In-Class Problems Week 11, Fri.

Problem 1.
[A Baseball Series]
The New York Yankees and the Boston Red Sox are playing a two-out-of-three series. (In other words, they play until one team has won two games. Then that team is declared the overall winner and the series ends.) Assume that the Red Sox win each game with probability $3/5$, regardless of the outcomes of previous games.

Answer the questions below using the four step method. You can use the same tree diagram for all three problems.

(a) What is the probability that a total of 3 games are played?

(b) What is the probability that the winner of the series loses the first game?

(c) What is the probability that the *correct* team wins the series?

Problem 2.
To determine which of two people gets a prize, a coin is flipped twice. If the flips are a Head and then a Tail, the first player wins. If the flips are a Tail and then a Head, the second player wins. However, if both coins land the same way, the flips don't count and whole the process starts over.

Assume that on each flip, a Head comes up with probability p, regardless of what happened on other flips. Use the four step method to find a simple formula for the probability that the first player wins. What is the probability that neither player wins?

Suggestions: The tree diagram and sample space are infinite, so you're not going to finish drawing the tree. Try drawing only enough to see a pattern. Summing all the winning outcome probabilities directly is difficult. However, a neat trick solves this problem and many others. Let s be the sum of all winning outcome probabilities in the whole tree. Notice that *you can write the sum of all the winning probabilities in certain subtrees as a function of* s. Use this observation to write an equation in s and then solve.

Problem 3.
Suppose there is a system with n components, and we know from past experience that any particular component will fail in a given year with probability p. That is, letting F_i be the event that the ith component fails within one year, we have

$$\Pr[F_i] = p$$

for $1 \le i \le n$. The *system* will fail if *any one* of its components fails. What can we say about the probability that the system will fail within one year?

Let F be the event that the system fails within one year. Without any additional assumptions, we can't get an exact answer for $\Pr[F]$. However, we can give useful upper and lower bounds, namely,

$$p \le \Pr[F] \le np. \tag{1}$$

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We may as well assume $p < 1/n$, since the upper bound is trivial otherwise. For example, if $n = 100$ and $p = 10^{-5}$, we conclude that there is at most one chance in 1000 of system failure within a year and at least one chance in 100,000.

Let's model this situation with the sample space $\mathcal{S} ::= \mathcal{P}(\{1, \ldots, n\})$ whose outcomes are subsets of positive integers $\leq n$, where $s \in \mathcal{S}$ corresponds to the indices of exactly those components that fail within one year. For example, $\{2, 5\}$ is the outcome that the second and fifth components failed within a year and none of the other components failed. So the outcome that the system did not fail corresponds to the emptyset, $\emptyset$.

(a) Show that the probability that the system fails could be as small as p by describing appropriate probabilities for the outcomes. Make sure to verify that the sum of your outcome probabilities is 1.

(b) Show that the probability that the system fails could actually be as large as np by describing appropriate probabilities for the outcomes. Make sure to verify that the sum of your outcome probabilities is 1.

(c) Prove inequality (1).

Problem 4.
Here are some handy rules for reasoning about probabilities that all follow directly from the Disjoint Sum Rule in the Appendix. Prove them.

$$\Pr[A - B] = \Pr[A] - \Pr[A \cap B] \qquad \text{(Difference Rule)}$$

$$\Pr[\overline{A}] = 1 - \Pr[A] \qquad \text{(Complement Rule)}$$

$$\Pr[A \cup B] = \Pr[A] + \Pr[B] - \Pr[A \cap B] \qquad \text{(Inclusion-Exclusion)}$$

$$\Pr[A \cup B] \leq \Pr[A] + \Pr[B]. \qquad \text{(2-event Union Bound)}$$

$$\text{If } A \subseteq B, \text{ then } \Pr[A] \leq \Pr[B]. \qquad \text{(Monotonicity)}$$

Appendix

The Four Step Method

This is a good approach to questions of the form, "What is the probability that ——-?" Intuition can be misleading, but this formal approach gives the right answer every time.

1. Find the sample space. (Use a tree diagram.)
2. Define events of interest. (Mark leaves corresponding to these events.)
3. Determine outcome probabilities:
   (a) Assign edge probabilities.
   (b) Compute outcome probabilities. (Multiply along root-to-leaf paths.)
4. Compute event probabilities. (Sum the probabilities of all outcomes in the event.)

Probability Spaces

A countable *sample space*, $\mathcal{S}$, is a nonempty countable set. An element $w \in \mathcal{S}$ is called an *outcome*. A subset of $\mathcal{S}$ is called an *event*.

A *probability space* consists of a sample space, $\mathcal{S}$, and a function $\Pr[] : \mathcal{S} \to [0, 1]$, called the *probability function*, such that

$$\sum_{w \in \mathcal{S}} \Pr[w] = 1.$$

For any event, $E \subseteq \mathcal{S}$, the *probability of* E is defined to be the sum of the probabilities of the outcomes in E:

$$\Pr[E] ::= \sum_{w \in E} \Pr[w].$$

Sum Rule & Union Bound

Let $E_0, E_1, \ldots$ be a (possibly infinite) sequence of events. These events are said to be *pairwise disjoint* if $E_i \cap E_j = \emptyset$ whenever $i \neq j$.

If these events are pairwise disjoint, then

$$\Pr[\bigcup_{n \geq 0} E_n] = \sum_{n \geq 0} \Pr[E_n]. \quad \text{(Disjoint Sum Rule)}$$

Even if they are not pairwise disjoint,

$$\Pr[\bigcup_{n \geq 0} E_n] \leq \sum_{i \geq n} \Pr[E_n]. \quad \text{(Union Bound)}$$

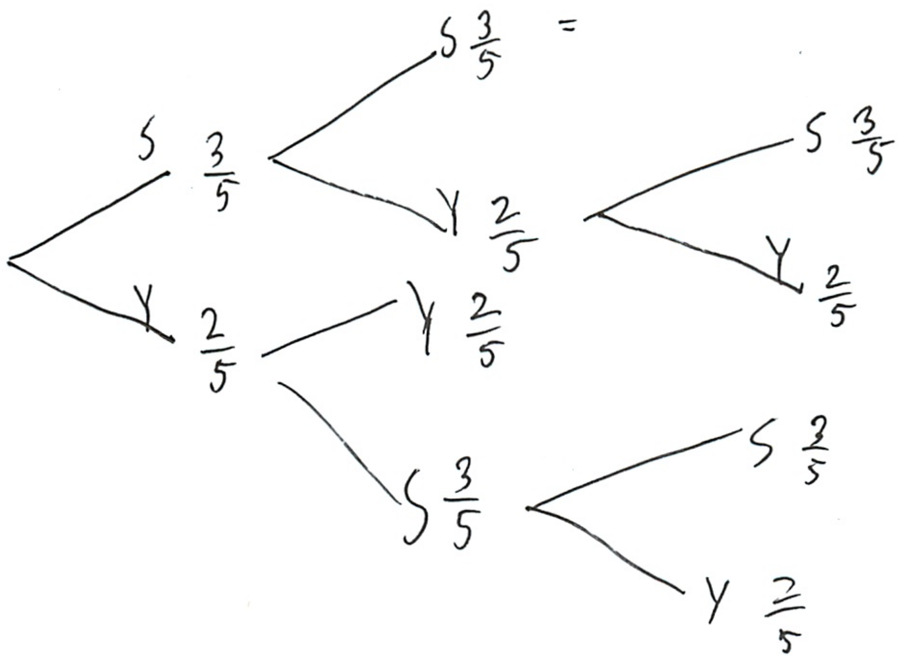
In Class 11 Fri

4/22

1. Baseball

~~Min~~ min 2 games

max 3 games



$\frac{3}{5}\cdot\frac{3}{5}$	S	$\frac{9}{25}$
$\frac{3}{5}\cdot\frac{2}{5}\cdot\frac{3}{5}$	S	$\frac{18}{125}$
$\frac{3}{5}\cdot\frac{2}{5}\cdot\frac{2}{5}$	Y	$\frac{12}{125}$
$\frac{2}{5}\cdot\frac{2}{5}$	Y	$\frac{4}{25}$
$\frac{2}{5}\cdot\frac{3}{5}\cdot\frac{3}{5}$	S	$\frac{18}{125}$
$\frac{2}{5}\cdot\frac{3}{5}\cdot\frac{2}{5}$	Y	$\frac{12}{125}$

c) S $\frac{9}{25} + \frac{18}{125} + \frac{18}{125} = \frac{81}{125} =$

Y $\frac{12}{125} + \frac{4}{25} + \frac{12}{125} = \frac{44}{125} =$

a) P(3 games played) - well can use same tables just assign diff events

$\frac{9}{25} + \frac{4}{25} = \frac{13}{25}$

②

b) Again diff events

$$\frac{12}{125} + \frac{18}{125} = \frac{30}{125} = \frac{6}{25}$$

---

2. Coin flipping

H → T 1 wins

T → H 2 wins

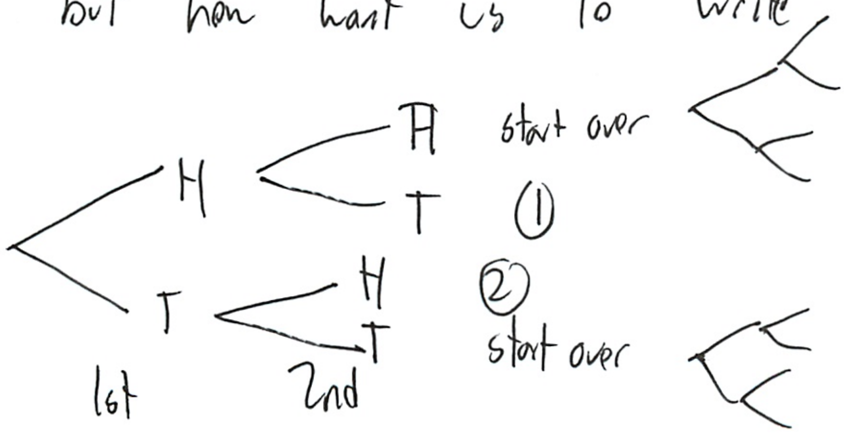
P(heads) = p

H → H start over

T → T

for 1 game? - yeah but start over goes ∞

but how want us to write as fn?



~~So ∅ $\frac{1}{2}$, $\frac{1}{2}$ (if fair)~~

no since p is p heads

③

1 winning is

$$p \cdot (1-p)$$

2 winning

$$(1-p)\,p$$

So actually is $\frac{1}{2}, \frac{1}{2}$

when it ends

prob don't have the proof they are looking for

3. n components

~~#~~ each fail ind w/ prob p

F_i is event that ith component fails

$$P(F_i) = p$$

for $1 \leq i \leq n$

System fails if any component fails.

Can't get exact ans for $P(F)$

But can get upper + lower bounds

$$p \leq P(F) \leq np$$

④

But isn't it p^n

no

but related

---

2 board) $p^2 + (1-p)^2$ * prof wants explination of this chance game repeats

For no player to win - game repeats indefitly

w/ prop $\lim_{n \to \infty} (1-2p+2p^2)^n = 0$

[if $p=0$ or $p=1$ neither player can win]

So player 1 wins w/ prob

$p(1-p)\left(1 + (1-2p+2p^2) + (1-2p+2p^2)^2 + \ldots\right)$

Since same sum so =, and $P(1)+P(2) = 1$ so must be $\frac{1}{2}$

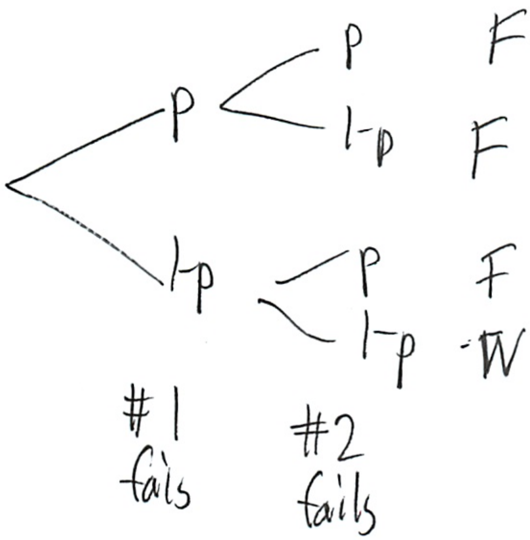
So $P(\text{player 1 wins}) = \frac{1}{2}$

how did i

(5)

3 cont

p is prob each fails



Oh $(1-p)^n$

So try # $.9 \cdot .9 \cdot .9 = .74$
about

---

4. Are axioms how to prove?

3cont. But why do they say all these things about ~~limits~~ bounds?

Oh we don't if ind or correlated

Or somewhere in between

Solutions to In-Class Problems Week 11, Fri.

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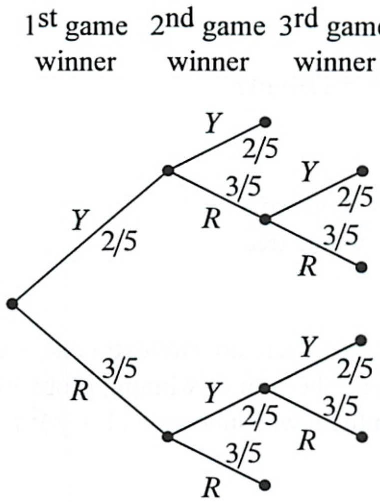
(a) What is the probability that a total of 3 games are played?

(b) What is the probability that the winner of the series loses the first game?

✓ Red Sox

(c) What is the probability that the *correct* team wins the series?

Solution. A tree diagram is worked out below. From the tree diagram, we get:



outcome	3 games played?	winner lost 1st game?	correct team wins?	probability
YY				4/25
YRY	✓			12/125
YRR	✓	✓	✓	18/125
RYY	✓	✓		12/125
RYR	✓		✓	18/125
RR			✓	9/25

$$\Pr[\text{3 games played}] = \frac{12}{125} + \frac{18}{125} + \frac{12}{125} + \frac{18}{125} = \frac{12}{25}$$

$$\Pr[\text{winner lost first game}] = \frac{18}{125} + \frac{12}{125} = \frac{6}{25}$$

$$\Pr[\text{correct team wins}] = \frac{18}{125} + \frac{18}{125} + \frac{9}{25} = \frac{81}{125}$$

■

Problem 2.
To determine which of two people gets a prize, a coin is flipped twice. If the flips are a Head and then a Tail,

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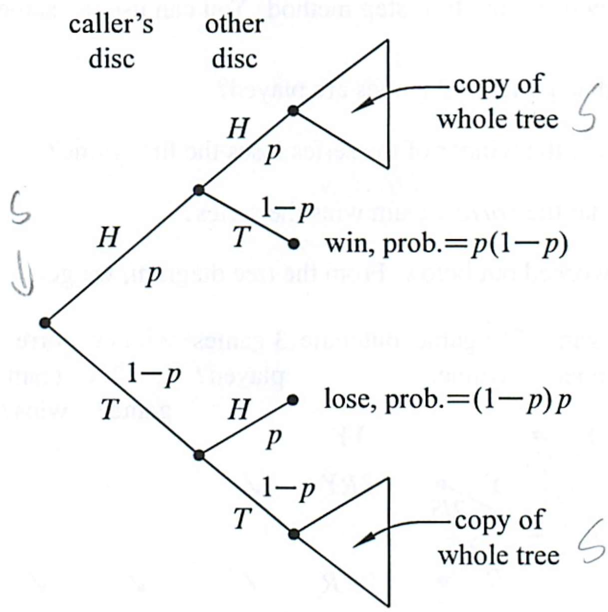
the first player wins. If the flips are a Tail and then a Head, the second player wins. However, if both coins land the same way, the flips don't count and whole the process starts over.

Assume that on each flip, a Head comes up with probability p, regardless of what happened on other flips. Use the four step method to find a simple formula for the probability that the first player wins. What is the probability that neither player wins?

Suggestions: The tree diagram and sample space are infinite, so you're not going to finish drawing the tree. Try drawing only enough to see a pattern. Summing all the winning outcome probabilities directly is difficult. However, a neat trick solves this problem and many others. Let s be the sum of all winning outcome probabilities in the whole tree. Notice that *you can write the sum of all the winning probabilities in certain subtrees as a function of s*. Use this observation to write an equation in s and then solve.

Solution. In the tree diagram below, the small triangles represent subtrees that are themselves complete copies of the whole tree.



I had just solved intuitively

Let s equal the sum of all winning probabilities in the whole tree. There are two extra edges with probability p on the path to each outcome in the top subtree. Therefore, the sum of winning probabilities in the upper tree is p^2s. Similarly, the sum of winning probabilities in the lower subtree is $(1-p)^2s$. This gives the equation:

Shortcut? ends +

$$s = p^2s + (1-p)^2s + p(1-p)$$

So write like this solve for s

The solution to this equation is $s = 1/2$, for all p between 0 and 1.

By symmetry, the probability that the first player loses is 1/2. This means that the event, if any, of flipping forever can only have probability zero.

Formally, the sample space is the (infinite) set of leaves of the tree, namely,

$$\mathcal{S} ::= \{\texttt{TT}, \texttt{HH}\}^* \cdot \{\texttt{HT}, \texttt{TH}\}$$

where $\{\texttt{TT}, \texttt{HH}\}^*$ denotes the set of strings formed by concatenating a sequence of HH's and TT's. For example,

$$\texttt{TTTTHHHT}, \texttt{HHTTTH}, \texttt{HHHHHHHHHT}, \texttt{HT} \in \mathcal{S}.$$

why does noted called

For any string $s \in \mathcal{S}$,

$$\Pr[s] ::= p^{\#\text{H's in } s}(1-p)^{\#\text{T's in } s}.$$

To verify that is defines a probability space, we must show that $\sum_{s\in\mathcal{S}} \Pr[s] = 1$. But the probability that two tosses match is $p^2 + (1-p)^2$, and that they don't match is $2p(1-p)$ so

$$\begin{aligned}
\sum_{s\in\mathcal{S}} \Pr[s] &= \sum_{n\geq 0} \sum_{|s|=2n+2} \Pr[s] \\
&= \sum_{n\geq 0} \left(p^2 + (1-p)^2\right)^n (2p(1-p)) \\
&= 2p(1-p) \sum_{n\geq 0} \left(p^2 + (1-p)^2\right)^n \\
&= \frac{2p(1-p)}{1-(p^2+(1-p)^2)} \\
&= \frac{2p(1-p)}{2p-2p^2} = 1.
\end{aligned}$$

■

Problem 3.

Suppose there is a system with n components, and we know from past experience that any particular component will fail in a given year with probability p. That is, letting F_i be the event that the ith component fails within one year, we have

$$\Pr[F_i] = p$$

for $1 \leq i \leq n$. The *system* will fail if *any one* of its components fails. What can we say about the probability that the system will fail within one year?

Let F be the event that the system fails within one year. Without any additional assumptions, we can't get an exact answer for $\Pr[F]$. However, we can give useful upper and lower bounds, namely,

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We may as well assume $p < 1/n$, since the upper bound is trivial otherwise. For example, if $n = 100$ and $p = 10^{-5}$, we conclude that there is at most one chance in 1000 of system failure within a year and at least one chance in 100,000.

Let's model this situation with the sample space $\mathcal{S} ::= \mathcal{P}(\{1,\ldots,n\})$ whose outcomes are subsets of positive integers $\leq n$, where $s \in \mathcal{S}$ corresponds to the indices of exactly those components that fail within one year. For example, $\{2,5\}$ is the outcome that the second and fifth components failed within a year and none of the other components failed. So the outcome that the system did not fail corresponds to the emptyset, $\emptyset$.

(a) Show that the probability that the system fails could be as small as p by describing appropriate probabilities for the outcomes. Make sure to verify that the sum of your outcome probabilities is 1.

Solution. There could be a probability p of system failure if all the individual failures occur together. That is, let $\Pr[\{1,\ldots,n\}] ::= p$, $\Pr[\emptyset] ::= 1-p$, and let the probability of all other outcomes be zero. So $F_i = \{s \in \mathcal{S} \mid i \in s\}$ and $\Pr[F_i] = 0 + 0 + \cdots + 0 + \Pr[\{1,\ldots,n\}] = \Pr[\{1,\ldots,n\}] = p$. Also, the only outcome with positive probability in F is $\{1,\ldots,n\}$, so $\Pr[F] = p$, as required. ■

(b) Show that the probability that the system fails could actually be as large as np by describing appropriate probabilities for the outcomes. Make sure to verify that the sum of your outcome probabilities is 1.

Solution. Suppose at most one component ever fails at a time. That is, $\Pr[\{i\}] = p$ for $1 \leq i \leq n$, $\Pr[\emptyset] = 1 - np$, and probability of all other outcomes is zero. The sum of the probabilities of all the outcomes is one, so this is a well-defined probability space. Also, the only outcome in F_i with positive probability is $\{i\}$, so $\Pr[F_i] = \Pr[\{i\}] = p$ as required. Finally, $\Pr[F] = np$ because $F = \{A \subseteq \{1, \ldots, n\} \mid A \neq \emptyset\}$, so F in particular contains all the n outcomes of the form $\{i\}$. ■

(c) Prove inequality (1).

Solution. $F = \bigcup_{i=1}^{n} F_i$ so

$$\begin{aligned}
p &= \Pr[F_1] && \text{(given)} \\
&\leq \Pr[F] && \text{(since } F_1 \subseteq F\text{)} \\
&= \Pr[\textstyle\bigcup F_i] && \text{(def. of } F\text{)} \\
&\leq \sum_{i=1}^{n} \Pr[F_i] && \text{(Union Bound)} \\
&= np.
\end{aligned}$$

■

Problem 4.
Here are some handy rules for reasoning about probabilities that all follow directly from the Disjoint Sum Rule. Prove them.

$$\Pr[A - B] = \Pr[A] - \Pr[A \cap B] \qquad \text{(Difference Rule)}$$

Solution. Any set A is the disjoint union of $A - B$ and $A \cap B$, so

$$\Pr[A] = \Pr[A - B] + \Pr[A \cap B]$$

by the Disjoint Sum Rule. ■

$$\Pr[\overline{A}] = 1 - \Pr[A] \qquad \text{(Complement Rule)}$$

Solution. $\overline{A} ::= S - A$, so by the Difference Rule

$$\Pr[\overline{A}] = \Pr[S] - \Pr[A] = 1 - \Pr[A].$$

■

$$\Pr[A \cup B] = \Pr[A] + \Pr[B] - \Pr[A \cap B] \qquad \text{(Inclusion-Exclusion)}$$

Clever think about it - don't just mentally skip over!

Solution. $A \cup B$ is the disjoint union of A and $B - A$ so

$$\begin{aligned}
\Pr[A \cup B] &= \Pr[A] + \Pr[B - A] && \text{(Disjoint Sum Rule)} \\
&= \Pr[A] + (\Pr[B] - \Pr[A \cap B]) && \text{(Difference Rule)}
\end{aligned}$$

■

go to some intermediate step

$$\Pr[A \cup B] \le \Pr[A] + \Pr[B]. \qquad \text{(2-event Union Bound)}$$

Solution. This follows immediately from Inclusion-Exclusion and the fact that $\Pr[A \cap B] \ge 0$. ■

$$\text{If } A \subseteq B, \text{ then } \Pr[A] \le \Pr[B]. \qquad \text{(Monotonicity)}$$

Solution.

$$\begin{aligned}
\Pr[A] &= \Pr[B] - (\Pr[B] - \Pr[A]) \\
&= \Pr[B] - (\Pr[B] - \Pr[A \cap B]) && \text{(since } A = A \cap B\text{)} \\
&= \Pr[B] - \Pr[B - A] && \text{(difference rule)} \\
&\le \Pr[B] && \text{(since } \Pr[B - A] \ge 0\text{).}
\end{aligned}$$

■

Appendix

The Four Step Method

This is a good approach to questions of the form, "What is the probability that ...?" Intuition can be misleading, but this formal approach explains exactly what's going on every time.

1. Find the sample space. (Use a tree diagram.)

2. Define events of interest. (Mark leaves corresponding to these events.)

3. Determine outcome probabilities:

   (a) Assign edge probabilities.

   (b) Compute outcome probabilities. (Multiply along root-to-leaf paths.)

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Probability Spaces

A countable *sample space*, $\mathcal{S}$, is a nonempty countable set. An element $w \in \mathcal{S}$ is called an *outcome*. A subset of $\mathcal{S}$ is called an *event*.

A *probability space* consists of a sample space, $\mathcal{S}$, and a function $\Pr[] : \mathcal{S} \to [0, 1]$, called the *probability function*, such that

$$\sum_{w \in \mathcal{S}} \Pr[w] = 1.$$

For any event, $E \subseteq \mathcal{S}$, the *probability of E* is defined to be the sum of the probabilities of the outcomes in E:

$$\Pr[E] ::= \sum_{w \in E} \Pr[w].$$

Sum Rule & Union Bound

Let $E_0, E_1, \ldots$ be a (possibly infinite) sequence of events. These events are said to be *pairwise disjoint* if $E_i \cap E_j = \emptyset$ whenever $i \neq j$.

If these events are pairwise disjoint, then

$$\Pr[\bigcup_{n \geq 0} E_n] = \sum_{n \geq 0} \Pr[E_n]. \qquad \text{(Disjoint Sum Rule)}$$

Even if they are not pairwise disjoint,

$$\Pr[\bigcup_{n \geq 0} E_n] \leq \sum_{i \geq n} \Pr[E_n]. \qquad \text{(Union Bound)}$$

4/25

Mathematics for Computer Science
MIT 6.042J/18.062J

Conditional Probability & Independence

Conditional Probability: A Fair Die

$$\Pr\{\text{roll } 1\} = \frac{|\{1\}|}{|\{1,2,3,4,5,6\}|} = \frac{1}{6}$$

"knowledge" changes probabilities:
Pr{roll 1 knowing rolled odd}

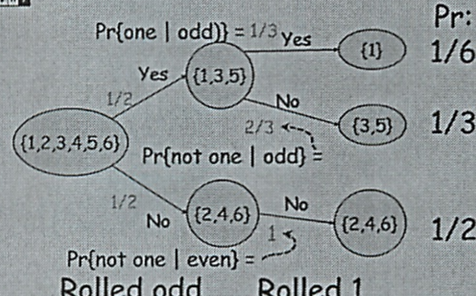
$$= \frac{|\{1\}|}{|\{1,3,5\}|} = \frac{1}{3}$$

Conditional Probability

$\Pr\{A|B\}$ is the probability of event A, *given* that event B has occurred:

$$\boxed{\Pr\{A \mid B\} ::= \frac{\Pr\{A \cap B\}}{\Pr\{B\}}}$$

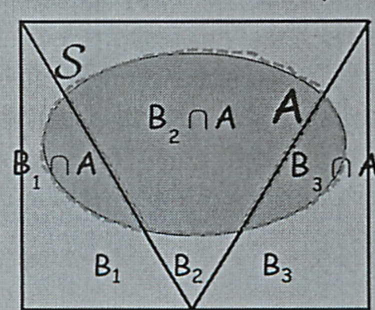
Conditional Probability: A Fair Die



Product Rule

$$\boxed{\begin{aligned}&\Pr\{A \cap B\} = \\ &\quad \Pr\{A \mid B\} \cdot \Pr\{B\}\end{aligned}}$$

Law of Total Probability



Law of Total Probability

$A = (B_1 \cap A) \cup (B_2 \cap A) \cup (B_3 \cap A)$

$$\Pr\{A\} = \Pr\{B_1 \cap A\} + \Pr\{B_2 \cap A\} + \Pr\{B_3 \cap A\}$$

Law of Total Probability

If $\mathcal{S}$ is disjoint union of $B_0, B_1, \ldots$

$$\Pr\{A\} = \sum_{i \geq 0} \Pr\{A \cap B_i\}$$

$$= \sum_{i \geq 0} \Pr\{A \mid B_i\} \cdot \Pr\{B_i\}$$

Conditional Probability: Monty Hall

Pr{ prize at 1 | picked 1 & goat at 2} $= \frac{1}{2}$ Really!

[picked 1 & goat at 2] =

{ (1,1,2), (1,1,3), (3,1,2)}

Conditional Probability: Monty Hall

Pr{ prize at 1 | picked 1 & goat at 2} $= \frac{1}{2}$

[picked 1 & goat at 2] =

$\{ \underbrace{(1,1,2)}_{Pr=1/18}, \underbrace{(1,1,3)}_{Pr=1/18}, \underbrace{(3,1,2)}_{Pr=1/9} \}$

Conditional Probability: Monty Hall

Seems that the contestant may as well stick, since the probability is 1/2 *given what he knows* when he chooses.

But wait, contestant knows more than goat at door 2: he knows

Carol opened door 2!

Conditional Probability: Monty Hall

Pr{ prize at 1 | picked 1 & Carol opens 2} = 1/3

[picked 1 & Carol opens 2] =

$\{ \underbrace{(1,1,2)}_{Pr=1/18}, \underbrace{(3,1,2)}_{Pr=1/9} \}$

$$\frac{1/18}{1/18 + 1/9} =$$

Independence

Definition 1:

Events A and B are independent iff

$$\Pr\{A\} = \Pr\{A \mid B\}.$$

Definition 2:

Events A and B are independent iff

$$\Pr\{A\} \cdot \Pr\{B\} = \Pr\{A \cap B\}.$$

Independent Events?

B: Baby born at Mass General Hospital between 1:00AM and 1:01AM.

F: Jupiter's moon IO is full.

Independent Events?

Does event B (baby born) have anything to do with event F (IO is full)?

Babies & Full Moons

My sweet Aunt Daisy believed in Astrology. She thought celestial events could influence babies.

We might say "nonsense," there's no effect.

But Daisy might be right

(for wrong reasons)

C:\42\pub\jup-radio_070115.htm

** INFORMATION FOR AMATEUR RADIO ASTRONOMERS ** JUPITER DECAMETRIC EMISSIONS ** JUPITER EPHEMERIS 01 Jul 1994, 0000UTC, Julian Day: 2449534.5, GMT Sidereal Time: 18h35m17s ....

C:\42\pub\jup-radio_070115.htm

SUMMARY: Jupiter's HF emissions are ...heard on earth when Jupiter's magnetic field "sweeps" the earth every 9h55m27s and at other times when Io's geometric position influences activity.

Babies & Full Moons

influence of IO's magnetic field changes with phases!
--might affect radios in ambulances, for example

Babies & Full Moons

So independence of B and F is actually unclear.

Deciding whether to treat them as independent is a matter of experiment, not Mathematics.

Mutual Independence

events $A_1, A_2, \ldots, A_n$ are mutually independent

iff $\Pr\{A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}\} =$

$\Pr\{A_{i_1}\} \cdot \Pr\{A_{i_2}\} \cdots \Pr\{A_{i_k}\}$

for all A_{i_j} ($2^n-(n+1)$ equations to check!)

Mutual Independence

Events $E_1, E_2, \ldots$ are k-way independent

iff every subset of k of them is mutually independent

Team Problems

Problems 1 – 3

6.042 Condi Prob + Ind 4/25

Seen it before

$$P(\text{roll } 1 \text{ on fair die}) = \frac{|\{1\}|}{|\{1,2,3,4,5,6\}|} = \frac{1}{6}$$

- Suppose

$$P(\text{roll } 1 \mid \text{rolled odd}) = \frac{|\{1\}|}{|\{1,3,5\}|} = \frac{1}{3}$$

$\mid$ ← given; rolled odd ← has occured

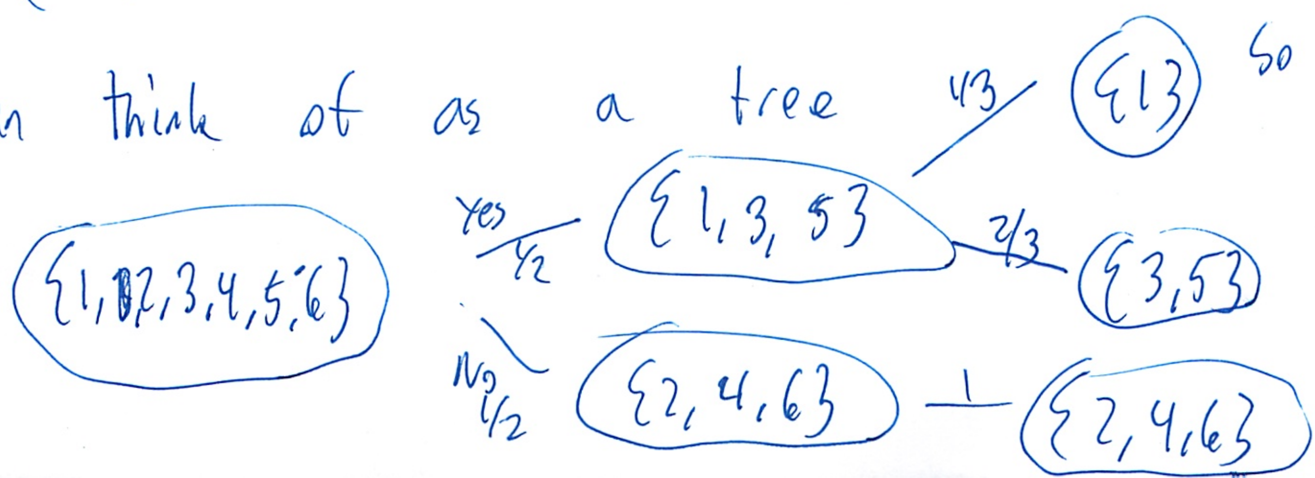
$$P\{A \mid B\} = \frac{P(A \cap B)}{P(B)}$$

↑ measure relative to size of B

(recognize much better than before 6.04))
(- improved math + intro to biz this year)

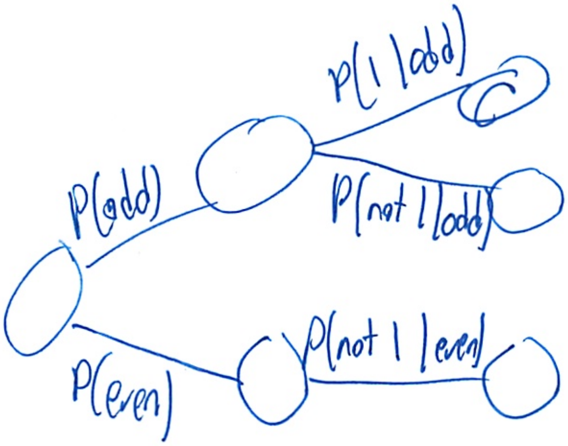
Can think of as a tree

So $\frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}$



2

The # on the branches were actually cond. prob!

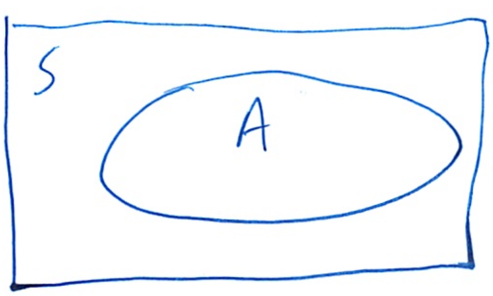


Product Rule

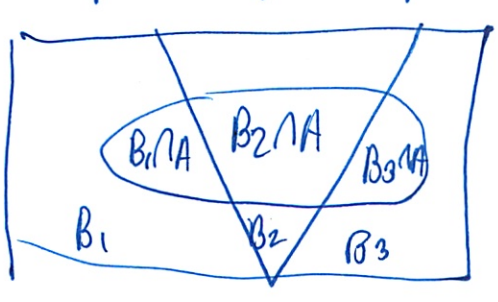
restatement of conditional

$$P(A \cap B) = P(A \mid B)\, P(B)$$

Law of Total Probability



Cut up sample space



③

So $A = (B_1 \cap A) \cup (B_2 \cap A) \cup (B_3 \cap A)$

- disjoint union

So add up - sum rule

$- \sum$ ...

Sometimes throws people off

$P(\text{prize } 1 \mid \text{picked } 1 + \text{goat at } 2\} = \frac{1}{2}$

?

$\underbrace{(1,1,2)}_{1/18}$, $\underbrace{(1,1,3)}_{1/18}$, $\underbrace{(3,1,2)}_{1/9}$ ← looks like $\frac{2}{3}$

prize at, picked, door opened

So these are actually <u>not</u> all the same

What ~~a~~ door they ~~should~~ show you does not matter - its always a goat

(4)

$P($ prize at 1 | picked 1 + 2 opened$) = 1/3$

$$[\underbrace{(1,1,2)}_{1/18}, \underbrace{(3,1,2)}_{1/9}]$$

but

$$\frac{1/18}{1/18 + 1/9} = \frac{1}{3}$$

Need to figure out what to condition on - OJ trial

Independence

fundamental, but subtle

knowing about one has no effect on other

A, B ind if

$$P(A) = P(A \mid B)$$

$$P(A) \cdot P(B) = P(A \cap B)$$

B = Baby born at MGH b/w 1AM 1:01AM

F = Jupiter's moon IO is full

(5)

Do they have any thing to do w/ each other ¿¿¿¿

Jupiter does mess up earth's magnetic field
Can affect noise on radio rushing to hospital ¿

Have to tell experimentally
- Collect stats

In real world - ~~based on~~ assume ind. a lot
- can make things wrong
- don't take it for granted
- need to know your assumptions

Math tells you how model behaves
- not which model to pick

Events ind if $P(A_1 \cap A_2 \cap A_3 \cap \dots \cap A_n)$

$$= P(A_1) \cdot P(A_2) \cdot P(A_3) \cdot \dots \cdot P(A_n)$$

⑥

A lot of eqn to check!

$$\left(2^n - (n+1)\right)$$

Events are <u>k-way ind.</u> if every subset of k events is mutually ind

- Could be 2-way ind, but not 3 ind or mutual ind

  ↑ often the case

  Can use a special methodology,

In-Class Problems Week 12, Mon.

Problem 1.
There are two decks of cards. One is complete, but the other is missing the ace of spades. Suppose you pick one of the two decks with equal probability and then select a card from that deck uniformly at random. What is the probability that you picked the complete deck, given that you selected the eight of hearts? Use the four-step method and a tree diagram.

Problem 2.
There are three prisoners in a maximum-security prison for fictional villains: the Evil Wizard Voldemort, the Dark Lord Sauron, and Little Bunny Foo-Foo. The parole board has declared that it will release two of the three, chosen uniformly at random, but has not yet released their names. Naturally, Sauron figures that he will be released to his home in Mordor, where the shadows lie, with probability $2/3$.

A guard offers to tell Sauron the name of one of the other prisoners who will be released (either Voldemort or Foo-Foo). Sauron knows the guard to be a truthful fellow. However, Sauron declines this offer. He reasons that if the guard says, for example, "Little Bunny Foo-Foo will be released", then his own probability of release will drop to $1/2$. This is because he will then know that either he or Voldemort will also be released, and these two events are equally likely.

Using a tree diagram and the four-step method, either prove that the Dark Lord Sauron has reasoned correctly or prove that he is wrong. Assume that if the guard has a choice of naming either Voldemort or Foo-Foo (because both are to be released), then he names one of the two uniformly at random.

Problem 3.
Suppose that you flip three fair, mutually independent coins. Define the following events:

- Let A be the event that *the first* coin is heads.
- Let B be the event that *the second* coin is heads.
- Let C be the event that *the third* coin is heads.
- Let D be the event that *an even number of* coins are heads.

(a) Use the four step method to determine the probability space for this experiment and the probability of each of A, B, C, D.

(b) Show that these events are not mutually independent.

(c) Show that they are 3-way independent.

Problem 4.
Let A, B, C be events. For each of the following statements, prove it or give a counterexample.

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(a) If A is independent of B, and A is independent of C, then A is independent of $B \cap C$.

(b) If A is independent of B, and A is independent of C, then A is independent of $B \cup C$.

(c) If A is independent of B, and A is independent of C, and A is independent of $B \cap C$, then A is independent of $B \cup C$.

Appendix

The Four Step Method

This is a good approach to questions of the form, "What is the probability that ——?" Intuition can be misleading, but this formal approach gives the right answer every time.

1. Find the sample space. (Use a tree diagram.)
2. Define events of interest. (Mark leaves corresponding to these events.)
3. Determine outcome probabilities:
   (a) Assign edge probabilities.
   (b) Compute outcome probabilities. (Multiply along root-to-leaf paths.)
4. Compute event probabilities. (Sum the probabilities of all outcomes in the event.)

Conditional Probabilitiy

For events E, F such that $\Pr[F] \neq 0$, the *conditional probability* of E given F is:

$$\Pr\left[E \mid F\right] ::= \frac{\Pr[E \cap F]}{\Pr[F]}$$

Law of Total Probability

Here is the Law stated for three sets: suppose E, F, G are pairwise disjoint events, and

$$A \subseteq E \cup F \cup G.$$

Then

$$\begin{aligned}\Pr[A] &= \Pr[A \cap E] + \Pr[A \cap F] + \Pr[A \cap G] \\ &= \Pr\left[A \mid E\right] \cdot \Pr[E] + \Pr\left[A \mid F\right] \cdot \Pr[F] + \Pr\left[A \mid G\right] \cdot \Pr[G].\end{aligned}$$

Independence

Events E, F are *independent* iff

$$\Pr[E \cap F] = \Pr[E] \cdot \Pr[F].$$

Events $E_1, E_2, \ldots, E_n$ are *mutually independent* if and only if

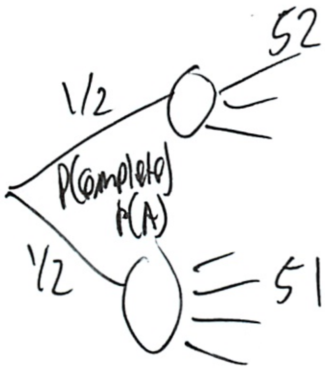
$$\Pr[\bigcap_{i \in J} E_i] = \prod_{i \in J} \Pr[E_i]$$

for all subsets $J \subseteq \{1, \ldots, n\}$.

Events $E_1, E_2, \ldots,$ are k-way independent iff every k of these events are mutually independent.

In-Class 12 Mon 4/25

1. p(complete | eight of hearts)



P(eight hearts) $= \left(\frac{1}{2}\cdot\frac{1}{52}\right)+\left(\frac{1}{51}\cdot\frac{1}{2}\right)$

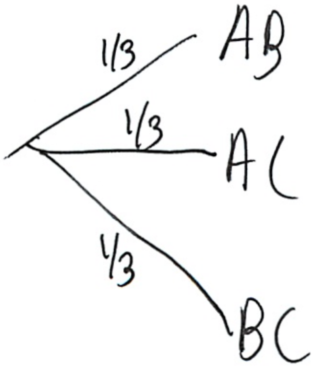
P(B)

$$P(A\mid B) = \frac{P(A\cap B)}{P(B)} = \frac{1/2 \cdot 1/52}{\left(\frac{1}{2}\cdot\frac{1}{52}\right)+\left(\frac{1}{51}\cdot\frac{1}{2}\right)}$$

duh! remember

2. No - knowing who does not affect his prob to be released

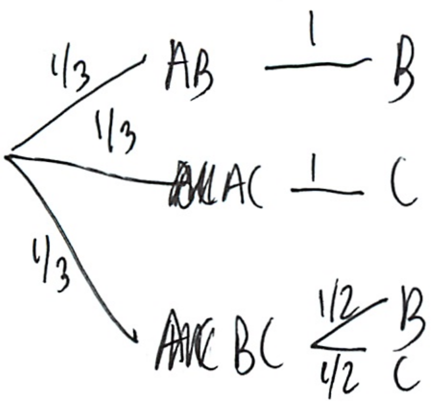
Chosen 1/2 ~~Chosen~~



So first is 2/3

Sauron = A

(2)



Chosen told

New P(he is released / what he is told)

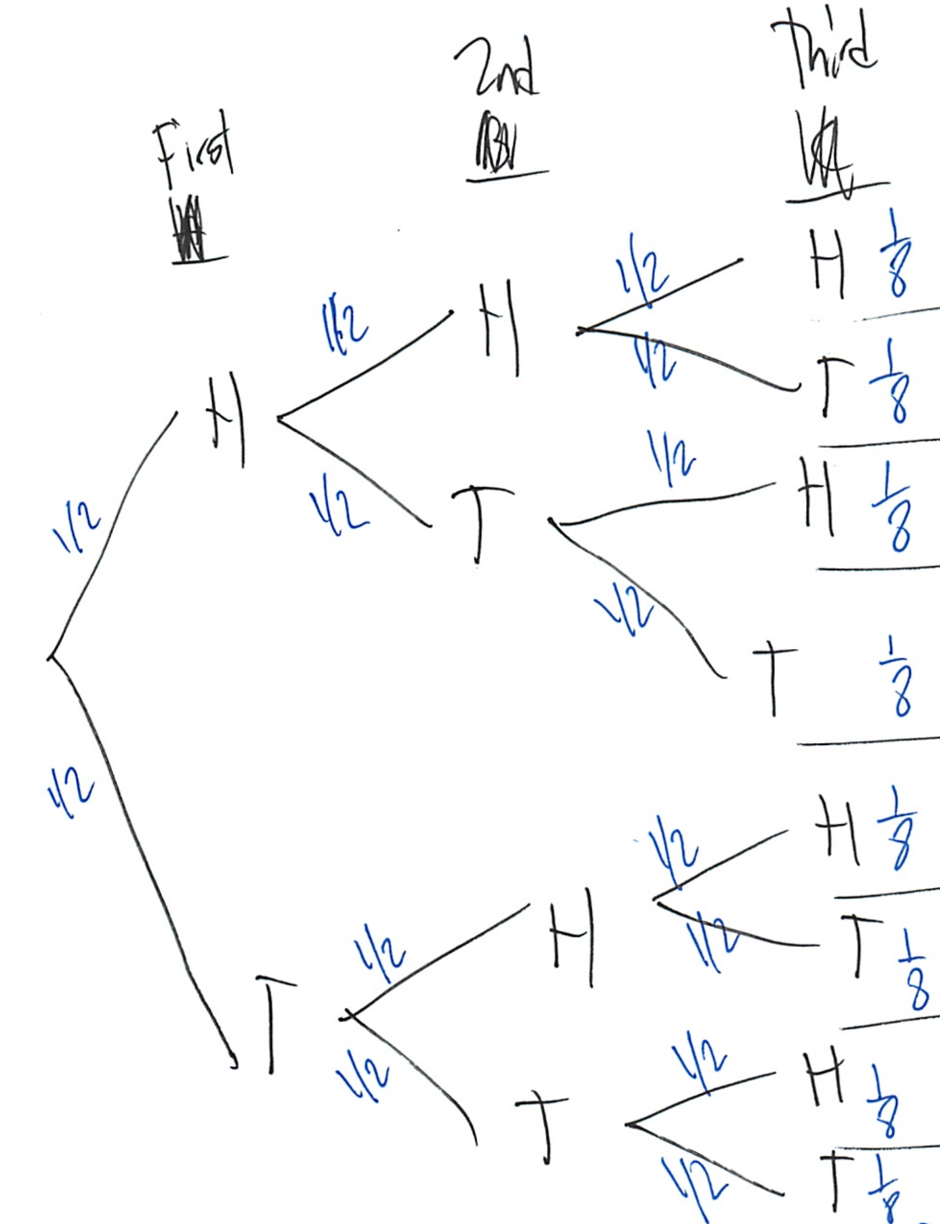
? what to put here

– no purpose since it does not chage anything

Is it his chance of being released or his knowledge / ~~alt~~ uncertainity.
For istance if told 2 – prob released still 2/3
But his knowledge is certain

3. A – first coin heads
B = 2nd coin heads
C = third coin
D = even # coin is heads

③



A	B	C	D
✓	✓	✓	
✓	✓		✓
✓		✓	✓
✓			
	✓	✓	✓
	✓		
		✓	

have it in case it was not all even/equal

b) $P(D \mid A)$ – since they control each other

A, B, C ind

How exactly to show

A, B, D mutually ind

(~~P(A)~~ $P(A \cap B \cap C) = P(A) P(B) P(C)$

Any set of 3 elements (thats ~~c~~)

(9)

4. A, B, C events

Prove or counter example

a. Ugg, here get into proving

How do you think about it?

· examples + counter examples?

Then how to write?

Start w/ equations

$P(A) \cdot P(B) \stackrel{?}{=} P(A \cap B)$

Allen did w/ die example

A = fare die roll is even

B = 1 or 2

C = 2 or 3

$P(A) = \frac{\{2,4,6\}}{\{1,2,3,4,5,6\}} = \frac{1}{2}$

$P(A|B) = \frac{\{2\}}{\{1,2\}} = \frac{1}{2}$

$P(A|C) = \frac{\{2\}}{\{2,3\}} = \frac{1}{2}$

5

$$P[A|B\cap C\} = \frac{2}{2} = 1 \neq \frac{1}{2}$$

$$P(A|B\cup C\} = \frac{\{2\}}{\{1,2,3\}} = \frac{1}{3} \neq \frac{1}{2}$$

So both were ~~counter examples~~ false - so could just provide the counter example

c) Given $P[A\cap B] = P[A]\,P[B]$

$P[A\cap C] = P[A]\,P[C]$

$P[A\cap B\cap C] = P[A]\,P[B\cap C]$

$$P[A\cap(B\cup C)] = P[A\cap(B + C - B\cup C)]$$

$$= P[A\cap B + A\cap C - A\cap B\cap C]$$

(6)

Should extend

$$P(A \text{ released} \mid \text{told } B) = \frac{\frac{1}{3}}{\frac{1}{3}+\frac{1}{6}} = \frac{2}{3}$$

not same thing

$$P(A \text{ released} \mid B \text{ released}) = \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{2}$$

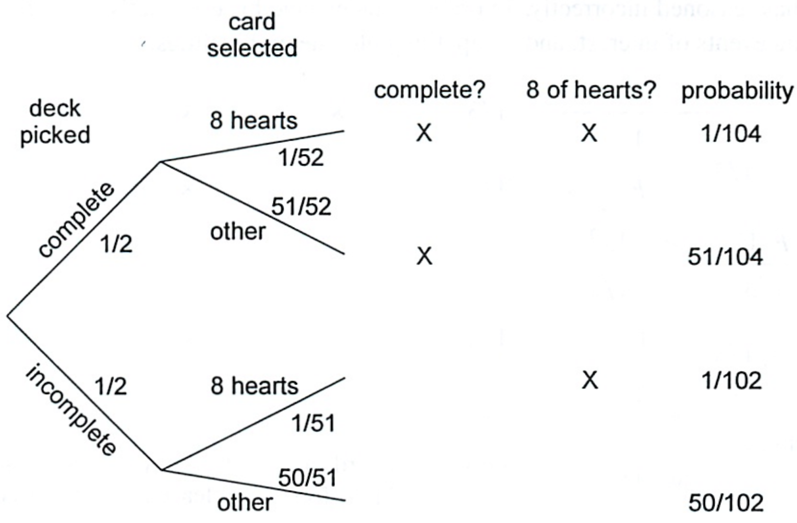
Solutions to In-Class Problems Week 12, Mon.

Problem 1.
There are two decks of cards. One is complete, but the other is missing the ace of spades. Suppose you pick one of the two decks with equal probability and then select a card from that deck uniformly at random. What is the probability that you picked the complete deck, given that you selected the eight of hearts? Use the four-step method and a tree diagram.

Solution. Let C be the event that you pick the complete deck, and let H be the event that you select the eight of hearts. In these terms, our aim is to compute:

$$\Pr[C \mid H] = \frac{\Pr[C \cap H]}{\Pr[H]}$$

A tree diagram is worked out below:



Now we can compute the desired conditional probability as follows:

$$\begin{aligned}
\Pr[C \mid H] &= \frac{\Pr[C \cap H]}{\Pr[H]} \\
&= \frac{\frac{1}{2} \cdot \frac{1}{52}}{\frac{1}{2} \cdot \frac{1}{52} + \frac{1}{2} \cdot \frac{1}{51}} \\
&= \frac{51}{103} \\
&= 0.495146\ldots
\end{aligned}$$

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Thus, if you selected the eight of hearts, then the deck you picked is less likely to be the complete one. It's worth thinking about how you might have arrived at this final conclusion without going through the detailed calculation.

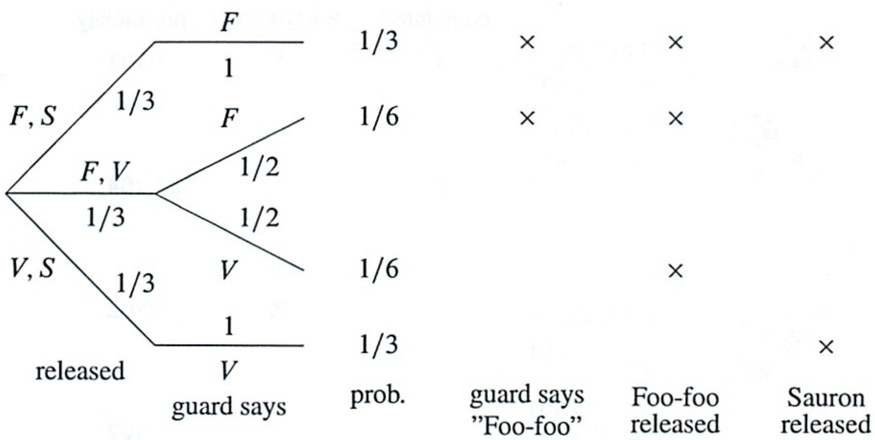
■

Problem 2.
There are three prisoners in a maximum-security prison for fictional villains: the Evil Wizard Voldemort, the Dark Lord Sauron, and Little Bunny Foo-Foo. The parole board has declared that it will release two of the three, chosen uniformly at random, but has not yet released their names. Naturally, Sauron figures that he will be released to his home in Mordor, where the shadows lie, with probability $2/3$.

A guard offers to tell Sauron the name of one of the other prisoners who will be released (either Voldemort or Foo-Foo). Sauron knows the guard to be a truthful fellow. However, Sauron declines this offer. He reasons that if the guard says, for example, "Little Bunny Foo-Foo will be released", then his own probability of release will drop to $1/2$. This is because he will then know that either he or Voldemort will also be released, and these two events are equally likely.

Using a tree diagram and the four-step method, either prove that the Dark Lord Sauron has reasoned correctly or prove that he is wrong. Assume that if the guard has a choice of naming either Voldemort or Foo-Foo (because both are to be released), then he names one of the two uniformly at random.

Solution. Sauron has reasoned incorrectly. In order to understand his error, let's begin by working out the sample space, noting events of interest, and computing outcome probabilities:



Define the events S, F, and "F" as follows:

$$\begin{aligned} \text{“}F\text{”} &= \text{Guard says Foo-Foo is released} \\ F &= \text{Foo-Foo is released} \\ S &= \text{Sauron is released} \end{aligned}$$

The outcomes in each of these events are noted in the tree diagram.

Sauron's error is in failing to realize that the event F (Foo-foo will be released) is different from the event "F" (the guard *says* Foo-foo will be released). In particular, the probability that Sauron is released, given that Foo-foo is released, is indeed $1/2$:

$$\begin{aligned}
\Pr\left[S \mid F\right] &= \frac{\Pr[S \cap F]}{\Pr[F]} \\
&= \frac{\frac{1}{3}}{\frac{1}{3} + \frac{1}{6} + \frac{1}{6}} \\
&= \frac{1}{2}
\end{aligned}$$

But the probability that Sauron is released given that the guard merely *says so* is still $2/3$:

$$\begin{aligned}
\Pr\left[S \mid \text{``}F\text{''}\right] &= \frac{\Pr[S \cap \text{``}F\text{''}]}{\Pr[\text{``}F\text{''}]} \\
&= \frac{\frac{1}{3}}{\frac{1}{3} + \frac{1}{6}} \\
&= \frac{2}{3}
\end{aligned}$$

So Sauron's probability of release is actually unchanged by the guard's statement. ■

Problem 3.
Suppose that you flip three fair, mutually independent coins. Define the following events:

- Let A be the event that *the first* coin is heads.
- Let B be the event that *the second* coin is heads.
- Let C be the event that *the third* coin is heads.
- Let D be the event that *an even number of* coins are heads.

(a) Use the four step method to determine the probability space for this experiment and the probability of each of A, B, C, D.

Solution. The tree is a binary tree with depth 3 and 8 leaves. The successive levels branching to show whether or not the successive events A, B, C occur. By definition of *fair* and *independent*, each branch out of a vertex is equally likely to be followed. So the probability space has as outcomes the eight length-3 strings of H's and T's, each of which has probability $(1/2)^3 = 1/8$.

Each of the events events A, B, C, D are true in four of the outcomes and hence has probability 1/2. ■

(b) Show that these events are not mutually independent.

Solution.

$$\Pr[A \cap B \cap C \cap D] = 0 \neq (1/2)^4 = \Pr[A] \cdot \Pr[B] \cdot \Pr[C] \cdot \Pr[D].$$

■

(c) Show that they are 3-way independent.

Solution. Because the coin tosses are mutually independent, we know:

$$\Pr[A \cap B \cap C] = \Pr[A] \cdot \Pr[B] \cdot \Pr[C].$$

What remains is to check that equality holds for the other subsets of three events: $\{A, B, D\}$, $\{A, C, D\}$, and $\{B, C, D\}$. By symmetry, again, we need only check one, say the first one.

$$\Pr[A \cap B \cap D] = \Pr[\{HHT\}] = \frac{1}{8}.$$

Since this is equal to $\Pr[A] \cdot \Pr[B] \cdot \Pr[D]$, these three events are independent.

We conclude that all four events are three-way independent. ■

Problem 4.
Let A, B, C be events. For each of the following statements, prove it or give a counterexample.

(a) If A is independent of B, and A is independent of C, then A is independent of $B \cap C$.

Solution. This is false. To see why, consider the usual random experiment of rolling a die and let A, B, C be the events that the die rolls less than 3, rolls an even number, or rolls a prime number, respectively, namely,

$$A = \{1, 2\} \qquad B = \{2, 4, 6\} \qquad C = \{2, 3, 5\}.$$

Then,

$$\Pr[A] = \tfrac{1}{3}, \qquad \Pr[B] = \Pr[C] = \tfrac{1}{2},$$

and

$$\Pr[A \cap B] = \Pr[A \cap C] = \Pr[B \cap C] = \Pr[A \cap B \cap C] = \Pr[\{2\}] = \tfrac{1}{6}.$$

Easily,

$$\Pr\left[A \mid B\right] = \Pr\left[A \mid C\right] = \tfrac{1}{6} / \tfrac{1}{2} = \tfrac{1}{3} = \Pr[A],$$

which proves A is independent of B and also independent of C. However,

$$\Pr\left[A \mid B \cap C\right] = \Pr[A \cap B \cap C] / \Pr[B \cap C] = \tfrac{1}{6} / \tfrac{1}{6} = 1 \neq \Pr[A],$$

which implies A is not independent of $B \cap C$. ■

(b) If A is independent of B, and A is independent of C, then A is independent of $B \cup C$.

Solution. This is false using the same example as for part (a), where

$$\Pr[B \cup C] = \tfrac{5}{6}, \qquad \Pr[A \cap (B \cup C)] = \tfrac{1}{6}.$$

Hence,

$$\Pr\left[A \mid B \cup C\right] = \tfrac{1}{6} / \tfrac{5}{6} = \tfrac{1}{5} \neq \Pr[A]$$

which again implies A is not independent of $B \cup C$. ■

(c) If A is independent of B, and A is independent of C, and A is independent of $B \cap C$, then A is independent of $B \cup C$.

Solution. This is true. To prove it, suppose A is independent of each of B, C, and $B \cap C$, so

$$\Pr[A \cap B] = \Pr[A]\Pr[B] \tag{1}$$
$$\Pr[A \cap C] = \Pr[A]\Pr[C] \tag{2}$$
$$\Pr[A \cap (B \cap C)] = \Pr[A]\Pr[B \cap C]. \tag{3}$$
$$\tag{4}$$

Then,

$$\begin{aligned}
\Pr[A \cap (B \cup C)] &= \Pr[(A \cap B) \cup (A \cap C)] \\
&= \Pr[A \cap B] + \Pr[A \cap C] - \Pr[(A \cap B) \cap (A \cap C)] && \text{(inclusion-exclusion)} \\
&= \Pr[A \cap B] + \Pr[A \cap C] - \Pr[A \cap B \cap C] \\
&= \Pr[A]\Pr[B] + \Pr[A]\Pr[C] - \Pr[A]\Pr[B \cap C] && \text{(by (1), (2), (3))} \\
&= \Pr[A](\Pr[B] + \Pr[C] - \Pr[B \cap C]) \\
&= \Pr[A]\Pr[B \cup C]. && \text{(inclusion-exclusion)}
\end{aligned}$$

■

4/27

Mathematics for Computer Science
MIT 6.042J/18.062J

Introduction to Random Variables

Guess the Bigger Number

Team 1:

- Write different integers between 0 and 7 on two pieces of paper
- Show to Team 2 face down

Team 2:

- Expose one paper and look at number
- Either *stick* or *switch* to other number

Team 2 wins if gets larger number

Strategy for Team 2

- pick a paper to expose, giving each paper equal probability.
- if exposed number is "small" then switch, otherwise stick. That is switch if $\leq$ threshold Z where Z is a random integer, $0 \leq Z \leq 6$.

Analysis of Team 2 Strategy

Case M: low $\leq Z <$ high

Team 2 wins in this case, so

Pr{Team 2 wins | M} = 1

and $\Pr\{M\} \geq \frac{1}{7}$

Analysis of Team 2 Strategy

Case H: high $\leq Z$

Team 2 will switch, so wins iff low card gets exposed

$\Pr\{\text{Team 2 wins} \mid H\} = \frac{1}{2}$

Analysis of Team 2 Strategy

Case L: $Z <$ low

Team 2 will stick, so wins iff high card gets exposed

$\Pr\{\text{Team 2 wins} \mid L\} = \frac{1}{2}$

Analysis of Team 2 Strategy

So $\geq 1/7$ of time, sure win.

Rest of time, win 1/2.

by Law of Total Probability

Analysis of Team 2 Strategy

So $\geq 1/7$ of time, sure win.

Rest of time, win 1/2.

Pr{Team 2 wins} =

Pr{win|Z good}·Pr{Z good} +

Pr{win|Z no good}·Pr{Z no good}

Analysis of Team 2 Strategy

So $\geq 1/7$ of time, sure win.

Rest of time, win 1/2.

Pr{Team 2 wins} $\geq$

$$\frac{1}{7}\cdot 1+\left(1-\frac{1}{7}\right)\cdot\frac{1}{2}=\frac{4}{7}$$

Analysis of Team 2 Strategy

Does not matter what Team 1 does!

Team 1 Strategy

...& Team 1 can play so

Pr{Team 2 wins} $\leq \frac{4}{7}$

whatever Team 2 does

Random Variables

Informally: an RV is a number produced by a *random process:*

- threshold variable Z
- number of larger card
- number of smaller card
- number of exposed card

Random Variables

Informally: an RV is a number produced by a *random process:*

- #hours to next system crash
- #faulty chips in production run
- avg # faulty chips in many runs
- #heads in n coin flips

Intro to Random Variables

Example: Flip three fair coins

C ::= # heads (**C**ount)

$$M ::= \begin{cases} 1 & \text{if all } \textbf{M}\text{atch,} \\ 0 & \text{otherwise.} \end{cases}$$

Intro to Random Variables

Specify events using values of variables

- $[C = 1]$ is event "exactly 1 head"

  $\Pr\{C = 1\} = 3/8$
- $\Pr\{C \geq 1\} = 7/8$
- $\Pr\{C \cdot M > 0\} = \Pr\{M > 0 \text{ and } C > 0\}$

  $= \Pr\{\text{all heads}\} = 1/8$

What is a Random Variable?

Formally,

$$R: S \to \mathbb{R}$$

Sample space

(usually)

Independent Variables

random variables R,S are independent iff

$[R = a], [S = b]$

are independent events for all a, b

Independent Variables

alternate version:

$\Pr\{R = a \text{ AND } S = b\} =$
$\Pr\{R = a\} \cdot \Pr\{S = b\}$

Independent Variables

Are C and M independent? NO:

$\Pr\{M=1\} \cdot \Pr\{C=1\} > 0$

$\Pr\{M=1 \text{ and } C=1\} = 0$

Indicator Variables

The indicator variable for event A:

$$I_A ::= \begin{cases} 1 & \text{if } A \text{ occurs,} \\ 0 & \text{if } A \text{ does not occur.} \end{cases}$$

(Sanity check:

I_A and I_B are independent iff A and B are independent.)

Mutally Independent Variables

Def: $R_1, R_2, \ldots, R_n$ are mutually indep RV's iff

$[R_1=a_1], [R_2=a_2], \ldots, [R_n=a_n]$

are mutually indep events for all $a_1, a_2, \ldots, a_n$

Mutally Independent Variables

$$\Pr\{R_1=a_1 \text{ AND } R_2=a_2 \text{ AND } \cdots \text{ AND } R_n=a_n\}$$
$$= \Pr\{R_1=a_1\} \cdot \Pr\{R_2=a_2\} \cdot \cdots \Pr\{R_n=a_n\}$$

Binomial Random Variable

$B_{n,p}$ ::= # heads in n mutually indep flips.

Coin may be biased. So 2 parameters

n ::= # flips, p ::= Pr{head}

C is binomial for 3 flips: C is $B_{3,1/2}$

for n=5, p=2/3

$\Pr\{HHTTH\} =$

$\Pr\{H\} \cdot \Pr\{H\} \cdot \Pr\{T\} \cdot \Pr\{T\} \cdot \Pr\{H\}$

(by independence)

Binomial Random Variable

$B_{n,p}$ ::= # heads in n mutually indep flips.

Coin may be biased. So 2 parameters

n ::= # flips, p ::= Pr{head}

C is binomial for 3 flips: C is $B_{3,1/2}$

for n=5, p=2/3

$\Pr\{HHTTH\} =$

$$\frac{2}{3} \cdot \frac{2}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{2}{3}$$

Binomial Random Variable

$B_{n,p}$ ::= # heads in n mutually indep flips.
Coin may be biased. So 2 parameters
n ::= # flips, p ::= Pr{head}
C is binomial for 3 flips: C is $B_{3,1/2}$
for n=5, p=2/3

$$\Pr\{HHTTH\} = \left(\frac{2}{3}\right)^3 \cdot \left(\frac{1}{3}\right)^2$$

Binomial Random Variable

$B_{n,p}$ ::= # heads in n mutually indep flips.
Coin may be biased. So 2 parameters
n ::= # flips, p ::= Pr{head}
Pr{each sequence w/i H's, n-i T's} =

$$p^i(1-p)^{n-i}$$

Binomial Random Variable

$B_{n,p}$ ::= # heads in n mutually indep flips.
Coin may be biased. So 2 parameters
n ::= # flips, p ::= Pr{head}
Pr{get i H's, n-i T's} = #seq's · pr[seq]

$$\binom{n}{i} p^i(1-p)^{n-i}$$

Binomial Random Variable

$B_{n,p}$ ::= # heads in n mutually indep flips.
Coin may be biased. So 2 parameters
n ::= # flips, p ::= Pr{head}
Pr{ $B_{n,p} = i$ } = #seq's · pr{seq}

$$\binom{n}{i} p^i(1-p)^{n-i}$$

Density & Distribution

Probability Density Function
of random variable R,

$$PDF_R(a) ::= \Pr\{R = a\}$$

so $$PDF_{B_{n,p}}(i) = \binom{n}{i} p^i(1-p)^{n-i}$$

Uniform Distribution

...all values equally likely.
"threshold" variable was uniform:

$$PDF_Z(i) ::= \Pr\{Z = i\} = \frac{1}{7}$$

for i = 0,1,...,6.

Uniform Distribution

R is uniform iff PDF_R is constant

D ::= outcome of fair die roll

$\Pr\{D=1\} = \Pr\{D=2\} = \cdots = \Pr\{D=6\} = 1/6$

S ::= 4-digit lottery number

$\Pr\{S = 0000\} = \Pr\{S = 0001\} = \cdots$

$= \Pr\{S = 9999\} = 1/10000$

Mutual Independence

Given mutually indep RV's $R_1, R_2, \ldots$

$[R_1=R_2]$ indep of $[R_3=R_4]$?

obviously!

Mutual Independence

Given mutually indep RV's $R_1, R_2, \ldots$

$[R_1=R_2]$ indep of $[R_3=R_2]$?

YES as long as one of the R_i's is uniform

Mutual Independence

Given mutually indep RV's $R_1, R_2, \ldots$

$[R_i=R_j]$ indep of $[R_k=R_l]$

for $(i,j) \neq (k,l)$ if one of the R's is uniform

they are pairwise indep

Mutual Independence

Given mutually indep RV's $R_1, R_2, \ldots$

not 3-way independent

$R_1=R_2$ and $R_3=R_2$

implies $R_1=R_3$

Team Problems

Problems

1—4

6.042 RVs

4/27

event

Outcome - set of events

RV - get a value out - not just yes, no

Guess The Bigger # Game

Team 1 writes diff int b/w 0, 7 on 2 pages

Show to Team 2 face down

Team 2 Picks a paper + looks at #

Can stick or switch

Team 2 wins if have larger #

So if 0 is picked - will always switch, always win

" 7 " stay, " "

If 6 prob larger #, so prob stick

But the minute you comm'it to that strategy

Team 1 knows this and puts 6, 7 together

But 7 not very good since 50-50 you will pick

②

Strategy for team 2

- game actually favors team 2
- even though team 2 appears to have an advantage

Switch if "small", otherwise ~~stick~~ stick

but what is "small"

Can change in a random way.

- random, probabalistic

Can be any # $0 \le z \le 6$

7 is def big

Choose z w/ = prob

Suppose Team 1 ...

$p($Team 2 wins$| n \} = 1$

low $\le z <$ high

and $p(n) = \frac{1}{7}$

When z is bad guess win half the time

Will always if turn over higher

$p($losing$)$

:-(lost track of argument)

3

You can play for a while + not ~~a~~ figure it out

Could they do better than $\frac{4}{7}$
Team 2

No since ~~ea~~ Team 1 has blocking strategy
- w/ $\frac{3}{7}$

RV

A RV is a # produced by random process

on card may not be random

But threshhold was cards

Could also do system crashing

RV $\Rightarrow$ # hrs ~~time~~ till system crashes

Flip 3 coins

$C = \#$ heads

$M = \begin{cases} 1 & \text{if match} \\ 0 & \text{otherwise} \end{cases}$ ← RV way of representing events

...

Can use to define all types of properties

(4)

Formally

$R: S \to \mathbb{R}$

a fn that sets a Real # for every possible value of R in the sample space

Independent

two RVs are ind iff

alt version $P(R = a \text{ AND } S = b) =$
$P(R = a) \cdot P(S = b)$

Indicator RVs

$$I_A = \begin{cases} 1 & \text{if } A \text{ occurs} \\ 0 & \text{if } \bar{A} \text{ occurs} \end{cases}$$

RV is generalization of event

2 ind. def of indpendence
- indicator RVs must be proven that ind.

I_A and I_B are ind if A, B ind

(5)

Are C, M ind ?

no, dep

if know one, you know the other

Mutual Ind

choose $V_1, V_2, \ldots$ ind ~~A~~ w/ = prob

for events $V_m = V_n$

any 2 are ind

but $[V_1 = V_2]$, $[V_2 = V_3]$, $[V_1 = V_3]$

are <u>not</u> mutually ind

Binomial RV

$B_{n,p}$ = # heads in n ind flips

Can you get 50 pennies on edge at once

Whats the prob of head

But most table had 33

2 to 1

Since way pennyness are built - are not fair

6

$n = \#$ flips $\qquad p = P(\text{heads})$

defines a RV C = binomial for 3 flips C is $B_{3,\frac{1}{2}}$

for $n=5$, $p=\frac{2}{3}$

not $\frac{1}{32}$

$$P(HHTTH)$$

by ind.

$$= P(H)\,P(H)\,P(T)\,P(T)\,P(H)$$

$$= \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{2}{3}$$

$$= p^i (1-p)^{n-i}$$

$P(i$ heads from n flips$) =$

\# seq w/ i heads, $n-i$ tails $\cdot$

prob of each one of them - is same

$$P(B=i) = \binom{n}{i} p^i (1-p)^{n-i}$$

should be on crib sheet

7

PDF

- slight abstraction of RV

$PDF_R(a) = P(R=a)$

Mapping b/w #s and $[0,1]$
no more sample space

CDF

Cumulative distribution function

$CDF_R(a) = P(R \le a)$

Uniform Dist

Threshhold variable was uniform

$PDF_Z(i) = \ldots$
for $i = 0, 1, \ldots, 6$

uniform means all variables = likely

I did not pay perfect attention to this lecture

In-Class Problems Week 12, Wed.

Guess the Bigger Number Game

Team 1:

- Write different integers between 0 and 7 on two pieces of paper.
- Put the papers face down on a table.

Team 2:

- Turn over one paper and look at the number on it.
- Either stick with this number or switch to the unseen other number.

Team 2 wins if it chooses the larger number.

Problem 1.
The analysis in section 17.3.3 implies that Team 2 has a strategy that wins 4/7 of the time no matter how Team 1 plays. Can Team 2 do better? The answer is "no," because Team 1 has a strategy that guarantees that it wins at least 3/7 of the time, no matter how Team 2 plays. Describe such a strategy for Team 1 and explain why it works.

Problem 2.
Suppose X_1, X_2, and X_3 are three mutually independent random variables, each having the uniform distribution

$$\Pr[X_i = k] \text{ equal to } 1/3 \text{ for each of } k = 1, 2, 3.$$

Let M be another random variable giving the maximum of these three random variables. What is the density function of M?

Problem 3. **(a)** Prove that if A and B are independent events, then so are A and $\overline{B}$.

(b) Let I_A and I_B be the indicator variables for events A and B. Prove that I_A and I_B are independent iff A and B are independent.

Hint: For any event, E, let $E^1 ::= E$ and $E^0 ::= \overline{E}$. So the event $[I_E = a]$ is the same as E^a.

Problem 4.
Let R, S, and T be random variables with the same codomain, V.

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(a) Suppose R is uniform —that is,

$$\Pr[R = b] = \frac{1}{|V|},$$

for all $b \in V$.

Suppose R is independent of S. We originally had the following argument in the class text:

> The probability that $R = S$ is the same as the probability that R takes whatever value S happens to have, therefore
>
> $$\Pr[R = S] = \frac{1}{|V|}. \tag{1}$$

Are you convinced by this argument? We decided to replace it by a reference to this problem. We'd like your advice on whether it should be put back it the text. Before advising us, write out a careful proof of (1).

Hint: The event $[R = S]$ is the same as the disjoint union of events $[R = b \text{ AND } S = b]$ for $b \in V$.

Definition. A random variable, R, is independent of a set $\{R_1, R_2, \ldots\}$ of random variables iff the event $[R = r]$ is independent of the event

$$[R_1 = r_1 \text{ AND } R_2 = r_2 \text{ AND} \cdots]$$

for all values $r, r_1, r_2, \ldots$.

(b) Now suppose R has a uniform distribution, and R is independent of $\{S, T\}$. How about this argument?

> The probability that $R = S$ is the same as the probability that R takes whatever value S and T happen to have in common, and this probability remains equal to $1/|V|$ by independence. Therefore the event $[R = S]$ is independent of $[S = T]$.

Write out a careful proof that $[R = S]$ is independent of $[S = T]$.

(c) Let $V = \{1, 2, 3\}$ and R, S, T take the following values with equal probability,

$$111, 211, 123, 223, 132, 232.$$

Verify that

1. R is independent of $\{S, T\}$,
2. The event $[R = S]$ is not independent of $[S = T]$.
3. S and T have a uniform distribution,

In Class 12 Wed

4/27

1. Can Team 2 do better?

No since team 1 has counter strategy that works $\frac{3}{7}$.

What is it?

Pick adj pairs w/ = prob

Team 2 will see # w/ = prob

So higher-lower is exactly $\frac{1}{2}$ each

0,7 100% chance winning

↑ $\frac{2}{14}$ ← 14 possible cards

2x as likely to see 1 than 0 w/ this strategy

Don't want # far apart

- ~~the~~ Team 2 lecture $\frac{4}{7}$ if close
- if further apart - gets worse

Team 2 can't get better

Team 1's strategy wins $\frac{3}{7}$ no matter what team 2 does

②

As $n \to \infty$ becomes fair

Or change roles every match

Or ~~...~~ ~~a~~ is diff

Or ~~...~~ honor multiplier

2. Write out all possibilities

$$\frac{1}{3^3} = \frac{1}{27} \leftarrow \text{prob outcomes}$$

3. ~~a~~ axiom? from 6.041?

b) again how to show?

2 board) $2^3 = 8$ possible outcomes $m \neq 3$

$$P(M=3) = 1 - \frac{8}{27} = \frac{19}{27}$$

If $M=1$ then $X_1 = X_2 = X_3 = 1 \to 1$ outcome

$$P(M=1) = \frac{1}{27}$$

Law Total Prob $P(M=2) = 1 - \frac{19}{27} - \frac{1}{27} = \frac{7}{27}$

So $PDF_M(a) = \begin{cases} \frac{1}{27} & a=1 \\ \frac{7}{27} & a=2 \\ 19/27 & a=3 \end{cases}$ 0 otherwise

3

3b E $\quad E' ::= E$

$E^0 ::= \overline{E}$) what? why?

4. R, S, T

a) suppose R uniform
ind S

Yeah

$[R = S]$ means $R = b$ and $S = b$ for $b \in N$

? what does this mean?
they have the same value?

Defn further down

Solutions to In-Class Problems Week 12, Wed.

Guess the Bigger Number Game

Team 1:

- Write different integers between 0 and 7 on two pieces of paper.
- Put the papers face down on a table.

Team 2:

- Turn over one paper and look at the number on it.
- Either stick with this number or switch to the unseen other number.

Team 2 wins if it chooses the larger number.

Problem 1.
The analysis in section 17.3.3 implies that Team 2 has a strategy that wins 4/7 of the time no matter how Team 1 plays. Can Team 2 do better? The answer is "no," because Team 1 has a strategy that guarantees that it wins at least 3/7 of the time, no matter how Team 2 plays. Describe such a strategy for Team 1 and explain why it works.

Solution. Team 1 should randomly choose a number $Z \in \{0, \dots, 6\}$ and write Z and $Z + 1$ on the pieces of paper with all numbers equally likely.

To see why this works, let N be the number on the paper that Team 2 turns over, and let OK be the event that $N \in \{1, \dots, 6\}$. So given event OK, that is, given that $N \in \{1, \dots, 6\}$, Team 1's strategy ensures that half the time N is the higher number and half the time N is the lower number. So given event OK, the probability that Team 1 wins is exactly 1/2 *no matter how Team 2 chooses to play* (stick or switch).

Now we claim that

$$\Pr[\text{OK}] = \frac{6}{7}, \tag{1}$$

which implies that the probability that Team 1 wins is indeed at least $(1/2)(6/7) = 3/7$.

To prove $\Pr[\text{OK}] = 6/7$, we can follow the four step method. (Note that we couldn't apply this method to model the behavior of Team 2, since we don't know how they may play, and so we can't let our analysis depend on what they do.)

The first level of the probability tree for this game will describe the value of Z: there are seven branches from the root with equal probability going to first level nodes corresponding to the seven possible values of Z. The second level of the tree describes the choice of the number, N: each of the seven first-level nodes has two branches with equal probability, one branch for the case that $N = Z$ and the other for the case that $N = Z + 1$. So there are 14 outcome (leaf) nodes at the second level of the tree, each with probability 1/14.

Now only two outcomes are not OK, namely, when $Z = 6$ and $N = 7$, and when $Z = 0$ and $N = 0$. Each of the other twelve outcomes is OK, and since each has probability 1/14, we conclude that $\Pr[\text{OK}] = 12/14 = 6/7$, as claimed. ■

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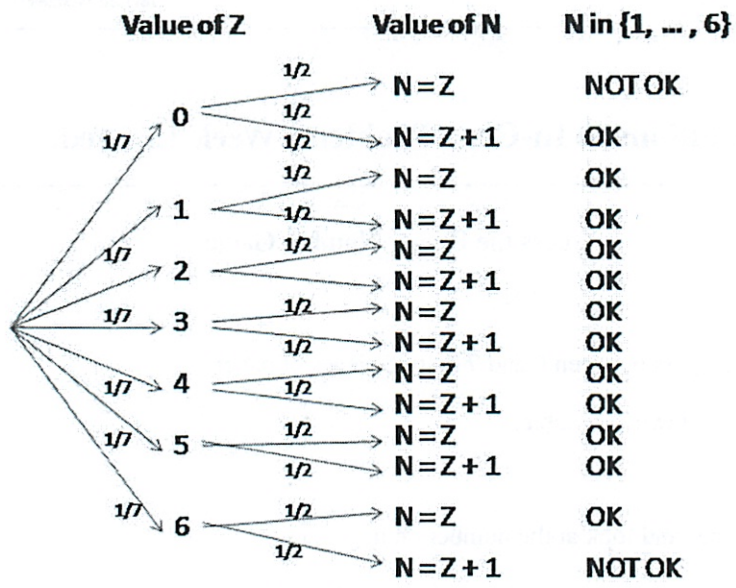

Problem 2.
Suppose X_1, X_2, and X_3 are three mutually independent random variables, each having the uniform distribution

$$\Pr[X_i = k] \text{ equal to } 1/3 \text{ for each of } k = 1, 2, 3.$$

Let M be another random variable giving the maximum of these three random variables. What is the density function of M?

Solution.

$$\mathrm{PDF}_M(1) = \frac{1}{27}$$
$$\mathrm{PDF}_M(2) = \frac{7}{27}$$
$$\mathrm{PDF}_M(3) = \frac{19}{27}$$

This can be hashed out by counting the possible outcomes. Alternatively, we can reason as follows:

The event $M = 1$ is the event that all three of the variables equal 1, and since they are mutually independent, we have

$$\Pr[M = 1] = \Pr[X_1 = 1] \cdot \Pr[X_2 = 1] \cdot \Pr[X_3 = 1] = \left(\frac{1}{3}\right)^3 = \frac{1}{27}.$$

To compute $\Pr[M = 2]$, we first compute $\Pr[M \leq 2]$. Now the event $[M \leq 2]$ is the event that all three of the variables is at most 2, so by mutual independence we have

$$\Pr[M \leq 2] = \Pr[X_1 \leq 2] \cdot \Pr[X_2 \leq 2] \cdot \Pr[X_3 \leq 2] = \left(\frac{2}{3}\right)^3 = \frac{8}{27}.$$

Therefore,

$$\Pr[M = 2] = \Pr[M \leq 2] - \Pr[M = 1] = \frac{8}{27} - \frac{1}{27} = \frac{7}{27}.$$

Finally,

$$\Pr[M = 3] \;=\; 1 - \Pr[M \le 2] = 1 - \frac{8}{27} = \frac{19}{27}.$$

■

Problem 3. (a) Prove that if A and B are independent events, then so are A and $\overline{B}$.

Solution. *Proof.*

$$\begin{aligned}
\Pr[A \cap \overline{B}] &= \Pr[A] - \Pr[A \cap B] && \text{(difference rule)} \\
&= \Pr[A] - \Pr[A] \cdot \Pr[B] && \text{(independence of } A \text{ and } B\text{)} \\
&= \Pr[A](1 - \Pr[B]) \\
&= \Pr[A] \cdot \Pr[\overline{B}] && \text{(complement rule).}
\end{aligned}$$

■

(b) Let I_A and I_B be the indicator variables for events A and B. Prove that I_A and I_B are independent iff A and B are independent.

Hint: For any event, E, let $E^1 ::= E$ and $E^0 ::= \overline{E}$. So the event $[I_E = a]$ is the same as E^a.

Solution. *Proof.* By part (a) and the fact that $\overline{\overline{E}} = E$, the following propositions are equivalent:

- A and B are independent,
- $\exists a, b \in \{0, 1\}.[A^a$ and B^b are independent],
- $\forall a, b \in \{0, 1\}.[A^a$ and B^b are independent].

Therefore, the following propositions are equivalent as well:

- I_A and I_B are independent,
- $\forall a, b \in \{0, 1\}$. [[$I_A = a$] and [$I_A = b$] are independent events] —by definition of independence for random variables,
- $\forall a, b \in \{0, 1\}$. [$A^a$ and B^b are independent],
- A and B are independent —(by part (a)).

■

Problem 4.
Let R, S, and T be random variables with the same codomain, V.

(a) Suppose R is uniform —that is,

$$\Pr[R = b] = \frac{1}{|V|},$$

for all $b \in V$.

Suppose R is independent of S. Originally this text had the following argument:

> The probability that $R = S$ is the same as the probability that R takes whatever value S happens to have, therefore
>
> $$\Pr[R = S] = \frac{1}{|V|}. \tag{2}$$

Are you convinced by this argument? We decided to replace it by a reference to this problem. We'd like your advice on whether it should be put back it the text. Before advising us, write out a careful proof of (2).

Hint: The event $[R = S]$ is the same as the disjoint union of events $[R = b \text{ AND } S = b]$ for $b \in V$.

Solution. *Proof.*

$$\begin{aligned}
\Pr[R = S] &= \Pr \bigcup_{b \in V} [R = b \text{ AND } S = b] \\
&= \sum_{b \in V} \Pr[R = b \text{ AND } S = b] && \text{(disjoint sum rule)} \\
&= \sum_{b \in V} \Pr[R = b] \cdot \Pr[S = b] && (R, S \text{ independent}) \\
&= \sum_{b \in V} \frac{1}{|V|} \cdot \Pr[S = b] && (R \text{ is uniform}) \\
&= \frac{1}{|V|} \cdot \sum_{b \in V} \Pr[S = b] \\
&= \frac{1}{|V|} \cdot 1 = \frac{1}{|V|}.
\end{aligned}$$

This proves (2). ■

We're now leaning toward putting the argument back in the text —along with a reference to a problem asking for the proof above.

■

Definition. A random variable, R, is independent of a set $\{R_1, R_2, \ldots\}$ of random variables iff the event $[R = r]$ is independent of the event

$$[R_1 = r_1 \text{ AND } R_2 = r_2 \text{ AND } \cdots]$$

for all values $r, r_1, r_2, \ldots$.

(b) Now suppose R has a uniform distribution, and R is independent of $\{S, T\}$. How about this argument?

> The probability that $R = S$ is the same as the probability that R takes whatever value S and T happen to have in common, and this probability remains equal to $1/|V|$ by independence. Therefore the event $[R = S]$ is independent of $[S = T]$.

Write out a careful proof that $[R = S]$ is independent of $[S = T]$.

Solution. We must show that:

$$\Pr[[R = S] \cap [S = T]] = \Pr[R = S] \cdot \Pr[S = T]. \tag{3}$$

Proof.

$$
\begin{aligned}
&\Pr[[R = S] \cap [S = T]] \\
&= \Pr[R = S \text{ AND } S = T] \\
&= \Pr\left[\bigcup_{b \in V} [R = b \text{ AND } S = b \text{ AND } T = b]\right] \\
&= \sum_{b \in V} \Pr[R = b \text{ AND } S = b \text{ AND } T = b] && \text{(disjoint sum rule)} \\
&= \sum_{b \in V} \Pr[R = b] \cdot \Pr[S = b \text{ AND } T = b] && \text{(R independent of $\{S, T\}$)} \\
&= \sum_{b \in V} \frac{1}{|V|} \cdot \Pr[S = b \text{ AND } T = b] && \text{(R is uniform)} \\
&= \frac{1}{|V|} \cdot \sum_{b \in V} \Pr[S = b \text{ AND } T = b] \\
&= \frac{1}{|V|} \cdot \Pr \bigcup_{b \in V} [S = b \text{ AND } T = b] && \text{(disjoint sum rule)} \\
&= \frac{1}{|V|} \cdot \Pr[S = T] \\
&= \Pr[R = S] \cdot \Pr[S = T], && \text{(part (a))}
\end{aligned}
$$

which proves (3). ■

(c) Let $V = \{1, 2, 3\}$ and R, S, T take the following values with equal probability,

$$111, 211, 123, 223, 132, 232.$$

Verify that

1. R is independent of $\{S, T\}$,
2. The event $[R = S]$ is not independent of $[S = T]$.
3. S and T have a uniform distribution,

Solution. To prove independence, note that $1, s, t$ is a possible sequence of values R, S, T iff $2, s, t$ and $1, t, s$ are also possible. This implies that

$$\Pr[R = 1] = \Pr[R = 2] = \frac{1}{2}.$$

It also implies that if s, t are possible values for S, T, then

$$\Pr[S = s \text{ AND } T = t] = \frac{1}{3}.$$

So if i, s, t are possible values for R, S, T, then

$$\Pr[R = i \text{ AND } S = s \text{ AND } T = t] = \frac{1}{6} = \frac{1}{2} \cdot \frac{1}{3} = \Pr[R = i] \cdot \Pr[S = s \text{ AND } T = t].$$

Likewise, if i, s, t are *not* possible values for R, S, T, then either $\Pr[R = i] = 0$ because $i = 3$, or else $\Pr[S = s \text{ AND } T = t] = 0$ because s, t are not possible values for S, T. So in this case

$$\Pr[R = i \text{ AND } S = s \text{ AND } T = t] = 0 = \Pr[R = i] \cdot \Pr[S = s \text{ AND } T = t].$$

This proves 1.

Finally, there are two outcomes out of six with $R = S$ and and two outcomes with with $S = T$, so $\Pr[R = S] = 1/3 = \Pr[S = T]$. But the only outcome in $[R = S] \cap [S = T]$ is 111, so

$$\Pr[[R = S] \cap [S = T]] = \frac{1}{6} \neq \frac{1}{9} = \Pr[R = S] \cdot \Pr[S = T].$$

This proves 2.

There are two outcomes with $S = i$ for each $i \in V$, so $\Pr[S = i] = 1/3$ for all $i \in V$, that is, S is uniform. Likewise for T. This proves 3.

■

Problem Set 10

Due: April 29

Reading: Chapter 16, Intro to Probability; Chapter 17, Random Variables

Problem 1.

[The Four-Door Deal]

Let's see what happens when *Let's Make a Deal* is played with **four** doors. A prize is hidden behind one of the four doors. Then the contestant picks a door. Next, the host opens an unpicked door that has no prize behind it. The contestant is allowed to stick with their original door or to switch to one of the two unopened, unpicked doors. The contestant wins if their final choice is the door hiding the prize.

Let's make the same assumptions as in the original problem:

1. The prize is equally likely to be behind each door.
2. The contestant is equally likely to pick each door initially, regardless of the prize's location.
3. The host is equally likely to reveal each door that does not conceal the prize and was not selected by the player.

Use The Four Step Method of Section 16.2 to find the following probabilities. The tree diagram may become awkwardly large, in which case just draw enough of it to make its structure clear.

(a) Contestant Stu, a sanitation engineer from Trenton, New Jersey, stays with his original door. What is the probability that Stu wins the prize?

(b) Contestant Zelda, an alien abduction researcher from Helena, Montana, switches to one of the remaining two doors with equal probability. What is the probability that Zelda wins the prize?

Now let's revise our assumptions about how contestants choose doors. Say the doors are labeled A, B, C, and D. Suppose that Carol always opens the *earliest* door possible (the door whose label is earliest in the alphabet) with the restriction that she can neither reveal the prize nor open the door that the player picked.

This gives contestant Mergatroid —an engineering student from Cambridge, MA —just a little more information about the location of the prize. Suppose that Mergatroid always switches to the earliest door, excluding his initial pick and the one Carol opened.

(c) What is the probability that Mergatroid wins the prize?

Problem 2.

You are organizing a neighborhood census and instruct your census takers to knock on doors and note the sex of any child that answers the knock. Assume that there are two children in a household and that girls and boys are equally likely to be children and to open the door.

A sample space for this experiment has outcomes that are triples whose first element is either `B` or `G` for the sex of the elder child, likewise for the second element and the sex of the younger child, and whose third coordinate is `E` or `Y` indicating whether the *e*lder child or *y*ounger child opened the door. For example, (`B`, `G`, `Y`) is the outcome that the elder child is a boy, the younger child is a girl, and the girl opened the door.

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(a) Let T be the event that the household has two girls, and O be the event that a girl opened the door. List the outcomes in T and O.

(b) What is the probability $\Pr\left[T \mid O\right]$, that both children are girls, given that a girl opened the door?

(c) Where is the mistake in the following argument?

> If a girl opens the door, then we know that there is at least one girl in the household. The probability that there is at least one girl is
>
> $$1 - \Pr[\text{both children are boys}] = 1 - (1/2 \times 1/2) = 3/4. \tag{1}$$
>
> So,
>
> $$\begin{aligned} &\Pr\left[T \mid \text{there is at least one girl in the household}\right] && (2)\\ &= \frac{\Pr[T \cap \text{there is at least one girl in the household}]}{\Pr[\text{there is at least one girl in the household}]} && (3)\\ &= \frac{\Pr[T]}{\Pr[\text{there is at least one girl in the household}]} && (4)\\ &= (1/4)/(3/4) = 1/3. && (5)\end{aligned}$$
>
> Therefore, given that a girl opened the door, the probability that there are two girls in the household is 1/3.

Problem 3.
There is a course —not Math for Computer Science, naturally —in which 10% of the assigned problems contain errors. If you ask a Teaching Assistant (TA) whether a problem has an error, then they will answer correctly 80% of the time. This 80% accuracy holds regardless of whether or not a problem has an error. Likewise when you ask a lecturer, but with only 75% accuracy.

We formulate this as an experiment of choosing one problem randomly and asking a particular TA and Lecturer about it. Define the following events:

$$\begin{aligned} E &::= \text{“the problem has an error,”}\\ T &::= \text{“the TA says the problem has an error,”}\\ L &::= \text{“the lecturer says the problem has an error.”}\end{aligned}$$

(a) Translate the description above into a precise set of equations involving conditional probabilities among the events E, T, and L.

(b) Suppose you have doubts about a problem and ask a TA about it, and they tell you that the problem is correct. To double-check, you ask a lecturer, who says that the problem has an error. Assuming that *the correctness of the lecturers' answer and the TA's answer are independent of each other, regardless of whether there is an error*[1], what is the probability that there is an error in the problem?

(c) Is the event that "the TA says that there is an error", independent of the event that "the lecturer says that there is an error"?

---

[1]This assumption is questionable: by and large, we would expect the lecturer and the TA's to spot the same glaring errors and to be fooled by the same subtle ones.

Problem 4.
Suppose you have a biased coin that has probability p of flipping heads. Let J be the number of heads in n independent coin flips. So J has the general binomial distribution:

$$\mathrm{PDF}_J(k) = \binom{n}{k} p^k q^{n-k}$$

where $q ::= 1 - p$.

(a) Show that

$$\begin{aligned} \mathrm{PDF}_J(k-1) &< \mathrm{PDF}_J(k) && \text{for } k < np + p, \\ \mathrm{PDF}_J(k-1) &> \mathrm{PDF}_J(k) && \text{for } k > np + p. \end{aligned}$$

(b) Conclude that the maximum value of PDF_J is asymptotically equal to

$$\frac{1}{\sqrt{2\pi npq}}.$$

Hint: For the asymptotic estimate, it's ok to assume that np is an integer, so by part (a), the maximum value is $\mathrm{PDF}_J(np)$. Use Stirling's formula (14.30):

$$n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n.$$

Problem 5.
Let R and S be independent random variables, and f and g be any functions such that $\mathrm{domain}(f) = \mathrm{codomain}(R)$ and $\mathrm{domain}(g) = \mathrm{codomain}(S)$. Prove that $f(R)$ and $g(S)$ are independent random variables. *Hint:* The event $[f(R) = a]$ is the disjoint union of all the events $[R = r]$ for r such that $f(r) = a$.

Doing P-Set 10

4/28

1. 4 Door Deal

Lets make a deal

-Just get started!

Prob - should be good at w/o reading at least non-proving section

-easy -did it no problem

think is right, but may be tech mistake

2. Census Boys and Girls

2 children

G, B = likly to exist + open door

(__ __ __)

elder younger who ans

These are fun!

Know So well, do diretly

---

actually c) what is wrong?

Seems right!

Oh I see

#3 How to do conditional probs w/ math

I think $\frac{.2}{.2+.9}$?

Read book ~~again!~~
1st time

Oh often charts help!!!

Useless

$$P(E_1 \cap E_2) = P(E_1)\,P(E_2 \mid E_1)$$

So $P(\text{no errors} \cap \text{TA says error}) = P(\text{no error}) \cdot P(\text{TA says error} \mid \text{no error})$

~~##~~ Oh just multiply
forgot exactly what to do

Why can't I get this intuitively
- 3rd time seing
- need a min to figure out

b) Fun - 3rd dimension
Make chart!

#4 Getting harder

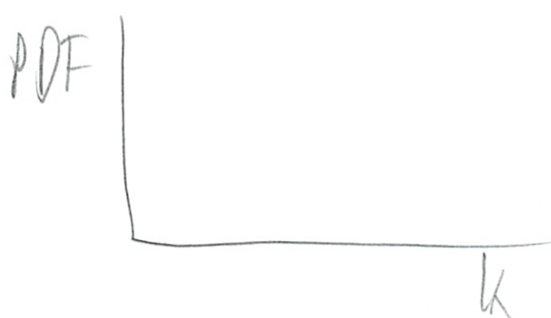
PDF

- δk still good w/

- But prooving :(

k is what - δh # to choose?

No dist over k = # heads
- not J



k - 1
t

but $k < np + p$

(# coins → n; prob → p; prob → + p; - One more add on)

I don't get what trying to say

How to show!!!

2

b) No clue either!

Give up

#5 Juang helped me w/

$f \to Q$
$g \to S$ ind

Hint $f(R) = a$ is disjoint union events $[R = r]$ for r such that $f(r) = a$

Two separate things?
I don't get what it is trying to show!

Have an ans now

4

Student's Solutions to Problem Set 10

Your name: Michael Plasmeier

Due date: April 29

Submission date: 4/29

Circle your TA/LA: Ali Nick Oscar Oshani **Table number** 12

Collaboration statement: Circle one of the two choices and provide all pertinent info.

1. I worked alone and only with course materials.
2. I collaborated on this assignment with:

   got help from:[1] iwdng7

   and referred to:[2]

DO NOT WRITE BELOW THIS LINE

Problem	Score
1	5
2	
3	
4	
5	
Total	

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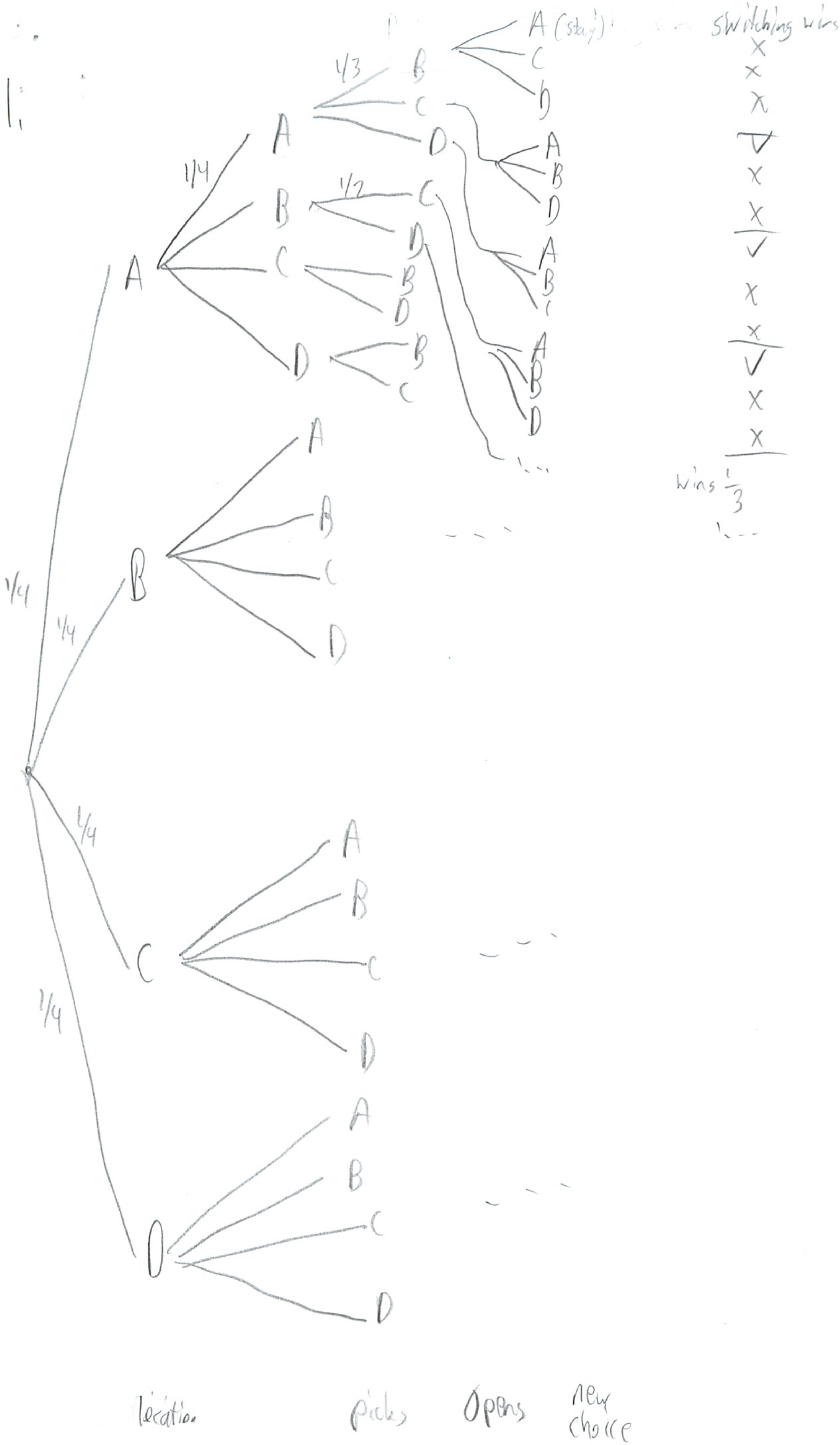
[1]People other than course staff.

[2]Give citations to texts and material other than the Spring '11 course materials.

#1.

5



A (stay)
switching wins
wins 1/3
1/4
1/3
1/2
location
picks
opens
new choice

(2)

So P(switching wins)?

$\frac{1}{4} \cdot 4$ so just look at A location

For one $\frac{1}{4}$

$\frac{1}{4} \cdot \frac{1}{3}$ chance switching does not win

$\frac{1}{4} \cdot \frac{2}{3} \cdot \frac{1}{3}$ switching wins

For other $\frac{3}{4}$

$\frac{3}{4} \cdot \frac{1}{2} \cdot 2 \cdot \frac{1}{3}$

either path

So $\frac{3}{4} \cdot \frac{1}{3} + \frac{1}{4} \cdot \frac{2}{3} \cdot \frac{1}{3}$

$= \frac{11}{36}$

a) Stu stays

$P(\text{win by staying}) = 1 - P(\text{win by switching})$

$= 1 - \frac{11}{36}$

$= \frac{25}{36}$

see sol'n

b) Zelda

$P(\text{switching}) =$ did above

$= \frac{11}{36}$

see sol'n

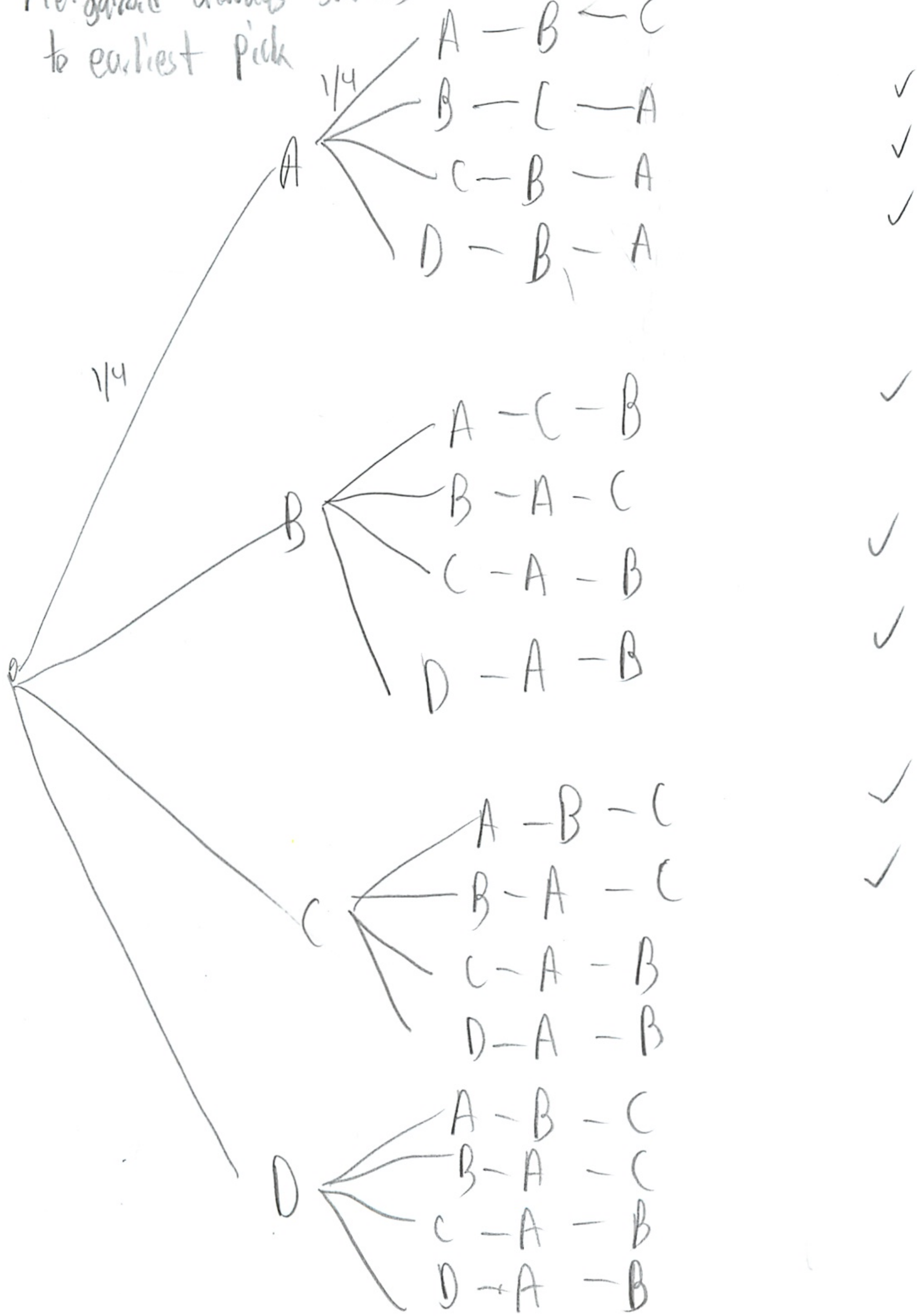
③

Now Carol opens smallest door possible

?Oh Carol works for the game!

Metgatroid always switches to earliest pick

wins



winner picks opens newchoice

(4)

$$P(\text{wins}_{\text{w/strategy}}) = 3\left(\frac{1}{4}\cdot\frac{1}{4}\right) + 3\left(\frac{1}{4}\cdot\frac{1}{4}\right) + 2\left(\frac{1}{4}\cdot\frac{1}{4}\right)$$

$$= 8\left(\frac{1}{16}\right)$$

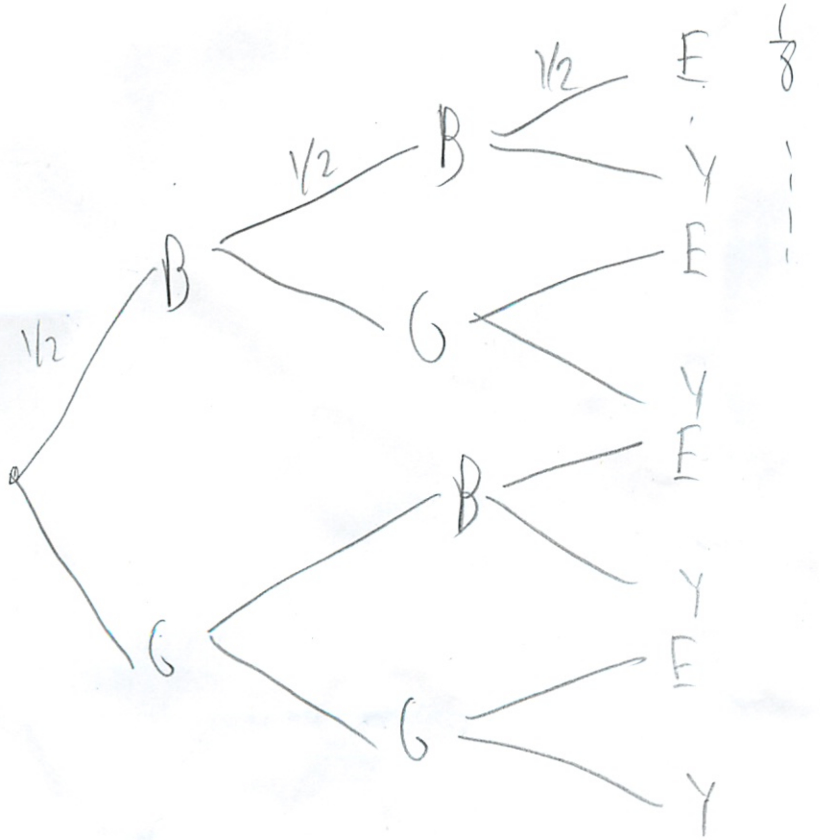
$$= \frac{1}{2}$$

Michael Plasmeier
Oshani
Table 12
P-Set 10

9

#2 Census Boys + Girls
2 children
G, B =ly likly to exist + open door

(___ , ___ , ___)
elder younger who ans



elder Younger Opens

2

a) T is 2 girls

O is girl opened

List outcomes

T) (G, G, E)

(G, G, Y)

O)

(B, G, Y)

(G, B, E)

(G, G, E)

(G, G, Y)

$4 \cdot \frac{1}{8} = \frac{1}{2}$

b) $P(T|O) = \dfrac{P(T \cap G)}{P(G)}$

$\underset{\text{T and}}{P(T \cap O)} = 2 \cdot \frac{1}{8} = \frac{1}{4}$ $\{(G, G, E) \, (G, G, Y)\}$

$P(O) = \frac{1}{2}$

$P(T|O) = \dfrac{1/4}{1/2} = \frac{1}{4} \cdot \frac{2}{1} = \frac{1}{2}$

③

c) What is mistake here?

If a girl opens, know at least one Girl.

$= 1 - P(\text{both boys}) = 1 - \frac{1}{4} = \frac{3}{4}$

Seems right so far

So $P(T \mid \geq \text{one girl})$

$$= \frac{P(T \cap \geq \text{one girl})}{P(\geq 1 \text{ girl})}$$

$$= \frac{P(T)}{P(\geq 1 \text{ girl})} \leftarrow \text{check this}$$

$P(T \cap \geq \text{one girl})$

is $\underline{I}$

(G, G, E)	(B, G, E)
(G, G, Y)	(B, G, Y)
	(G, B, E)
	(G, B, Y)
	(G, G, E)
	(G, G, Y)

$\neq P[T]$

g) ...

$$\therefore = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{4} \cdot \frac{4}{3} = \frac{1}{3}$$

So given that girl opend door, the prob 2 girls is 1/3

(that ≥ 1 girl in house

Oh what is P(girl opens door)

(B, G, Y)

(G, B, E)

(G, G, E)

(G, G, Y)

1/2 not $\frac{3}{4}$

$P(\text{at least one girl}) = \frac{3}{4}$) $\neq$

but $P(\text{girl opens door}) = \frac{1}{2}$

$\frac{1}{3}$ is prob [2 girls $|$ ≥ 1 girl in house]

$$P[\text{2 girls} \mid \text{girl opens}] = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{4} \cdot \frac{2}{1} = \frac{1}{2}$$

Yeah makes sense of 4 possibilities, true in 2 cases

(B,G,Y) x (G,G,E) ✓

(G,B,E) x (G,G,Y) ✓

try being more concise.
read the official solution.

Michael Plasmeier

Oshani

Table 12

10

P-Set #10

#3 10% problems error = E

80% accuracy identifing error TA = T

75% Lecturer = L

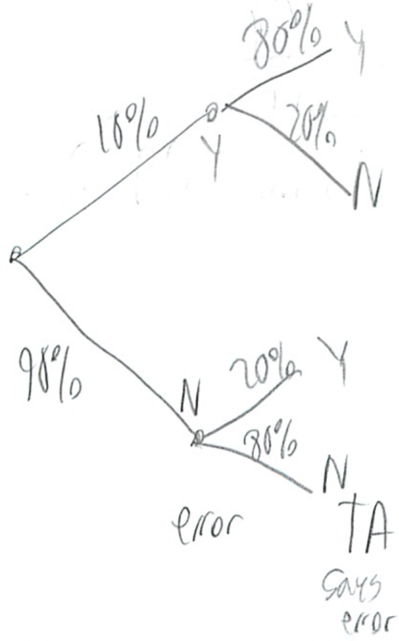
a) Equations

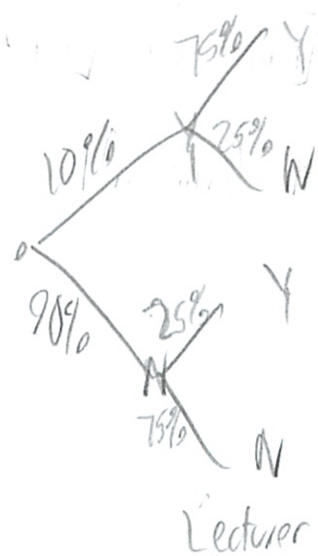
E: P(error) = 10%

T: P(TA says has error) =

= P(TA says error | no error) + P(TA says error | error)

20% 90% 80% 10%





(?)

$= 20\% \cdot 90\% + 88\% \cdot 10\%$

$= 26\%$

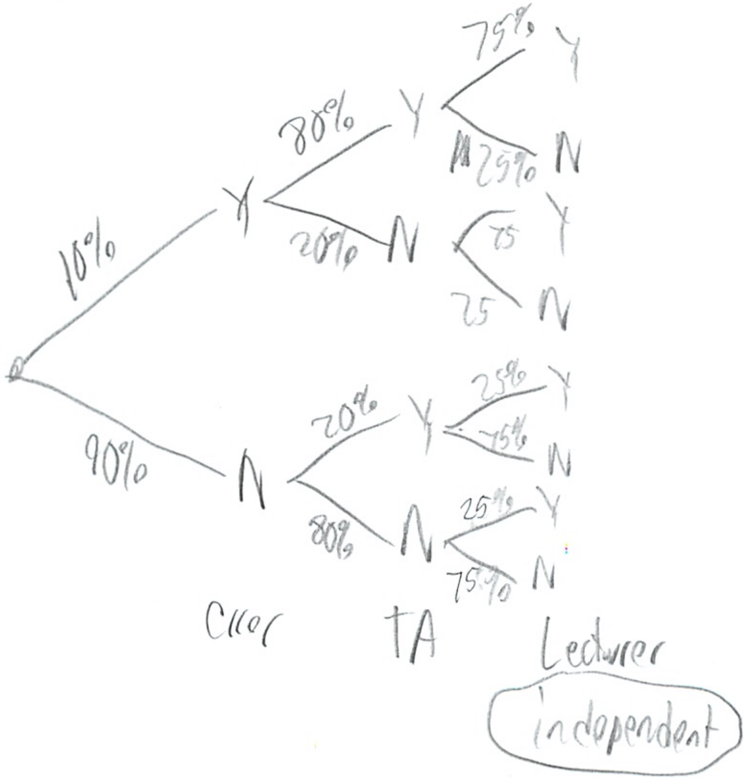
L: P(lecturer says problem has error)

$= P(\text{Lecturer says error} \mid \text{no error}) + P(\text{Lecturer says error} \mid \text{error})$

$= 25\% \cdot 90\% + 75\% \cdot 10\%$

$= 30\%$

b) Suppose double check w/ lecturer



P(error | TA says N AND Lecturer says Y)

(3)

$$= \frac{P(\text{error} \cap \text{TA}: N \cap \text{Lecturer}: Y)}{P(\text{TA}: N \cap \text{Lecturer}: Y)}$$

$$= \frac{.10 \cdot .20 \cdot .75}{(.10 \cdot .20 \cdot .75 + .9 \cdot .80 \cdot .25)}$$

$$\approx 7.692\% \approx \frac{1}{13}$$

c) Are "TA says error" and "Lecturer says error" ind?

No! If there is indeed an error then both are likely to agree, So if one says there is an error then the other one is likly to concur.

Michael Plasmeier

Oshani

Table 12

P-set 10

6

#4. Biased coin

$p = P(\text{heads})$

$J = \#$ of heads in n flips

$$PDF_J(k) = \binom{n}{k} p^k q^{n-k}$$

$; q = 1-p$

a) Show that $PDF_J(k-1) < PDF_J(k)$ for $k < np+p$

$PDF_J(k-1) > PDF_J(k)$ for $k > np+p$

The General Binomial Distribution

$$\binom{n}{k-1} p^{k-1} q^{n-k-1} < \binom{n}{k} p^k q^{n-k} \quad k < np+p$$

$$\frac{n!}{k-1!\, n-k-1!} p^{k-1} q^{n-k-1} < \frac{n!}{k!\, n-k!} p^k q^{n-k}$$

$k < np+p$

②.

Or howabout another way to think about it.

$PDF_J(k-1)$ is the PDF shifted one to the left

This will be lower for each value of k if $k < np + p$. But what is $np + p$? The mean? It gets smaller as the number of flips ↑ or p ↓. It could also be written as $(n+1)p$

③

b) Conclude max value PDF_J asy = to

$$\frac{1}{\sqrt{2\pi npq}}$$

Hint - assume np is an integer so max value is $PDF_J(np)$
Can use sterling formula

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

What does this represent?

-4

?

see solutions

Michael Plasmeier

O'Shani

Table 12

P-Set 10

9

#5. Event $[f(R)=a]$ is disjoint union of all the events $R=r$ for r s.t. $f(R)=a$

$$P[f(R)=a] = \bigcup_{r=1}^{R} P(r) \text{ s.t. } f(r)=a$$

disjoint so can sum

$$= \sum_{\{r \mid f(r)=a\}} P(R=r)$$

? put conditions here

; The rest seems kinda obvious - they share no common letters

$$\left.\begin{array}{l} f \to R \\ g \to S \end{array}\right\} \text{independent}$$

$[f(R)=a$ AND $g(S)=b]$ is disjoint union of the events $[R=r$ and $S=s]$ for (r,s) s.t.
$f(r)=a$
$g(s)=b$

②

So split it up

$$P(f(R)=a \text{ AND } g(S)=b)$$

$$= \cancel{P(f(R)=a) \cdot P(g(S)=b)}$$

can't do yet!
need to prove ind

Use model from earlier

$$= \sum_{\{(r,s) \mid f(r)=a \text{ AND } g(s)=b\}} P(R=r \text{ and } S=s)$$

disjoint so can split

$$= \sum_{\{(r,s) \mid f(r)=a \text{ AND } g(s)=b\}} P(R=r) \text{ AND } P(S=s)$$

why is this step possible?

$$= \sum_{\{(r,s) \mid f(r)=a \text{ AND } g(s)=b\}} P(R=r) \cdot P(S=s)$$

Split further

$$= \sum_{\{r \mid f(r)=a\}} P(R=r) \cdot \sum_{\{s \mid f(s)=b\}} P(S=s)$$

(3)

$$= \sum_{\{r \mid f(r) = a\}} P(Q = r) \cdot P(f(S) = b)$$

$$= P(f(r) = a) \cdot P(f(S) = b)$$

So ind.

Solutions to Problem Set 10

Reading: Chapter 16, Intro to Probability; Chapter 17, Random Variables

Problem 1.
[The Four-Door Deal]

Let's see what happens when *Let's Make a Deal* is played with **four** doors. A prize is hidden behind one of the four doors. Then the contestant picks a door. Next, the host opens an unpicked door that has no prize behind it. The contestant is allowed to stick with their original door or to switch to one of the two unopened, unpicked doors. The contestant wins if their final choice is the door hiding the prize.

Let's make the same assumptions as in the original problem:

1. The prize is equally likely to be behind each door.
2. The contestant is equally likely to pick each door initially, regardless of the prize's location.
3. The host is equally likely to reveal each door that does not conceal the prize and was not selected by the player.

Use The Four Step Method of Section 16.2 to find the following probabilities. The tree diagram may become awkwardly large, in which case just draw enough of it to make its structure clear.

(a) Contestant Stu, a sanitation engineer from Trenton, New Jersey, stays with his original door. What is the probability that Stu wins the prize?

Solution. A partial tree diagram is shown below. The remaining subtrees are symmetric to the fully-expanded subtree. The probability that Stu wins the prize is:

$$\Pr[\text{Stu wins}] = 4 \cdot \left(\frac{1}{48} + \frac{1}{48} + \frac{1}{48}\right) = \frac{1}{4}$$

We multiply by 4 to account for the four subtrees, of which we've only drawn one.

Notice that we expanded the tree out to the third ("door revealed") level to spell out the outcomes, but in this case we could, in fact, have stopped at the second level ("player's initial guess"). This follows because the win/lose outcome is determined by the prize location and Stu's selected door, regardless of what happens after that. ■

(b) Contestant Zelda, an alien abduction researcher from Helena, Montana, switches to one of the remaining two doors with equal probability. What is the probability that Zelda wins the prize?

Solution. A partial tree diagram is worked out below. The probability that Zelda wins the prize is:

$$\Pr[\text{Zelda wins}] = 4 \cdot \left(\frac{1}{64} + \frac{1}{64} + \frac{1}{64} + \frac{1}{64} + \frac{1}{64} + \frac{1}{64}\right) = \frac{3}{8}$$

■

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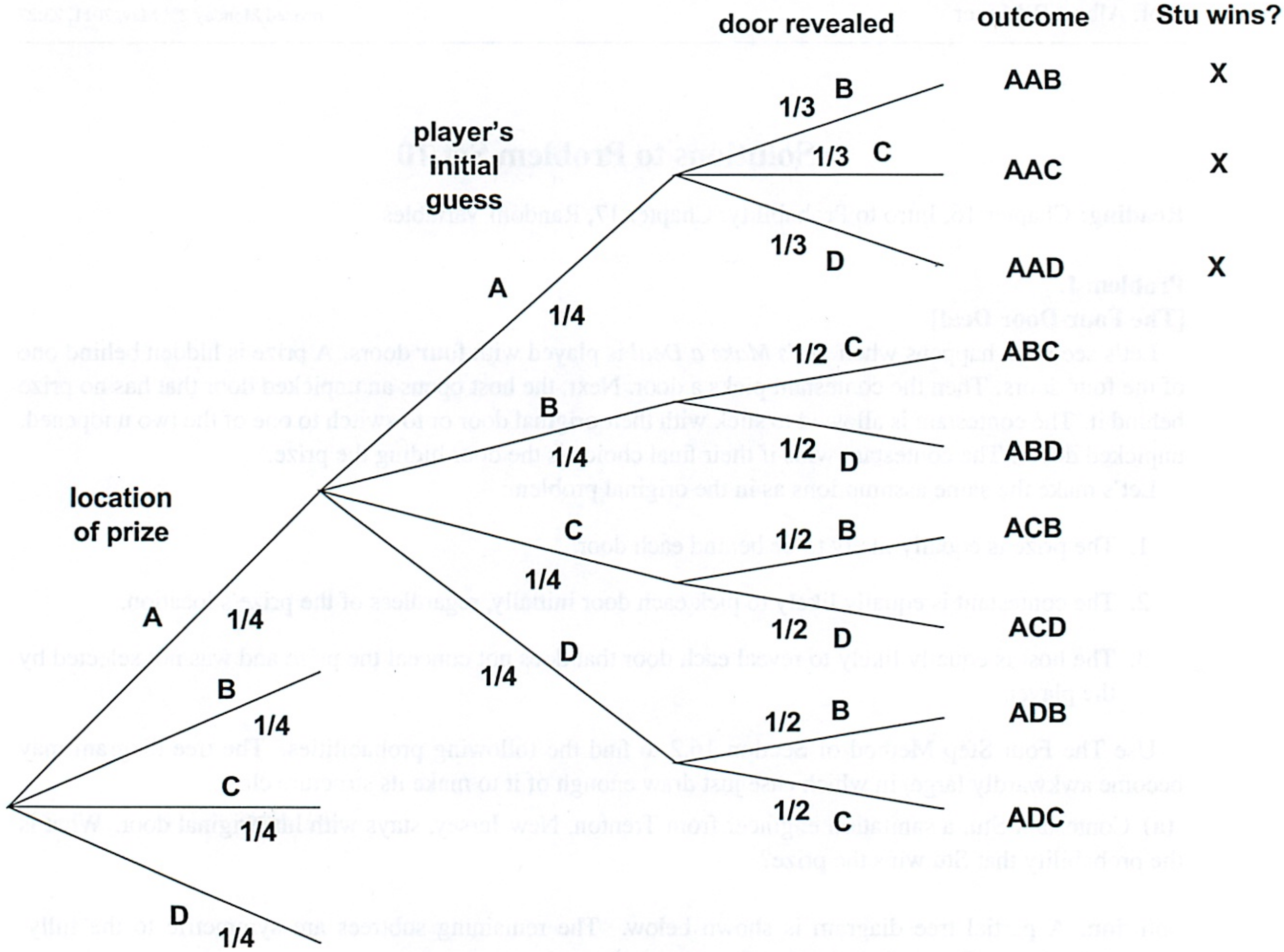
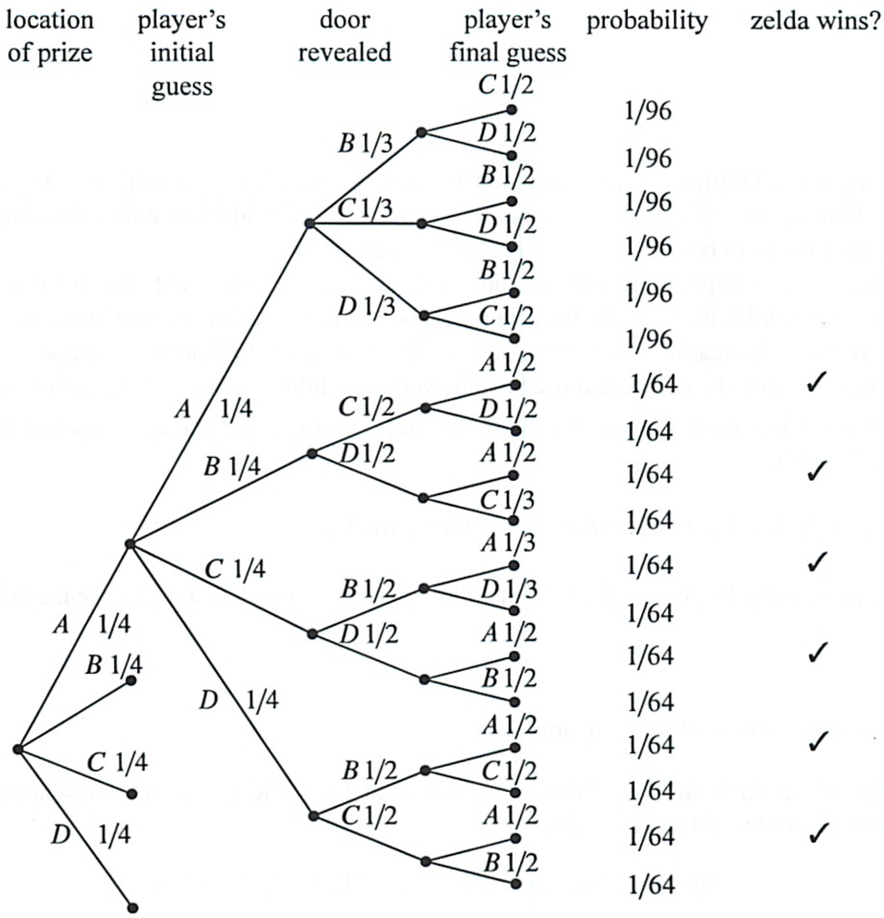


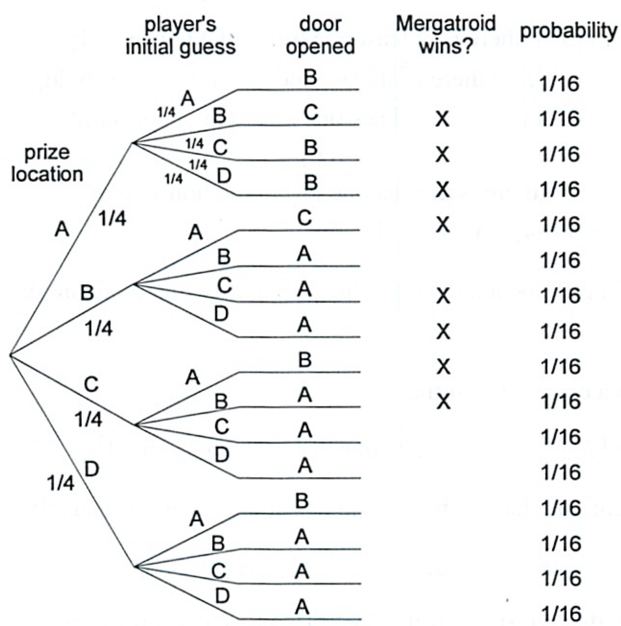
Now let's revise our assumptions about how contestants choose doors. Say the doors are labeled A, B, C, and D. Suppose that Carol always opens the *earliest* door possible (the door whose label is earliest in the alphabet) with the restriction that she can neither reveal the prize nor open the door that the player picked.

This gives contestant Mergatroid —an engineering student from Cambridge, MA —just a little more information about the location of the prize. Suppose that Mergatroid always switches to the earliest door, excluding his initial pick and the one Carol opened.

(c) What is the probability that Mergatroid wins the prize?

Solution. A tree diagram is worked out below.





The probability that Mergatroid wins is:

$$\Pr[\text{win}] = 8 \cdot \frac{1}{16} = \frac{1}{2}$$

■

Problem 2.
You are organizing a neighborhood census and instruct your census takers to knock on doors and note the sex of any child that answers the knock. Assume that there are two children in a household and that girls and boys are equally likely to be children and to open the door.

A sample space for this experiment has outcomes that are triples whose first element is either B or G for the sex of the elder child, likewise for the second element and the sex of the younger child, and whose third coordinate is E or Y indicating whether the *e*lder child or *y*ounger child opened the door. For example, (B, G, Y) is the outcome that the elder child is a boy, the younger child is a girl, and the girl opened the door.

(a) Let T be the event that the household has two girls, and O be the event that a girl opened the door. List the outcomes in T and O.

Solution. $T = \{GGE, GGY\}$, $O = \{GGE, GGY, GBE, BGY\}$ ■

(b) What is the probability $\Pr\left[T \mid O\right]$, that both children are girls, given that a girl opened the door?

Solution. 1/2 ■

(c) Where is the mistake in the following argument?

> If a girl opens the door, then we know that there is at least one girl in the household. The probability that there is at least one girl is
>
> $$1 - \Pr[\text{both children are boys}] = 1 - (1/2 \times 1/2) = 3/4. \tag{1}$$
>
> So,
>
> $$\begin{aligned} &\Pr\left[T \mid \text{there is at least one girl in the household}\right] && (2)\\ &= \frac{\Pr[T \cap \text{there is at least one girl in the household}]}{\Pr[\text{there is at least one girl in the household}]} && (3)\\ &= \frac{\Pr[T]}{\Pr[\text{there is at least one girl in the household}]} && (4)\\ &= (1/4)/(3/4) = 1/3. && (5)\end{aligned}$$
>
> Therefore, given that a girl opened the door, the probability that there are two girls in the household is 1/3.

Solution. The argument is a correct proof that

$$\Pr\left[T \mid \text{there is at least one girl in the household}\right] = 1/3.$$

The problem is that the event, H, that the household has at least one girl, namely,

$$H ::= \{\texttt{GGE}, \texttt{GGY}, \texttt{GBE}, \texttt{GBY}, \texttt{BGE}, \texttt{BGY}\},$$

is not equal to the event, O, that a girl opens the door. These two events differ:

$$H - O = \{\texttt{BGE}, \texttt{GBY}\},$$

and their probabilities are different. So the fallacy is in the final conclusion where the value of $\Pr\left[T \mid H\right]$ is taken to be the same as the value $\Pr\left[T \mid O\right]$. Actually, $\Pr\left[T \mid O\right] = 1/2$. ■

Problem 3.

There is a course —not Math for Computer Science, naturally —in which 10% of the assigned problems contain errors. If you ask a Teaching Assistant (TA) whether a problem has an error, then they will answer correctly 80% of the time. This 80% accuracy holds regardless of whether or not a problem has an error. Likewise when you ask a lecturer, but with only 75% accuracy.

We formulate this as an experiment of choosing one problem randomly and asking a particular TA and Lecturer about it. Define the following events:

$$\begin{aligned} E &::= \text{"the problem has an error,"} \\ T &::= \text{"the TA says the problem has an error,"} \\ L &::= \text{"the lecturer says the problem has an error."} \end{aligned}$$

(a) Translate the description above into a precise set of equations involving conditional probabilities among the events E, T, and L.

Solution. The assumptions above tell us:

$$\begin{aligned} \Pr[E] &= \frac{10}{100} = \frac{1}{10}, \\ \Pr\left[T \mid E\right] = \Pr\left[\overline{T} \mid \overline{E}\right] &= \frac{80}{100} = \frac{4}{5}, \\ \Pr\left[L \mid E\right] = \Pr\left[\overline{L} \mid \overline{E}\right] &= \frac{75}{100} = \frac{3}{4}, \end{aligned}$$

Also, T and L are independent given E, and given $\overline{E}$:

$$\begin{aligned} \Pr\left[T \cap L \mid E\right] &= \Pr\left[T \mid E\right]\Pr\left[L \mid E\right] \\ \Pr\left[T \cap L \mid \overline{E}\right] &= \Pr\left[T \mid \overline{E}\right]\Pr\left[L \mid \overline{E}\right] \end{aligned}$$

Note that while we know that T and L are independent *given* E or *given* $\overline{E}$, they are not independent by themselves, see part (c). ■

(b) Suppose you have doubts about a problem and ask a TA about it, and they tell you that the problem is correct. To double-check, you ask a lecturer, who says that the problem has an error. Assuming that *the correctness of the lecturers' answer and the TA's answer are independent of each other, regardless of whether there is an error*[1], what is the probability that there is an error in the problem?

Solution. We want to calculate

$$\Pr\left[E \mid \overline{T} \cap L\right].$$

From the definition of conditional probability (this is known as *Bayes' rule*):

$$\Pr\left[E \mid \overline{T} \cap L\right] = \Pr[E]\frac{\Pr\left[\overline{T} \cap L \mid E\right]}{\Pr[\overline{T} \cap L]}. \tag{6}$$

[1] This assumption is questionable: by and large, we would expect the lecturer and the TA's to spot the same glaring errors and to be fooled by the same subtle ones.

By the independence assumptions, we have:

$$\begin{aligned}
\Pr\left[\overline{T} \cap L \mid E\right] &= \Pr\left[\overline{T} \mid E\right]\Pr\left[L \mid E\right] = \frac{1}{5}\frac{3}{4} = \frac{3}{20}, \\
\Pr\left[\overline{T} \cap L \mid \overline{E}\right] &= \Pr\left[\overline{T} \mid \overline{E}\right]\Pr\left[L \mid \overline{E}\right] = \frac{4}{5} \cdot \frac{1}{4} = \frac{1}{5}, \\
\Pr[\overline{T} \cap L] &= \Pr\left[\overline{T} \cap L \mid E\right]\Pr[E] + \Pr\left[\overline{T} \cap L \mid \overline{E}\right]\Pr[\overline{E}] \\
&= \frac{3}{20} \cdot \frac{1}{10} + \frac{1}{5} \cdot \frac{9}{10} = \frac{39}{200}.
\end{aligned}$$

Substituting these values in equation (6), we get

$$\Pr\left[E \mid \overline{T} \cap L\right] = \frac{1}{10} \cdot \frac{3/20}{39/200} = \frac{1}{13} \approx 0.077.$$

So this contradictory information has decreased the probability of an error from 10% to about 7.7%.

The calculations here support the common-sense rule that when two people make contradictory statements, you should be influenced more by the most "authoritative" person —the one who is right more often. But note that this does not mean that you should *believe* in what the most authoritative person says, since the probability of an error remains uncomfortably large. ■

(c) Is the event that "the TA says that there is an error", independent of the event that "the lecturer says that there is an error"?

Solution. The answer is no. Because the TA is usually right, when the TA says that the problem has an error, the likelihood that there really is an error is increased. But the lecturer is also usually right, so increasing the likelihood of there *being* an error also increases the likelihood that the lecturer will *report* an error.

We verify this informal argument by actually calculating the probability of each of these events and their conjunction, and observing that the probability that the two events occur is different from the product of the probabilities. Let events E, T, L be as above.

$$\begin{aligned}
\Pr[T] &= \Pr[T \cap E] + \Pr[T \cap \overline{E}] \\
&= \Pr\left[T \mid E\right]\Pr[E] + \Pr\left[T \mid \overline{E}\right]\Pr[\overline{E}] \\
&= \frac{4}{5}\frac{1}{10} + (1 - \frac{4}{5})(1 - \frac{1}{10}) = \frac{13}{50}, \\
\Pr[L] &= \Pr[L \cap E] + \Pr[L \cap \overline{E}] \\
&= \frac{3}{4}\frac{1}{10} + (1 - \frac{3}{4})(1 - \frac{1}{10}) = \frac{3}{10}, \\
\Pr[L \cap T] &= \Pr[L \cap T \cap E] + \Pr[L \cap T \cap \overline{E}] \\
&= \Pr\left[L \cap T \mid E\right]\Pr[E] + \Pr\left[L \cap T \mid \overline{E}\right]\Pr[\overline{E}] \\
&= \Pr\left[L \mid E\right]\Pr\left[T \mid E\right]\Pr[E] + \Pr\left[L \mid \overline{E}\right]\Pr\left[T \mid \overline{E}\right]\Pr[\overline{E}] \\
&= \frac{3}{4} \cdot \frac{4}{5} \cdot \frac{1}{10} + (1 - \frac{3}{4})(1 - \frac{4}{5}) \cdot (1 - \frac{1}{10}) = \frac{105}{1000} = 0.105,
\end{aligned}$$

which is higher than

$$\Pr[L]\Pr[T] = \frac{3}{10} \cdot \frac{13}{50} = .078.$$

■

Problem 4.
Suppose you have a biased coin that has probability p of flipping heads. Let J be the number of heads in n independent coin flips. So J has the general binomial distribution:

$$\mathrm{PDF}_J(k) = \binom{n}{k} p^k q^{n-k}$$

where $q ::= 1 - p$.

(a) Show that

$$\begin{aligned} \mathrm{PDF}_J(k-1) &< \mathrm{PDF}_J(k) \qquad && \text{for } k < np + p, \\ \mathrm{PDF}_J(k-1) &> \mathrm{PDF}_J(k) \qquad && \text{for } k > np + p. \end{aligned}$$

Solution. Consider the ratio of the probability of k heads over the probability of $k-1$ heads.

$$\begin{aligned} \frac{\mathrm{PDF}_J(k)}{\mathrm{PDF}_J(k-1)} &= \frac{\binom{n}{k} p^k q^{n-k}}{\binom{n}{k-1} p^{k-1} q^{n-k+1}} \\ &= \frac{\frac{n!}{k!\,(n-k)!}}{\frac{n!}{(k-1)!\,(n-k+1)!}} \frac{p}{q} \\ &= \frac{(n-k+1)p}{kq} \end{aligned}$$

This fraction is greater than 1 precisely when $(n-k+1)p > kq = k(1-p)$, that is when $k < np + p$. So for $k < np + p$, the probability of k heads increases as k increases, and for $k > np + p$, the probability decreases as k increases. ■

(b) Conclude that the maximum value of PDF_J is asymptotically equal to

$$\frac{1}{\sqrt{2\pi npq}}.$$

Hint: For the asymptotic estimate, it's ok to assume that np is an integer, so by part (a), the maximum value is $\mathrm{PDF}_J(np)$. Use Stirling's formula (14.30):

$$n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n.$$

Solution.

$$
\begin{aligned}
\mathrm{PDF}_J(np) &::= \binom{n}{np} p^{np} q^{n-np} \\
&= \frac{n!}{(np)!\,(nq)!} p^{np} q^{nq} \\
&\sim \frac{\left(\frac{n}{e}\right)^n \sqrt{2\pi n}}{\left(\left(\frac{np}{e}\right)^{np} \sqrt{2\pi np}\right)\left(\left(\frac{nq}{e}\right)^{nq} \sqrt{2\pi nq}\right)} p^{np} q^{nq} \\
&= \frac{\frac{n^n}{e^n}\sqrt{2\pi n}}{\left(\frac{n^{np}p^{np}}{e^{np}}\sqrt{2\pi np}\right)\left(\frac{n^{nq}q^{nq}}{e^{nq}}\sqrt{2\pi nq}\right)} p^{np} q^{nq} \\
&= \frac{\frac{n^n}{e^n}\sqrt{2\pi n}}{\frac{n^{np+nq}p^{np}q^{nq}}{e^{np+nq}}\sqrt{2\pi np}\sqrt{2\pi nq}} p^{np} q^{nq} p^{np} q^{nq} \\
&= \frac{\frac{n^n}{e^n}\sqrt{2\pi n}}{\frac{n^n}{e^n}\sqrt{2\pi np}\sqrt{2\pi nq}} \\
&= \frac{1}{\sqrt{2\pi npq}}.
\end{aligned}
$$

■

Problem 5.
Let R and S be independent random variables, and f and g be any functions such that $\text{domain}(f) = \text{codomain}(R)$ and $\text{domain}(g) = \text{codomain}(S)$. Prove that $f(R)$ and $g(S)$ are independent random variables. *Hint:* The event $[f(R) = a]$ is the disjoint union of all the events $[R = r]$ for r such that $f(r) = a$.

Solution. By the hint and the Sum Rule

$$\Pr[f(R) = a] = \sum_{\{r \mid f(r)=a\}} \Pr[R = r].$$

Also, the event $[f(R) = a \text{ AND } g(S) = b]$ is the disjoint union of the events $[R = r \text{ AND } S = s]$ for pairs (r, s) such that $f(r) = a$ and $g(s) = b$. Hence,

$$
\begin{aligned}
&\Pr[f(R) = a \text{ AND } g(S) = b] \\
&= \sum_{\{(r,s) \mid f(r)=a \text{ AND } g(s)=b\}} \Pr[R = r \text{ AND } S = s] \\
&= \sum_{\{(r,s) \mid f(r)=a \text{ AND } g(s)=b\}} \Pr[R = r] \cdot \Pr[S = s] \qquad [R, S \text{ independent}] \\
&= \sum_{\{s \mid g(s)=b\}} \Pr[S = s] \left(\sum_{\{r \mid f(r)=a\}} \Pr[R = r] \right) \\
&= \sum_{\{s \mid g(s)=b\}} \Pr[S = s](\Pr[f(R) = a]) \\
&= \Pr[f(R) = a] \sum_{\{s \mid g(s)=b\}} \Pr[S = s] \\
&= \Pr[f(R) = a] \cdot \Pr[g(S) = b].
\end{aligned}
$$

■