Massachusetts Institute of Technology
6.042J/18.062J, Spring '11: Mathematics for Computer Science
Prof. Albert R Meyer

April 1
revised Thursday 31st March, 2011, 15:10

# In-Class Problems Week 8, Fri.

**Problem 1.**
Figures 1–4 show different pictures of planar graphs.

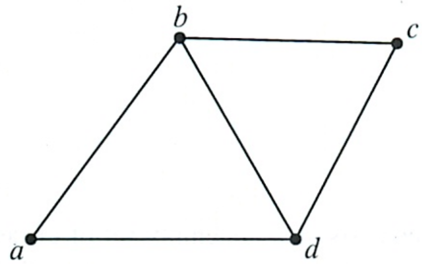



figure 1

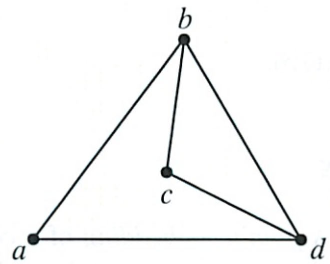



figure 2

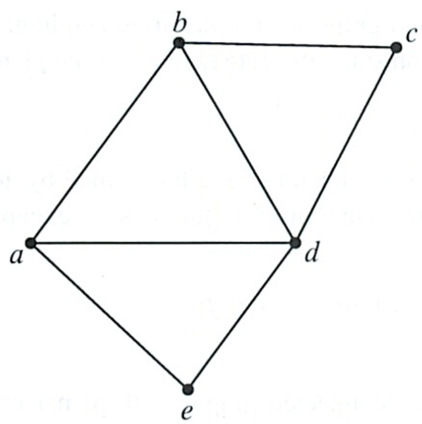



figure 3

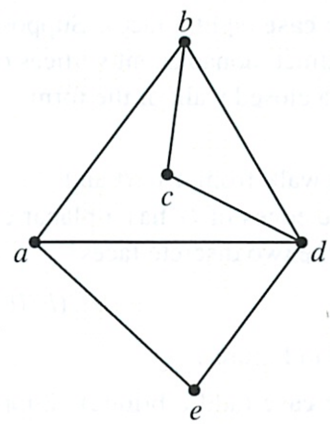



figure 4

**(a)** For each picture, describe its discrete faces (closed walks that define the region borders).

**(b)** Which of the pictured graphs are isomorphic? Which pictures represent the same *planar embedding*?—that is, they have the same discrete faces.

**(c)** Describe a way to construct the embedding in Figure 4 according to the recursive Definition 12.2.2 of planar embedding. For each application of a constructor rule, be sure to indicate the faces (cycles) to which the rule was applied and the cycles which result from the application.

**Problem 2.**
Prove the following assertions by structural induction on the definition of planar embedding.

**(a)** In a planar embedding of a graph, each edge occurs exactly twice in the faces of the embedding.

**(b)** In a planar embedding of a connected graph with at least three vertices, each face is of length at least three.

**Problem 3.**
A simple graph is *triangle-free* when it has no cycle of length three.

**(a)** Prove for any connected triangle-free planar graph with $v > 2$ vertices and $e$ edges,

$$e \leq 2v - 4. \tag{1}$$

*Hint:* Similar to the proof that $e \leq 3v - 6$. Use Problem 2.

**(b)** Show that any connected triangle-free planar graph has at least one vertex of degree three or less.

**(c)** Prove by induction on the number of vertices that any connected triangle-free planar graph is 4-colorable.

*Hint:* use part (b).

# Appendix

**Definition.** A *planar embedding* of a *connected* graph consists of a nonempty set of closed walks of the graph called the *discrete faces* of the embedding. Planar embeddings are defined recursively as follows:

**Base case**: If $G$ is a graph consisting of a single vertex, $v$, then a planar embedding of $G$ has one discrete face, namely, the length zero closed walk, $v$.

**Constructor case** (split a face): Suppose $G$ is a connected graph with a planar embedding, and suppose $a$ and $b$ are distinct, nonadjacent vertices of $G$ that appear on some discrete face, $\gamma$, of the planar embedding. That is, $\gamma$ is a closed walk of the form

$$\alpha \widehat{\ } \beta$$

where $\alpha$ is a walk from $a$ to $b$ and $\beta$ is a walk from $b$ to $a$.[1] Then the graph obtained by adding the edge $\langle a—b \rangle$ to the edges of $G$ has a planar embedding with the same discrete faces as $G$, except that face $\gamma$ is replaced by the two discrete faces[2]

$$\alpha \widehat{\ } (b\ \langle b—a \rangle\ a) \quad \text{and} \quad (a\ \langle a—b \rangle\ b) \widehat{\ } \beta$$

as illustrated in Figure 1.

**Constructor case** (add a bridge): Suppose $G$ and $H$ are connected graphs with planar embeddings and disjoint sets of vertices. Let $\gamma$ be a discrete face of the embedding of $G$ and suppose that $\gamma$ begins and ends at vertex $a$.

Similarly, let $\delta$ be a discrete face of the embedding of $H$ that begins and ends at vertex $b$.

Then the graph obtained by connecting $G$ and $H$ with a new edge, $\langle a—b \rangle$, has a planar embedding whose discrete faces are the union of the discrete faces of $G$ and $H$, except that faces $\gamma$ and $\delta$ are replaced by one new face

$$\gamma \widehat{\ } (a\ \langle a—b \rangle\ b) \widehat{\ } \delta \widehat{\ } (b\ \langle b—a \rangle\ a).$$

This is illustrated in Figure 2, where the vertex sequences of the faces of $G$ and $H$ are:

$$G : \{axyza,\ axya,\ ayza\} \qquad H : \{btuvwb,\ btvwb,\ tuvt\},$$

and after adding the bridge $\langle a—b \rangle$, there is a single connected graph whose faces have the vertex sequences

$$\{axyzabtuvwba,\ axya,\ ayza,\ btvwb,\ tuvt\}.$$

[1] If a walk **f** ends with a vertex, $v$, and a walk **r** starts with the same vertex, $v$, their merge, $\mathbf{f} \widehat{\ } \mathbf{r}$, is the walk that starts with **f** and continues with **r**. Two walks can only be merged if the first ends with the same vertex, $v$, that the second one starts with.

[2] There is a minor exception to this definition of embedding in the special case when $G$ is a line graph beginning with $a$ and ending with $b$. In this case the cycles into which $\gamma$ splits are actually the same. That's because adding edge $\langle a—b \rangle$ creates a cycle that divides the plane into "inner" and "outer" continuous faces that are both bordered by this cycle. In order to maintain the correspondence between continuous faces and discrete faces in this case, we define the two discrete faces of the embedding to be two "copies" of this same cycle.

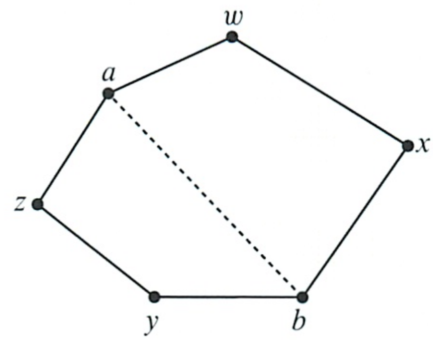



**Figure 1** The "split a face" case: $awxbyza$ splits into $awxyba$ and $abyza$.

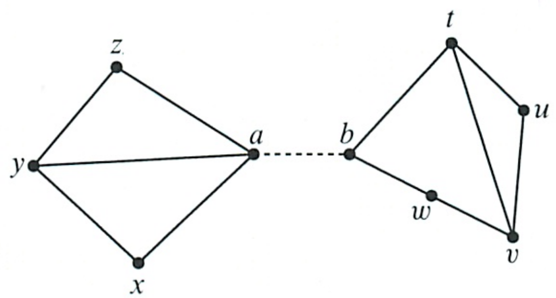



**Figure 2** The "add a bridge" case.

**Theorem 3.1** (Euler's Formula). *If a connected graph has a planar embedding, then*

$$v - e + f = 2$$

*where $v$ is the number of vertices, $e$ is the number of edges, and $f$ is the number of faces.*

**Corollary 3.2.** *Suppose a connected planar graph has $v \geq 3$ vertices and $e$ edges. Then*

$$e \leq 3v - 6.$$

*Proof.* By definition, a connected graph is planar iff it has a planar embedding. So suppose a connected graph with $v$ vertices and $e$ edges has a planar embedding with $f$ faces. By Problem 2.a, every edge is traversed exactly twice by the face boundaries. So the sum of the lengths of the face boundaries is exactly $2e$. Also by Problem 2.b, when $v \geq 3$, each face boundary is of length at least three, so this sum is at least $3f$. This implies that

$$3f \leq 2e. \tag{2}$$

But $f = e - v + 2$ by Euler's formula, and substituting into (2) gives

$$\begin{aligned} 3(e - v + 2) &\leq 2e \\ e - 3v + 6 &\leq 0 \\ e &\leq 3v - 6 \end{aligned}$$

■

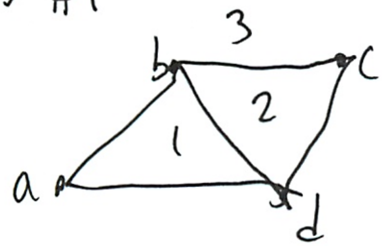
**Corollary 3.3.** *$K_5$ is not planar.*

*Proof.*

$$e = 10 > 9 = 3v - 6.$$

■

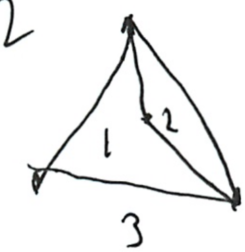
1a) #1




a b d a
b c d b
a b c d a

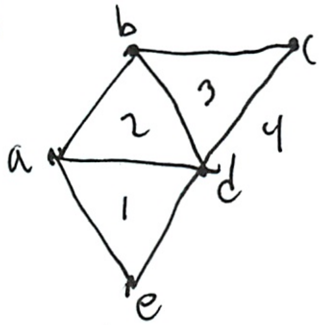
nothing called "outerface"

#2



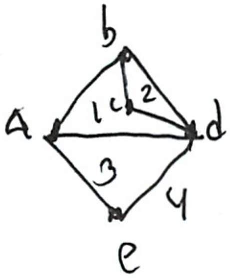

a b c d a
b c d b
a b d a

#3




a b d a
a d e a
b c d b
a b c d e a

#4



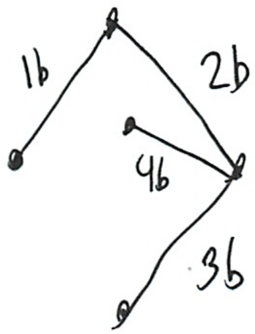

a b c d a
b d c b
a d e a
a b d e a

b) isomorphic - 1, 2
3, 4

planar embedding - same discrete faces 1, 2
~~3, 4~~ Not! - added triangle

②

c) Build #4 w/ adding rule

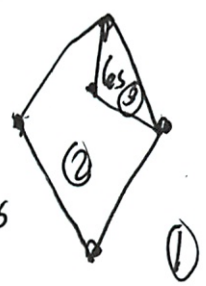



← all one face

"but how do you close now

Can draw line b/w w/ the split rule

- (all pics showed like that, but use def)




5s splits ① into ① ②

6s splits ② into ② ③

to have a planar embedding – must be isomorphic?
(pair of graphs)

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Rest of lecture

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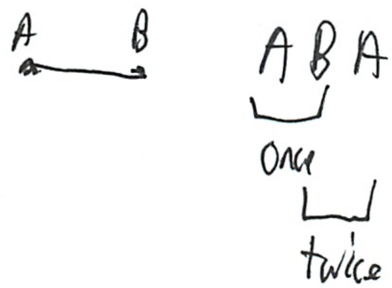
(3)

2. Prove by structural induction on def planar embedding

a) One for each

<u>Base</u> V=1 does not count ?

<u>Base V=2</u>




? prob won't need ~~since with~~ - since could also do w/ constructor

<u>Constructor add bridge</u>

- what I showed above except use more general lang

A, ~~A~~B could have also been connected components, not just single ?(w/ multiple vertices)

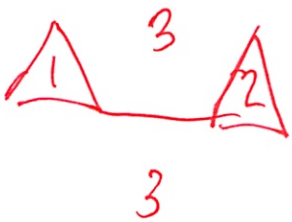
(4)

not allowed to connect later

- must bild up w/ 1 connected component example


- Cald Connect if each was built in its own place


B/c when you connect, you lose a face



(5)

3. A simple graph is $\Delta$-free if has no cycles of lenght 3

a) Prove for any connected triangle free planar graph w/ $v > 2$ verticies

$$e \leq 2v - 4$$

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# Solutions to In-Class Problems Week 8, Fri.

**Problem 1.**
Figures 1–4 show different pictures of planar graphs.

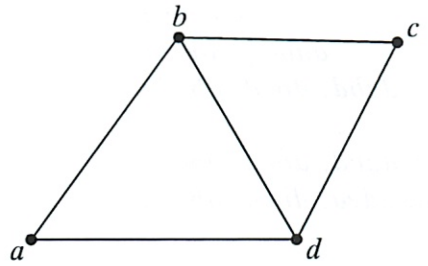



figure 1

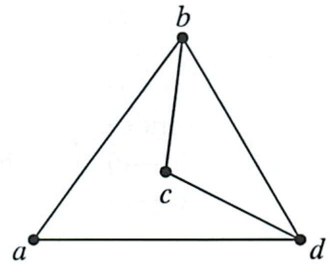



figure 2

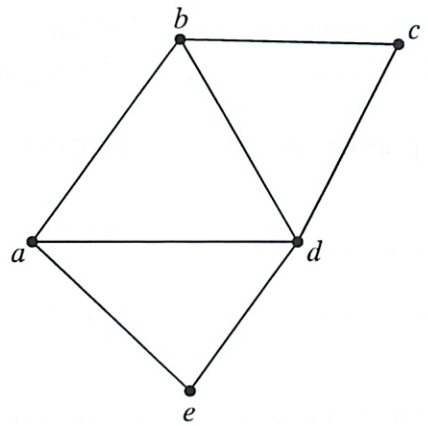



figure 3

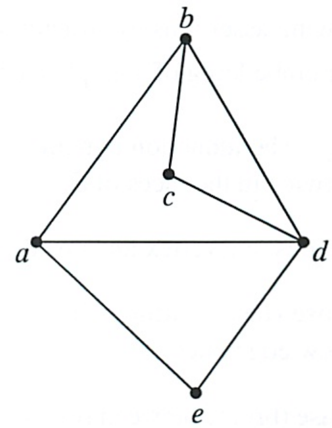



figure 4

**(a)** For each picture, describe its discrete faces (closed walks that define the region borders).

**Solution.** Figs 1 & 2: abda, bcdb, abcda. Fig 3: abcdea, adea,abda,bcdb. Fig 4: abcda, abdea, bdcb, adea. ■

**(b)** Which of the pictured graphs are isomorphic? Which pictures represent the same *planar embedding*?—that is, they have the same discrete faces.

**Solution.** Figs 1 & 2 have the same faces, so are different pictures of the *same* planar drawing. Figs 3 & 4 both have four faces, but they are different, for example, Fig 3 has a face with 5 edges, but the longest face in Fig 4 has 4 edges. ■

**(c)** Describe a way to construct the embedding in Figure 4 according to the recursive Definition **??** of planar embedding. For each application of a constructor rule, be sure to indicate the faces (cycles) to which the rule was applied and the cycles which result from the application.

**Solution.** Here's one way. (The constructor steps could actually be done in any order.)

| recursive step | | faces |
|---|---|---|
| vertex $a$ | (base case) | $a$ |
| vertex $b$ | (base) | $b$ |
| $\langle a—b\rangle$ | (bridge) | $aba$ |
| vertex $c$ | (base) | $c$ |
| $\langle b—c\rangle$ | (bridge) | $abcba$ |
| vertex $d$ | (base) | $d$ |
| $\langle c—d\rangle$ | (bridge) | $abcdcba$ |
| $\langle a—d\rangle$ | (split) | $dabcd,\ dabcd$ |
| $\langle b—d\rangle$ | (split) | $dabd,\ dbcd,\ abcda$ |
| vertex $e$ | (base) | $e$ |
| $\langle d—e\rangle$ | (bridge) | $dedabd,\ dbcd,\ abcda$ |
| $\langle a—e\rangle$ | (split) | $abdea,\ adea,\ dbcd,\ abcda$ |

■

**Problem 2.**
Prove the following assertions by structural induction on the definition of planar embedding.

**(a)** In a planar embedding of a graph, each edge occurs exactly twice in the faces of the embedding.

**Solution.** *Proof.* The induction hypothesis is that if $\mathcal{E}$ is a planar embedding of a graph, then each edge is occurs exactly twice in the faces of $\mathcal{E}$.

**Base case:** There is one vertex and no edges, so this case holds vacuously.

**Constructor case** (face-splitting): The only change is that one face of $\mathcal{E}$ splits into two new faces, each including the new edge once.

**Constructor case** (bridge between two connected graphs): The only change is that two faces merge into one face that has two occurrences of the new bridging edge. So the occurrences of other edges are unchanged, and the new edge occurs twice in the new face.

So in any case, all edges of $\mathcal{E}$ are occur exactly twice. This completes the proof of the Constructor case. We conclude by structural induction that for all planar embeddings, $\mathcal{E}$, then each edge occurs exactly twice in the faces of $\mathcal{E}$.

■

**(b)** In a planar embedding of a connected graph with at least three vertices, each face is of length at least three.

**Solution.** *Proof.* The induction hypothesis is that if $\mathcal{E}$ is a planar embedding of a graph with at least three vertices, then all faces in $\mathcal{E}$ are of length at least three.

**Base case:** There is one vertex, so this case holds vacuously.

**Constructor case:** (face-splitting) An edge $\langle a—b\rangle$ is added between nonadjacent vertices $a, b$ on the same face. This face is replaced by two new faces of the form $abc\ldots a$ and $abd\ldots a$ where $c \neq d$ are vertices different from $a$ and $b$. So both new faces are of length at least 3; no other faces change.

**Constructor case:** (bridge between two connected graphs)

**case 1:** (both graphs have one vertex). Connecting these graphs with a bridge gives a graph with fewer than three vertices, so this case holds vacuously.

**case 2:** (one graph has exactly two vertices and the other has at most two vertices). Connecting these graphs with a bridge yields a line graph of length two or three whose unique embedding is a cycle of length four or six going from one end of the graph to the other and back. In any case, the one face has length more than three.

**case 3:** (one graph has at most two vertices and the other has at least three vertices). Connecting replaces the face of the vertex graph with at most two vertices and a face of the other graph with a face of length at least $2 + 3 = 5$, and leaves all other faces unchanged. So all faces are indeed of length at least three.

**case 4:** (both graphs have at least three vertices). Connecting replaces two faces of length at least three by a single face of length at least $2 + 3 + 3 = 8$, and leaves all other faces unchanged. So all faces are indeed of length at least three.

So in any case, all faces of connected planar embedding of graphs with at least three vertices are indeed of length at least three. This completes the proof of the Constructor case and the structural induction.

■

**Problem 3.**
A simple graph is *triangle-free* when it has no cycle of length three.

**(a)** Prove for any connected triangle-free planar graph with $v > 2$ vertices and $e$ edges,

$$e \leq 2v - 4. \tag{1}$$

*Hint:* Similar to the proof that $e \leq 3v - 6$. Use Problem 2.

**Solution.** The proof that $e \leq 2v - 4$ for any connected triangle-free planar graph $G$ with more than two vertices is identical to the proof of the same inequality for bipartite graph planar graphs:

*Proof.* By Problem 2.b, every face is of length at least 3. But in a triangle-free graph there are no faces of size 3, so all must be of length at least 4.

Each edge is occurs exactly twice in the faces, so

$$2e = \sum_{f \in \text{faces}} \text{length}(f) \geq \sum_{f \in \text{faces}} 4 = 4f. \tag{2}$$

By Euler's formula, $f = e - v + 2$, so substituting for $f$ in (2), yields

$$2e \geq 4(e - v + 2),$$

which simplifies to (1).

■

**(b)** Show that any connected triangle-free planar graph has at least one vertex of degree three or less.

**Solution.** If $v \leq 4$, *all* vertices have degree at most three, so the claim is immediate for $v \leq 4$.

Also, by the Handshaking Lemma, the sum of degrees is $2e$ so the average degree is $2e/v$. By part (a), $2e/v \leq (4v - 8)/v < 4$ for $v > 2$. But the average degree can be less than 4 only if at least one vertex has degree less than 4.

It follows that for all $v > 0$, there is a vertex of degree three or less. ■

**(c)** Prove by induction on the number of vertices that any connected triangle-free planar graph is 4-colorable.

*Hint:* use part (b).

**Solution.**

*Proof.* By strong induction on the number of vertices with the induction hypothesis that if a graph is connected, planar and triangle-free then it is 4-colorable.

**base case:** A planar graph with a single vertex is trivially connected, triangle-free and 1-colorable.

**inductive step:** Any connected triangle-free planar graph $G$ with 2 or more vertices has a vertex of degree 3 or less. Removing this vertex and any incident edges results in a graph $H$ whose connected components are subgraphs of a planar graph and therefore planar. They are also triangle-free since removing vertices/edges from a graph with no triangles cannot create triangles. Since the components have strictly fewer vertices than $G$, the induction hypothesis implies each connected component is 4-colorable and thus $H$ is 4-colorable.

A 4-coloring of $G$ is then given by a 4-coloring of $H$ where the removed vertex is colored with a color not used for the (at most 3) adjacent vertices. ■

## Appendix

**Definition.** A *planar embedding* of a *connected* graph consists of a nonempty set of closed walks of the graph called the *discrete faces* of the embedding. Planar embeddings are defined recursively as follows:

**Base case:** If $G$ is a graph consisting of a single vertex, $v$, then a planar embedding of $G$ has one discrete face, namely, the length zero closed walk, $v$.

**Constructor case** (split a face): Suppose $G$ is a connected graph with a planar embedding, and suppose $a$ and $b$ are distinct, nonadjacent vertices of $G$ that appear on some discrete face, $\gamma$, of the planar embedding. That is, $\gamma$ is a closed walk of the form

$$\alpha \frown \beta$$

where $\alpha$ is a walk from $a$ to $b$ and $\beta$ is a walk from $b$ to $a$.[1] Then the graph obtained by adding the edge $\langle a\text{—}b \rangle$ to the edges of $G$ has a planar embedding with the same discrete faces as $G$, except that face $\gamma$ is replaced by the two discrete faces[2]

$$\alpha \frown (b\ \langle b\text{—}a \rangle\ a) \quad \text{and} \quad (a\ \langle a\text{—}b \rangle\ b) \frown \beta$$

as illustrated in Figure 1.

**Constructor case** (add a bridge): Suppose $G$ and $H$ are connected graphs with planar embeddings and disjoint sets of vertices. Let $\gamma$ be a discrete face of the embedding of $G$ and suppose that $\gamma$ begins and ends at vertex $a$.

Similarly, let $\delta$ be a discrete face of the embedding of $H$ that begins and ends at vertex $b$.

[1] If a walk $\mathbf{f}$ ends with a vertex, $v$, and a walk $\mathbf{r}$ starts with the same vertex, $v$, their merge, $\mathbf{f} \frown \mathbf{r}$, is the walk that starts with $\mathbf{f}$ and continues with $\mathbf{r}$. Two walks can only be merged if the first ends with the same vertex, $v$, that the second one starts with.

[2] There is a minor exception to this definition of embedding in the special case when $G$ is a line graph beginning with $a$ and ending with $b$. In this case the cycles into which $\gamma$ splits are actually the same. That's because adding edge $\langle a\text{—}b \rangle$ creates a cycle that divides the plane into "inner" and "outer" continuous faces that are both bordered by this cycle. In order to maintain the correspondence between continuous faces and discrete faces in this case, we define the two discrete faces of the embedding to be two "copies" of this same cycle.

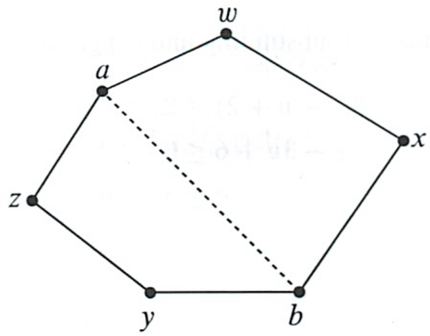



**Figure 1** The "split a face" case: $awxbyza$ splits into $awxyba$ and $abyza$.

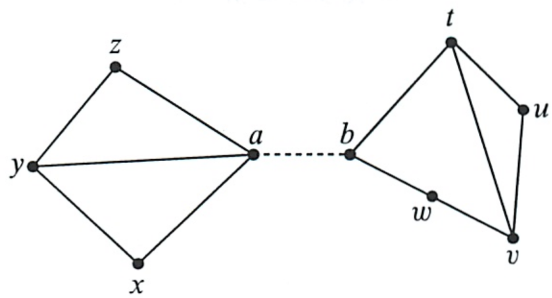



**Figure 2** The "add a bridge" case.

Then the graph obtained by connecting $G$ and $H$ with a new edge, $\langle a—b \rangle$, has a planar embedding whose discrete faces are the union of the discrete faces of $G$ and $H$, except that faces $\gamma$ and $\delta$ are replaced by one new face

$$\gamma\widehat{\ }(a\ \langle a—b \rangle\ b)\widehat{\ }\delta\widehat{\ }(b\ \langle b—a \rangle\ a).$$

This is illustrated in Figure 2, where the vertex sequences of the faces of $G$ and $H$ are:

$$G : \{axyza,\ axya,\ ayza\} \qquad H : \{btuvwb,\ btvwb,\ tuvt\},$$

and after adding the bridge $\langle a—b \rangle$, there is a single connected graph whose faces have the vertex sequences

$$\{axyzabtuvwba,\ axya,\ ayza,\ btvwb,\ tuvt\}.$$

**Theorem 3.1** (Euler's Formula). *If a connected graph has a planar embedding, then*

$$v - e + f = 2$$

*where $v$ is the number of vertices, $e$ is the number of edges, and $f$ is the number of faces.*

**Corollary 3.2.** *Suppose a connected planar graph has $v \geq 3$ vertices and $e$ edges. Then*

$$e \leq 3v - 6.$$

*Proof.* By definition, a connected graph is planar iff it has a planar embedding. So suppose a connected graph with $v$ vertices and $e$ edges has a planar embedding with $f$ faces. By Problem 2.a, every edge is traversed exactly twice by the face boundaries. So the sum of the lengths of the face boundaries is exactly $2e$. Also by Problem 2.b, when $v \geq 3$, each face boundary is of length at least three, so this sum is at least $3f$. This implies that

$$3f \leq 2e. \tag{3}$$

But $f = e - v + 2$ by Euler's formula, and substituting into (3) gives

$$\begin{aligned} 3(e - v + 2) &\leq 2e \\ e - 3v + 6 &\leq 0 \\ e &\leq 3v - 6 \end{aligned}$$

■

**Corollary 3.3.** *$K_5$ is not planar.*

*Proof.*

$$e = 10 > 9 = 3v - 6.$$

■

# TP 8



## TP.8.1 Faces of Planar embedding

What are the faces here?

1. abcda
   efge
   abcefgecda ✓

2. rstur
   rstvxyxvwvtur ✓

## TP 8.2 Planar Graphs

A planar graph has 7 more edges than vertices. How many faces does it have?

$v - e + f = 2$

$1 - 8 + f = 2$

$-7 + f = 2$

$f = 9$ ✓

②

## TP 8.3 Annuities

r = 4%

10,000

$$\sum_{T=1}^{\infty} \frac{10,000}{(1+.04)^T}$$ perpituity

I know its $\frac{10,000}{.04}$ from 16.40(

250,000 ✓

---

## TP 8.4 Summation

$$\sum_{i=1}^{\infty} i^p$$ Converges to finite value iff $p < a$

Part a) value of a

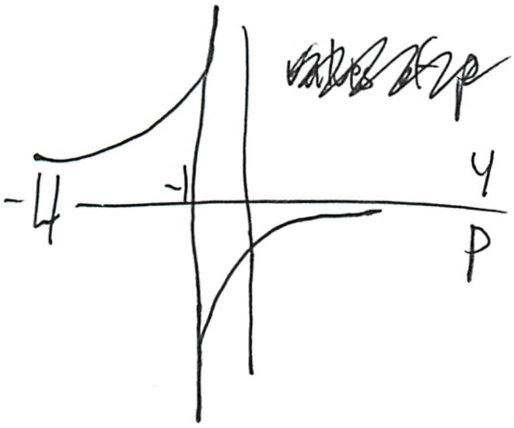
- is this top p 385?

a = 1 ⊗

But what can it be?

(3)

Did in Wolfram alpha




-1 ✓

↑ but how do you know

its been a long time since I did this

Part 2 proof

Which would be good proof for a?

1. Find a closed form for $\int_1^\infty x^p \, dx$

↑ what does this mean again

wp can be answered
- like a #

I have no clue what is best

2. Closed form $\int_1^\infty i^x \, dx$

3. Induction on $p$

4. induction on $n$ $\sum_{i=1}^{n} i^p$

~~8~~ (9)

5. Compare series term by term w/ harmonic seq

Is multiple ans

#1 is $-\frac{1}{p+1}$ if $Re(p) \leq -1$
real

will condition tells us

#2 does not converge

Answer 1, 5

#1 Sum is $\oplus\left(\int_1^\infty x^p \, dx\right)$

for $p \neq -1$ indefinite integral is $\frac{x^{p+1}}{p+1}$

another familiar word I forget
nothing on top or bottom
1st theorem of Calculus uses $\int$

If $p < -1$ then $p+1 < 0$ so $\lim_{x \to \infty} x^{p+1} = 0$ ~~$\frac{x^{p+1}}{p+1}$~~

~~d~~ definite integral from $1 \to \infty$ evaluates to $-\frac{1}{(p+1)}$ Here sum bounded from above - since increasing, finite limit so converges

(5)

if $p > -1$ then $p+1 > 0$ so $\lim_{x \to \infty} x^{p+1} = \infty$

diverges

$p = -1$ indefinite integral is $\log x$ which also approaches $\infty$ as $x \to \infty$ so diverges

#4 incorrect - needs ideas from inductive step
- so induction is moot

#5 correct

For $p = -1$ the sum is the harmonic series which we know does not converge. Since term $i^p$ is increasing in $p$ for $i > 1$, sum will be larger and also diverge for $p > -1$

---

TP 8.5 Stirling's Formula

$$\frac{(2n)!}{2^{2n}(n!)^2}$$ will come up later in class

What is asy $\simeq$ to?

(6)

So Stirling formula

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\epsilon(n)}$$

$$\sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

So

$$\frac{\sqrt{2\pi \cdot 2n}\left(\frac{2n}{e}\right)^{2n}}{2^{2n}\left(\sqrt{2\pi n}\left(\frac{n}{e}\right)^n\right)^2}$$

$$\frac{2\sqrt{\pi n}\left(\frac{2n}{e}\right)^{2n}}{2^{2n}\left(\sqrt{2\pi n}\right)\left(\frac{n}{e}\right)^{2n}}$$

I can't do that

- This is a ton of algebra I don't feel like doing

Wolfram alpha $\frac{2^{-2n}(2n)!}{(n!)^2}$

(7)

$$\frac{4^{-n}(2n)!}{(n!)^2}$$

~~Multiple ans?~~

Goes to 0 as $n \to \infty$

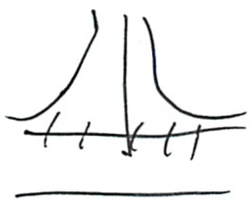
$\frac{1}{\sqrt{\pi n}}$ - no does not go to 0

← this is it (a)

So perhaps?

$\frac{1}{\sqrt{2\pi n}}$ - no same

$\sqrt{\frac{2}{\pi n}}$

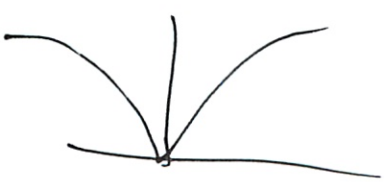


← not to 0

$2^n\sqrt{2\pi n}$

No!

$\sqrt{2\pi n}$



Worse

(8)

$$\frac{(2n)!}{2^{2n}(n!)} \sim \frac{(2n/e)^{2n}\sqrt{2\pi 2n}}{2^{2n}\left[(n/e)^{n}\sqrt{2\pi n}\right.}$$

$$= \frac{2^{2n}(n/e)^{2n}\sqrt{2\pi 2n}}{2^{2n}(n/e)^{2n}\left[\sqrt{2\pi n}\right]^{2}}$$

$$= \frac{\sqrt{2\pi 2n}}{\left[\sqrt{2\pi n}\right]^{2}}$$

$$= \frac{\sqrt{2}\,\sqrt{2\pi n}}{\left[\sqrt{2\pi n}\right]^{2}}$$

$$= \frac{\sqrt{2}}{\sqrt{2\pi n}}$$

$$= \frac{1}{\sqrt{\pi n}}$$

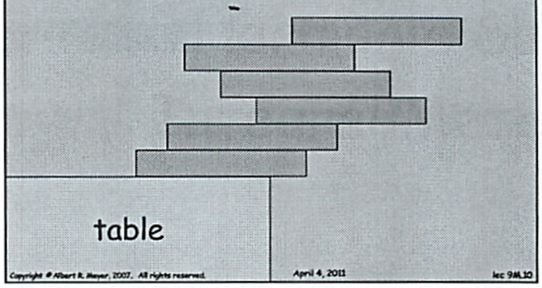
4/4

## Mathematics for Computer Science
MIT 6.042J/18.062J

# Harmonic Sum Integral Method

Albert R Meyer, April 4, 2011 lec 9M.1

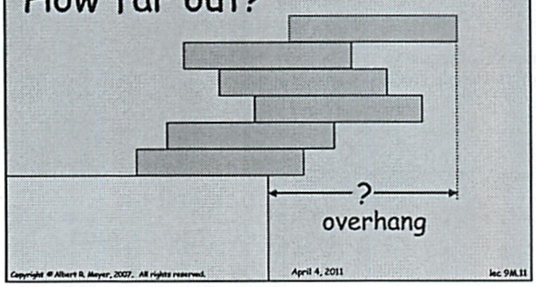
## Book Stacking




 April 4, 2011 lec 9M.10

## Book Stacking

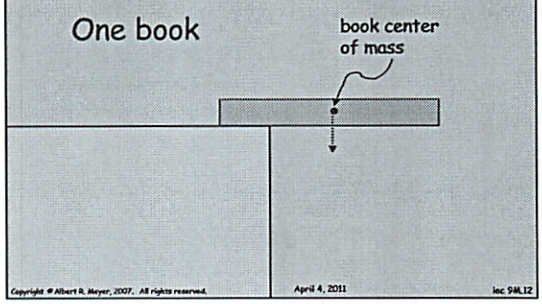
How far out?




 April 4, 2011 lec 9M.11

## Book Stacking

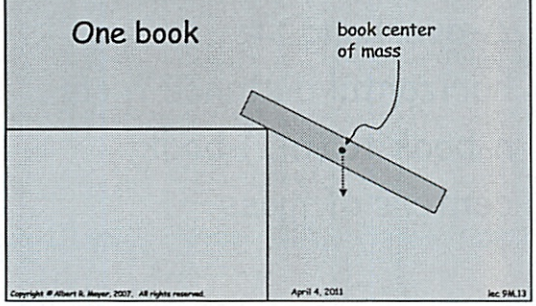
One book




 April 4, 2011 lec 9M.12

## Book Stacking

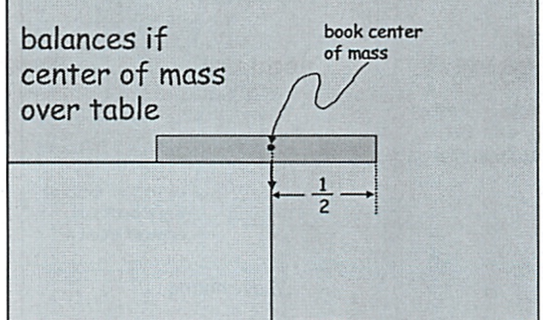
One book



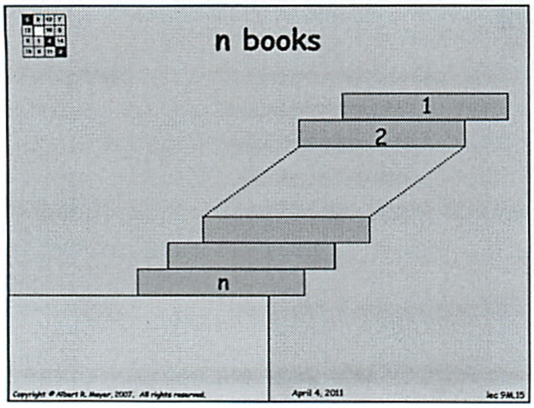

 April 4, 2011 lec 9M.13

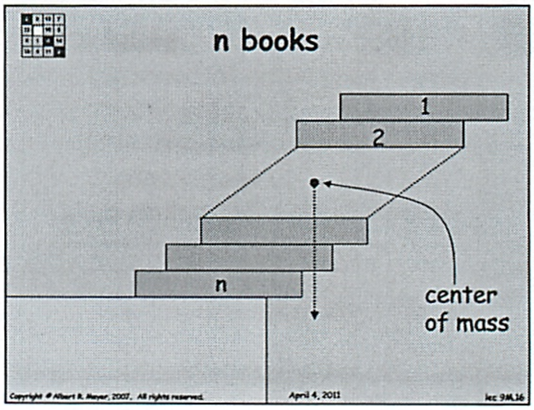
## Book Stacking

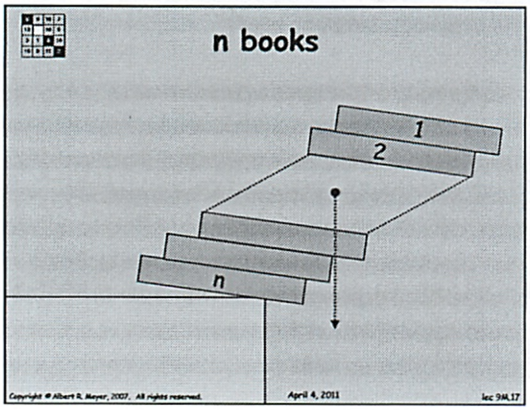
balances if center of mass over table

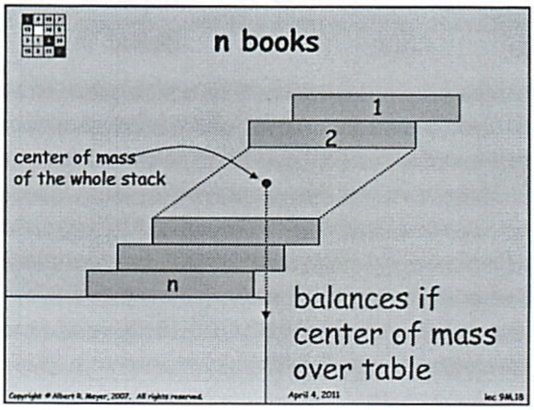


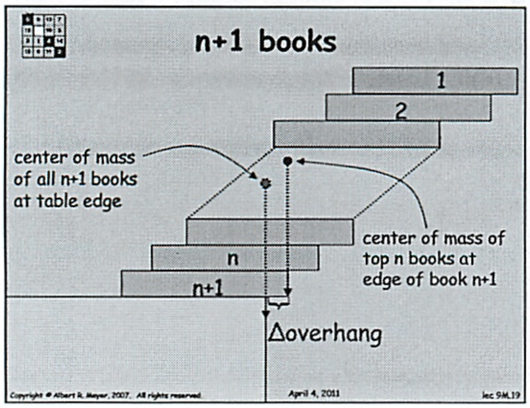

 April 4, 2011 lec 9M.14



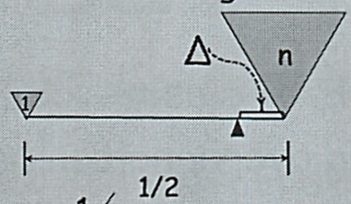




$\Delta$-overhang ::=
horizontal distance from
n-book to (n+1)-book
centers of mass

 lec 9M.20

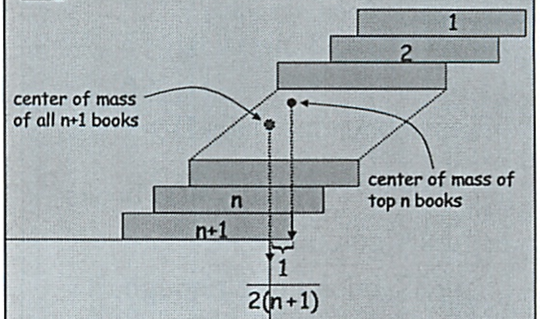
## Δ-overhang




$$\Delta = \frac{\frac{1}{2}}{n+1} = \frac{1}{2(n+1)}$$



## n+1 books






## Book stacking summary

$B_n$ ::= overhang of n books

$B_1 = 1/2$

$$B_{n+1} = B_n + \frac{1}{2(n+1)}$$

$$B_n = \frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right)$$



## Harmonic Sums

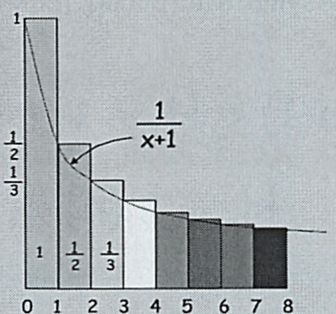
$$H_n ::= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

$n^{th}$ Harmonic number

$$B_n = H_n/2$$



## Integral estimate for $H_n$






## Integral estimate for $H_n$

$H_n$ = area of rectangles

> area under $1/(x+1)$ =

$$\int_0^n \frac{1}{x+1}\,dx = \int_1^{n+1} \frac{1}{x}\,dx = \ln(n+1)$$

## Book stacking

for overhang 3, need $B_n \geq 3$

$H_n \geq 6$

integral bound: $\ln(n+1) \geq 6$

so ok with $n \geq \lceil e^6 - 1 \rceil = 403$ books

actually calculate $H_n$:

227 books are enough.



## Book stacking

$H_n \to \infty$ as $n \to \infty$,

so overhang can be as big as desired!

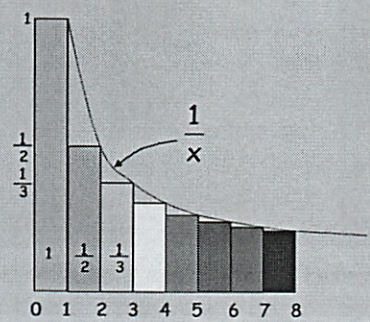


## CD cases over the edge

43 cases high --top 4 cases completely off the table --1.8 or 1.9 case-lengths



## Upper bound for $H_n$






## Upper bound for $H_n$

$$H_n < 1 + \int_1^n \frac{1}{x}\,dx$$

$$= 1 + \ln(n)$$



## Asymptotic bound for $H_n$

$$\ln(n+1) < H_n < 1 + \ln(n)$$

$$H_n \sim \ln(n)$$

## Asymptotic Equivalence

*Def:* $f(n) \sim g(n)$

$$\lim_{n \to \infty} \left( \frac{f(n)}{g(n)} \right) = 1$$



## Asymptotic Equivalence ~

*Example:* $(n^2 + n) \sim n^2$

*pf:*

$$\lim_{n \to \infty} \frac{n^2 + n}{n^2} = \lim_{n \to \infty} \left( 1 + \frac{1}{n} \right) = 1$$



## Team Problems

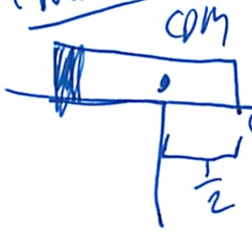
# Problems

# 1—4

(5 min late)

Book stacking
- so does not fall off table

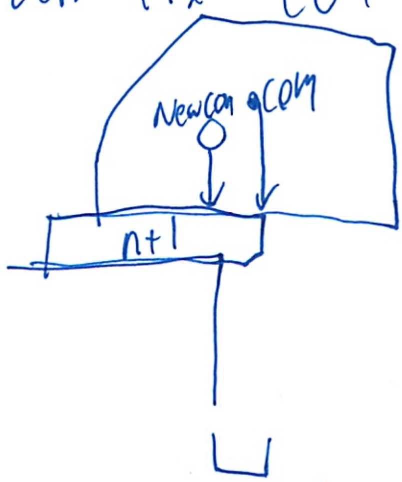
1 book




Max overhang = $\frac{1}{2}$

n books
take avg of all books COM
- avg COM must be on table to balance

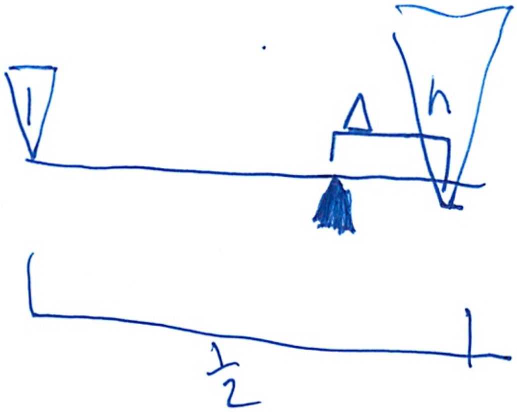
n+1 books
Lets fix COM of top n books at edge of book n+1




So New COM is still stable

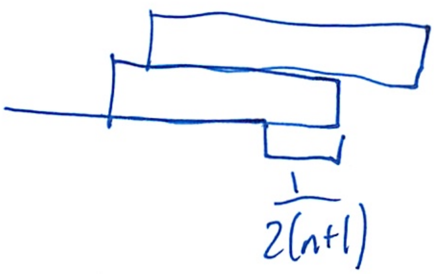
this is the new overhang $\Delta$ overhang

2) Want it to balance




$$\Delta = \frac{\frac{1}{2}}{n+1} = \frac{1}{2(n+1)}$$

That is distance




Recursive construction

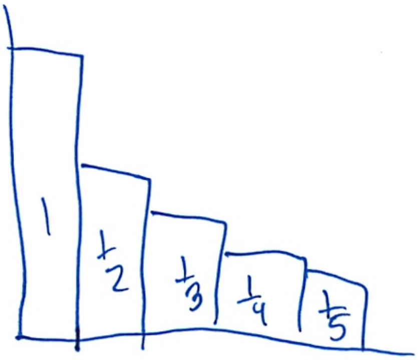
(did not copy)

③

## Harmonic sum

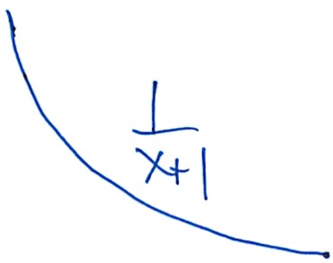
$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$

- $\frac{1}{2}$ of the previous




No nice closed form
Need to estimate
- use sum of size of rectangles

By turning sum into integration
(proof by picture)



$\frac{1}{x+1}$ is lower-bound on our area

$H_n$ = area rectangles > $\frac{1}{x+1}$ area rectangles

$\int_0^n \frac{1}{x+1}\,dn$ ... did not see

(4)

log grows $\infty$ - so can always put more books at

for overhang 3 - need estimate 403 books

actual is 227 -
- calculate w/ sum

hard to actually do w/ books - compress
do w/ CD cases

estimater is upper bound

$\frac{1}{x}$ + area first rectangle (1)

$H_n < 1 + \frac{1}{x}$

So

$$\ln(n+1) < H_n < 1 + \ln(n)$$

estimate by integration

$H_n \sim \ln(n)$

"asymptotic to" - means ratio goes to 1 in limit

(5)

<u>Def</u> ... missed ...

Used to see which parts are dominating the growth

# In-Class Problems Week 9, Mon.

**Problem 1.**
You've seen this neat trick for evaluating a geometric sum:

$$\begin{aligned} S &= 1 + z + z^2 + \ldots + z^n \\ zS &= z + z^2 + \ldots + z^n + z^{n+1} \\ S - zS &= 1 - z^{n+1} \\ S &= \frac{1 - z^{n+1}}{1 - z} \end{aligned}$$

Use the same approach to find a closed-form expression for this sum:

$$T = 1z + 2z^2 + 3z^3 + \ldots + nz^n$$

**Problem 2.**
An explorer is trying to reach the Holy Grail, which she believes is located in a desert shrine $d$ days walk from the nearest oasis. In the desert heat, the explorer must drink continuously. She can carry at most 1 gallon of water, which is enough for 1 day. However, she is free to make multiple trips carrying up to a gallon each time to create water caches out in the desert.

For example, if the shrine were $2/3$ of a day's walk into the desert, then she could recover the Holy Grail after two days using the following strategy. She leaves the oasis with 1 gallon of water, travels $1/3$ day into the desert, caches $1/3$ gallon, and then walks back to the oasis—arriving just as her water supply runs out. Then she picks up another gallon of water at the oasis, walks $1/3$ day into the desert, tops off her water supply by taking the $1/3$ gallon in her cache, walks the remaining $1/3$ day to the shrine, grabs the Holy Grail, and then walks for $2/3$ of a day back to the oasis—again arriving with no water to spare.

But what if the shrine were located farther away?

**(a)** What is the most distant point that the explorer can reach and then return to the oasis if she takes a total of only 1 gallon from the oasis?

**(b)** What is the most distant point the explorer can reach and still return to the oasis if she takes a total of only 2 gallons from the oasis? No proof is required; just do the best you can.

**(c)** The explorer will travel using a recursive strategy to go far into the desert and back drawing a total of $n$ gallons of water from the oasis. Her strategy is to build up a cache of $n-1$ gallons, plus enough to get home, a certain fraction of a day's distance into the desert. On the last delivery to the cache, instead of returning home, she proceeds recursively with her $n-1$ gallon strategy to go farther into the desert and return to the cache. At this point, the cache has just enough water left to get her home.

Prove that with $n$ gallons of water, this strategy will get her $H_n/2$ days into the desert and back, where $H_n$ is the $n$th Harmonic number:

$$H_n ::= \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

Conclude that she can reach the shrine, however far it is from the oasis.

**(d)** Suppose that the shrine is $d = 10$ days walk into the desert. Use the asymptotic approximation $H_n \sim \ln n$ to show that it will take more than a million years for the explorer to recover the Holy Grail.

**Problem 3.**
There is a number $a$ such that $\sum_{i=1}^{\infty} i^p$ converges iff $p < a$. What is the value of $a$? Prove it.

**Problem 4.**
Suppose $f, g : \mathbb{N}^+ \to \mathbb{N}^+$ and $f \sim g$.

**(a)** Prove that $2f \sim 2g$.

**(b)** Prove that $f^2 \sim g^2$.

**(c)** Give examples of $f$ and $g$ such that $2^f \not\sim 2^g$.

# In Class 9 Mon



1. We've seen trick for evaluating geo. sum
   Do it for

$$T = 1z + 2z^2 + 3z^3 + \dots + nz^n$$

Try z2

$$zT = z^2 + 2z^3 + 3z^4 + \dots + nz^{n+1}$$

~~$T - zT =$ no thats not very nice~~

$T + 1$?

Our group did

$$T - zT = 1z + 1z^2 + 1z^3 + \dots - nz^{n-1}$$

$$= \frac{z^{n+1}}{1-z}$$

$$- nz^{n+1}$$

2a. $\frac{1}{2}$ day

b $\frac{3}{4}$ day

(2)

Oh I see it can be represented w/ Geometric seq
Find the closed form to know how far she can go

$$H_n = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$$

She can get $\frac{H_n}{2}$ days into the dessert

d) Suppose shrine is $d = 10$ days into desert
Use asymptotic approx $H_n \sim \ln(n)$ to show
it will take more than a million years.

Well makes sense - b/c additions get smaller
+ smaller each day - as go further out
need more + more water

Treat cache as oasis
Build up to $n-1$ gallon

③

3. Tutor problem! Except now you actually have to prove it

$\sum_{i=1}^{\infty} i^p$ converges if $p < a$

What is a?

$-1$

just copy ans

Sum is $\Theta\left(\int_1^\infty x^p \, dx\right)$

for $p \neq -1$ ind. integral is $\frac{x^{p+1}}{p+1}$

- If $p < -1$ then $p+1 < 0$ so $\lim_{x\to\infty} x^{p+1} = 0$
  definite integral from $1 \to \infty$ evals to $-\frac{1}{(p+1)}$
  Here sum bounded from above - since increasing, finite limit, so it converges
- If $p > -1$ then $p+1 > 0$ so $\lim_{x\to\infty} x^{p+1} = \infty$
  so diverges

(4)

$p=-1$ indef. int. is $\log x$, which also approaches $\infty$ as $x \to \infty$, so diverges

4. Suppose $f, g$ $N^+ \to N^+$ and $f \sim g$

a. Prove $2f \sim 2g$

so this is ratios

$f \sim g$ means $\frac{f}{g} \approx 1$

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = 1$$

So

$$\lim_{x \to \infty} \frac{2f}{2g} = 1$$

$\frac{2}{2}$ cancles

b) $\lim \frac{f^2}{g^2}$ take $\sqrt{\ }$ of ea $\frac{f}{g} = 1$

↑ ? legal move on its own

(5)

c) Give 2 counter examples s.t.

$$2^f \not\approx 2^g$$

That is not allowed mathwise

But what could you say? for a counter-example?

---

2. Our board

a) $\frac{1}{2}$

b) $\frac{3}{4}$

c) For the explorer to deposit $n-1$ gallons at a position, using $n$ gallons, he needs to make $n$ trips drinking a total of 1 gallon ~~on each trip~~. Hence, each trip will have to be $\frac{1}{n}$ days long round trip or $\frac{1}{2n}$ days long 1-way.

Doing this recursively, the first cache always has $n-1$ at dist $\frac{1}{2n}$. 2nd has $n-2$ at $\frac{1}{2n-1}$

(c)

So total distance is

$$\sum_{i=0}^{n-1} \frac{1}{2n-i} = \frac{1}{2n} + \frac{1}{2n-2} + \frac{1}{2n-4} + \dots + \frac{1}{6} + \frac{1}{4} + \frac{1}{2}$$

$$= \frac{1}{2}\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}\right)$$

$$= \frac{1}{2} H_n$$

d) $d = \frac{1}{2} H_n$

$d = 10 \approx \frac{1}{2} \ln n$

$n = e^{20} = 4.8 E 8$ days $= 1.7 E 6$ years

= Very big!

---

3. Prove $\sum_{i=1}^{\infty} i^p$ converges if $p < -1$

<u>Case 1</u> $p > -1$

Then $\lim_{p \to \infty}$

... forget it

(7)

Case $p=-1$

$\sum_{i=1}^{\infty} i^p$ is the harmonic series

---

4. a) Like I had it

b) $\lim_{n\to\infty} \frac{f(n)^2}{g(n)^2} = \lim \frac{f(n)}{g(n)} \cdot \frac{f(n)}{g(n)}$

$f \sim g$ iff $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$

See how that was written! pay attention to detail

as long as $g(n) \neq 0$

So if $f \sim g$ $\lim_{n\to\infty} \frac{f(n)}{g(n)} \cdot \frac{f(n)}{g(n)} = 1 \cdot 1 = 1$

so $f \sim g \to f^2 \sim g^2$

c) $n \sim n+1$, but $\lim_{n\to\infty} \frac{2^{n+1}}{2^n} = 2 \cdot \frac{2^n}{2^n} = 2$

# Solutions to In-Class Problems Week 9, Mon.

**Problem 1.**
An explorer is trying to reach the Holy Grail, which she believes is located in a desert shrine $d$ days walk from the nearest oasis. In the desert heat, the explorer must drink continuously. She can carry at most 1 gallon of water, which is enough for 1 day. However, she is free to make multiple trips carrying up to a gallon each time to create water caches out in the desert.

For example, if the shrine were $2/3$ of a day's walk into the desert, then she could recover the Holy Grail after two days using the following strategy. She leaves the oasis with 1 gallon of water, travels $1/3$ day into the desert, caches $1/3$ gallon, and then walks back to the oasis—arriving just as her water supply runs out. Then she picks up another gallon of water at the oasis, walks $1/3$ day into the desert, tops off her water supply by taking the $1/3$ gallon in her cache, walks the remaining $1/3$ day to the shrine, grabs the Holy Grail, and then walks for $2/3$ of a day back to the oasis—again arriving with no water to spare.

But what if the shrine were located farther away?

**(a)** What is the most distant point that the explorer can reach and then return to the oasis if she takes a total of only 1 gallon from the oasis?

**Solution.** At best she can walk $1/2$ day into the desert and then walk back. ■

**(b)** What is the most distant point the explorer can reach and still return to the oasis if she takes a total of only 2 gallons from the oasis? No proof is required; just do the best you can.

**Solution.** The explorer walks $1/4$ day into the desert, drops $1/2$ gallon, then walks home. Next, she walks $1/4$ day into the desert, picks up $1/4$ gallon from her cache, walks an additional $1/2$ day out and back, then picks up another $1/4$ gallon from her cache and walks home. Thus, her maximum distance from the oasis is $3/4$ of a day's walk. ■

**(c)** The explorer will travel using a recursive strategy to go far into the desert and back drawing a total of $n$ gallons of water from the oasis. Her strategy is to build up a cache of $n-1$ gallons, plus enough to get home, a certain fraction of a day's distance into the desert. On the last delivery to the cache, instead of returning home, she proceeds recursively with her $n-1$ gallon strategy to go farther into the desert and return to the cache. At this point, the cache has just enough water left to get her home.

Prove that with $n$ gallons of water, this strategy will get her $H_n/2$ days into the desert and back, where $H_n$ is the $n$th Harmonic number:

$$H_n ::= \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

Conclude that she can reach the shrine, however far it is from the oasis.

**Solution.** To build up the first cache of $n-1$ gallons, she should make $n$ trips $1/(2n)$ days into the desert, dropping off $(n-1)/n$ gallons each time. Before she leaves the cache for the last time, she has $n-1$ gallons plus enough for the walk home. Then she applies her $(n-1)$-day strategy. So letting $D_n$ be her maximum distance into the desert and back, we have

$$D_n = \frac{1}{2n} + D_{n-1}.$$

So

$$
\begin{aligned}
D_n &= \frac{1}{2n} + \frac{1}{2(n-1)} + \frac{1}{2(n-2)} + \cdots + \frac{1}{2 \cdot 2} + \frac{1}{2 \cdot 1} \\
&= \frac{1}{2}\left(\frac{1}{n} + \frac{1}{(n-1)} + \frac{1}{(n-2)} + \cdots + \frac{1}{\cdot 2} + \frac{1}{\cdot 1}\right) \\
&= \frac{H_n}{2}.
\end{aligned}
$$

■

**(d)** Suppose that the shrine is $d = 10$ days walk into the desert. Use the asymptotic approximation $H_n \sim \ln n$ to show that it will take more than a million years for the explorer to recover the Holy Grail.

**Solution.** She obtains the Grail when:

$$\frac{H_n}{2} \approx \frac{\ln n}{2} \geq 10.$$

This requires $n \geq e^{20} = 4.8 \cdot 10^8$ days $> 1.329M$ years.

■

**Problem 2.**
There is a number $a$ such that $\sum_{i=1}^{\infty} i^p$ converges iff $p < a$. What is the value of $a$? Prove it.

**Solution.** $a = -1$.

For $p = -1$, the sum is the harmonic series which we know does not converge. Since the term $i^p$ is increasing in $p$ for $i > 1$, the sum will be larger, and hence also diverge for $p > -1$.

For $p < -1$ there exists an $\epsilon > 0$ such that $p = -(1 + \epsilon)$. By the integral method,

$$
\begin{aligned}
\sum_{i=1}^{\infty} i^{-(1+\epsilon)} &\leq 1 + \int_1^{\infty} x^{-(1+\epsilon)}\, dx \\
&= 1 + \epsilon^{-1} - \epsilon^{-1} \lim_{\alpha \to \infty} \alpha^{-\epsilon} \\
&= 1 + \epsilon^{-1} \\
&< \infty
\end{aligned}
$$

Hence the sum is bounded above, and since it is increasing, it has a finite limit, that is, it converges. ■

**Problem 3.**
Suppose $f, g : \mathbb{N}^+ \to \mathbb{N}^+$ and $f \sim g$.

**(a)** Prove that $2f \sim 2g$.

**Solution.**

$$\frac{2f}{2g} = \frac{f}{g},$$

so they have the same limit as $n \to \infty$.

■

**(b)** Prove that $f^2 \sim g^2$.

**Solution.**

$$\lim_{n\to\infty} \frac{f(n)^2}{g(n)^2} = \lim_{n\to\infty} \frac{f(n)}{g(n)} \cdot \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{f(n)}{g(n)} \cdot \lim_{n\to\infty} \frac{f(n)}{g(n)} = 1 \cdot 1 = 1.$$

■

**(c)** Give examples of $f$ and $g$ such that $2^f \not\sim 2^g$.

**Solution.**

$$f(n) ::= n + 1$$
$$g(n) ::= n.$$

Then $f \sim g$ since $\lim(n+1)/n = 1$, but $2^f = 2^{n+1} = 2 \cdot 2^n = 2 \cdot 2^g$ so

$$\lim \frac{2^f}{2^g} = 2 \neq 1.$$

■

**Problem 4.**
You've seen this neat trick for evaluating a geometric sum:

$$\begin{aligned} S &= 1 + z + z^2 + \ldots + z^n \\ zS &= z + z^2 + \ldots + z^n + z^{n+1} \\ S - zS &= 1 - z^{n+1} \\ S &= \frac{1 - z^{n+1}}{1 - z} \end{aligned}$$

Use the same approach to find a closed-form expression for this sum:

$$T = 1z + 2z^2 + 3z^3 + \ldots + nz^n$$

**Solution.**

$$\begin{aligned} zT &= 1z^2 + 2z^3 + 3z^4 + \ldots + nz^{n+1} \\ T - zT &= z + z^2 + z^3 + \ldots + z^n - nz^{n+1} \\ &= \frac{1 - z^{n+1}}{1 - z} - 1 - nz^{n+1} \\ T &= \frac{1 - z^{n+1}}{(1 - z)^2} - \frac{1 + nz^{n+1}}{1 - z} \end{aligned}$$

■

# 6.042 Miniquiz 4



graph = network

directed = di = 1 way = arrows

DAG = directed, acyclic [no cycles]

dots = nodes = verticies

$e = \langle u \to v \rangle$

$\text{in deg}(v) ::= |\{e \in E(G) \mid \text{head}(e) = v\}|$

$\text{out deg}(v) ::= |\{e \in E(G) \mid \text{tail}(e) = v\}|$

$\sum_{v \in V(G)} \text{indeg}(v) = \sum_{v \in V(G)} \text{outdeg}(v)$

V(G) = verticies  E(G) = edges

walk - can repeat points

path - all pts must be unique

merge = $f \hat{v} r$ combine 2 walks

distance = lenght of shortest path

Adj Matrix $(A_G)_{ij}$ if $\langle v_i \to v_j \rangle \in V(G)$

$(A_G)^k$ count of lenght k walks b/w = $C_{uv}$ for a certain point

$u G^* v$ is a path  $G^+$ = pos lenght $G_n$ path

transitive $(aRb \text{ AND } bRc) \to (aRc)$ for every $a,b,c \in A$

reflexive iff aRa for all $a \in A$
- true always b/c 0 lenght path

Can compose relations
$a(R \circ S)c ::= \exists b \in B (aSb) \text{ AND } (bRc)$

Closed walk - starts + ends same vertex

Cycle = closed walk w/ distinctive verticies lenght > 2

asymmetric iff $aRb \to \text{NOT}(bRa)$ for all $a,b \in A$
no self-loops

irreflexive no + lenght path from any vertex to itself
NOT (aRa) for all $a \in A$

Strict partial order = trans + asymmetric
if pos path relation of a DAG

weak partial order - can also be =
aRb iff (aSb OR a=b)

Antisymmetric $aRb \to \text{NOT}(bRa)$ for all $a \neq b \in A$ (self loops allowed)

WPO = transitive, reflexive, antisymetric
= $\preceq$ or $\subseteq$

$\subseteq$ = subset

isomorphic if relation preserving bij

total - always 1 arrow $\forall x \neq y \in A (xRy \text{ or } yRx)$

product order $R_1 \times R_2$
domain $(R_1 \times R_2)$ = domain $(R_1) \times$ domain $(R_2)$
codomain " = co " $\times$ co "

$(a_1, a_2)(R_1 \times R_2)(b_1, b_2)$ iff $[a_1 R_1 b_1$ and $a_2 R_2 b_2]$
- where both are true

topological sort  $a \preceq b \to a \sqsubseteq b$
partial  total

antichain - all items incomparable

equivilance = reflexive, symmetric, transitive

$\subset$ = proper subset  $A \subset B$ mens B has everything + more
- asymmetric
- transitive

SPO - transitive + asymmetric  DAG
< less then, ranked higher

WPO - same as SPO except aRa always holds
$\subseteq$ on sets  $\leq$ on $\mathbb{R}$
- reflexive  for all
- transitive
- antisymmetric

total - like a path/chain

Symmetric $\forall x,y \in A \; xRy \to yRx$
- arrow in both dirs

Simple graphs - undireted (no arrows)
V—W = undirected edge
no self loops (from u to u)
two pts adjacent if edge
edge is incident to end pts
deg(v) = # edges incident to vertice

$\sum_{x \in M} \deg(x) = \sum_{y \in F} \deg(y)$

Handshake sum of deg of vertices = 2 x # edges

$k_n$ = complete graph - every arrow
$L_n$ = line graph
- if add 1 edge would have cycle

$2|E| = \sum_{v \in V} \deg(v)$

isomorphism is a edge-preserving bij $f: V(G) \to V(H)$ s.t.
$u - v \in E(G)$ iff $f(u) - f(v) \in E(H)$ for all $u, v \in V(G)$

biparte - can split into 2 graps

Matching cond - every subset men likes at least as large as subset of men

matching - set M of edges G s.t. no vertex is incident to ≥ 1 edge in M.

Covers - if all vertices included = perfect

bottleneck $|S| > |N(S)|$
neighbors

Hall's Theorem: Matching in G (biparte) that covers L(G) iff no subset of L(G) is a bottleneck

if degree constrained - is a matching

degree constrained $\deg(l) \geq \deg(r)$ for all $l, r$

regular - each node has same degree

Every reg biparte graph has perfect matching

Stable - no rouge couples - pair that likes each other more

if w is off m's list w has suitor she prefers over m

men = optimal  termination #remaining
girls = pessimal  names strictly ↓

Coloring - adj vertices diff color

$\chi()$ = chromatic # = min # colors

$\chi(k_n) = n$  $\chi(\text{biparte}) = 2$

$\chi(C_{even}) = 2$  $\chi(C_{odd}) = 3$

$\chi(\text{Max degree } k) = k+1$

Subgraphs

Connected - every pair verticies connected
Connected components - path exists somewhere

k-edge connected = # edges can remove till splits
- called cut edge

Tree - connected acyclic graph
Connected component of trees = forest
leaf = node w/ deg(1)

1. Each connected subgraph = tree
2. Unique simple path b/w every pair of vertices
3. Adding edge b/w nonadj nodes creates a cycle
4. Removing any edge - disconnects
   └ All edges = cut edges
5. If ≥ 2 vertices ≥ 2 leaves
6. # vertices = # edges + 1

Spanning tree - min # of lines so all vertices still connected

if branches weighted → Min-weight tree (MST)

---

Planar - no lines crossing

drawing - one particular set of curves

face - continuous
- but divide up into discrete
- don't forget outside

bridge

dangle

discrete face = planar embeddings
- either split a face or add a bridge

Euler's Formula $v - e + f = 2$
- proof w/ the 2 constructors

---

$e \leq 3v - 6$ limit of planar

minor - delete vertices, edges, merge verticies

every planar graph has degree ≤ 5
- so 5-colorable

---

At most 5 regular polyhedra

Power set -- set of all subsets

So $P\{1,2,3\} = \{1\},\{2\},\{3\}$
$\{1,2\}\{2,3\}\{3,1\}$
$\{1,2,3\}$

# Miniquiz 4

4/5

Week 7 Mon - Week 9 Fri

Topics:

- Partial order
- Simple ~~deg~~ graph degrees
- isomorphism
- stable marriage
- Mating ritual
- ~~gr~~ graph connectivity
  - trees
  - coloring
  - planar graphs

---

Actually most was post - SB

Write lemmas - they seemed to be most useful

# Mini-Quiz Apr. 6

**Your name:** Michael Plasmeier

**Circle the name of your TA and write your table number**:

Ali Nick Oscar Oshani **Table number** 12

- This quiz is **closed book**. Total time is 30 minutes.
- Write your solutions in the space provided. If you need more space, write on the back of the sheet containing the problem. Please keep your entire answer to a problem on that problem's page.
- GOOD LUCK!

**DO NOT WRITE BELOW THIS LINE**

| Problem | Points | Grade | Grader |
|---|---|---|---|
| 1 | 6 | 3 | OS |
| 2 | 3 | 0 | NJ |
| 3 | 3 | 2 | OM |
| 4 | 5 | 2 | AK |
| 5 | 3 | 2 | OS |
| Total | 20 | 9 | OS |

**Problem 1 (6 points). (a)** A simple graph has 8 vertices and 24 edges. What is the average degree per vertex?

✓ Handshake: $\sum deg = 2|E|$

$8 \cdot avg = 2 \cdot 24$

$avg = \frac{48}{8} = 6$ ✓

**(b)** A connected planar simple graph has 5 more edges than it has vertices. How many faces does it have?

✓ Euler's $v - e + f = 2$

$1 - 6 + f = 2$

$-5 + f = 2$

$f = 7$ ✓

**(c)** A connected simple graph has one more vertex than it has edges. Is it necessarily planar?

~~$2 - 1 + f = 2$~~

Because of the condition that $v > 2$ this does not always hold – only when $v > 2$ proved w/ base + constructor

Base $e \leq 3v - 6$

say $v = 3$, $e = 2$

$2 \leq 3(3) - 6$

$2 \leq 12$ ✓ can work

Iterative

$v = +1$

$e = +1$

will work – right hand side ↑ at factor of 3 compared to left

?

Try $v = 2$ $e = 1$

$1 \leq 3(2) - 6$

$1 \leq 0$ ⊗ won't work.

Must $v \geq 3$

**(d)** If your answer to the previous part was *yes*, then how many faces can such a graph have? If your answer was *no*, then give an example of a nonplanar connected simple graph whose vertices outnumber its edges by one.

$v - e + f = 2$

So $f = 2 - v + e$

when $v = 3$ $e = 2$

$f = 2 - 3 + 2$

$f = 1$

$v = 4$ $e = 3$

$f = 2 - 4 + 3$

$= 1$ ✓

$v = 5$ $e = 4$

$f = 2 - 5 + 4$

$= 1$

How it holds when $v > 2$

**(e)** Consider the graph shown in Figure 1. How many distinct isomorphisms exist between this graph and itself? (Include the identity isomorphism.)

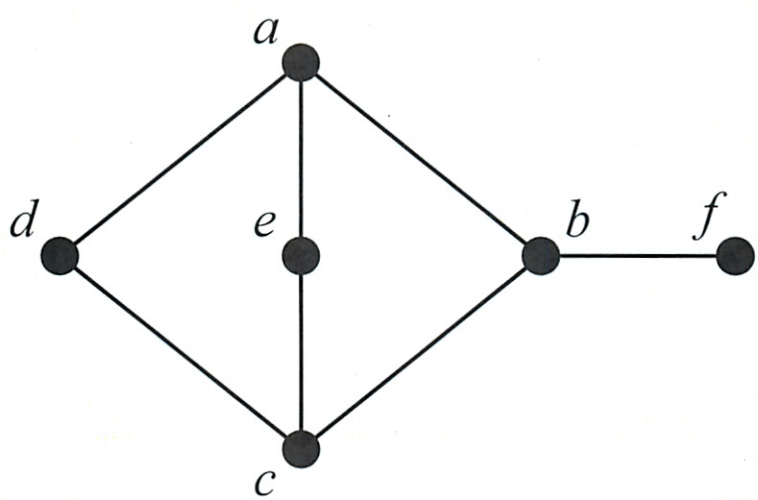



**Figure 1**

Just 1 by definition ✗

- since can't move or relable anything

a → a
b → b
c → c
d → d
e → e
f → f

**Problem 2 (3 points).**
The $n$-dimensional hypercube, $H_n$, is a simple graph whose vertices are the binary strings of length $n$. Two vertices are adjacent if and only if they differ in exactly one bit. Consider for example $H_3$, shown in Figure 2. (Here, vertices 111 and 011 are adjacent because they differ only in the first bit, while vertices 101 and 011 are not adjacent because they differ in both the first and second bits.) Hemming distance

Explain why it is impossible to find two spanning trees of $H_3$ that have no edges in common.

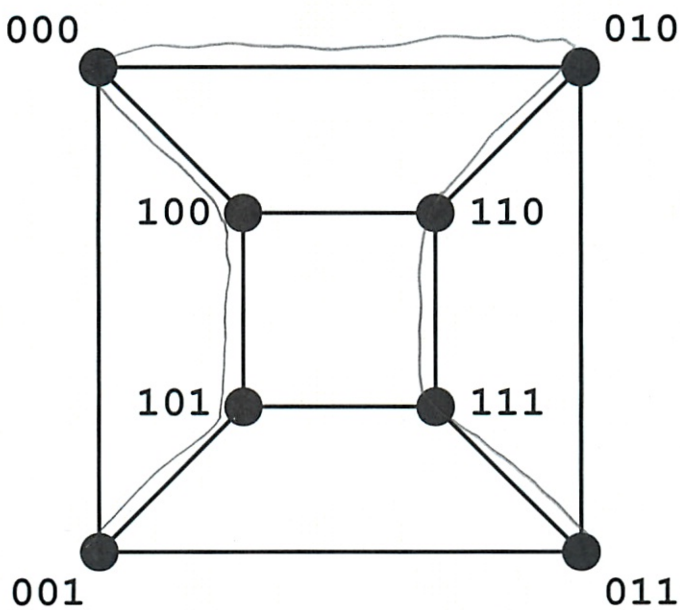
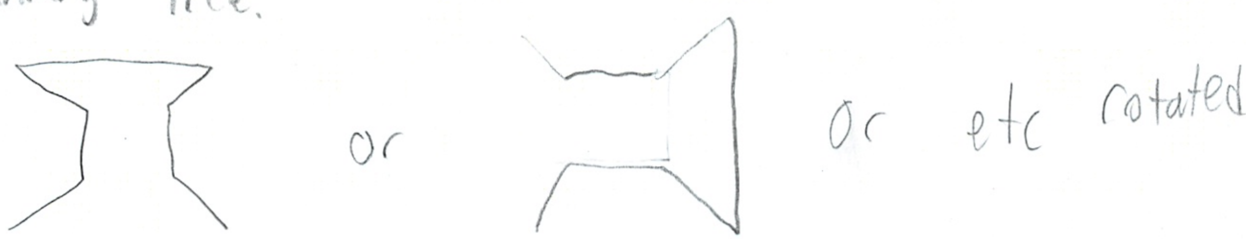



**Figure 2** $H_3$.

0/3

Once you start in a certain way, there are very limited choices as to what you can do next in the spanning tree.




Its just the same pattern rotated
Each point can is degree 3, so there are a limited # of cut edges possible to find different spanning trees. that's not a general argument

**Problem 3 (3 points).**

✓ Consider the graph shown in Figure 3. Determine a valid coloring of the graph, using as few colors as possible. (Simply write your proposed color for each vertex next to that vertex. You may use *R* for red, *G* for green, etc.)

Will use #s

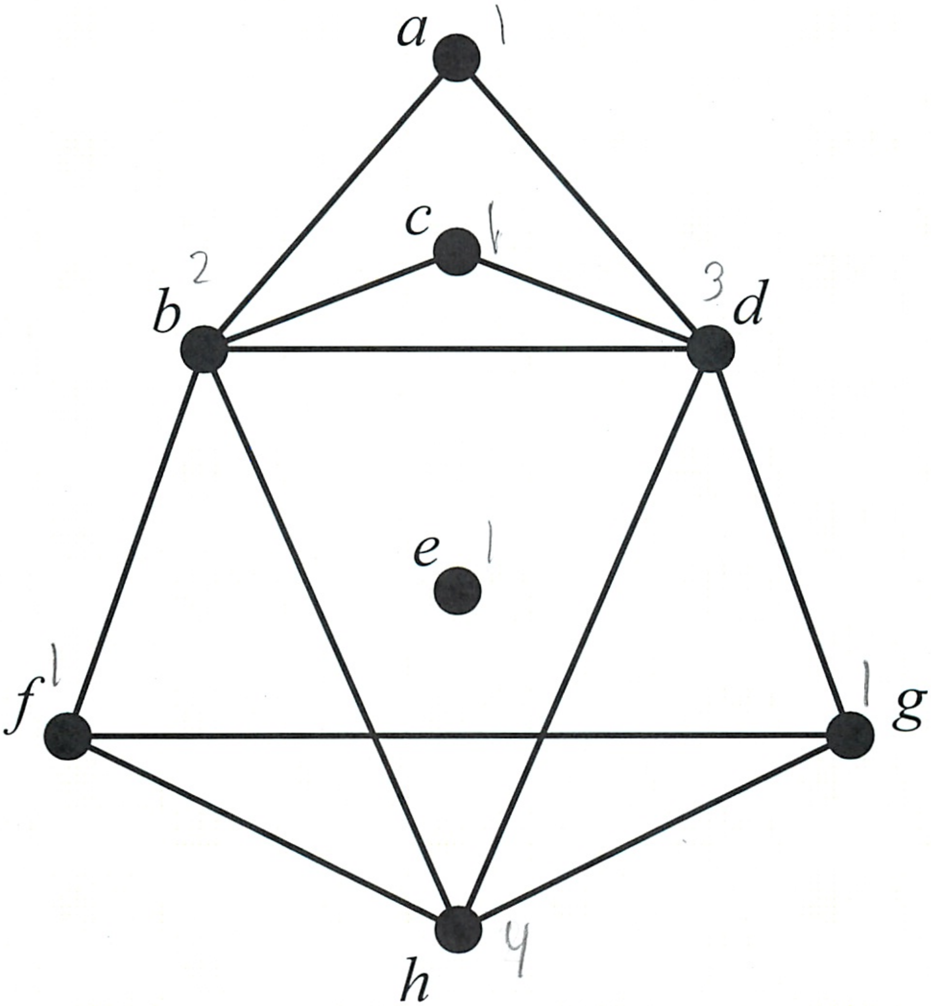



**Figure 3**

Say
1 = Red
2 = Green
3 = Yellow
4 = Orange

Since max degree = 4

−1 3 is enough

**Problem 4 (5 points). (a)** Consider the bipartite graph $G$ in Figure 4. Is it possible to find a matching that covers $L(G)$? If yes, explain what property of the graph guarantees the existence of such a matching. (Show that the graph exhibits this property and what this implies. Full credit will not be given for merely identifying a matching.) If no, identify a bottleneck that prevents a matching.

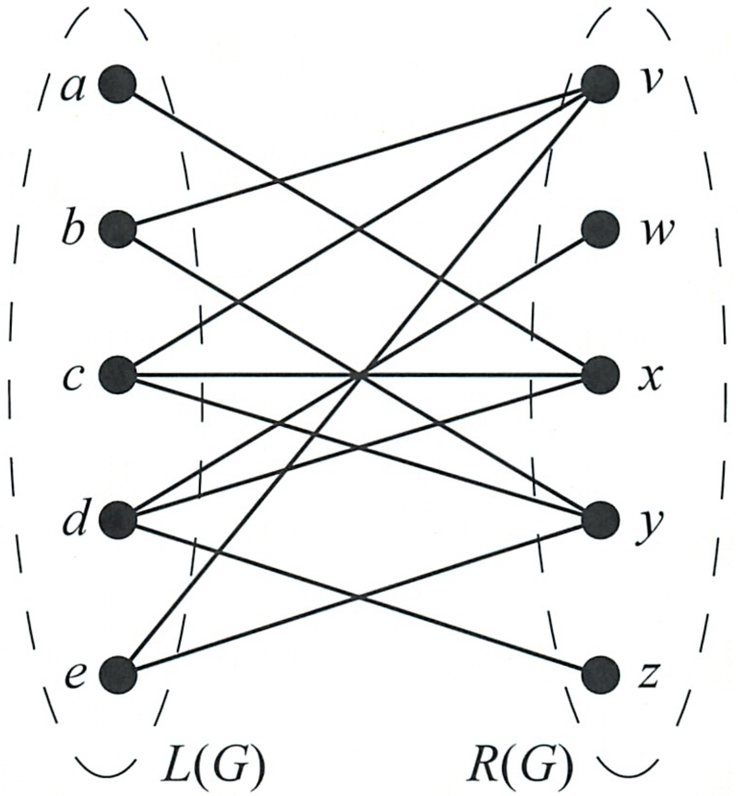



**Figure 4** $G$.

Matching - set of M edges G s.t. no vertex is incident to ≥ 1 edges in M

Matching Condition - every subset of L(G) is connected to at least as large a subset of R(G)

bottle neck $|S| > |N(S)|$ neighbors too

Covers - all vertices included (perfect)

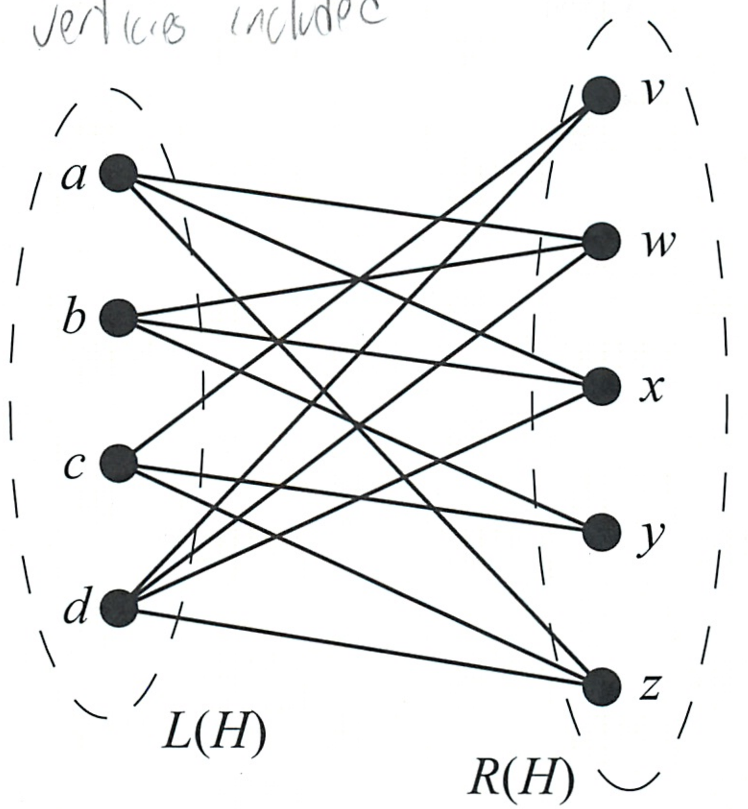
Hall's theorm - Matting in G (bipartite) that covers L(G) if no subset of L(G) is a bottleneck.

There is no bottleneck. For ~~for~~ all subsets of L(G) there exists a subset of equal or larger size in R(G)

√

**(b)** Consider the bipartite graph $H$ in Figure 5. Is it possible to find a matching that covers $L(H)$? If yes, explain what property of the graph guarantees the existence of such a matching. (Show that the graph exhibits this property and what this implies. Full credit will not be given for merely identifying a matching.) If no, identify a bottleneck that prevents a matching.

Covers - all verticies included




2

**Figure 5** $H$.

Yes. Since the deg(l) ≥ deg(r) for all l∈L and r∈R so that it is degree constrained

This means there is a matching that covers ~~the entire graph,~~ No! L(H).

See def in previous page.

**Problem 5 (3 points).**
In the Mating Ritual, suppose Tiger is one of the boys and Elin is one of the girls. Which of the following are preserved invariants **in general**?

2/3

1. Tiger is Elin's only suitor.
2. On Tiger's current list, the girl whom he prefers to all the others is his optimal wife[1].
3. Elin's name has been crossed off by Tiger and by everyone whom she prefers to him.

1. We don't know that w/ info we have been given.

2. Yes, true. Of the names remaining on the list (the current names) the name at the top would be the girl he perfers to all others. This is defined as his optimal wife.
He stays with this girl till they get married or she kicks him out and then he is off the list (no longer on current list)

3. False. Everyone who Elin perfers to Tiger has no relation to who Tiger crosses off his list. Elin's name is crossed off by Tiger when she rejects him. There is no relation between Elin's name on Tiger's list and Elin's personal preferences.

[1] His *optimal wife* in the usual sense: Given some particular instance of the Stable Marriage Problem, consider all possible stable perfect matchings, including that which is generated by the Mating Ritual. In each of these, Tiger has a wife. Of these "possible wives," he prefers one to all others. This girl, to whom he is married in one of the matchings but not necessarily all of them, is his *optimal wife*.

# Solutions to Mini-Quiz Apr. 6

**Problem 1 (6 points). (a)** A simple graph has 8 vertices and 24 edges. What is the average degree per vertex?

**Solution.** By the Handshaking Lemma, the sum of the degrees of the vertices in any graph is equal to twice the number of edges. So in this case, the sum of the degrees of the vertices is $2 \times 24 = 48$. With 8 vertices, the average degree per vertex is $\frac{48}{8} = 6$. ■

**(b)** A connected planar simple graph has 5 more edges than it has vertices. How many faces does it have?

**Solution.** Denoting the number of vertices by $v$, the number of edges by $e$, and the number of faces by $f$, Euler's Formula states that $v - e + f = 2$. But here, $e = v + 5$. Substituting gives $v - (v + 5) + f = 2$ and hence $f = 7$. ■

**(c)** A connected simple graph has one more vertex than it has edges. Is it necessarily planar?

**Solution.** Let $G$ denote any such graph. Now, any graph with $v$ vertices but fewer than $v - 1$ edges cannot possibly be connected. So every edge in $G$ is a cut edge, and therefore $G$ is acyclic. So $G$ is a tree and must be planar. ■

**(d)** If your answer to the previous part was *yes*, then how many faces can such a graph have? If your answer was *no*, then give an example of a nonplanar connected simple graph whose vertices outnumber its edges by one.

**Solution.** Since the graph is connected and acyclic, it only has one face. ■

**(e)** Consider the graph shown in Figure 1. How many distinct isomorphisms exist between this graph and itself? (Include the identity isomorphism.)

**Solution.** Only vertex $f$ has degree 1, so in any self-isomorphism, $f$ must map to itself. $b$ is the only vertex to be adjacent to a degree-1 vertex, so $b$ must also map to itself. $a$ and $c$ are both degree-3 vertices, and $d$ and $e$ are both degree-2 vertices. It is clear from examining the graph that $a$ can be mapped to $c$ and $c$ to $a$, or each of $a$ and $c$ can be mapped to itself. Independently, and similarly, $d$ can be mapped to $e$ and $e$ to $d$, or each of $d$ and $e$ can be mapped to itself. The only possible isomorphisms, then, are obtained by choosing one of the two possible mappings for $a$ and $c$ and, independently, one of the two possible mappings for $d$ and $e$. The result is $2 \times 2 = 4$ possible isomorphisms. ■

**Problem 2 (3 points).**
The $n$-dimensional hypercube, $H_n$, is a simple graph whose vertices are the binary strings of length $n$. Two vertices are adjacent if and only if they differ in exactly one bit. Consider for example $H_3$, shown in Figure 2. (Here, vertices `111` and `011` are adjacent because they differ only in the first bit, while vertices `101` and `011` are not adjacent because they differ in both the first and second bits.)

Explain why it is impossible to find two spanning trees of $H_3$ that have no edges in common.

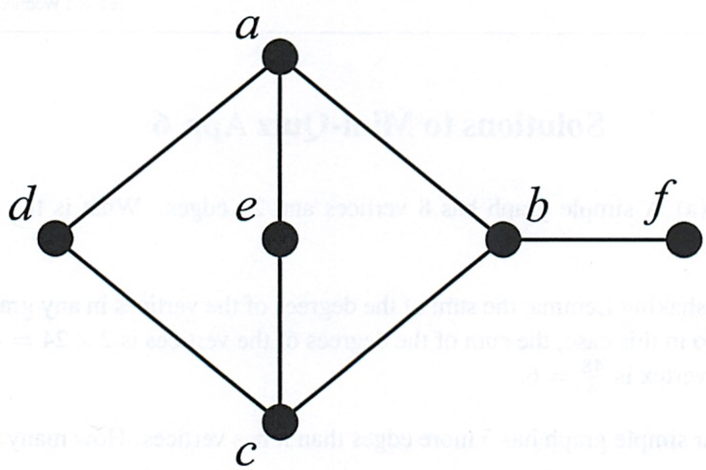



**Figure 1**

**Solution.** $H_3$ has 8 vertices, so any spanning tree must have $8 - 1 = 7$ edges. But $H_3$ has only 12 edges, so any two sets of 7 edges must overlap. ■

**Problem 3 (3 points).**
Consider the graph shown in Figure 3. Determine a valid coloring of the graph, using as few colors as possible. (Simply write your proposed color for each vertex next to that vertex. You may use $R$ for red, $G$ for green, etc.)

**Solution.** There are odd-length cycles in the graph, so at least three colors will be needed. So assume that three colors are sufficient. (If we encounter a contradiction under this assumption, we will need to use more colors.) Start with the length-3 cycle $abda$. All of its vertices must be colored differently, so assign red to $a$, blue to $b$, and green to $d$. The length-3 cycle $bdhb$ now forces $h$ to be colored red. $f$ must now be colored green and $g$ must be colored blue. The coloring is valid so far. $c$ is adjacent to a blue vertex and a green vertex, and no others, it must be colored red. Finally, $e$ is not adjacent to any other vertices, so it can be assigned any of the three colors. Choosing red for $e$, the result is shown in Figure 4. There is no pair of like-colored adjacent vertices, so this coloring is valid. ■

**Problem 4 (5 points). (a)** Consider the bipartite graph $G$ in Figure 5. Is it possible to find a matching that covers $L(G)$? If yes, explain what property of the graph guarantees the existence of such a matching. (Show that the graph exhibits this property and what this implies. Full credit will not be given for merely identifying a matching.) If no, identify a bottleneck that prevents a matching.

**Solution.** It is not possible. One bottleneck is $S = \{a, b, c, e\}$, since $N(S) = \{v, x, y\}$ and hence $|S| = 4 > 3 = |N(S)|$. (It is easy to see that there are no bottlenecks of size 1, 2, 3, or 5.) ■

**(b)** Consider the bipartite graph $H$ in Figure 6. Is it possible to find a matching that covers $L(H)$? If yes, explain what property of the graph guarantees the existence of such a matching. (Show that the graph exhibits this property and what this implies. Full credit will not be given for merely identifying a matching.) If no, identify a bottleneck that prevents a matching.

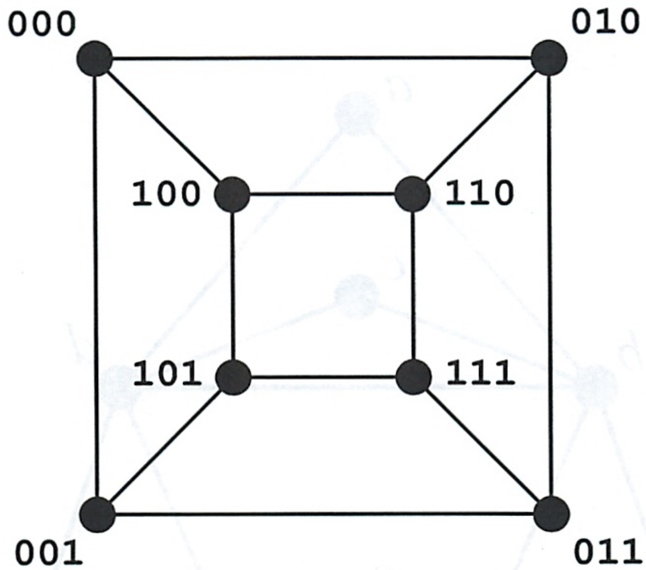



**Figure 2** $H_3$.

**Solution.** A matching is guaranteed to exist. Each vertex in $L(H)$ has degree at least 3, while each vertex in $R(H)$ has degree at most 3. Consequently, the graph is degree-constrained. There are therefore no bottlenecks and a matching must exist by Hall's Theorem. ■

**Problem 5 (3 points).**
In the Mating Ritual, suppose Tiger is one of the boys and Elin is one of the girls. Which of the following are preserved invariants **in general**?

1. Tiger is Elin's only suitor.
2. On Tiger's current list, the girl whom he prefers to all the others is his optimal wife[1].
3. Elin's name has been crossed off by Tiger and by everyone whom she prefers to him.

**Solution.** The statements that are preserved invariants in general appear in boldface below:

1. Tiger is Elin's only suitor. (This would certainly make Tiger Elin's favorite that day, but one or more of the boys who got rejected by another girl that day may visit Elin the following day.)
2. **On Tiger's current list, the girl whom he prefers to all the others is his optimal wife.** (The Mating Ritual gives each boy his optimal wife. Tiger must therefore ultimately marry his optimal wife, so once she becomes the most preferred girl on his list – and thus the girl he is serenading – she must remain the top girl on his list.)
3. **Elin's name has been crossed off by Tiger and by everyone whom she prefers to him.** (Note that this is a preserved invariant because it cannot ever be true. Were it true on some day, Tiger would have crossed Elin's name off his list, so he would end up marrying a woman he finds less desirable.

[1] His *optimal wife* in the usual sense: Given some particular instance of the Stable Marriage Problem, consider all possible stable perfect matchings, including that which is generated by the Mating Ritual. In each of these, Tiger has a wife. Of these "possible wives," he prefers one to all the others. This girl, to whom he is married in one of the matchings but not necessarily all of them, is his *optimal wife*.

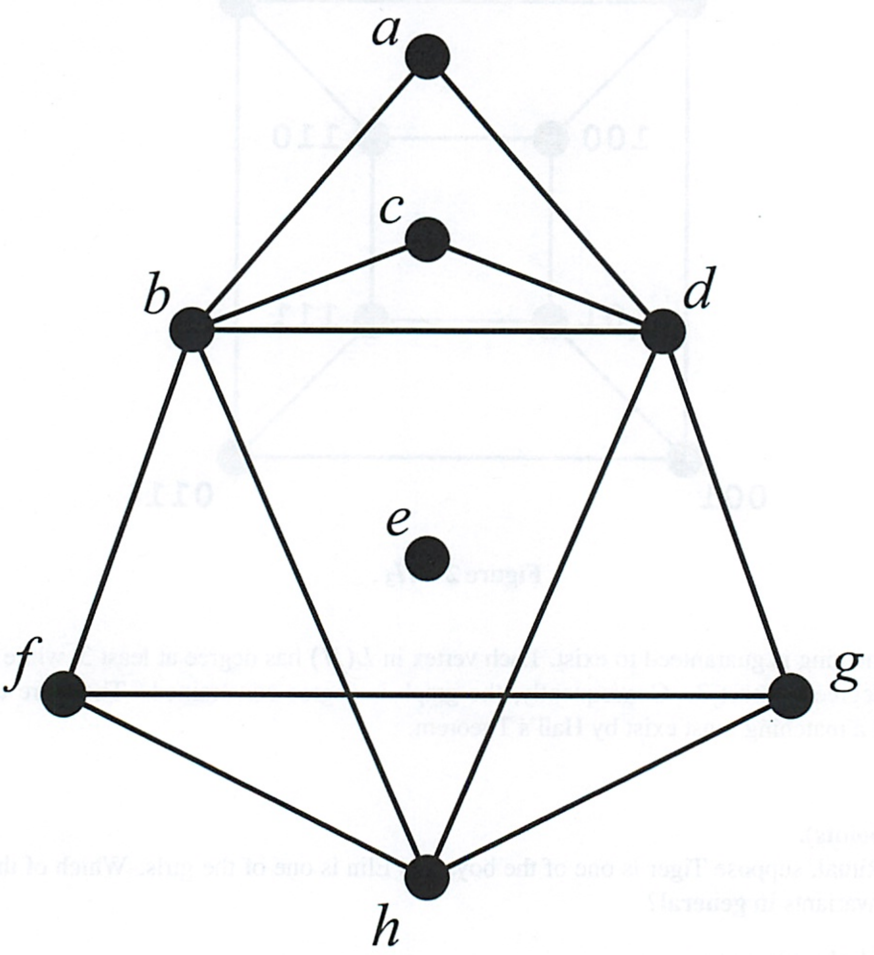



**Figure 3**

She would also have removed from contention everyone she finds more desirable than Tiger. So she would end up marrying someone she finds less desirable than Tiger. Consequently, Tiger and Elin would constitute would a rogue couple. Another way to think about it is this: If Elin's name was crossed off by Tiger and all the boys Elin prefers to him, then she must have a current favorite whom she prefers to all of them. But Tiger and his betters in Elin's eyes are the top boys on her list: there is no one she prefers to them.)

■

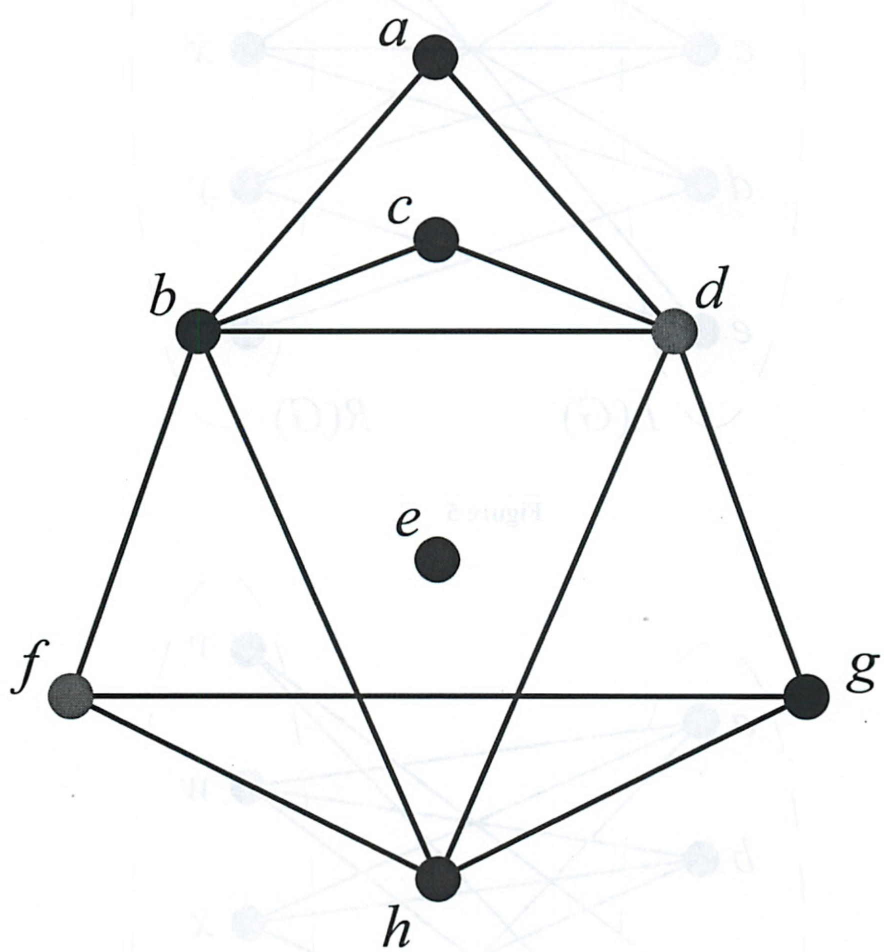



**Figure 4** A valid coloring.

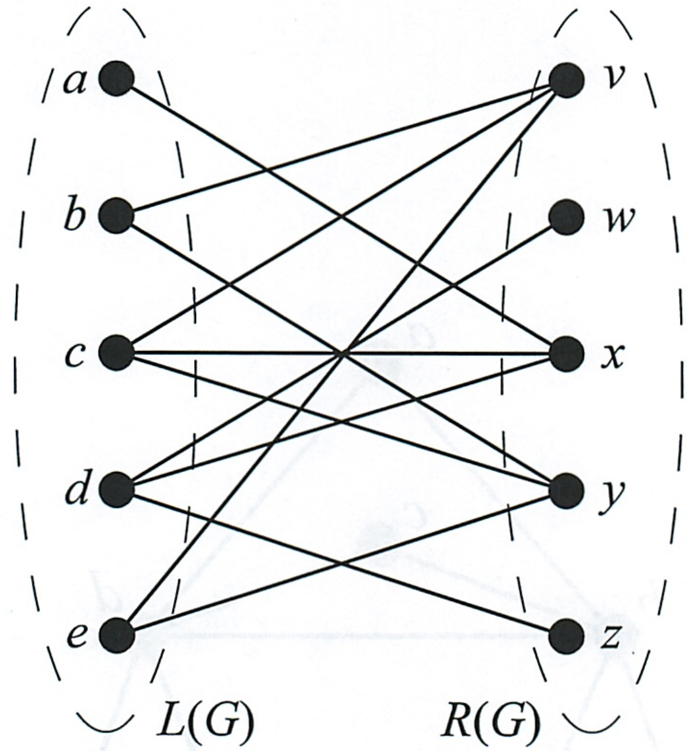



**Figure 5** *G*.

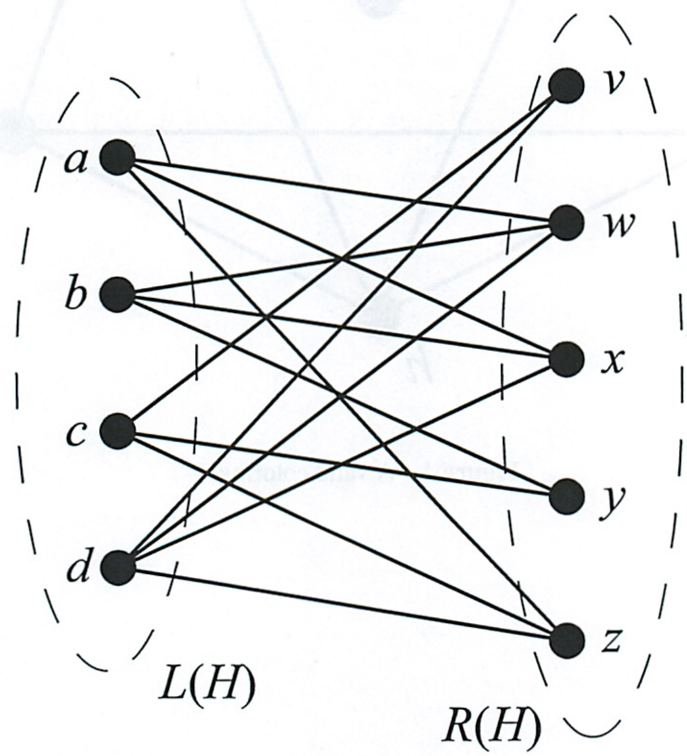
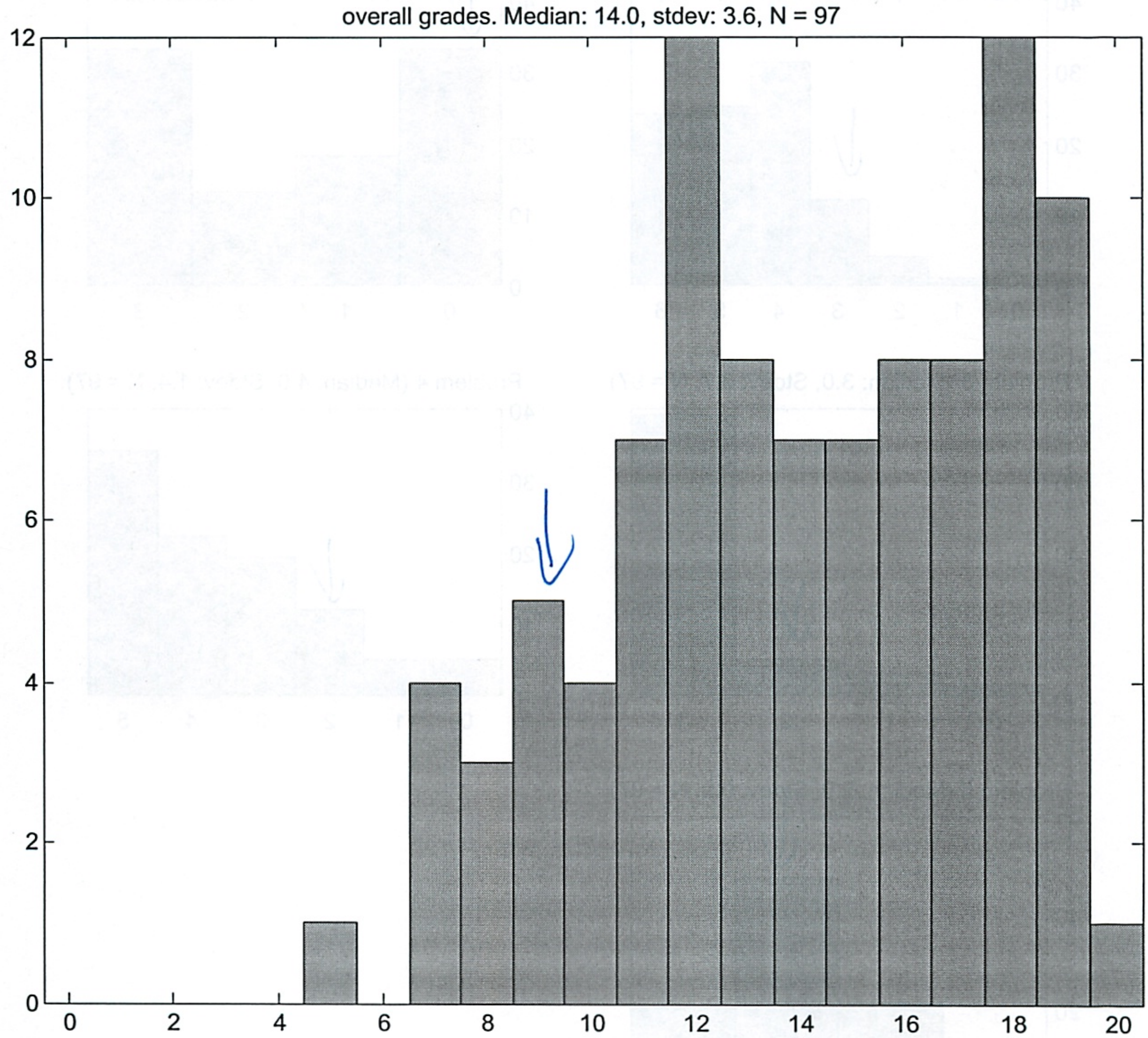
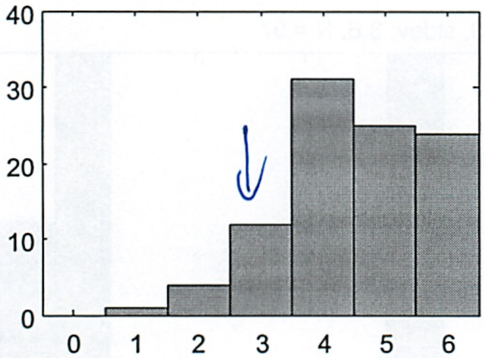


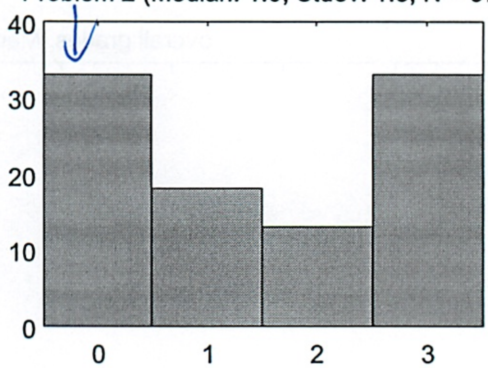

**Figure 6** *H*.


overall grades. Median: 14.0, stdev: 3.6, N = 97
0
2
4
6
8
10
12
0
2
4
6
8
10
12
14
16
18
20

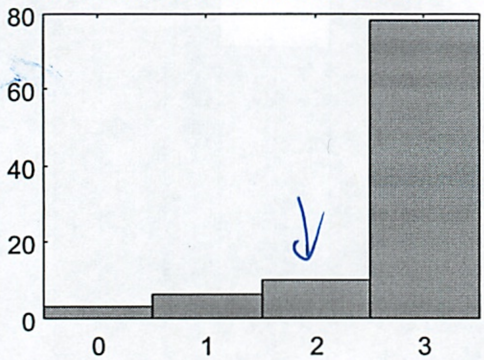
Problem 1 (Median: 5.0, Stdev: 1.1, N = 97)



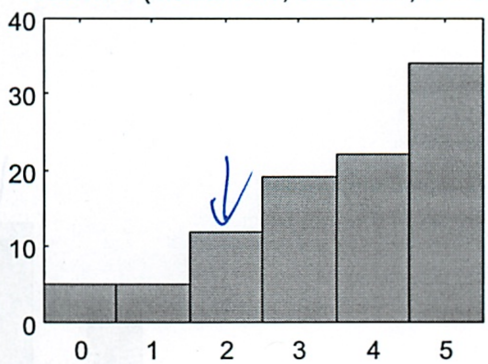

Problem 2 (Median: 1.0, Stdev: 1.3, N = 97)




Problem 3 (Median: 3.0, Stdev: 0.7, N = 97)




Problem 4 (Median: 4.0, Stdev: 1.4, N = 97)




Problem 5 (Median: 2.0, Stdev: 0.9, N = 97)