wedding Day.

Partial Correctness:

Everyone is married.

Marriages are stable.

Stable Marriage: termination

total # remaining names on boys' lists:

strictly decreasing

& $\mathbb{N}$-valued

So $\exists$ Wedding Day

Mating Ritual

Different girls have different favorites, because boys serenade one girl at a time.

Mating Ritual: girls improve

Lemma:

A girl's favorite tomorrow will be at least as desirable to her as today's.

...because today's favorite will stay until she rejects him for someone better.

Mating Ritual: boys get worse

Lemma:

A boy's favorite tomorrow will be no more desirable to him than today's.

...because boys work straight down their lists.

Mating Ritual: invariant

If G is not on B's list, then she has a better current favorite.

Proof: When G rejected B she had a better suitor (her favorite that day), and her favorites never get worse.

On Wedding Day

Each girl has ≤ 1 suitor.
(by def of wedding day)

Each boy is married, or has no girls on his list.

Mating Ritual: everyone marries

Everyone is married on wedding day

Proof: By contradiction.

If B is not married, his list is empty. By invariant, all girls have favorites better than B -- so they do have a favorite. That is, all girls are married, so all boys are married.

Mating Ritual: stable marriages

Marriages are Stable:

Bob won't be in rogue couple with case 1: a girl G on his final list, since he's already married to the best of them.

Mating Ritual: stable marriages

Marriages are Stable:

Bob won't be in rogue couple with case 2: a girl G not on his list, since by invariant, G likes her spouse better than Bob.

Mating Ritual

Who does better, boys or girls?

Girls' suitors get better, and boys' sweethearts get worse, so girls do better?

No!

Boy Optimal

Mating Ritual is Optimal for all boys at once.

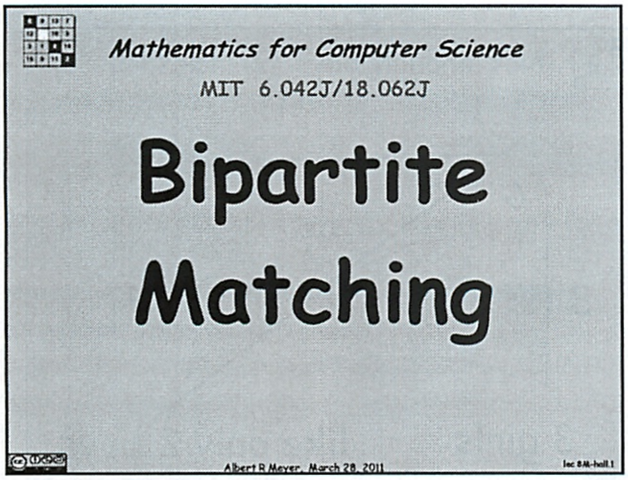
Pessimal for all girls.

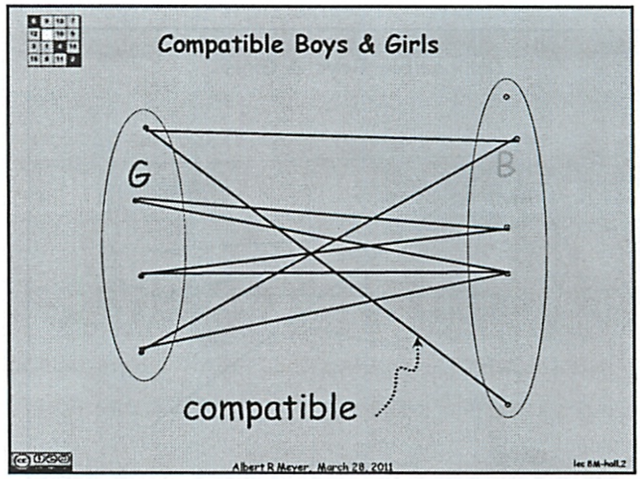
Stable Marriage

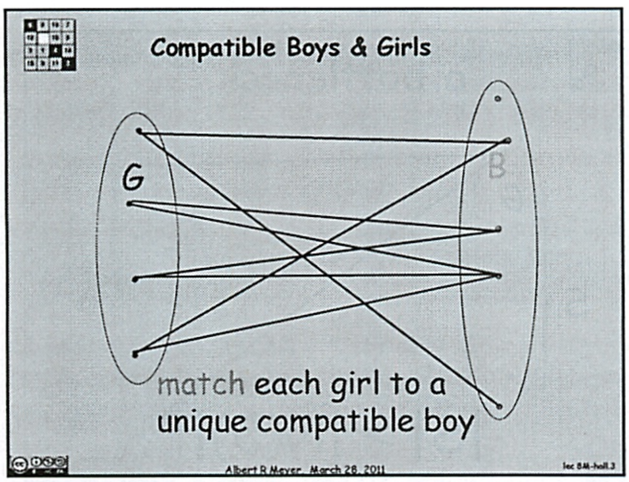
More questions, rich theory

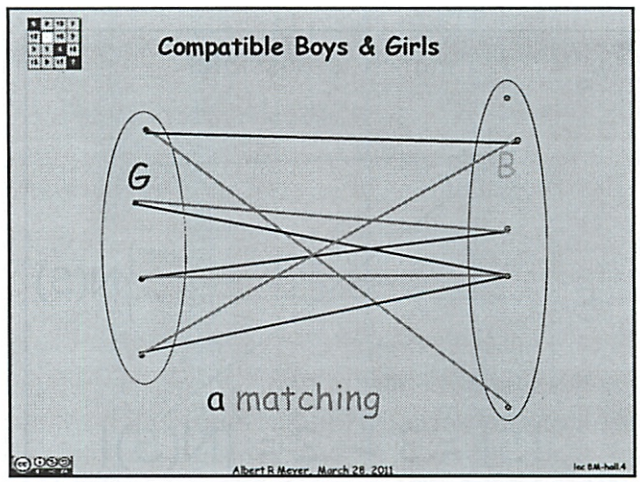
- other stable marriages possible? -can be many
- do better by lying?
  boys -No! girls -Yes!

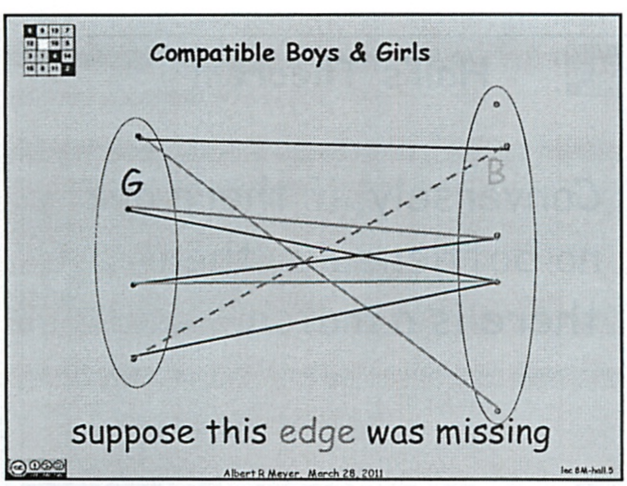
3/28
As well

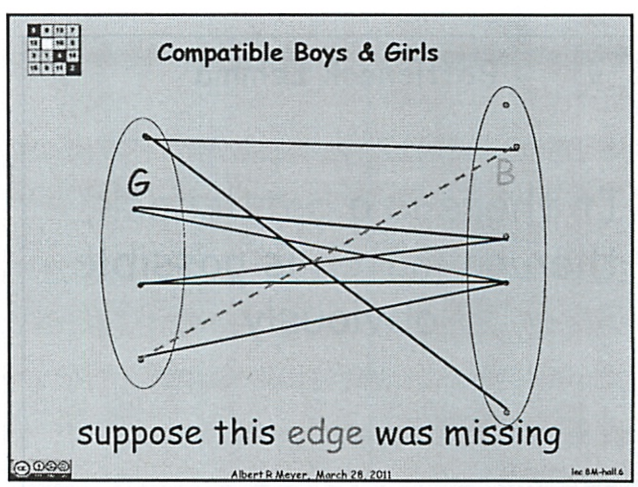

Mathematics for Computer Science
MIT 6.042J/18.062J
Bipartite
Matching
Albert R Meyer, March 28, 2011
lec 8M-hall.1


Compatible Boys & Girls
G
B
compatible
Albert R Meyer, March 28, 2011
lec 8M-hall.2

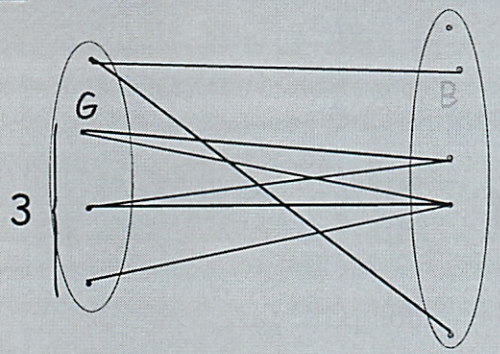

Compatible Boys & Girls
G
B
match each girl to a
unique compatible boy
Albert R Meyer, March 28, 2011
lec 8M-hall.3


Compatible Boys & Girls
G
B
a matching
Albert R Meyer, March 28, 2011
lec 8M-hall.4

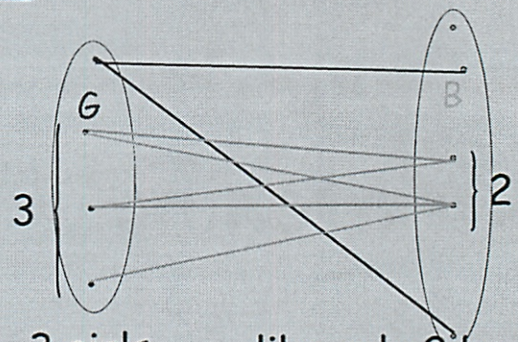

Compatible Boys & Girls
G
B
suppose this edge was missing
Albert R Meyer, March 28, 2011
lec 8M-hall.5


Compatible Boys & Girls
G
B
suppose this edge was missing
Albert R Meyer, March 28, 2011
lec 8M-hall.6

Compatible Boys & Girls

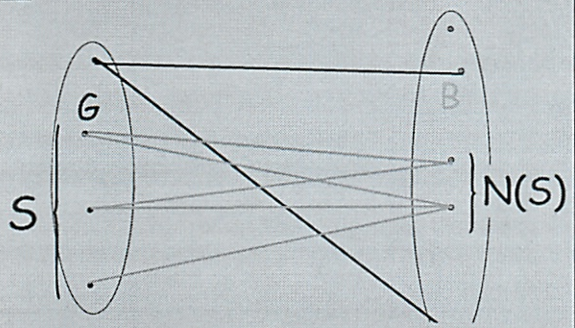


Not enough boys for these girls!



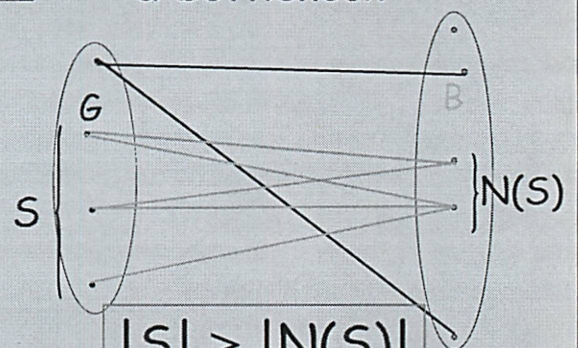
3 girls like only 2 boys

No match is possible!



$|S| = 3 > 2 = |N(S)|$

a bottleneck



$\boxed{|S| > |N(S)|}$

Bottleneck Lemma

If there is a bottleneck, then no match is possible, obviously.

Hall's Theorem

Conversely, if there are no bottlenecks, then there is a match.

Hall's Theorem

Hall's condition

If $|S| \leq |N(S)|$ for all sets of girls, S, then there is a match.

(proof in Notes)

How to verify no bottlenecks?

fairly efficient matching procedure is known
(explained in algorithms subjects)

...but there is a special situation that ensures a match...

How to verify no bottlenecks?

If every girl likes $\geq$ d boys, and every boy likes $\leq$ d girls, then no bottlenecks.

a degree-constrained bipartite graph

Team Problems

Problems

1—4

6.042 3/28

(7 min late)

Marrage day will exist b/c # matches flat or down

Lemma: girls improve

- her fav tomorrow is as least as good as todays
- guy will stay until he sends away

Boy working down list

- tomorrow no more desirable

So invarients

- if G is not on B's list - then she has a better current favorite

Weding day - girl ≤ 1 boy

and boy has 0 people on list if not married

- must have stayed home
  - if all girls rejected him
- all girls have better fav
- that they married
- same # B, G, so contridiction
- all boys must be married

(2)

All marrages stable

- All boys - girl married to on top of list
- For girl to reject boy, must have a more fav boy

Is boy optimal, girl pessimal
- worst deal for girl

Can be multiple stable marriages

Hospital Residencies

Roommates

Can anyone improve shot by lying?
boys - No - already doing best
girls - Yes - can fake for girl optimal

Bipartite Matching

diff # boys + girls
Split all verticies into 2 graps
edges only run b/w points
matching - one possible compatability
- every vertex degree 0 or 1

(3)

Is there always a possible matching

But in this example 3 girls only like 2 boys

- not enough to go around

Call subset girls S

boys $N(S)$

$|S| > |N(S)|$ ← = <u>bottleneck</u>

- no possible matching

set of vertices are <u>Covered</u> - every vertex has 1 edge

- matching is <u>perfect</u> (all the vertices are covered)) both property of matching

---

<u>Halls' Theorm</u> - converse, non obvious

If no bottlenecks ← Halls' Theorm $(|S| \leq |N(S)|)$ for all sets of girls S then there is a <u>match</u>

How to check no bottlenecks?

- Other class

But can verify:

If every girl likes $\geq d$ boys, and every boy likes $\leq d$ girls then no bottlenecks

In-Class Problems Week 8, Mon.

Problem 1.
Four Students want separate assignments to four VI-A Companies. Here are their preference rankings:

Student	Companies
Albert:	HP, Bellcore, AT&T, Draper
Nick:	AT&T, Bellcore, Draper, HP
Oshani:	HP, Draper, AT&T, Bellcore
Ali:	Draper, AT&T, Bellcore, HP

Company	Students
AT&T:	Ali, Albert, Oshani, Nick
Bellcore:	Oshani, Nick, Albert, Ali
HP:	Ali, Oshani, Albert, Nick
Draper:	Nick, Ali, Oshani, Albert

(a) Use the Mating Ritual to find *two* stable assignments of Students to Companies.

(b) Describe a simple procedure to determine whether any given stable marriage problem has a unique solution, that is, only one possible stable matching.

Problem 2.
A preserved invariant of the Mating ritual is:

> For every girl, G, and every boy, B, if G is crossed off B's list, then G has a favorite suitor and she prefers him over B.

Use the invariant to prove that the Mating Algorithm produces stable marriages. (Don't look up the proof in the Notes or slides.)

Problem 3.
Because of the incredible popularity of Math for Computer Science, Rajeev decides to give up on regular office hours. Instead, each student can join some study groups. Each group must choose a representative to talk to the staff, but there is a staff rule that a student can only represent one group. The problem is to find a representative from each group while obeying the staff rule.

(a) Explain how to model the delegate selection problem as a bipartite matching problem.

(b) The staff's records show that no student is a member of more than 4 groups, and all the groups must have at least 4 members. That's enough to guarantee there is a proper delegate selection. Explain.

Problem 4.
Overworked and over-caffeinated, the Teaching Assistant's (TA's) decide to oust the lecturer and teach their own recitations. They will run a recitation session at 4 different times in the same room. There are exactly 20 chairs to which a student can be assigned in each recitation. Each student has provided the TA's with a list of the recitation sessions her schedule allows and no student's schedule conflicts with all 4 sessions. The TA's must assign each student to a chair during recitation at a time she can attend, if such an assignment is possible.

Describe how to model this situation as a matching problem. Be sure to specify what the vertices/edges should be and briefly describe how a matching would determine seat assignments for each student in a recitation that does not conflict with his schedule. This is a *modeling problem* —you need not determine whether a match is always possible.

Appendix: The Mating Ritual

The *Mating Ritual* takes place over several days. The following events happen each day:

Morning: Each girl stands on her balcony. Each boy stands under the balcony of his favorite among the girls on his list, and he serenades her. If a boy has no girls left on his list, he stays home and does his 6.042 homework.

Afternoon: Each girl who has one or more suitors serenading her, says to her favorite suitor, "We might get engaged. Come back tomorrow." To the others, she says, "No. I will never marry you! Take a hike!"

Evening: Any boy who is told by a girl to take a hike, crosses that girl off his list.

Termination condition: When every girl has at most one suitor, the ritual ends with each girl marrying her suitor, if she has one.

In-Class Problems Week 8, Mon.

Additional Problem

Problem 1.
Suppose that Harry is one of the boys and Alice is one of the girls in the *Mating Ritual*. Which of the properties below are preserved invariants? Why?

a. Alice is the only girl on Harry's list.

b. There is a girl who does not have any boys serenading her.

c. If Alice is not on Harry's list, then Alice has a suitor that she prefers to Harry.

d. Alice is crossed off Harry's list and Harry prefers Alice to anyone he is serenading.

e. If Alice is on Harry's list, then she prefers to Harry to any suitor she has.

In Class Week 8 Mon

3/28

1a. Find two stable matchings using Maiting Ritual
- Similar qu previous p-set

~~Students~~ Companies

AT+T	Bell	HP	Draper
Nick		Alb, Oshani	Ali
"	Albert	~~Alb~~ Oshani	

done

Switch

Al	Nick	Oshani	Ali
	Draper	Bellcorp	AT+T, HP
	"	Bell, HP	AT+T
	Draper, Bell	HP	AT+T
	Bell	HP	AT+T, Draper
AT+T	Bell	HP	Draper

②

b) ~~Same as was discussed~~ in class?

No! This is that there is only 1 stable matching

Our board: Run ritual 2x (like I did) w/ changing who is on balcony and who is suitor.

If same - only 1 possible ans

How does this prove?

2. Persevered invarient: For every G and B - if G is crossed off B's list than G has a ~~fav~~ suitor she perfers to B

Prove ~~matching~~ that Maiting algorthm produces stable marriages

- Ok this is eaiser than proving this is invarient

Prob have to prove ~~that~~ also that B has no better.

So because crosses off - she likes someone better and that person stays with her. Her prospects only improve. Boys go to fav girl that has not rejected them. If was a girl they liked better - then they would go to her - but they don't.

(3)

On our board - our group is doing a more formal proof

3. No more OH - instead groups of 4
   - well some #
   - has one rep
   - and student can only rep one group

a) How to model as Biparte problem?

b) ~~XX~~ All groups ≥ 4 people
   No student in more than 4 ~~four~~ groups
   - That guarentees a proper delegation selection. How?

What is worse case trying to prove?
   - group 4 people
     A student in 3 groups

     ABCD ABCD ABCD
      one rep possible

     ABCD, ABCE ABCF
        again

     But how to write formally?

Perhaps do a first

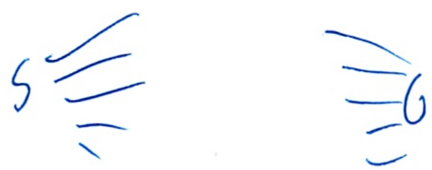
(4)

a)

b) No bottlenecks no matching) TA
this ~~ks~~ meats no bottleneck)

a) Group one - student vertices
" two - student group "
Edge if member
So if can cover - would be a delegate for each group

b)



$|e| \leq 4$ $|e| \geq 4$

4. First group - students
2nd - seat / time pair
-ie 12 pm seat #1
1 pm " #1
12 pm " #2

There is an edge if it fits
If it "covers" set of students then each student can be accomodate
The subset of students

Do you have to say sepertly seats ~~[illegible]~~ ≤ 1 person

(5)

Additional Problem 1)

Which are preserved invariants?

a) Alice is only girl on Harrys list?
No, false

b) ~~∅~~ Lonly girl
not always true

c) False - prefs ~~do~~ not need ~~to~~ be symetric

d) True - otherwise he would be serenading Alice or other girl higher on his original, not-crossed out list.

e) False, see c

Remember preserved invarient means once true it stays true

False → False Ok
True → True Ok
False → True Great
True → False NO!

6

Redo

3b) For a subset of groups are at least $4n$ ~~[illegible]~~ incident edges. Since each student is incident at most 4 ~~[illegible]~~ edges - there must be at least n students adj to this subset of groups.

So cannot be a bottleneck

Since there are no bottleneck Halls Theorem says a matching exists so is a proper delegate section.

---

add. problem 1)

a b d T

b e F

Solutions to In-Class Problems Week 8, Mon.

Note: Only problems 1, 2, 4, and 5 were originally assigned. The additional problem later handed out in class is here listed as problem 3.

Problem 1.
Four Students want separate assignments to four VI-A Companies. Here are their preference rankings:

Student	Companies
Albert:	HP, Bellcore, AT&T, Draper
Nick:	AT&T, Bellcore, Draper, HP
Oshani:	HP, Draper, AT&T, Bellcore
Ali:	Draper, AT&T, Bellcore, HP

Company	Students
AT&T:	Ali, Albert, Oshani, Nick
Bellcore:	Oshani, Nick, Albert, Ali
HP:	Ali, Oshani, Albert, Nick
Draper:	Nick, Ali, Oshani, Albert

(a) Use the Mating Ritual to find *two* stable assignments of Students to Companies.

Solution. Treat Students as Boys and the result is the following assignment:

Student	Companies	Rank in the original list
Albert:	Bellcore	2
Nick:	AT&T	1
Oshani:	HP	1
Ali:	Draper	1

Treat Companies as Boys and the result is the following assignment:

Company	Students	Rank in the original list
AT&T:	Albert	2
Bellcore:	Nick	2
HP:	Oshani	2
Draper:	Ali	2

■

(b) Describe a simple procedure to determine whether any given stable marriage problem has a unique solution, that is, only one possible stable matching.

Solution. See if the Mating Ritual with Boys as suitors yields the same solution as the algorithm with Girls as suitors. These two marriage assignments are boy-optimal and boy-pessimal, respective. Obviously, if every boy's optimal and pessimal choices are the same, then every boy has an unique choice. The solution is unique.

■

Problem 2.
A preserved invariant of the Mating ritual is:

> For every girl, G, and every boy, B, if G is crossed off B's list, then G has a favorite suitor and she prefers him over B.

Use the invariant to prove that the Mating Algorithm produces stable marriages. (Don't look up the proof in the Notes or slides.)

Solution. *Proof.* Let Brad be some boy and Jen be any girl that he is *not* married to on the last day of the Mating Ritual. We claim that Brad and Jen are not a rogue couple. Since Brad is an arbitrary boy, it follows that no boy is part of a rogue couple. Hence the marriages on the last day are stable.

To prove the claim, we consider two cases:

Case 1. Jen is not on Brad's list. Then by invariant P, we know that Jen prefers her husband to Brad. So she's not going to run off with Brad: the claim holds in this case.

Case 2. Otherwise, Jen is on Brad's list. But since Brad is not married to Jen, he must be choosing to serenade his wife instead of Jen, so he must prefer his wife. So he's not going to run off with Jen: the claim also holds in this case. ■

Problem 3.
Suppose that Harry is one of the boys and Alice is one of the girls in the *Mating Ritual*. Which of the properties below are preserved invariants? Why?

a. Alice is the only girl on Harry's list.

b. There is a girl who does not have any boys serenading her.

c. If Alice is not on Harry's list, then Alice has a suitor that she prefers to Harry.

d. Alice is crossed off Harry's list and Harry prefers Alice to anyone he is serenading.

e. If Alice is on Harry's list, then she prefers to Harry to any suitor she has.

Solution. The 1st, 3rd, and 4th are preserved invariants.

a. A preserved invariant; no girl will be added to Harry's list. If Alice got crossed off, there would be no one for Harry to marry. So she must remain as the sole girl on his list. **Reminder**: A *preserved invariant* need not be true all the time, as in this example. It only needs to stay true once it first becomes true.

b. Not preserved; a girl may not have a suitor on the first day —if, for example, she's not at the top of any boy's list —but every girl is guaranteed to have one at the end, namely, her husband.

c. A preserved invariant; this is the basic invariant used to verify the Ritual.

d. A preserved invariant; Harry crosses off the girls in his order of preference, so if Alice is crossed off, Harry likes her better than anybody that's left.

e. Not preserved. Suppose the preferences among two couples and a third boy are:

Harry:	Alice,	Elvira,	...
Billy:	Elvira,	Alice,	...
Wilfred:	Elvira,	...	
Alice:	Billy,	Harry,	...
Elvira:	Wilfred,	Billy,	...

The alleged invariant is true on the first day since Harry is Alice's only suitor. But Elvira rejects Billy in favor of Wilfred on the first afternoon, so on the second day, Billy and Harry are serenading Alice. Since Alice prefers Billy to Harry, the alleged invariant is no longer true, so it was not preserved.

■

Problem 4.
Because of the incredible popularity of Math for Computer Science, Rajeev decides to give up on regular office hours. Instead, each student can join some study groups. Each group must choose a representative to talk to the staff, but there is a staff rule that a student can only represent one group. The problem is to find a representative from each group while obeying the staff rule.

(a) Explain how to model the delegate selection problem as a bipartite matching problem.

Solution. Define a bipartite graph with the study groups as one set of vertices and students in the groups as the other set of vertices. A group and a student are adjacent exactly when the student belongs to the group. Now a matching of study groups to students will give a proper selection of delegates: every group will have a delegate, and every delegate will represent exactly one club. ■

(b) The staff's records show that no student is a member of more than 4 groups, and all the groups must have at least 4 members. That's enough to guarantee there is a proper delegate selection. Explain.

Solution. The degree of every group is at least 4, and the degree of every student is at most 4, so the graph is *degree-constrained* (Def. 11.5.5) which implies there will be no bottlenecks to prevent a matching. Hall's Theorem then guarantees a matching. ■

Problem 5.
Overworked and over-caffeinated, the Teaching Assistant's (TA's) decide to oust the lecturer and teach their own recitations. They will run a recitation session at 4 different times in the same room. There are exactly 20 chairs to which a student can be assigned in each recitation. Each student has provided the TA's with a list of the recitation sessions her schedule allows and no student's schedule conflicts with all 4 sessions. The TA's must assign each student to a chair during recitation at a time she can attend, if such an assignment is possible.

Describe how to model this situation as a matching problem. Be sure to specify what the vertices/edges should be and briefly describe how a matching would determine seat assignments for each student in a recitation that does not conflict with his schedule. This is a *modeling problem* —you need not determine whether a match is always possible.

Solution. There will be one vertex for each student, and 20 vertices for each recitation time slot (one for each chair). There is an edge between a student and all chair vertices for a particular recitation time slot if that time slot does not conflict with her schedule. A matching for the students assigns a student to a chair in a recitation that he can attend and assigns at most 20 students to any recitation.

It is possible to assign the students to recitations iff a matching exists. ■

Problem Set 6

Due: March 30

Reading: Chapter 9.5–9.9, Partial Orders; Chapter 11–11.6, Simple Graphs.
Skip Chapter 10, Communication Nets, which will not be covered this term.

Problem 1.
Let R_1, R_2 be binary relations on the same set, A. A relational property is preserved under product, if $R_1 \times R_2$ has the property whenever both R_1 and R_2 have the property.

(a) Verify that each of the following properties are preserved under product.

1. reflexivity,
2. antisymmetry,
3. transitivity.

(b) Verify that if *either* of R_1 or R_2 is irreflexive, then so is $R_1 \times R_2$.

Note that it now follows immediately that if if R_1 and R_2 are partial orders and at least one of them is strict, then $R_1 \times R_2$ is a strict partial order.

Problem 2.
The most famous application of stable matching was in assigning graduating medical students to hospital residencies. Each hospital has a preference ranking of students and each student has a preference order of hospitals, but unlike the setup in the notes where there are an equal number of boys and girls and monogamous marriages, hospitals generally have differing numbers of available residencies, and the total number of residencies may not equal the number of graduating students. Modify the definition of stable matching so it applies in this situation, and explain how to modify the Mating Ritual so it yields stable assignments of students to residencies.

Briefly indicate what, if any, modifications of the preserved invariant used to verify the original Mating are needed to verify this one for hospitals and students.

Problem 3.
Scholars through the ages have identified *twenty* fundamental human virtues: honesty, generosity, loyalty, prudence, completing the weekly course reading-response, etc. At the beginning of the term, every student in Math for Computer Science possessed exactly *eight* of these virtues. Furthermore, every student was unique; that is, no two students possessed exactly the same set of virtues. The Math for Computer Science course staff must select *one* additional virtue to impart to each student by the end of the term. Prove that there is a way to select an additional virtue for each student so that every student is unique at the end of the term as well.

Suggestion: Use Hall's theorem. Try various interpretations for the vertices on the left and right sides of your bipartite graph.

Problem 4.
Determine which among the four graphs pictured in the Figures are isomorphic. If two of these graphs are isomorphic, describe an isomorphism between them. If they are not, give a property that is preserved under isomorphism such that one graph has the property, but the other does not. For at least one of the properties you choose, *prove* that it is indeed preserved under isomorphism (you only need prove one of them).

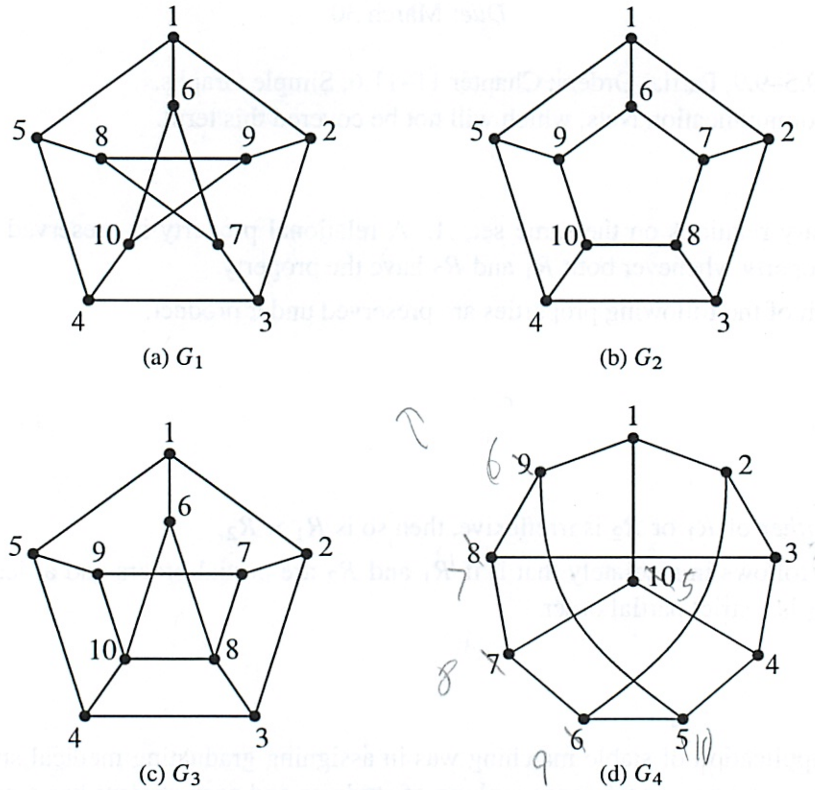


Figure 1 Which graphs are isomorphic?

Problem 5. **(a)** For any vertex, v, in a graph, let $N(v)$ be the set of *neighbors* of v, namely, the vertices adjacent to v:

$$N(v) ::= \{u \mid u\text{—}v \text{ is an edge of the graph}\}.$$

Suppose f is an isomorphism from graph G to graph H. Prove that $f(N(v)) = N(f(v))$.

Your proof should follow by simple reasoning using the definitions of isomorphism and neighbors—no pictures or handwaving.

Hint: Prove by a chain of iff's that

$$h \in N(f(v)) \quad \text{iff} \quad h \in f(N(v))$$

for every $h \in V_H$. Use the fact that $h = f(u)$ for some $u \in V_G$.

(b) Conclude that if G and H are isomorphic graphs, then for each $k \in \mathbb{N}$, they have the same number of degree k vertices.

Doing P-Set 6 3/28

1. Two relations on set

relational property preserved under product

if have both

So what is this

↑ "concatinate"

9.9 product orders

$a R_1 b_1$ and $a_2 R_2 b_2$

What is this exactly

$a_1 \xrightarrow{R_1} b_1$ and

$a_2 \xrightarrow{R_2} b_2$

WP: Cartesaian product

C, C is this same thing {suits} x {Ace, king, ... 10, 9, ...}

is the 52 cards

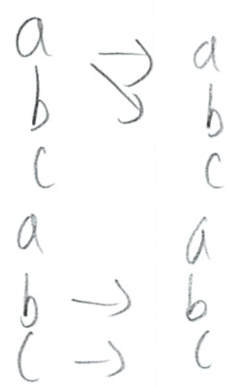
But this is a relation

- ~~sets~~ domain, codomain, graph
- basically an arrow

2

So like

Would be



?

or

Wish he did an example
One in book very unclear
So one relation younger
other shorter
So combined is both younger and shorter?
Both must have property
But how to show this problem is asking
Reflexitivity
- if aRa for all a∈A
But how to really explain?

3

2. How to modify

Was thinking about this

How much detail do they want?

(4)

3. 20 fund. human virtues

Each student 8 of these at start

" " unique set of virtue

Can an (1) addi. virtue be added so still unique

no

My first thought is we are adding - so of course more possibilities

What do we have now

1	1
2	2
3	3
⋮	⋮
20	20

20^8 possibilities

- oh repeats can't count

$20 \cdot 19 \cdot 18 \cdot 17 \cdot 16 \cdot 15 \cdot 14 \cdot 13$

Not prob so don't need $\binom{20}{8}$

Or is this it?

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

5

$\binom{20}{8} = 125970$

$= \dfrac{20!}{8! \times 12!}$

$20 \cdot \ldots \cdot 13 = 5$ billion

– much larger

What is right?

$\binom{20}{8}$ is ~~?? prob that 8 chosen?~~

No the choose notation likely right

My guess – why wrong

1 round

20 possible

2 round

$20 \cdot 19$ possible

AB
AC
AD
} 19

...

BA
BC
BD
} 19

20 times

= 380

$\binom{20}{2} = 190$ ↗ So 1 half of above

(6)

Oh duh since

$$AB = BA$$

So ( ) is correct

But they suggest Hall's theorm

- which is set of women liked by men if at least one man likes

So every subset of ~~men~~ women must be ~~as~~ size ≥ ~~every~~ every subset of men

So what are matches?

I can solve w/ ( ) - but that is not what they want

- we are not supposed to have learned yet

How would you solve like they want?

? student → virtue

- but nothing we learned is about # of likes from each

Or student → virtue set 125,450

then student → new virtue set 167,960

? Or need something more ~~[illegible]~~ - not the added item.

Don't foget Hall

7

4. isomophic

- Same arrows
- diff labling + layout

- same graph isomophic
- describe
- don't get if not
- and 2 of 4 or what? (guess just why not)
- prove one property

Count verticies

G_1	G_2	G_3	G_4
3	3	3	3
3	3	3	3
3	3	3	3
3	3	3	3
3	3	3	3
3	3	3	3
3	3	3	3
3	3	4	3
3	3	3	3
3	3	4	3

Oh but same # of verticies is not definitive
Must map vertices

(8)

G_1 G_2

1 → 1

. . . .

5 → 5

6 → 6

~~6→5~~

7 - goes to 6, 3, ~~[illegible]~~ 8

: so no ~~[illegible]~~ isomorphism :

G_4

How to check exactly:

---

def: edge preserving bijection

isomorphism is the bij

two graphs isomorphic if isomorphism b/w them

transitive (duh)

Graph "preserved under isomorphism"

Look at preserved properties

└ what is that?

not in book

WP: property preserved under all isomorphisms

(9)

So its no use in finding isomorphisms

Graph isomorphism property is hard

Can use matlab to check?

Too much work

I found one while doing write up

Hope writeup is enough

(10)

5. $N(v)$ is neighbors $N(v) ::= \{u \mid u-v \text{ is edge}\}$

f is isomophism

Prove $f(N(v)) = N(f(v))$

? isomorphism of neighbors

? neighbors of isomophism

- using def of iso + neighbors

(I get it - but how to write exactly?)

Use chain of iffs

$h \in N(f(v))$ iff $h \in f(N(v))$

for all $h \in V_H$

Use $h = f(u)$ for some $u \in V_G$

How to actually write?

This is what I don't get at all in this class

Ask Matt

He took awhile to figure out - but solved it in 3 lines

Still really don't get

- Study!

~~[illegible]~~ How get lines 1 + 2? ~~[illegible]~~ Can you make it any more basic?

11

3/29

Matt's Help

Subset of ele ~~that~~ in G would be N(v)

H is in N(v) if neighbor
def

Then iff f(H) is in f(N(v))
H is el of N(v)

"i can just assume"

~~the~~ Each h is = to f(v)

Start H=f(u)
~~N~~ u is N of some v
~~know that~~
then sine u - v
and H = f(u)
v is in f(v)
def of isomorphism
H is in N(f(u)) iff f^{-1}(G) is in N(v)

(bii?)

~~if u, v are N then~~

$f(u) \in N(f(v))$ IFF $u \in N(v)$ Def. of Isomorphism

$u \in N(v)$ IFF $f(u) \in f(N(v))$

let $h = f(u)$ for some $u \in V,$

$h \in N(f(v))$ IFF $h \in f(N(v))$

~~So $f(N(v))$~~

(12)

Now b

- just conclude?

I am going to expand with some stuff

Still looking back - why do you need the h = part?

I will not be able to remember this!

I think I did pretty good on this

Massachusetts Institute of Technology
6.042J/18.062J, Spring '11: Mathematics for Computer Science
Prof. Albert R Meyer

Solutions cover sheet
March 14

Student's Solutions to Problem Set 6

Your name: Michael Plasmeier

Due date: March 30

Submission date: 3/30

Circle your TA/LA: Ali Nick Oscar Oshani

Table 12

Collaboration statement: Circle one of the two choices and provide all pertinent info.

1. I worked alone and only with course materials.

2. I collaborated on this assignment with:

   got help from:[1] Matt Fawk

   and referred to:[2] wikipedia,
   Cartesian product - not right topic
   Binomical coefficent
   Graph property
   Graph isomorphism

DO NOT WRITE BELOW THIS LINE

Problem	Score
1	
2	
3	
4	
5	
Total	

[1]People other than course staff.

[2]Give citations to texts and material other than the Spring '11 course materials.

1. a. Reflexive
   - aRa for all $a \in A$

10

This means that every item has a self arrow, at least

If this is true for both R_1 and R_2 then it will be true for $R_1 \times R_2$

~~Rough example.~~

~~$a \to a$ $\quad$ $a \to a$~~
~~$b \to b$ $\quad$ $b \to b$~~
~~$c \to c$ $\quad$ $c \to c$~~

~~$a \to a \to a$~~
~~$b \to b \to b$~~
~~$c \to c \to c$~~

If there is a self arrow for each item - in both of the relations - there will be a self arrow in the combied item

$$(a_1, a_2)\,(R_1 \times R_2)\,(b_1, b_2) \text{ iff } [a_1 R_1 b_1 \text{ and } a_2 R_2 b_2]$$

This is because $R_1 \times R_2$ means the edge must be in both relations - it must satisfy both conditions - ie younger + shorter (example from book)

(2)

antisymmetry -

aRb IMPLIES NOT (bRa) for all $a \neq b \in A$

This bacially means arrows are only allowed in one direction.

$\rightarrow$ or $\leftarrow$ NOT $\leftrightarrow$

If one relation is antisymmetric and the other relation is antisymetric then the whole thing will be antisymmetric.

Actually could say only one relation needs to be antisymmetric to make the $R_1 \times R_2$ antisymmetric - right?

- because the one relation "breaks" the ability to go backwards

[There is at most one edge between two points but can be self loops]

6.

(3)

transitivity –

$$\forall x, y, z \in A. \ (xRy \text{ and } yRz) \text{ IMPLIES } xRz$$

To me this is the definition of a product order. –

Well wait – no the example is something different.

Only have one domain and codomain – and must satisfy both conditions – younger AND shorter

Never-the less. this condition still applies inside.

If there is a positive lenght path from u to v then there simply can be an edge from u to v.

If both R_1 and R_2 have it, then $R_1 \times R_2$ will have it because $R_1 \times R_2$ is the arrows that satisfy both conditions.

(4.)

b. If either R_1 or R_2 is irreflexive then so is $R_1 \times R_2$

R is irreflexive when NOT $[\exists x \in A \quad x R x]$

Basically it means there can not be no self loops,

$R_1 \times R_2$ means that both conditions need to be true
- ie younger and shorter

If one of the relations does not have self-loops
- ie is irreflexive then $R_1 \times R_2$ will not have it

Michael Plasmeier
Oshani
Table 12

10

#2 We can modify The Mating Ritual so that we can assign students to hospital residencies,

However since # students ≠ # spots then we do not guarentee that every student (if # students > # spots) is placed or every spot (if # students < # spots) is filled.

One way to think of it is each spot is a seperate "balcony" and that all spots in a hosital have the same pref list However this has some problems. Which spot/"balcony" should students stand under at a given hospital? — this does work.

A better approach would be that hospitals are one balcony but they keep their top N students where N is the number of spots at their certain hospital. If a student is not prefered and can't get a spot, then they go to their next choice hospital. Also this plan allows hospitals to have a diff. # of spots.

(2)

Remember stable matching means no hospital perfers a
Student more AND student perfers that hospital more
This is still true.

Also true is Lemm 11.6.4: For every hospital h and student s
if h is crossed of s's list then h has N students
it perfers over s

Students get their optimal matching
Hospitals " " pessimal

Michael Plasmeier
Oshani
Table 12

P-Set 6

#3 There are $\binom{20}{8}$ possible combos of the 8 virtues, This means that there are 125,970 possible combos — far larger than the size of 6.042.

When you add a virtue there are $\binom{20}{9}$ combos. This is 167,960 possible combos.

can you guarantee this for every subset of students?

Since 167,960 ≥ 125,970 it means that a unique solution can still be found, since there are now more possibilities. An extra virtue can always be imported.

each student can only end up with certain set of 9 virtues

Alterantivly we can represent this as a biparte graph

Students	→	Virtue Set of 8
~100		125,970
↑ exactly 1 arrow out		↑ 0 or 1 arrow in

5

Then the bigger virtue set

Students	→	Virtue Set of 9
		167,960
↑ can still exactly 1 arrow out		↑ still 0 or 1 arrow in since more possibilities

2

By Halls Theorm/ Matching Principle

The # of virtue sets must be $\geq$ # of students for every possible subset of students

Michael Plasmeier

Oshani

Table 12

P-Set 6

4

#4. First off we know that G_3 is not isomophic because all verticies in G_1, G_2, G_4 have a degree of 3. Some verticies in G_3 have degree 4.

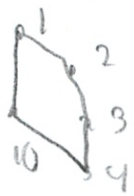
The other 3 remain canidates for isomorphism. However I was not able to find a match between some of them.

G_1 and G_2 - If you would unravel the star into a pentagon (which would normally be possible) it would break the ring.

Can you be more precise?

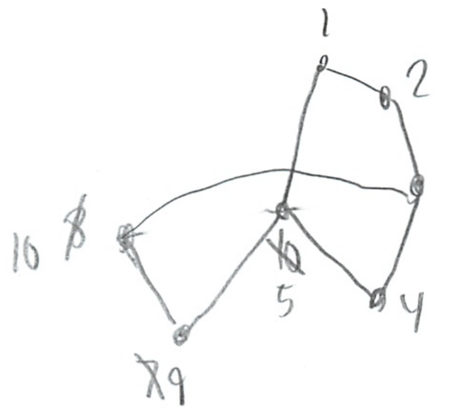
G_2 and G_4 - I could not find a way to map this. In G_4 what would be the ring. There needs to be a path that includes 5 vertices where the third path is to one of the other points included.

I was able to make such a path in G_4



(2)

However then point 10, which I labled as 5 did not connect well. I did mark 7 as 9 and 8 as 10 but then I was forced to mark 3 as 8. 3 should be part of the original ring, so I can not lable it so there is not an isomorphism

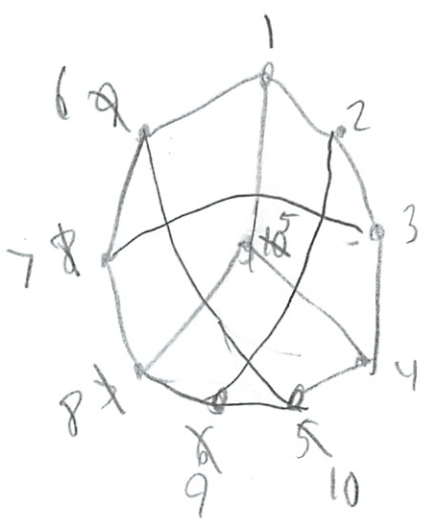


3 ← would have to be 8
contridiction

<u>G_1 and G_4</u> - I was able to find an isomorphism between this. Again I labled 1, 2, 3, 4, 10 as 1, 2, 3, 4, 5. I then sought to lable the rest. In G_1 5 is connected to 1, 4, 8. I already had 1, 4 so the third line from 5 in G_4 must be labled 8. (it was 7) 8 then connects to 7 and 9 in G_1. I saw that 8 in G_4 also connected to 3, just like 7 did in G_1 so I labled 8 as 7.

(3).

This let me fill in 9 as 6 in G_4 - wich links to 2 - like in G_1. I then contined orand G_1 and Saw that 9 was linked to 10. This must be 5 On G_4. 10 on G_1 links to 6. This must be 9 on G_4. All points have been relabled and Confirmed sucessfully so an isomophism exists



proof of property?

In general, you don't need to give a story, it obscures your solution.

Michael Plasmeier

Oshani

Table 12

P-set 6

1/7 #5 Proof $f(N(v)) = N(f(v))$

same statement since using iff

$f(u) \in N(f(v))$ iff $u \in N(v)$ Def of Isomorphism

$u \in N(v)$ iff $f(u) \in f(N(v))$

no actual proof

Let $h = f(u)$ for some $u \in V_G$

$h \in N(f(v))$ iff $h \in f(N(v))$ for all $h \in V_H$

3/3 b. So if G, H are isomorphic, then for each $k \in \mathbb{N}$ they have the same # of degree k vertices

- property of isomorphism
- We showed above that for every $h \in V_H$ that $h = f(v)$ for some $v \in V_G$. This means that all the vertices need to isomorphic - and have some match in degree

Solutions to Problem Set 6

Reading: Chapter 9.5–9.9, Partial Orders; Chapter **??–??**, Simple Graphs.
Skip Chapter 10, Communication Nets, which will not be covered this term.

Problem 1.
Let R_1, R_2 be binary relations on the same set, A. A relational property is preserved under product, if $R_1 \times R_2$ has the property whenever both R_1 and R_2 have the property.

(a) Verify that each of the following properties are preserved under product.

1. reflexivity,
2. antisymmetry,
3. transitivity.

Solution. These facts follows directly from the definitions. We'll write out just the case of antisymmetry. So suppose R_1, R_2 are antisymmetric.

Proof. To prove $R_1 \times R_2$ is antisymmetric, suppose

$$(r_1, r_2)\ [R_1 \times R_2]\ (s_1, s_2) \quad \text{and also} \tag{1}$$
$$(s_1, s_2)\ [R_1 \times R_2]\ (r_1, r_2). \tag{2}$$

We need to show that $(r_1, s_1) = (r_2, s_2)$.

By (1) and the definition of $R_1 \times R_2$, we know that $r_i\ R_i\ s_i$ for $i = 1, 2$. Similarly, by (1) $s_i\ R_i\ r_i$. Since R_i is antisymmetric, it follows that $r_i = s_i$ for $i = 1, 2$. That is, $(r_1, s_1) = (r_2, s_2)$. ∎

(b) Verify that if *either* of R_1 or R_2 is irreflexive, then so is $R_1 \times R_2$.

Solution. We may as well assume R_1 is irreflexive. This means that $\text{NOT}(r_1\ R_1\ r_1)$ for every $r_1 \in \text{domain}(R_1)$. So by definition of relational product,

$$\text{NOT}[(r_1, r_2)\ [R_1 \times R_2]\ (r_1, s_2)]$$

for all $r_1 \in \text{domain}(R_1)$ and $r_2, s_2 \in \text{domain}(R_2)$. In particular

$$\text{NOT}[(r_1, r_2)\ [R_1 \times R_2]\ (r_1, r_2)],$$

which implies that $R_1 \times R_2$ is irreflexive. ∎

Note that it now follows immediately that if if R_1 and R_2 are partial orders and at least one of them is strict, then $R_1 \times R_2$ is a strict partial order.

Problem 2.
The most famous application of stable matching was in assigning graduating medical students to hospital residencies. Each hospital has a preference ranking of students and each student has a preference order of hospitals, but unlike the setup in the notes where there are an equal number of boys and girls and monogamous marriages, hospitals generally have differing numbers of available residencies, and the total number of residencies may not equal the number of graduating students. Modify the definition of stable matching so it applies in this situation, and explain how to modify the Mating Ritual so it yields stable assignments of students to residencies.

Briefly indicate what, if any, modifications of the preserved invariant used to verify the original Mating are needed to verify this one for hospitals and students.

Solution. The Mating Ritual can be applied to this situation by letting the students be the boys and each of the *residencies* (not the hospitals) be the girls.

A matching is an assignment of students to residencies (an injection, A : students $\rightarrow$ residencies) such that every student has a residency (A is total), or every residency has an assigned student (A is a surjection). A stable assignment is one with no *rogue couples*, where a rogue couple is a hospital student pair (H, S) such that S is not assigned to one of the residencies at H, which she prefers over her current assignment, and

- H has some students assigned to some of its residencies and prefers S to at least one of its assigned students, or
- H has none of its residencies assigned,

■

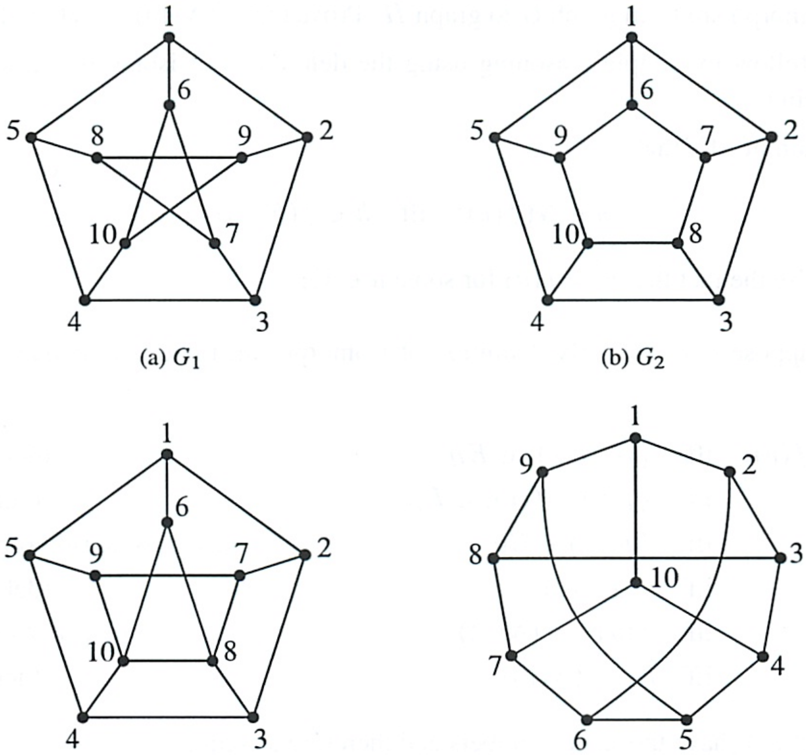
Problem 3.
Scholars through the ages have identified *twenty* fundamental human virtues: honesty, generosity, loyalty, prudence, completing the weekly course reading-response, etc. At the beginning of the term, every student in Math for Computer Science possessed exactly *eight* of these virtues. Furthermore, every student was unique; that is, no two students possessed exactly the same set of virtues. The Math for Computer Science course staff must select *one* additional virtue to impart to each student by the end of the term. Prove that there is a way to select an additional virtue for each student so that every student is unique at the end of the term as well.

Suggestion: Use Hall's theorem. Try various interpretations for the vertices on the left and right sides of your bipartite graph.

Solution. Construct a bipartite graph G as follows. The vertices on on the left are all students and the virtues on the right are all subset of nine virtues. There is an edge between a student and a set of 9 virtues if the student already has 8 of those virtues.

Each vertex on the left has degree 12, since each student can learn one of 12 additional virtues. The vertices on the right have degree at most 9, since each set of 9 virtues has only 9 subsets of size 8. So this bipartite graph is degree-constrained, and therefore, by Lemma **??**, there is a matching for the students. Thus, if each student is taught the additional virtue in the set of 9 virtues with whom he or she is matched, then every student is unique at the end of the term. ■

Problem 4.
Determine which among the four graphs pictured in the Figures are isomorphic. If two of these graphs are isomorphic, describe an isomorphism between them. If they are not, give a property that is preserved under



(a) G_1 (b) G_2 (c) G_3 (d) G_4

Figure 1 Which graphs are isomorphic?

isomorphism such that one graph has the property, but the other does not. For at least one of the properties you choose, *prove* that it is indeed preserved under isomorphism (you only need prove one of them).

Solution. G_1 and G_3 are isomorphic. In particular, the function $f : V_1 \to V_3$ is an isomomorphism, where

$f(1) = 1$	$f(2) = 2$	$f(3) = 3$	$f(4) = 8$	$f(5) = 9$
$f(6) = 10$	$f(7) = 4$	$f(8) = 5$	$f(9) = 6$	$f(10) = 7$

G_1 and G_4 are not isomorphic to G_2: G_2 has a vertex of degree four and neither G_1 nor G_4 has one.

G_1 and G_4 are not isomorphic: G_4 has a cycle of length four and G_1 does not.

There are many examples of properties preserved under graph isomorphism. For example, we will prove that the degree of each vertex is preserved under isomorphism.

Let G and H be isomorphic graphs. Since they are isomorphic, there is an edge-preserving bijection between the vertices of G and H:

$$f(u) \in V(H) \longleftrightarrow f(u) \in V(G)$$

We let the set of vertices adjacent to u be $N(u)$. Because f is an edge-preserving bijection, there is an edge from $f(u)$ to a vertex $f(k)$ iff $k \in N(u)$. Thus $|N(f(u))| = |N(u)|$ and the degree of each vertex is preserved under isomorphism. ■

Problem 5. (a) For any vertex, v, in a graph, let $N(v)$ be the set of *neighbors* of v, namely, the vertices adjacent to v:

$$N(v) ::= \{u \mid \langle u\text{—}v\rangle \text{ is an edge of the graph}\}.$$

Suppose f is an isomorphism from graph G to graph H. Prove that $f(N(v)) = N(f(v))$.

Your proof should follow by simple reasoning using the definitions of isomorphism and neighbors—no pictures or handwaving.

Hint: Prove by a chain of iff's that

$$h \in N(f(v)) \quad \text{iff} \quad h \in f(N(v))$$

for every $h \in V_H$. Use the fact that $h = f(u)$ for some $u \in V_G$.

Solution. *Proof.* Suppose $h \in V_H$. By definition of isomorphism, there is a unique $u \in V_G$ such that $f(u) = h$. Then

$$\begin{aligned}
h \in N(f(v)) \quad &\text{iff} \quad \langle h\text{—}f(v)\rangle \in E_H && \text{(def of } N) \\
&\text{iff} \quad \langle f(u)\text{—}f(v)\rangle \in E_H && \text{(def of } u) \\
&\text{iff} \quad \langle u\text{—}v\rangle \in E_V && \text{(since } f \text{ is an isomorphism)} \\
&\text{iff} \quad u \in N(v) && \text{(def of } N) \\
&\text{iff} \quad f(u) \in f(N(v)) && \text{(def of } f\text{-image)} \\
&\text{iff} \quad h \in f(N(v)) && \text{(def of } u)
\end{aligned}$$

So $N(f(v))$ and $f(N(v))$ have the same members and therefore are equal.

■

(b) Conclude that if G and H are isomorphic graphs, then for each $k \in \mathbb{N}$, they have the same number of degree k vertices.

Solution. By definition, $\deg(v) = |N(v)|$. Since an isomorphism is a bijection, any set of vertices and its image under an isomorphism will be the same size (by the Mapping Rule from Week 2 Notes), so part (a) implies that an isomorphism, f, maps degree k vertices to degree k vertices. This means that the image under f of the set of degree k vertices of G is precisely the set of degree k vertices of H. So by the Mapping Rule again, there are the same number of degree k vertices in G and H. ■

3/30

Mathematics for Computer Science

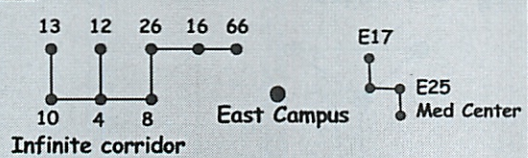
MIT 6.042J/18.062J

Graph Connectivity
Trees & Coloring

Connected Components

Every graph consists of separate connected pieces (subgraphs) called connected components

Connected Components



3 connected components

the more connected components, the more "broken up" the graph is.

actually wrong now

Connected Components

The connected component of vertex v ::=

$\{w \mid v \text{ and } w \text{ are connected}\}$

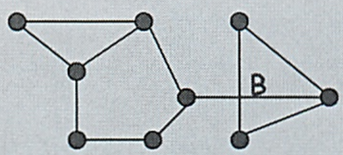
Connected Components

So a graph is connected iff it has only 1 connected component

Cut Edges

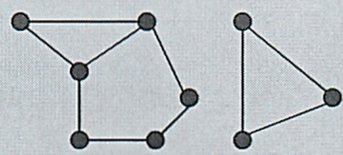
An edge is a cut edge if removing it from the graph disconnects two vertices.

Cut Edges



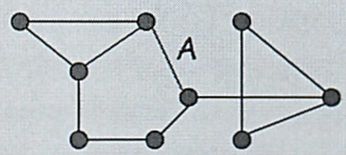
B is a cut edge

Cut Edges



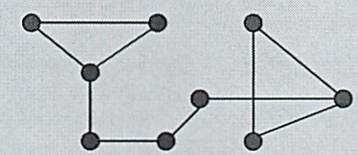
deleting B gives two components

Cut Edges



A is *not* a cut edge

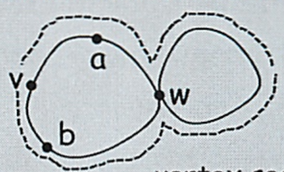
Cut Edges



still connected with edge A deleted

Closed Walks

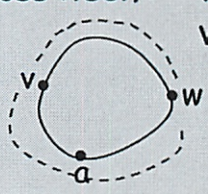
A closed walk is a walk that begins and ends with the same vertex



vertex sequence:
v···b···w···w···a···v

Cycles

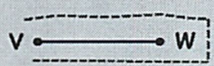
A cycle is a closed walk of length > 2 that doesn't cross itself: vertex sequence:
v···a···w···v



also:
w···a···v···w

Cycles

length $>$ 2 implies that going back & forth over an edge is not a cycle



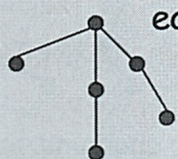
Cut Edges and Cycles

Lemma: An edge is a not a cut edge iff it is on a cycle.

Trees

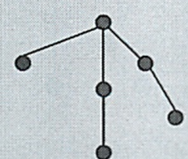
A tree is a connected graph with no cycles.

equivalently:

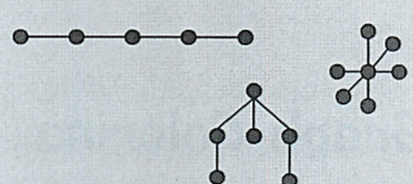


Trees

A tree is a connected graph with every edge a cut edge.



More Trees



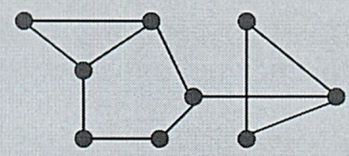
Other Tree Definitions

- graph with a unique path between any 2 vertices
- connected graph with n vertices and n-1 edges
- an edge-maximal acyclic graph

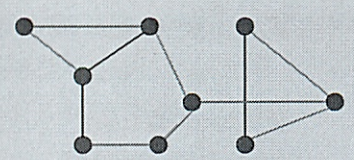
Spanning Trees

A *spanning tree* of a graph G is any subgraph T that is a tree and contains all the vertices of G.

Spanning Trees

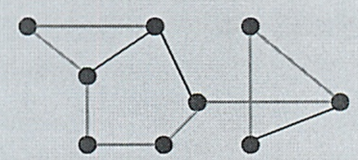


Spanning Trees



a spanning tree

Spanning Trees



another spanning tree
(can have many)

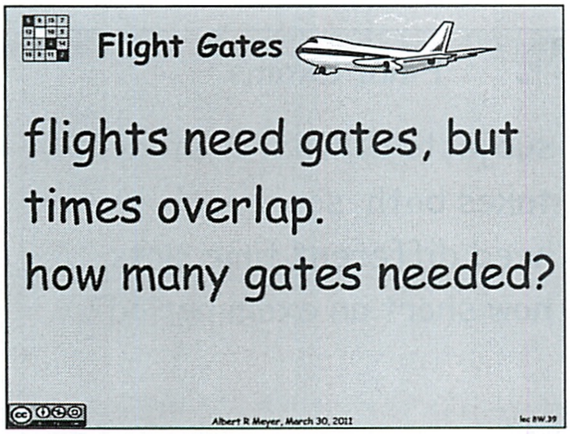
Spanning Trees

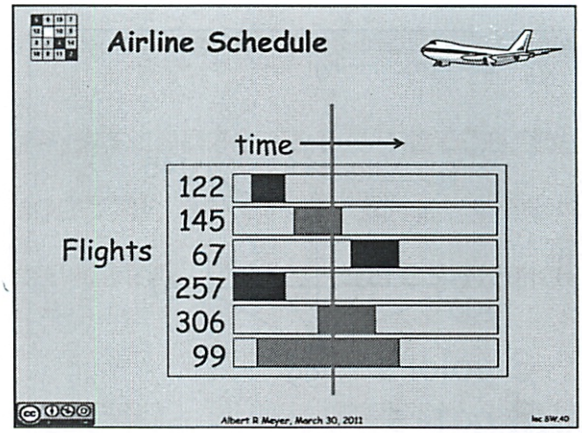
Lemma: G connected implies G has a spanning tree

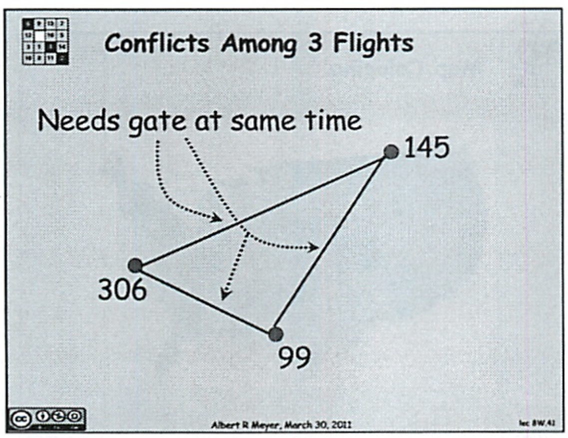
Pf: Among connected subgraphs with all the vertices of G: those with the fewest edges are spanning trees. (Why?)

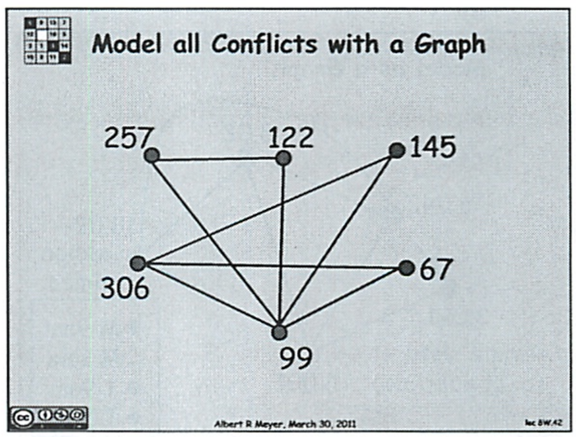
Mathematics for Computer Science
MIT 6.042J/18.062J

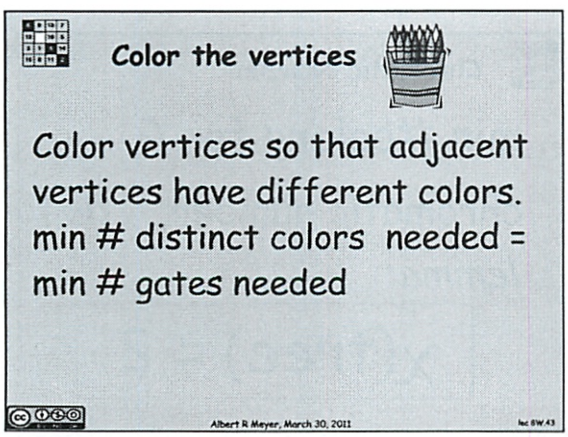
Graph Coloring

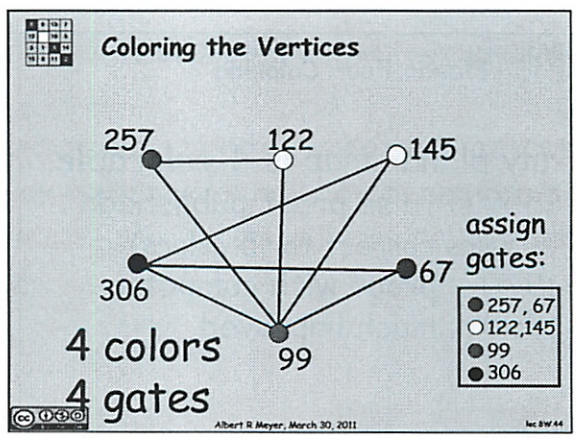

Flight Gates
flights need gates, but
times overlap.
how many gates needed?
Albert R Meyer, March 30, 2011


Airline Schedule
time
Flights
122
145
67
257
306
99
Albert R Meyer, March 30, 2011

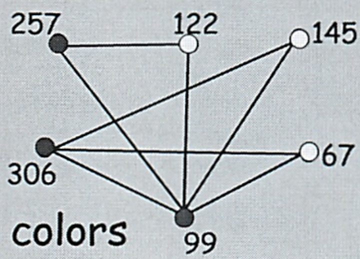

Conflicts Among 3 Flights
Needs gate at same time
145
306
99
Albert R Meyer, March 30, 2011


Model all Conflicts with a Graph
257
122
145
306
67
99
Albert R Meyer, March 30, 2011


Color the vertices
Color vertices so that adjacent
vertices have different colors.
min # distinct colors needed =
min # gates needed
Albert R Meyer, March 30, 2011


Coloring the Vertices
257
122
145
assign
gates:
306
67
257, 67
122,145
99
306
4 colors
99
4 gates
Albert R Meyer, March 30, 2011

Better coloring

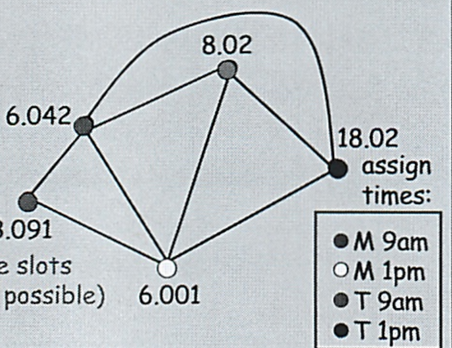


3 colors
3 gates

Final Exams

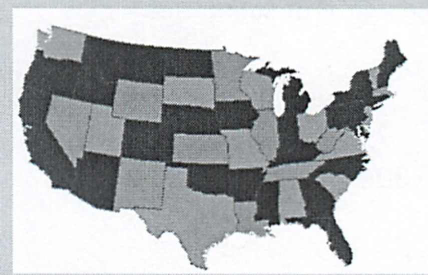
subjects conflict if student takes both, so
need different time slots.
how short an exam period?

Model as a Graph



4 time slots
(best possible)

Map Coloring



Planar Four Coloring

any planar map is 4-colorable.
1850's: false proof published
(was correct for 5 colors).
1970's: proof with computer
1990's: much improved

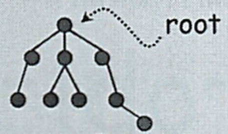
Chromatic Number

min #colors for G is
chromatic number, $\chi(G)$

lemma:

$$\boxed{\chi(\text{tree}) = 2}$$

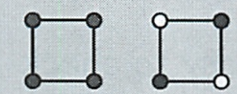
Trees are 2-colorable



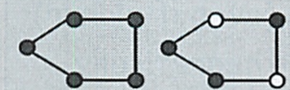
Pick any vertex as "root."
if (unique) path from root is
even length: ●
odd length: ●

 lec 8W.53

Simple Cycles



$\chi(C_{\text{even}}) = 2$



$\chi(C_{\text{odd}}) = 3$

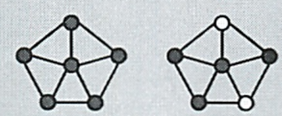
 lec 8W.54

Complete Graph K_5

$\chi(K_n) = n$

 lec 8W.55

The Wheel W_n



W_5

$\chi(W_{\text{odd}}) = 4$
$\chi(W_{\text{even}}) = 3$

 lec 8W.56

Bounded Degree

all degrees $\leq$ k, implies

$\chi(G) \leq k+1$

very simple algorithm...

 lec 8W.57

"Greedy" Coloring

...color vertices in any order.
next vertex gets a color different from its neighbors.

$\leq$ k neighbors, so
k+1 colors always work

 lec 8W.58

coloring arbitrary graphs

2-colorable? --easy to check
3-colorable? --hard to check
(even if planar)

find $\chi(G)$? --theoretically no harder than 3-color, but harder in practice

Team Problems

Problems
1—4

6.042 Graph Conectivity + Coloring

3/30

In general graph not connected

Fom groups that are connected "connected compents"

Not all components connected to each other

Def v

$\{w \mid v$ and w are connected$\}$

Connected if only 1 connected coomponent

Cut edge - if remove 2 vertices are no longer conected

now ≥ 2 connected components

Closed walk - walk begins + ends at same pt

- can ~~growth~~ go through pts multiple times
  - not a cycle
- need to specify begining, end, stops

Cycle - special closed walk

- lenght ≥ 2
- does not cross itself
- don't think of begining, end, or direction
- "hunk" of graph

Lemma: An edge is not a cut edge iff it is on a ~~simple~~ cycle

 can still get anywhere

- kinda follows from def

Trees - connected graph w/ no cycles
- break any edge, it falls apart
- unique ~~pa~~ path b/w any 2 points
- graph w/ 1 vertex 0 edges is a tree
- connected graph w/ n vertices + $n-1$ edges
  - is proved in notes

Spanning Tree - minimal set of edges that allow everything to be connected
- ~~AAA~~ purple subgraph on slides
- can have multiple
- Cool algebra to calc how many spanning trees are

③

Lemma G connected → G has spanning trees

- the one w/ the fewest edges is the spanning tree

Graph Coloring

- scheduling
- resolving conflicts
- how many gates are needed?

- draw edge b/w flights on ground at same time
  - at some moment

Color the vertices so adj vertices have diff colors
- each gate diff color
- then that min # of colors is ~~an~~ min # gates needed

May not color right - his inital try had 4 colors
Did again for 3
Problem to find min # colors

Final Exam scheduling

How short an exam period can you get away w/?

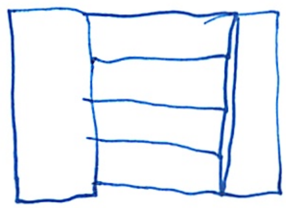
(4)

Do graph coloring

Also map coloring
- if have border - diff colors
- Corners don't count

Planar Map can always be done in 4 colors

(? Wait



Oh wait is right)

Needs 600 cases for computer to check

Min # colors for G is

Chromatic # $\chi(G)$
?chi

Trees are 2-Colorable
- One color per level
- or more abstractly distance from root
- even + odd

Cycles even lenght

$\chi = 2$

Cycles odd $\chi = 3$

(5)

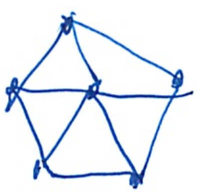
Messiest K_5 Complete graph

- Since every vertex adj to each one
- So 5 colors - each one is different

Comple K_n

~~dt~~ $\chi(K_n) = n$

Wheel W_n



circle w/ axle in middle

4 colors

- 3 for odd cycle
- axel 4th color

$\chi(W_{odd}) = 4$

$\chi(W_{even}) = 3$

Greedy Assignment

Assign something ~~tha~~ that does not conflict w/ neighbor

$\leq k$

(too fast!)

(6)

2 colorable check

- easy

3 colorable check

- very hard, Millenium prize
- even if planer
  - know 4 is enough

Can translate graphs into not, or gates if find SAT solution

$\chi(G)$

- theoretically as hard as 3-color
- pragmatically

In-Class Problems Week 8, Wed.

Problem 1.

False Claim. *If every vertex in a graph has positive degree, then the graph is connected.*

(a) Prove that this Claim is indeed false by providing a counterexample.

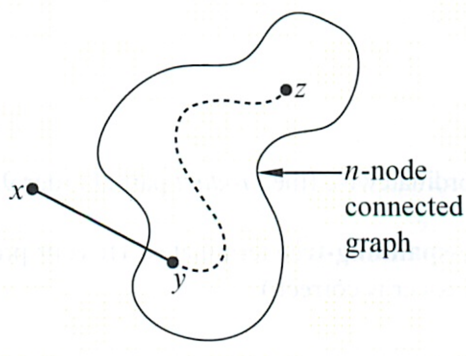
(b) Since the Claim is false, there must be an logical mistake in the following bogus proof. Pinpoint the *first* logical mistake (unjustified step) in the proof.

Bogus proof. We prove the Claim above by induction. Let $P(n)$ be the proposition that if every vertex in an n-vertex graph has positive degree, then the graph is connected.

Base cases: ($n \leq 2$). In a graph with 1 vertex, that vertex cannot have positive degree, so $P(1)$ holds vacuously.

$P(2)$ holds because there is only one graph with two vertices of positive degree, namely, the graph with an edge between the vertices, and this graph is connected.

Inductive step: We must show that $P(n)$ implies $P(n+1)$ for all $n \geq 2$. Consider an n-vertex graph in which every vertex has positive degree. By the assumption $P(n)$, this graph is connected; that is, there is a path between every pair of vertices. Now we add one more vertex x to obtain an $(n+1)$-vertex graph:



All that remains is to check that there is a path from x to every other vertex z. Since x has positive degree, there is an edge from x to some other vertex, y. Thus, we can obtain a path from x to z by going from x to y and then following the path from y to z. This proves $P(n+1)$.

By the principle of induction, $P(n)$ is true for all $n \geq 0$, which proves the Claim.

■

Problem 2.

Procedure **create-spanning-tree**

Given a simple graph G, keep applying the following operations to the graph until no operation applies:

1. If an edge $\langle u—v \rangle$ of G is on a cycle, then delete $\langle u—v \rangle$.
2. If vertices u and v of G are not connected, then add the edge $\langle u—v \rangle$.

Assume the vertices of G are the integers $1, 2, \ldots, n$ for some $n \geq 2$. Procedure **create-spanning-tree** can be modeled as a state machine whose states are all possible simple graphs with vertices $1, 2, \ldots, n$. The start state is G, and the final states are the graphs on which no operation is possible.

(a) Let G be the graph with vertices $\{1, 2, 3, 4\}$ and edges

$$\{\langle 1—2 \rangle, \langle 3—4 \rangle\}$$

What are the possible final states reachable from start state G? Draw them.

(b) Prove that any final state of must be a tree on the vertices.

(c) For any state, G', let e be the number of edges in G', c be the number of connected components it has, and s be the number of cycles. For each of the derived variables below, indicate the *strongest* of the properties that it is guaranteed to satisfy, no matter what the starting graph G is and be prepared to briefly explain your answer.

The choices for properties are: *constant*, *strictly increasing*, *strictly decreasing*, *weakly increasing*, *weakly decreasing*, *none of these*. The derived variables are

(i) e

(ii) c

(iii) s

(iv) $e - s$

(v) $c + e$

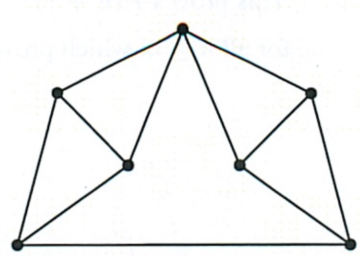
(vi) $3c + 2e$

(vii) $c + s$

(viii) (c, e), partially ordered coordinatewise (the *product* partial order 9.9.1).

(d) Prove that procedure **create-spanning-tree** terminates. (If your proof depends on one of the answers to part (c), you must prove that answer is correct.)

Problem 3.
Let G be the graph below[1]. Carefully explain why $\chi(G) = 4$.



[1] From *Discrete Mathematics*, Lovász, Pelikan, and Vesztergombi. Springer, 2003. Exercise 13.3.1

Problem 4.

A portion of a computer program consists of a sequence of calculations where the results are stored in variables, like this:

$$\begin{array}{rrcl}
 & \text{Inputs:} & & a, b \\
\text{Step 1.} & c & = & a + b \\
2. & d & = & a * c \\
3. & e & = & c + 3 \\
4. & f & = & c - e \\
5. & g & = & a + f \\
6. & h & = & f + 1 \\
 & \text{Outputs:} & & d, g, h
\end{array}$$

A computer can perform such calculations most quickly if the value of each variable is stored in a *register*, a chunk of very fast memory inside the microprocessor. Programming language compilers face the problem of assigning each variable in a program to a register. Computers usually have few registers, however, so they must be used wisely and reused often. This is called the *register allocation* problem.

In the example above, variables a and b must be assigned different registers, because they hold distinct input values. Furthermore, c and d must be assigned different registers; if they used the same one, then the value of c would be overwritten in the second step and we'd get the wrong answer in the third step. On the other hand, variables b and d may use the same register; after the first step, we no longer need b and can overwrite the register that holds its value. Also, f and h may use the same register; once $f + 1$ is evaluated in the last step, the register holding the value of f can be overwritten.(Assume that the computer carries out each step in the order listed and that each step is completed before the next is begun.)

(a) Recast the register allocation problem as a question about graph coloring. What do the vertices correspond to? Under what conditions should there be an edge between two vertices? Construct the graph corresponding to the example above.

(b) Color your graph using as few colors as you can. Call the computer's registers $R1$, $R2$, etc. Describe the assignment of variables to registers implied by your coloring. How many registers do you need?

(c) Suppose that a variable is assigned a value more than once, as in the code snippet below:

$$\begin{aligned}
&\ldots \\
t &= r + s \\
u &= t * 3 \\
t &= m - k \\
v &= t + u \\
&\ldots
\end{aligned}$$

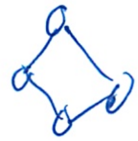
How might you cope with this complication?

InClass 8Wed

Connected - ~~Path from every vertex to every other vertex~~

Total - ~~vert~~ edge from every vertex to every vertex

Connected - every vertex has at least one edge



1.a Prove false by counter example

i. Isn't that true

Is true - as in that must be true for connected

But exception

> 0 degree ~~→~~ → connected

Connected → > 0 degree

b) Must be logical mistake in proof

Well if keep adding a ~~line~~ point and line to last added point then it would work

But ~~even~~ counterexample does not do this

Meyer: I want to know exactly which step it went wrong

②

- does not matter which edge you connect to in proof - this is the issue

Meyer: The QED

- no line in here that is wrong
- proving wrong thing
- are graphs w >0 degree that can't be built that way
- build up error
- Induction - think about N+1
  - break up into smaller pieces you understand
  - must be sure built every possible graph

What I put!

Lectre

---

2. Create a spanning Tree proc given simple graph G
   1. If edge (u-v) is on cycle, delete
   2. If verticies u, v not connected add ⟨u-v⟩

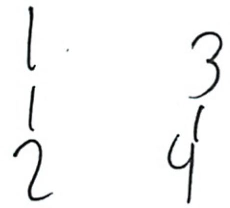
③

Assume vertices 1, 2, ... n for some $n \geq 2$

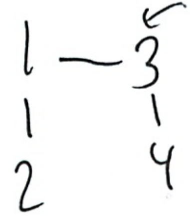
Can model as SM

- all possible graphs that can be constructed

a) Let G be ~~{1, 2, 3, 4}~~

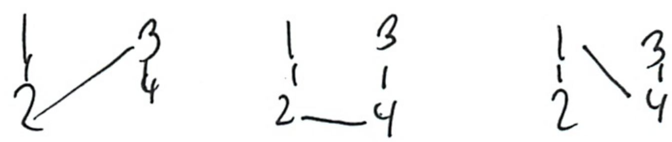


What are possible states



← not connected means in general connected ~~e~~ guessing or directly (ie ~~...~~ edge b/w them)

Also



b) Prove that final state must be a tree

; def of tree

Check def of tree and use

A final state reached when proc terminates

- when no cycles in graph & all vertices connected

Theirfore all final states are connected graphs w/o cycles

→ So all final states are trees

(4)

That def feels to me as cheating

c) Which property guarenteed to satisfy

- no matter starting graph

So basically - what happens to variables

e none

c weakly ↓

s " ↓

e - s " ↑

c + e " ↓

3c + 2e strictly ↓

c + s " ↓

(c, e) ?

Strictly - always ↓

Weakly - ↓ or $\underbrace{\text{stays same}}_{\text{weakly adds this possibility}}$

d) Prove terminates

- that one of these quantaties comes to what the def. said

$c = 1 \quad s = 0$

$c \geq 1 \quad s \geq 0$ keeps ↓

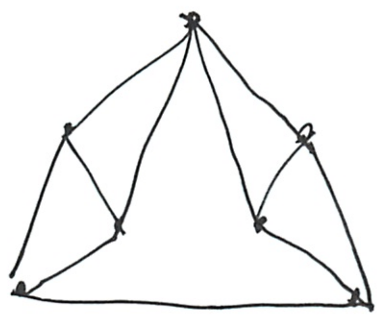
connected components always ↓ to 1 - so must terminate

c + s strongly decreasing - one or other must go down

(5)

3. Why $\chi(G) = 4$

graph given



easy to show 4 colors

but how to prove 3 colors

¿ Like I did on P-set - show example ?

No Just proves that one did not work

Not that there could be something that works

Solutions to In-Class Problems Week 8, Wed.

Problem 1.

False Claim. *If every vertex in a graph has positive degree, then the graph is connected.*

(a) Prove that this Claim is indeed false by providing a counterexample.

Solution. There are many counterexamples; here is one:

■

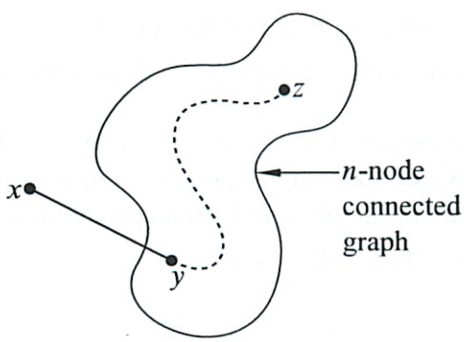
(b) Since the Claim is false, there must be an logical mistake in the following bogus proof. Pinpoint the *first* logical mistake (unjustified step) in the proof.

Bogus proof. We prove the Claim above by induction. Let $P(n)$ be the proposition that if every vertex in an n-vertex graph has positive degree, then the graph is connected.

Base cases: ($n \leq 2$). In a graph with 1 vertex, that vertex cannot have positive degree, so $P(1)$ holds vacuously.

$P(2)$ holds because there is only one graph with two vertices of positive degree, namely, the graph with an edge between the vertices, and this graph is connected.

Inductive step: We must show that $P(n)$ implies $P(n+1)$ for all $n \geq 2$. Consider an n-vertex graph in which every vertex has positive degree. By the assumption $P(n)$, this graph is connected; that is, there is a path between every pair of vertices. Now we add one more vertex x to obtain an $(n+1)$-vertex graph:



All that remains is to check that there is a path from x to every other vertex z. Since x has positive degree, there is an edge from x to some other vertex, y. Thus, we can obtain a path from x to z by going from x to y and then following the path from y to z. This proves $P(n+1)$.

By the principle of induction, $P(n)$ is true for all $n \geq 0$, which proves the Claim.

■

Solution. This one is tricky: the proof is actually a good proof of something else. The first error in the proof is only in the final statement of the inductive step: "This proves $P(n+1)$".

The issue is that to prove $P(n+1)$, *every* $(n+1)$-vertex positive-degree graph must be shown to be connected. But the proof doesn't show this. Instead, it shows that every $(n+1)$-vertex positive-degree graph *that can be built up by adding a vertex of positive degree to an n-vertex connected graph*, is connected.

The problem is that *not every* $(n+1)$-vertex positive-degree graph can be built up in this way. The counterexample above illustrates this: there is no way to build that 4-vertex positive-degree graph from a 3-vertex positive-degree graph.

More generally, this is an example of "buildup error". This error arises from a faulty assumption that every size $n+1$ graph with some property can be "built up" in some particular way from a size n graph with the same property. (This assumption is correct for some properties, but incorrect for others—such as the one in the argument above.)

One way to avoid an accidental build-up error is to use a "shrink down, grow back" process in the inductive step: start with a size $n+1$ graph, remove a vertex (or edge), apply the inductive hypothesis $P(n)$ to the smaller graph, and then add back the vertex (or edge) and argue that $P(n+1)$ holds. Let's see what would have happened if we'd tried to prove the claim above by this method:

Inductive step: We must show that $P(n)$ implies $P(n+1)$ for all $n \geq 1$. Consider an $(n+1)$-vertex graph G in which every vertex has degree at least 1. Remove an arbitrary vertex v, leaving an n-vertex graph G' in which every vertex has degree... uh-oh!

The reduced graph G' might contain a vertex of degree 0, making the inductive hypothesis $P(n)$ inapplicable! We are stuck—and properly so, since the claim is false! ■

Problem 2.

Procedure **create-spanning-tree**

Given a simple graph G, keep applying the following operations to the graph until no operation applies:

1. If an edge $\langle u—v \rangle$ of G is on a cycle, then delete $\langle u—v \rangle$.
2. If vertices u and v of G are not connected, then add the edge $\langle u—v \rangle$.

Assume the vertices of G are the integers $1, 2, \ldots, n$ for some $n \geq 2$. Procedure **create-spanning-tree** can be modeled as a state machine whose states are all possible simple graphs with vertices $1, 2, \ldots, n$. The start state is G, and the final states are the graphs on which no operation is possible.

(a) Let G be the graph with vertices $\{1, 2, 3, 4\}$ and edges

$$\{\langle 1—2 \rangle, \langle 3—4 \rangle\}$$

What are the possible final states reachable from start state G? Draw them.

Solution. It's not possible to delete any edge. The procedure can only add an edge connecting exactly one of vertices 1 or 2 to exactly one of vertices 3 or 4, and then terminate. So there are four possible final states. ■

(b) Prove that any final state of must be a tree on the vertices.

Solution. We use the characterization of a tree as an acyclic connected graph.

A final state must be connected, because otherwise there would be two unconnected vertices, and then a transition adding the edge between them would be possible, contradicting finality of the state.

A final state can't have a cycle, because deleting any edge on the cycle would be a possible transition. ■

(c) For any state, G', let e be the number of edges in G', c be the number of connected components it has, and s be the number of cycles. For each of the derived variables below, indicate the *strongest* of the properties that it is guaranteed to satisfy, no matter what the starting graph G is and be prepared to briefly explain your answer.

The choices for properties are: *constant*, *strictly increasing*, *strictly decreasing*, *weakly increasing*, *weakly decreasing*, *none of these*. The derived variables are

(i) e

Solution. none of these ■

(ii) c

Solution. weakly decreasing ■

(iii) s

Solution. weakly decreasing ■

(iv) $e - s$

Solution. weakly increasing ■

(v) $c + e$

Solution. weakly decreasing ■

(vi) $3c + 2e$

Solution. strictly decreasing ■

(vii) $c + s$

Solution. strictly decreasing ■

(viii) (c, e), partially ordered coordinatewise (the *product* partial order 9.9.1).

Solution. none of these ■

(d) Prove that procedure **create-spanning-tree** terminates. (If your proof depends on one of the answers to part (c), you must prove that answer is correct.)

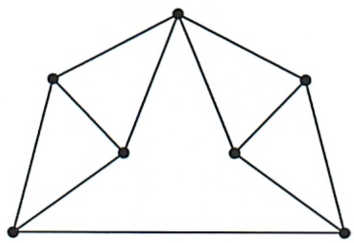
Solution. If a value (a *derived variable*) associated with a process state is nonnegative integer-valued and decreases at each step, then the process terminates after at most as many steps as the initial value of the quantity. So we need only identify such a derived variable. There are two in the list above, namely (vi) and (vii).

To show that the variable (vi) strictly decreases, note that the rule for deleting an edge ensures that the connectedness relation does not change, so neither does the number of connected components c. Meanwhile the number of edges e decreases by one when an edge is deleted. Therefore the variable $3c + 2e$ decreases by 2. The rule for adding an edge ensures that the number of connected components c decreases by one and the number of edges e increases by one. Therefore the variable $3c + 2e$ decreases by 1.

To show that the variable (vii) strictly decreases, note that the rule for deleting an edge ensures that the number of connected components c does not change and the number of cycles s decreases by n, where $n \geq 1$. Therefore the variable $c + s$ decreases by n. The rule for adding an edge ensures that the number of connected components c decreases by one and the number of cycles s does not change. Therefore the variable $c + s$ decreases by one.

■

Problem 3.
Let G be the graph below[1]. Carefully explain why $\chi(G) = 4$.



Solution. Four colors are sufficient, so $\chi(G) \leq 4$.

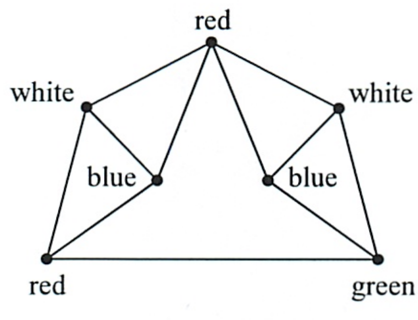


Figure 1 A 4-coloring of the Graph

Now assume $\chi(G) = 3$. We may assume the top vertex is colored red. The top two triangles require 3 colors each, and since they share the top red vertex, they must have the other two colors, white and blue, at their bases, as in Figure 1. Now the bottom two vertices are both adjacent to vertices colored white and blue, and cannot have the same color since they are adjacent, so there is no alternative but to color one with a third color and the other with a fourth color, contradicting the assumption that 3 colors are enough. Hence, $\chi(G) > 3$. This together with the coloring of Figure 1 implies that $\chi(G) = 4$. ■

[1]From *Discrete Mathematics*, Lovász, Pelikan, and Vesztergombi. Springer, 2003. Exercise 13.3.1

Problem 4.
A portion of a computer program consists of a sequence of calculations where the results are stored in variables, like this:

$$\begin{array}{rrcl} & \text{Inputs:} & & a, b \\ \text{Step 1.} & c & = & a + b \\ 2. & d & = & a * c \\ 3. & e & = & c + 3 \\ 4. & f & = & c - e \\ 5. & g & = & a + f \\ 6. & h & = & f + 1 \\ & \text{Outputs:} & & d, g, h \end{array}$$

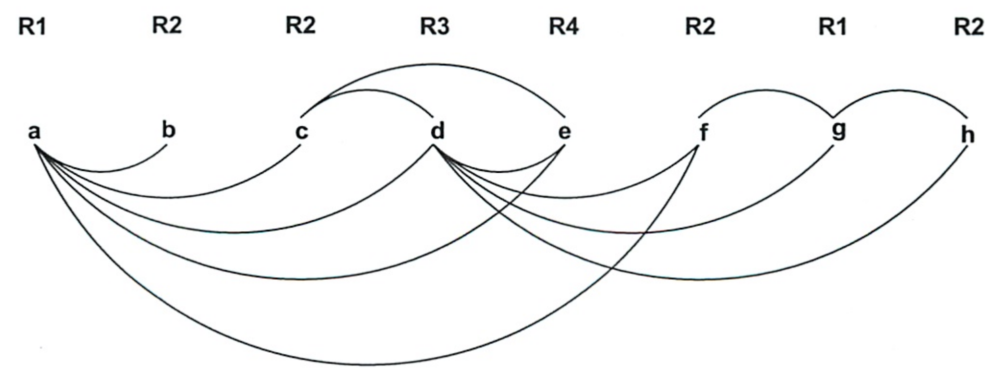
A computer can perform such calculations most quickly if the value of each variable is stored in a *register*, a chunk of very fast memory inside the microprocessor. Programming language compilers face the problem of assigning each variable in a program to a register. Computers usually have few registers, however, so they must be used wisely and reused often. This is called the *register allocation* problem.

In the example above, variables a and b must be assigned different registers, because they hold distinct input values. Furthermore, c and d must be assigned different registers; if they used the same one, then the value of c would be overwritten in the second step and we'd get the wrong answer in the third step. On the other hand, variables b and d may use the same register; after the first step, we no longer need b and can overwrite the register that holds its value. Also, f and h may use the same register; once $f + 1$ is evaluated in the last step, the register holding the value of f can be overwritten.(Assume that the computer carries out each step in the order listed and that each step is completed before the next is begun.)

(a) Recast the register allocation problem as a question about graph coloring. What do the vertices correspond to? Under what conditions should there be an edge between two vertices? Construct the graph corresponding to the example above.

Solution. There is one vertex for each variable. An edge between two vertices indicates that the values of the variables must be stored in different registers.

We can classify each appearance of a variable in the program as either an *assignment* or a *use*. In particular, an appearance is an assignment if the variable is on the left side of an equation or on the "Inputs" line. An appearance of a variable is a use if the variable is on the right side of an equation or on the "Outputs" line. The *lifetime of a variable* is the segment of code extending from the initial assignment of the variable until the last use.[2] There is an edge between two variables if their lifetimes overlap. This rule generates the following graph:



■

[2]This definition is for the case that each variable is assigned at most once (see part (c)).

(b) Color your graph using as few colors as you can. Call the computer's registers $R1$, $R2$, etc. Describe the assignment of variables to registers implied by your coloring. How many registers do you need?

Solution. Four registers are needed.

One possible assignment of variables to registers is indicated in the figure above. In general, coloring a graph using the minimum number of colors is quite difficult; no efficient procedure is known. However, the register allocation problem always leads to an *interval graph*, and optimal colorings for interval graphs are always easy to find. This makes it easy for compilers to allocate a minimum number of registers. ■

(c) Suppose that a variable is assigned a value more than once, as in the code snippet below:

$$
\begin{aligned}
&\dots\\
t &= r + s\\
u &= t * 3\\
t &= m - k\\
v &= t + u\\
&\dots
\end{aligned}
$$

How might you cope with this complication?

Solution. Each time a variable is reassigned, we could regard it as a completely new variable. Then we would regard the example as equivalent to the following:

$$
\begin{aligned}
&\dots\\
t &= r + s\\
u &= t * 3\\
t' &= m - k\\
v &= t' + u\\
&\dots
\end{aligned}
$$

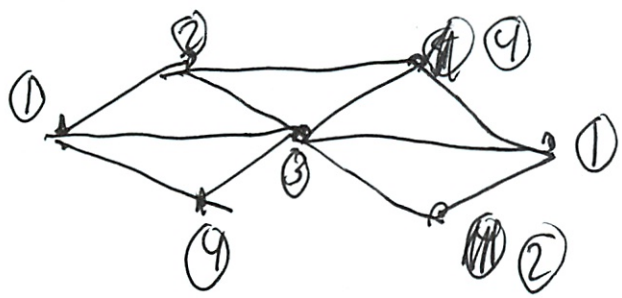
We can now proceed with graph construction and coloring as before. ■

Tutor 7 Cont

TP7.7 Coloring

Chromatic #

- need to do manually - no better way



can do (1)-(2)-(1)

just not (1)-(1)

4 ⊗

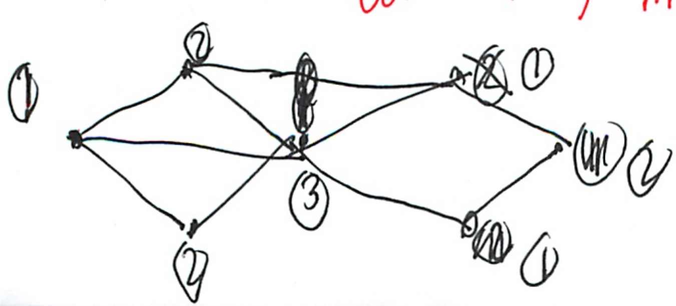
5 ⊗

3 ✓ - triangles need 3

~~What do~~

What does it mean for colors to be sufficent?

Use two for outer rim, third for center



← could do that

2

TP 7.8 Which are Trees?

- ? or forests of trees?

1

2

~~3~~

4

1 2 4 x

2 4 ✓ Guess forests don't count

TP 7.9 Leaves

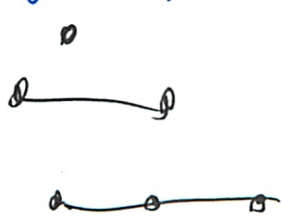
1. What is possible # of vertices where everything is leaf

leaf = degree 1

1

2

~~3~~



2 x

1,2 ✓ oh wanted all possible

(3)

b. Smallest possible # leaves in trees w/ 99 vertices

c) Largest is easy 98

- does not need to be binary

b)

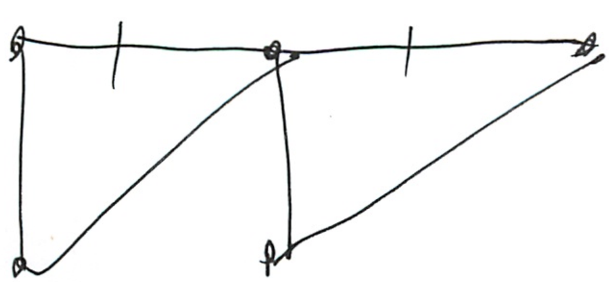
2

TP 7.10 Graph Colloring

$\chi(\text{Tree}) = 2$

TP. 7.11 Spanning Trees

Find a spanning tree



(4)

TP 7.12 Graph Algorithm

Graph G
- with vertices V
  edges E

Mark edges if no path marked edges b/w

(sounds like Spanning tree)

1. Pres. Inv. and also hold for start state
   1. No
   2. ✓
   3. ~~3 ✓~~
   4. Not always

2,3 ✗

2 ✓

2. Derived variables - how do they change?

unmarked edges - strictly ↓ ✓

marked edges " ↑ ✓

unmarked edges + marked edges = constant ✓

5

marked − # unmarked

↑ − ↓

+1 − −1

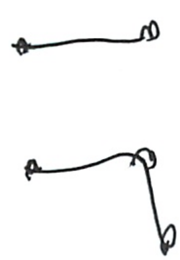
+2 strictly ↑ ✓

connected components − only marked edges

i weakly ↓ ✗

well first

then



could also

weakly ↑ ✗

but then they connect at some pt?

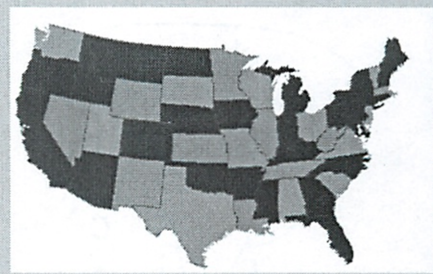
strictly ↓ ✓ *a vertice sitting by itself is a connected component*

4/1

Mathematics for Computer Science
MIT 6.042J/18.062J

Planar Graphs

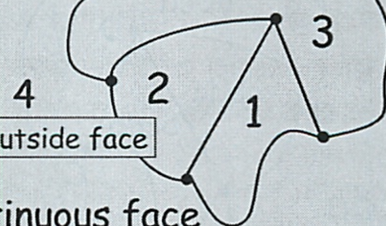
Planar Graphs



Planar Graphs

A graph is planar if there is a way to draw it in the plane without edges crossing.

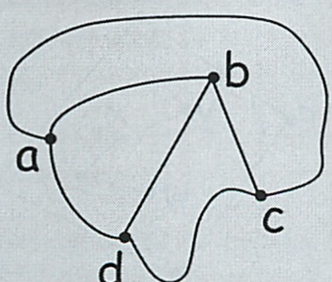
4 Continuous Faces here



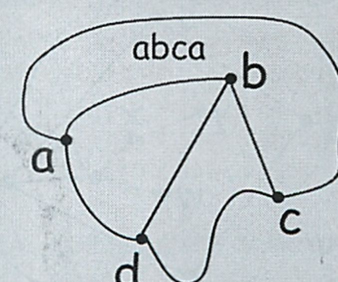
the outside face

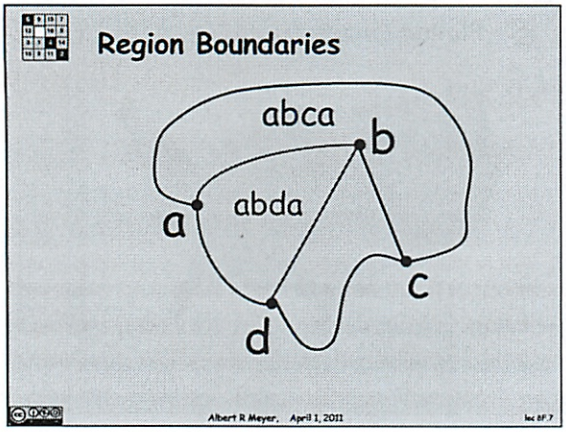
continuous face
::=connected region

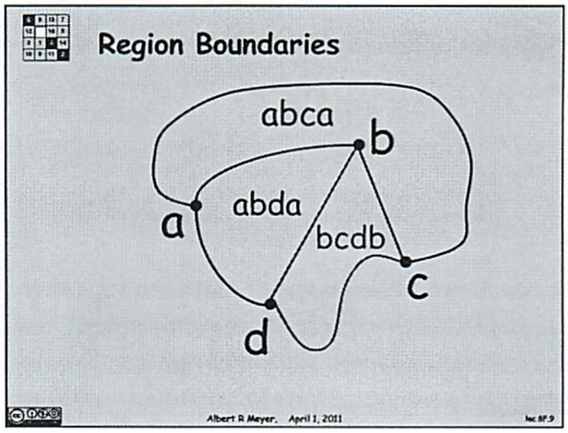
Region Boundaries

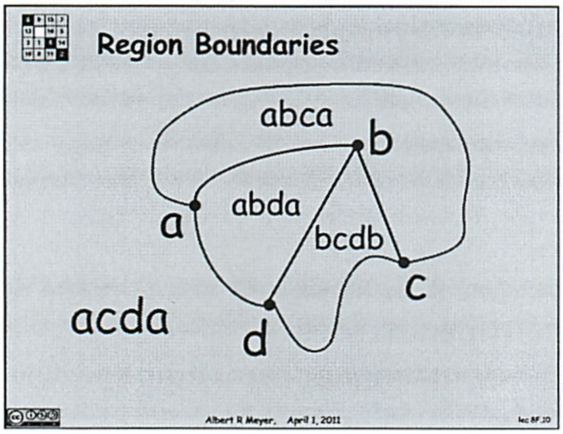


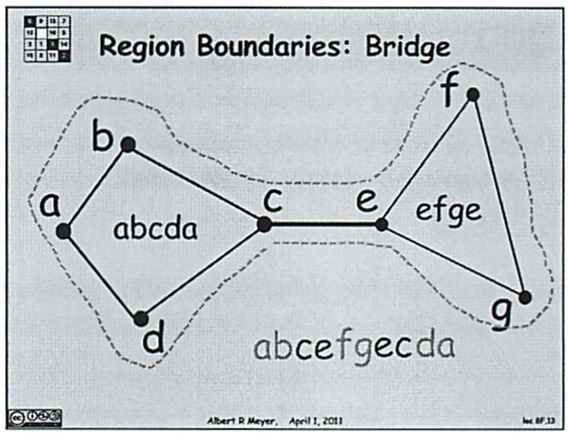
Region Boundaries

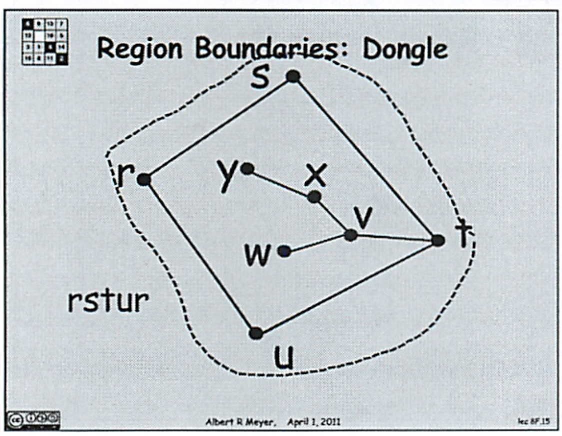


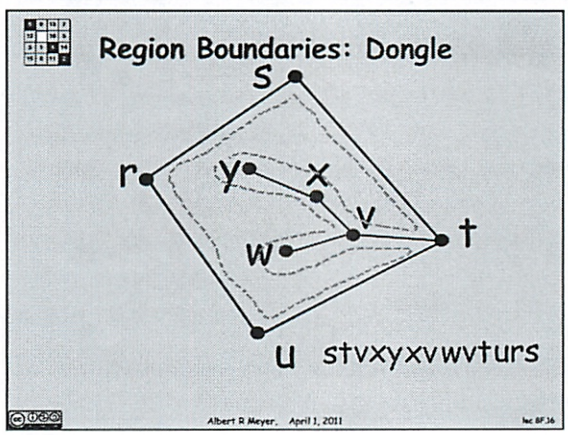

Region Boundaries
abca
abda
a
b
c
d
Albert R Meyer, April 1, 2011


Region Boundaries
abca
abda
bcdb
a
b
c
d
Albert R Meyer, April 1, 2011


Region Boundaries
abca
abda
bcdb
acda
a
b
c
d
Albert R Meyer, April 1, 2011


Region Boundaries: Bridge
abcda
efge
abcefgecda
a
b
c
d
e
f
g
Albert R Meyer, April 1, 2011

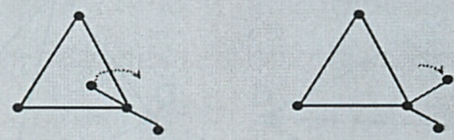

Region Boundaries: Dongle
rstur
s
r
y
x
v
w
t
u
Albert R Meyer, April 1, 2011


Region Boundaries: Dongle
stvxyxvwvturs
s
r
y
x
v
w
t
u
Albert R Meyer, April 1, 2011

Planar Embedding

A planar embedding is a connected graph *along with* its face boundary walks (same graph may have different embeddings)

Same graph, different embeddings



2 length 5 faces

length 3 face
length 7 face

Recursive Def: Planar Embeddings

Base: a graph consisting of
- single vertex, v,
- with a single face:
  length 0 walk from v to v,

is a PE.

v● v

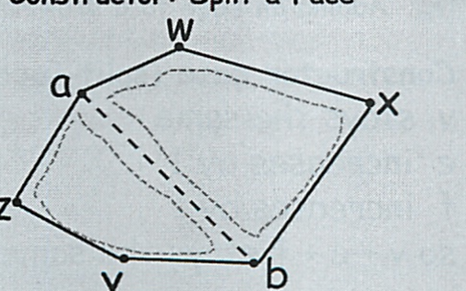
graph face

Adding an edge to an embedding

Two constructor cases:

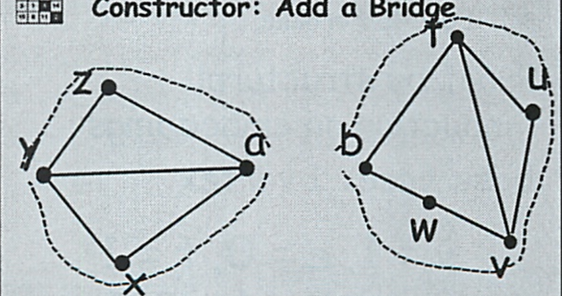
1) Add edge across a face (splits face in two)
2) Add bridge between connected components (merges 2 outer faces)

Constructor: Split a Face



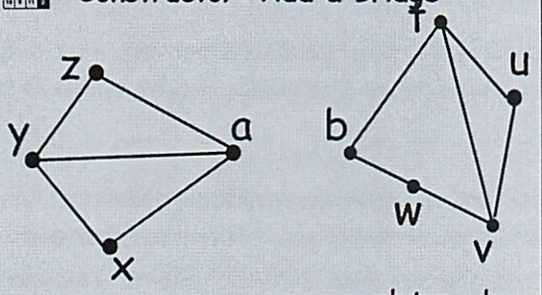
awxbyza → awxba, abyza

Constructor: Add a Bridge



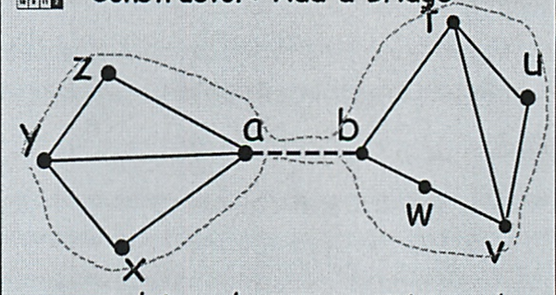
axyza btuvwb

Constructor: Add a Bridge



axyza, btuvwb

Constructor: Add a Bridge



axyza, btuvwb $\rightarrow$ axyzabtuvwba

Team Problem

Problem 1

Euler's Formula

If a planar embedding has v vertices, e edges, and f faces, then

$$v - e + f = 2$$

Euler's Formula

Proof by structural induction on embeddings:

base case: 1 vertex

$$v = 1, e = 0, f = 1$$
$$1 - 0 + 1 = 2$$ O.K.

Adding an edge to a drawing

Constructor case (split face):

v stays the same
e increases by 1
f increases by 1
so v - e + f stays the same

O.K.

Adding an edge to a drawing

Constructor case (add bridge):

$$\begin{array}{rl} v = & v_1 + v_2 \\ -e = & -(e_1 + e_2 + 1) \\ f = & f_1 + f_2 - 1 \\ \hline 2 = & 2 + 2 - 2 \end{array}$$

(two outer faces merge into one)

O.K.

Planar Properties

- an edge appears twice on faces
- face length ≥ 3 (for $v \geq 3$)

$$3(e-v+2) = 3f \leq 2e$$

combining with Euler

$$\boxed{e \leq 3v-6}$$

Planar Properties

$$e \leq 3v-6$$

Cor: K_5 is not planar

pf: $v = 5, e = 10$

$$10 \not\leq 3 \cdot 5 - 6$$

Planar Properties

$$e \leq 3v-6$$

Cor: Every planar graph has a vertex of degree ≤ 5

pf: suppose all degrees ≥ 6

Then

$6v \leq \sum$ degrees $= 2e \leq 6v-12$

contradiction!

Planar Properties

Cor: Every planar graph has a vertex of degree ≤ 5

Therefore,

every planar graph is 6-colorable

Euler's Formula

Cor: There are at most 5 regular polyhedra (proof in Notes)

Team Problems

Problems 2 & 3

Albert R Meyer, April 1, 2011 lec 8F.35

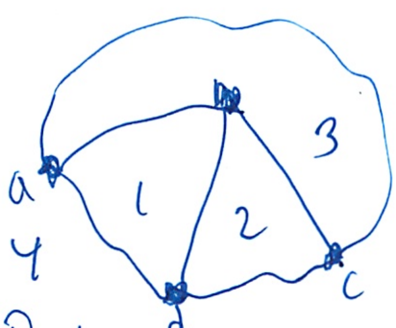
Planar Graphs

- "beautiful"
- but won't build on it later
- map = planar graph
  - vertices + edges
  - but drawn in plane w/o edges crossing each other
  - usually ~~tough~~ thought of as state is vertex
  - edge b/w if they have a + length border

* can't draw ~~each~~ w/o edges cross

divide up into smaller regions

- continuous faces



Outside face - to ∞

(2)

But want to think about it discretly

- Seq of vertices along region

3 is abca 2 bcdb

1 " abda 4 acda

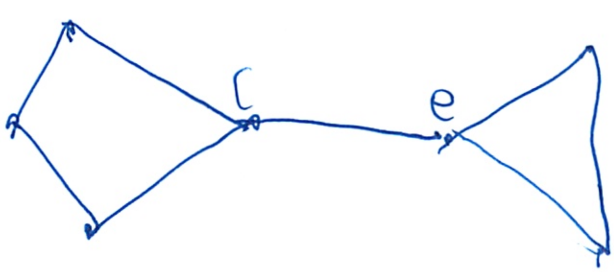
~~eta~~

does not matter where start, what dir

~~region~~

when nice - region boundries are all cycles

Sometimes <u>bridges</u>

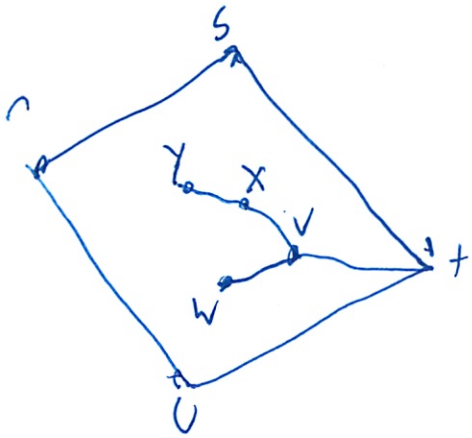


but now outer boundries are messy

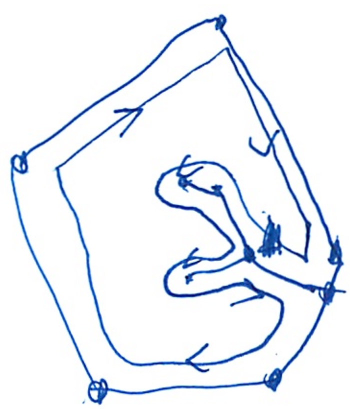
are closed walks - need to cross vertices and edges

(3)

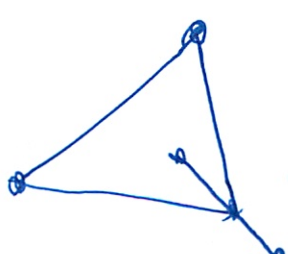
dangles



Have to backtrack
- every edge on dangle visited twice



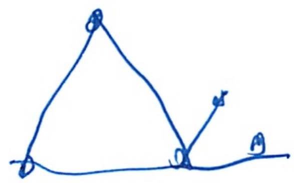
Planar embedding - connected graph along w/ its face boundary walks



2 lenght 5 faces

(4)

But could puller inner dongle outside - isomorphism



lenght 3 face
7 face

but diff embeddings

Could insist tripple connected to get rid of bridges + dongles

Define ~~Plane~~ Planar Embeddings recursivly

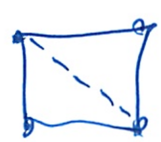
Base

V

single face : lenght 0 walk V→V

Constructor

1. Add edge accross face (splits face in two)



2. Add a bridge b/w. 2 connected components

5

- which is combining 2 faces into one big face

Problem 1

---

embedding - a bunch of closed walks

Now make use of def

Euler's Formula

$$\underset{\text{vertices}}{V} - \underset{\text{edges}}{e} + \underset{\text{faces}}{f} = 2$$

- is an invarient
- satisfied law of planar embeddings
  - Only embeddings have faces

Base

$V = 1$
$e = 0$
$f = 1$

$1 - 0 + 1 = 2$ ✓

6

Constructor 1 (split)

V stays same

e ↑ 1

f ↑ 1

Only one face changes → split - get one more since added edge

So

$0 + 1 - 1 = 0$ so same ✓

Constructor 2 (bridge)

$V = V_1 + V_2$

V_1 ↑ first graph

V_2 ↑ 2nd graph

$e = e_1 + e_2 + 1$

$f = f_1 + f_2 - 1$

add it up

$2 + 2 - 2 = 2$ ✓

□

7

2 preserved planer graph propetices

- an edge appears twice on faces
- from def
- ~~for $v \geq 3$ - doesn't work for degenerate graphs~~

So total face lenght $= 2e$

- face lenghth ≥ 3 (for $v \geq 3$ - does not work degenerate)

So $3f \leq 2e$

- connect w/ Euler's theorm

and $\boxed{e \leq 3v - 6}$

Cor: K_5 is not planar

- can't drawing it is not a proof!
- but can use the invarient

$v = 5 \quad e = 10$

$10 \leq 3(5) - 6$ X

Contridiction! so ~~false~~ true

8

Corollary Every planar graph has a vertex of degrees ≤ 5

Pf Suppose all degrees ≥ 6

Then $6v \leq \sum$ degrees

$= 2e \leq 6v - 12$

Therefore every graph is 6-colorable

Cor There are at most 5 regular polyhedra

- in textbook
- Eulers + degree constraint

Applied result in CS

In-Class Problems Week 8, Fri.

Problem 1.
Figures 1–4 show different pictures of planar graphs.

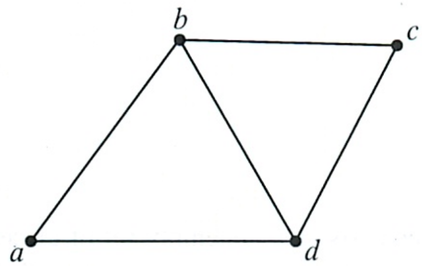


figure 1

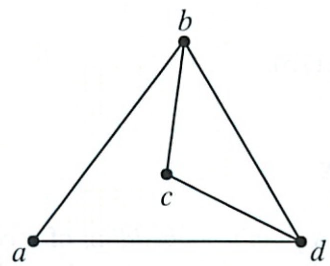


figure 2

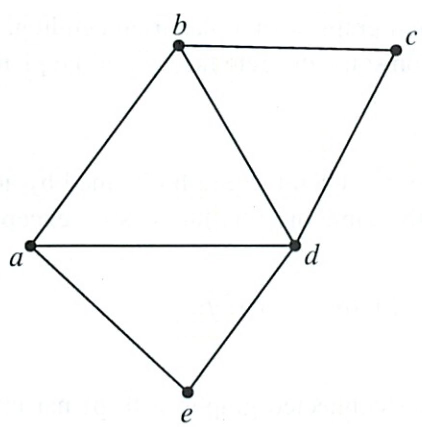


figure 3

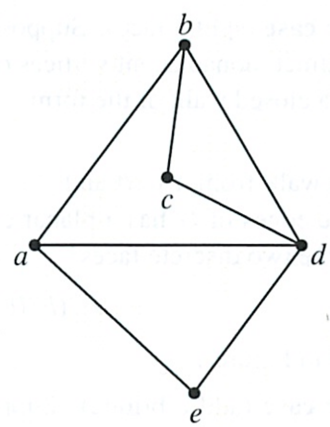


figure 4

(a) For each picture, describe its discrete faces (closed walks that define the region borders).

(b) Which of the pictured graphs are isomorphic? Which pictures represent the same *planar embedding*?—that is, they have the same discrete faces.

(c) Describe a way to construct the embedding in Figure 4 according to the recursive Definition 12.2.2 of planar embedding. For each application of a constructor rule, be sure to indicate the faces (cycles) to which the rule was applied and the cycles which result from the application.

Problem 2.
Prove the following assertions by structural induction on the definition of planar embedding.

(a) In a planar embedding of a graph, each edge occurs exactly twice in the faces of the embedding.

(b) In a planar embedding of a connected graph with at least three vertices, each face is of length at least three.

Problem 3.
A simple graph is *triangle-free* when it has no cycle of length three.

(a) Prove for any connected triangle-free planar graph with $v > 2$ vertices and e edges,

$$e \leq 2v - 4. \tag{1}$$

Hint: Similar to the proof that $e \leq 3v - 6$. Use Problem 2.

(b) Show that any connected triangle-free planar graph has at least one vertex of degree three or less.

(c) Prove by induction on the number of vertices that any connected triangle-free planar graph is 4-colorable.

Hint: use part (b).

Appendix

Definition. A *planar embedding* of a *connected* graph consists of a nonempty set of closed walks of the graph called the *discrete faces* of the embedding. Planar embeddings are defined recursively as follows:

Base case: If G is a graph consisting of a single vertex, v, then a planar embedding of G has one discrete face, namely, the length zero closed walk, v.

Constructor case (split a face): Suppose G is a connected graph with a planar embedding, and suppose a and b are distinct, nonadjacent vertices of G that appear on some discrete face, γ, of the planar embedding. That is, γ is a closed walk of the form

$$\alpha \frown \beta$$

where α is a walk from a to b and β is a walk from b to a.[1] Then the graph obtained by adding the edge $\langle a—b \rangle$ to the edges of G has a planar embedding with the same discrete faces as G, except that face γ is replaced by the two discrete faces[2]

$$\alpha \frown (b \ \langle b—a \rangle \ a) \quad \text{and} \quad (a \ \langle a—b \rangle \ b) \frown \beta$$

as illustrated in Figure 1.

Constructor case (add a bridge): Suppose G and H are connected graphs with planar embeddings and disjoint sets of vertices. Let γ be a discrete face of the embedding of G and suppose that γ begins and ends at vertex a.

Similarly, let δ be a discrete face of the embedding of H that begins and ends at vertex b.

Then the graph obtained by connecting G and H with a new edge, $\langle a—b \rangle$, has a planar embedding whose discrete faces are the union of the discrete faces of G and H, except that faces γ and δ are replaced by one new face

$$\gamma \frown (a \ \langle a—b \rangle \ b) \frown \delta \frown (b \ \langle b—a \rangle \ a).$$

This is illustrated in Figure 2, where the vertex sequences of the faces of G and H are:

$$G : \{axyza, \ axya, \ ayza\} \qquad H : \{btuvwb, \ btvwb, \ tuvt\},$$

and after adding the bridge $\langle a—b \rangle$, there is a single connected graph whose faces have the vertex sequences

$$\{axyzabtuvwba, \ axya, \ ayza, \ btvwb, \ tuvt\}.$$

[1] If a walk $\mathbf{f}$ ends with a vertex, v, and a walk $\mathbf{r}$ starts with the same vertex, v, their merge, $\mathbf{f} \frown \mathbf{r}$, is the walk that starts with $\mathbf{f}$ and continues with $\mathbf{r}$. Two walks can only be merged if the first ends with the same vertex, v, that the second one starts with.

[2] There is a minor exception to this definition of embedding in the special case when G is a line graph beginning with a and ending with b. In this case the cycles into which γ splits are actually the same. That's because adding edge $\langle a—b \rangle$ creates a cycle that divides the plane into "inner" and "outer" continuous faces that are both bordered by this cycle. In order to maintain the correspondence between continuous faces and discrete faces in this case, we define the two discrete faces of the embedding to be two "copies" of this same cycle.

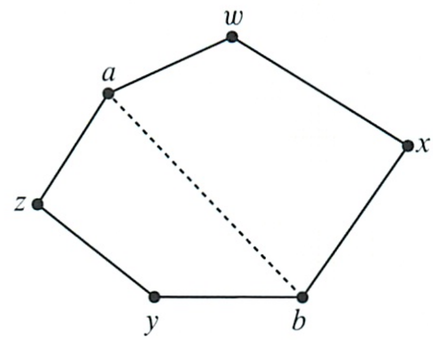


Figure 1 The "split a face" case: $awxbyza$ splits into $awxyba$ and $abyza$.

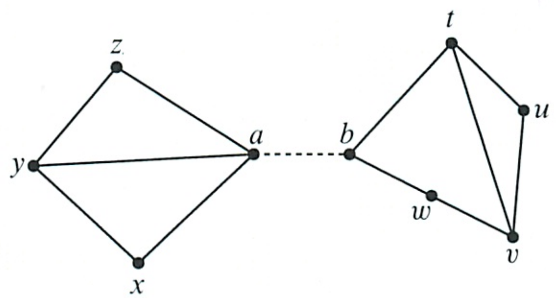


Figure 2 The "add a bridge" case.

Theorem 3.1 (Euler's Formula). *If a connected graph has a planar embedding, then*

$$v - e + f = 2$$

where v is the number of vertices, e is the number of edges, and f is the number of faces.

Corollary 3.2. *Suppose a connected planar graph has $v \geq 3$ vertices and e edges. Then*

$$e \leq 3v - 6.$$

Proof. By definition, a connected graph is planar iff it has a planar embedding. So suppose a connected graph with v vertices and e edges has a planar embedding with f faces. By Problem 2.a, every edge is traversed exactly twice by the face boundaries. So the sum of the lengths of the face boundaries is exactly $2e$. Also by Problem 2.b, when $v \geq 3$, each face boundary is of length at least three, so this sum is at least $3f$. This implies that

$$3f \leq 2e. \tag{2}$$

But $f = e - v + 2$ by Euler's formula, and substituting into (2) gives

$$\begin{aligned} 3(e - v + 2) &\leq 2e \\ e - 3v + 6 &\leq 0 \\ e &\leq 3v - 6 \end{aligned}$$

■

Corollary 3.3. *K_5 is not planar.*

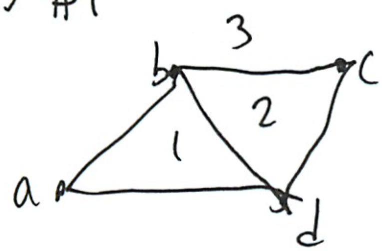
Proof.

$$e = 10 > 9 = 3v - 6.$$

■

In Class 8 Fri 4/1

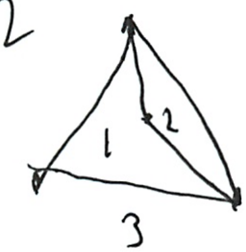
1a.) #1



a b d a
b c d b
a b c d a

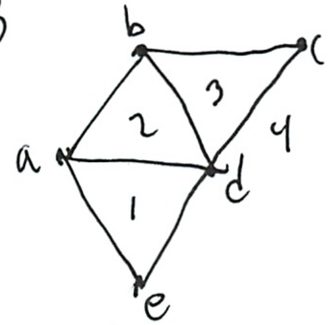
nothing called "outerface"

#2



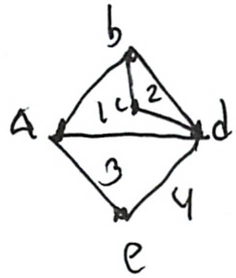
a b c d a
b c d b
a b d a

#3



a b d a
a d e a
b c d b
a b c d e a

#4



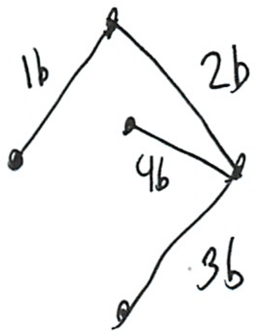
a b c d a
b d c b
a d e a
a b d e a

b) isomorphic - 1, 2
3, 4

planar embedding - same discrete faces 1, 2
~~3, 4~~ Not! -added triangle

2

c) Build #4 w/ adding rule

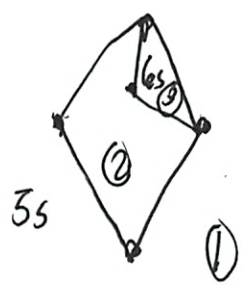


← all one face

↳ but how do you close now

Can draw line b/w w/ the split rule

- (all pics showed like that, but use def)



5s splits 1 into 1 2

6s splits 2 into 2 3

to have a planar embedding - must be isomorphic?
pair of graphs

Rest of lecture

---

③

2. Prove by structural induction on def planar embedding

a) One for each

Base $V=1$ does not count ;

Base $V=2$

$\underbrace{A\ B}_{\text{once}}$ A

A $\underbrace{B\ A}_{\text{twice}}$

? prob won't need ~~since will~~ -since could also do w/ constructor

Constructor add bridge

- what I showed above except use more general lang

A, ~~A~~B could have also been connected components, not just single ? (w/ multiple vertices)

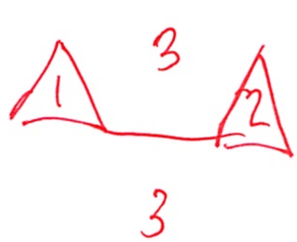
(4)

not allowed
to connect later

- must bild up w/ 1 connected component example

- Cald connect ~~[illegible]~~ if each was built in its own place

B/c when you connect, you lose a face



(5)

3. A simple graph is Δ-free if has no cycles of lenght 3

a) Prove for any connected triangle free planar graph w/ $V > 2$ verticies

$$e \leq 2v - 4$$

Massachusetts Institute of Technology
6.042J/18.062J, Spring '11: Mathematics for Computer Science — April 1
Prof. Albert R Meyer — revised Friday 1st April, 2011, 12:54

Solutions to In-Class Problems Week 8, Fri.

Problem 1.
Figures 1–4 show different pictures of planar graphs.

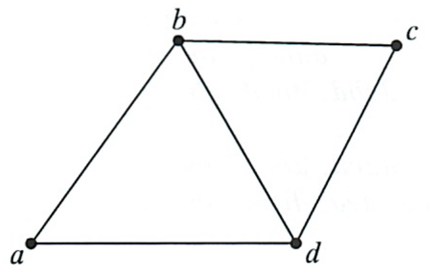


figure 1

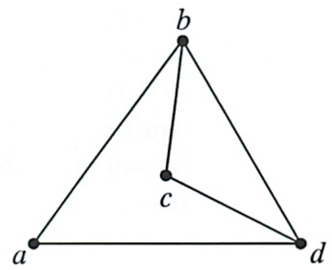


figure 2

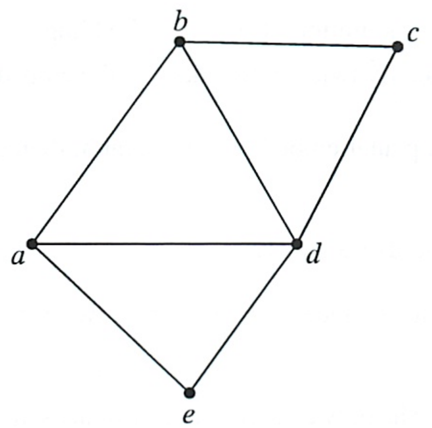


figure 3

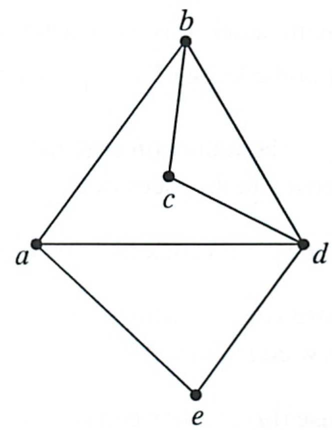


figure 4

(a) For each picture, describe its discrete faces (closed walks that define the region borders).

Solution. Figs 1 & 2: abda, bcdb, abcda. Fig 3: abcdea, adea,abda,bcdb. Fig 4: abcda, abdea, bdcb, adea. ■

(b) Which of the pictured graphs are isomorphic? Which pictures represent the same *planar embedding*?—that is, they have the same discrete faces.

Solution. Figs 1 & 2 have the same faces, so are different pictures of the *same* planar drawing. Figs 3 & 4 both have four faces, but they are different, for example, Fig 3 has a face with 5 edges, but the longest face in Fig 4 has 4 edges. ■

(c) Describe a way to construct the embedding in Figure 4 according to the recursive Definition **??** of planar embedding. For each application of a constructor rule, be sure to indicate the faces (cycles) to which the rule was applied and the cycles which result from the application.

Solution. Here's one way. (The constructor steps could actually be done in any order.)

recursive step		faces
vertex a	(base case)	a
vertex b	(base)	b
$\langle a—b \rangle$	(bridge)	aba
vertex c	(base)	c
$\langle b—c \rangle$	(bridge)	$abcba$
vertex d	(base)	d
$\langle c—d \rangle$	(bridge)	$abcdcba$
$\langle a—d \rangle$	(split)	$dabcd,\ dabcd$
$\langle b—d \rangle$	(split)	$dabd,\ dbcd,\ abcda$
vertex e	(base)	e
$\langle d—e \rangle$	(bridge)	$dedabd,\ dbcd,\ abcda$
$\langle a—e \rangle$	(split)	$abdea,\ adea,\ dbcd,\ abcda$

■

Problem 2.
Prove the following assertions by structural induction on the definition of planar embedding.

(a) In a planar embedding of a graph, each edge occurs exactly twice in the faces of the embedding.

Solution. *Proof.* The induction hypothesis is that if $\mathcal{E}$ is a planar embedding of a graph, then each edge is occurs exactly twice in the faces of $\mathcal{E}$.

Base case: There is one vertex and no edges, so this case holds vacuously.

Constructor case (face-splitting): The only change is that one face of $\mathcal{E}$ splits into two new faces, each including the new edge once.

Constructor case (bridge between two connected graphs): The only change is that two faces merge into one face that has two occurrences of the new bridging edge. So the occurrences of other edges are unchanged, and the new edge occurs twice in the new face.

So in any case, all edges of $\mathcal{E}$ are occur exactly twice. This completes the proof of the Constructor case. We conclude by structural induction that for all planar embeddings, $\mathcal{E}$, then each edge occurs exactly twice in the faces of $\mathcal{E}$.

■

(b) In a planar embedding of a connected graph with at least three vertices, each face is of length at least three.

Solution. *Proof.* The induction hypothesis is that if $\mathcal{E}$ is a planar embedding of a graph with at least three vertices, then all faces in $\mathcal{E}$ are of length at least three.

Base case: There is one vertex, so this case holds vacuously.

Constructor case: (face-splitting) An edge $\langle a—b \rangle$ is added between nonadjacent vertices a, b on the same face. This face is replaced by two new faces of the form $abc \ldots a$ and $abd \ldots a$ where $c \neq d$ are vertices different from a and b. So both new faces are of length at least 3; no other faces change.

Constructor case: (bridge between two connected graphs)

case 1: (both graphs have one vertex). Connecting these graphs with a bridge gives a graph with fewer than three vertices, so this case holds vacuously.

case 2: (one graph has exactly two vertices and the other has at most two vertices). Connecting these graphs with a bridge yields a line graph of length two or three whose unique embedding is a cycle of length four or six going from one end of the graph to the other and back. In any case, the one face has length more than three.

case 3: (one graph has at most two vertices and the other has at least three vertices). Connecting replaces the face of the vertex graph with at most two vertices and a face of the other graph with a face of length at least $2 + 3 = 5$, and leaves all other faces unchanged. So all faces are indeed of length at least three.

case 4: (both graphs have at least three vertices). Connecting replaces two faces of length at least three by a single face of length at least $2 + 3 + 3 = 8$, and leaves all other faces unchanged. So all faces are indeed of length at least three.

So in any case, all faces of connected planar embedding of graphs with at least three vertices are indeed of length at least three. This completes the proof of the Constructor case and the structural induction.

■

Problem 3.
A simple graph is *triangle-free* when it has no cycle of length three.

(a) Prove for any connected triangle-free planar graph with $v > 2$ vertices and e edges,

$$e \leq 2v - 4. \tag{1}$$

Hint: Similar to the proof that $e \leq 3v - 6$. Use Problem 2.

Solution. The proof that $e \leq 2v - 4$ for any connected triangle-free planar graph G with more than two vertices is identical to the proof of the same inequality for bipartite graph planar graphs:

Proof. By Problem 2.b, every face is of length at least 3. But in a triangle-free graph there are no faces of size 3, so all must be of length at least 4.

Each edge is occurs exactly twice in the faces, so

$$2e = \sum_{f \in \text{faces}} \text{length}(f) \geq \sum_{f \in \text{faces}} 4 = 4f. \tag{2}$$

By Euler's formula, $f = e - v + 2$, so substituting for f in (2), yields

$$2e \geq 4(e - v + 2),$$

which simplifies to (1).

■

(b) Show that any connected triangle-free planar graph has at least one vertex of degree three or less.

Solution. If $v \leq 4$, *all* vertices have degree at most three, so the claim is immediate for $v \leq 4$.

Also, by the Handshaking Lemma, the sum of degrees is $2e$ so the average degree is $2e/v$. By part (a), $2e/v \leq (4v - 8)/v < 4$ for $v > 2$. But the average degree can be less than 4 only if at least one vertex has degree less than 4.

It follows that for all $v > 0$, there is a vertex of degree three or less. ■

(c) Prove by induction on the number of vertices that any connected triangle-free planar graph is 4-colorable.

Hint: use part (b).

Solution.

Proof. By strong induction on the number of vertices with the induction hypothesis that if a graph is connected, planar and triangle-free then it is 4-colorable.

base case: A planar graph with a single vertex is trivially connected, triangle-free and 1-colorable.

inductive step: Any connected triangle-free planar graph G with 2 or more vertices has a vertex of degree 3 or less. Removing this vertex and any incident edges results in a graph H whose connected components are subgraphs of a planar graph and therefore planar. They are also triangle-free since removing vertices/edges from a graph with no triangles cannot create triangles. Since the components have strictly fewer vertices than G, the induction hypothesis implies each connected component is 4-colorable and thus H is 4-colorable.

A 4-coloring of G is then given by a 4-coloring of H where the removed vertex is colored with a color not used for the (at most 3) adjacent vertices. ■

Appendix

Definition. A *planar embedding* of a *connected* graph consists of a nonempty set of closed walks of the graph called the *discrete faces* of the embedding. Planar embeddings are defined recursively as follows:

Base case: If G is a graph consisting of a single vertex, v, then a planar embedding of G has one discrete face, namely, the length zero closed walk, v.

Constructor case (split a face): Suppose G is a connected graph with a planar embedding, and suppose a and b are distinct, nonadjacent vertices of G that appear on some discrete face, γ, of the planar embedding. That is, γ is a closed walk of the form

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where α is a walk from a to b and β is a walk from b to a.[1] Then the graph obtained by adding the edge $\langle a\text{—}b\rangle$ to the edges of G has a planar embedding with the same discrete faces as G, except that face γ is replaced by the two discrete faces[2]

$$\alpha \frown (b\ \langle b\text{—}a\rangle\ a) \quad \text{and} \quad (a\ \langle a\text{—}b\rangle\ b) \frown \beta$$

as illustrated in Figure 1.

Constructor case (add a bridge): Suppose G and H are connected graphs with planar embeddings and disjoint sets of vertices. Let γ be a discrete face of the embedding of G and suppose that γ begins and ends at vertex a.

Similarly, let δ be a discrete face of the embedding of H that begins and ends at vertex b.

[1] If a walk $\mathbf{f}$ ends with a vertex, v, and a walk $\mathbf{r}$ starts with the same vertex, v, their merge, $\mathbf{f} \frown \mathbf{r}$, is the walk that starts with $\mathbf{f}$ and continues with $\mathbf{r}$. Two walks can only be merged if the first ends with the same vertex, v, that the second one starts with.

[2] There is a minor exception to this definition of embedding in the special case when G is a line graph beginning with a and ending with b. In this case the cycles into which γ splits are actually the same. That's because adding edge $\langle a\text{—}b\rangle$ creates a cycle that divides the plane into "inner" and "outer" continuous faces that are both bordered by this cycle. In order to maintain the correspondence between continuous faces and discrete faces in this case, we define the two discrete faces of the embedding to be two "copies" of this same cycle.

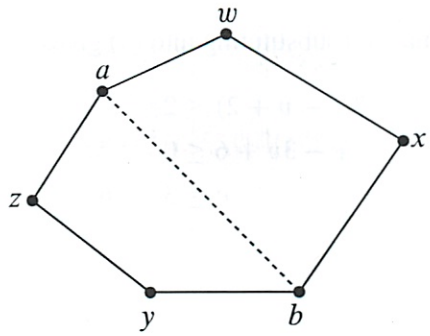


Figure 1 The "split a face" case: $awxbyza$ splits into $awxyba$ and $abyza$.

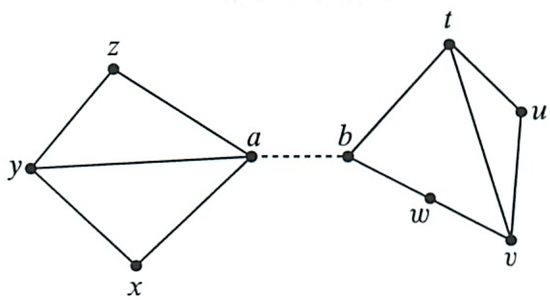


Figure 2 The "add a bridge" case.

Then the graph obtained by connecting G and H with a new edge, $\langle a\text{—}b\rangle$, has a planar embedding whose discrete faces are the union of the discrete faces of G and H, except that faces γ and δ are replaced by one new face

$$\gamma\frown(a\ \langle a\text{—}b\rangle\ b)\frown\delta\frown(b\ \langle b\text{—}a\rangle\ a).$$

This is illustrated in Figure 2, where the vertex sequences of the faces of G and H are:

$$G : \{axyza,\ axya,\ ayza\} \qquad H : \{btuvwb,\ btvwb,\ tuvt\},$$

and after adding the bridge $\langle a\text{—}b\rangle$, there is a single connected graph whose faces have the vertex sequences

$$\{axyzabtuvwba,\ axya,\ ayza,\ btvwb,\ tuvt\}.$$

Theorem 3.1 (Euler's Formula). *If a connected graph has a planar embedding, then*

$$v - e + f = 2$$

where v is the number of vertices, e is the number of edges, and f is the number of faces.

Corollary 3.2. *Suppose a connected planar graph has $v \geq 3$ vertices and e edges. Then*

$$e \leq 3v - 6.$$

Proof. By definition, a connected graph is planar iff it has a planar embedding. So suppose a connected graph with v vertices and e edges has a planar embedding with f faces. By Problem 2.a, every edge is traversed exactly twice by the face boundaries. So the sum of the lengths of the face boundaries is exactly $2e$. Also by Problem 2.b, when $v \geq 3$, each face boundary is of length at least three, so this sum is at least $3f$. This implies that

$$3f \leq 2e. \tag{3}$$

But $f = e - v + 2$ by Euler's formula, and substituting into (3) gives

$$\begin{aligned} 3(e - v + 2) &\leq 2e \\ e - 3v + 6 &\leq 0 \\ e &\leq 3v - 6 \end{aligned}$$

■

Corollary 3.3. *K_5 is not planar.*

Proof.

$$e = 10 > 9 = 3v - 6.$$

■

TP 8

4/4

TP.8.1 Faces of Planar embedding

What are the faces here?

1. abcda
   efge
   abcefgecda ✓

2. rstur
   rstvxyxvwvtur ✓

TP 8.2 Planar Graphs

A planar graph has 7 more edges than vertices. How many faces does it have?

$v - e + f = 2$

$1 - 8 + f = 2$

$-7 + f = 2$

$f = 9$ ✓

②

TP 8.3 Annuities

$r = 4\%$

10,000

$$\sum_{T=1}^{\infty} \frac{10,000}{(1+.04)^T}$$ perpituity

I know its $\frac{10,000}{.04}$ from 16.401

250,000 ✓

TP 8.4 Summation

$$\sum_{i=1}^{\infty} i^p$$ Converges to finite value iff $p < a$

Part a) value of a

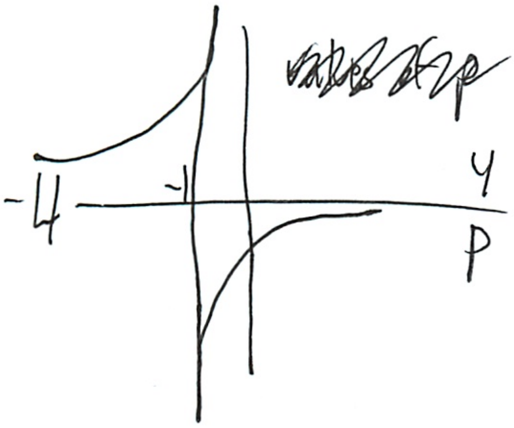
- is this top p 385?

$a = 1$ ⓧ

But what can it be?

③

Did in Wolfram alpha



-1 ✓

but how do you know

its been a long time since I did this

Part 2 proof

Which would be good proof for a?

1. Find a closed form for $\int_1^\infty x^p \, dx$

what does this mean again

wp can be answered

- like a #

I have no clue what is best

2. Closed form $\int_1^\infty i^x \, dx$
3. Induction on p
4. induction on n $\sum_{i=1}^{n} i^p$

~~8~~ 9

5. Compare series term by term w/ harmonic seq

Is multiple ans

#1 is $-\frac{1}{p+1}$ if $\underbrace{Re(p) \leq -1}_{\text{will condition tells us}}$
real

#2 does not converge

Answer 1, 5

#1 Sum is $\left(\int_1^\infty x^p \, dx \right)$

for $p \neq -1$ indefinite integral is $\frac{x^{p+1}}{p+1}$
another familiar word I forget
nothing on top or botton
1st theron of Calculus uses $\int$

If $p < -1$ then $p+1 < 0$ so $\lim_{x \to \infty} x^{p+1} = 0$
definite integral from $1 \to \infty$ evaluates to $-\frac{1}{(p+1)}$ Here sum bounded from above - since increasing, finite limit so converges

(5)

if $p > -1$ then $p+1 > 0$ so $\lim_{x \to \infty} x^{p+1} = \infty$

diverges

$p = -1$ indefinite integral is $\log x$ which also approaches ∞ as $x \to \infty$ so diverges

#4 incorrect - needs ideas from inductive step

- so induction is moot

#5 correct

For $p = -1$ the sum is the harmonic series which we know does not converge. Since term i^p is increasing in p for $i > 1$, sum will be larger and also diverge for $p > -1$

TP 8.5 Stirling's Formula

$$\frac{(2n)!}{2^{2n}(n!)^2}$$ will come up later in class

What is asy $\simeq$ to?

⑥

So Stirling formula

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\epsilon(n)}$$

$$\sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

So

$$\frac{\sqrt{2\pi \cdot 2n}\left(\frac{2n}{e}\right)^{2n}}{2^{2n}\left(\sqrt{2\pi n}\left(\frac{n}{e}\right)^n\right)^2}$$

$$\frac{2\cancel{2}\sqrt{\pi n}\left(\frac{2n}{e}\right)^{2n}}{2\cancel{2}^{2n}\left(\sqrt{2\pi n}\right)\left(\frac{n}{e}\right)^{2n}}$$

I can't do that

- This is a ton of algebra I don't feel like doing

Wolfram alpha $\frac{2^{-2n}(2n)!}{(n!)^2}$

7

$$\frac{4^{-n}(2n)!}{(n!)^2}$$

~~Multiple ans~~

Goes to 0 as $n \to \infty$

$\frac{1}{\sqrt{\pi n}}$ - no does not go to 0

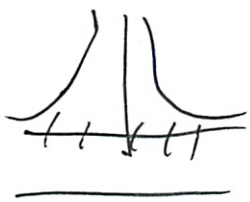
this is it

a

So perhaps.

$\frac{1}{\sqrt{2\pi n}}$ - no same

$\sqrt{\frac{2}{\pi n}}$

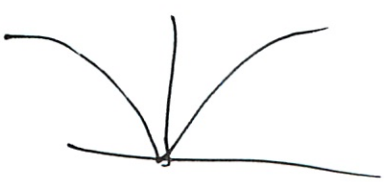


not to 0

$2^n\sqrt{2\pi n}$

No!

$\sqrt{2\pi n}$



worse

8

$$\frac{(2n)!}{2^{2n}(n!)} \approx \frac{(2n/e)^{2n}\sqrt{2\pi 2n}}{2^{2n}\left[(n/e)^{n}\sqrt{2\pi n}\right.}$$

$$= \frac{2^{2n}(n/e)^{2n}\sqrt{2\pi 2n}}{2^{2n}(n/e)^{2n}\left[\sqrt{2\pi n}\right]^2}$$

$$= \frac{\sqrt{2\pi 2n}}{\left[\sqrt{2\pi n}\right]^2}$$

$$= \frac{\sqrt{2}\sqrt{2\pi n}}{\left[\sqrt{2\pi n}\right]^2}$$

$$= \frac{\sqrt{2}}{\sqrt{2\pi n}}$$

$$= \frac{1}{\sqrt{\pi n}}$$

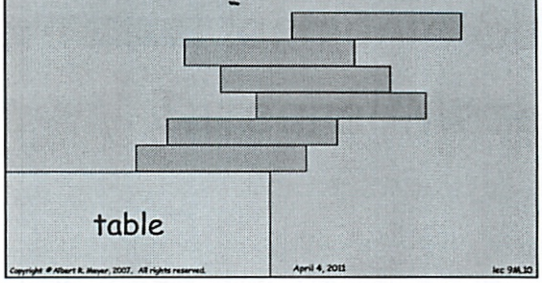
4/4

Mathematics for Computer Science
MIT 6.042J/18.062J

Harmonic Sum Integral Method

Albert R Meyer, April 4, 2011 lec 9M.1

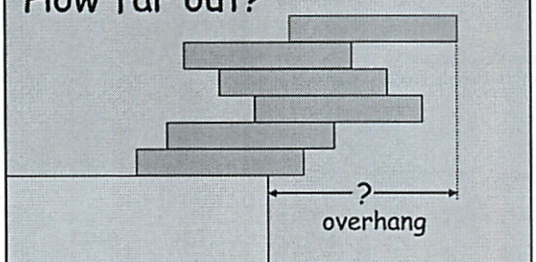
Book Stacking



 April 4, 2011 lec 9M.10

Book Stacking

How far out?

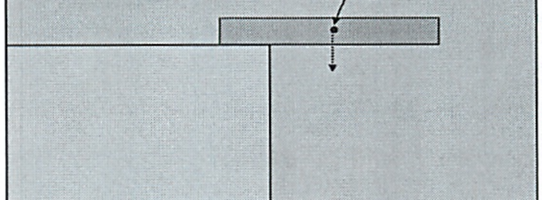


 April 4, 2011 lec 9M.11

Book Stacking

One book

book center of mass

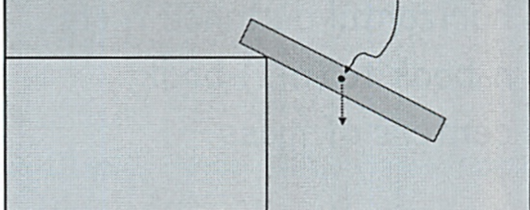


 April 4, 2011 lec 9M.12

Book Stacking

One book

book center of mass

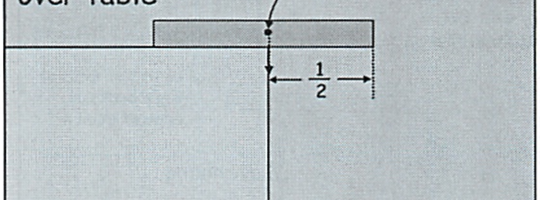


 April 4, 2011 lec 9M.13

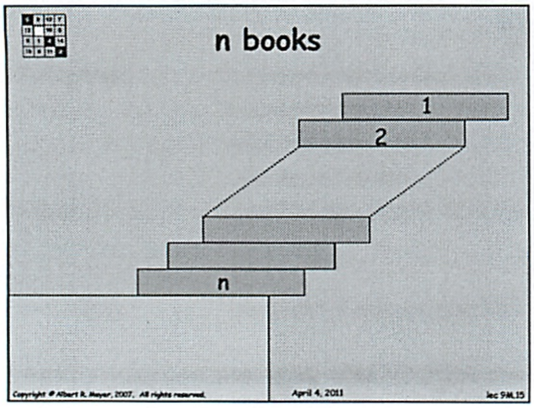
Book Stacking

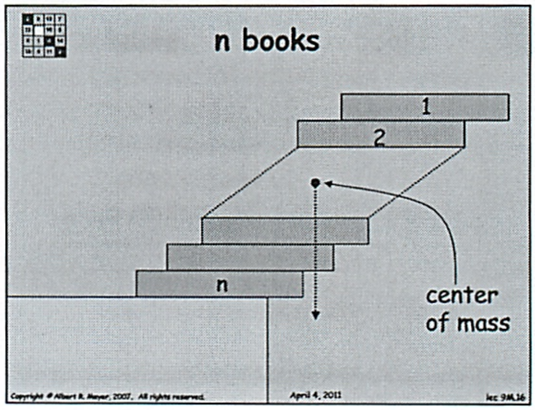
balances if center of mass over table

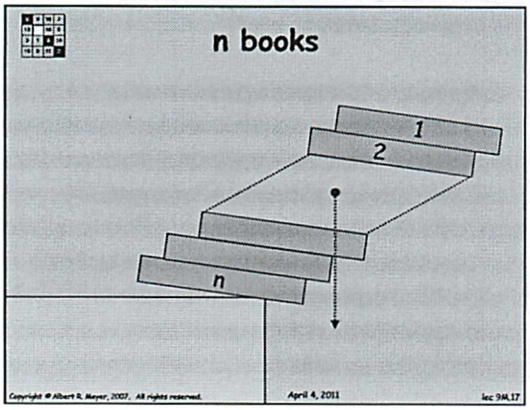
book center of mass

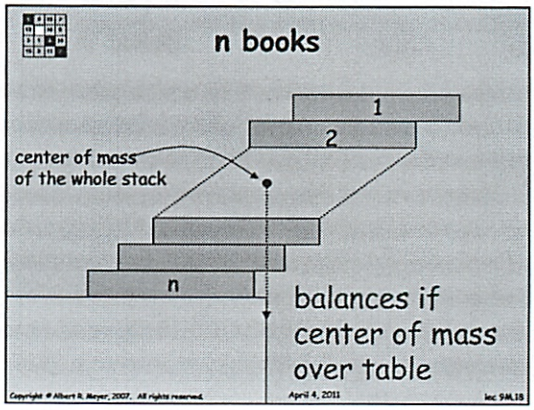


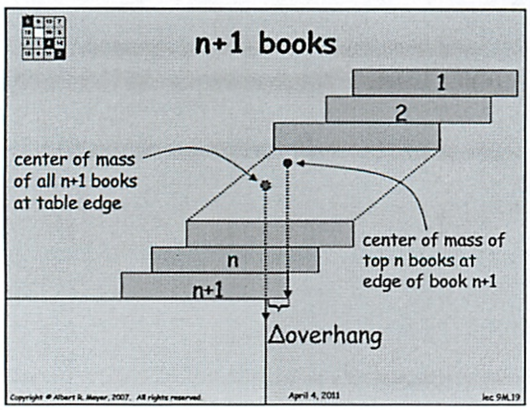
 April 4, 2011 lec 9M.14







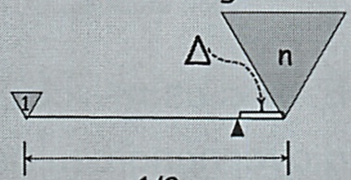




Δ-overhang ::=
horizontal distance from
n-book to (n+1)-book
centers of mass

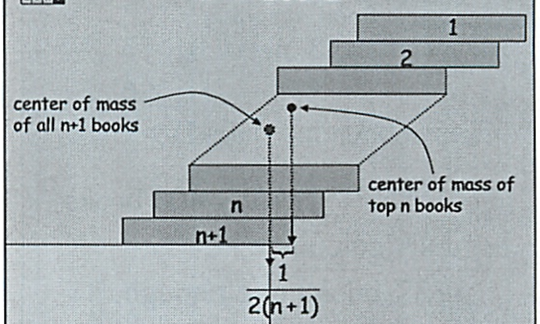
Albert R Meyer, April 4, 2011 lec 9M.20

Δ-overhang



$$\Delta = \frac{\frac{1}{2}}{n+1} = \frac{1}{2(n+1)}$$

n+1 books



Book stacking summary

B_n ::= overhang of n books

$B_1 = 1/2$

$$B_{n+1} = B_n + \frac{1}{2(n+1)}$$

$$B_n = \frac{1}{2}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)$$

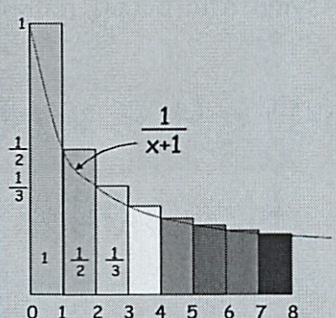
Harmonic Sums

$$H_n ::= 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$$

n^{th} Harmonic number

$$B_n = H_n/2$$

Integral estimate for H_n



Integral estimate for H_n

H_n = area of rectangles

> area under $1/(x+1)$ =

$$\int_0^n \frac{1}{x+1}\,dx = \int_1^{n+1} \frac{1}{x}\,dx = \ln(n+1)$$

Book stacking

for overhang 3, need $B_n \geq 3$

$H_n \geq 6$

integral bound: $\ln(n+1) \geq 6$

so ok with $n \geq \lceil e^6 - 1 \rceil = 403$ books

actually calculate H_n:

227 books are enough.

Book stacking

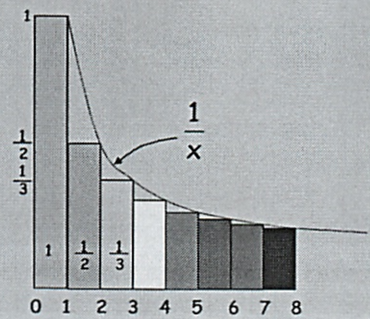
$H_n \to \infty$ as $n \to \infty$,

so overhang can be as big as desired!

CD cases over the edge

43 cases high --top 4 cases completely off the table --1.8 or 1.9 case-lengths

Upper bound for H_n



Upper bound for H_n

$$H_n < 1 + \int_1^n \frac{1}{x}\,dx$$

$$= 1 + \ln(n)$$

Asymptotic bound for H_n

$$\ln(n+1) < H_n < 1 + \ln(n)$$

$$H_n \sim \ln(n)$$

Asymptotic Equivalence

Def: $f(n) \sim g(n)$

$$\lim_{n\to\infty}\left(\frac{f(n)}{g(n)}\right)=1$$

Asymptotic Equivalence ~

Example: $(n^2 + n) \sim n^2$

pf:

$$\lim_{n\to\infty}\frac{n^2+n}{n^2}=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)=1$$

Team Problems

Problems

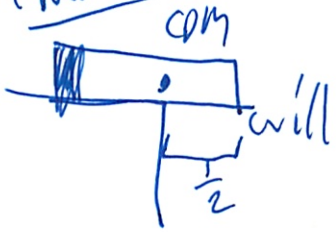
1—4

(5 min late)

Book stacking

- so does not fall off table

1 book



Max overhang = $\frac{1}{2}$

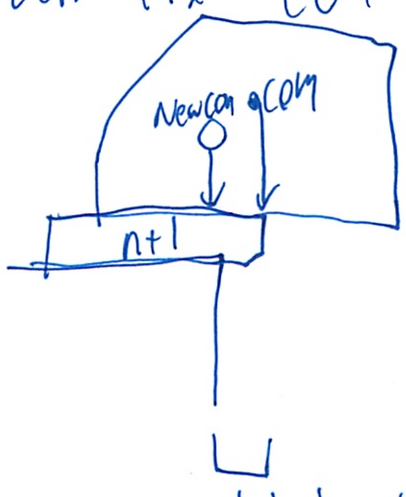
n books

take avg of all books COM

- avg COM must be on table to balance

n+1 books

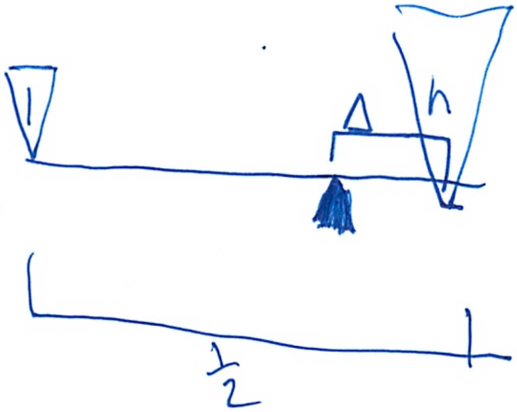
Lets fix COM of top n books at edge of book n+1



So New COM is still stable

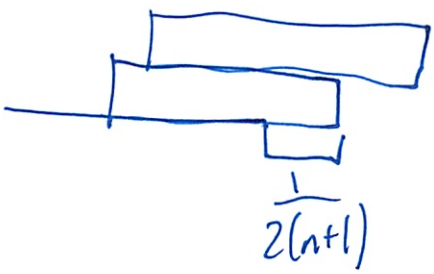
this is the new overhang Δ overhang

2) Want it to balance



$$\Delta = \frac{\frac{1}{2}}{n+1} = \frac{1}{2(n+1)}$$

That is distance



Recursive construction

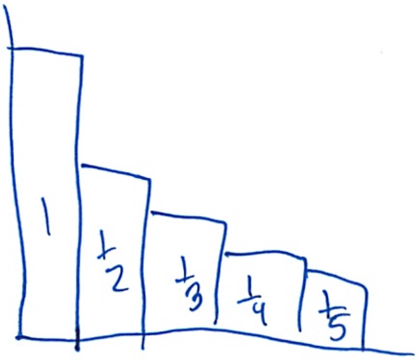
(did not copy)

(3)

Harmonic sum

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

- $\frac{1}{2}$ of the previous



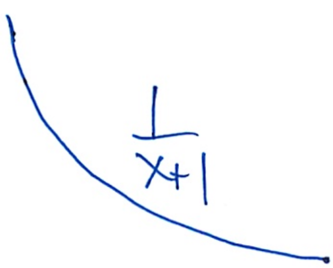
No nice closed form

Need to estimate

- use sum of size of rectangles

By turning sum into integration

(proof by picture)



$\frac{1}{x+1}$ is <u>lower-band</u> on our area

H_n = area rectangles $>$ $\frac{1}{x+1}$ area rectangles

$$\int_0^n \frac{1}{x+1}\,dn$$

... did not see

(4)

log grows ∞ - so can always put more books at

for overhang 3 - need ~~~~ 403 books
estimate

actual is 227 -
- calculate w/ sum

hard to actually do w/ books - compress

do w/ CD cases

estimates ~~~~ upper bound

$\frac{1}{x}$ + area first rectangle (1)

$H_n < 1 + \frac{1}{x}$

So

$\ln(n+1) < H_n < 1 + \ln(n)$

estimate by integration

$H_n \sim \ln(n)$

"asymptotic to" - means ratio goes to 1 in limit

(5)

Def ... missed ...

Used to see which parts are dominating the growth

In-Class Problems Week 9, Mon.

Problem 1.
You've seen this neat trick for evaluating a geometric sum:

$$\begin{aligned} S &= 1 + z + z^2 + \ldots + z^n \\ zS &= z + z^2 + \ldots + z^n + z^{n+1} \\ S - zS &= 1 - z^{n+1} \\ S &= \frac{1 - z^{n+1}}{1 - z} \end{aligned}$$

Use the same approach to find a closed-form expression for this sum:

$$T = 1z + 2z^2 + 3z^3 + \ldots + nz^n$$

Problem 2.
An explorer is trying to reach the Holy Grail, which she believes is located in a desert shrine d days walk from the nearest oasis. In the desert heat, the explorer must drink continuously. She can carry at most 1 gallon of water, which is enough for 1 day. However, she is free to make multiple trips carrying up to a gallon each time to create water caches out in the desert.

For example, if the shrine were $2/3$ of a day's walk into the desert, then she could recover the Holy Grail after two days using the following strategy. She leaves the oasis with 1 gallon of water, travels $1/3$ day into the desert, caches $1/3$ gallon, and then walks back to the oasis—arriving just as her water supply runs out. Then she picks up another gallon of water at the oasis, walks $1/3$ day into the desert, tops off her water supply by taking the $1/3$ gallon in her cache, walks the remaining $1/3$ day to the shrine, grabs the Holy Grail, and then walks for $2/3$ of a day back to the oasis—again arriving with no water to spare.

But what if the shrine were located farther away?

(a) What is the most distant point that the explorer can reach and then return to the oasis if she takes a total of only 1 gallon from the oasis?

(b) What is the most distant point the explorer can reach and still return to the oasis if she takes a total of only 2 gallons from the oasis? No proof is required; just do the best you can.

(c) The explorer will travel using a recursive strategy to go far into the desert and back drawing a total of n gallons of water from the oasis. Her strategy is to build up a cache of $n-1$ gallons, plus enough to get home, a certain fraction of a day's distance into the desert. On the last delivery to the cache, instead of returning home, she proceeds recursively with her $n-1$ gallon strategy to go farther into the desert and return to the cache. At this point, the cache has just enough water left to get her home.

Prove that with n gallons of water, this strategy will get her $H_n/2$ days into the desert and back, where H_n is the nth Harmonic number:

$$H_n ::= \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

Conclude that she can reach the shrine, however far it is from the oasis.

(d) Suppose that the shrine is $d = 10$ days walk into the desert. Use the asymptotic approximation $H_n \sim \ln n$ to show that it will take more than a million years for the explorer to recover the Holy Grail.

Problem 3.
There is a number a such that $\sum_{i=1}^{\infty} i^p$ converges iff $p < a$. What is the value of a? Prove it.

Problem 4.
Suppose $f, g : \mathbb{N}^+ \to \mathbb{N}^+$ and $f \sim g$.

(a) Prove that $2f \sim 2g$.

(b) Prove that $f^2 \sim g^2$.

(c) Give examples of f and g such that $2^f \not\sim 2^g$.

In Class 9 Mon

4/4

1. We've seen trick for evaluating geo. sum
Do it for

$$T = 1z + 2z^2 + 3z^3 + \dots + nz^n$$

Try z2

$$zT = z^2 + 2z^3 + 3z^4 + \dots + nz^{n+1}$$

~~$T - zT =$ no thats not very nice~~

~~$T + 1$?~~

Our group did

$$T - zT = 1z + 1z^2 + 1z^3 + \dots - nz^{n-1}$$

$$= \frac{z^{n+1}}{1-z}$$

$$- nz^{n+1}$$

2a. $\frac{1}{2}$ day

b $\frac{3}{4}$ day

(2)

Oh I see it can be represented w/ Geometric seq
Find the closed form to know how far she can go

$$H_n = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$$

She can get $\frac{H_n}{2}$ days into the dessert

d) Suppose shrine is $d = 10$ days into desert
Use asymptotic approx $H_n \sim \ln(n)$ to show
it will take more than a million years.

Well makes sense - b/c additions get smaller
+ smaller each day - as go further out
need more + more water

Treat cache as oasis
Build up to $n-1$ gallon

(3)

3. Tutor problem! except now you actually have to prove it

$\sum_{i=1}^{\infty} i^p$ converges if $p < a$

What is a?

-1

- just copy ans

Sum is $(+)$ $\left(\int_1^{\infty} x^p \, dx\right)$

for $p \neq -1$ ind. integral is $\frac{x^{p+1}}{p+1}$

- If $p < -1$ then $p+1 < 0$ So $\lim_{x \to \infty} x^{p+1} = 0$

definite integral from $1 \to \infty$ evals to $-\frac{1}{(p+1)}$

Here sum bounded from above - since increasing, finite limit, so it converges

- If $p > -1$ then $p+1 > 0$ so $\lim_{x \to \infty} x^{p+1} = \infty$

so diverges

(4)

$p=-1$ indef. int. is $\log x$, which also approaches ∞ as $x \to \infty$, so diverges

4. Suppose f, g $N^+ \to N^+$ and $f \sim g$

a. Prove $2f \sim 2g$

so this is ratios

$f \sim g$ means $\frac{f}{g} \approx 1$

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = 1$$

So

$$\lim_{x \to \infty} \frac{2f}{2g} = 1$$

$\frac{2}{2}$ cancles

b) $\lim \frac{f^2}{g^2}$ take $\sqrt{\ }$ of ea $\frac{f}{g} = 1$

? legal move on its own

(5)

c) Give 2 counter examples s.t.

$$2^f \not\approx 2^g$$

That is not allowed mathwise

But what could you say? for a counter-example?

---

2. Our board

a) $\frac{1}{2}$

b) $\frac{3}{4}$

c) For the explorer to deposit $n-1$ gallons at a position, using n gallons, he needs to make n trips drinking a total of 1 gallon ~~on each trip~~. Hence, each trip will have to be $\frac{1}{n}$ days long round trip or $\frac{1}{2n}$ days long 1-way.

Doing this recursively, the first cache always has $n-1$ at dist $\frac{1}{2n}$. 2nd has $n-2$ at $\frac{1}{2n-1}$

⑥

So total distance is

$$\sum_{i=0}^{n-1} \frac{1}{2n-i} = \frac{1}{2n} + \frac{1}{2n-2} + \frac{1}{2n-\cancel{4}4} + \dots + \frac{1}{6}+\frac{1}{4}+\frac{1}{2}$$

$$= \frac{1}{2}\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}\right)$$

$$= \frac{1}{2} H_n$$

d) $d = \frac{1}{2} H_n$

$d = 10 \approx \frac{1}{2} \ln n$

$n = e^{20} = 4.8 \text{ E } 9$ days $= 1.7 \text{ E } 6$ years

= Very big!

---

3. Prove $\sum_{i=1}^{\infty} i^p$ converges if $p < -1$

Case 1 $p > -1$

Then $\lim_{p \to \infty}$

. . . forget it

(7)

Case $p=-1$

$\sum_{i=1}^{\infty} i^p$ is the harmonic series

4, a) Like I had it

b) $\lim_{n\to\infty} \frac{f(n)^2}{g(n)^2} = \lim \frac{f(n)}{g(n)} \cdot \frac{f(n)}{g(n)}$

$f \sim g$ iff $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$

$\boxed{\text{See how that was written! pay attention to detail}}$

as long as $g(n) \neq 0$

So if $f \sim g$ $\lim_{n\to\infty} \frac{f(n)}{g(n)} \cdot \frac{f(n)}{g(n)} = 1 \cdot 1 = 1$

so $f \sim g \to f^2 \sim g^2$

c) $n \sim n+1$, but $\lim_{n\to\infty} \frac{2^{n+1}}{2^n} = \frac{2 \cdot 2^n}{2^n} = 2$

Solutions to In-Class Problems Week 9, Mon.

Problem 1.
An explorer is trying to reach the Holy Grail, which she believes is located in a desert shrine d days walk from the nearest oasis. In the desert heat, the explorer must drink continuously. She can carry at most 1 gallon of water, which is enough for 1 day. However, she is free to make multiple trips carrying up to a gallon each time to create water caches out in the desert.

For example, if the shrine were $2/3$ of a day's walk into the desert, then she could recover the Holy Grail after two days using the following strategy. She leaves the oasis with 1 gallon of water, travels $1/3$ day into the desert, caches $1/3$ gallon, and then walks back to the oasis—arriving just as her water supply runs out. Then she picks up another gallon of water at the oasis, walks $1/3$ day into the desert, tops off her water supply by taking the $1/3$ gallon in her cache, walks the remaining $1/3$ day to the shrine, grabs the Holy Grail, and then walks for $2/3$ of a day back to the oasis—again arriving with no water to spare.

But what if the shrine were located farther away?

(a) What is the most distant point that the explorer can reach and then return to the oasis if she takes a total of only 1 gallon from the oasis?

Solution. At best she can walk $1/2$ day into the desert and then walk back. ■

(b) What is the most distant point the explorer can reach and still return to the oasis if she takes a total of only 2 gallons from the oasis? No proof is required; just do the best you can.

Solution. The explorer walks $1/4$ day into the desert, drops $1/2$ gallon, then walks home. Next, she walks $1/4$ day into the desert, picks up $1/4$ gallon from her cache, walks an additional $1/2$ day out and back, then picks up another $1/4$ gallon from her cache and walks home. Thus, her maximum distance from the oasis is $3/4$ of a day's walk. ■

(c) The explorer will travel using a recursive strategy to go far into the desert and back drawing a total of n gallons of water from the oasis. Her strategy is to build up a cache of $n-1$ gallons, plus enough to get home, a certain fraction of a day's distance into the desert. On the last delivery to the cache, instead of returning home, she proceeds recursively with her $n-1$ gallon strategy to go farther into the desert and return to the cache. At this point, the cache has just enough water left to get her home.

Prove that with n gallons of water, this strategy will get her $H_n/2$ days into the desert and back, where H_n is the nth Harmonic number:

$$H_n ::= \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

Conclude that she can reach the shrine, however far it is from the oasis.

Solution. To build up the first cache of $n-1$ gallons, she should make n trips $1/(2n)$ days into the desert, dropping off $(n-1)/n$ gallons each time. Before she leaves the cache for the last time, she has $n-1$ gallons plus enough for the walk home. Then she applies her $(n-1)$-day strategy. So letting D_n be her maximum distance into the desert and back, we have

$$D_n = \frac{1}{2n} + D_{n-1}.$$

So

$$\begin{aligned} D_n &= \frac{1}{2n} + \frac{1}{2(n-1)} + \frac{1}{2(n-2)} + \cdots + \frac{1}{2\cdot 2} + \frac{1}{2\cdot 1} \\ &= \frac{1}{2}\left(\frac{1}{n} + \frac{1}{(n-1)} + \frac{1}{(n-2)} + \cdots + \frac{1}{\cdot 2} + \frac{1}{\cdot 1}\right) \\ &= \frac{H_n}{2}. \end{aligned}$$

■

(d) Suppose that the shrine is $d = 10$ days walk into the desert. Use the asymptotic approximation $H_n \sim \ln n$ to show that it will take more than a million years for the explorer to recover the Holy Grail.

Solution. She obtains the Grail when:

$$\frac{H_n}{2} \approx \frac{\ln n}{2} \geq 10.$$

This requires $n \geq e^{20} = 4.8 \cdot 10^8$ days $> 1.329M$ years.

■

Problem 2.
There is a number a such that $\sum_{i=1}^{\infty} i^p$ converges iff $p < a$. What is the value of a? Prove it.

Solution. $a = -1$.

For $p = -1$, the sum is the harmonic series which we know does not converge. Since the term i^p is increasing in p for $i > 1$, the sum will be larger, and hence also diverge for $p > -1$.

For $p < -1$ there exists an $\epsilon > 0$ such that $p = -(1+\epsilon)$. By the integral method,

$$\begin{aligned} \sum_{i=1}^{\infty} i^{-(1+\epsilon)} &\leq 1 + \int_1^{\infty} x^{-(1+\epsilon)}\, dx \\ &= 1 + \epsilon^{-1} - \epsilon^{-1} \lim_{\alpha\to\infty} \alpha^{-\epsilon} \\ &= 1 + \epsilon^{-1} \\ &< \infty \end{aligned}$$

Hence the sum is bounded above, and since it is increasing, it has a finite limit, that is, it converges. ■

Problem 3.
Suppose $f, g : \mathbb{N}^+ \to \mathbb{N}^+$ and $f \sim g$.

(a) Prove that $2f \sim 2g$.

Solution.

$$\frac{2f}{2g} = \frac{f}{g},$$

so they have the same limit as $n \to \infty$.

■

(b) Prove that $f^2 \sim g^2$.

Solution.
$$\lim_{n\to\infty} \frac{f(n)^2}{g(n)^2} = \lim_{n\to\infty} \frac{f(n)}{g(n)} \cdot \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{f(n)}{g(n)} \cdot \lim_{n\to\infty} \frac{f(n)}{g(n)} = 1 \cdot 1 = 1.$$

■

(c) Give examples of f and g such that $2^f \not\sim 2^g$.

Solution.

$$\begin{aligned} f(n) &::= n + 1 \\ g(n) &::= n. \end{aligned}$$

Then $f \sim g$ since $\lim(n+1)/n = 1$, but $2^f = 2^{n+1} = 2 \cdot 2^n = 2 \cdot 2^g$ so

$$\lim \frac{2^f}{2^g} = 2 \neq 1.$$

■

Problem 4.
You've seen this neat trick for evaluating a geometric sum:

$$\begin{aligned} S &= 1 + z + z^2 + \ldots + z^n \\ zS &= z + z^2 + \ldots + z^n + z^{n+1} \\ S - zS &= 1 - z^{n+1} \\ S &= \frac{1 - z^{n+1}}{1 - z} \end{aligned}$$

Use the same approach to find a closed-form expression for this sum:

$$T = 1z + 2z^2 + 3z^3 + \ldots + nz^n$$

Solution.

$$\begin{aligned} zT &= 1z^2 + 2z^3 + 3z^4 + \ldots + nz^{n+1} \\ T - zT &= z + z^2 + z^3 + \ldots + z^n - nz^{n+1} \\ &= \frac{1 - z^{n+1}}{1 - z} - 1 - nz^{n+1} \\ T &= \frac{1 - z^{n+1}}{(1-z)^2} - \frac{1 + nz^{n+1}}{1 - z} \end{aligned}$$

■

6.042 Miniquiz 4

4/5

graph = network

directed = di = 1 way = arrows

DAG = directed, acyclic [no cycles]

dots = nodes = verticies

$e = \langle u \to v \rangle$

$\text{indeg}(v) ::= |\{e \in E(G) \mid \text{head}(e) = v\}|$

$\text{outdeg}(v) ::= |\{e \in E(G) \mid \text{tail}(e) = v\}|$

$$\sum_{v \in V(G)} \text{indeg}(v) = \sum_{v \in V(G)} \text{outdeg}(v)$$

$V(G)$ = verticies $E(G)$ = edges

walk - can repeat points

path - all pts must be unique

merge = $f \hat{v} r$ combine 2 walks

distance = lenght of shortest path

Adj Matrix $(A_G)_{ij}$ if $\langle v_i \to v_j \rangle \in V(G)$

$(A_G)^k$ count of lenght k walks b/w

= C_{uv} for a certain point

$u G^* v$ is a path G^+ = pos lenght G_n path

transitive $(aRb \text{ AND } bRc) \to (aRc)$ for every $a,b,c \in A$

reflexive iff aRa for all $a \in A$

- true always b/c 0 lenght path

Can compose relations

$a(R \circ S)c ::= \exists b \in B\, (aSb) \text{ AND } (bRc)$

Closed walk - starts + ends same vertex

Cycle = closed walk w/ distinctive verticies lenght ≥ 2

asymmetric iff $aRb \to \text{NOT}(bRa)$ for all $a,b \in A$ no self-loops

irreflexive no + lenght path from any vertex to itself

$\text{NOT}(aRa)$ for all $a \in A$

Strict partial order = trans + asymmetric

if pos path relation of a DAG

weak partial order - can also be =

aRb iff $(aSb \text{ OR } a=b)$

Antisymmetric $aRb \to \text{NOT}(bRa)$ for all $a \neq b \in A$ (self loops allowed)

WPO = transitive, reflexive, antisymetric

= $\preceq$ or $\subseteq$

$\subset$ = subset

isomorphic if relation preserving bij

total - always 1 arrow $\forall x \neq y \in A\, (xRy \text{ or } yRx)$

product order $R_1 \times R_2$

$\text{domain}(R_1 \times R_2) = \text{domain}(R_1) \times \text{domain}(R_2)$

codomain " = co " × co "

$(a_1, a_2)(R_1 \times R_2)(b_1, b_2)$ iff $[a_1 R_1 b_1$ and $a_2 R_2 b_2]$

- where both are true

topological sort $a \preceq b \to a \sqsubseteq b$ (partial) (total)

antichain - all items incomparable

equivilance = reflexive, symmetric, transitive

$\subset$ = proper subset $A \subset B$ means B has everything + more

- asymmetric
- transitive

SPO - transitive + asymmetric   DAG

$<$ less then, ranked higher

WPO - same as SPO except aRa always holds

$\subseteq$ on sets $\leq$ on $\mathbb{R}$

- reflexive   for all
- transitive
- antisymmetric

total - like a path/chain

Symmetric $\forall x,y \in A\; xRy \to yRx$

- arrow in both dirs

Simple graphs - undirected (no arrows)

$v - w$ = undirected edge

no self loops (from u to u)

two pts adjacent if edge

edge is incident to end pts

$\deg(v)$ = # edges incident to vertice

$$\sum_{x \in M} \deg(x) = \sum_{y \in F} \deg(y)$$

Handshake sum of deg of vertices = 2 × # edges

K_n = complete graph - every arrow $\boxed{2|E| = \sum_{v \in V} \deg(v)}$

L_n = line graph

- if add 1 edge would have cycle

isomorphism is a edge-preserving bij $f: V(G) \to V(H)$ s.t.

$u - v \in E(G)$ iff $f(u) - f(v) \in E(H)$ for all $u,v \in V(G)$

biparte - can split into 2 graps

Matching cond - every subset men likes at least as large as subset of men

matching - set M of edges G s.t. no vertex is incident to ≥ 1 edge in M.

covers - if all vertices included = perfect

bottleneck $|S| > |N(S)|$ neighbors

Hall's Theorem: Matching in G (biparite) that covers $L(G)$ iff no subset of $L(G)$ is a bottleneck

if degree constrained - is a matching

degree constrained $\deg(l) \geq \deg(r)$ for all l, r

regular - each node has same degree

Every reg biparite graph has perfect matching

Stable - no rogue couples - pair that likes each other more

if w is off m's list w has suitor she prefers over m

men = optimal   termination # remaining names strictly ↓

girls = pessimal

Coloring - adj vertices diff color

$\chi()$ = chromatic # = min # colors

$\chi(K_n) = n$ $\chi(\text{biparite}) = 2$

$\chi(C_{even}) = 2$ $\chi(C_{odd}) = 3$

$\chi(\text{max degree } k) = k+1$

Subgraphs

connected - every pair verticies connected - path exists somewhere

connected components

k-edge connected = # edges can remove till splits

- called cut edge

Tree - connected acyclic graph

Connected component of trees = forest

leaf = node w/ deg(1)

1. Each connected subgraph = tree
2. Unique simple path b/w every pair of vertices
3. Adding edge b/w nonadj nodes creates a cycle
4. Removing any edge - disconnects
   - All edges = cut edges
5. If ≥ 2 vertices ≥ 2 leaves
6. # vertices = #edges +1

Spanning tree - min # of lines so all vertices still connected

if branches weighted → Min-weight tree (MST)

Planar - no lines crossing

drawing - one particular set of curves

face - continuous
- but divide up into discrete
- don't forget outside

bridge

dangle

discrete face = planar embeddings
- either split a face or add a bridge

Euler's Formula $v - e + f = 2$
- proof w/ the 2 constructors

$e \leq 3v - 6$ limit of planar

minor - delete vertices, edges, merge verticies

every planar graph has degree ≤ 5
- so 5-colorable

At most 5 regular polyhedra

Power set -- set of all subsets

So $P\{1,2,3\} = \{1\}, \{2\}, \{3\}$
$\{1,2\} \{2,3\} \{3,1\}$
$\{1,2,3\}$

Miniquiz 4

4/5

Week 7 Mon - Week 9 Fri

Topics:

- Partial order
- Simple ~~[illegible]~~ graph degrees
- isomorphism
- stable marriage
- Mating ritual
- ~~[illegible]~~ graph connectivity
  - trees
  - coloring
  - planar graphs

Actually most was post - SB

Write lemmas - they seemed to be most useful

Mini-Quiz Apr. 6

Your name: Michael Plasmeier

Circle the name of your TA and write your table number:

Ali Nick Oscar Oshani **Table number** 12

- This quiz is **closed book**. Total time is 30 minutes.
- Write your solutions in the space provided. If you need more space, write on the back of the sheet containing the problem. Please keep your entire answer to a problem on that problem's page.
- GOOD LUCK!

DO NOT WRITE BELOW THIS LINE

Problem	Points	Grade	Grader
1	6	3	OS
2	3	0	NJ
3	3	2	OM
4	5	2	AK
5	3	2	OS
Total	20	9	OS

Problem 1 (6 points). (a) A simple graph has 8 vertices and 24 edges. What is the average degree per vertex?

✓ Handshake = $\sum deg = 2|E|$

$8 \cdot avg = 2 \cdot 24$

$avg = \frac{48}{8} = 6$ ✓

(b) A connected planar simple graph has 5 more edges than it has vertices. How many faces does it have?

✓ Euler's $v - e + f = 2$

$1 - 6 + f = 2$

$-5 + f = 2$

$f = 7$ ✓

(c) A connected simple graph has one more vertex than it has edges. Is it necessarily planar?

~~$2 - 1 + f = 2$~~

Because of the condition that $v > 2$ this does not always hold - only when $v > 2$ proved w/ base + constructor

Base $e \leq 3v - 6$

say $v = 3$, $e = 2$

$2 \leq 3(3) - 6$

$2 \leq 12$ ✓ can work

Iterative

$v = +1$

$e = +1$

will work - right hand side ↑ at factor of 3 compared to left

?

Try $v = 2$ $e = 1$

$1 \leq 3(2) - 6$

$1 \leq 0$ ⊗ won't work.

Must $v \geq 3$

(d) If your answer to the previous part was *yes*, then how many faces can such a graph have? If your answer was *no*, then give an example of a nonplanar connected simple graph whose vertices outnumber its edges by one.

$v - e + f = 2$

So $f = 2 - v + e$

How it holds when $v > 2$

when $v = 3$ $e = 2$

$f = 2 - 3 + 2$

$f = 1$

$v = 4$ $e = 3$

$f = 2 - 4 + 3$

$= 1$ ✓

$v = 5$ $e = 4$

$f = 2 - 5 + 4$

$= 1$

(e) Consider the graph shown in Figure 1. How many distinct isomorphisms exist between this graph and itself? (Include the identity isomorphism.)

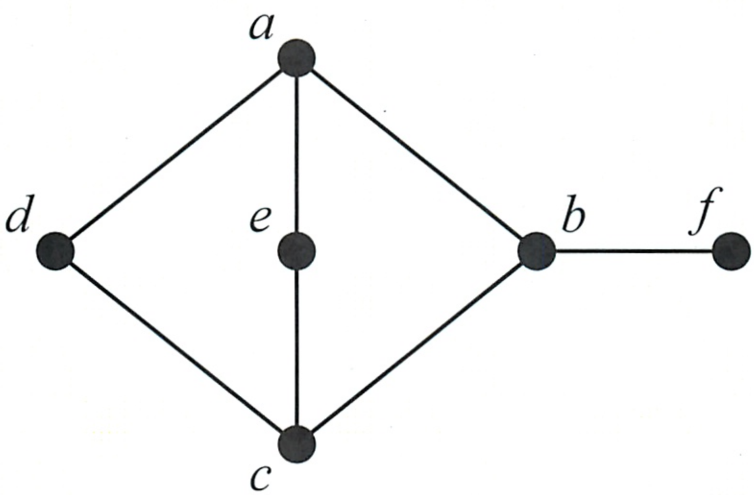


Figure 1

Just 1 by definition ✗

- since can't move or relable anything

a → a
b → b
c → c
d → d
e → e
f → f

Problem 2 (3 points).
The n-dimensional hypercube, H_n, is a simple graph whose vertices are the binary strings of length n. Two vertices are adjacent if and only if they differ in exactly one bit. Consider for example H_3, shown in Figure 2. (Here, vertices `111` and `011` are adjacent because they differ only in the first bit, while vertices `101` and `011` are not adjacent because they differ in both the first and second bits.) Hemming distance

Explain why it is impossible to find two spanning trees of H_3 that have no edges in common.

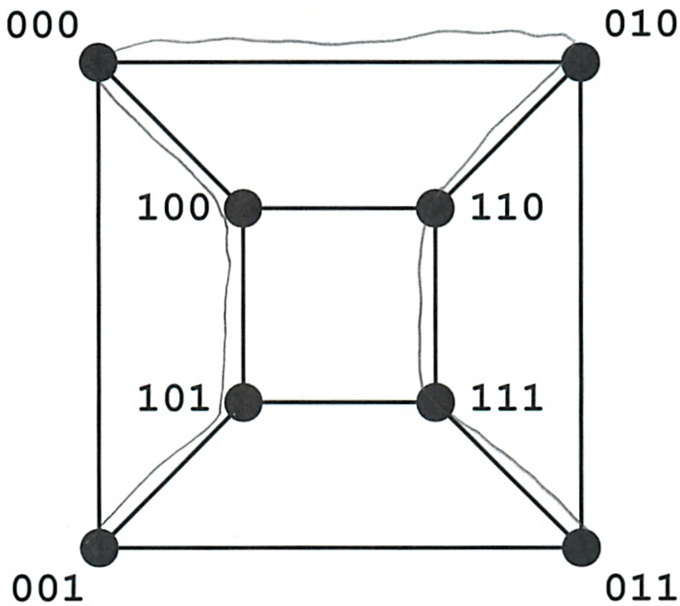
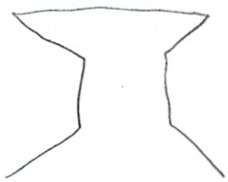


Figure 2 H_3.

0/3

Once you stat in a certain way, there are very limited Choices as to what you can do next in the Spanning tree.



or

or etc rotated

Its just the same pattern rotated
Each point ~~can~~ is degree 3, so there are a limited # of cut edges possible to find different Spanning trees. that's not a general argument

Problem 3 (3 points).

Consider the graph shown in Figure 3. Determine a valid coloring of the graph, using as few colors as possible. (Simply write your proposed color for each vertex next to that vertex. You may use *R* for red, *G* for green, etc.)

Will use #s

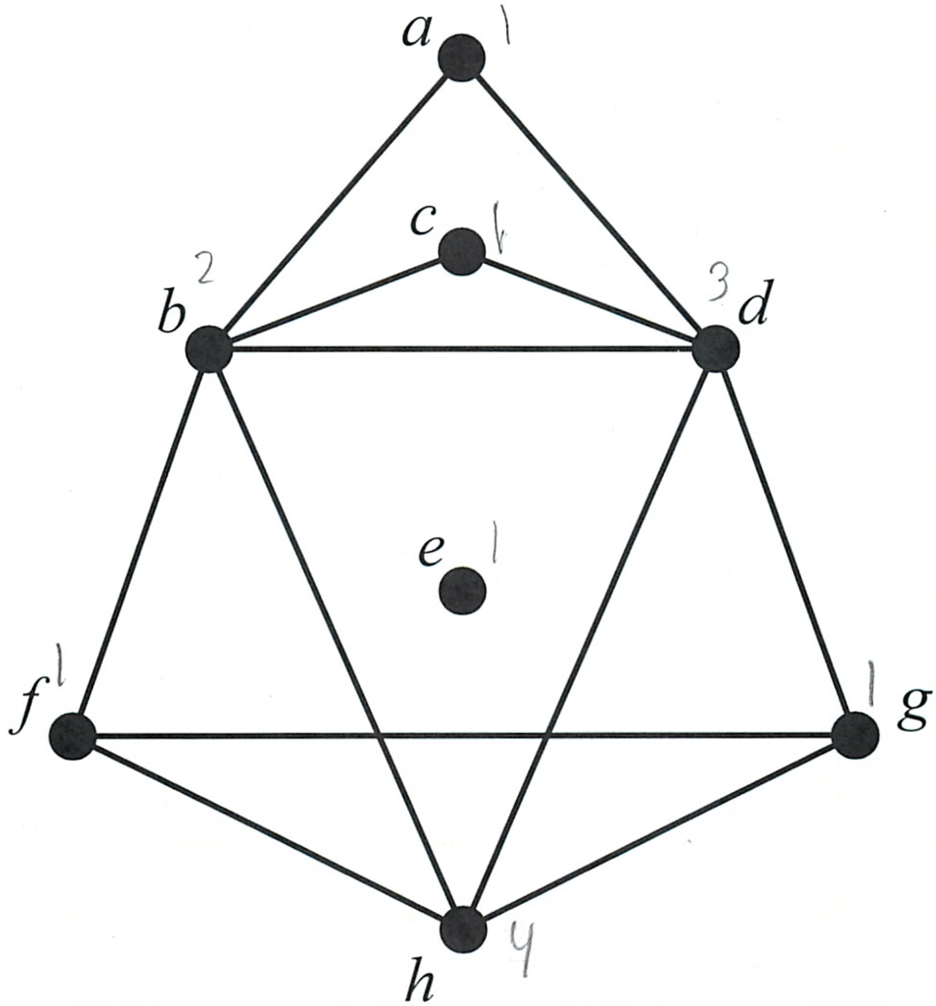


Figure 3

Say
1=Red
2=Green
3=Yellow
4=Orange

Since max degree =4

−1 3 is enoug

Problem 4 (5 points). (a) Consider the bipartite graph G in Figure 4. Is it possible to find a matching that covers $L(G)$? If yes, explain what property of the graph guarantees the existence of such a matching. (Show that the graph exhibits this property and what this implies. Full credit will not be given for merely identifying a matching.) If no, identify a bottleneck that prevents a matching.

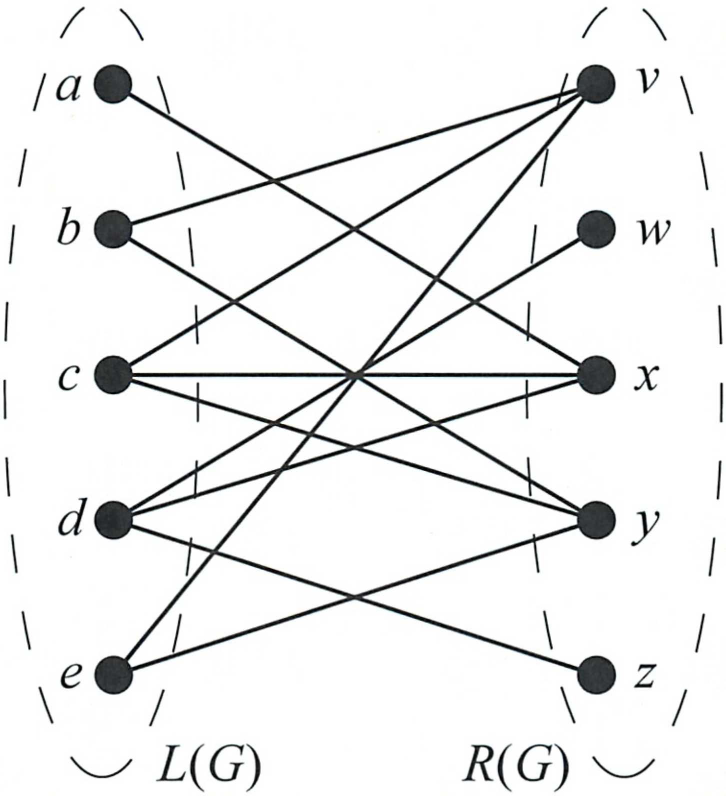


Figure 4 G.

$\emptyset$

Matching - set of M edges G s.t. no vertex is incident to ≥ 1 edges in M

Matching Condition - every subset of L(G) is connected to at least as large a subset of R(G)

bottle neck $|S| > |N(S)|$ neighbors too

Covers - all vertices included (perfect)

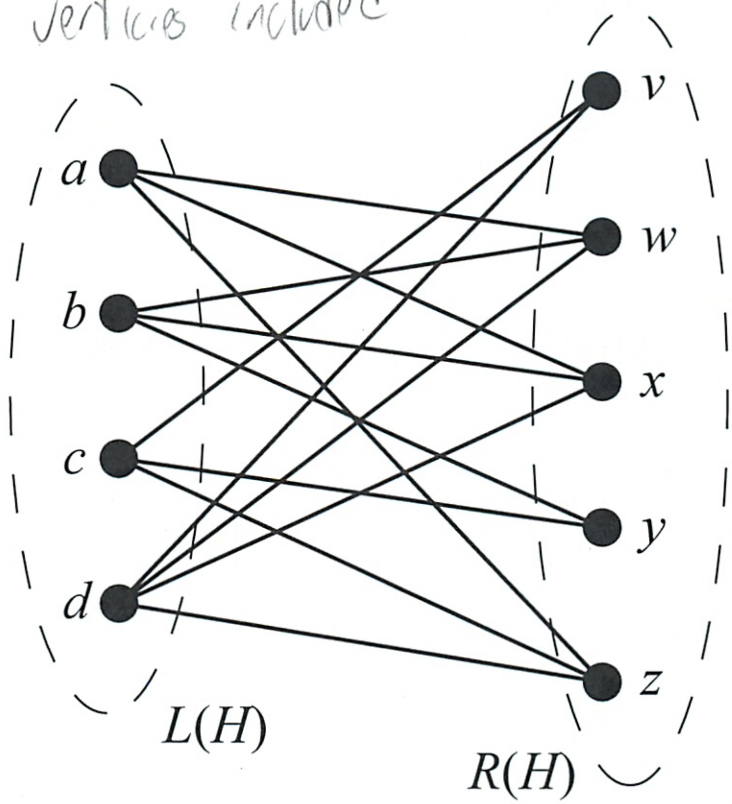
Hall's theorm - Matting in G (bipartite) that covers L(G) if no subset of L(G) is a bottleneck.

There is no bottleneck. For ~~for~~ all subsets of L(G) there exists a subset of equal or larger size in R(G)

√

(b) Consider the bipartite graph H in Figure 5. Is it possible to find a matching that covers $L(H)$? If yes, explain what property of the graph guarantees the existence of such a matching. (Show that the graph exhibits this property and what this implies. Full credit will not be given for merely identifying a matching.) If no, identify a bottleneck that prevents a matching.

Covers - all verticies included



2

Figure 5 H.

Yes. Since the deg(l) ≥ deg(r) for all l ∈ L and r ∈ R so that it is degree constrained

This means there is a matching that covers ~~the entire graph,~~ No! L(H).

See def in previous page.

Problem 5 (3 points).
In the Mating Ritual, suppose Tiger is one of the boys and Elin is one of the girls. Which of the following are preserved invariants **in general**?

2/3

1. Tiger is Elin's only suitor.
2. On Tiger's current list, the girl whom he prefers to all the others is his optimal wife[1].
3. Elin's name has been crossed off by Tiger and by everyone whom she prefers to him.

1. We don't know that w/ info we have been given.

2. Yes, true. Of the names remaining on the list (the current names) the name at the top would be the girl he perfers to all others. This is defined as his optimal wife.
He stays with this girl till they get married or she kicks him out and then he is off the list (no longer on current list)

3. False. Everyone who Elin perfers to Tiger has no relation to who Tiger crosses off his list. Elin's name is crossed off by Tiger when she rejects him. There is no relation between Elins name on Tigers list and Elin's personal preferences.

[1]His *optimal wife* in the usual sense: Given some particular instance of the Stable Marriage Problem, consider all possible stable perfect matchings, including that which is generated by the Mating Ritual. In each of these, Tiger has a wife. Of these "possible wives," he prefers one to all others. This girl, to whom he is married in one of the matchings but not necessarily all of them, is his *optimal wife*.

Solutions to Mini-Quiz Apr. 6

Problem 1 (6 points). (a) A simple graph has 8 vertices and 24 edges. What is the average degree per vertex?

Solution. By the Handshaking Lemma, the sum of the degrees of the vertices in any graph is equal to twice the number of edges. So in this case, the sum of the degrees of the vertices is $2 \times 24 = 48$. With 8 vertices, the average degree per vertex is $\frac{48}{8} = 6$. ∎

(b) A connected planar simple graph has 5 more edges than it has vertices. How many faces does it have?

Solution. Denoting the number of vertices by v, the number of edges by e, and the number of faces by f, Euler's Formula states that $v - e + f = 2$. But here, $e = v + 5$. Substituting gives $v - (v + 5) + f = 2$ and hence $f = 7$. ∎

(c) A connected simple graph has one more vertex than it has edges. Is it necessarily planar?

Solution. Let G denote any such graph. Now, any graph with v vertices but fewer than $v - 1$ edges cannot possibly be connected. So every edge in G is a cut edge, and therefore G is acyclic. So G is a tree and must be planar. ∎

(d) If your answer to the previous part was *yes*, then how many faces can such a graph have? If your answer was *no*, then give an example of a nonplanar connected simple graph whose vertices outnumber its edges by one.

Solution. Since the graph is connected and acyclic, it only has one face. ∎

(e) Consider the graph shown in Figure 1. How many distinct isomorphisms exist between this graph and itself? (Include the identity isomorphism.)

Solution. Only vertex f has degree 1, so in any self-isomorphism, f must map to itself. b is the only vertex to be adjacent to a degree-1 vertex, so b must also map to itself. a and c are both degree-3 vertices, and d and e are both degree-2 vertices. It is clear from examining the graph that a can be mapped to c and c to a, or each of a and c can be mapped to itself. Independently, and similarly, d can be mapped to e and e to d, or each of d and e can be mapped to itself. The only possible isomorphisms, then, are obtained by choosing one of the two possible mappings for a and c and, independently, one of the two possible mappings for d and e. The result is $2 \times 2 = 4$ possible isomorphisms. ∎

Problem 2 (3 points).
The n-dimensional hypercube, H_n, is a simple graph whose vertices are the binary strings of length n. Two vertices are adjacent if and only if they differ in exactly one bit. Consider for example H_3, shown in Figure 2. (Here, vertices `111` and `011` are adjacent because they differ only in the first bit, while vertices `101` and `011` are not adjacent because they differ in both the first and second bits.)

Explain why it is impossible to find two spanning trees of H_3 that have no edges in common.

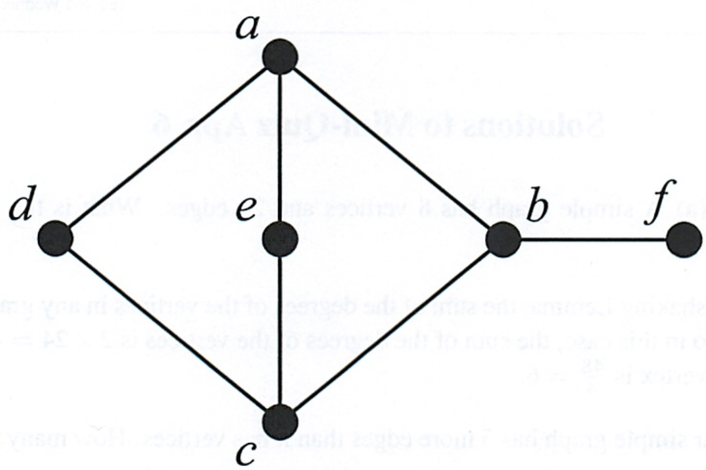


Figure 1

Solution. H_3 has 8 vertices, so any spanning tree must have $8 - 1 = 7$ edges. But H_3 has only 12 edges, so any two sets of 7 edges must overlap. ■

Problem 3 (3 points).
Consider the graph shown in Figure 3. Determine a valid coloring of the graph, using as few colors as possible. (Simply write your proposed color for each vertex next to that vertex. You may use R for red, G for green, etc.)

Solution. There are odd-length cycles in the graph, so at least three colors will be needed. So assume that three colors are sufficient. (If we encounter a contradiction under this assumption, we will need to use more colors.) Start with the length-3 cycle $abda$. All of its vertices must be colored differently, so assign red to a, blue to b, and green to d. The length-3 cycle $bdhb$ now forces h to be colored red. f must now be colored green and g must be colored blue. The coloring is valid so far. c is adjacent to a blue vertex and a green vertex, and no others, it must be colored red. Finally, e is not adjacent to any other vertices, so it can be assigned any of the three colors. Choosing red for e, the result is shown in Figure 4. There is no pair of like-colored adjacent vertices, so this coloring is valid. ■

Problem 4 (5 points). (a) Consider the bipartite graph G in Figure 5. Is it possible to find a matching that covers $L(G)$? If yes, explain what property of the graph guarantees the existence of such a matching. (Show that the graph exhibits this property and what this implies. Full credit will not be given for merely identifying a matching.) If no, identify a bottleneck that prevents a matching.

Solution. It is not possible. One bottleneck is $S = \{a, b, c, e\}$, since $N(S) = \{v, x, y\}$ and hence $|S| = 4 > 3 = |N(S)|$. (It is easy to see that there are no bottlenecks of size 1, 2, 3, or 5.) ■

(b) Consider the bipartite graph H in Figure 6. Is it possible to find a matching that covers $L(H)$? If yes, explain what property of the graph guarantees the existence of such a matching. (Show that the graph exhibits this property and what this implies. Full credit will not be given for merely identifying a matching.) If no, identify a bottleneck that prevents a matching.

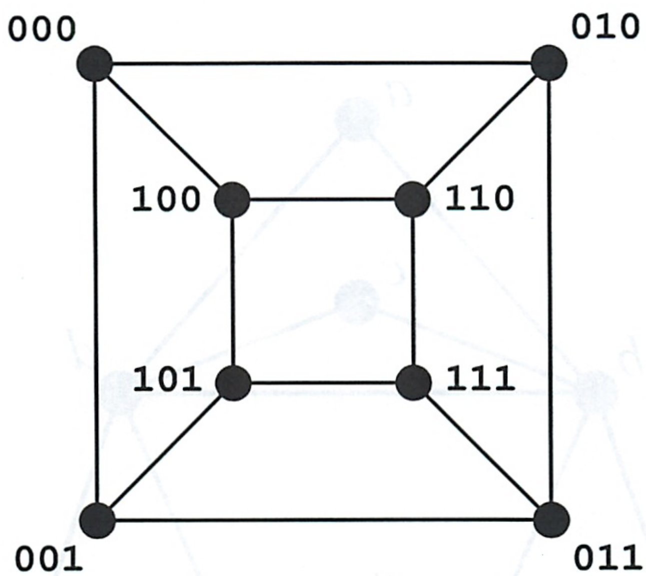


Figure 2 H_3.

Solution. A matching is guaranteed to exist. Each vertex in $L(H)$ has degree at least 3, while each vertex in $R(H)$ has degree at most 3. Consequently, the graph is degree-constrained. There are therefore no bottlenecks and a matching must exist by Hall's Theorem. ■

Problem 5 (3 points).
In the Mating Ritual, suppose Tiger is one of the boys and Elin is one of the girls. Which of the following are preserved invariants **in general**?

1. Tiger is Elin's only suitor.
2. On Tiger's current list, the girl whom he prefers to all the others is his optimal wife[1].
3. Elin's name has been crossed off by Tiger and by everyone whom she prefers to him.

Solution. The statements that are preserved invariants in general appear in boldface below:

1. Tiger is Elin's only suitor. (This would certainly make Tiger Elin's favorite that day, but one or more of the boys who got rejected by another girl that day may visit Elin the following day.)
2. **On Tiger's current list, the girl whom he prefers to all the others is his optimal wife.** (The Mating Ritual gives each boy his optimal wife. Tiger must therefore ultimately marry his optimal wife, so once she becomes the most preferred girl on his list – and thus the girl he is serenading – she must remain the top girl on his list.)
3. **Elin's name has been crossed off by Tiger and by everyone whom she prefers to him.** (Note that this is a preserved invariant because it cannot ever be true. Were it true on some day, Tiger would have crossed Elin's name off his list, so he would end up marrying a woman he finds less desirable.

[1] His *optimal wife* in the usual sense: Given some particular instance of the Stable Marriage Problem, consider all possible stable perfect matchings, including that which is generated by the Mating Ritual. In each of these, Tiger has a wife. Of these "possible wives," he prefers one to all the others. This girl, to whom he is married in one of the matchings but not necessarily all of them, is his *optimal wife*.

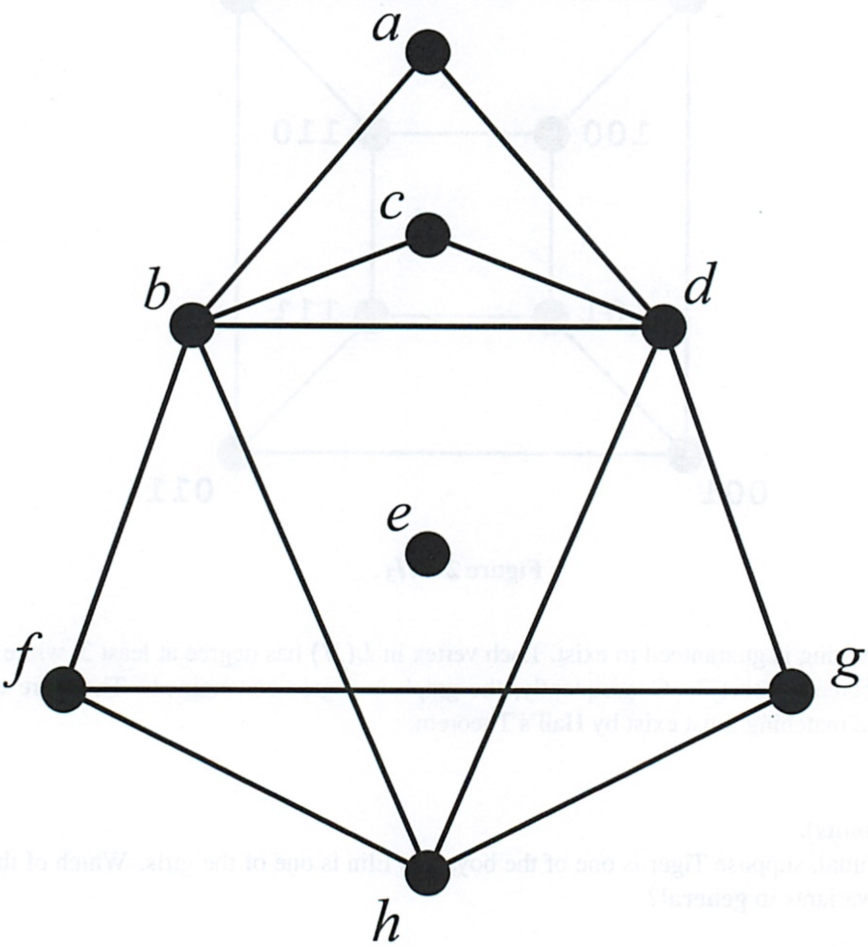


Figure 3

She would also have removed from contention everyone she finds more desirable than Tiger. So she would end up marrying someone she finds less desirable than Tiger. Consequently, Tiger and Elin would constitute would a rogue couple. Another way to think about it is this: If Elin's name was crossed off by Tiger and all the boys Elin prefers to him, then she must have a current favorite whom she prefers to all of them. But Tiger and his betters in Elin's eyes are the top boys on her list: there is no one she prefers to them.)

■

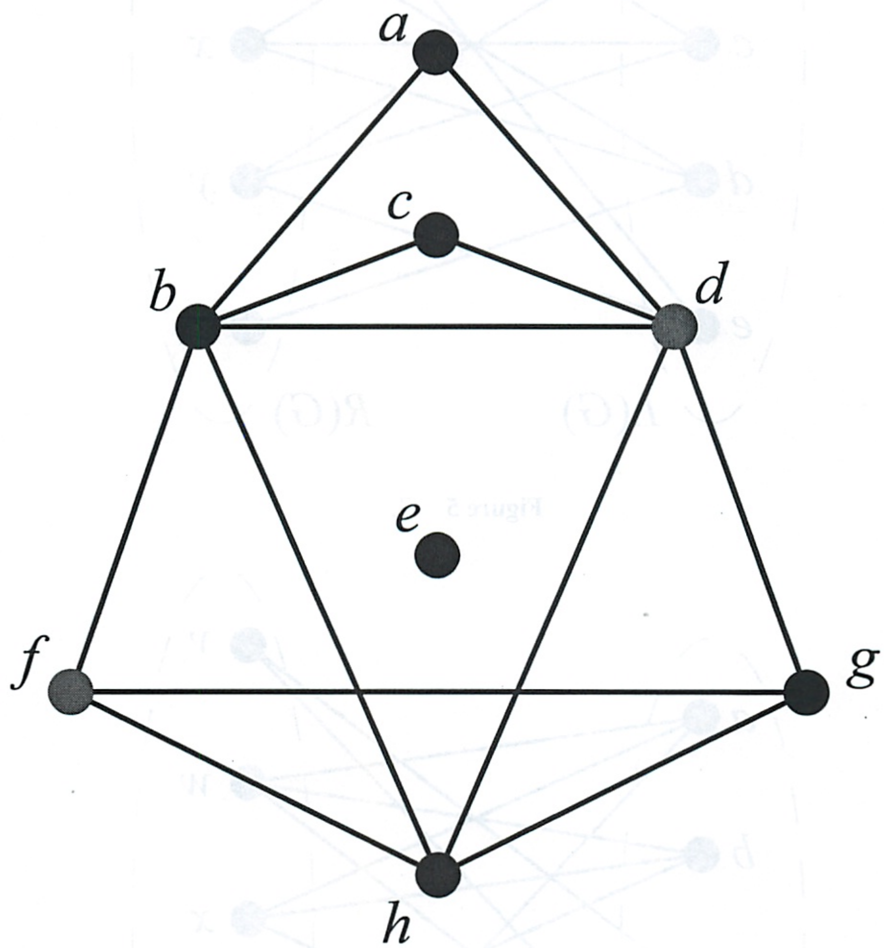


Figure 4 A valid coloring.

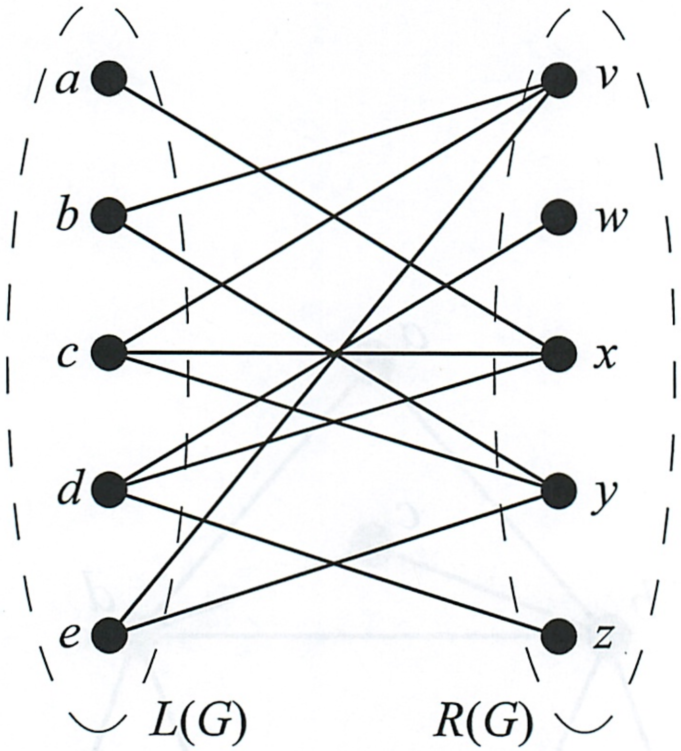


Figure 5 G.

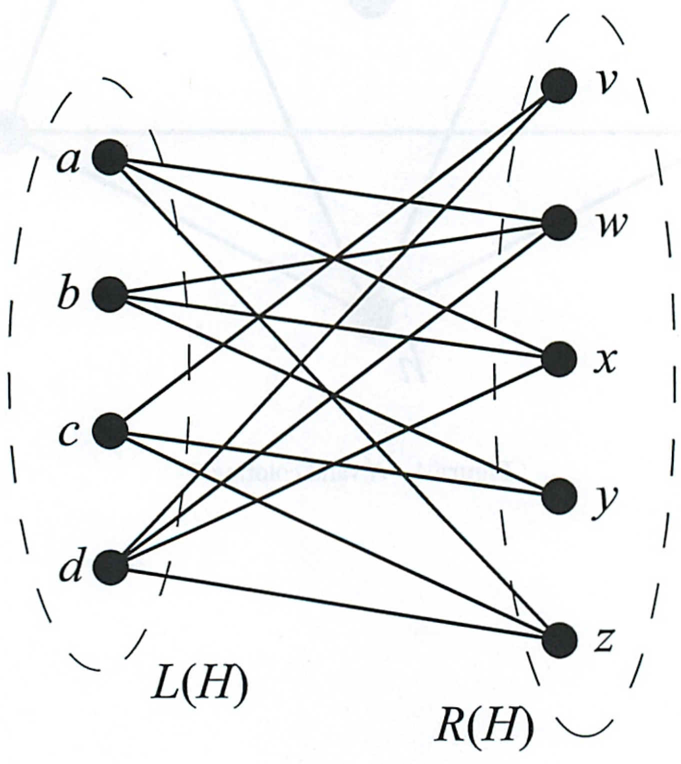
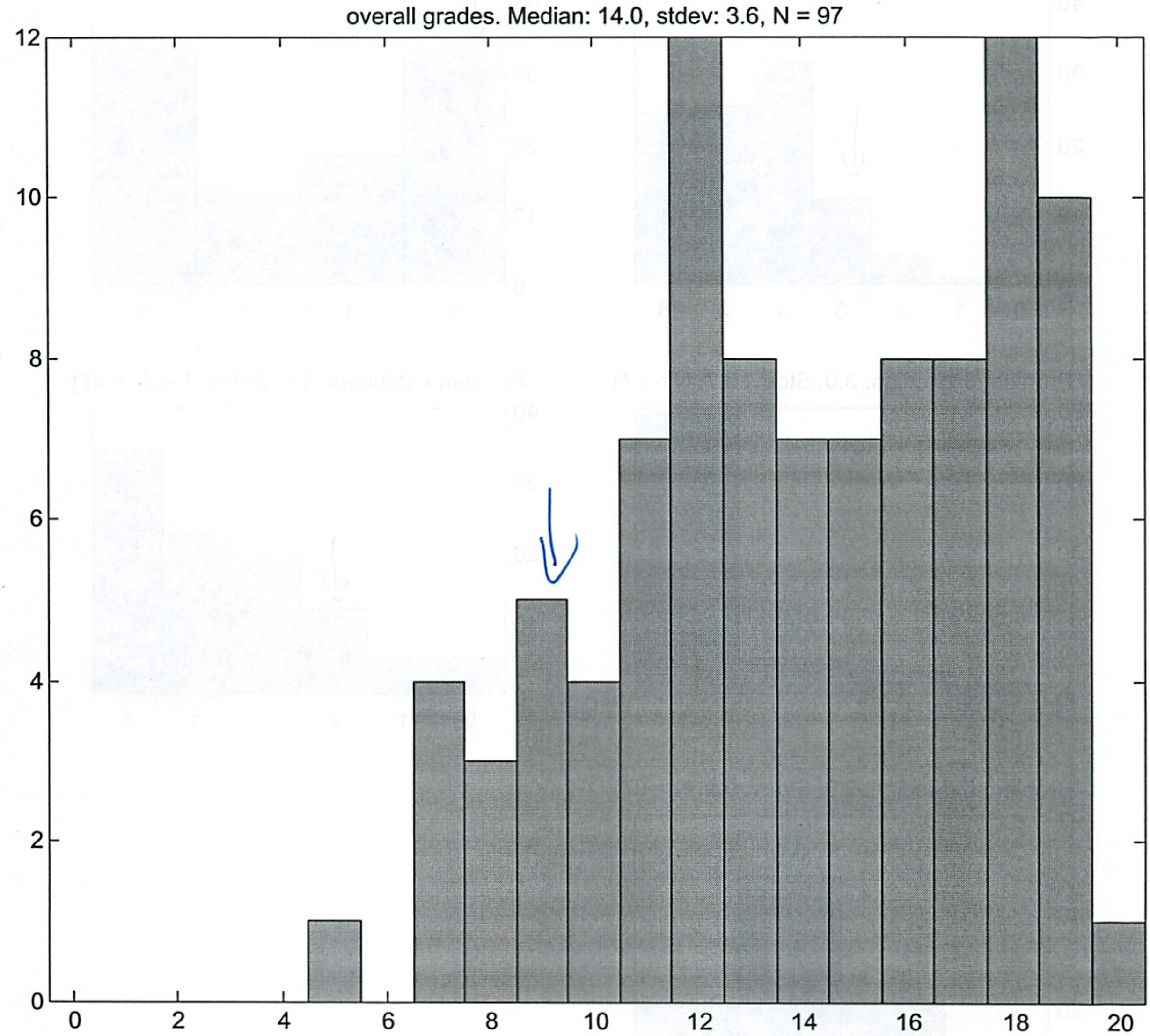
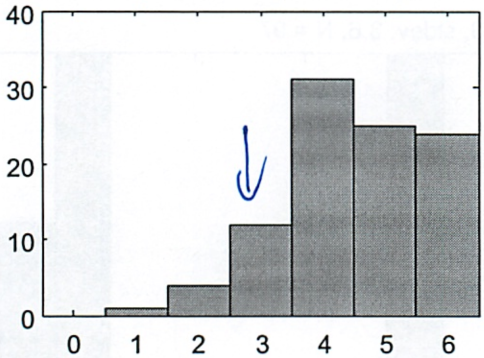


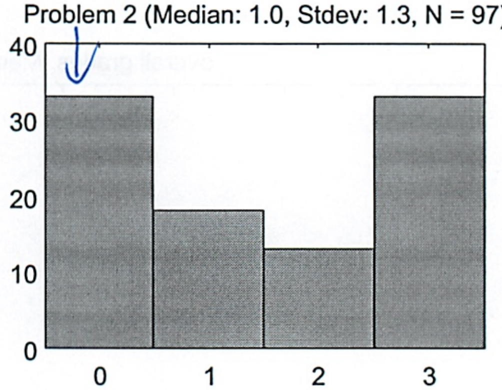
Figure 6 H.


overall grades. Median: 14.0, stdev: 3.6, N = 97
0
2
4
6
8
10
12
0
2
4
6
8
10
12
14
16
18
20

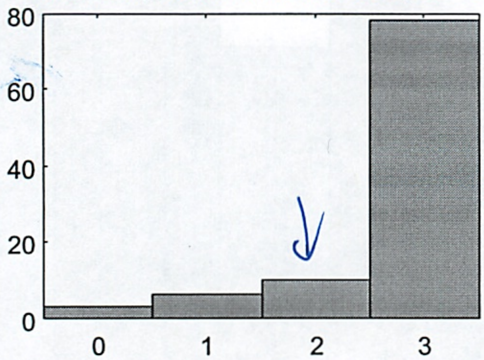
Problem 1 (Median: 5.0, Stdev: 1.1, N = 97)



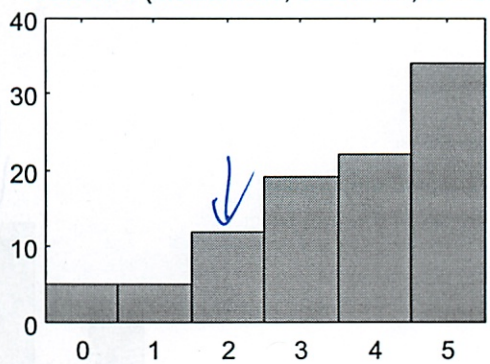
Problem 2 (Median: 1.0, Stdev: 1.3, N = 97)



Problem 3 (Median: 3.0, Stdev: 0.7, N = 97)



Problem 4 (Median: 4.0, Stdev: 1.4, N = 97)



Problem 5 (Median: 2.0, Stdev: 0.9, N = 97)