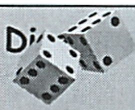
4/29

Mathematics for Computer Science
MIT 6.042J/18.062J

Great Expectations

Carnival Dice



choose a number from 1 to 6, then roll 3 fair dice:

win $1 for each match

lose $1 if no match

Carnival Dice

Example: choose 5, then

roll 2,3,4: lose $1

roll 5,4,6: win $1

roll 5,4,5: win $2

roll 5,5,5: win $3

Carnival Dice

Is this a fair game?

Carnival Dice

# matches	probability	$ won
0	125/216	-1
1	75/216	1
2	15/216	2
3	1/216	3

Carnival Dice

so every 216 games, expect
0 matches about 125 times
1 match about 75 times
2 matches about 15 times
3 matches about once

Carnival Dice

So on average expect to win:

$125 \cdot$ $\boxed{\text{NOT fair!}}$ $1 \cdot 3$

$= -\frac{17}{216} \approx -8 \text{ cents}$

Carnival Dice

You can "expect" to lose 8 cents per play. But you never actually lose 8 cents on any single play, this is just your average loss.

Expected Value

The expected value of random variable R is the average value of R --with values weighted by their probabilities

Expected Value

The expected value of random variable R is

$$\boxed{E[R] ::= \sum v \cdot \Pr\{R = v\}}$$

so E[\$win in Carnival] $= -\frac{17}{216}$

Expected Value

An equivalent definition

$$\boxed{E[R] = \sum_{\omega \in S} R(\omega) \cdot \Pr\{\omega\}}$$

both forms are useful

Expected Value

also called

mean value, mean, or expectation

Indicator Variables

The indicator variable for event A:

$$I_A ::= \begin{cases} 1 & \text{if } A \text{ occurs,} \\ 0 & \text{if } A \text{ does not occur.} \end{cases}$$

$$\Pr\{I_A = 1\} = \Pr\{A\}$$

Expectation of indicator I_A

$$\boxed{E[I_A]} ::= 1 \cdot \Pr\{I_A=1\} + 0 \cdot \Pr\{I_A=0\}$$

$$= \Pr\{I_A=1\}$$

$$\boxed{= \Pr\{A\}}$$

Expected #Heads

n independent flips of a coin with bias p for Heads. How many Heads expected?

$$E[B_{n,p}] ::= \sum_{k=0}^{n} k \cdot \Pr\{k \text{ Heads}\} = \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k}$$

Expected #Heads

Binomial thm & differentiating gives a closed formula, but simpler approach is coming

$$E[B_{n,p}] ::= \sum_{k=0}^{n} k \cdot \Pr\{k \text{ Heads}\} = \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k}$$

Law of Total Expectation

Def: conditional expectation

$$E[R \mid A] ::= \sum v \cdot \Pr\{R = v \mid A\}$$

$$\boxed{E[R] = E[R \mid A] \cdot \Pr\{A\} + E[R \mid \overline{A}] \cdot \Pr\{\overline{A}\}}$$

good for reasoning by cases

Expected #Heads

Let e(n) ::= expected #H's in n flips.

= 1 + e(n-1)   if 1st flip H

= e(n-1)   if 1st flip T

by Total Expectation:

$$e(n) = [1 + e(n-1)] \cdot p + e(n-1) \cdot q$$

$$e(n) = e(n-1)+p = e(n-2)+2p \cdots$$

Expected #Heads

Let e(n) ::= expected #H's in n flips.

$= 1 + e(n-1)$ if 1st flip H

$= e(n-1)$ if 1st flip T

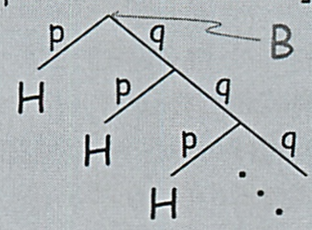
by Total Expectation:

$$e(n) = [1 + e(n-1)]\cdot p + e(n-1)\cdot q$$

$$e(n) = e(n-1)+p = \boxed{np = E\left[B_{n,p}\right]}$$

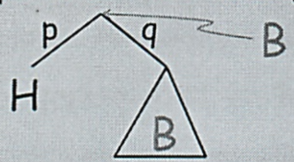
Mean Time to "Failure"

E[# flips until first head]?



Mean Time to "Failure"

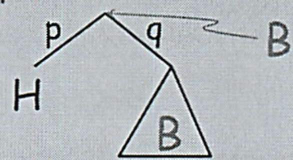
E[# flips until first head]?



now use Total Expectation

Mean Time to "Failure"

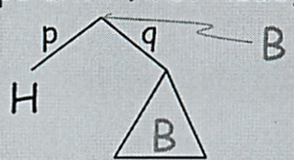
E[# flips until first head]?



$$E = \underbrace{E[\#\ |1^{st}\text{ is H}]}_{1}\cdot p + \underbrace{E[\#\ |1^{st}\text{ is T}]}_{1+E}\cdot q$$

Mean Time to "Failure"

E[# flips until first head]?



$$E = 1 \cdot p + [E+1]\cdot (1-p)$$

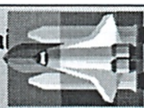
now solve for E

Mean Time to "Failure"

E[# flips until first head]

$$\boxed{= \frac{1}{p}}$$

Mean Time to Fai



application: if space station Mir has 1/150,000 chance of exploding in any given hour, after how may hours do we expect it to explode?

150,000 hours $\approx$ 17 years

Linearity of Expectation

R,S random variables, a,b constants

$$\boxed{E[aR + bS] = aE[R] + bE[S]}$$

even if R,S are dependent

Expected #Heads

$$\#H's = H_1 + H_2 + \cdots + H_n$$

where H_i is indicator for Head on ith flip

Expected #Heads

$$\boxed{E\left[\#H's\right]} = E\left[H_1 + H_2 + \cdots + H_n\right]$$

so by linearity

$$= E\left[H_1\right] + E\left[H_2\right] + \cdots + E\left[H_n\right]$$

$$= n \cdot E\left[H_1\right] \boxed{= np}$$

Expected #hats returned

n men each check their hat. Hats get scrambled so pr{man #i gets own hat back}

$$= 1/n$$

How many men do we expect will get their hat back?

Expected #hats returned

Let R_i be indicator for man #i getting his own hat back.
R_i and R_j are not independent!

Expected #hats returned

Let R_i be indicator for man #i getting his own hat back.

But E[# hats returned] =

$E[\sum_i R_i] = \sum_i E[R_i] = \sum_i \Pr\{R_i=1\} = \sum_i 1/n = n(1/n) = 1$

Expectation & Independence

for $\boxed{\text{independent } R,S}$

$$\boxed{E[R \cdot S] = E[R] \cdot E[S]}$$

Team Problems

Problems 1 — 4

6.042 Expectations

4/29

RVs

- have an expectation

Carnival Die game

(did not hear the rules...).

- is it a fair game?
- calc probs w/ tree
- for each # of matches

RV	Prob	$
0	$\frac{125}{216}$	-1
1	$\frac{75}{216}$	+1
2	$\frac{15}{216}$	+2
3	$\frac{1}{216}$	+3

So for every 216 games expect

0 matches 125 times

⋮

Where will we be

$$125 \cdot (-1) + 75 \cdot 1 + 15 \cdot 2 + 1 \cdot 3 = -17$$

②

So on avg, per game

$= -\frac{17}{216} = -8$ cents

Not fair!

fair is avg $= 0$

Never actlly lose 8 cents

— but what happens

E[ ] value is weighted avg of possible values
w/ prob

$$\boxed{E[R] = \sum v \cdot P(R=v)}$$

$E[\text{carnival}] = -\frac{17}{216} \approx -8$ cents

Another def - set-wise

$$\boxed{E[R] = \sum_{w \in S} R(w) \cdot P(w)}$$

3

Also called "mean" or "expectation"

Indicator RV

- make events a special case of RV

$$I_A = \begin{cases} 1 & \text{if } A \text{ happens} \\ 0 & \text{otherwise} \end{cases}$$

$$P(\cancel{A}\; I_A = 1) = P(A)$$

$$E[I_A] = 1 \cdot P(I_A = 1) + 0 \cdot P(I_A = 0)$$
$$= P(I_A = 1)$$
$$= P(A)$$

Expected # of heads

n flips

$p = P(\text{Heads})$

$E[\#\text{ heads}] = ?$

$$E[B_{n,p}] = \sum_{k=0}^{n} k \cdot P(k \text{ heads})$$

(4)

$$= \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k}$$

Can nock off w/ binomial theorm
differentate

Conditional Expectation

$$E[R \mid A] = \sum v \, P[R = v \mid A]$$

$$E[R] = E[R \mid A] \, P(A) + P(R \mid \bar{A}) \, P(\bar{A})$$

law of total ~~prob~~ expectation

for A and $\bar{A}$

$e(n)$ = expected # of heads in n flips

Analize by cases

$$= 1 + e(n-1) \quad \text{if 1st flip H}$$

remaining flips

$$= e(n-1) \quad \text{if 1st flip T}$$

$$e(n) = p[1 + e(n-1)] + q[e(n-1)]$$

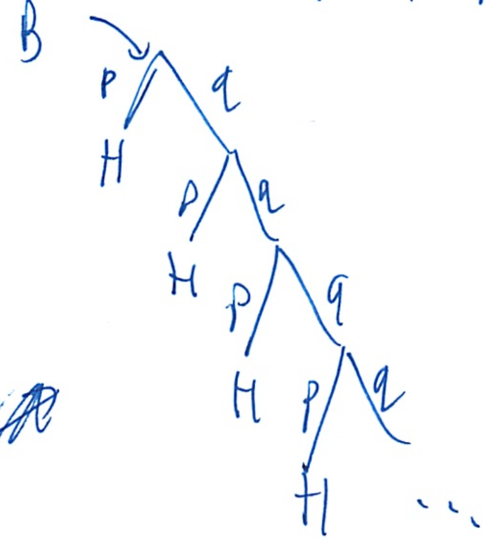
$$= e(n-1) + p = e(n-2) + 2p, \ \ldots, \ = np$$

5

$E[\#\text{ of flips to 1st head}] =$

↑ mean time to failure

how long will sys run before it crashes



assuming p of heads

in ind

can do recursive def

Now use total expectation theorm

$$E[\# \mid \text{1st is } H]\cdot p + E[\# \mid \text{1st is } T]\cdot q$$

$$E = 1\cdot p + [E+1]\cdot q$$

$$= \frac{1}{p}$$

6

Expectation is linear fn on variables

$$E[aR+bS] = aE[R] + bE[S]$$

generalizes to many variables

not abt dep, ind for R, S

Expected # of heads

$$\#H's = H_1 + H_2 + \ldots + H_n$$

where H_i is indicator for head on ith flip

$$E[\#H's] = E[H_1 + H_2 + \ldots + H_n]$$
$$= E[H_1] + E[H_2] + \ldots E[H_n]$$
$$= n \cdot E[H_1]$$
$$= np$$

Suppose n hats at coat check have been scabled - What is P(person gets correct hat)

(7)

Let R_i is ith hat returned to owner

R_i and R_j are <u>not</u> ind.

- What happens to first affects last
- Or Chineese Banquet table

- ~~It~~ either they all get hat back or none

$E[\text{\# hats returned}] =$

$$E[\textstyle\sum_i R_i]$$

$$= \sum_i E[R_i]$$

$$= \sum_i P(R_i = 1)$$

$$= \sum_i \frac{1}{n}$$

$$= n \frac{1}{n}$$

$$= 1$$

8

So assume that people did not stack hats???

? prof confused

On avg 1 person gets their hat back

For ind R, S

$$E[R \cdot S] = E[R] \cdot E[S]$$

- we will describe proof

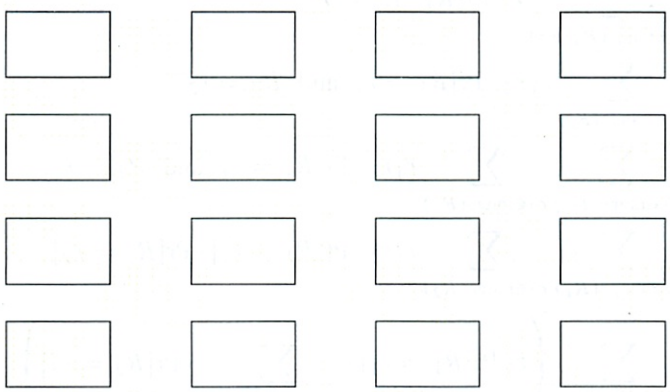
In-Class Problems Week 12, Fri.

Problem 1.
Let's see what it takes to make Carnival Dice fair. Here's the game with payoff parameter k: make three independent rolls of a fair die. If you roll a six

- no times, then you lose 1 dollar.
- exactly once, then you win 1 dollar.
- exactly twice, then you win two dollars.
- all three times, then you win k dollars.

For what value of k is this game fair?

Problem 2.
A classroom has sixteen desks in a 4×4 arrangement as shown below.



If there is a girl in front, behind, to the left, or to the right of a boy, then the two of them *flirt*. One student may be in multiple flirting couples; for example, a student in a corner of the classroom can flirt with up to two others, while a student in the center can flirt with as many as four others. Suppose that desks are occupied by boys and girls with equal probability and mutually independently. What is the expected number of flirting couples? *Hint:* Linearity.

Problem 3. (a) Suppose we flip a fair coin and let N_{TT} be the number of flips until the first time two Tails in a row appear. What is $\text{Ex}[N_{TT}]$?

Hint: Let D be the tree diagram for this process. Explain why

$$D = H \cdot D + T \cdot (H \cdot D + T).$$

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Use the **Law of Total Expectation**: Let R be a random variable and $A_1, A_2, \ldots,$ be a partition of the sample space. Then

$$\text{Ex}[R] = \sum_i \text{Ex}[R \mid A_i] \Pr[A_i].$$

(b) Suppose we flip a fair coin until a Tail immediately followed by a Head come up. What is the expectation of the number N_{TH} of flips we perform?

(c) Suppose we now play a game: flip a fair coin until either TT or TH first occurs. You win if TT comes up first, lose if TH comes up first. Since TT takes 50% longer on average to turn up, your opponent agrees that he has the advantage. So you tell him you're willing to play if you pay him \$5 when he wins, but he merely pays you a 20% premium, that is, \$6, when you win.

If you do this, you're sneakily taking advantage of your opponent's untrained intuition, since you've gotten him to agree to unfair odds. What is your expected profit per game?

Problem 4.
Justify each line of the following proof that if R_1 and R_2 are *independent*, then

$$\text{Ex}[R_1 \cdot R_2] = \text{Ex}[R_1] \cdot \text{Ex}[R_2].$$

Proof.

$$\begin{aligned}
&\text{Ex}[R_1 \cdot R_2] \\
&= \sum_{r \in \text{range}(R_1 \cdot R_2)} r \cdot \Pr[R_1 \cdot R_2 = r] \\
&= \sum_{r_i \in \text{range}(R_i)} r_1 r_2 \cdot \Pr[R_1 = r_1 \text{ and } R_2 = r_2] \\
&= \sum_{r_1 \in \text{range}(R_1)} \sum_{r_2 \in \text{range}(R_2)} r_1 r_2 \cdot \Pr[R_1 = r_1 \text{ and } R_2 = r_2] \\
&= \sum_{r_1 \in \text{range}(R_1)} \sum_{r_2 \in \text{range}(R_2)} r_1 r_2 \cdot \Pr[R_1 = r_1] \cdot \Pr[R_2 = r_2] \\
&= \sum_{r_1 \in \text{range}(R_1)} \left(r_1 \Pr[R_1 = r_1] \cdot \sum_{r_2 \in \text{range}(R_2)} r_2 \Pr[R_2 = r_2] \right) \\
&= \sum_{r_1 \in \text{range}(R_1)} r_1 \Pr[R_1 = r_1] \cdot \text{Ex}[R_2] \\
&= \text{Ex}[R_2] \cdot \sum_{r_1 \in \text{range}(R_1)} r_1 \Pr[R_1 = r_1] \\
&= \text{Ex}[R_2] \cdot \text{Ex}[R_1].
\end{aligned}$$

■

In Class 12 Fri

4/28

1. Make carnival fair

For what value of k is it fair

$$0 = -1\left(\frac{125}{216}\right) + 1\left(\frac{75}{216}\right) + 2\left(\frac{15}{216}\right) + k\left(\frac{1}{216}\right)$$

solve for k

$$0 = -\frac{5}{54} + \frac{k}{216}$$

$$\frac{5}{54} = \frac{k}{216}$$

$$54k = 5 \cdot 216$$

$$k = \frac{5 \cdot 216}{54}$$

$$k = 20$$ ✓

2. Desks

Hard - were to start

2x2 grid and generalize?

Max 24 couples

$\frac{1}{2}$ prob BB BG GB BG they flirt

X ✓ ✓ X

So 12

← 6.041 - should go back & reread

② But couples are ind ?!

? if they were couples (but not what is what!)

tells you nothing about!

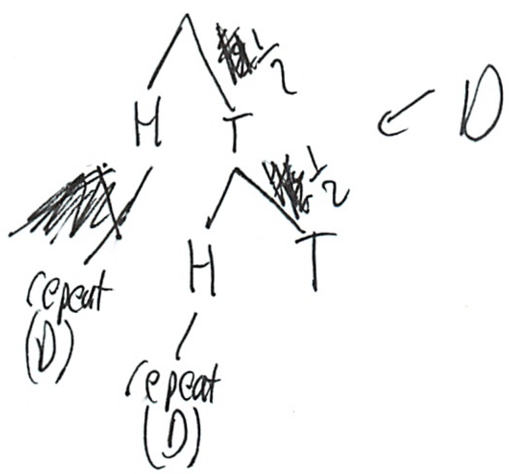
3. Flip fair coin

N_{TT} = # flips until 2 tails appear

- Same as 6.041 issues

What is $E[N_{TT}]$

They rec. a tree diagram



$$D = H \cdot D + T(H \cdot D + T)$$

since repeat is here

and here

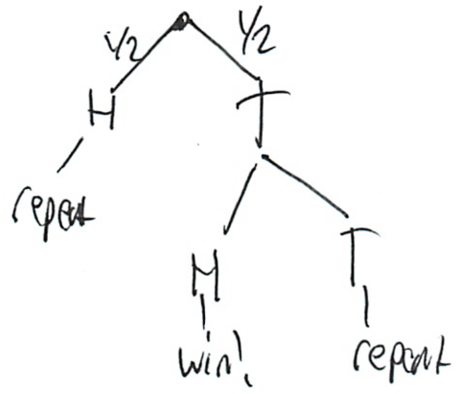
a win

Use <u>Law of Total Expectation</u>

$$E[R] = \sum_i E[R \mid A_i] P[A_i]$$

(3)

b) Now N_{TH}



? Same as a) ?

c) Flip till TT or TH

TT = win

TH = loss

Since TT takes 50% longer to show up

?? really ??

Win = 6

Loss = −5

What is E[profit]?

You have odds stacked in your favor

(4)

3a again) Never really ans ~~a~~

$$E = \frac{1}{2} \cdot E + \frac{1}{2}\left(\frac{1}{2} \cdot E + \frac{1}{2}\right)$$

$$E = \frac{1}{2}E + \frac{1}{4}E + \frac{1}{4}$$

$$E\left(1 - \frac{1}{2} - \frac{1}{4}\right) = \frac{1}{4}$$

$$\frac{1}{4}E = \frac{1}{4}$$

$\cdot 4 \quad \cdot 4$

$$E = 1$$

Wrong

Can't do that!

E[# flips]

need to x by # of flips

~~3b) is it diff~~
again

$$E = 1 \cdot \frac{1}{2} \cdot E + 2 \cdot \frac{1}{2}\left(\frac{1}{2} \cdot E + \frac{1}{2}\right)$$

$$E = \frac{1}{2}E + \frac{1}{2}E + \frac{1}{2}$$

$$E = 1E + \frac{1}{2}$$

Can't do that!

(5)

2a board) $E = \frac{1}{2}(E+1) + \frac{1}{4}(E+2) + \frac{1}{2}(2)$

$= \frac{3}{4}E + \frac{3}{2}$

$= 6$

2a Allan)

$E[\text{asorbtion}_T] = \frac{1}{2}S + 1$

$S = \frac{1}{2}S + \frac{1}{2}T + 1$

Solve → $S = T + 2$

$\frac{1}{2}T = 2$

$T = 4$

$S = T + 2$

$= 6$

6

3b board)

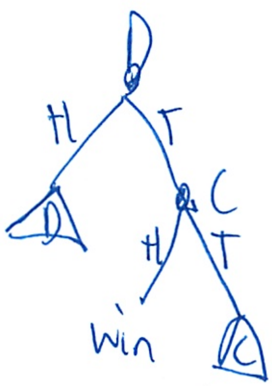
$$E = \frac{1}{2}(E+2) + \frac{1}{4}(2) + \frac{1}{4}(E+1)$$

$$= \frac{1}{2}E + 2$$

$$= 4$$

Interesting thats its a flip

But its not! Look at carefully!



goes to diff places

TA: this is ~~wrong~~ incomplete
how get rid of C?
not obvious!

Prof: not that big a deal
Solve C in terms of C

This is just a Markov chain!

(2)

$$E(D) = \frac{1}{2}(E(D)+1) + \frac{1}{2}(E(C)+1)$$

$$E[C] = \frac{1}{2}(1) + \frac{1}{2}(E(C)+1)$$

$$= \frac{1}{2}E(C) + 1$$

$$= 2$$

$$E[D] = \frac{1}{2}E[D] + \frac{1}{2} + 3/2$$

$$\frac{1}{2}E[D] = 2$$

$$E[D] = 4$$

Prof Think about the contridiction I fell for

Massachusetts Institute of Technology
6.042J/18.062J, Spring '11: Mathematics for Computer Science
Prof. Albert R Meyer

April 29
revised Wednesday 27th April, 2011, 01:05

Solutions to In-Class Problems Week 12, Fri.

Problem 1.
Let's see what it takes to make Carnival Dice fair. Here's the game with payoff parameter k: make three independent rolls of a fair die. If you roll a six

- no times, then you lose 1 dollar.
- exactly once, then you win 1 dollar.
- exactly twice, then you win two dollars.
- all three times, then you win k dollars.

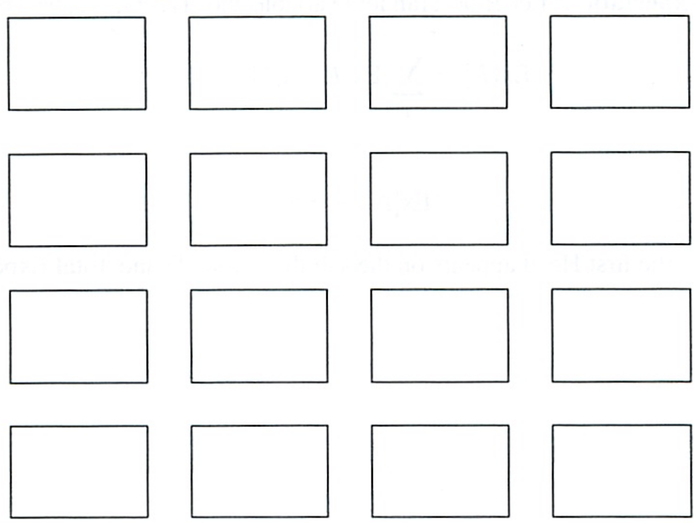
For what value of k is this game fair?

Solution. Let the random variable P be your payoff. Then we can compute $\text{Ex}[P]$ as follows:

$$\begin{aligned}
\text{Ex}[P] &= -1 \cdot \Pr[0 \text{ sixes}] + 1 \cdot \Pr[1 \text{ six}] + 2 \cdot \Pr[2 \text{ sixes}] + k \cdot \Pr[3 \text{ sixes}] \\
&= -1 \cdot \left(\frac{5}{6}\right)^3 + 1 \cdot 3\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^2 + 2 \cdot 3\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right) + k \cdot \left(\frac{1}{6}\right)^3 \\
&= \frac{-125 + 75 + 30 + k}{216}
\end{aligned}$$

The game is fair when $\text{Ex}[P] = 0$. This happens when $k = 20$. ■

Problem 2.
A classroom has sixteen desks arranged as shown below.



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If there is a girl in front, behind, to the left, or to the right of a boy, then the two of them *flirt*. One student may be in multiple flirting couples; for example, a student in a corner of the classroom can flirt with up to two others, while a student in the center can flirt with as many as four others. Suppose that desks are occupied by boys and girls with equal probability and mutually independently. What is the expected number of flirting couples? *Hint:* Linearity.

Solution. First, let's count the number of pairs of adjacent desks. There are three in each row and three in each column. Since there are four rows and four columns, there are $3 \cdot 4 + 3 \cdot 4 = 24$ pairs of adjacent desks.

Number these pairs of adjacent desks from 1 to 24. Let F_i be an indicator for the event that occupants of the desks in the i-th pair are flirting. The probability we want is then:

$$\begin{aligned}
\text{Ex}[\sum_{i=1}^{24} F_i] &= \sum_{i=1}^{24} \text{Ex}[F_i] && \text{(linearity of Ex}[\cdot]) \\
&= \sum_{i=1}^{24} \Pr[F_i = 1] && (F_i \text{ is an indicator})
\end{aligned}$$

The occupants of adjacent desks are flirting iff they are of opposite sexes, which happens with probability 1/2, that is, $\Pr[F_i = 1] = 1/2$. Plugging this into the previous expression gives:

$$\text{Ex}[\sum_{i=1}^{24} F_i] = \sum_{i=1}^{24} \Pr[F_i = 1] = 24 \cdot \frac{1}{2} = 12$$

oh fairly simple, I thought would be much more complex!

■

Problem 3. **(a)** Suppose we flip a fair coin and let N_{TT} be the number of flips until the first time two Tails in a row appear. What is $\text{Ex}[N_{\text{TT}}]$?

Hint: Let D be the tree diagram for this process. Explain why

$$D = H \cdot D + T \cdot (H \cdot D + T).$$

Use the **Law of Total Expectation**: Let R be a random variable and $A_1, A_2, \ldots,$ be a partition of the sample space. Then

$$\text{Ex}[R] = \sum_i \text{Ex}[R \mid A_i] \Pr[A_i].$$

Solution.

$$\text{Ex}[N_{\text{TT}}] = 6.$$

Let H_k be the event that the first Head appears on the kth flip. From D and Total Expectation:

$$
\begin{aligned}
&\mathrm{Ex}[N_{\mathrm{TT}}] \\
&= \mathrm{Ex}[N_{\mathrm{TT}} \mid H_1] \cdot \Pr[H_1] + \mathrm{Ex}[N_{\mathrm{TT}} \mid \overline{H_1}] \cdot \Pr[\overline{H_1}] \\
&= (1 + \mathrm{Ex}[N_{\mathrm{TT}}]) \cdot \frac{1}{2} \\
&\qquad + \left(\left(\mathrm{Ex}[N_{\mathrm{TT}} \mid H_2] \Pr\left[H_2 \mid \overline{H_1}\right]\right) + \left(\mathrm{Ex}[N_{\mathrm{TT}} \mid \overline{H_1} \cap \overline{H_2}]\right) \Pr\left[\overline{H_2} \mid \overline{H_1}\right]\right) \cdot \frac{1}{2} \\
&= (1 + \mathrm{Ex}[N_{\mathrm{TT}}]) \cdot \frac{1}{2} + \left((2 + \mathrm{Ex}[N_{\mathrm{TT}}]) \cdot \frac{1}{2} + 2 \cdot \frac{1}{2}\right) \cdot \frac{1}{2} \\
&= \frac{1}{2} + \frac{\mathrm{Ex}[N_{\mathrm{TT}}]}{2} + (2 + \mathrm{Ex}[N_{\mathrm{TT}}]) \cdot \frac{1}{4} + \frac{1}{2} \\
&= \frac{3}{2} + \frac{3\,\mathrm{Ex}[N_{\mathrm{TT}}]}{4}
\end{aligned}
$$

So

$$\frac{\mathrm{Ex}[N_{\mathrm{TT}}]}{4} = \frac{3}{2}.$$

■

(b) Suppose we flip a fair coin until a Tail immediately followed by a Head come up. What is the expectation of the number N_{TH} of flips we perform?

Solution.

$$\mathrm{Ex}[N_{\mathrm{TH}}] = 4.$$

This time the tree diagram is $C ::= H \cdot C + T \cdot B$ where the subtree $B ::= H + T \cdot B$.

So

$$\mathrm{Ex}[N_{\mathrm{TH}}] = (1 + \mathrm{Ex}[N_{\mathrm{TH}}]) \cdot \frac{1}{2} + (1 + \mathrm{Ex}[N_B]) \cdot \frac{1}{2}$$

where N_B is the expected number of flips in the B subtree. But

$$\mathrm{Ex}[N_B] = 1 \cdot \frac{1}{2} + (1 + \mathrm{Ex}[N_B]) \cdot \frac{1}{2}.$$

That is, $\mathrm{Ex}[N_B] = 2$. Hence,

$$\mathrm{Ex}[N_{\mathrm{TH}}] = \frac{1}{2} + \frac{\mathrm{Ex}[N_{\mathrm{TH}}]}{2} + \frac{1}{2} + \frac{2}{2}$$

which implies $\mathrm{Ex}[N_{\mathrm{TH}}] = 4$. ■

(c) Suppose we now play a game: flip a fair coin until either `TT` or `TH` first occurs. You win if `TT` comes up first, lose if `TH` comes up first. Since `TT` takes 50% longer on average to turn up, your opponent agrees that he has the advantage. So you tell him you're willing to play if you pay him \$5 when he wins, but he merely pays you a 20% premium, that is, \$6, when you win.

If you do this, you're sneakily taking advantage of your opponent's untrained intuition, since you've gotten him to agree to unfair odds. What is your expected profit per game?

Solution. It's easy to see that both `TT` and `TH` are equally likely to show up first. (Every game play consists of a sequence of `H`'s followed by a `T`, after which the game ends with a `T` or an `H`, with equal probability.) So your expected profit is

$$\frac{1}{2} \cdot 6 + \frac{1}{2} \cdot (-5)$$

dollars, that is 50 cents per game. So leap to play. ■

Problem 4.
Justify each line of the following proof that if R_1 and R_2 are *independent*, then

$$\text{Ex}[R_1 \cdot R_2] = \text{Ex}[R_1] \cdot \text{Ex}[R_2].$$

Proof.

$$\begin{aligned}
&\text{Ex}[R_1 \cdot R_2] \\
&= \sum_{r \in \text{range}(R_1 \cdot R_2)} r \cdot \Pr[R_1 \cdot R_2 = r] \\
&= \sum_{r_i \in \text{range}(R_i)} r_1 r_2 \cdot \Pr[R_1 = r_1 \text{ and } R_2 = r_2] \\
&= \sum_{r_1 \in \text{range}(R_1)} \sum_{r_2 \in \text{range}(R_2)} r_1 r_2 \cdot \Pr[R_1 = r_1 \text{ and } R_2 = r_2] \\
&= \sum_{r_1 \in \text{range}(R_1)} \sum_{r_2 \in \text{range}(R_2)} r_1 r_2 \cdot \Pr[R_1 = r_1] \cdot \Pr[R_2 = r_2] \\
&= \sum_{r_1 \in \text{range}(R_1)} \left(r_1 \Pr[R_1 = r_1] \cdot \sum_{r_2 \in \text{range}(R_2)} r_2 \Pr[R_2 = r_2] \right) \\
&= \sum_{r_1 \in \text{range}(R_1)} r_1 \Pr[R_1 = r_1] \cdot \text{Ex}[R_2] \\
&= \text{Ex}[R_2] \cdot \sum_{r_1 \in \text{range}(R_1)} r_1 \Pr[R_1 = r_1] \\
&= \text{Ex}[R_2] \cdot \text{Ex}[R_1].
\end{aligned}$$

■

Solution. Note that the event $[R_1 \cdot R_2 = r]$ is the disjoint union of events $[R_1 = r_1 \text{ AND } R_2 = r_2]$ such that $r_i \in \text{range}(R_i)$ for $i = 1, 2$ and $r_1 r_2 = r$.

Proof.

$$
\begin{aligned}
&\mathrm{Ex}[R_1 \cdot R_2] \\
&::= \sum_{r \in \mathrm{range}(R_1 \cdot R_2)} r \cdot \Pr[R_1 \cdot R_2 = r] && \text{(by definition)} \\
&= \sum_{r_i \in \mathrm{range}(R_i)} r_1 r_2 \cdot \Pr[R_1 = r_1 \text{ AND } R_2 = r_2] && \text{(remarked above)} \\
&= \sum_{r_1 \in \mathrm{range}(R_1)} \sum_{r_2 \in \mathrm{range}(R_2)} r_1 r_2 \cdot \Pr[R_1 = r_1 \text{ AND } R_2 = r_2] && \text{(ordering terms in the sum)} \\
&= \sum_{r_1 \in \mathrm{range}(R_1)} \sum_{r_2 \in \mathrm{range}(R_2)} r_1 r_2 \cdot \Pr[R_1 = r_1] \cdot \Pr[R_2 = r_2] && \text{(independence of } R_1, R_2\text{)} \\
&= \sum_{r_1 \in \mathrm{range}(R_1)} \left(r_1 \Pr[R_1 = r_1] \cdot \sum_{r_2 \in \mathrm{range}(R_2)} r_2 \Pr[R_2 = r_2] \right) && \text{(factor out } r_1 \Pr[R_1 = r_1]\text{)} \\
&= \sum_{r_1 \in \mathrm{range}(R_1)} r_1 \Pr[R_1 = r_1] \cdot \mathrm{Ex}[R_2] && \text{(def of Ex}[R_2]\text{)} \\
&= \mathrm{Ex}[R_2] \cdot \sum_{r_1 \in \mathrm{range}(R_1)} r_1 \Pr[R_1 = r_1] && \text{(factor out Ex}[R_2]\text{)} \\
&= \mathrm{Ex}[R_2] \cdot \mathrm{Ex}[R_1]. && \text{(def of Ex}[R_1]\text{)}
\end{aligned}
$$

■

Tutor 11 5/1

11.1 Conditional Prob

A = event A L lectures $P(A) = .8$

B = event I attend lecture $P(B) = .4$

$P(B \mid A) = .3$

$P(A \mid B) = ?$

do from memor

$\frac{P(A \cap B)}{P(B)}$ ← but what is this

oh

oh $\frac{P(B \cap A)}{P(A)} = .3$

$\frac{x}{.8} = .3$

$x = .24$

So $\frac{.24}{.4} = .6$

②

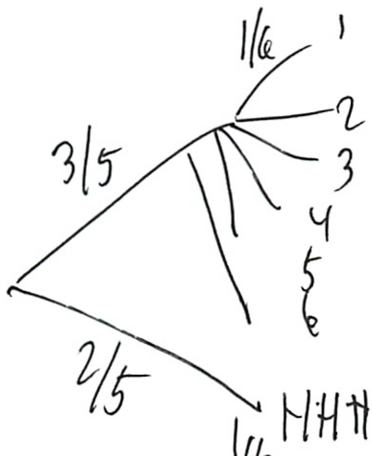
TP 11.2 A Random

First flip coin $P(\text{heads}) = \frac{3}{5}$ ← $F\begin{cases}1 & \text{heads}\\ 0 & \text{tails}\end{cases}$

If heads roll die, return

If tails flip ~~a~~ fair coin 3x, return heads ×2

N = # return



$\frac{1}{10}$

HHH	1/20	6
HTH		4
HHT		4
THH		4
HTT		2
THT		2
TTH		2
TTT		0

So

$$N = \begin{cases} 0 & 1/20 \\ 1 & 1/10 \\ 2 & 1/4 \\ 3 & 1/10 \\ 4 & 1/4 \\ 5 & 1/10 \\ 6 & 3/20 \end{cases}$$

0 otherwise

✓

(3)

Now given $F=0$

oh just look at original tree

$$\frac{1/20}{2/5} = \frac{1}{8}$$

↑given original

Now other way – add possibilities

$$\frac{\frac{1}{20}}{\frac{1}{20} + \frac{1}{10}} = \frac{\frac{1}{20}}{\frac{3}{20}} = \frac{1}{20} \cdot \frac{20}{3} = \frac{1}{3}$$

~~a~~ Now have to actually add the #s

~~dwell~~ So for top part add #, ~~so~~ from $2 \to 7$

$$N+T = \begin{cases} 0 & 1/20 \\ 1 & 0 \\ 2 & 1/4 \\ 3 & 1/10 \\ 4 & 1/4 \\ 5 & 1/10 \\ 6 & 3/20 \\ 7 & 1/10 \end{cases}$$

Woot perfect so far!

(4)

TP 11.3 Independence

x = 3 bit string

E = x odd # of 1s

F = x starts w/ 1

G = x starts w/ 0

H = x ends w/ 1

Part 1: Ind of 2 events

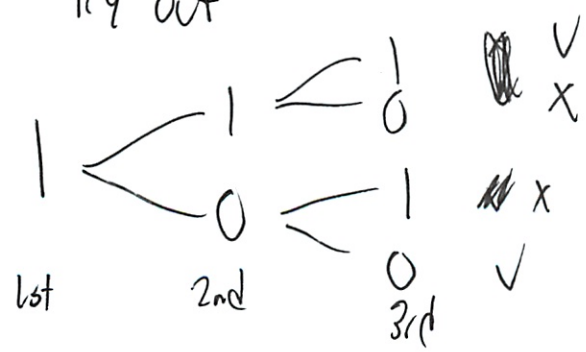
Which pairs are ind?

- oh tough to make sure have all

EF or EG - no?

↑ try out

EF



actually split Yes

EG

actually do yes

(5)

Guess H is same (w/ E) Yes

FG x No!

FH Yes – has nothing to do

GH Yes ✓

Part 2 Pairwise + Mutual Ind

1. Are the events ~~E, G, H~~ E, F, H pairwise ind?

– ie ~~da~~ is each pair ind

EF ✓

EH ✓ Yes ✓

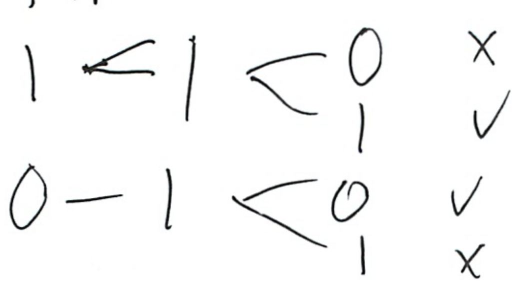
FH ✓

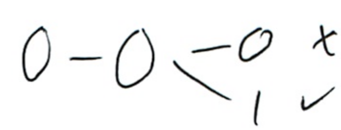
2. Are E, F, H mutually ind?

All together E, F, H?

No, ~~b~~ if knew F, H, would you know E

Say F, H true





Yes! ✓

(6)

Part 3: More

a) F, G, H are:

Not mutual

and not pair F, G

b) E, G, H:

both - well mutually

c) E, F, G

F, G, no!

11.4 Binomial Board Breaking

5 ft boards

if can break w/ $p = .8$

each board ind

1a.) P (Bruce breaks exactly 2 out of 5 boards)

$.8 \cdot .8 \cdot .2 \cdot .2 \cdot .2 = .00512$ X

but any order - ? do we care about that?

Yes $\binom{5}{2}$

7

Why is that again?

- I need to add all those up?

(5 choose 2) = 10

↑ So is multiplicative (result larger)

.0512 ✓

b) At most 3 boards

So $P(0) + P(1) + P(2) + P(3)$

$= \binom{5}{0} .2^5 + \binom{5}{1} .8 \cdot .2^4 + \binom{5}{2} .8^2 \cdot .2^3 + \binom{5}{3} .8^3 \cdot .2^2$

$= .00032 + .00064 + .0512 + .2048$

$= .2656$ ✗

Earlier $\underbrace{\binom{5}{2}}_{\#\text{ seq}} \underbrace{.8^2 \; .2^3}_{\text{prob of each seq}}$

Yeah CDF says $\sum_{i=0}^{k}$

But this should be right, unless math error

.26272 ✓ was math error

(8)

c) E[# boards broken]

diff section

(I've seen all this stuff before — but not memorized

well get cheat sheet)

17.4.10

Yeah just weighted avg

$= 0 \cdot .00032 + 1 \cdot .0064 + 2 \cdot .0512 + 3 \cdot .2048$

$+ 4 \cdot .4096 + 5 \cdot .32768$

$= 4$ ← no dividing needed ✓

~~ND~~

11.5 Great Expectations

1. What the E[sum of # when roll 6 sided die + 12 sided die]

 ; how to automate this

 ; oh linearity — both individually

 Only have 1 6 sided die — add differently

 still think will be 7 — high # ignored ; (X)

 guess

9

1
2
3
4
5
6

1
2
3
4
5
6
7
8
9
10
11
12

Sn

2	|
3	||
4	|||
5	||||
6	
7	
8	
9	
10	
11	
12	
13	
14	
15	||||
16	|||
17	||
18	|

, was there a better way?

(10)

b) PC 1 RV $\{1, \ldots, 99\}$ uniform

2 RV $\{1, \ldots, 999\}$ uniform

Roll a fair die. If 5 up, use PC 1, otherwise 2

-need to know the trick!

slight under 499

(did not really read section)

linearity

So $E[1,99]$ is $\frac{b-a}{2}$ < 6.041!

49

499

So

$$\frac{1}{6} \cdot 49 + \frac{5}{6} 499 =$$

424 ⊗

Can you weight them like that?

Why not?

(11)

c) What if multiplied

$49 \cdot \cancel{40}449 \stackrel{?}{=} 24451$

? does not seem right

(x) Its not

Give up

b) 425.0

$E[\text{generated}] = P(\text{roll } 5) \cdot E[\cancel{\#} PC\ 1] + P(\text{not } 5)\, E\{PC_2\}$

$= \frac{1}{6} \cdot 50 + \frac{5}{6} \cdot 500$

?

but how's it 50?

I emailed in -

c) 25000

$E[PC_1 \cdot PC_2] = E[PC_1] \cdot E[PC_2] = 50 \cdot 500$

$= 25000$

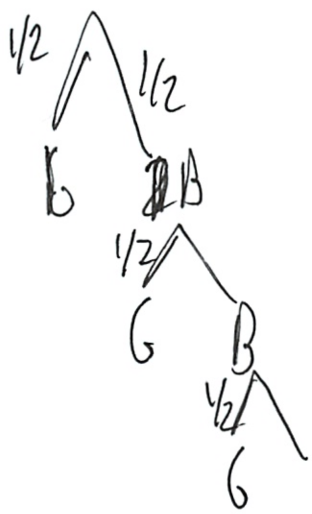
Another one I contest

(12) (I am doing good on this section!)

TP 11.6 Until We Have a Girl

Keep having child till girl
(Like in China, etc except w/ boys)

1. What is $PDF_B(i)$?
   - ? what is the notation they are using?
   - anyway try it 1st



$\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \ldots$

? what is long form of this
never good at that ...

WA $\sum_{k=1}^{\infty} 2^{-k} = 1$

? Oh duh has to add to 1

But they want abstract expression
i = # children already have?

$\left(\frac{1}{2}\right)^{i}$ or $\left(\frac{1}{2}\right)^{i+1}$
? since $\frac{1}{2}^0 = 1$ which is wrong!

✓

(13)

b) What is CDF

Σ PDFs up to that point

So that sum I had

And my long form from WA

↑ no

Do WA again

$$\sum_{k=1}^{x} 2^{-k} = 2^{-x}(2^x - 1)$$

↑ I should pratice doing this manual

~~Remember~~

$1 - \left(\frac{1}{2}\right)^{i+1}$ is what they had

c) E[# of boys]

= mean time till "failure"

(didn't read that either)

$$E[C] = \underbrace{E[C \mid A]}_{1 = b/c \text{ 1st step}} P(A) + E[C \mid \bar{A}] P(\bar{A})$$

↑ does not crash 1st step

$$E[C \mid \bar{A}] = 1 + E[C]$$

(14)

$$E[C] = 1 \cdot p + (1 + E[C])(1-p)$$
$$= p + 1 - p + (1-p) E[C]$$
$$= 1 + (1-p) E[C]$$

← look carefully at later

Rearrange

$$1 = E[C] - (1-p) E[C]$$
$$= p E[C]$$

So

$$E[C] = \frac{1}{p}$$

← oh simple

$$\frac{1}{.5} = 2$$

Think said in book

# of boys to have = above $-1 = 1$

Done - Got all right, except that one I emailed in on

Looked up $E[\text{uniform}] = \frac{a+b}{2}$

Mathematics for Computer Science
MIT 6.042J/18.062J

Deviation from the Mean

Example: IQ

IQ measure was constructed so that

average IQ = 100.

What fraction of the people can possibly have an IQ $\geq$ 300?

...at most 1/3

IQ Higher than 300?

If more than 1/3 have IQ $\geq$ 300, then
avg $> (1/3)\cdot 300 > 100$!
--a contradiction

IQ Higher than x?

In general,

$$\Pr\{IQ \geq x\} \leq \frac{100}{x}$$

IQ Higher than x?

Besides mean = 100,
we used only one fact about the distribution of IQ:
IQ is always nonnegative

Markov Bound

If R is nonnegative, then

$$\boxed{\Pr\{R \geq x\} \leq \frac{E[R]}{x}}$$

for $x > 0$

Markov Bound

- Weak
- Obvious
- Useful anyway

IQ ≥ 300, again

Suppose we are given that IQ is always ≥ 50?
Get a better bound using
(IQ - 50)
since this is now ≥ 0.

IQ ≥ 300, again

$$\Pr\{IQ \geq 300\} =$$

$$\Pr\{IQ - 50 \geq 300 - 50\}$$

$$\leq \frac{100-50}{300-50} = \frac{1}{5}$$

Improving the Markov Bound

$$\Pr\{|R-\mu| \geq x\}$$

$$= \Pr\{(R-\mu)^2 \geq x^2\}$$

by Markov:

$$\leq \frac{E[(R-\mu)^2]}{x^2}$$

variance of R

Chebyshev Bound

$$\boxed{\Pr\{|R-\mu| \geq x\} \leq \frac{Var[R]}{x^2}}$$

$$Var[R] ::= E[(R-\mu)^2]$$

$$\sigma_R ::= \sqrt{Var[R]}$$

Standard Deviation

$$\boxed{\Pr\{|R-\mu| \geq x\} \leq \frac{\sigma^2}{x^2}}$$

R probably not many σ's from μ:

further than	σ	Pr ≤ 1
	2σ	Pr ≤ 1/4
	3σ	Pr ≤ 1/9
	4σ	Pr ≤ 1/16

Variance of an Indicator

I an indicator with E[I]=p:

$$\boxed{\mathrm{Var}[I]} ::= E\left[(I-p)^2\right]$$
$$= E\left[I^2\right] - 2pE\left[I\right] + p^2$$
$$= E\left[I\right] - 2p\cdot p + p^2$$
$$= p - 2p^2 + p^2 \boxed{= pq}$$

Albert R Meyer, May 2, 2011 lec 13M.37

Calculating Variance

$$\mathrm{Var}[aR+b] = a^2\,\mathrm{Var}[R]$$
$$\mathrm{Var}[R] = E[R^2] - \left(E[R]\right)^2$$

simple proofs applying linearity of E[] to the def of Var[]

Albert R Meyer, May 2, 2011 lec 13M.38

Calculating Variance

Pairwise Independent Additivity

$$\mathrm{Var}[R_1 + R_2 + \cdots + R_n]$$
$$= \mathrm{Var}[R_1] + \mathrm{Var}[R_2] + \cdots + \mathrm{Var}[R_n]$$

providing $R_1, R_2, \ldots, R_n$ are pairwise independent

again, a simple proof applying linearity of E[] to the def of Var[]

Albert R Meyer, May 2, 2011 lec 13M.46

Jacob D. Bernoulli (1659–1705)

Even the stupidest man –by some instinct of nature *per se* and by no previous instruction (this is truly amazing) –knows for sure that the more observations ...that are taken, the less the danger will be of straying from the mark.

---*Ars Conjectandi* (The Art of Guessing), 1713*

*taken from Grinstead & Snell, http://www.dartmouth.edu/~chance/teaching_aids/books_articles/probability_book/book.html *Introduction to Probability*, American Mathematical Society, p. 310.

Albert R Meyer, May 2, 2011 lec 13M.48

Jacob D. Bernoulli (1659–1705)

It certainly remains to be inquired whether after the number of observations has been increased, the probability…of obtaining the true ratio…finally exceeds any given degree of certainty; or whether the problem has, so to speak, its own asymptote –that is, whether some degree of certainty is given which one can never exceed.

Albert R Meyer, May 2, 2011 lec 13M.49

Repeated Trials

Random var R with mean μ

n independent observations

$$R_1, \cdots, R_n$$

Albert R Meyer, May 2, 2011 lec 13M.57

Repeated Trials

take average:

$$A_n ::= \frac{R_1 + R_2 + \cdots + R_n}{n}$$

Bernoulli question: is it probably close to μ if n is big

$$\Pr\{|A_n - \mu| \le \delta\} = ?$$

Jacob D. Bernoulli (1659 - 1705)

Therefore, this is the problem which I now set forth and make known after I have pondered over it for twenty years. Both its novelty and its very great usefulness, coupled with its just as great difficulty, can exceed in weight and value all the remaining chapters of this thesis.

Weak Law of Large Numbers

$$\lim_{n\to\infty} \Pr\{|A_n - \mu| \le \delta\} = 1$$

$$\boxed{\lim_{n\to\infty} \Pr\{|A_n - \mu| > \delta\} = 0}$$

Weak Law of Large Numbers

will follow easily by Chebyshev & variance properties

$$\boxed{\lim_{n\to\infty} \Pr\{|A_n - \mu| > \delta\} = 0}$$

Repeated Trials

$$\boxed{E[A_n]} ::= E\left[\frac{R_1 + R_2 + \cdots + R_n}{n}\right]$$

$$= \frac{E[R_1] + E[R_2] + \cdots + E[R_n]}{n}$$

$$= \frac{n\mu}{n} \boxed{= \mu}$$

Weak Law of Large Numbers

So by Chebyshev

$$\Pr\{|A_n - \mu| > \delta\} \le \frac{\mathrm{Var}[A_n]}{\delta^2}$$

need only show

$\mathrm{Var}[A_n] \to 0$ as $n \to \infty$

Repeated Trials

$$\boxed{Var[A_n]} = Var\left[\frac{(R_1 + R_2 + \cdots + R_n)}{n}\right]$$

$$= \frac{(Var[R_1] + Var[R_2] + \cdots + Var[R_n])}{n^2}$$

$$QED = \frac{n\sigma^2}{n^2} = \boxed{\frac{\sigma^2}{n}} \longrightarrow 0$$

Albert R Meyer, May 2, 2011 lec 13M.66

Analysis of the Proof

proof only used that $R_1,\ldots,R_n$ have

- same mean
- same variance
- & variances add
  — which follows from pairwise independence

Albert R Meyer, May 2, 2011 lec 13M.67

Pairwise Independent Sampling

Theorem:
Let $R_1,\ldots,R_n$ be pairwise independent random vars with the same finite mean μ and variance σ^2. Let $A_n ::= (R_1 + R_2 + \cdots + R_n)/n$. Then

$$\boxed{\Pr\{|A_n - \mu| > \delta\} \le \frac{1}{n}\left(\frac{\sigma}{\delta}\right)^2}$$

Albert R Meyer, May 2, 2011 lec 13M.68

Pairwise Independent Sampling

The punchline:
we now know how big a sample is needed to estimate the mean of any* random variable within any* desired tolerance with any* desired probability

*variance $< \infty$, tolerance > 0, probability < 1

Albert R Meyer, May 2, 2011 lec 13M.69

Birthday Pairs

D ::= # pairs with matching b'days among *n* people in a *d*-day year

$$D = \sum_{1 \le i < j \le n} M_{ij}$$

M_{ij} ::= indicator that ith & jth birthdays match

Albert R Meyer, May 2, 2011 lec 13M.70

Birthday Pairs

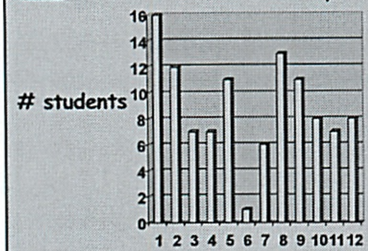
$$E[M_{ij}] = 1/d$$

so by linearity of E[]

$$E[D] = \sum_{1 \le i < j \le n} E[M_{ij}] = \binom{n}{2} \cdot \frac{1}{d}$$

Albert R Meyer, May 2, 2011 lec 13M.71

Actual Distribution by Month



Uniformity assumption not true but pretend it is

Birthday Pairs

Have data on 91 students

$$E[D] = \binom{91}{2} \cdot \frac{1}{365} \approx 11.2$$

Pairwise Independence

[Albert and Sonya have same b'day] is independent of [Albert and Olga have same b'day]

that is, $E_{Alice,Bob}$ & $E_{Alice,Carol}$ are independent

(pairwise, but not 3-way: $E_{Bob,Carol}$ depends on other two)

Birthday Pairs

$\mathrm{Var}[M_{ij}] = (1/365)(1 - 1/365)$

so by prwise linearity of Var[]

$$\mathrm{Var}[D] = \sum \mathrm{Var}[M_{ij}]$$

$$= \binom{91}{2} \cdot \frac{1}{365} \cdot \left(1 - \frac{1}{365}\right) \approx 11.2$$

Birthday Pairs

$$\mathrm{Var}[M] \approx 11.1$$

$$\sigma_M < 4$$

Birthday Predictions

Chebyshev:

$\Pr\{11.2 \pm 2\sigma \text{ pairs}\} > 1 - (1/2)^2 = 3/4$

4 to 20 pairs 75% of the time

We actually found *exactly* 13 pairs (& no triples)

Spring '11 Matching Birthdays

1. Jan 20
2. Jan 22
3. Jan 23
4. Apr 04
5. May 12
6. May 14
7. May 28
8. Jul 23
9. Sep 19
10. Oct 22
11. Nov 02
12. Nov 13
13. Nov 18

Team Problems

Problems 1–4

6.042 Deviations from The Mean

5/2

(4 min late)

Avg IQ = 100

What fraction can possibly have IQ ≥ 300?

- at most $\frac{1}{3}$

  if avg $> (\frac{1}{3})\, 300 > 100$

  Contridiction

General

$$P(IQ \geq x) \leq \frac{100}{x}$$

- make IQ a RV

(IQs can't be nonnegitive)

$$P(R \geq x) \leq \frac{E[R]}{x} \quad \text{for } x > 0$$

(Markov's theorm

②

Pretty weak bound

- since very few people have IQ of 300

obvious

but very useful anyway

Can stenghten:

Suppose told IQ is ≥ 50

So have new RV: $IQ - 50$

So $P(IQ \geq 300) = P(IQ - 50 \geq 300 - 50)$

$$\leq \frac{100 - 50}{300 - 50} = \frac{1}{5}$$

new, better upper bound

So get better bounds if you have a lower bound

③ Further improving Markov Bound

$$P(|R - \mu| \geq x)$$

$$= P((R - \mu)^2 \geq x^2)$$

nonneg RV
can get
directly

Apply Markov to this

by Markov:

$$\leq \frac{E[(R-\mu)^2]}{x^2}$$ ← Var of R

- measures how unbalanced the dist of R is

Restated as Chebyshev Bound

$$P(|R - \mu| \geq x) \leq \frac{Var(R)}{x^2}$$

Can increase power to ↑ accuracy

But often 4th power is ∞ on some RV which are "normal" when squared

9

Have to look at ∞ series to see how they behave

$Var(R) = E[(R-\mu)^2]$

$\sigma_R = \sqrt{Var(R)}$

$var^2 = \sigma$

St dev bands

$$P(|R-\mu| \geq x) \leq \frac{\sigma^2}{x^2}$$

R is prob. not many σ from μ

σ $P \leq 1$

2σ $P \leq 1/4$

3σ $P \leq 1/9$

4σ $P \leq 1/16$ ↓ quadratically

Sometimes faster

- binomial → exponentially

but this is a bound

3)

I is an indicator w/ $E[I]=p$

$\mathrm{Var}(I) = \underset{\text{by def}}{E[(I-p)^2]}$ by def

$= E[I^2] - 2pE[I] + p^2$ expand

$= E[I] - 2p \cdot p + p^2$ linearity of expectations

$= p - 2p^2 + p^2$

$= pq$
$q=(1-p)$

Calculating Var

$\mathrm{Var}(aR+b) = \cancel{}\, a^2 \,\mathrm{Var}(R)$

? multiplicative factor ?

$\mathrm{Var}(R) = E[R^2] - (E[R])^2$

(6)

Var is not linear

but easy to calc if you have pairwise ind. variables

$Var[R_1 + R_2 + \dots R_n]$ if pairwise ind

$= Var(R_1) + Var(R_2) + \dots + Var(R_n)$

- only dealing w/ terms that are power of 2

Story

Bernoulli came up w/ this

Said everyone could tell more obs - get closer to mean

Can you have a degree of certainty?

RV R w/ mean μ

n ind. obs

take avg $A_n = \dfrac{R_1 + R_2 + \dots + R_n}{n}$

Bernoulli q: Is it prob close to μ if n is big

$P(|A_n - \mu| \le \delta) = ?$

Berlouni took years to find this

No limit to prob

$$\lim_{n \to \infty} P(|A_n - \mu| \le \delta) = 1$$

↑ can be as close to 1 as you want, if n is big enough, WLLN!

Follows from Chebchev + var

$$E[A_n] = E\left[\frac{R_1 + R_2 + \ldots + R_n}{n}\right]$$

$$= \frac{E[R_1] + E[R_2] + \ldots + E[R_n]}{n}$$

$$= \frac{n\mu}{n}$$

$$= \mu$$

(8)

So by Chebyshev

$$P(|A_n - \mu| > \delta) \leq \frac{Var(A_n)}{\delta^2}$$

need only show

$Var(A_n) \to 0$ as $n \to \infty$

Repeated trials

$$Var[A_n] = Var\left[\frac{(R_1 + R_2 + \dots + R_n)}{n}\right]$$

$$= \frac{(Var[R_1] + Var[R_2] + \dots + Var[R_n])}{n^2}$$

$$= \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}$$ ← var of original RV r

Proves WLLN ?

- same mean
- " var
- their var add

9

$$P(|A_n - \mu| \geq \delta) \leq \frac{1}{n}\left(\frac{\sigma}{\delta}\right)^2$$

We now ~~kn~~ now how big a sample needs to be in order to estimate

---

Birthday Pairs

D = # matching b-days of n- people d-day year

$$= \sum_{1 \leq i \leq j \leq n} M_{ij}$$ indicator ith jth match

So $E[M_{ij}] = \frac{1}{d}$ by linearity of $E[\,]$

each persons bday

$$E[D] = \sum_{1 \leq i \leq j \leq n} E[M_{ij}] = \binom{n}{2}\frac{1}{d}$$

10

Birthdays are not really evenly distributed
Pretend it is uniform

$E[D] = \binom{91}{2} \frac{1}{365} = 11.2$

matching birthdays

M_{ij} are pairwise ind

$var(M_{ij}) = \frac{1}{365} \cdot \frac{1}{365}$

$var(D) = \sum var(M_{ij})$

...

So $var(M) \approx 11.1$
is < 4

So Chebshev

$p(11.2 \pm 20 \text{ pairs}) > 1 - \left(\frac{1}{2}\right)^2 = 3/4$

b/w 4 and 20

will occur $\geq \frac{3}{4}$ of the time

He found 13 matches

In-Class Problems Week 13, Mon.

Problem 1.

A herd of cows is stricken by an outbreak of *cold cow disease*. The disease lowers the normal body temperature of a cow, and a cow will die if its temperature goes below 90 degrees F. The disease epidemic is so intense that it lowered the average temperature of the herd to 85 degrees. Body temperatures as low as 70 degrees, **but no lower**, were actually found in the herd.

(a) Prove that at most 3/4 of the cows could have survived.

Hint: Let T be the temperature of a random cow. Make use of Markov's bound.

(b) Suppose there are 400 cows in the herd. Show that the bound of part (a) is best possible by giving an example set of temperatures for the cows so that the average herd temperature is 85, and with probability 3/4, a randomly chosen cow will have a high enough temperature to survive.

Problem 2.

A gambler plays 120 hands of draw poker, 60 hands of black jack, and 20 hands of stud poker per day. He wins a hand of draw poker with probability 1/6, a hand of black jack with probability 1/2, and a hand of stud poker with probability 1/5.

(a) What is the expected number of hands the gambler wins in a day?

(b) What would the Markov bound be on the probability that the gambler will win at least 108 hands on a given day? $\frac{1}{2}$

(c) Assume the outcomes of the card games are pairwise independent. What is the variance in the number of hands won per day?

(d) What would the Chebyshev bound be on the probability that the gambler will win at least 108 hands on a given day? You may answer with a numerical expression that is not completely evaluated.

Problem 3.

The proof of the Pairwise Independent Sampling Theorem 18.5.1 was given for a sequence $R_1, R_2, \ldots$ of pairwise independent random variables with the same mean and variance.

The theorem generalizes straighforwardly to sequences of pairwise independent random variables, possibly with *different* distributions, as long as all their variances are bounded by some constant.

Theorem (Generalized Pairwise Independent Sampling). *Let $X_1, X_2, \ldots$ be a sequence of pairwise independent random variables such that* $\text{Var}[X_i] \le b$ *for some $b \ge 0$ and all $i \ge 1$. Let*

$$A_n ::= \frac{X_1 + X_2 + \cdots + X_n}{n},$$
$$\mu_n ::= \text{Ex}[A_n].$$

Then for every $\epsilon > 0$,

$$\Pr[|A_n - \mu_n| > \epsilon] \le \frac{b}{\epsilon^2} \cdot \frac{1}{n}. \tag{1}$$

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(a) Prove the Generalized Pairwise Independent Sampling Theorem.

(b) Conclude that the following holds:
Corollary (Generalized Weak Law of Large Numbers). *For every* $\epsilon > 0$,

$$\lim_{n\to\infty} \Pr[|A_n - \mu_n| \le \epsilon] = 1.$$

Problem 4.
For any random variable, R, with mean, μ, and standard deviation, σ, the Chebyshev Bound says that for any real number $x > 0$,

$$\Pr[|R - \mu| \ge x] \le \left(\frac{\sigma}{x}\right)^2.$$

Show that for any real number, μ, and real numbers $x \ge \sigma > 0$, there is an R for which the Chebyshev Bound is tight, that is,

$$\Pr[|R| \ge x] = \left(\frac{\sigma}{x}\right)^2. \tag{2}$$

Hint: First assume $\mu = 0$ and let R only take values $0, -x$, and x.

Pairwise Independent Sampling

Let R be a random variable, and a a constant. Then

$$\text{Var}[aR] = a^2 \, \text{Var}[R]. \tag{3}$$

Theorem (Pairwise Independent Sampling). *Let $G_1, \ldots, G_n$ be pairwise independent variables with the same mean, μ, and deviation, σ. Define*

$$S_n ::= \sum_{i=1}^{n} G_i.$$

Then

$$\Pr[\left|\frac{S_n}{n} - \mu\right| \geq x] \leq \frac{1}{n}\left(\frac{\sigma}{x}\right)^2.$$

Proof.

$$\begin{aligned}
\text{Ex}\left[\frac{S_n}{n}\right] &= \text{Ex}\left[\frac{\sum_{i=1}^{n} G_i}{n}\right] && \text{(def of } S_n) \\
&= \frac{\sum_{i=1}^{n} \text{Ex}[G_i]}{n} && \text{(linearity of expectation)} \\
&= \frac{\sum_{i=1}^{n} \mu}{n} \\
&= \frac{n\mu}{n} = \mu.
\end{aligned}$$

$$\begin{aligned}
\text{Var}\left[\frac{S_n}{n}\right] &= \left(\frac{1}{n}\right)^2 \text{Var}[S_n] && \text{(by (3))} \\
&= \frac{1}{n^2} \text{Var}\left[\sum_{i=1}^{n} G_i\right] && \text{(def of } S_n) \\
&= \frac{1}{n^2} \sum_{i=1}^{n} \text{Var}[G_i] && \text{(pairwise independent additivity)} \\
&= \frac{1}{n^2} \cdot n\sigma^2 = \frac{\sigma^2}{n}. && \text{(4)}
\end{aligned}$$

This is enough to apply Chebyshev's Theorem and conclude:

$$\begin{aligned}
\Pr\left|\frac{S_n}{n} - \mu\right| \geq x &\leq \frac{\text{Var}\left[S_n/n\right]}{x^2}. && \text{(Chebyshev's bound)} \\
&= \frac{\sigma^2/n}{x^2} && \text{(by (4))} \\
&= \frac{1}{n}\left(\frac{\sigma}{x}\right)^2.
\end{aligned}$$

■

In-Class 13 Mon

3/2

1. Cold cow disease

max $= 90°$

avg $85°$

lowest $70°$

So markov inequality w/ improvement

$T =$ temp of random cow

$\cancel{\leq \underline{E[(R-\mu)^2]}}$

$P(T - 70 \geq 90 - 70)$

$$\leq \frac{85-70}{90-70} = \frac{15}{20} = \frac{3}{4}$$

not really going off formula

b) 400 cows in heard

Show w/ examples w/ $P(\cancel{\ }) = \frac{3}{4}$ a randomly chosen cow will survive

(2)

Yeah that is some property I forget name of

~~BB~~ But why show an example

Arrange cows so max will die

~~B~~ 300 at 90°
100 at 70° $\updownarrow$ have to split ~~bottom~~ ~~so~~ to the 2 extreams

2. 120 hands of poker
60 bj
20 stud poker

P(~~##~~ wins | poker) = $\frac{1}{6}$

$\frac{1}{2}$

$\frac{1}{5}$

a) E[# wins] = ?

Conditional Expectations

$$E[W] = \frac{1}{6} \cdot \frac{120}{200} + \frac{1}{2} \cdot \frac{60}{200} + \frac{1}{5} \cdot \frac{20}{200}$$

$\bar{0}.27$ ↑ why ?!

54

~~b)~~ ↑ frac hands correct
not # hands correct - trying to be too smart

(3)

b) ~~Van~~ Markov bound   108 hands

$$P(\text{wins} \geq 108) \leq \frac{54}{108}$$

$$\frac{1}{2}$$

c) Pairwise ind

Var = ?

$$E[R^2] - E[R]^2$$

$$\underset{?}{E[R^2]} \qquad \underset{54^2}{E[R]^2}$$

$$\left(120 \cdot \tfrac{1}{6}\right)^2 + \left(60 \cdot \tfrac{1}{2}\right)^2 + \left(20 \cdot \tfrac{1}{3}\right)^2$$

if pairwise ind, the var just adds

Solutions to In-Class Problems Week 13, Mon.

Problem 1.
A herd of cows is stricken by an outbreak of *cold cow disease*. The disease lowers the normal body temperature of a cow, and a cow will die if its temperature goes below 90 degrees F. The disease epidemic is so intense that it lowered the average temperature of the herd to 85 degrees. Body temperatures as low as 70 degrees, **but no lower**, were actually found in the herd.

(a) Prove that at most 3/4 of the cows could have survived.

Hint: Let T be the temperature of a random cow. Make use of Markov's bound.

Solution. Let T be the temperature of a random cow. Then the fraction of cows that survive is the probability that $T \geq 90$, and $\text{Ex}[T]$ is the average temperature of the herd.

Applying Markov's Bound to T:

$$\Pr[T \geq 90] =\leq \frac{\text{Ex}[T]}{90} = \frac{85}{90} = \frac{17}{18}.$$

But $17/18 > 3/4$, so this bound is not good enough.

Instead, we apply Markov's Bound to $T - 70$:

$$\Pr[T \geq 90] = \Pr[T - 70 \geq 20] \leq \frac{\text{Ex}[T - 70]}{20} = (85 - 70)/20 = 3/4.$$

■

(b) Suppose there are 400 cows in the herd. Show that the bound of part (a) is best possible by giving an example set of temperatures for the cows so that the average herd temperature is 85, and with probability 3/4, a randomly chosen cow will have a high enough temperature to survive.

Solution. Let 100 cows have temperature 70 degrees and 300 have 90 degrees. So the probability that a random cow has a high enough temperature to survive is exactly 3/4. Also, the mean temperature is

$$(1/4)70 + (3/4)90 = 85.$$

So this distribution of temperatures satisfies the conditions under which the Markov bound implies that the probability of having a high enough temperature to survive cannot be larger than 3/4. ■

Problem 2.
A gambler plays 120 hands of draw poker, 60 hands of black jack, and 20 hands of stud poker per day. He wins a hand of draw poker with probability 1/6, a hand of black jack with probability 1/2, and a hand of stud poker with probability 1/5.

(a) What is the expected number of hands the gambler wins in a day?

Solution. $120(1/6) + 60(1/2) + 20(1/5) = 54.$ ■

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(b) What would the Markov bound be on the probability that the gambler will win at least 108 hands on a given day?

Solution. The expected number of games won is 54, so by Markov, $\Pr[R \geq 108] \leq 54/108 = 1/2$. ■

(c) Assume the outcomes of the card games are pairwise independent. What is the variance in the number of hands won per day?

Solution. The variance can also be calculated using linearity of variance. For an individual hand the variance is $p(1-p)$ where p is the probability of winning. Therefore the variance is

$$120(1/6)(5/6) + 60(1/2)(1/2) + 20(1/5)(4/5) = 523/15 = 34\,\tfrac{13}{15}.$$

■

(d) What would the Chebyshev bound be on the probability that the gambler will win at least 108 hands on a given day? You may answer with a numerical expression that is not completely evaluated.

Solution.

$$\Pr[R \geq 108] = \Pr[R - 54 \geq 54] \leq \Pr[|R - 54| \geq 54] \leq \frac{\text{Var}[R]}{54^2} = \frac{523}{15(54)^2} \approx 0.01196.$$

■

Problem 3.
The proof of the Pairwise Independent Sampling Theorem 18.5.1 was given for a sequence $R_1, R_2, \ldots$ of pairwise independent random variables with the same mean and variance.

The theorem generalizes straighforwardly to sequences of pairwise independent random variables, possibly with *different* distributions, as long as all their variances are bounded by some constant.

Theorem (Generalized Pairwise Independent Sampling). *Let $X_1, X_2, \ldots$ be a sequence of pairwise independent random variables such that $\text{Var}[X_i] \leq b$ for some $b \geq 0$ and all $i \geq 1$. Let*

$$A_n ::= \frac{X_1 + X_2 + \cdots + X_n}{n},$$
$$\mu_n ::= \text{Ex}[A_n].$$

Then for every $\epsilon > 0$,

$$\Pr[|A_n - \mu_n| > \epsilon] \leq \frac{b}{\epsilon^2} \cdot \frac{1}{n}. \tag{1}$$

(a) Prove the Generalized Pairwise Independent Sampling Theorem.

Solution. Essentially identical to the proof of Theorem 18.5.1 in the text, except that G gets replaced by X and $\text{Var}[G_i]$ by b, with the equality where the b is first used becoming $\leq$. ■

(b) Conclude that the following holds:
Corollary (Generalized Weak Law of Large Numbers). *For every $\epsilon > 0$,*

$$\lim_{n\to\infty} \Pr[|A_n - \mu_n| \leq \epsilon] = 1.$$

Solution.

$$\begin{aligned}\Pr[|A_n - \mu_n| \le \epsilon] &= 1 - \Pr[|A_n - \mu_n| > \epsilon] \\ &\ge 1 - b/(n\epsilon^2) \qquad \text{(by (1))},\end{aligned}$$

and for any fixed ϵ, this last term approaches 1 as n approaches infinity. ■

Problem 4.
For any random variable, R, with mean, μ, and standard deviation, σ, the Chebyshev Bound says that for any real number $x > 0$,

$$\Pr[|R - \mu| \ge x] \le \left(\frac{\sigma}{x}\right)^2.$$

Show that for any real number, μ, and real numbers $x \ge \sigma > 0$, there is an R for which the Chebyshev Bound is tight, that is,

$$\Pr[|R| \ge x] = \left(\frac{\sigma}{x}\right)^2. \tag{2}$$

Hint: First assume $\mu = 0$ and let R only take values $0, -x$, and x.

Solution. From the hint, we aim to find an R with $\text{Ex}[R] = 0$ and $\text{Var}[R] = \sigma^2$ that satisfies equation (2).
Using the further hint that R takes only values $0, -x, x$, we have

$$0 = \text{Ex}[R] = x \Pr[R = x] - x \Pr[R = -x] = x\,(\Pr[R = x] - \Pr[R = -x])$$

so

$$\Pr[R = x] = \Pr[R = -x], \tag{3}$$

since $x > 0$. Also,

$$\sigma^2 = \text{Ex}[R^2] = x^2 \Pr[R = -x] + x^2 \Pr[R = x] = 2x^2 \Pr[R = x],$$

so

$$\Pr[R = x] = \frac{\sigma^2}{2x^2}.$$

This implies

$$\Pr[R = 0] = 1 - 2\Pr[R = x] = 1 - \left(\frac{\sigma}{x}\right)^2,$$

which completely determines the distribution of R. Moreover,

$$\Pr[|R| \ge x] = \Pr[R = -x] + \Pr[R = x] = 2\Pr[R = x] = \left(\frac{\sigma}{x}\right)^2$$

which confirms (2).

Finally, given μ, x, and σ, if we let $R' ::= R + \mu$, then R' will be the desired random variable for which the Chebyshev Bound is tight. ■

Pairwise Independent Sampling

Let R be a random variable, and a a constant. Then

$$\operatorname{Var}[aR] = a^2 \operatorname{Var}[R]. \tag{4}$$

Theorem (Pairwise Independent Sampling). *Let $G_1, \dots, G_n$ be pairwise independent variables with the same mean, μ, and deviation, σ. Define*

$$S_n ::= \sum_{i=1}^{n} G_i.$$

Then

$$\Pr[\left|\frac{S_n}{n} - \mu\right| \geq x] \leq \frac{1}{n}\left(\frac{\sigma}{x}\right)^2.$$

Proof.

$$\begin{aligned}
\operatorname{Ex}\left[\frac{S_n}{n}\right] &= \operatorname{Ex}\left[\frac{\sum_{i=1}^n G_i}{n}\right] && \text{(def of } S_n\text{)}\\
&= \frac{\sum_{i=1}^n \operatorname{Ex}[G_i]}{n} && \text{(linearity of expectation)}\\
&= \frac{\sum_{i=1}^n \mu}{n}\\
&= \frac{n\mu}{n} = \mu.
\end{aligned}$$

$$\begin{aligned}
\operatorname{Var}\left[\frac{S_n}{n}\right] &= \left(\frac{1}{n}\right)^2 \operatorname{Var}[S_n] && \text{(by (4))}\\
&= \frac{1}{n^2}\operatorname{Var}\left[\sum_{i=1}^n G_i\right] && \text{(def of } S_n\text{)}\\
&= \frac{1}{n^2}\sum_{i=1}^n \operatorname{Var}[G_i] && \text{(pairwise independent additivity)}\\
&= \frac{1}{n^2}\cdot n\sigma^2 = \frac{\sigma^2}{n}. && (5)
\end{aligned}$$

This is enough to apply Chebyshev's Theorem and conclude:

$$\begin{aligned}
\Pr\left|\frac{S_n}{n} - \mu\right| \geq x &\leq \frac{\operatorname{Var}[S_n/n]}{x^2}. && \text{(Chebyshev's bound)}\\
&= \frac{\sigma^2/n}{x^2} && \text{(by (5))}\\
&= \frac{1}{n}\left(\frac{\sigma}{x}\right)^2.
\end{aligned}$$

■

Miniquiz 6 5/3

Should do better on this
- review / read for 1st time! book

Mon was binomial + combitorial proofs
Is the stuff in between on the exam?

Have more time - will also read PPT + problems
instead of starting to study 1 hr before bed
Perhaps do a few problems
10W3

10W3 Combintorial proof
S = set of all lengnt n-seq 0, 1, one *

Own So 2^n is set lenght n-seq 0, 1

2^{n-1} remove for star 'n for star anywhere

$\sum_{k=1}^{n} k\binom{n}{k}$ So $1 \cdot \cancel{\binom{n}{1}} + 2\binom{n}{2} + 3\binom{n}{3} \cdots + n\cancel{\binom{n}{n}}\ 1$
n

? What is this?

②

Number of 1s

So all the possibilities $\binom{n}{i}$ where 1 can be

i but what is other side

Look

<u>bij</u> ← use this trick to actually show relationship

Let $P = \{0, \ldots, n-1\} \times \{0,1\}^{n-1}$

bij P to S map (k, x) → so but *

after kth bit

↑ a more math way of what I said

but use this more math way of talking

Every seq contains b/w 1 and n non ~~0~~ zero entries

Oh so ( ) is # <u>non zero</u> and then

the k is where the * is

I almost had it!

(3)

b) Now prove (1) w/ Binomial theorm to $(1+x)^n$ + tak deriv

Try $\sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$

$\uparrow 1$ $\uparrow$ anything $= 1$

$\sum_{k=0}^{n} \binom{n}{k} x^k$

; take deriv, look

$n(1+x)^{n-1} = \sum_{k=0}^{n} k \binom{n}{k} x^{k-1}$

$= \frac{1}{x} \sum_{k=0}^{n} k \binom{n}{k} x^k$

Letting $x = 1$ in (3) yields (1)

? what does this show?

- don't think I remember how to take deriv $(1+x)^n$

oh $n(1+x)^{n-1}$

binom - don't do anything

? why

well deriv w/ respect to what

constant in front

$x^{k-1} = x^k x^{-1} = \frac{x^k}{x}$

4

Got to know your basic math stuff!

Work through def of Expectation 17.4.4

R is defed in S

$$E[R] = \sum_{w \in S} R(w)\,P(w) \qquad \text{def}$$

$$= \sum_{x \in range(R)} \sum_{w \in [R=x]} R(w)\,P(w) \qquad \text{open/dist sins}$$

$$= \sum_{x \in Range(R)} \sum_{w \in R(x)} x\,R(w) \qquad \text{(def of event } R=x\text{)}$$

$$= \sum_{x \in Range(R)} x \left(\sum_{w \in [R=x]} P(w) \right) \qquad \text{dist. over inner sum}$$

$$= \sum_{x \in range(R)} x\,P[R=x] \qquad \text{def } P[R=x]$$

did not follow

have to realy understand what is going on

(5)

Part of proof: showing $\sum_{s \in S} P(s) = 1$

$$= \sum_{n \geq 0} \sum_{|s|=2n+2} P(s)$$

$$= \sum_{n \geq 0} (p^2 + (1-p)^2)^n (2p(1-p))$$

↑ can occur as often as it needs to

$$= 2p(1-p) \sum_{n \geq 0} (p^2 + (1-p)^2)^n$$

↑ add up all possible values of n

$$= \frac{2 \cdot p(1-p)}{1 - (p^2 + (1-p)^2)}$$

← expanding sum from prior class

$$= \frac{2p(1-p)}{2p - 2p^2}$$

$$= 1$$

6

Try copying some proofs to learn

If $A \subseteq B$ $P(A) \leq P(B)$

$$P(A) = P(B) - (P(B) - P(A))$$

$$= P(B) - (P(B) - P(A \cap B))$$

$$= P(B) - P(B - A) \quad \text{diff rule}$$

$$\leq P(B) \quad \text{minus something?}$$

all these seem kinda weak

but I should be able to crank out!

Check for events being ind - like I had done

$$P(A \cap \bar{B}) = P(A) - P(A \cap B)$$

$$= P(A) - P(A) \cdot P(B)$$

$$= P(A)(1 - P(B))$$

$$= P(A) \cdot P(\bar{B})$$

Oh neat

Just need to see

Still kinda glossed over heavy math/proof stuff

6.042 Cheat Sheet 6

5/3

Binomial Theorem

binomial = sum of 2 terms $a+b$

One term for each seq of a, b

terms is $\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$ (bookkeeper)

Can expand terms to

$$\sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^{k}$$

Can have **multinomial** for all $n \in N$

$$\sum_{\substack{k_1, \ldots, k_m \in N \\ k_1 + k_2 + \ldots + k_m = n}} \binom{n}{k_1, k_2, \ldots, k_m} z_1^{k_1} z_2^{k_2} \cdots$$

Pascal's Identity

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

Combinatorial Proof tell a story

1. Define a set S
2. Show $|S| = n$ by 1 way
3. Show $|S| = m$ by other way
4. Conclude $n = m$

Probability

Draw a tree!

outcome

events = set of outcomes

4 step method

1. Find the sample space
   - draw the tree
2. Find the events of interest
   - set of outcomes we are looking at
3. Determine outcome probabilities
   - multiply along tree
4. Compute event probabilities
   - add up the outcomes

For any $n \geq 2$ there is a set of n dice, for any n-node diagraph w/ exactly 1 directed edge b/w every 2 distinct nodes, there is a # of rolls k s.t. the sum of k rolls of the ith die is $>$ sum for jth die w/ $P(>) \geq \frac{1}{2}$ iff edge $i \to j$ in graph

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$P[w] \geq 0$ for all $w \in S$

$$\sum_{w \in S} P(w) = 1$$

For event $E \subseteq S$ $P(E) = \sum_{w \in E} P(w)$ Uniform $P(E) = \frac{|E|}{|S|}$

Sum Rule

$$P\left(\bigcup_{n \in N} E_n\right) = \sum_{n \in N} P(E_n) \quad \text{disjoint}$$

Complement Rule $P(\overline{A}) = 1 - P(A)$

Diff Rule $P(B - A) = P(B) - P(A \cap B)$

In-Ex $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Boole's Ineq $P(A \cup B) \leq P(A) + P(B)$

Monotonicity If $A \subseteq B$ then $P(A) \leq P(B)$

Union Bound $P(E_1 \cup E_2 \cup \ldots \cup E_n) \leq P(E_1) + \ldots + P(E_n)$

∞ Prob Space same ∞ sums as before

$S\{T^n H \mid n \in N\}$ $\quad P(T^n H) = \frac{1}{2^{n+1}}$

Conditional $P(A \mid B) = \frac{P(A \cap B)}{P(B)}$

Product Rule $P(A \cap B) = P(B \mid A)\, P(A)$

Always draw tree!

Law of Total Prob $P(A) = P(A \mid E) \cdot P(E) + P(A \mid \overline{E}) P(\overline{E})$

$$= \sum_{i=1}^{n} P(A \mid E_i) P(E_i)$$

Independence $P(A \mid B) = P(A)$

If disjoint $\Rightarrow$ know **not** ind trick!

$P(A \cap B) = P(A) \cdot P(B)$

$$P\left(\bigcap_{j \in S} E_j\right) = \prod_{j \in S} P(E_j)$$

k-wise ind if every set k ind

pairwise ind = 2-wise ind

mutually ind = all subsets ind

$2^n - (n+1)$ to check

Random Variables map outcomes to #

Bernoulli = if 1 or 0

Independence $P(C = x_1 \text{ AND } M = x_2) = P(C = x_1) \cdot P(M = x_2)$

Two events ind if indicator variables ind

PDF

$$PDF_R(x) = \begin{cases} P(R = x) & \text{if } x \in range(R) \\ 0 & \text{if } x \notin range(R) \end{cases}$$

$$\sum_{x \in Range} PDF_R(x) = 1$$

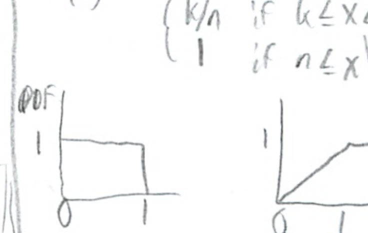
$$CDF_R(x) = P(R \leq x) = \sum_{y \leq x} P(R = y) = \sum_{y \leq x} PDF_R(y)$$

Binomial $f_p(0) = p$ $\quad f_p(1) = 1 - p$

$$F_p(x) = \begin{cases} 0 & \text{if } x < 0 \\ p & \text{if } 0 \leq x < 1 \\ 1 & \text{if } 1 \leq x \end{cases}$$

Uniform $f: V \to [0,1]$ $f(v) = \frac{1}{n}$ for all $v \in V$

$$F(x) = \begin{cases} 0 & \text{if } x < 1 \\ k/n & \text{if } k \leq x < k+1 \text{ for } 1 \leq k < n \\ 1 & \text{if } n \leq x \end{cases}$$

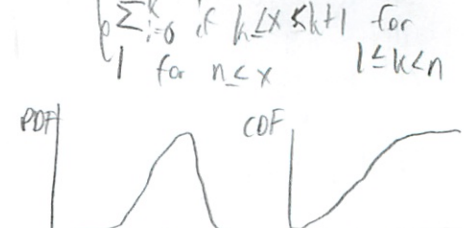


Binomial k # heads in n flips

$\binom{n}{k} 2^{-n}$

seq, prob of each seq

$$F(n) = \begin{cases} 0 & \text{if } x < 1 \\ \sum_{i=0}^{k} & \text{if } k \leq x \leq k+1 \text{ for } 1 \leq k < n \\ 1 & \text{for } n < x \end{cases}$$



General Binomial Dist Fn if coins biased $P(\text{heads}) = p$

$$f_{n,p} = \underbrace{\binom{n}{k}}_{\#\text{ seqs}} \underbrace{p^k (1-p)^{n-k}}_{\text{prob of each seq}}$$

$$F_{n,p}(x) = \begin{cases} 0 & \text{if } x < 1 \\ \sum_{i=0}^{k} \binom{n}{i} p^i (1-p)^{n-i} & \text{for } k \le x \le k+1 \text{ for } 1 \le k < n \\ 1 & \text{if } n \le x \end{cases}$$

Expectations aka mean or avg

$$E[R] = \sum_{w \in S} R(w)\, P(w)$$

weighted avg of value

$$E[\text{uniform}] = \sum_{i=1}^{n} i \cdot \frac{1}{n} = \frac{n(n+1)}{2n} = \frac{n+1}{2} = \frac{b+a}{2}$$

$$E[I_A] = p(A)$$

Indicator! 1 or 0

Median $P(R \le x) \le \frac{1}{2}$ and $p(R \ge x) < \frac{1}{2}$

Cond. Expectation Same as before

Law Total Ex $E[R] = \sum_i E[R|A_i] P(A_i)$

Mean Time to Failure

$$E[C] = 1 \cdot p + (1 + E[C])(1-p)$$
$$= p + 1 - p + (1-p) E[C]$$
$$= 1 + (1-p) E[C]$$

$$1 = E[C] - (1-p)E[C] = pE[x]$$
$$E[C] = \frac{1}{p}$$

In gambling - weight the payoffs

Linearity of Expectations

$$E[R_1 + R_2] = E[R_1] + E[R_2]$$
$$E[a_1 R_1 + a_2 R_2] = a_1 E[R_1] + a_2 E[R_2]$$

ind or not!

Sum of indicator RV to show 1 person gets hat back

Sum the prob of each event occuring

$$E[\text{Binomial}] = np$$

Coupon Collector

$P(\text{we have already}) = \frac{k}{n}$

So $P(\text{new}) = 1 - \frac{k}{n} = \frac{n-k}{n}$

So $E[\#\text{ meals till new}] = \frac{n}{n-k}$

Sum these up

$$E[T] = E[X_0] + E[X_1] + \dots E[X_{n-1}]$$
$$= \frac{n}{n-0} + \frac{n}{n-1} + \dots \frac{n}{3} + \frac{n}{2} + \frac{n}{1}$$
$$= n\left(\frac{1}{n} + \frac{1}{n-1} + \dots \frac{1}{3} + \frac{1}{2} + \frac{1}{1}\right)$$
$$= n\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n-1} + \frac{1}{n}\right)$$
$$= n H_n$$
$$\sim n \ln(n)$$

If ∞ sum converges

$$E\left[\sum_{i=0}^{\infty} R_i\right] = \sum_{i=0}^{\infty} E[R_i]$$

$E[R^2] = (E[R])^2$ only if ind

If A, B ind, $A, \bar{B}$ ind

$$P(A \cap \bar{B}) = P(A) - P(A \cap B)$$
$$= P(A) - P(A) \cdot P(B)$$
$$= P(A)(1 - P(B))$$
$$= P(A)\, P(\bar{B})$$

Mini-Quiz May 5

Your name: Michael Plasmeier

Circle the name of your TA and write your table number:

Ali Nick Oscar Oshani **Table number** 12

- This quiz is **closed book**. Total time is 30 minutes.
- Write your solutions in the space provided. If you need more space, write on the back of the sheet containing the problem. Please keep your entire answer to a problem on that problem's page.
- GOOD LUCK!

DO NOT WRITE BELOW THIS LINE

Problem	Points	Grade	Grader
1	6	1	OS
2	6	4	OM
3	4	1	NJ
4	4	4	AK
Total	20	10	

Avg = 12.5

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$\boxed{\begin{matrix} 1 & 2 \\ 3 & 4 \end{matrix}}$

Problem 1 (6 points).
Suppose there are 4 desks in a classroom, laid out in the corners of a square with corners 1 2 3 and 4.

Each desk is occupied by a male with probability $p > 0$ or a female with probability $q ::= 1 - p > 0$. A male and a female *flirt* when they occupy desks in adjacent corners of the square. Let $I_{12}, I_{23}, I_{34}, I_{41}$ be the indicator variables that there is a flirting couple at the indicated adjacent desks.

(a) Show that if $p = q$ then the events $I_{12} = 1$ and $I_{23} = 1$ are independent.

Each desk has a M, F determined independently w/ P() = p

0/2

So $P(I_{12} = 1) = \dfrac{2pq}{p^2 + 2pq + q^2}$ $P(I_{12} = 1 \text{ AND } I_{23} = 1) = P(I_{12} = 1) \cdot P(I_{23} = 1)$

$\dfrac{2p_1q_1}{p^2+2pq+q^2} \cdot \dfrac{2p_2q_2}{p^2+2pq+q^2} = \dfrac{2pq}{p^2+2pq+q^2} + \dfrac{2pq}{p^2+2pq+q^2}$

Same for other one I_{23}

Over

(b) Show rigorously that if the events $I_{12} = 1$ and $I_{23} = 1$ are independent then $p = q$. Hint: work from the definition of independence, set up an equation and solve.

$\dfrac{2pq}{p^2 + 2pq + q^2}$ Otherwise, would be bias - and would not be ind

If you have B G Then the third one can be B or G w/o bias of before, But if one gender is more biased,

0/2

then the two I_{12}, I_{23} are not as likely to be evenly split - will be biased

Two events ind iff indicator variables ind

(c) What is the expected number of flirting couples in terms of p and q?

$E = \dfrac{1}{n} = 2$

1/2

more rigorous

$$\frac{p_1 q_2 + p_2 q_1}{p_1 p_2 + p_1 q_2 + p_2 q_1 + q_1 q_2}$$

$p_1 = p_2$ as defined in problem

$q_1 = q_2$

So can simplify

One does not effect the other

Each child determined ind, means each couple picked independently

Problem 2 (6 points).

Consider the following 2 player game. A coin is tossed repeatedly. Turns alternate between the two players. The game stops after the first Heads come up. If the first time the coin came up Heads is during one of player 1's turns, player 1 wins. On the other hand, if the first time the coin came up Heads is during one of player 2's turns then player 2 wins.

Assume p = prob heads $= \frac{1}{2}$

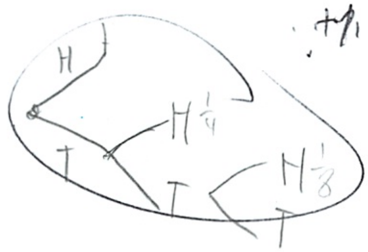
(a) What is the expected number of turns N until the game ends?

$S = p \cdot 1 + q(1+s)$

Mean time to failure $= \frac{1}{p} = \frac{1}{1/2} = 2$

(b) What is the probability p_1 that player 1 wins? (Hint: draw an event tree)

1 2 1 $+p_1$



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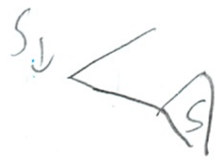


1 $p^1 + p^3 + p^5 + \ldots$

2 $p^2 + p^4 + p^6$

1 $\sum_{n = \text{odd}\#} p^n$

2 $\sum_{n \text{ even}\#} p^n$

→ over

(c) What is $\text{Ex}[N \mid 1]$, the expected number N of rounds in the game given player 1 wins? You can assume that the game ends with probability 1 and that $\text{Ex}[N \mid 2] = \text{Ex}[N \mid 1] + 1$. Hint: Law of total Expectation.

$E[N] = E[N \mid 1]\, P(1) + E(N \mid 2)\, P(2)$

$\frac{1}{p} = E[N \mid 1] \sum_{n \in \text{odd}} p^n + (E[N \mid 1] + 1) \sum_{n \in \text{even}\#} p^n$

+1

$E[N \mid 1] = \dfrac{(E[N \mid 1] + 1) \sum_{n = \text{even}\#} p^n}{\sum_{n = \text{odd}\#} p^n}$

$E(N \mid 1) = \dfrac{(E(N \mid 1) + 1)\, p_1}{p_2}$

Over

1) $\frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \dots$

2) $\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \dots$

forgot sums formula - not this unit

$$\frac{1}{p} = p_2 E[N|1] + p_1 E[N|1] + p_1$$

$$\frac{1}{p} = E[N|1](p_1 + p_2) + p_1$$

$$E[N|1] = \frac{\frac{1}{p} - p_1}{p_1 + p_2}$$

Problem 3 (4 points). (a) Write the term of $(x+y)^{40}$ which includes x^3.

$(x+y)^{40} = x^0 y^{40} + x^1 y^{39} + \dots + x^3 y^{37}$

you need the binom coefficients too though. study more

$\binom{40}{3} x^3 y^{37}$

(b) Write the term of $(x+y+z)^{40}$ which includes x^3y^5.

$x^3 y^5 z^{32}$

(c) Give a combinatorial proof that

$$\sum_{i=0}^{n} \binom{n}{i} = 2^n.$$

2^3

$\binom{3}{0} + \binom{3}{1} + \binom{3}{2} + \binom{3}{3}$

Hint: Begin by finding a set whose cardinality is equal to the right hand side of the equation.

RHS 2^n is the number of bit strings (either 1 or 0) of length n

good

LHS Is all of the possible combinations of 1s, $\frac{1}{2}$.

i is the number of 1s. The binomial term is the # of possible combinations of bit strings with i 1s. Add up all possible 1s.

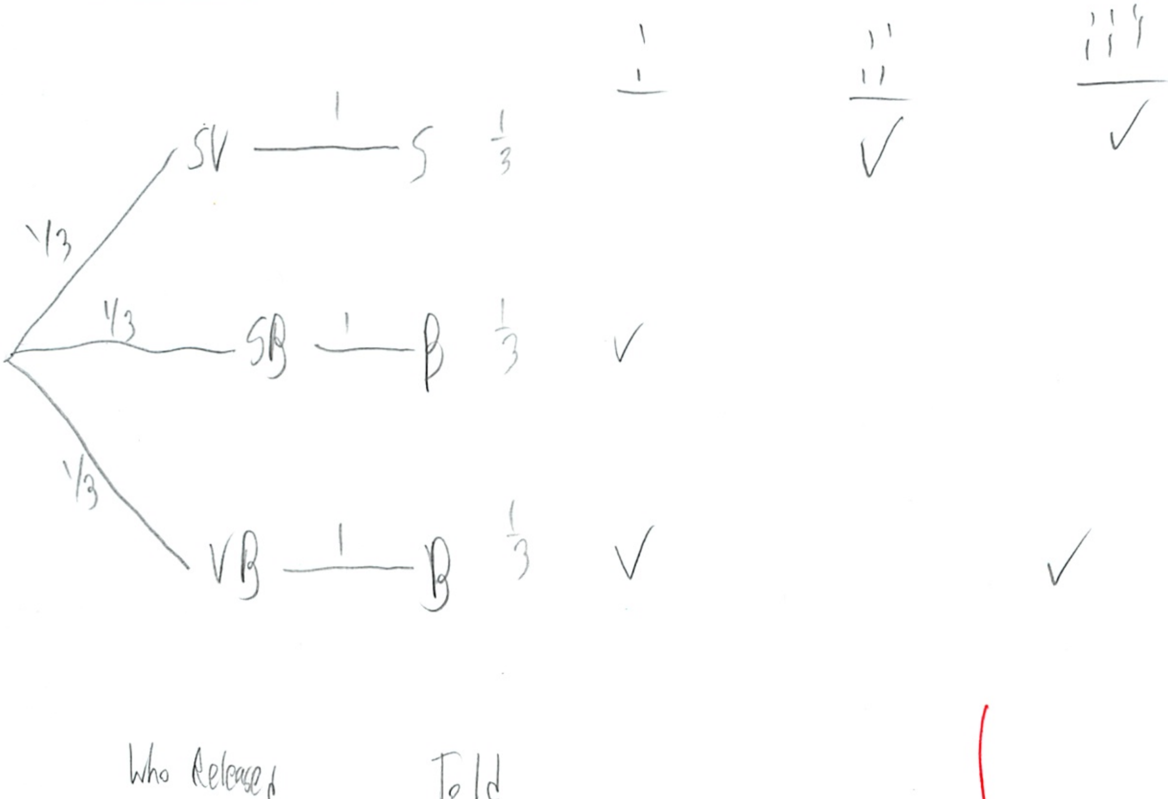
Conc The LHS = RHS, so both count the same thing.

seems like you have right idea but you say things ~~which~~ ~~understand~~ which don't have any clear meaning

S V B

Problem 4 (4 points).
We revisit Sauron, Voldemort, and Bunny Foo Foo as in the class problem. As before, the guard is going to release exactly two of the three prisoners; he's equally likely to release any set of two prisoners. The guard offers to tell Voldemort the name of one of the prisoners to be released. The guards rule for which name he chooses:
1. The guard will never say that Voldemort will be released.
2. If both Foo Foo and Sauron are getting released, the guard will always give Foo Foos name.
Were interested in which characters are released, and in which character the guard says will be released.

(a) Draw a tree to represent the sample space. Indicate, in your drawing, which outcomes correspond to the following events:
i. The guard tells Voldemort that Foo Foo will be released
ii. The guard tells Voldemort that Sauron will be released
iii. Voldemort is released



(b) What is the probability that Voldemort is released, given that the guard says Foo-foo will be released?

$$P(iii \mid i) = \frac{P(iii \cap i)}{P(i)} = \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{3} \cdot \frac{3}{2} = \frac{1}{2}$$

(c) What is the probability Voldemort is released, given that the guard says Sauron will be released?

$$P(\text{iii} \mid \text{ii}) = \frac{P(\text{iii} \cap \text{ii})}{P(\text{ii})} = \frac{\frac{1}{3}}{\frac{1}{3}} = 1$$

(d) Use the above calculations, and the Law of Total Probability, to find the total probability that Voldemort will be released.

$$P(\text{iii}) = P(\text{iii} \mid \text{i})\,P(\text{i}) + P(\text{iii} \mid \text{ii})\,P(\text{ii})$$

$$\frac{1}{2} \cdot \frac{2}{3} + 1 \cdot \frac{1}{3}$$

$$= \frac{2}{6} + \frac{1}{3}$$

$$= \frac{2}{3}$$

Solutions to Mini-Quiz May 4

Problem 1 (6 points).
Suppose there are 4 desks in a classroom, laid out in the corners of a square with corners 1 2 3 and 4.

Each desk is occupied by a male with probability $p > 0$ or a female with probability $q ::= 1 - p > 0$. A male and a female *flirt* when they occupy desks in adjacent corners of the square. Let $I_{12}, I_{23}, I_{34}, I_{41}$ be the indicator variables that there is a flirting couple at the indicated adjacent desks.

(a) Show that if $p = q$ then the events $I_{12} = 1$ and $I_{23} = 1$ are independent.

Solution. If $p = q = 1/2$ then $\Pr[I_{12} = 1] = \Pr[I_{23} = 1] = 1/2$ and $\Pr[I_{12} = 1\&\ I_{23} = 1]$ can be calculated from the fact that only F-M-F and M-F-M are possible when both couples are flirting. In that case, we have $\Pr[I_{12} = 1\&\ I_{23} = 1] = 2/8 = 1/4 = \Pr[I_{12} = 1] \cdot \Pr[I_{12} = 1]$. ■

(b) Show rigorously that if the events $I_{12} = 1$ and $I_{23} = 1$ are independent then $p = q$. Hint: work from the definition of independence, set up an equation and solve.

Solution. We can again compare $\Pr[I_{12} = 1\&\ I_{23} = 1]$ and $\Pr[I_{12} = 1] \cdot \Pr[I_{23} = 1]$.

As in the previous part, $I_{12} = 1\&\ I_{23} = 1$ only happen when we have a pattern of F-M-F or M-F-M for students 1 2 and 3 respectively. These occur with total probablity $p^2q + pq^2$. On the other hand, I_{12} happens with probability $2pq$ total, accouting for the two patterns possible, M-F and F-M. Hence, I_{12} and I_{23} are independent iff $p^2q + pq^2 = pq(p + q) = 4p^2q^2$. By manipulating the expression we get $p + q = 4pq$. Recall $p + q = 1$. Hence, we are dealing with $1 = 4p - 4p^2$. The equation can be factored into $(2p - 1)^2 = 0$, yielding $p = 1/2$. ■

(c) What is the expected number of flirting couples in terms of p and q?

Solution. The expected number of couples is $8pq$ by linearity of expectation. ■

Problem 2 (6 points).
Consider the following 2 player game. A coin is tossed repeatedly. Turns alternate between the two players. The game stops after the first Heads come up. If the first time the coin came up Heads is during one of player 1's turns, player 1 wins. On the other hand, if the first time the coin came up Heads is during one of player 2's turns then player 2 wins.

(a) What is the expected number of turns N until the game ends?

Solution. This is just mean time to failure (a Head), so by Lemma 17.4.8, the expected number of steps is $\text{Ex}[N] = 1/(1/2) = 2$. ■

(b) What is the probability p_1 that player 1 wins? *Hint:* draw an event tree.

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Solution. The tree can be described by $A = H_1 + T_1(H_2 + T_2 A)$. The probability of winning can be found via the law of total probability.

$p_1 = (1/2) \cdot 1 + (1/2)(1/2 \cdot 0 + 1/2 \cdot p_1)$

Hence $(3/4) \cdot p_1 = 1/2$, so $p_1 = 2/3$

■

(c) What is $\text{Ex}[N \mid 1]$, the expected number N of rounds in the game given player 1 wins? You can assume that the game ends with probability 1 and that $\text{Ex}[N \mid 2] = \text{Ex}[N \mid 1] + 1$. *Hint:* Law of total Expectation.

Solution. From the law of total expectation, we know $\text{Ex}[N] = \text{Ex}[N \mid 1]p_1 + \text{Ex}[N \mid 2]p_2$. Now we know $p_1 = 2/3$, $p_2 = 1/3$ and $\text{Ex}[N] = 2$ and the hint.

We get $(2/3 + 1/3)\,\text{Ex}[N \mid 1] = 2 - 1/3$ so $\text{Ex}[N \mid 1] = 5/3$. ■

Problem 3 (4 points). (a) Write the term of $(x + y)^{40}$ which includes x^3.

Solution.

$$\binom{40}{3} x^3 y^{37}.$$

■

(b) Write the term of $(x + y + z)^{40}$ which includes $x^3 y^5$.

Solution.

$$\binom{40}{3, 5, 32} x^3 y^5 z^{32}$$

■

(c) Give a combinatorial proof that

$$\sum_{i=0}^{n} \binom{n}{i} = 2^n.$$

Hint: Begin by finding a set whose cardinality is equal to the right hand side of the equation.

Solution. Count the number of n-length bit strings. For the LHS, we consider the ith term of the sum to represent the bit strings which have i zeros. ■

Problem 4 (4 points).
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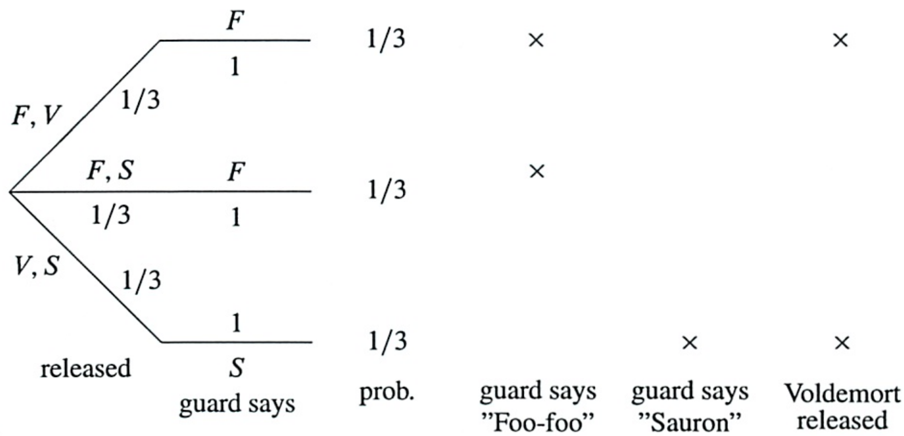


Solution. ■

(b) What is the probability that Voldemort is released, given that the guard says Foo-foo will be released?

Solution. $\frac{1}{2}$ ■

(c) What is the probability Voldemort is released, given that the guard says Sauron will be released?

Solution. 1 ■

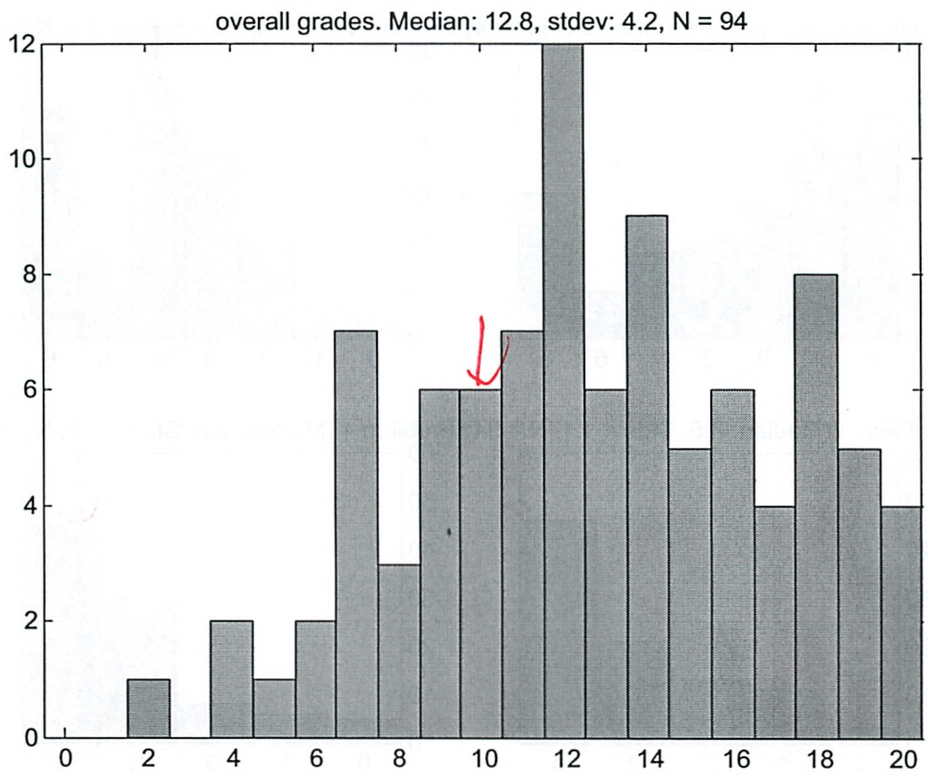
(d) Use the above calculations, and the Law of Total Probability, to find the total probability that Voldemort will be released.

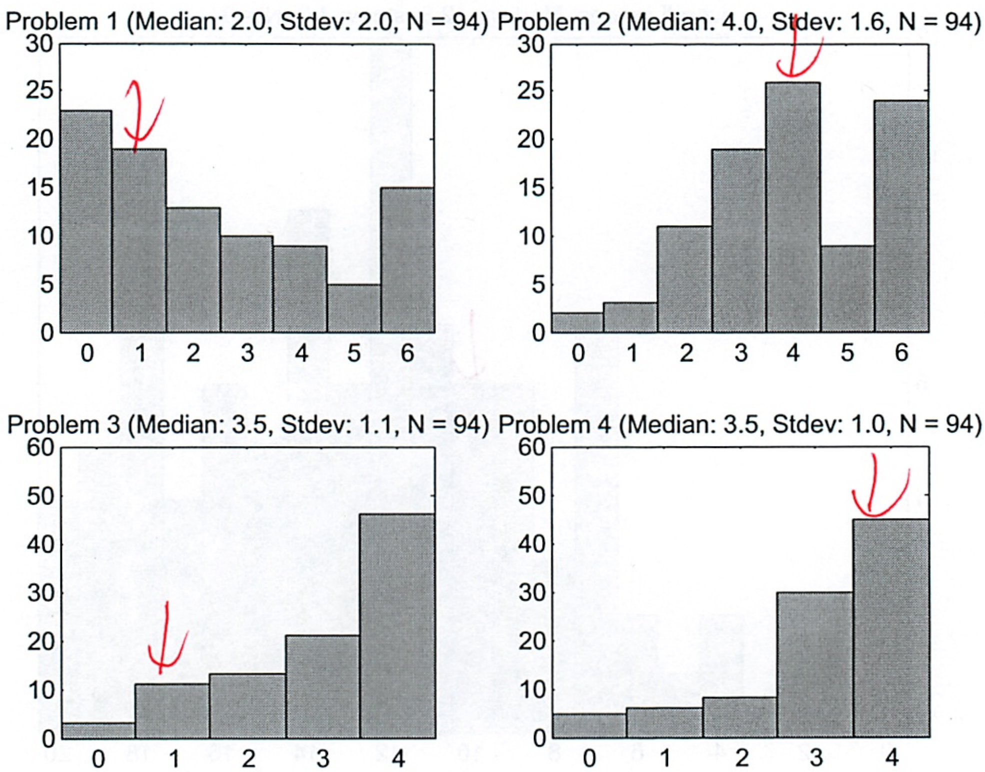
Solution. Still 2/3, by law of total probability.

$$\begin{aligned}\Pr[\texttt{V released}] &= \Pr\left[\texttt{V released} \mid \texttt{says foofoo}\right] \cdot \Pr[\texttt{says foofoo}] \\ &\quad + \Pr\left[\texttt{V released} \mid \texttt{says sauron}\right] \cdot \Pr[\texttt{says sauron}] \\ &=\end{aligned}$$

■

MQ 6




Problem 1 (Median: 2.0, Stdev: 2.0, N = 94)
Problem 2 (Median: 4.0, Stdev: 1.6, N = 94)
Problem 3 (Median: 3.5, Stdev: 1.1, N = 94)
Problem 4 (Median: 3.5, Stdev: 1.0, N = 94)